J. DIFFERENTIAL GEOMETRY
49 (1998) 1-74

FLOER HOMOLOGY AND ARNOLD CONJECTURE

GANG LIU & GANG TIAN

1. Introduction

Let V be a closed symplectic manifold with a symplectic form ω. This means that ω is a closed non-degenerate two-form. Because of the non-degeneracy of ω, with any time-dependent periodical Hamiltonian function $H : V \times S^1 \to \mathbf{R}$, we can associate a θ-dependent vector field X_{H_θ} given by:

$$\omega(X_{H_\theta}, \cdot) = dH_\theta,$$

where $\theta \in \mathbf{R}$ is the usual angular coordinate of S^1 and $H_\theta = H|_{S^1\times\{\theta\}}$. Consider the Hamiltonian equation:

$$\frac{dz}{d\theta} = X_{H_\theta}(z). \tag{0.1}$$

Let $\mathcal{P}(H)$ be the set of periodic-1 solutions of (0.1). Clearly $\mathcal{P}(H)$ is one to one correspondence to the set of fixed points of the time-1 flows ϕ_1^H of V associated to (0.1). For a "generic" choice of H, the graph $\Gamma_{\phi_1^H}$ of ϕ_1^H is transversal to the diagonal $\triangle_V$ in $V \times V$. It follows that $\mathcal{P}(H)$ is finite in this case. We refer this as a nondegenerate case. By the Lefschetz fixed point theorem, the algebraic cardinality of $\mathcal{P}(H)$ is just the Euler characteristic $\chi(V)$ of V, which is the alternating sum of the Betti number $b_i(V)$ of V. However, it has been conjectured by V.I. Arnold in [1] that the geometric cardinality of $\mathcal{P}(H)$ should satisfy a Morse inequality, $\#\mathcal{P}(H) \geq \sum_i b_i(V)$. This yields a much stronger estimate than what is expected by algebraic topology and reflects the remarkable symplectic rigidity (see [2] and [9]). This famous conjecture

Received September 24, 1996, and, in revised form, July 21, 1997.

has been a major driving force for the developments of various theory and techniques in symplectic topology and many special cases have been proved. The first breakthrough was made by Conley and Zehnder in 1982, who proved the conjecture for the torus T^{2n} with the standard symplectic structure. In the subsequent years, this result was extended by Floer and Sikorav to certain other quotients of $\mathbf{R}^n$, which include all the two dimensional orientable surfaces. When ϕ_1^H is C^0-close to the identity, the conjecture was proved to be true in general by Weinstein (See [21]). However, despite of various interesting results, the subject did not achieve a unified framework until the advent of Floer homology. In 1985, Gromov introduced the idea of using J-holomorphic curves in symplectic topology, which yields many important new results in the subject. In the very next year, combining the variational method previously used by Conley and Zehnder with the theory of J-holomorphic curves, Floer introduced his celebrated Floer homology theory for closed monotonic symplectic manifolds, and consequently proved the Arnold conjecture for this class of symplectic manifolds. Later on, this result was extended by Hofer and Salamon in [10] to semi-positive case with a mild extra restriction on the minimal Chern number, which includes all Calabi-Yau manifolds. Soon after that, this extra condition was removed by Ono in [17]. However, there were serious obstructions to extend the Floer homology, hence to prove the Arnold conjecture in general, because of the appearance of multiply covered J-holomorphic curves with negative first Chern class. Unlike the semi-positive case, the moduli spaces used in the general case to construct Floer chain complex are not compact any more. Their natural compactifcation may contain strata whose dimension may be greater than that of the moduli space itself. It was completely unclear whether or not a cohomology theory of Floer-type could be ever assembled from Hamiltonian systems because of these wild strata. One has to develop a new method of counting contributions from those strata in the boundary of the natural compactification of the moduli space, so that a cohomology theory of Floer-type can be well-defined and consequently, the Arnold conjecture can be proved. In fact, a similar difficulty also appeared in establishing a mathematical theory of quantum cohomology and GW-invariants beyond the scope of semi-positive symplectic manifolds.

Recent development in theory of GW-invariants casts new light on the subject and reveals the possibility to overcome the difficulty. In 1995, J. Li and the second author of this paper introduced the method of constructing virtual moduli cycles in the setting of algebraic geometry (see [12]). Their idea is to use the global two-term free resolutions of the deformation-obstruction complexes. Inspired by this, J. Li and the second author of this paper constructed virtual moduli cycles and defined G-W invariants for general symplectic manifolds in [13], while we developed in this paper a different method of constructing virtual moduli cycles in our dynamic setting of Hamiltonian system. As a consequence of this, we extended Floer (co-)homology to all symplectic manifolds without any positivity assumption and proved Arnold conjecture in general. One of the main techniques of this paper is the gluing of J-holomorphic curves for which the transversality may fall. Gluing of J-holomorphic curves under transversality assumption was developed before in [11] and [19]. The method we used in this paper was based on the work of the first author in [11]. The method in [19] can also be adapted here.

To motivate our construction, we first need to introduce some ideas and notations prevailed in previous Floer (co)homology theory.

Recall that the question of finding 1-periodic orbits of (0.1) has a variational formulation.

Let $\mathcal{L}$ be the space of contractible loops in V and $\widetilde{\mathcal{L}}$ be its universal covering with covering group $\pi_2(V)$. Each element $[z,w]$ of $\widetilde{\mathcal{L}}$ can be represented by a C^∞-map $w: D^2 \to V$ with boundary value $z = w|_{\partial D^2 = S^1}$. We denote this representation by (z,w).

However, we will introduce a weaker relation for the definition of $\mathcal{L}$, namely, we define $[z_1,w_1] \sim [z_2,w_2]$ if $z_1 = z_2$ and w_1 and w_2 are homologous to each other. Under this equivalence relation, we have $\mathcal{L}(V) = \widetilde{\mathcal{L}}(V)/\Gamma$, where Γ is the image of $\pi_2(V)$ under the Hurewicz map $\pi_2(V) \to H_2(V)$. The symplectic action functional $a_H : \widetilde{\mathcal{L}} \to \mathbf{R}$ is defined by

$$a_H([z,w]) = \int_{D^2} w^*\omega + \int_{S^1} H_\theta(z(\theta))d\theta.$$

The critical points of a_H are just those $[z,w]$ with z being the 1-periodic solution of (0.1). We will use $\tilde{\mathcal{P}}(H)$ to denote the set of critical points of a_H, which is just the "lifting" of $\mathcal{P}(H)$ in $\tilde{\mathcal{L}}(V)$.

Let J be a ω-compatible almost complex structure in the sense that for any $x, y \in T_vV, \omega(Jx,Jy) = \omega(x,y)$ and the symmetric bilinear form $g_J(x,y) = \omega(x,Jy)$ is positive on T_vV for any $v \in V$. Clearly, g_J is a J-invariant Riemannian metric on V, which, in turn, induces an L^2-metric on $\mathcal{L}(V)$. With respect to this metric, a gradient flow line of a_H is just a connecting orbit $f : \mathbf{R} \times S^1 \to V$ with bounded energy, satisfying the equation $\bar{\partial}_{J,H} f = 0$ and the limit condition along the ends of $\mathbf{R} \times S^1$, namely, $\lim_{s \to \pm\infty} f(s, \theta) = z^{\pm}(\theta)$, where $\bar{\partial}_{J,H} f \in \Gamma(\wedge^{0,1}(f^* TV))$ is given by

$$\bar{\partial}_{J,H} f(\frac{\partial}{\partial s}) = \frac{\partial f}{\partial s} + J(f) \frac{\partial f}{\partial \theta} + \nabla_x H(f, \theta)$$

and $z^{\pm} \in \mathcal{P}(H)$. We will use $\widetilde{\mathcal{M}}^{D_T}(J, H; \tilde{z}^-, \tilde{z}^+)$ to denote the space of the connecting orbits defined as above with $\tilde{z}^- \# f = \tilde{z}^+$. Now the energy $E(f)$ of f is defined by

$$E(f) = \frac{1}{2} \int_{S^1} \int_{\mathbf{R}} (|\frac{\partial f}{\partial s}|^2 + |\frac{\partial f}{\partial \theta} - X_{H_\theta}(f)|^2) ds d\theta,$$

and any element $f \in \widetilde{\mathcal{M}}^{D_T}(J, H; \tilde{z}^-, \tilde{z}^+)$ has a fixed energy

$$E(f) = a_H(\tilde{z}^+) - a_H(\tilde{z}^-).$$

Let

$$\mathcal{M}^{D_T}(J, H; \tilde{z}^-, \tilde{z}^+) = \widetilde{\mathcal{M}}^{D_T}(J, H; \tilde{z}^-, \tilde{z}^+)/\mathbf{R}$$

be the moduli space of unparametrized connecting orbits, where $\mathbf{R}$ acts on $\widetilde{\mathcal{M}}^{D_T}(J, H; \tilde{z}^-, \tilde{z}^+)$ by s-translations.

For a "generic" choice of (J, H), $\widetilde{\mathcal{M}}^{D_T}(J, H; \tilde{z}^-, \tilde{z}^+)$ is a smooth manifold of dimension $\mu(\tilde{z}^+) - \mu(\tilde{z}^-)$, where $\mu : \tilde{\mathbf{P}}(H) \to \mathbf{Z}$ is the Conley-Zehnder index.

With such data one can attempt to develop a Morse theory for a_H to get an estimate on $\#\mathbf{P}(H)$. The "classical" Floer cohomology is just a such device constructed for some ideal situations, such as in the case of semi-positive symplectic manifolds.

The idea is to construct a chain complex $(C^*(H), \delta_{J,H})$, whose homology $H^*(C^*(H), \delta_{J,H})$ is isomorphic to $H^*(V)$, in such a way that $C^*(H)$ is generated by the elements of $\tilde{\mathbf{P}}(H)$ as a $\mathbf{Q}$-vector space, and the coboundary operator $\delta_{J,H}$ is defined by "counting" the number of discrete connecting orbits. More precisely, we define $C^*(H) = \oplus_k C^k(H)$, and any element $\xi \in C^k(H)$ is a formal sum $\xi = \sum_{\mu(\tilde{z})=k} \xi_{\tilde{z}} \cdot \tilde{z}$ with $\xi_{\tilde{z}} \in \mathbf{Q}$, such that for any $c > 0$,

$$\#\{\tilde{z} \,|\, \xi_{\tilde{z}} \neq 0, \quad a_H(\tilde{z}) \leq c\} < \infty.$$

In general $C^*(H)$ is of course infinite dimensional over $\mathbf{Q}$, but it is a finite dimensional vector space over the Novikov ring $\wedge_\omega$, which is a field in our case (see the relative definition in Section 5). In fact the dimension of $C^*(H)$ over $\wedge_\omega$ is just $\#\mathbf{P}(H)$.

Now $\delta_{J,H}: C^k \to C^{k+1}$ is defined by

$$\delta_{J,H}(\tilde{x}) = \sum_{\mu(\tilde{y})=\mu(\tilde{x})+1} n(\tilde{x},\tilde{y})\tilde{y}$$

for any $\tilde{x} \in C^k$, where $n(\tilde{x},\tilde{y})$ is the oriented number $\#\mathcal{M}^{D_T}(J,H;\tilde{x},\tilde{y})$.

If we have

(i) $n(\tilde{x},\tilde{y})$ is finite when $\mu(\tilde{y}) - \mu(\tilde{x}) = 1$;

(ii) $\sum_{\mu(\tilde{y})=k+1} n(\tilde{x},\tilde{y}) \cdot n(\tilde{y},\tilde{z}) = 0$ for any $\tilde{x} \in C^k$ and $\tilde{z} \in C^{k+2}$,

then $\delta_{J,H}$ is well-defined and $(C^*(H), \delta_{J,H}))$ is a chain complex. The "classical" Floer cohomology is just the homology of $(C^*(H), \delta_{J,H})$ for a generic (J,H) when (V,ω) is semi-positive so that (i) and (ii) hold.

Note that the left-hand side of (ii) can be interpreted as the (oriented) number of pairs of "broken" connecting orbits between $\tilde{x}$ and $\tilde{z}$ when $\mu(\tilde{z}) - \mu(\tilde{x}) = 2$. In the "ideal" situation, the space

$$\cup_{\mu(\tilde{y})=k+1}\mathcal{M}^{D_T}(J,H;\tilde{x},\tilde{y}) \times \mathcal{M}^{D_T}(J,H;\tilde{y},\tilde{z})$$

of such "broken" connecting orbits is just the "boundary" of $\mathcal{M}^{D_T}(J,H;\tilde{x},\tilde{z})$. We denote its union with $\mathcal{M}^{D_T}(J,H;\tilde{x},\tilde{z})$ by $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{z})$. One can show that it is compact. In fact, in this ideal case, this compact moduli space of "broken" connecting orbits coincides with the moduli space of stable (J,H)-maps connecting $\tilde{x}$ and $\tilde{z}$ (See the definition in Sec.2). Therefore, we have a "good" compactification of $\mathcal{M}^{D_T}(J,H;\tilde{x},\tilde{z})$ with boundary components of codimension 1. Putting this in a more algebraic form, we can summarize the "classical" Floer cohomology (for good cases) in the following statement:

(iii) When $\tilde{x} \in C^k$, $\tilde{z} \in C^{k+2}$, the moduli space $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{z})$ of stable (J,H)-maps connecting $\tilde{x}$ and $\tilde{z}$ is compact and is a one-dimensional manifold with boundary. It can be viewed as a relative virtual 1-cycle with

$$\partial\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{z}) = \cup_{\mu(\tilde{y})=k+1}\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y}) \times \overline{\mathcal{M}}(J,H;\tilde{y},\tilde{z}).$$

Clearly, (i) and (ii) follow from (iii).

Now for a general closed symplectic manifold other than semi-positive ones, the natural compactification $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{z})$ of $\mathcal{M}^{D_T}(J,H;\tilde{x},\tilde{z})$, the stable compactification, contains not only those "broken" connecting orbits as above, but also some bubbles of J-holomorphic spheres. A "boundary" component of the stable compactification containing some multiply covered bubbles with negative first Chern class may have a higher dimension than that of $\mathcal{M}^{D_T}(J,H;\tilde{x},\tilde{z})$ itself. Consequently both (i) and (ii) may fail.

To overcome this difficulty, we will construct a virtual moduli $\mathbf{Q}$-cycle $C(\overline{\mathcal{M}}^{\nu}(J,H;\tilde{x},\tilde{z}))$ such that its underlying moduli space $\overline{\mathcal{M}}^{\nu}(J,H;\tilde{x},\tilde{z})$ is compact. Here ν stands for certain "generic" perturbation of the $\bar{\partial}_{J,H}$-operator. Below is the outline of our construction.

The construction consists of two parts, local and global one. Firstly, note that $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$ consists of all unparametrized stable (J,H)-maps connecting $\tilde{x}$ and $\tilde{y}$, which is contained in the infinite dimensional space $\mathcal{B}(\tilde{x},\tilde{y})$ of unparametrized stable L_k^p-maps, $k-\frac{2}{p}>1$ (see the relevant definitions on page 17).There is an infinite dimensional "bundle" $\mathcal{L}\to\mathcal{B}(\tilde{x},\tilde{y})$ with each fiber $\mathcal{L}_{[f]}$ of $[f]\in\mathcal{B}(\tilde{x},\tilde{y})$ consisting of all L_{k-1}^p-sections of the bundle $\wedge^{0,1}(f^*TV), f\in[f]$, modulo the equivalence relation induced by reparametrization of the domains. In general, we do not expect to get any useful smooth structure for $\mathcal{B}(\tilde{x},\tilde{y})$ due to the non-compactness of reparametrization group. However, there exists an open set $W\subset\mathcal{B}(\tilde{x},\tilde{y})$ such that $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})\subset W$ and W is a stratified Banach orbifold, called partially smooth orbifold, stratified according to the topological types of the domains of the stable maps. In fact , $W=\cup_{i=1}^m W_i$ and each $W_i=W(f_i)$ is an open neighborhood in $\mathcal{B}(\tilde{x},\tilde{y})$ of $[f_i]$, with $[f_i]\in\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$, such that W_i is uniformized by $\pi_i^W:\widetilde{W}_i=\widetilde{W}(f_i;\mathbf{H}_i)\to W_i$ with a finite automorphism group Γ_i, where the uniformizer $\widetilde{W}_i$ consists of all those stable L_k^p-maps in the neighborhood $\widetilde{W}_i(f_i)=\pi^{-1}(W_i)$ of f_i which send their marked points into some particularly constructed family of local hypersurfaces $\mathbf{H}_i$. Let $\widetilde{\mathcal{L}}_i=\pi_i^*(\mathcal{L})\to\widetilde{W}_i$. Then

$$\pi_i=(\pi_i^L,\pi_i^W):(\widetilde{\mathcal{L}}_i,\widetilde{W}_i)\to(\mathcal{L}_i,W_i),\quad i=1,\cdots,m,$$

gives rise to a uniformizing system for the orbifold bundle $(\mathcal{L}|_W,W)$. $\overline{\partial}_{J,H}$-operator can be interpreted as a section of $\widetilde{\mathcal{L}}_i$ which is Γ_i- equivariant. Let $\widetilde{\mathcal{M}}_i$ be the zero set of $\overline{\partial}_{J,H}$ in $\widetilde{W}_i$. Then $\mathcal{M}_i=\pi_i^W(\widetilde{\mathcal{M}}_i)$ is just

$$\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})\cap W_i.$$

Now the wrong dimension of the boundary of the stable compactification simply means that $\overline{\partial}_{J,H}$ is not a transversal section yet in some

$\widetilde{W}_i$ though we have chosen a "generic" pair (J, H). In terms of this orbifold structure, the non-sufficiency of perturbing (J, H) to achieve transversality for $\overline{\partial}_{J,H}$ is quite easy to understand. It is simply because that any perturbation in $\mathcal{J} \times \mathcal{H}$ of the set of pairs (J, H) will yield a Γ_i-equivariant change in $\widetilde{W}_i$, but for a given pair (J, H), which is effective in the sense that there exists some u, say, in $\widetilde{W}_i$, such that $\overline{\partial}_{J,H} u = 0$, the cokernel $R_i(u)$ of the linearization of $\overline{\partial}_{J,H}$ at u may not be generated by Γ_i -invariant sections of $\widetilde{\mathcal{L}}_i$. Because of this, our remedy for this non-transversality problem becomes quite plain at least locally. What we need to do is to choose a "generic" perturbation ν_i in $R_i = R(f_i)$, which in general may not be generated by Γ_i-invariant sections of $\widetilde{\mathcal{L}}_i$, and consider the ν_i-perturbed section $\overline{\partial}_{J,H} + \nu_i : \widetilde{W}_i \to \widetilde{\mathcal{L}}_i$. It directly follows from the construction that this new section is transversal to zero section and the local moduli space

$$\widetilde{\mathcal{M}}_i^{\nu_i} = (\overline{\partial}_{J,H} + \nu_i)^{-1}(0)$$

and its projection $\mathcal{M}_i^{\nu_i}$ to W_i certainly have the right dimension at all their stratum as expected by the index theorem.

To complete our construction of the virtual moduli $\mathbf{Q}$-cycle $C(\overline{\mathcal{M}}^\nu(J, H; \tilde{x}, \tilde{y}))$, we need to globalize the above construction. The main difficulty here is how to transform each non-equivariant section ν_i in $\widetilde{W}_i$ into $\widetilde{W}_j$ when $W_i \cap W_j$ is not empty.

In order to get such a transformation, let $W_{ij} = W_i \cap W_j$ and consider

$$\pi_i : \widetilde{W}_{\hat{i}j} = \pi_i^{-1}(W_{ij}) \to W_{ij}$$

and

$$\pi_j : \widetilde{W}_{i\hat{j}} = \pi_j^{-1}(W_{ij}) \to W_{ij}.$$

We define $\widetilde{W}_{ij}^{\Gamma_{ij}}$ to be their fiber product $\widetilde{W}_{\hat{i}j} \times_{W_{ij}} \widetilde{W}_{i\hat{j}}$ over W_{ij}, which in some sense can be thought of as a substitute for "$\widetilde{W}_i \cap \widetilde{W}_j$". In general, let $\mathcal{N}$ be the nerve of the covering $\mathcal{W} = \{W_i; i = 1, \cdots, m\}$. For each

$$W_I = W_{i_1, \cdots, i_n} = W_{i_1} \cap W_{i_2} \cdots \cap W_{i_n}$$

with $I = (i_1, \cdots, i_n) \in \mathcal{N}$, let

$$\widetilde{W}_{i_1, \cdots, \hat{i}_k, \cdots, i_n} = \pi_k^{-1}(W_I).$$

Then we have n "finite" morphisms

$$\pi_k : \widetilde{W}_{i_1, \cdots, \hat{i}_k, \cdots, i_n} \to W_I$$

with automorphism group Γ_k. As above, we define $\widetilde{W}_I^{\Gamma_I}$ as the fiber product of $\widetilde{W}_{i_1,\cdots,\hat{i}_k,\cdots,i_n}(1 \le k \le n)$ over $W_{i_1,\cdots,i_n}$, where

$$\Gamma_I = \Gamma_{i_1} \times \Gamma_{i_2} \cdots \times \Gamma_{i_n}.$$

Obviously Γ_I acts on $\widetilde{W}_I^{\Gamma_I}$ and $\pi_I^W \cdot \sigma = \pi_I^W$ for any $\sigma \in \Gamma_I$, where π_I^W is the natural projection from $\widetilde{W}_I^{\Gamma_I}$ to W_I. We have a similar construction $\pi_I^{\mathcal{L}} : \widetilde{\mathcal{L}}_I^{\Gamma_I} \to \mathcal{L}_I$ for "bundles" $\mathcal{L}_I = (\pi^{\mathcal{L}})^{-1}(W_I)$. If $J \subset I \in \mathcal{N}$, there exists a morphism

$$\pi_J^I : (\widetilde{\mathcal{L}}_I^{\Gamma_I}, \widetilde{W}_I^{\Gamma_I}) \to (\widetilde{\mathcal{L}}_J^{\Gamma_J}, \widetilde{W}_J^{\Gamma_J})$$

such that

(i) $\pi_J^W \circ \pi_J^I = E_J^I \circ \pi_I^W$, where $E_J^I : W_I \to W_J$ is the inclusion;

(ii) $\#((\pi_J^I)^{-1}(u))$ is N_I/N_J for a generic u, where $N_I = |\Gamma_I|$.

Now we can construct an open subset $V_I \subset W_I$ for each $I \in \mathcal{N}$ to remove those "extra" overlaps between these W_I's (see detail in Section 4). By replacing W_I and all induced construction above by V_I's, we get a system of morphisms of bundles:

$$\{\pi^I : (\widetilde{E}_I^{\Gamma_I}, \widetilde{V}_I^{\Gamma_I}) \to (E_I, V_I); I \in \mathcal{N}\}.$$

Note that each $\widetilde{W}_I^{\Gamma_I}$ and $\widetilde{V}_I^{\Gamma_I}$ are not (partially) smooth manifolds, but rather (partially) smooth varieties (see the definition on page 59). We now use above system $(\widetilde{E}^{\Gamma}, \widetilde{V}^{\Gamma})$ to globalize our local moduli space $\widetilde{\mathcal{M}}_i^{\nu_i}$. Since these $(\widetilde{E}_I^{\Gamma_I}, \widetilde{V}_I^{\Gamma_I})$ relate to each other by those "semi-global" morphisms π_J^I, a global section s of the system $(\widetilde{E}^{\Gamma}, \widetilde{V}^{\Gamma})$ can be defined as a collection of sections $\{s_I; I \in \mathcal{N}\}$ of $(\widetilde{E}_I^{\Gamma_I}, \widetilde{V}_I^{\Gamma_I})$ such that $(\pi_I^J)^* s_I = s_J$ is valid over smooth points in their overlap. Clearly, $\overline{\partial}_{J,H}$ gives rise to a global section of this system, and each element $\nu_i \in R(f_i)$ can be transformed as a global section of the system by using these π_I^W's to lift it into a collection of sections of $(\widetilde{E}_I^{\Gamma_I}, \widetilde{V}_I^{\Gamma_I}), I \in \mathcal{N}$. Let $R = \oplus_{i=1}^m R(f_i)$. We will prove in Section 4 that for a "generic" choice of $\nu \in R$, $\overline{\partial}_{J,H} + \nu$ is a transversal global section of $(\widetilde{E}_I^{\Gamma_I}, \widetilde{V}_I^{\Gamma_I})$.

Now

$$(\overline{\partial}_{J,H} + \nu)^{-1}(0) = \{(\overline{\partial}_{J,H} + \nu_I)^{-1}(0); I \in \mathcal{N}\}$$

are certainly compatible with each other. Let

$$\widetilde{\mathcal{M}}_I^{\nu} = (\overline{\partial}_{J,H} + \nu_I)^{-1}(0)$$

and $\mathcal{M}^\nu_I = \pi_I(\widetilde{\mathcal{M}}^\nu_I)$. Then $\mathcal{M}^\nu = \{\mathcal{M}^\nu_I; I \in \mathcal{N}\}$ is the compact moduli space induced by ν underlying the virtual moduli $\mathbf{Q}$-cycle that we are looking for. The resulting relative virtual cycle is "formally" defined to be

$$C(\overline{\mathcal{M}}^\nu) = \sum_{I \in \mathcal{N}} \frac{1}{N_I} \widetilde{\mathcal{M}}^{\nu_I}_I$$

(see the precise definition on page 64). The following theorem, which is proved in Sec.4, serves as a technique base of this paper.

Theorem 1.1. *The above $C(\overline{\mathcal{M}}^\nu(\tilde{x}, \tilde{z}))$ is a rational cycle in $\mathcal{B}(\tilde{x}, \tilde{z})$ of dimension $\mu(\tilde{z}) - \mu(\tilde{x}) - 1$. Moreover, we have*

$$\partial(C(\overline{\mathcal{M}}^\nu(\tilde{x}, \tilde{z})) = \cup_{\tilde{y}} C(\overline{\mathcal{M}}^\nu(\tilde{x}, \tilde{y})) \times C(\overline{\mathcal{M}}^\nu(\tilde{y}, \tilde{z})).$$

In the case that $\mu(\tilde{y}) - \mu(\tilde{x}) = 1$, it follows from this that $\mathcal{M}^\nu(\tilde{x}, \tilde{y})$ is a finite set, and oriented number $\#(C(\mathcal{M}^\nu(\tilde{x}, \tilde{y}))) \in \mathbf{Q}$ is well-defined. If we define $n(\tilde{x}, \tilde{y}) = \#(C(\mathcal{M}^\nu(\tilde{x}, \tilde{y})))$, it is easy to see that (i) and (ii) will follow from above theorem. With this new interpretation of $n(\tilde{x}, \tilde{y})$, we now can extend Floer (co-) homology to all closed symplectic manifolds by the very same formulae as before. By using a parametrized version of above theorem we can prove that the resulting Floer cohomology $FH^*(V, \omega; J, H, \nu)$ is independent of the parameter (J, H, ν). In fact, with certain suitable modification of the above theorem, we can also define both the intrinsic and exterior multiplicative structures in the Floer cohomology for all closed symplectic manifolds, which were only defined for semi-positive case before.

This paper is organized as follows.

In Section 2, we will define the moduli space $\overline{\mathcal{M}}(J, H; \tilde{x}, \tilde{y})$ of stable (J, H)-maps connecting $\tilde{x}$ and $\tilde{y}$ and its ambient space $\mathcal{B}(\tilde{x}, \tilde{y})$ of stable L^p_k-maps, $k - \frac{2}{p} > 1$. We then prove in Lemmas 2.6 and 2.7 that for each $[f] \in \overline{\mathcal{M}}(J, H; \tilde{x}, \tilde{y})$, there is an open neighborhood $W(f)$ of $[f]$ in $\mathcal{B}(\tilde{x}, \tilde{y})$, which is a (partially) smooth orbifold with a (partially) smooth uniformizer $\widetilde{W}(f; \mathbf{H})$ and automorphism group Γ_f, and an orbifold bundle $\mathcal{L}(f)$ over $W(f)$ with uniformizer $\widetilde{\mathcal{L}}(f)$. The $\overline{\partial}_{J,H}$-operator gives rise to a Γ_f-equivariant section of $\widetilde{\mathcal{L}}(f)$.

In Section 3, we will establish the main local transversality of $\overline{\partial}_{J,H}$-section perturbed by a "generic" section ν of the finite dimensional "obstruction" bundle $R(f)$. The main technical part of this section is the main estimate in Proposition 3.1. As a consequence of the transversality of $\overline{\partial}_{J,H} + \nu$, in Lemma 3.9 we will prove that the local perturbed moduli space $\widetilde{\mathcal{M}}^{\nu}(f)$ has the "right" dimension as expected by index theorem for each of its stratum. Another corollary of the transversality is the gluing construction of Proposition 3.2 and Corollary 3.3 which will serve as a basis for comparing the strong L_k^p-topology for $\overline{\mathcal{M}}(J, H; \tilde{x}, \tilde{y})$ defined in this section with the weak C^{∞}-topology used before by Floer and Gromov.

Section 4 is devoted to globalize above local moduli space $\widetilde{\mathcal{M}}^{\nu}(f)$. We will give the details of our construction of the relative virtual moduli $\mathbf{Q}$-cycle $C(\mathcal{M}^{\nu}(J, H; \tilde{x}, \tilde{y}))$ sketched in this introduction.

In Section 5, we will use the theory which we developed in the previous sections to extend Floer cohomology $FH^*(V, \omega; J, H, \nu)$ to a general closed symplectic manifold and prove that it is invariant with respect to the parameter (J, H, ν). We conclude our proof of Arnold conjecture in Theorem 5.3 and Corollary 5.4 showing that $FH^*(V, \omega; J, H, \nu)$ is isomorphic to $H^*(V) \otimes \wedge_{\omega}$.

During the preparation of this paper, we learned that Fukaya and Ono obtained a different proof of the Arnold Conjecture in [8].

The authors are grateful to referees for their suggestions for improving the writing of the paper. In paticular, we are very grateful to one of the referees, who pointed out that the parametrized moduli space introduced in the first version was not needed. Though the proof in the present version of this paper is the same as the first version, this suggestion of the referee makes our presentation much more clear and simpler.

2. Moduli space of stable maps

In this section we will define the moduli space $\overline{\mathcal{M}}(J, H; \tilde{x}, \tilde{y})$ of stable (J, H)-maps and its ambient space $\mathcal{B}(\tilde{x}, \tilde{y})$ of stable L_k^p-maps connecting $\tilde{x}$ and $\tilde{y}$. Near $\overline{\mathcal{M}}(J, H; \tilde{x}, \tilde{y})$, $\mathcal{B}(\tilde{x}, \tilde{y})$ has a (partially) smooth orbifold structure. Locally, this amounts to say that for each stable (J, H)-map $[f]$, there exists an open neighborhood $U(f)$ of f in $\mathcal{B}(\tilde{x}, \tilde{y})$ such that $U(f)$ is uniformized by a connected (partially) smooth manifold $\widetilde{U}(f; \mathbf{H})$. Over each uniformizer $\widetilde{U}(f; \mathbf{H})$, we will define a Banach bundle $\mathcal{L}(f)$. The $\overline{\partial}_{J,H}$-operator is an equivariant section of the bundle, which is smooth on each strata of $\widetilde{U}(f; \mathbf{H})$.

2.1 Moduli space $\overline{\mathcal{FM}}_{0,k}$ of stable curves

We start with a description of the domains of our stable maps and the structure of the space $\overline{\mathcal{FM}}_{0,k}$ of the collection of these domains which we call $\mathcal{F}$-stable curves. Due to the appearance of the inhomogeneous term in perturbed $\overline{\partial}_J$-operator, the moduli space $\mathcal{FM}_{0,k}$ and its compactification $\overline{\mathcal{FM}}_{0,k}$ that we use in the paper are different from $\mathcal{M}_{0,k}$ of stable curves and its Degline-Mumford compactification $\overline{\mathcal{M}}_{0,k}$.

Recall that a k-pointed genus zero curve $(\Sigma, x_1, \cdots, x_k)$ is said to be stable in the sense of Deligne-Mumford if geometrically Σ can be obtained by joining pairwise its L components $\Sigma_l = S^2$, $l = 1, \cdots, L$, at some distinguished points, called double points, then adding the marked points x_i, $i = 1, \cdots, k$, away from the double points. The stability condition means that on each Σ_l there are at least three marked or double points. Note that the components Σ_l, $l = 1, \cdots, L$ form an open string for a genus zero stable curve.

A genus zero k-pointed stable curve $(\Sigma, x_1, \cdots, x_k)$ is said to be $\mathcal{F}$-stable if it satisfies the condition that we describe now. We divide the components of Σ into principal components $\Sigma_{p,i}$, $i = 1, \cdots, L_1$ and bubble components $\Sigma_{b,j}$, $j = 1, \cdots, L_2$. Each $\Sigma_{p,i}$ has two particular double points $y_{i,\pm\infty}$ except for $i = 1, L_1$, where one of these $y_{\pm\infty}$ is a marked points. But we will distinguish these two marked points from those x_i's. All principal components together form a chain such that $y_{i+1,+\infty} = y_{i,-\infty}$, $i = 1, \cdots, L_1 - 1$. There is particularly chosen "marked line" $L_{p,i}$ connecting $y_{i,\pm\infty}$ on each $\Sigma_{p,i}$. Because of this, we may identify each $(\Sigma_{p,i}; y_{i,-\infty}, y_{i,+\infty})$ with $(\mathbf{R} \times S^1)$ canonically modulo $\mathbf{R}$-translation with $L_{p,i}$ corresponding to $\theta = 0$.

From now on, we will simply use Σ_p and Σ_b to denote the principal and bubble components of Σ respectively whenever the context is clear even abuse of notation may occur. This usage is also applicable to all relevant quantities and constructions.

Two such curves $(\Sigma_1; x_1^1, \cdots, x_k^1)$ and $(\Sigma_2; x_1^2, \cdots, x_k^2)$ are said to be equivalent if there is a homeomorphism $\phi : \Sigma_1 \longrightarrow \Sigma_2$ preserving marked points and marked lines such that the restriction of ϕ to each component of Σ_1 is holomorphic. The resulting equivalent class, denoted by $[\Sigma_1]$, is called a $\mathcal{F}$-stable curve.

There is an obvious "forgetting marking" procedure that sends $\Sigma = (\Sigma;\ x_1,\ \cdots,\ x_k)$ to Σ^u by simply ignoring all marked points x_i, $i = 1, \cdots, k$. A component Σ_l of Σ is said to be free if it is not stable after forgetting the markings. Hence a free principal components Σ_p does not have any double points other than $y_{p,\pm\infty}$, and a free bubble component has either one or two double points. We can get a stable curve Σ^s with minimal number of markings from Σ^u by adding one or two marked points to each of its free components. To obtain a unique result, we require that the marked point added to a free principal component Σ_p to stabilize it is on L_p. Note that the automorphism group G_p of each free principal component Σ_p consists of all $\mathbf{R}$-translations, and the automorphism group G_b of each free bubble component Σ_b consists of all holomorphic maps of Σ_b that preserves the double points of Σ_b. We will use G_Σ to denote the automorphism group of Σ which consists of all holomorphic isomorphisms of Σ after forgetting its marked points. Note that G_Σ may interchange different components of Σ and we refer it as reparametrization group. It contains G_p and G_b's as subgroups.

Since the components of Σ form a "tree", if we think the chain of the principal components as the "roots" of the "tree" ,we can associate to each bubble component Σ_b an unique principal component of its "root", and we denote it by $\Sigma_{b,p}$.

A $\mathcal{F}$-stable curve is said to be smooth if it only has one principle component. We define the moduli space $\mathcal{F}\mathcal{M}_{0,k}$ of k-pointed $\mathcal{F}$-stable curves to be the collection of equivalent class of all smooth $\mathcal{F}$-stable curves Σ with k marked points x_i, $i = 1, \cdots, k$ and two ends $y_{\pm\infty}$. Equipped with the obvious smooth structure, $\mathcal{F}\mathcal{M}_{0,k}$ is a smooth manifold of real dimension $2(k-1)+1$, and there is a S^1-fibration $S^1 \longrightarrow \mathcal{F}\mathcal{M}_{0,k} \longrightarrow \mathcal{M}_{0,k+2}$, which corresponds to the procedure of forgetting the marking line L_Σ of Σ.

To obtain a compactification $\overline{\mathcal{F}\mathcal{M}}_{0,k}$ of $\mathcal{F}\mathcal{M}_{0,k}$, we let some of the x_i's of Σ go together or go to the ends $y_{\pm\infty}$. As the case of Deligne-Mumford compactification, this intuitive process corresponds to a degeneration of Σ into a k-points $\mathcal{F}$-stable curve. Therefore, $\overline{\mathcal{F}\mathcal{M}}_{0,k}$ is just the set of all k-points $\mathcal{F}$-stable curves described above. This will become clearer after we describe the local structure of $\overline{\mathcal{F}\mathcal{M}}_{0,k}$ in a moment. Note that unlike Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,k}$ of $\mathcal{M}_{0,k}$, $\overline{\mathcal{F}\mathcal{M}}_{0,k}$ is not a smooth manifold but has "boundary" and "corners". However, $\overline{\mathcal{F}\mathcal{M}}_{0,k}$ can be decomposed as a finite union of smooth manifolds according to the topological type of Σ. We describe it now.

The topological type of Σ is determined by its intersection pattern $I = I_\Sigma$, which is simply a pairwise correspondence of the distinguished points of the smooth resolution $\widetilde{\Sigma}$ of Σ that corresponds to the double points of Σ. It is clear that

$$\mathcal{I} = \{ I_\Sigma \quad | \quad \Sigma \in \overline{\mathcal{FM}}_{0,k} \}$$

is finite and

$$\mathcal{FM}^I_{0,k} = \{ \Sigma \,|\, \Sigma \in \overline{\mathcal{FM}}_{0,k}, \quad I_\Sigma = I \}$$

is a smooth manifold. One has the obvious decomposition

$$\overline{\mathcal{FM}}_{0,k} = \cup_{I \in \mathcal{I}} \mathcal{FM}^I_{0,k}.$$

The motivation to introduce the space $\overline{\mathcal{FM}}_{0,k}$ comes from the fact that its elements appear naturally as domains of the stable compactification of the moduli space of connecting orbits. Our main concern therefore is only the underlying set Σ^u of Σ, or rather the Σ^s which is Σ equipped with minimal number of marked points needed for stability. However in order to understand the change of the topological type of the domains in the stable compactification, it is necessary to include those stable curves which have extra markings. Nevertheless we can always start with the case that $\Sigma = \Sigma^s$.

Fixing $I = I_{\Sigma^s}$ and a point $\Sigma \in \mathcal{FM}^I_{0,k}$, we will give a concrete local description of the "universal" family of stable curves over $\overline{\mathcal{FM}}_{0,k}$, whose projection to $\overline{\mathcal{FM}}_{0,k}$ gives rise a local coordinate of Σ in $\overline{\mathcal{FM}}_{0,k}$.

Let $K_l + 3$ be the number of marked or double points on the component Σ_l of Σ. Since $\Sigma = \Sigma^s$, there are at most two marked points on Σ_l, so that the first K_l distinguished points can be arranged as double points that we will call $d_{l,k}$. Here for each principal component Σ_p, we have ordered its two ends $y_{p,\pm\infty}$ as the last two distinguished points. Let q_p be the third from the last distinguished point of the free principal component Σ_p. We may use the automorphism group G_b of the bubble component Σ_b to bring the last three distinguished points to the standard position of $0, 1$, and ∞ of S^2 and G_p of principal component of Σ_p to bring q_p to the central circle $\{s = 0\}$.

Because of the assumption that Σ^s has no extra markings, the locations of above points for the nearby Σ' serve as a local coordinate (uniformizer) of $\mathcal{FM}^I_{0,k}$ near Σ. More precisely, if $\alpha_{l,k} \in D_\delta(d_{l,k})$ is the complex coordinate of the δ-disc centering at $d_{l,k}$, and $\theta_p \in I_\delta(q_p)$ is the argument parameter in the δ-interval of $S^1 = \{s = 0\}$ centered at q_p, then the collection $(\alpha, \theta) = (\alpha_{l,k}, \theta_p)$ is the local coordinate of $\mathcal{FM}^I_{0,k}$ near Σ. We will denote the corresponding curve by $\Sigma_{(\alpha,\theta)}$.

Now for each double point of $d'_{l,k} = d'_{l'.k'}$ of $\Sigma' = \Sigma_{(\alpha,\theta)}$ with $I(l,k) = (l',k')$, we associate a complex gluing parameter $t_{l,k} = t_{l',k'} \in D_\delta$

of the δ-disc centering at the origin of $\mathbf{C}$, and for each pair of ends $y_{p,+\infty} = y_{p+1,+\infty}$ of Σ'_p and Σ'_{p+1}, a real gluing parameter $\tau_p \in [0, \delta]$. Note that here we have used $d'_{l,k}$ to denote all double points of Σ'_l rather than just the first K_l ones. Let $(\alpha, \theta, t, \tau)$ be the totality of $(\alpha_{l,k}, \theta_p, t_{l,k}, \tau_p)$. Then the corresponding curve $\Sigma_{(\alpha,\theta,t,\tau)}$ can be obtained from $\Sigma_{(\alpha,\theta)}$ by the following gluing procedure: for each double point $d'_{l,k} = d'_{l',k'}$ of $\Sigma_{(\alpha,\theta)}$ with coordinate $\alpha_{l,k}$ and $\alpha_{l',k'}$, consider the discs $D_{\delta_1}(\alpha_{l,k}) \subset \Sigma'_l$ and $D'_{\delta_1}(\alpha_{l',k'}) \subset \Sigma'_{l'}$ with coordinate $w_{l,k}$ and $w_{l',k'}$ respectively. Let (s, ϕ) be the corresponding cylindrical coordinate, i.e., $w = e^{-2\pi(s+i\phi)}$. We cut off the discs $\{(s_{l,k}, \phi_{l,k}) | s_{l,k} > -\log|t_{l,k}|\}$ in $D_{\delta_1}(\alpha_{l,k})$ and $\{(s_{l',k'}, \phi_{l',k'}) | s_{l',k'} > -\log|t_{l',k'}|\}$ in $D_{\delta_1}(\alpha_{l',k'})$. Then gluing back the remaining parts of $D_{\delta_1}(\alpha_{l,k})$ and $D_{\delta_1}(\alpha_{l',k'})$ along their boundary by the formula $\phi_{l,k} = \phi_{l',k'} + arg(t_{l,k})$. A similar and simpler gluing process is applied to the real parameter τ for gluing along ends $y_{p,\pm\infty}$ of the principal component of $\Sigma_{(\alpha,\theta)}$. We denote the resulting curve by $\Sigma_{(\alpha,\theta,t,\tau)}$, which is an element of $\overline{\mathcal{FM}}_{0,k}$ near Σ. The parameter $(\alpha, \theta, t, \tau)$ serves as a " cornered" coordinate chart of $\overline{\mathcal{FM}}_{0,k}$ near Σ, and $\Sigma_{(\alpha,\theta,t,\tau)}$ forms the universal curve over it.

Note that $\Sigma_{(\alpha,\theta)} \in \mathcal{FM}^I_{0,k}$ if and only if $t = 0$ and $\tau = 0$. Similarly, letting some of components of (t, τ) be zero, we get various curves in $\mathcal{FM}^{I_1}_{0,k}$ with $I_1 > I$. Here the partial order among the intersection patterns is the obvious one, i.e., $I_1 > I$ if the topological type Σ_{I_1} can be obtained from Σ_I through gluing. In particular, for $\Sigma_{(\alpha,\theta,t,\tau)}$ on the top strata of $\mathcal{FM}_{0,k}$, none of any component of t and τ is zero.

2.2 Moduli space of stable maps

Now let (V, ω) be a closed symplectic manifold with a ω-compatible almost complex structure J and a time-dependent Hamiltonian function $H : V \times S^1 \longrightarrow \mathbf{R}$. For generic H, the set $\mathbf{P}(H)$ of 1-periodic orbits of the Hamiltonian equation of H is finite. Let $\widetilde{\mathbf{P}}(H)$ be the corresponding "lifting" in the universal covering $\widetilde{\mathcal{L}}(V)$ of the contractible space $\mathcal{L}(V)$ of loops of V. Hence, each element $\tilde{z} \in \widetilde{\mathbf{P}}(H)$ is a map $w : D^2 \longrightarrow V$ such that $z = w|_{\partial D^2} \in \mathbf{P}(H)$. We will still call $\tilde{z}$ a closed orbit.

Recall that each principal components Σ_p of a stable curve Σ has two particular double points $y_{p,+\infty}$ and $y_{p,-\infty}$. Let y_∞ be the collection of all such double points of Σ.

Given a $\mathcal{F}$-stable curve Σ, a map $f : \Sigma \backslash \{y_\infty\} \longrightarrow V$ is said to be a stable (J, H)-map if there exist $L_1 + 1$ closed orbits $\tilde{z}_p$, $p = 0, 1, \cdots, L_1$, such that:

(A) On each principal component Σ_p with cylindrical coordinate (s,θ), f_p satisfies the equation of connecting orbits between z_{p-1} and z_p. More precisely, this means (i) $\frac{\partial f_p}{\partial s} + J(f_p)\frac{\partial f_p}{\partial \theta} = \nabla_2 H(\theta, f_p)$, (ii) $\lim_{s\to -\infty} f_p(s,\theta) = z_{p-1}(\theta)$, $\lim_{s\to +\infty} f_p(s,\theta) = z_p(\theta)$.

(B) On each bubble component Σ_b, f_b is J-holomorphic.

(C) $[\tilde{z}_p] = [\tilde{z}_{p-1}] + [f_p] + \sum_b [f_{b,p}]$ as relative homology class of (V, z_p), where the domain of $f_{b,k}$ is $\Sigma_{b,k}$ defined in Sec.2.1.

(D) All homotopically trivial principal components or homologically trivial bubble components are not free.

By somewhat abuse the notation, we will use Σ to denote the domain of f and write $f : \Sigma \longrightarrow V$ instead of $f : \Sigma \backslash \{y_\infty\} \longrightarrow V$.

Note that the last requirement is imposed in order to rule out the possibility of producing a "ghost" bubble through a sequence of rescaling of f at any given point of Σ, which will certainly result in a non-Hausdorff moduli space that can not be compactified in any reasonable topology.

Two stable maps f_1 and f_2 are equivalent if there exists a equivalence ϕ of their domain Σ_1 and Σ_2 such that $f_2 = f_1 \circ \phi$. We will use (f) to denote the resulting equivalence class of f and call both f and (f) parametrized stable maps. The unparametrized stable curve $[f]$ is obtained from (f) by forgetting marked points of (f) first, and then quotienting out the actions of the reparametrization group G_Σ. In other words, $[f]$ is just the isomorphism class of f under the holomorphis identification of domains without any marked points.

We also need the notion of stable L_k^p-maps with the Sobolev index $k - \frac{2}{p} > 1$. A stable L_k^p-map $f : \Sigma \to V$ is simply a L_k^p-map on each component Σ_l of the stable curve Σ such that only (C) and (D) above hold and that each principal component f_p satisfies an exponential decay condition along its ends z_{p-1} and z_p given by

$$\int\int_{\mathbf{R}\times S^1} e^{\epsilon_0 |s|}(|\xi_{p-1}^{(m)}|^p + |\xi_p^{(m)}|^p) ds d\theta < \infty,$$

for $m = 0, 1, \cdots, k$, where ξ_p is defined by $f_p(s,\theta) = \exp_{z_p}\xi_p(s,\theta)$ for s sufficiently large, and ξ_{p-1} is defined in a similar way. Here $0 < \epsilon_0 < 1$, which is fixed throughout this paper.

All other notions and notation which we introduced for stable (J,H)-maps above are also applicable to stable L_k^p-maps.

We remark here that the only meaningful objects for us are those unparametrized stable maps. However in order to understand the deformation of such maps under the topological change of their domains, we need to choose a representative f for $[f]$ and to stabilize its domain Σ_f by adding minimal number of markings. Then we deform the domain Σ_f in $\overline{\mathcal{FM}}_{0,k}$ through the universal curve there and get the corresponding deformation of f through parametrized stable maps with certain constraints at their marked points. The domains of this deformation will have extra markings, and they only serve as an intermediate objects. We get the deformation of unparametrized stable maps by sending those parametrized stable maps to their equivalent class by simply forgetting their marked points.

Most of the rest of this subsection will be devoted to giving the precise constructions sketched in above remark.

Each stable map (f) or $[f]$ determines an intersection pattern D_f of f which contains the following data: (a) the intersection pattern I_Σ of the domain Σ of f; (b) the relative homotopy class of each principal component f_p determined by the two ends $\tilde{z}_{p-1}$ and $\tilde{z}_p$ of f_p and homology class represented by each bubble component f_b in $H_2(V, \mathbf{Z})$.

An intersection pattern D is said to be effective if $D = D_f$ with f being stable (J, H)-map. For such an intersection pattern we define its energy to be

$$E(D_f) = E(f) = \sum_p E(f_p) + \sum_b \int_{S^2} f_b^* \omega,$$

and consider the set

$$\mathbf{D}^e = \{D \mid E(D) < e,\ D \text{ is effective } \}$$

of effective intersection patterns of bounded energy.

From Floer-Gromov compactification theorem for cuspidal maps it follows that there exists a constant $\epsilon > 0$ depending only on (V, ω, J, H) such that for each non-trivial component f_l of a stable (J, H)-map f, $E(f_l) > \epsilon$. Therefore if $E(f) < e$, f has at most c/ϵ non-trivial components. To see the finiteness of $\mathbf{D}^e$, we need to get a uniform bound on the number of ghost bubble components. To this end, note that after forgetting extra markings, there are only one or two marked point(s) on each free component which is non-trivial. This implies that there are at most $2e/\epsilon$ marked points. This in turn determines the number of double points, hence, the number of the components of ghost bubbles. From the above analysis follows

Lemma 2.1. *The set* $\mathbf{D}^e$ *is finite.*

Now we can define various moduli spaces of stable maps. Let

$$\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y}) = \{[f] \mid f\, is\, stable\, (J,H)-map,\ \tilde{z}_0=\tilde{x},\ \tilde{z}_{L_1+1}=\tilde{y}\},$$

where $\tilde{z}_0$ and $\tilde{z}_{L_1+1}$ are the first and the last end of the chain of the principal component of f;

$$\mathcal{M}^D(J,H,\tilde{x},\tilde{y}) = \{[f] \mid f \in \overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y}),\ D_f = D\},$$

and

$$\begin{aligned} \mathcal{B}^e(\tilde{x},\tilde{y}) &= \mathcal{B}^{p,e}_k(\tilde{x},\tilde{y}) \\ &= \{[f] \mid f\ is\ L^p_k - stable,\ E(f)<e,\ D_f\ is\ effective\}, \\ \mathcal{B}^{D,e}(\tilde{x},\tilde{y}) &= \{[f] \mid [f]\in \mathcal{B}^e(\tilde{x},\tilde{y}), D_f = D\}. \end{aligned}$$

Note that from (C) of the definition of stable (J,H)-map it follows that the energy $E(f)$ is bounded for any $[f]\in\overline{\mathcal{M}}(J,H,\tilde{x},\tilde{y})$. Therefore $\overline{\mathcal{M}}(J,H,\tilde{x},\tilde{y}) \subset \mathcal{B}^e(\tilde{x},\tilde{y})$ for e large enough. We will choose such an e once for all and omit the superscript e for the moduli spaces of stable L^p_k-maps in the rest of this paper.

The moduli space

$$\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y}) = \cup_D \mathcal{M}^D(J,H;\tilde{x},\tilde{y})$$

is the stable compactification of the moduli space $\mathcal{M}^{D_T}(J,H;\tilde{x},\tilde{y})$, which is the moduli space of connecting orbits between $\tilde{x}$ and $\tilde{y}$. Here we use D_T to denote the top strata $\mathcal{FM}_{0,k}$. However, the boundary component $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y}) \setminus \mathcal{M}^{D_T}(J,H;\tilde{x},\tilde{y})$ of this compactification may have higher dimension than the dimension of $\mathcal{M}^{D_T}(J,H;\tilde{x},\tilde{y})$ itself. Our motivation to introduce these moduli spaces of stable L^p_k-maps, inside which $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$ appears as the zero set of a certain section induced by the $\bar{\partial}_{J,H}$- operator, is to use them as an ambient space to alter the defining sections of $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$ to get a new compact moduli space with "right" boundary. To this end, we need to understand the smooth and topology structure of these spaces. We start with those spaces whose elements have a fixed intersection pattern D, and therefore a fixed intersection pattern I of their domains.

Given $[f]\in\mathcal{B}^D(\tilde{x},\tilde{y})$, let $f\in[f]$ be a representative with domain $\Sigma\in\mathcal{FM}^I_{0,k}$. Note that here we have chosen minimal number of marked points to stabilize Σ. Recall that in this case, a neighborhood of Σ in

$\mathcal{FM}_{0,k}^{I}$ can be parametrized by parameters $(\alpha, \theta) = (\alpha_{l,k}, \theta_p)$, where $(\alpha_{l,k}, \theta_p) \in D_\delta(d_{l,k}) \times I_\delta(q_p)$ for $k = 1, \cdots, K_l$ and $p = 1, \cdots, L_1$ (see the relevant definitions in 2.1). For simplicity, we may assume that there is no free principal components; therefore, θ_p is a parameter for some double points. Now consider the mapping space

$$\mathcal{M}ap(\Sigma_l, V) = \{g_l \mid g_l : \Sigma_l \to V, \ \|g_l\|_{k,p} < \infty\}, \quad l = 1, \cdots, L.$$

In the case that Σ_l is a principal component, we understand that the L_k^p-norm has been exponentially weighted along the ends of Σ_l as in the corresponding part of the definition of stable L_k^p-maps in 2.1.

Let $U(f_l)$ be a neighborhood of f_l in $\mathcal{M}ap(\Sigma_l, V)$. Set $U = \prod_{l=1}^{L} U(f_l)$ and $K = \sum_{l=1}^{L} K_L$. We define the evaluation map

$$e_D : U \times (D_\delta)^K \times I_\delta^{L_1} \longrightarrow V^{K+L_1+3L_2}$$

given by:

$$\begin{aligned}(g_1, \cdots, g_L, \cdots, \alpha_{l,k}, \cdots, \theta_p, \cdots, \beta_{b,j}, \cdots)& \\ \to (\cdots, g_l(\alpha_{l,k}), \cdots, g_p(\theta_p), \cdots, g_b(\beta_{b,j}), \cdots),&\end{aligned}$$

where $\beta_{b,j}$, $j = 1, 2, 3$, are the last three distinguished points of the bubble components Σ_b, which are already brought to the standard positions by the group $SL(2, \mathbf{C})$.

The multi-diagonal $\Delta_I \subset V^{K+L_1+3L_2}$ is defined in an obvious way, determined by the intersection pattern I. For instance, if $I(\{l, k\}) = \{b, j\}$, then the component $\alpha_{l,k} = \beta_{b,j}$ in Δ_I. Clearly, e_D is transversal to Δ_I, and $e_D^{-1}(\Delta_I)$ is a Banach manifold. Now the reparametrization group G_{Σ_g} acts on $g \in e_D^{-1}(\Delta_I)$. Let $\pi : e_D^{-1}(\Delta_I) \to e_D^{-1}(\Delta_I)/G_\Sigma$ be the quotient map. The moduli space space $\mathcal{B}^D(\tilde{x}, \tilde{y})$ of unparametrized stable maps can be topologized by using the quotient topology. Because of the non-compactness of the reparametrization group, we do not expect to have a good structure of $\mathcal{B}^D(\tilde{x}, \tilde{y})$. However, near the moduli space $\mathcal{M}^D(J, H; \tilde{x}, \tilde{y})$ of stable (J, H)-maps, the action G has a "good" slicing, which implies that $\mathcal{B}^D(\tilde{x}, \tilde{y})$ has an orbifold structure in a neighborhood of $\mathcal{M}^D(\tilde{x}, \tilde{y})$.

To describe this slicing, we assume that $[f]$ and its representative $f \in [f]$ above are stable (J, H)-maps. Let f_p, $p = 1, \cdots, P$, and f_b, $b = 1, \cdots, B$ are its free principal and bubble components. Each free bubble component has one or two marked points. For simplicity, we may assume that there is only one such point. We may choose $(0, 0)$ and 0

as the marked points for all f_p and f_b respectively, and assume also that they are generic points. This implies that for each principle component f_p, $f_p|_{\mathbf{R}\times\{0\}}$ is an embedding near $(0,0)$ and $f(s,0)\neq f(0,0)$ if $s\neq 0$ for generic J and for each bubble component f_b, f_b is a local embedding near 0. Now choose a hypersurface $\mathbf{H}_p$ of codimension 1 locally near $f_p(0,0)$ for each free component f_p, $p=1,\cdots,P$, such that $f_p|_{\mathbf{R}\times\{0\}}$ is transversal to $\mathbf{H}_p$ at $(0,0)$ and a hypersurface $\mathbf{H}_b$ of codimension 2 locally near $f_b(0)$ for each free bubble component f_b $b=1,\cdots,B$, such that f_b is transversal to $\mathbf{H}_b$ at 0. Let $\mathbf{H}=\prod_p \mathbf{H}_p\times\prod_b \mathbf{H}_b$.

Given a small $\epsilon>0$, we define the ϵ-neighborhood of f in $e_D^{-1}(\Delta_I)$, $\widetilde{U}_\epsilon^D(f)=\{\,g\,|\,\|g-f\|_{\mathcal{B}^D}<\epsilon\,\}$, where the norm $\|g-f\|_{\mathcal{B}^D}=\sum_{l=1}^L\|g_l-f_l\|_{k,p}+$ summation of distances of corresponding double points on each Σ_l.

Since the Sobolev index $k-\frac{2}{p}>1$, from Sobolev embedding theorem and the assumption of exponential decay along the ends of each principal component g_p it follows that when ϵ is small enough, $g_p|_{\mathbf{R}\times\{0\}}$ is transversal to $\mathbf{H}_p$ and has one and only one intersection point with $\mathbf{H}_p$ for any $g\in\widetilde{U}_\epsilon^D(f)$. Therefore, if we define

$$\widetilde{U}_\epsilon^D(f,\mathbf{H})=\{\,g\,|\,g\in\widetilde{U}_\epsilon^D(f),\ g_p(0,0)\in\mathbf{H}_p,\ g_b(0)\in\mathbf{H}_b\},$$

where p running through from 1 to P and b from 1 to B, then we already get a slicing of $\widetilde{U}_\epsilon^D(f)$ for those group actions of $\prod_{p=1}^P G_p$. Because of this, we only need to deal with bubble component g_b.

According to [15], each bubble component g_b can be factorized as $f_b=\tilde{f}_b\circ\pi_b$ where $\pi_b:\Sigma_b\to\Sigma_b$ is a n_b-fold branched covering of S^2, and $\tilde{f}_b$ is a simple J-holomorphic map in the sense that it is an embedding away from finite singular or double points of $\tilde{f}_b$. Let $f_b^{-1}(f_b(0))=\{w_1=0,\cdots,w_{n_b}\}$, then we have

Lemma 2.2. *When ϵ, δ small enough, for any $g\in\widetilde{U}_\epsilon^D(f)$, there exist exactly n_b points, $w_1(g_b),\cdots,w_{n_b}(g_b)$ such that $w_i(g_b)\in D_\delta(w_i)$, $i=1,\cdots,n_b$, and $g_b^{-1}(\mathbf{H}_b)=\{w_i(g_b)\}$. Moreover, if $\epsilon\to 0$, we can choose $\delta\to 0$ also.*

Proof. We choose a coordinate chart W_b of V near $f_b(0)$ such that $W_b=\mathbf{H}_b\oplus\mathbf{R}^2$. Let $h_b:W_b\to\mathbf{R}^2$ be the projective of W_b to the second factor $\mathbf{R}^2$ of the direct sum. The assumption that $k-\frac{2}{p}>1$ implies that g_b is C^1-close to f_b. So is $h\circ g_b$ to $h\circ f_b$. Since $h\circ f_b(w_i)=0$ and $\|D(h_b\circ f_b)(w_i)\|=1$, it follows that there exists a constant $r>1$ such that $|h\circ g_b(w_i)|<r\epsilon$, and $\frac{1}{r}<\|D(h_b\circ g_b)(w_i)\|<r$. Now Picard method

for implicit function theorem implies that there exists a δ depending only on r and the norm of the second order expansion of $h_b \circ g_b$, which we may assume to be uniformly bounded, such that there is one and only one zero of $h_b \circ g_b$ in each $D_\delta(w_i)$, $i = 1, \cdots, n_b$. It is easy to see that these are the only intersection points of g_b and $\mathbf{H}_b$ when ϵ is small enough.

The last statement also follows from Picard method. q.e.d.

Lemma 2.3. *Let $U_\epsilon^D(f, \mathbf{H})$ be the image of $\widetilde{U}_\epsilon^D(f, \mathbf{H})$ under the projection of the quotient map π. Then $U_\epsilon^D(f, \mathbf{H})$ contains an open neighborhood of $[f]$ in $\mathcal{B}^D(\tilde{x}, \tilde{y})$.*

Proof. As before we only need to deal with free bubble component g_b, and we still assume, for simplicity, that g_b has only one free parameter. We only need to prove that when $\epsilon_1 << \epsilon$, for any $g \in \widetilde{U}_{\epsilon_1}^D(f)$, there exists $\phi = (\phi_l)$, with

$$\phi_l \in SL(2, \mathbf{C}; 1, \infty) = \{\, \psi \,|\, \psi \in SL(2, \mathbf{C}), \quad \psi(1) = 1, \ \psi(\infty) = \infty\}$$

such that $g \circ \phi \in \widetilde{U}_\epsilon^D(f, \mathbf{H})$. It follows from Lemma 2.2, that there exists $w_1(g_b) \in D_\delta(0)$ for some $\delta > 0$, such that $g_b(w_1(g_b)) \in \mathbf{H}_b$ and that when $\epsilon_1 \to 0$, $\delta \to 0$ also. Now define ϕ_b to be the automorphism of S^2 preserving $1, \infty$ and sending 0 to $w_1(g_b)$. Then $g \circ \phi \in \widetilde{U}_\epsilon^D(f, \mathbf{H})$ when ϵ_1 and hence δ are small. q.e.d.

Our slicing $\widetilde{U}_\epsilon^D(f, \mathbf{H})$ does not give a local coordinate of $B^D(\tilde{x}, \tilde{y})$ near $[f]$. There are further equivalence relations of finite order among the elements of $\widetilde{U}_\epsilon^D(f, \mathbf{H})$, which can be described by extending the action of the automorphism group Γ_f of f, where $\Gamma_f = \{\, \phi \,|\, \phi \in G_\Sigma, f \circ \phi = f\}$. Note that Γ_f is a finite group. It is generated by the subgroup $\widetilde{\Gamma}_f = \prod_{l=1}^L \Gamma_f^l$ together with those elements in G_Σ which permute the components of Σ and preserve f. Here $\Gamma_f^l = \{\, \phi_l \,|\, f_l \circ \phi_l = f_l, \ \phi_l$ preserves double points of $f_l\}$.

Since $f_l = \tilde{f}_l \circ \pi$ with each $\tilde{f}_l$ being an embedding essentially, it follows that $f_l \circ \phi = f_l$ if and only if $\pi \circ \phi_l = \pi$ and ϕ_l preserves double points of Σ_l. Therefore, Γ_f^l is a subgroup of the finite automorphism group of the branched covering of S^2, whose elements fix at least one or two distinguished points of S^2. This implies

Lemma 2.4. *Each Γ_f^l is a cyclic group consisting of "rotations".*

We now only describe how to extend the actions of $\widetilde{\Gamma}_f$ to $\widetilde{U}_\epsilon^D(f, \mathbf{H})$. The extension for general case is essentially same. Since we will do this componentwisely, for simplicity, we may assume that g has only one component with one free parameter.

Let ϕ_i be the automorphism of S^2 such that $\phi_i(w_1) = w_i$, $i = 1, \cdots, n$, $\phi_i(1) = 1$, $\phi_(\infty) = \infty$, where $w_1 = 0$ as before. Then $\widetilde{\Gamma}_f$ consists of all those ϕ_i such that $f \circ \phi_i = f$. We may assume that they are the first m ϕ_i's. Set $r = \min_{i>m} \|f - f \circ \phi_i\|$. For any $g \in \widetilde{U}^D_\epsilon(f, \mathbf{H})$, we define ϕ^g_i of automorphism of S^2 by setting $\phi^g_i(w_1) = w_i(g)$ and $\phi_i(1) = 1$, $\phi_i(\infty) = \infty$. It is easy to see that when $\epsilon << \epsilon_1 << r$, $g \circ \phi^g_i \in \widetilde{U}^D_{\epsilon_1}(f, \mathbf{H})$ if and only $i \leq m$. This gives rise to an action of $\widetilde{\Gamma}_f$ on $\widetilde{U}^D_{\epsilon_1}(f, \mathbf{H})$ given by: $g * \phi = g \circ \phi^g$ for $g \in \widetilde{U}^D_{\epsilon_1}(f, \mathbf{H})$, $\phi \in \widetilde{\Gamma}_f$, when it is defined. It is clear that given any two elements g_1 and g_2 of $\widetilde{U}^D_{\epsilon_1}(f, \mathbf{H})$, g_1 and g_2 are equivalent if and only if there exists a $\phi \in \Gamma_f$ such that $g_1 = g_2 * \phi$. If we replace $\widetilde{U}^D_{\epsilon_1}(f; \mathbf{H})$ by the Γ_f-invariant subset $\widetilde{W}^D_\epsilon(f, \mathbf{H}) = \cup_{\phi \in \Gamma_f} \phi(\widetilde{U}^D_\epsilon(f; \mathbf{H}))$, we have

Lemma 2.5. *The action defined above is a smooth right action on $\widetilde{W}^D_\epsilon(f; \mathbf{H})$, and $\widetilde{W}^D_\epsilon(f, \mathbf{H})/\Gamma_f$ is homomorphic to a neighborhood of $[f]$ in $\mathcal{B}_D(\tilde{x}, \tilde{y})$.*

Having completed the description of the local orbifold structure of $\mathcal{B}^D(\tilde{x}, \tilde{y})$ near points of $\mathcal{M}^D(J, H; \tilde{x}, \tilde{y})$ for a fixed intersection pattern D, we now turn to the same question for

$$\mathcal{B}(\tilde{x}, \tilde{y}) = \cup_D \mathcal{B}^D(\tilde{x}, \tilde{y}).$$

Given $[f] \in \mathcal{B}^D(\tilde{x}, \tilde{y})$ with $f \in [f]$, let Σ be the domain of f with intersection pattern I. Then a δ-neighborhood of Σ in $\mathcal{FM}^I_{0,k}$ or in $\overline{\mathcal{FM}}_{0,k}$ can be described by parameters $(\alpha, \theta) = (\alpha_{l,k}, \theta_p)$ with $\alpha_{l,k} \in D_\delta(d_{l,k})$ and $\theta_p \in I_\delta(q_p)$ in the former case and parameters $(\alpha, \theta, t, \tau)$ with $\|t\|$, $|\tau| < \delta$, in the latter case. Define the "base points"

$$f_{(\alpha,\theta)} : \Sigma_{(\alpha,\theta)} \to V$$

and

$$f_{(\alpha,\theta,t,\tau)} : \Sigma_{(\alpha,\theta,t,\tau)} \to V$$

as follows.

We define $f_{(\alpha,\theta)} = f \circ \phi_{(\alpha,\theta)}$ with $\phi_{(\alpha,\theta)} : \Sigma_{(\alpha,\theta)} \to \Sigma$ of a diffeomorphism between them. More precisely, fix a $r >> \delta > 0$, define $\phi_{(\alpha,\theta)}$ to be identity on $\Sigma_{(\alpha,\theta)} \setminus (\cup D_r(d_{l,k}) \cup \{I_r(q_p) \times I_r\})$ and to be the obvious "rotation" of S^2 or S^1 on each $D_\delta(d_{l,k})$ or $I_\delta(q_p) \times I_\delta$, which brings $\alpha_{l,k}$ to $d_{l,k}$ and θ_p to q_p. Since $r > 2\delta$, the image of $\phi_{(\alpha,\theta)}$ restricting to

$D_\delta(d) \times (I_\delta(q) \times I_\delta)$ is contained in $D_r(d) \times (I_r(q) \times I_r)$. Therefore, we can easily extend $\phi_{(\alpha,\theta)}$ to all $\Sigma_{\alpha,\theta}$. It follows from the construction that when δ is small enough, $\phi_{(\alpha,\theta)}$ is C^m-close to identity, for any given m; therefore $f_{(\alpha,\theta)}$ is L^p_k-close to f.

We define the pre-gluing $f_{(\alpha,\theta,t,\tau)}$ of $f_{(\alpha,\theta)}$ with gluing parameter (t, τ) as follows. Recall that $\Sigma_{(\alpha,\theta,t,\tau)}$ can be obtained from $\Sigma_{(\alpha,\theta)}$ by cutting off $\{(s,\phi) \mid s > -\log|t_{l,k}|\}$ of each pair $D_\delta(\alpha_{l,k})$ and $D_\delta(\alpha_{l',k'})$, and by cutting off $\{\, (s,\phi) \mid s > \frac{1}{\tau}\}$ of each successive pair of principal component Σ_{p-1} and Σ_p, and then gluing back along their boundaries. Let $T^m_{l,k}$ and T^n_p be the annulus in $\Sigma_{(\alpha,\theta,t,\tau)}$ defined by

$$T^m_{l,k} = \{(s_{l,k}, \phi_{l,k}) \mid s_{l,k} > -\log t_{l,k} - m\}$$

and

$$T^n_p = \{(s_p, \phi_p) \mid s_p > \frac{1}{\tau} - n\}.$$

Then we define:

(1) $f_{(\alpha,\theta,t,\tau)} \equiv f_{(\alpha,\theta)}$ on $\Sigma_{(\alpha,\theta,t,\tau)} \setminus \cup_{\{l,k\}} T^2_{l,k} \cup_p T^2_p$.

(2) $f_{(\alpha,\theta,t,\tau)}(s,\phi) = \exp_{f_{(\alpha,\theta)}}(\beta(s)\cdot \xi_{(\alpha,\theta)}(s,\phi))$ on $\cup_{l,k} T^2_{l,k} \cup_p T^2_p$, where $\xi_{(\alpha,\theta)}(s,\phi)$ is defined by

$$\begin{cases} f_{(\alpha,\theta)}(s_{l,k}, \phi_{l,k}) &= \exp_{f_{(\alpha,\theta)}(\alpha_{l,k})} \xi_{(\alpha,\theta)}(s_{l,k}, \phi_{l,k}), \\ f_{(\alpha,\theta)}(s_p, \phi_p) &= \exp_{f_{(\alpha,\theta)}(z_p(\phi_p))} \xi_{(\alpha,\theta)}(s_p, \phi_p), \end{cases}$$

and β is a cut-off function which is equal to identity outside $\cup_{l,k} T^2_{l,k} \cup T^2_p$.

Now for a fixed $(\alpha, \theta, t, \tau)$, we define

$$\widetilde{U}^{(\alpha,\theta,t,\tau)}_\epsilon(f, \mathbf{H}) = \{\, g \mid g : \Sigma_{(\alpha,\theta,t,\tau)} \to V,\ ||g - f_{(\alpha,\theta,t,\tau)}||_{k,p} < \epsilon, g(x) \in \mathbf{H}\},$$

where the L^p_k-norm is measured with respect to the induced "spherical" and "cylindrical" metric on $\Sigma_{(\alpha,\theta,t,\tau)}$. Here we use x to denote the collection of marked points $x_{l,k}$ of $\Sigma_g = \Sigma_{(\alpha,\theta,t,\tau)}$, each of which comes from some components Σ_l of Σ via the gluing. The notation $g(x) \in \mathbf{H}$ simply means that $g(x_{l,k}) \in \mathbf{H}_{l,k}$. By letting the parameter $(\alpha, \theta, t, \tau)$ varies with $||(\alpha, \theta, t, \tau)|| < \epsilon$, we may define $\widetilde{U}_\epsilon(f, \mathbf{H}) = \cup_{||(\alpha,\theta,t,\tau)|| < \epsilon} \widetilde{U}^{(\alpha,\theta,t,\tau)}_\epsilon(f, \mathbf{H})$. The quotient map $\pi : \widetilde{U}_\epsilon(f, \mathbf{H}) \to \mathcal{B}(\tilde{x}, \tilde{y})$ can be defined as follows. For each $g \in \widetilde{U}_\epsilon(f, \mathbf{H})$, we forget all the markings of the domain Σ_g first, and then send g to its equivalent class $[g]$ of unparametried stable map, that is $\pi(g) = [g]$. Let $U_\epsilon(f, \mathbf{H}) = \pi(\widetilde{U}_\epsilon(f, \mathbf{H}))$ be the image of $\widetilde{U}_\epsilon(f, \mathbf{H})$ in $\mathcal{B}(\tilde{x}, \tilde{y})$, which forms open neighborhood of f in $\mathcal{B}(\tilde{x}, \tilde{y})$.

Now let $[f]$ varies in a neighboorhood of $\mathcal{M}(J,H;\tilde{x},\tilde{y})$ in $\mathcal{B}(\tilde{x},\tilde{y})$, we define its (strong) L_k^p-topology to be the topology generated by $U_\epsilon(f,\mathbf{H})$.

The local structure of $\mathcal{B}(x,y)$ near a stable (J,H)-map $[f]$ is stated in the following lemma.

Lemma 2.6. *The action Γ_f on $\widetilde{U}^D_{\epsilon_1}(f,\mathbf{H})$ can be extended to $\widetilde{U}_{\epsilon_1}(f,\mathbf{H})$. It is a continuous right action on $\widetilde{U}_{\epsilon_1}(f,\mathbf{H})$ when defined and smooth on each strata $\widetilde{U}^{D'}_{\epsilon_1}(f,\mathbf{H})$ for $D'\geq D$. The natural projection π of the quotient map*

$$\widetilde{U}_{\epsilon_1}(f,\mathbf{H})\to\mathcal{B}(\tilde{x},\tilde{y})$$

commutes with Γ_f-actions. Moreover,

$$\overline{\pi}:\widetilde{U}_{\epsilon_1}(f,\mathbf{H})/\Gamma_f\to\mathcal{B}(\tilde{x},\tilde{y})$$

is a homeomorphism from $\widetilde{U}_{\epsilon_1}(f,\mathbf{H})/\Gamma_f$ to a neighborhood of $[f]$ in $\mathcal{B}(\tilde{x},\tilde{y})$.

Proof. As before we only consider how to extend the actions of $\widetilde{\Gamma}_f$. For simplicity as before, we may assume that each free bubble component f_b of f contains only one free parameter $x_b=0$, and $f_b^{-1}(f_b(0))=\{w_{b,1},\cdots,w_{b,n_b}\}$. Recall that in this case $\widetilde{\Gamma}_f=\{\phi_{b,1},\cdots,\phi_{b,m_b}\}$ is determined by $\phi_{b_i}(0)=w_{b,i}$, $i\leq m_b\leq n_b$ when each $\phi_{b,i}$ is considered as an automorphism of Σ_b. Now the proof of Lemma 2.2 can be easily adapted here and it implies that when ϵ_1 and δ are small enough, for any $g\in\widetilde{U}_{\epsilon_1}(f)$ with domain$\Sigma_{(\alpha,\theta,t,\tau)}$, there exist $\sum_b n_b$ points, $w_{b,i}(g)\in D_\delta(w_{b,i})$,$i=1,\cdots,n_b$, such that $g^{-1}(\mathbf{H})=\{w_{b,i}(g)\}$. Here we have considered points $w_{b,i}$ of Σ as points of $\Sigma_{(\alpha,\theta,t,\tau)}$ through the gluing construction.

Now for each b, choose $i\in\{1,\cdots,n_b\}$, say, $i=1$, and consider

$$(\Sigma_{(\alpha,\theta,t,\tau)};w_{1,1}(g),\cdots,w_{k,1}(g)).$$

From our construction of the "universal curve" parametrized by $\overline{\mathcal{FM}}_{0,k}$ it follows that there exists a

$$(\Sigma_{(\alpha',\theta',t',\tau')};x_1,\cdots,x_k)$$

with x_b coming from $x_b=0$ on Σ_b, $b=1,\cdots,k$ and an equivalence

$$\phi:(\Sigma_{(\alpha',\theta',t',\tau')};x_1,\cdots,x_k)\to(\Sigma_{(\alpha,\theta,t,\tau)};w_{1,1}(g),\cdots,w_{k,1}(g)).$$

Now $g\circ\phi\in\widetilde{U}_\epsilon(f,\mathbf{H})$. This proves that up to identifying domains, each $g\in\widetilde{U}_{\epsilon_1}(f)$ is equivalent to an element in $\widetilde{U}_\epsilon(f,\mathbf{H})$ where $\epsilon_1<<\epsilon$.

In the same way, we define the action of

$$\phi_{i_1,i_2,\cdots,i_k} = (\phi_{1,i_1}, \phi_{2,i_2}, \cdots, \phi_{k,i_k})$$

of $\widetilde{\Gamma}_f$ acting on g by

$$g * \phi_{i_1,\cdots,i_k} = g \circ \phi_{i_1,\cdots,i_k},$$

where

$$\phi_{i_1,\cdots,i_k} : (\Sigma_{(\alpha',\theta',t',\tau')}, x_1, \cdots, x_k) \to (\Sigma_{(\alpha,\theta,t,\tau)}; w_{1,i_1}(g), \cdots, w_{k,i_k}(g))$$

is defined similarly as above. In this way, we extend the group action of $\widetilde{\Gamma}_f$ to $\widetilde{U}_\epsilon(f, \mathbf{H})$.

The rest of the proof follows easily from above. We leave it to the readers. q.e.d.

As before, we can get Γ_f-invariant set $\widetilde{W}_\epsilon(f; \mathbf{H})$ by taking the union of Γ_f-image of $\widetilde{U}_\epsilon(f; \mathbf{H})$.

Now we can define locally the bundles $\widetilde{\mathcal{L}}_D(f)$ and $\widetilde{\mathcal{L}}(f)$ over $\widetilde{W}^D_\epsilon(f, \mathbf{H})$ and $\widetilde{W}_\epsilon(f, \mathbf{H})$ as follows. For each $g \in \widetilde{W}^D_\epsilon(f, \mathbf{H})$ or $\widetilde{W}_\epsilon(f, \mathbf{H})$, the fiber

$$(\widetilde{\mathcal{L}}_D(f))_g = (\widetilde{\mathcal{L}}(f))_g = \{\, \xi \mid \xi \in L^p_{k-1}(\wedge^{0,1}(g^*TV))\},$$

where the L^p_{k-1}-norm is measured with respect to the "standard" metric on the domain $\Sigma_g = \Sigma_{(\alpha,\theta,t,\tau)}$ induced by the gluing construction from the metric on Σ which is "spherical" on Σ_b and cylindrical on Σ_p.

It is clear that for a fixed D, $\widetilde{\mathcal{L}}_D(f)$ is a locally trivial Banach bundle over $\widetilde{W}^D_\epsilon(f; \mathbf{H})$, and $\widetilde{\mathcal{L}}(f)$ is locally trivial only when restricted to each strata $\widetilde{W}^D_\epsilon(f; \mathbf{H})$ of $\widetilde{W}_\epsilon(f; \mathbf{H})$. Because of this local triviality, the topology of $\widetilde{\mathcal{L}}(f)$, when restricted to each strata of $\widetilde{W}^D_\epsilon(f; \mathbf{H})$, is well-defined. We will not attempt to specify the topology of $\widetilde{\mathcal{L}}(f)$ over $\widetilde{W}_\epsilon(f; \mathbf{H})$, since the objects which we are intrested in are just the moduli spaces of stable or perturbed stable (J, H)-maps and their topology will be specified by the gluing construction of next section when the intersection patterns of the domains change. What relevant to our later construction are the following "sub-bundles", though we do not really use it. Let $\widetilde{\mathcal{L}}_{D,\delta}(f)$ be the sub-bundle of $\widetilde{\mathcal{L}}_D(f)$ defined by

$$(\widetilde{\mathcal{L}}_{D,\delta}(f))_g = \{\xi \mid \xi \in (\widetilde{\mathcal{L}}_D(f))_g,\ \xi = 0 \text{ on each } \delta\text{-discs around double points}\}.$$

Let $\widetilde{W}^D_{\epsilon,\delta}(f,\mathbf{H}) = \bar\partial^{-1}_{J,H}(\widetilde{\mathcal{L}}_{D,\delta}(f))$. We have the restricted bundle $\widetilde{\mathcal{L}}_{D,\delta}(f) \to \widetilde{W}^D_{\epsilon,\delta}(f,\mathbf{H})$. Now if $D \leq D_1$, we can certainly use parallel transformation to move the fiber of $\widetilde{\mathcal{L}}_{D,\delta}(f)$ over some point in $\widetilde{W}^D_{\epsilon,\delta}(f,\mathbf{H})$ into the fibers of $\widetilde{\mathcal{L}}_{D_1,\delta_1}(f)$ over a neighborhood of the given point in $\widetilde{W}^{D_1}_{\epsilon,\delta_1}(f,\mathbf{H})$, when $\delta_1 << \delta$. These parallel transformations give rise a topology for the union

$$\widetilde{\mathcal{L}}^0(f) = \cup_{D,\delta}\widetilde{\mathcal{L}}_{D,\delta}(f) \to \widetilde{W}^0_\epsilon(f,\mathbf{H}) = \cup_{D,\delta}\widetilde{W}^D_{\epsilon,\delta}(f,\mathbf{H}).$$

The Γ_f actions on $\widetilde{W}^D_\epsilon(f;\mathbf{H})$ and $\widetilde{W}_\epsilon(f;\mathbf{H})$ can be lifted to the bundles via pull-back. Let $W^D_\epsilon(f;\mathbf{H}) = \widetilde{W}^D_\epsilon(f;\mathbf{H})/\Gamma_f$ and $W_\epsilon(f;\mathbf{H}) = \widetilde{W}_\epsilon(f;\mathbf{H})/\Gamma_f$. Then $\mathcal{L}_D(f) = \widetilde{\mathcal{L}}_D(f)/\Gamma_f$ and $\mathcal{L}(f) = \widetilde{\mathcal{L}}(f)/\Gamma_f$ are orbifold bundles over them.

Now for each principal component g_p of g, the Hamiltonian function $H : V \times S^1 \to \mathbf{R}$ gives rise to a section $S_H(g_p)$ of $\wedge^{0,1}(g_p^*TV)$ given by

$$(s,\theta) \mapsto \frac{1}{2}(\nabla_x H + J \circ \nabla_x H \circ i)(g_p(s,\theta),\theta)ds,$$

and we define $S_H(g_b) \equiv 0$ for each bubble component. Clearly S_H is a section of $\widetilde{\mathcal{L}}(f)$ and is smooth on each strata $\widetilde{W}^D(f)$. We have

Lemma 2.7. *The $\bar\partial_{J,H}$-operator gives rise to a Γ_f-equivariant section, still denoted by $\bar\partial_{J,H}$, of the bundle $\widetilde{\mathcal{L}}(f) \to \widetilde{W}_\epsilon(f;\mathbf{H})$ given by $g \mapsto \bar\partial_J g + S_H(g)$. It is smooth on each strata $\widetilde{W}^D_\epsilon(f;\mathbf{H})$, and continuous when restricted to $\widetilde{\mathcal{L}}^0(f) \to \widetilde{W}^0_\epsilon(f;\mathbf{H})$.*

The zero sets $\bar\partial^{-1}_{J,H}(0)$ in $\widetilde{W}^D(f;\mathbf{H})$ and $\widetilde{W}(f;\mathbf{H})$, when projected to $\mathcal{B}^D(\tilde x,\tilde y)$ and $\mathcal{B}(\tilde x,\tilde y)$ are just $\mathcal{M}^D(J,H,\tilde x,\tilde y) \cap W^D_\epsilon(f;\mathbf{H})$ and $\overline{\mathcal{M}}(J,H;\tilde x,\tilde y) \cap W_\epsilon(f;\mathbf{H})$.

Proof. The proof is straightforward. q.e.d.

3. Transversality and gluing

In this section, we will study the transversality of the linearization $L_{(\alpha,\theta,t,\tau)}$ of the $\overline{\partial}_{J,H}$-operator at some "approximate" stable (J,H)-map $f_{(\alpha,\theta,t,\tau)}$. Because of the possible appearance of multiple covered J-holomorphic spheres in the stable (J,H)-map $f_{(0,0,0,0,)}$, $L_{(\alpha,\theta,t,\tau)}$ is not surjective in general. However, we will prove in Proposition 3.1 that the transversality can be achieved modulo $R_{(\alpha,\theta,t,\tau)}$ of a finite dimensional vector space. Using this, we will construct a local moduli space of perturbed stable (J, H)-maps which has right dimension on each of its stratum. As another application of the transversality, we will extend the technique of gluing J-holomorphic curves developed in [11] and [19] to the case of gluing (J, H)-maps for which the transversality may not hold.

3.1 Transversality of $L_{\alpha,\theta}(f)$

Let f be a stable (J, H)-map with intersection pattern D as before. We start with the case of fixed intersection pattern D. Consider the local uniformizer

$$\widetilde{W}_\epsilon^D(f, \mathbf{H}) = \cup_{(\alpha,\theta)\in D} \widetilde{W}_\epsilon^{(\alpha,\theta)}(f_{(\alpha,\theta)}, H)$$

and the bundle $\widetilde{\mathcal{L}}^D(f) = \cup_{(\alpha,\theta)\in D} \widetilde{\mathcal{L}}^{(\alpha,\theta)}(f)$ over it. We can give a coordinate chart of $\widetilde{W}_\epsilon^D(f, \mathbf{H})$ and a trivialization of $\widetilde{\mathcal{L}}_D(f)$ as following: when (α, θ) is fixed, we define

$$V_\epsilon^{(\alpha,\theta)} = \{\, \xi = \xi_{(\alpha,\theta)} \,|\, \xi \in L_k^p(f_{(\alpha,\theta)}^* TV, h),\, \|\xi\|_{k,p} < \epsilon\},$$

where $\xi \in L_k^p(f_{(\alpha,\theta)}^* TV, h)$ means that $\xi_l(x_{l,j}) \in h_{l,j}$, and $x_{l,j}$ is a marked point of a component $\Sigma_{(\alpha,\theta)}^l$ of $\Sigma_{(\alpha,\theta)}$, and $h_{l,j}$ is the tangent space of $\mathbf{H}_{l,j}$ at $x_{l,j}$, $j = 1, \cdots, k_l$, $\sum k_l = k$. It is clear that $V_\epsilon^{(\alpha,\theta)}$ is smooth coordinate chart for $\widetilde{W}_{\epsilon_1}^{(\alpha,\theta)}(f_{(\alpha,\theta)}, \mathbf{H})$ near $f_{(\alpha,\theta)}$ via exponential map $\exp_f^{(\alpha,\theta)} : V_\epsilon^{(\alpha,\theta)} \to \widetilde{W}_{\epsilon_1}^{(\alpha,\theta)}(f_{(\alpha,\theta)}, \mathbf{H})$ given by $\xi \mapsto \exp_{f_{(\alpha,\theta)}} \xi$ when $\epsilon << \epsilon_1$. Note that here we have assumed that all $\mathbf{H}_{l,j}$ are geodesic submanifolds of (V, ω, J) under the induced metric g_J of ω and J. The coordinate chart for $\widetilde{W}_{\epsilon_1}^D(f, \mathbf{H})$ is given by

$$\begin{aligned} \exp_{D,f} &= \{\exp_f^{(\alpha,\theta)}\} : \\ V_\epsilon &= \cup_{(\alpha,\theta)\in D} V_\epsilon^{(\alpha,\theta)} \to \widetilde{W}^D(f, \mathbf{H}) = \cup_{(\alpha,\theta)\in D} \widetilde{W}^{(\alpha,\theta)}(f, \mathbf{H}). \end{aligned}$$

Note that V_ϵ splits as $V_\epsilon^{(0,0)} \times \Lambda_\epsilon$, where $\Lambda_\epsilon = \{(\alpha, \theta) \,|\, (\alpha, \theta) \in D,\, \|\alpha\|, |\theta| < \epsilon\}$. To see this, recall that we have defined $f_{(\alpha,\theta)} = f \circ \phi_{(\alpha,\theta)}$, with

$$\phi_{(\alpha,\theta)} : \Sigma_{(\alpha,\theta)} \to \Sigma,$$

which brings those distinguished points in $\Sigma_{(\alpha,\theta)}$ parametrized by (α, θ) to the corresponding points in Σ, and is equal to identity outside a neighborhood of these points.

Thus the pull-back

$$\phi^*_{(\alpha,\theta)} : L^p_k(f^*TV, h) \to L^p_k(f^*_{(\alpha,\theta)}TV, h)$$

gives a diffeomorphism $V_\epsilon^{(0,0)} \to V_\epsilon^{(\alpha,\theta)}$, and

$$\phi^* = \{\phi^*_{(\alpha,\theta)}\} : V_\epsilon^{(0,0)} \times \Lambda_\epsilon \to V_\epsilon$$

is the required splitting. Now $\exp_{D,f} \circ \phi^*$ gives the local coordinate of $\widetilde{W}^D_{\epsilon_1}(f, \mathbf{H})$ in terms of $V_\epsilon^{(0,0)} \times \Lambda_\epsilon$.

The trivialization of the bundle $\widetilde{\mathcal{L}}^{(\alpha,\theta)} \to \widetilde{W}_\epsilon^{(\alpha,\theta)}(f, \mathbf{H})$ can be obtained by a J-invariant connection ∇ as usual(cf. [16] or [19] for detail). We use $\psi_{(\alpha,\theta)}$ to denote the resulting trivialization:

$$\widetilde{W}_\epsilon^{(\alpha,\theta)}(f, \mathbf{H}) \times L^p_{k-1}(\wedge^{0,1}(f^*_{(\alpha,\theta)}T(M)) \to \widetilde{\mathcal{L}}^{(\alpha,\theta)}(f).$$

Hence, to obtain a trivialization of $\widetilde{\mathcal{L}}_D(f)$, we only need to identify the "central fiber" $L^p_{k-1}(\wedge^{0,1}(f^*_{(\alpha,\theta)}T(M))$ of $\widetilde{\mathcal{L}}^{(\alpha,\theta)}(f)$ with $\widetilde{\mathcal{L}}^p_{k-1}(\wedge^{0,1}(f^*T(M))$ of $\widetilde{\mathcal{L}}^{(0,0)}(f)$.

To this end, observe that $\phi_{(\alpha,\theta)} : \Sigma_{(\alpha,\theta)} \to \Sigma$ also gives a identification of

$$\wedge^1(f^*TV) \to \wedge^1(f^*_{(\alpha,\theta)}TV)$$

via pulling back $\phi^*_{(\alpha,\theta)}$. However, since $\phi_{(\alpha,\theta)}$ is not holomorphic in those annulus around its double points, the image of $\wedge^{0,1}(f^*TV)$ of $\phi^*_{(\alpha,\theta)}$ may not be in $\wedge^{0,1}(f^*_{(\alpha,\theta)}TM)$. Let

$$\pi_2 : \wedge^1 = \wedge^{1,0} \oplus \wedge^{01} \to \wedge^{0,1}$$

be the projections of the second factors. Then $\pi_2 \circ \phi^*_{(\alpha,\theta)}$ induces a $\mathbf{R}$ -linear map from $L^p_{k-1}(\wedge^{0,1}(f^*TV))$ to $L^p_{k-1}(\wedge^{0,1}(f^*_{(\alpha,\theta)}TV))$, which is an isomorphism when ϵ is small enough. Let $\gamma_{(\alpha,\theta)}$ denote

$$\psi_{(\alpha,\theta)} \circ (Id \times \pi_2 \circ \phi^*_{(\alpha,\theta)}) : \widetilde{W}_\epsilon^{(\alpha,\theta)}(f, \mathbf{H}) \times L^p_{k-1}(\wedge^{0,1}(f^*TM)) \to \widetilde{\mathcal{L}}^{(\alpha,\theta)}(f),$$

and $\gamma_D = \{\gamma_{(\alpha,\theta)}\} : \widetilde{W}^D(f, \mathbf{H}) \times L^p_{k-1}(\wedge^{0,1}(f^*TM)) \to \widetilde{\mathcal{L}}_D(f)$.

Now in terms of these local charts and trivialization the $\bar\partial_{J,H}$-section:

$$\widetilde{W}_\epsilon^{(\alpha,\theta)}(f, \mathbf{H}) \to \widetilde{\mathcal{L}}^{(\alpha,\theta)}$$

becomes a function $F_{(\alpha,\theta)}$ between Banach spaces

$$F_{(\alpha,\theta)} : V_\epsilon^{(0,0)} \to L^p_{k-1}(\wedge^{0,1}(f^*TM))$$

for a fixed (α, θ).

Let $L_{(\alpha,\theta)} = DF_{(\alpha,\theta)}(0)$ be the derivative of $F_{(\alpha,\theta)}$ at 0, and $L = DF_{(0,0)}(0)$. A direct calculation shows that (see [16] and [19] for detail.)

$$L(\xi_p) = \nabla\xi_p + J(f) \circ \nabla\xi_p \circ i + \frac{1}{4} N_J(\bar{\partial}_{J,H} f, \xi_p) + \nabla_{\xi_p} S_H,$$

where $S_H = (\frac{1}{2}\nabla_x H + \frac{1}{2} J \circ \nabla_x H \circ i)ds$, ξ_p is a component of ξ over a principal component f_p, and $L(\xi_b)$ is the same as above except deleting the last zero order term involving Hamiltonian perturbation.

The very same formula can also be established for $\widetilde{L}_{(\alpha,\theta)} = D\widetilde{F}_{(\alpha,\theta)}(0)$, where $\widetilde{F}_{(\alpha,\theta)}$ is defined by $F_{(\alpha,\theta)} = \gamma^{-1}_{(\alpha,\theta)} \circ \widetilde{F}_{(\alpha,\theta)} \circ \phi^*_{(\alpha,\theta)}$. From the definition of $\phi_{(\alpha,\theta)}$, which is identity outside a small δ-neighborhood of the double points of $\Sigma_{(\alpha,\theta)}$, it follows that for each ξ_p over the principal component f_p,

$$\begin{aligned} L_{(\alpha,\theta)}(\xi_p) &= DF_{(\alpha,\theta)}(0)(\xi_p) = D\gamma^{-1}_{(\alpha,\theta)} \circ D\widetilde{F}_{(\alpha,\theta)}(0) \circ D\phi^*_{(\alpha,\theta)}(0) \\ &= \nabla\xi_p + J(f_{\alpha,\theta)}) \circ \nabla\xi_p \circ i \\ &\quad + \frac{1}{4}\gamma^{-1}_{(\alpha,\theta)} \circ N_J((\bar{\partial}_{J,H} f_{(\alpha,\theta)}), \phi^*_{(\alpha,\theta)}(\xi_p)) \\ &\quad + \gamma^{-1}_{(\alpha,\theta)}\{\nabla_{\phi^*_{(\alpha,\theta)}(\xi_p)} S_H\} + A_{((\alpha,\theta))}(\xi_p), \end{aligned}$$

where $A_{(\alpha,\theta)}$ is a zero-order "matrix" operator, which is concentrated in same annulus inside the ϵ- neighborhood of double points Σ, and $\|A_{(\alpha,\theta)}\|_{C^m}$ is uniformly bounded with respect to (α, θ) for any fixed $m > 0$. As before, $L_{(\alpha,\theta)}(\xi_b)$ is the same as above but deleting the terms concerning Hamiltonian function H. Form these formulas; $L_{_{(\alpha,\theta)}}$ can be thought as a small deformation of L when ϵ is small. Therefore, we only need to establish required transversality property for L, and the corresponding result for $L_{_{(\alpha,\theta)})}$ will follow when ϵ is small.

Now it is well-known that the operator L is not surjective even for "generic" choice of (J, H) when some of bubble components of f_b of f is multiply covered. However, since each

$$L_l : L^p_k(f_l^*TM, h_l) \to L^p_{k-1}(\wedge^{0,1}(f_l^*TM))$$

is a linear elliptic operator, hence Fredholm, there is only a finite dimensional cokernel $\widetilde{K}_l = \widetilde{K}_l(f)$, which can be identified with

$ker(L_l^*) \subset L_{k-1}^p(\wedge^{0,1}(f_l^*TM))$ with respect to the metrics induced from Σ_l of Σ defined before. Here L_l is induced from L over the corresponding sections, but without the constraints given by those matching conditions for sections of $L_k^p(f_l^*TV, h_l)$ at double points. For later use of extending the vector space $\widetilde{K}_l(f)$ to a vector bundle over $\widetilde{W}_\epsilon(f;\mathbf{H})$, we choose a cut-off function β_l defined on Σ_l, vanishing at each bubble point of Σ and being equal to identity outside a δ_1-neighborhood of each double point, and define

$$K_l = K_l(f) = \{\,\xi \mid \xi = \beta \cdot \eta,\ \eta \in \widetilde{K}_l\}.$$

Since the unique continuation principle holds for the solution set $\widetilde{K}_l$ of L_l^*, it is clear that $dim\widetilde{K}_l(f) = dimK_l(f)$. When δ_1 is small enough, the projection

$$\pi_1 : L_{k-1}^p(\wedge^{0,1}(f^*TV)) = \widetilde{K} \oplus Im(L) \to \widetilde{K}$$

restricting to $K \subset L_{k-1}^p(\wedge^{0,1}(f^*TV))$ gives an isomorphism of K and $\widetilde{K}$. It follows that $L_{k-1}^p(\wedge^{0,1}(f^*TV)) = K \oplus Im(L)$ for δ_1 small enough. Since for generic (J,H), L_p is surjective for each principal component f_p, hence $\widetilde{K}_p = K_p \equiv 0$. We only need to consider bubble components, and we define

$$K = \oplus_{l=1}^{L} K_l = \oplus_{b=1}^{L_2} K_b \subset L_{k-1}^p(\wedge^{0,1}(f^*TM)).$$

It is clear that

$$L_l \oplus I_l : L_k^p(f_l^*TM, h_l) \oplus K_l \to L_{k-1}^p(\wedge^{0,1}(f_l^*TV))$$

is surjective. However, because of the restraints at double points of Σ_f,

$$L \oplus I : L_k^p(f^*TV, h) \oplus K \to L_{k-1}^p(\wedge^{0,1}(f^*TV))$$

may not be surjective, where

$$I = \oplus_{l=1}^{L} I_l : \oplus_{l=1}^{L} K_l \to L_{k-1}^p(\wedge^{0,1} f^*(TV)) = \oplus_{l=1}^{L}(\wedge^{0,1}(f^*TV)).$$

To achieve surjectivity, we need to enlarge the domain of $L \oplus I$ as following:

Let $d_{l,r}, r = 1, \cdots, R_l$ be the double points of the component Σ_l of Σ. The intersection pattern I of Σ determines a pairwise correspondence among the double points of Σ. We denote $I(d_{l,r})$ by $d_{l',r'}$. Let $N_l = kerL_l$ and $N = \prod_{l=1}^{L} N_l$. Consider the evaluation map

$$e_I : N \subset \prod_{l=1}^{L} L_k^p(f_l^* TV, h_l) \to T = \prod_{\{l,r\},\{l',r'\}} (T_{l,r} \times T_{l',r'}),$$

given by

$$(\xi_1, \cdots, \xi_l, \cdots, \xi_L) \to (\cdots, \xi_l(d_{l,r}), \xi_{l'}(d_{l',r'}), \cdots),$$

where $T_{l,r} = T_{l',r'} = T_{f_l(d_{l,r}}V = T_{f_{l'}(d_{l',r'})}V$.

Let $\triangle_I \subset T$ be the multi-diagonal determined by I. Assume that $\widetilde{N} + \triangle_I \subset T$ is a proper subspace, hence the dimension d of $T/(\widetilde{N} + \triangle_I)$ is not 0. Here $\widetilde{N}$ is the image of N under evaluation map e_I. We now construct a d-dimensional subspace Ω of $\prod_{l=1}^{L} L_k^p(f_l^* TV, h_l)$ such that $\Omega \cap N = \{0\}$ and that $e_I : \Omega \oplus N \to T$ is transversal to $\triangle_I$. In fact for the dimension reason, $e_I(\Omega) \oplus (\widetilde{N} + \triangle_I) = T$.

To this end, observe that for any proper subspace S of T which contains $\triangle_I$, there exists at least one pair of double point $d_{l,r}$ and $d_{l',r'}$ such that none of $T_{l,r}$ and $T_{l',r'}$ is contained in S. Note that among Σ_l and $\Sigma_{l'}$, at least one of them, say Σ_l, is a bubble component. Now according to [16], there exists a family of J_t-holomorphic curves $f_l^t : \Sigma_l \to V$ such that:

(1) $f_l^0 = f_l$;

(2) $(\frac{\partial}{\partial t} f_l^t)_{t=0}(d_{l,r}) = t_{l,r} \in T_{l,r}$ for any given $t_{l,r}$ in $T_{l,r}$;

(3) $f_l^t = f_l$ outside a small prescribed neighborhood of $d_{l,r}$ and is J-holomorphic in a smaller neighborhood of $d_{l,r}$.

Taking its linearized form, we conclude that for any $t_{l,r} \in T_{l,r}$, there exists a $\xi \in L_k^p(f_l^* TV)$ such that $\xi(d_{l,r}) = t_{l,r}$ and ξ is equal to zero outside a small neighborhood around $d_{l,r}$. Combining this with the above observation, we can easily construct Ω inductively. Let $\widetilde{\Omega} = L(\Omega)$, where $L = \oplus_{l=1} L_l$. Since $\Omega \cap N = \{0\}$, $dim\widetilde{\Omega} = d$ and L is an isomorphism of Ω and $\widetilde{\Omega}$. Now let

$$J : \widetilde{\Omega} \to L_{k-1}^p(\wedge^{0,1}(f^* TV))$$

be the inclusion. We have

Lemma 3.1.

$$L\oplus I\oplus J : L^p_k(f^*TV,h)\oplus K\oplus\widetilde{\Omega}\to L^p_{k-1}(\wedge^{0,1}(f^*TV))$$

is surjective, and the kernel of this map is the same as N of the kernel L.

Proof. Consider

$$\oplus_l L_l\oplus I\oplus J : \oplus_l L^p_k(f_l^*TV,h_l)\oplus K\oplus\widetilde{\Omega}\to L^p_{k-1}(\wedge^{0,1}f^*TV).$$

To prove the surjectivity of $L\oplus I\oplus J$, it suffices to prove that $Ker(\oplus_l L_l\oplus I\oplus J)$ is transversal to $\triangle_I\subset T$ under the evaluation map e_I acting on the first factor of the kernel. But

$$\begin{aligned} Ker(\oplus_l L_l\oplus I\oplus J) &= Ker(\oplus_l L\oplus J)\\ &= \{\,(\xi,-\sum_l L_l(\xi))\mid \sum_l L_l(\xi)\in\widetilde{\Omega}\}\\ &\cong \Omega\oplus N,\end{aligned}$$

which is transversal to $\triangle_I$ under e_I by our construction.

Now $\xi\in Ker(\oplus_l L_l\oplus I\oplus J)\cong\Omega\oplus N$ belongs to $Ker(L\oplus I\oplus J)$ if and only if $e_I(\xi)\in\triangle_I$. But it follows our construction that $e_I:\Omega\to e_I(\Omega)$ is an isomorphism and $e_I(\Omega)\cap(\triangle_I+e_I(N))=\{0\}$. This implies that if $(\xi,\eta)\in\Omega\oplus N$ such that $e_I(\xi)+e_I(\eta)=\gamma\in\triangle_I$, then

$$e_I(\xi)=\gamma-e_I(\eta)\in e_I(\Omega)\cap(\triangle_I+e_I(N))=\{0\}.$$

Hence $\xi=0$ and $e_I(\eta)\in\triangle_I$. q.e.d.

Let $K\oplus\widetilde{\Omega}=R\subset L^p_{k-1}(\wedge^{0,1}(f^*TV))$. We can extend R over $\widetilde{W}_\epsilon^{(0,0)}(f,\mathbf{H})$ and $\widetilde{W}_\epsilon^D(f,\mathbf{H})=\cup_{(\alpha,\theta)\in D}\widetilde{W}_\epsilon^{(\alpha,\theta)}(f,\mathbf{H})$ with $D=D(f)$ of the intersection pattern f, by using the trivialization of $\widetilde{\mathcal{L}}_D(f)$ over these spaces introduced before this subsection, as long as we know how to extend it over "base point" $f_{(\alpha,\theta)}$, $(\alpha,\theta)\in D$. But, because of the way that we constructed $R(f)$, each element of R vanishes in a neighborhood of double points of Σ_f, and hence can also be regarded as an element of $L^p_{k-1}(\wedge^{0,1}(f^*_{(\alpha,\theta)}TV))$ when $\|\alpha\|$, $\|\theta\|$ is small enough. If $dimR=r$, then we get a r-dimensional vector bundle R over $\widetilde{W}_\epsilon^D(f,\mathbf{H})$. We will use R_g or $R(g)$ to denote the fiber of R over $g\in\widetilde{W}_\epsilon^D(f,\mathbf{H})$.

Corollary 3.1. *When $\|(\alpha,\theta)\|$ is small enough,*

$$L_{(\alpha,\theta)} \oplus M_{(\alpha,\theta)} : L^p_k(f^*_{(\alpha,\theta)}TV, h) \oplus R(f_{(\alpha,\theta)}) \to L^p_{k-1}(\wedge^{0,1} f^*_{(\alpha,\theta)}(TV))$$

is surjective, where

$$M_{(\alpha,\theta)} : R(f_{(\alpha,\theta)}) \hookrightarrow L^p_{k-1}(\wedge^{0,1} f^*_{(\alpha,\theta)}(TV))$$

is the inclusion.

Even though the moduli space $\widetilde{\mathcal{M}}^D(J,H;\tilde{x},\tilde{y}) \cap \widetilde{W}^D_\epsilon(f,\mathbf{H})$ may not be a manifold, if we replace the equation $\bar{\partial}_{J,H} g = 0$ by a weaker form $\bar{\partial}_{J,H} g \in R$ in $\widetilde{W}^D_\epsilon(f,\mathbf{H})$, then from the implicit function theorem and the above Lemma and its corollary follows

Corollary 3.2. *When ϵ is small enough,*

$$\widetilde{\mathcal{M}}^D_{R,\epsilon}(J,H;\tilde{x},\tilde{y}) = \{\, g \mid g \in \widetilde{W}^D_\epsilon(f,\mathbf{H}),\ \bar{\partial}_{J,H} \in R\,\}$$

is a smooth manifold of dimension $r + Index(L_D)$.

3.2 Main estimate for $L_{(\alpha,\theta,t,\tau)}$

Now we can extend all notions in previous subsection, which only involve a fixed intersection pattern of domain parametrized by (α,θ), to incorporate the changes of the topological type of the the domains described by the gluing parameter (t,τ). Since this extension is straightforward, we only summarize up the result here.

For a fixed parameter (α,θ,t,τ),

$$V^{(\alpha,\theta,t,\tau)}_\epsilon = \{\, \xi \mid \xi \in L^p_k(f^*_{(\alpha,\theta,t,\tau)}TV, h),\ \|\xi\|_{k,p} < \epsilon\}$$

gives a coordinate of $\widetilde{W}^{(\alpha,\theta,t,\tau)}_\epsilon(f,\mathbf{H})$ via exponential map. The trivialization of $\widetilde{\mathcal{L}}^{(\alpha,\theta,t,\tau)}(f) \to \widetilde{W}^{(\alpha,\theta,t,\tau)}_\epsilon(f,\mathbf{H})$ via a J-parallel connection is the same as before. With respect to this coordinate and this trivialization, the section

$$\bar{\partial}_{J,H} : \widetilde{W}^{(\alpha,\theta,t,\tau)}_\epsilon(f,\mathbf{H}) \to \widetilde{\mathcal{L}}^{(\alpha,\theta,t,\tau)}(f)$$

becomes a function

$$F_{(\alpha,\theta,t,\tau)} : V^{(\alpha,\theta,t,\tau)}_\epsilon \to L^p_{k-1}(\wedge^{0,1}(f^*_{(\alpha,\theta,t,\tau)}TV))$$

whose linearization

$$L_{(\alpha,\theta,t,\tau)} = DF_{(\alpha,\theta,t,\tau)}(0) : L^p_k(f^*_{(\alpha,\theta,t,\tau)}TV, h) \to L^p_{k-1}(\wedge^{0,1} f^*_{(\alpha,\theta,t,\tau)}TV))$$

is an elliptic operator.

Note that all Sobolev L^p_k-norm above are measured with respect to the "standard" metric of $\Sigma_{(\alpha,\theta,t,\tau)}$ obtained from the metric of $\Sigma_{(\alpha,\theta)}$ via gluing, and the metric on $\Sigma_{(\alpha,\theta)}$ is the cylindrical one on each principal component and spherical one on each bubble component. Our goal of this subsection is to obtain a uniform estimate for the right inverse of the operator $L_{(\alpha,\theta,t,\tau)} \oplus M_{(\alpha,\theta,t,\tau)}$ with respect to the parameters

$$(\alpha, \theta, t, \tau) \in \Lambda_\delta = \{(\alpha, \theta, t, \tau) \mid \|(\alpha, \theta, t, \tau)\| < \delta\}.$$

To simplify our notation, throughout this subsection we will use (y, v) to denote the parameter $(\alpha, \theta, t, \tau)$ with $y = (\alpha, \theta)$ and $v = (t, \tau)$, wherever the context is clear.

To obtain the desired uniform estimate we have to use some exponential weighted equivalent norms on $L^p_k(f^*_{(y,v)}TV, h)$ and $L^p_{k-1}(\wedge^{0,1}(f^*_{(y,v)}TV))$ that we describe now. Note that along each cylindrical end of principal component of $f_{(\alpha,\theta)}$, the metrics on the above two spaces are already exponentially weighted. Therefore, we only need to consider the double points of $\Sigma_{(\alpha,\theta)}$, rather than those $y'_{\pm,\infty}$. To simplify our notation, we may assume that $(\alpha, \theta) = (0, 0)$ and use Σ to denote $\Sigma_{(0,0)}$. Let $d_{l,r}, r = 1.\cdots, R_l$, be the double points of Σ_l, and $D_{\delta_1}(d_{l,r})$ be the δ_1-discs centered at $d_{l,r}$ with cylindrical coordinate $(s_{l,r}, \phi_{l,r})$. Here, $d_{l,r}$ has been considered as a cylindrical end with $s_{l,r} = \pm\infty$, and $\partial D_{\delta_1}(d_{l,r})$ corresponds to $\{s_{l,r} = 0\}$. Now if $I(l, r) = (l', r')$, we use $A_{l,r} = A_{l',r'}$ to denote the annulus in $\Sigma_{(0,0,t,\tau)}$ which is the union of the sets $\{(s_{l,r}, \phi_{l,r}) \mid 0 < s_{l,r} < -\log t_{l,r}\}$ and $\{(s_{l',r'}, \phi_{l',r'}) \mid 0 < s_{l',r'} < -\log t_{l',r'}\}$. Let $T_{l,r} = T_{f(d_{l,r})}V$. We may assume that $f(D_\delta(d_{l,r}))$ is contained in the image of normal coordinate $N_{l,r} \subset T_{l,r}$ at $f(d_{l,r})$. Therefore, each vector $\tilde{u} \in T_{l,r}$ can be thought of as a vector field over $N_{l,r}$ and hence a vector field u of $f^*_{(y,v)}TV$. Here we have used a cut-off function supported in $A_{l,r}$ to extend u to a section of $f^*_{y,v}TV$, still denoted by u. Let $S^1_{l,r} = S^1_{l',r'}$ be the "central circle" in $A_{l,r}$ with coordinate,

$$s_{l,r} = -\log|t_{l,r}| \quad or \quad s_{l',r'} = -\log|t_{l',r'}|,$$

then $f_{(y,v)}(S^1_{l,r}) \equiv f(d_{l,r})$. Therefore, for any $\xi \in L^p_k(f^*_{(y,v)}TV)$, we define

$\tilde{\xi}^0 = (\tilde{\xi}^0_{l,r})$ with each $\tilde{\xi}^0_{l,r} \in T_{l,r}$ given by:

$$\tilde{\xi}^0_{l,r} = \int_{S^1_{l,r}} \xi d\phi_{l,r}.$$

Let $\xi^0_{l,r}$ be the corresponding C^∞-section of $f^*_{(y,v)}TV$. When δ_1 is small enough, we may assume that the annulus $A_{l,r}$ are mutually disjoint. Set

$$\xi^0 = \sum_{l,r} \xi^0_{l,r}$$

and $\xi^1 = \xi - \xi^0$.

Definition 3.1. Let $0 < \mu < 1$. For any η in $L^p_{k-1}(\wedge^{0,1}(f^*_{(y,v)}TV))$ and ξ in $L^p_k(f^*_{(y,v)}TV, h)$, we define

$$\|\eta\|_{\chi;k-1,p} = \|\eta\|_{k-1,p;\mu} = \|e^{\mu\cdot s} \cdot \eta\|_{k-1,p}$$

and

$$\|\xi\|_{\chi;k,p} = \|\xi^1\|_{k,p;\mu} + |\xi^0| = \|e^{\mu\cdot s}\xi\|_{k,p} + |\xi^0|,$$

where $|\xi^0| = |\tilde{\xi}^0|$, and $e^{\mu\cdot s}$ is equal to $e^{\mu\cdot s_{l,r}}$ and $s^{\mu\cdot s_{l',r'}}$ on $A_{l,r}$ of $\Sigma_{(y,v)}$ and constant on $\Sigma_{(y,v)} \setminus \cup_{l,r} A_{l,r}$. For $(\xi, \zeta) \in L^p_k(f^*_{(y,v)}TV) \oplus R(f_{(y,v)})$, define $\|(\xi,\zeta)\|_{\chi;k,p} = \|\xi\|_{\chi;k,p} + |\zeta|$.

Proposition 3.1. *The operator* $L_{(y,v)} \oplus M_{(y,v)}$:

$$L^p_k(f^*_{(y,v)}TV, h) \oplus R(f_{(y,v)}) \to L^p_{k-1}(\wedge^{0,1}(f^*_{(y,v)}TV))$$

has a right inverse G under the above norm in the sense that there exists a constant $c = c(f)$ depending only on f but not on the parameter $(y, v) \in \Lambda_\delta$ such that for δ small enough,

$$\|G_{(y,v)}\eta\|_{\chi;k,p} < c(f)\|\eta\|_{\chi;k-1,p}$$

for any $\eta \in L^p_{k-1}(\wedge^{0,1}(f^*_{(y,v)}TV))$.

Proof. Let $N_{(y,v)}$ be the asymptotic kernel of $L_{(y,v)} \oplus M_{(y,v)}$, which is obtained from the kernel N of $L \oplus M$ by multiplying each element $\tilde{n}$ of N a cut-off function $\beta_{(\alpha,\theta,t,\tau)} = \beta_{(0,0,t,0)}$, denoted by β_t, defined by

$$\begin{cases} \beta_t(x) = 1 & if\ x \in \Sigma \setminus (U_1 \cup U_2), \\ \beta_t(x) = 0 & if\ the\ s_{l,r}\ or\ s_{l',r'}\ coordinate\ of\ x > -\log t_{l,r} - 3, \end{cases}$$

where

$$U_1 = \cup_{\{l,r\}} \{(s_{l,r}, \phi_{l,r}) \mid s_{l,r} > -\log t_{l,r} - 4\}$$

and

$$U_2 = \cup_{\{l',r'\}} \{(s_{l',r'}, \phi_{l',r'}) \mid s_{l',r'} > -\log t_{l',r'} - 4\}.$$

Now by our construction of $f_{(y,v)}$, $f_{(y,v)}(x) = f(d_{l,r})$ if the $(s_{l,r}, t_{l,r})$ or $(s_{l',r'}, t_{l',r'})$ coordinate of $x > -\log t_{l,r} - 2$. This implies that

$$\beta_{(y,v)} \cdot \tilde{n} \in L^p_k(f^*_{(y,v)} TV, h).$$

Let $N^*_{(y,v)}$ be the L^2-orthogonal complement of $N_{(y,v)}$ in $L^p_k(f^*_{(y,v)} TV, h)$, where the L^2-norm is defined with respect to the "standard" metric on $\Sigma_{(y,v)}$.

Set

$$C_{(y,v)} = N^*_{(y,v)} \oplus R(f_{(y,v)}).$$

Because that the index of $L_{(y,v)}$ is the same as the index of L, we only need to prove that for any $c = (\xi, \gamma) \in C_{(y,v)}$, there exists a constant $C = C(f)$ such that

$$\|(\xi, \gamma)\|_{\chi;k,p} \leq C \|L_{(y,v)}(\xi) + \gamma\|_{\chi;k-1,p}$$

for $|(y,v)|$ small enough.

If this is not true, then there exists a sequence of

$$c_{(y,v)} = (\xi_{(y,v)}, \gamma_{(y,v)}) \in C_{(y,v)}$$

with $|(y,v)| \to 0$, such that

(1) $\|c_{(y,v)}\|_{\chi;k,p} = \|\xi^1_{(y,v)}\|_{k,p,\mu} + |\xi^0_{(y,v)}| + |\gamma_{(y,v)}| = 1$;

(2) $\|L_{(y,v)}(\xi_{(y,v)}) + \gamma_{(y,v)}\|_{\chi;k-1,p} \to 0$, when $(y,v) \to 0$.

We will prove that (1) and (2) contradict each other.

In the proof, we will repeatedly use the following facts:

Lemma 3.2. *Let B be a Banach space with a norm $\|\cdot\|_B$ and $\Psi : B \to \mathbf{R}^+$ be a convex continuous function. If $\{x_{(y,v)}\}$ is a sequence in B such that $x_{(y,v)} \to x$ weakly for some $x \in B$, then $\Psi(x) \leq \liminf_{(y,v)} \Psi(x_{(y,v)})$.*

We will apply this when Ψ is continuous semi-norm with respect to $\|\cdot\|_B$.

Lemma 3.3. *(1) and (2) above imply that there exists a sequence*

$$\{(\xi_{(y,v)}, \gamma_{(y,v)})\}$$

such that

$$|\xi^0_{(y,v)}| + |\gamma_{(y,v)}| \to 0,$$

when $(y, v) \to 0$.

Proof. By definition,

$$|\xi^0_{(y,v)}| = |\tilde{\xi}^0_{(y,v)}|.$$

From (1), we know that

$$|\tilde{\xi}^0_{(y,v)}| + |\gamma_{(y,v)}| \le 1.$$

This implies that there exists a convergent subsequence $\{(\tilde{\xi}^0_{(y,v)}, \gamma_{(y,v)})\}$, with limit $\tilde{\xi}^0_0 \in T = \prod_{l,r} T_{l,r}$ and $\gamma_0 \in R(f)$. We only need to prove that $\tilde{\xi}^0 = 0$, $\gamma_0 = 0$. The idea of the proof is to construct an element $\xi \in N^*$, such that $\xi(d_{l,r}) = (\tilde{\xi}^0)_{l,r}$ and $L(\xi) + \gamma_0 = 0$. But this latter condition gives that $\gamma_0 = 0$ and $\xi \in N$, hence $\xi \in N \cap N^* = \{0\}$. Therefore, $\tilde{\xi}^0 = 0$.

To this end, we define ξ^0_0 from $\tilde{\xi}^0_0$ in the same way as we define $\xi^0_{(y,v)}$ from $\tilde{\xi}^0_{(y,v)}$. Thus $\xi^0_0 \in \Gamma(f^*(TV), h)$. It is clear that as $(y, v) \to 0$, $\xi^0_{(y,v)}$ is locally C^∞-convergent to ξ^0_0 in $\Sigma \setminus \cup_{l,r}\{d_{l,r}\}$.

Given $R > 0$, let D_R be the domain in $\Sigma_{(y,v)}$ (or in Σ) with complement

$$D^-_R = \cup_{l,r}\{(s_{l,r}, \phi_{l,r}) \mid s_{l,r} \ge R\}.$$

From (1), we know that $\|\xi^0_{(y,v)}\| \le 1$. This implies that for any $R > 0$, there exists a $C(R)$ depending on R such that $\|\xi^1_{(y,v)}\|_{k,p} \le C(R)$ for all (y, v). Note that when (y, v) is small enough, all these $\xi^1_{(y,v)}|_{D_R}$ lie in the same space for a fixed R by our construction of $f_{(y,v)}$. Therefore, $\xi^1_{(y,v)}|_{D_R} \to \xi^1_{0,;R}$ weakly in L^p_k-space for some $\xi^1_{o;R} \in L^p_k(f|_{D_R})$. By letting $R \to \infty$ and taking a diagonal subsequence, we conclude that by a standard Sobolev embedding argument that all these $\xi^1_{0;R}$'s can be pasted together to yield a single section $\xi^1_0 \in L^p_{k,loc}(f, h)$ such that

$$\xi^1_{(y,v)}|_{D_R} \to \xi^1_0|_{D_R} = \xi^1_{0;R}$$

weakly in L^p_k-space. Note here ξ^1_0 is only defined on the smooth part of Σ_f.

Let $\xi_0 = \xi_0^0 + \xi_0^1$. Then $\xi_{(y,v)}|_{D_R} \to \xi_0|_{D_R}$ weakly in L_k^p-space. Therefore,

$$L_{(y,v)}\xi_{(y,v)}|_{D_R} \to L\xi_0|_{D_R}$$

weakly in L_{k-1}^p -space. Our assumption (2) implies that

$$\|(L_{(y,v)}\xi_{(y,v)} + \gamma_{(y,v)})|_{D_R}\|_{k-1,p} \to 0$$

as $(y,v) \to 0$ for any fixed $R > 0$. From Lemma 3.2, we conclude that

$$\|(L(\xi_0) + \gamma_0)|_{D_R}\|_{k-1,p} \le \liminf_{(y,v)\to 0} \|(L_{(y,v)}\xi_{(y,v)} + \gamma_{(y,v)})|_{D_R}\|_{k-1,p} = 0.$$

It follows that $(L\xi_0 + \gamma_0)|_{D_R} = 0$ for any $R > 0$, hence $L\xi_0 + \gamma_0 = 0$. By our construction γ_0 vanishes on $D_{\delta_1}(d) = \cup_{l,r} D_{\delta_1}(d_{l,r})$ of a δ_1-neighborhood of double points of Σ for some small $\delta_1 > 0$. It follows that $L(\xi_0) = 0$ on $D_{\delta_1}(d) \setminus \{d\}$. This together with the fact that $\|\xi_0|_{D_{\delta_1}(d)\setminus\{d\}}\|_{L^2}$ is bounded, which follows from our assumption (1) and Lemma 3.2, implies that for each component $(\xi_0)_l$ of ξ_0 the singularity of $(\xi_0)_l$ at $d_{l,r}$, $r = 1, \cdots, R_l$, is removable. Therefore each $(\xi_0)_l$ extends to a section of $L_k^p(f_l^*TV)$. However, to prove that $\xi_0 \in L_k^p(f^*TV, h)$, we need to prove that , for any pair $d_{l,r} = d_{l',r'}$, $(\xi_0)_l(d_{l,r}) = (\xi_0)_{l'}(d_{l',r'})$. Note that ξ_0^0 is already smooth, therefore in the cylindrical coordinate $(s_{l,r}, \phi_{l,r})$ near $d_{l,r}$, $r = 1, \cdots, R_l$, all these three sections $(\xi_0)_l$, $(\xi_0^0)_l$ and $(\xi_0^1)_l$ of the bundle f_l^*TV over $\Sigma_l \setminus \cup_{r=1}^{R_l} d_{l,r}$ are convergent uniformly with respect to $\phi_{l,r}$ as $s_{l,r} \to \infty$. Combining this with the fact that

$$\|(\xi_0^1)_l\|_{0,p;\mu} \le \liminf_{(y,v)\to 0} \|\xi_{(y,v)}^1\|_{0,p;\mu} \le 1,$$

which follows from (1) and Lemma 3.2, we conclude that

$$\lim_{s_r\to\infty} (\xi_0^1)_l = 0 \quad , r = 1, \cdots, R_l.$$

Therefore,

$$\begin{aligned} (\xi_0)_l(d_{l,r}) &= \lim_{s_{l,r}\to\infty} (\xi_0)_l(s_{l,r}, \phi_{l,r}) \\ &= \lim_{s_{l,r}\to\infty} (\xi_0^0)_l(s_{l,r}, \phi_{l,r}) = (\tilde{\xi}_0^0)_{l,r}. \end{aligned}$$

Now the same calculation also applies to $(\xi_0)_{l'}$ with the same conclusion that

$$(\xi_0)_{l'}(d_{l',r'}) = (\tilde{\xi}_0^0)_{l',r'}.$$

Now ξ_0 extends to a section of $L_k^p(f^*TV, h)$ such that $L\xi_0 + \gamma_0 = 0$. This proves that $\gamma_0 = 0$ and $\xi_0 \in N$. To see $\xi_0 \in N^*$, note that each $\xi_{(y,v)} \in N^*_{(y,v)}$. Then the conclusion follows from the construction of $N_{(y,v)}$ and Sobolev embedding theorem. Therefore, $\xi_0 \equiv 0$, and $\tilde{\xi}^0 = \xi_0(d)$ is also equal to zero as shown above. q.e.d.

Now $L_{(y,v)}$ is a first order operator with a zero order term that exponentially decays along each double point considered as a cylindrical ends. This together with the fact that $\xi^0_{(y,v)}$ is essentially a "constant" vector field with $|\xi^0_{(y,v)}| \to 0$ as $(y,v) \to 0$ proved above yields

Lemma 3.4. *The condition (1) and (2) in Lemma 3.3 also implies that*

$$\lim_{(y,v)\to 0} \|L_{(y,v)}\xi^0_{(y,v)}\|_{k-1,p;\mu} = 0.$$

From Lemma 3.3 and Lemma 3.4, we may assume that

$$\begin{array}{ll} (I) & \|\xi^1_{(y,v)}\|_{k,p;\mu} = 1, \\ (II) & \|L_{(y,v)}\xi^1_{(y,v)}\|_{k-1,p;\mu} \to 0 \quad as\ (y,v) \to 0. \end{array}$$

Now we need to prove that (I) and (II) contradict each other. To do this, we need to have an estimate of $\xi^1_{(y,v)}$ on those middle annulus $A_{l,r} = A_{l',r'}$ with $\{l', r'\} = I(l,r)$, where $A_{l,r} \subset \Sigma_{(y,v)}$ is defined as before, but instead of using the coordinate $(s_{l,r}, \phi_{l,r})$ and $(s_{l',r'}, \phi_{l',r'})$, we introduce a new cylindrical coordinate $(\gamma_{l,r}, \phi_{l,r})$ on $A_{l,r}$. Since the estimate can be done for each $A_{l,r}$ separately, we will suppress all subscriptions in all notation introduced.

Let β be a cut-off function on $\Sigma_{(y,v)}$ which is supported in

$$-3 < \gamma < 3$$

and equal to 1 on $-2 < \gamma < 2$.

Lemma 3.5.

$$\lim_{(y,v)\to 0} \|\beta\xi^1_{(y,v)}\|_{k,p;\mu} = 0.$$

Proof. Let $T_{(y,v)} = -\log T$ be the length of the cylindrical coordinate along γ-direction. Define

$$\zeta_{(y,v)} : [-T_{(y,v)}, T_{(y,v)}] \times S^1 \to T_{f(d)}V$$

by

$$D_{\exp_{f(d)}}(\tilde{f}_{(y,v)}(\gamma,\phi))(\zeta_{(y,v)}(\gamma,\phi)) = e^{T_{(y,v)}\cdot\mu}\xi^1_{(y,v)}(\gamma,\phi).$$

Extend $\zeta_{(y,v)}$ trivially over the whole cylinder. Then from (I) there exists a constant c such that

$$\|e^{-\mu|\gamma|}\cdot\zeta_{(y,v)}\|_p \quad\leq\quad c. \tag{0.2}$$

for all (y,v). Let $\zeta_{(y,v);R}$ be the restriction of $\zeta_{(y,v)}$ to the domain $\mathbf{Z}_R=[-\mathbf{R},\mathbf{R}]\times S^1$. Hence from (I) again, there exists a constant $C(R)$ depending on R such that $\|\zeta_{(y,v);R}\|_{k,p}\leq C(R)$ for all (y,v). Therefore, as $(y,v)\to 0$,

$$\zeta_{(y,v);R}\quad\to\quad\zeta_{0;R}, \tag{0.3}$$

weakly in $L^p_k(\mathbf{Z}_R,T_{f(d)}V)$ for some $\zeta_{0;R}\in L^p_k(\mathbf{Z}_R,T_{f(d)}V)$. By the same reason in the proof of Lemma 3.3, we have that all these $\zeta_{0;R}$'s agree with each other on their overlaps to form a single element

$$\zeta_0\in L^p_{k,loc}(\mathbf{R}^1\times S^1,T_{f(d)}V)$$

such that $\zeta_0|_{\mathbf{Z}_R}=\zeta_{0;R}$. Now (0.3) implies that when $(y,v)\to 0$,

$$\bar\partial_{J_0}\zeta_{(y,v);R}\quad\to\quad\bar\partial_{J_0}\zeta_{0;R} \tag{0.4}$$

weakly in $L^p_{k-1}(\mathbf{Z}_R,T_{f(d)}V)$ when $\bar\partial_{J_0}$ is the standard Cauchy-Riemann operator.

Let $\widetilde{L}_{(y,v)}$ be the lifting of $L_{(y,v)}$ under $\exp_{f(d)}$, and $\widetilde{J}$ be the corresponding lifting of the almost complex structure J. Then $\widetilde{L}_{(y,v)}$ is of the form

$$\begin{aligned}\widetilde{L}_{(y,v)}(\zeta_{(y,v);R}) \quad=\quad & \frac{\partial\zeta_{(y,v);R}}{\partial\gamma}+J_0\frac{\partial\zeta_{(y,v);R}}{\partial\phi}\\ & +(\widetilde{J}-J_0)(\tilde f_{(y,v)})\frac{\partial\zeta_{(y,v);R}}{\partial\phi}+A_{(y,v);R}\zeta_{(y,v);R},\end{aligned}$$

where $A_{(y,v);R}$ is the restriction to $\mathbf{Z}_R$ of some zero order operator $A_{(y,v)}$. It is clear that when R is fixed,

$$\lim_{(y,v)\to 0}|A_{(y,v);R}|=0,\quad\text{and}\quad\lim_{(y,v)\to 0}|\widetilde{J}-J_0|_{\tilde f_{(y,v);R}}|=0.$$

From (II) we have

$$\lim_{(y,v)\to 0}\|\widetilde{L}_{(y,v)}\zeta_{(y,v);R}\|_{k-1,p}=0.$$

Hence

$$\lim_{(y,v)\to 0} \|\bar\partial_{J_0}\zeta_{(y,v);R}\|_{k-1,p} \quad = \quad 0. \tag{0.5}$$

By this,(0.4) and Lemma 3.2 we obtain

$$\|\bar\partial_{J_0}\zeta_{0;R}\|_{k-1,p} \le \liminf_{(y,v)\to 0} \|\bar\partial_{J_0}\zeta_{(y,v);R}\| = 0.$$

Thus $\bar\partial_{J_0}\zeta_{0;R} = 0$ for any $R > 0$, and therefore

$$\bar\partial_{J_0}\zeta_0 \quad = \quad 0. \tag{0.6}$$

Now (I) implies that

$$\|\zeta_{(y,v);R}\|_{0,p;(-\mu)} = \|e^{-\mu|\gamma|}\zeta_{(y,v)}\|_p$$

is bounded independently on R, that $\|\zeta_0\|_{0,p;(-\mu)} < \infty$. This together with (0.6) and the fact that the constant Fourier component of $\zeta_0|_{\{0\}\times S^1}$ is zero lead to that $\zeta_0 = 0$. By Sobolev embedding theorem we conclude that for any fixed $R > 0$, $\zeta_{(y,v);R}$ is C^{k-1}-convergent to zero. Therefore, when $(y,v) \to 0$,

$$\begin{aligned} \|\beta\zeta_{(y,v)}\|_{k,p} &\le C\|\bar\partial_{J_0}(\beta\zeta_{(y,v)})\|_{k-1,p} && (0.7)\\ &\le C(\|\beta'\zeta_{(y,v)}\|_{k-1,p} + \|\beta\bar\partial_{J_0}\zeta_{(y,v)}\|_{k-1,p}) \to 0. && (0.8)\end{aligned}$$

This implies that

$$\lim_{(y,v)\to 0} e^{\mu T((y,v))}\|\beta\xi^1_{(y,v)}\|_{k,p} = 0.$$

Hence,

$$\lim_{(y,v)\to 0} \|\beta\xi^1_{(y,v)}\|_{k,p;\mu} = 0.$$

q.e.d.

Finishing the proof of Proposition 3.1

Let $L^p_{k;\mu}(f^*TV,h)$ and $L^p_{k-1;\mu}(\wedge^{0,1}(f^*TV))$ be the weighted Sobolev spaces of sections of (f^*TV,h) and $\wedge^{0,1}(f^*TV)$ over $\Sigma\setminus\cup_{l,r}\{d_{l,r}\}$ with cylindrical ends near each $d_{l,r}$.

It is well-known (Ref. [4] and [14]) that when $0<\mu<1$,

$$L = L_\mu : L^p_{k;\mu}(f^*TV,h) \to L^p_{k-1,p;\mu}(\wedge^{0,1}(f^*TV))$$

is Fredholm. It follows that there exists a constant $c = c(f)$ such that for any $\xi \in L^p_{k;\mu}(f^*TV, h)$,

$$\|\xi\|_{k,p;\mu} \le c(\|L_\mu \xi\|_{k-1,p;\mu} + |\xi_{N_\mu}|),$$

where ξ_{N_μ} is the L^2-projection of ξ to the kernel N_μ of L_μ. Here the L^2-norm is still measured with respect to the "standard" metric of $\Sigma \setminus \cup_{l,r}\{d_{l,r}\}$ induced from Σ. Because of the exponential norm imposed on $L^p_{k;\mu}(f^*TV, h)$, the removable singularity theorem is also applicable to each element of N_μ, and thus $N_\mu \subset N$.

Now $(1-\beta)\xi^1_{(y,v)}$ is in $L^p_{k;\mu}(f^*TV, h)$. Therefore, there exists a constant C independent of (y, v) such that

$$\begin{aligned}
&\|(1-\beta)\xi^1_{(y,v)}\|_{k,p;\mu} \\
&\quad \le C\{\|L_\mu((1-\beta)\xi^1_{(y,v)})\|_{k-1,p;\mu} + |(1-\beta)(\xi^1_{(y,v)})_{N_\mu}|\} \\
&\quad \le C\{\|L_{(y,v)}((1-\beta)\xi^1_{(y,v)})\|_{k-1,p;\mu} \\
&\qquad + C\lim_{(y,v)\to 0} |((1-\beta)\xi^1_{(y,v)})_{N_{(y,v)}}| \\
&\quad \le C\{2\|L_{(y,v)}\xi^1_{(y,v)}\|_{k-1.p;\mu} + \|\beta'\xi^1_{(y,v)}\|_{k-1,p;\mu} \\
&\qquad + C\lim_{(y,v)\to 0} |(\beta\xi^1_{(y,v)})_{N_{(y,v)}}| \to 0,
\end{aligned}$$

when $(y, v) \to 0$.

Hence,

$$\|\xi^1_{(y,v)}\|_{k,p;\mu} \le \|(1-\beta)\xi^1_{(y,v)}\|_{k,p;\mu} + \|\beta\xi^1_{(y,v)}\|_{k,p;\mu} \to 0,$$

when $(y, v) \to 0$. This contradicts to (I). q.e.d.

3.3 Gluing

Now a direct computation shows that the pre-gluing $f_{(y,v)}$ is an asymptotic solution of $\bar\partial_{J,H} g = 0$ when f is a stable (J, H)-map. More precisely, we have

Lemma 3.6.

$$\lim_{(y,v)\to 0} \|\bar\partial_{J,H} f_{(y,v)}\|_{\chi;k-1,p} = 0.$$

To do gluing, we also need an estimate on the second order term $Q_{(y,v)}$ in the Taylor expansion of $F_{(y,v)}$:

$$V^{(y,v)}_\epsilon \subset L^p_k(f^*_{(y,v)}TV, h) \to L^p_{k-1}(\wedge^{0,1}(f^*_{(y,v)}TV)),$$

where $Q_{(y,v)}$ is defined by

$$F_{(y,v)}(\xi) = F_{(y,v)}(0) + L_{(y,v)}(\xi) + Q_{(y,v)}(\xi).$$

Lemma 3.7. *There exists a constant $C_1 = C_1(f)$ only depending on f such that for any $\xi_{(y,v)}, \eta_{(y,v)} \in L^p_k(f^*_{(y,v)}TV, h)$,*

$$(i) \qquad \|Q(\xi_{(y,v)})\|_{\chi;k-1,p} \le C_1\|\xi_{(y,v)}\|_\infty\|\xi\|_{\chi;k,p};$$

$$(ii) \qquad \|Q(\xi_{(y,v)}) - Q(\eta_{(y,v)})\|_{\chi;k-1,p}$$
$$\le C(\|\xi_{(y,v)}\|_{x;k,p} + \|\eta_{(y,v)}\|_{\chi;k,p})\|\xi_{(y,v)} - \eta_{(y,v)}\|_{\chi;k,p}.$$

Proof. The corresponding statement was proved in [4] when $k = 1$, and $1 - \frac{2}{p} > 0$. The general case here follows from that by a direct induction argument. q.e.d.

Lemma 3.8 (Picard method). *Assume that a smooth map $f : E \to F$ from Banach spaces $(E, \|\cdot\|)$ to F has a Taylor expansion*

$$f(\xi) = f(0) + Df(0)\xi + Q(\xi)$$

such that $Df(0)$ has a finite dimensional kernel and a right inverse G satisfying

$$\|GQ(\xi) - GQ(\eta)\| \le C(\|\xi\| + \|\eta\|)\|\xi - \eta\|$$

for some constant C. Let $\delta_1 = \frac{1}{8C}$. If $\|G \circ f(0)\| \le \frac{\delta_1}{2}$, then the zero set of f in $B_{\delta_1} = \{\xi, \mid \|\xi\| < \delta_1\}$ is a smooth manifold of dimension equal to the dimension of $kerDf(0)$. In fact, if

$$K_{\delta_1} = \{\xi \,|\xi \in kerDf(0), \|\xi\| < \delta_1\}$$

and $K^\perp = G(F)$, then there exists a smooth function

$$\phi : K_{\delta_1} \to K^\perp$$

such that $f(\xi + \phi(\xi)) = 0$ and all zeros of f in B_{δ_1} are of the form $\xi + \phi(\xi)$.

The proof of this Lemma is an elementary application of Banach's fixed point theorem (see [4]). Applying this to our case, we get the following gluing construction over a local uniformizer $\widetilde{W}_\epsilon(f, \mathbf{H})$.

Recall that given a stable (J, H)-map $[f] \in \mathcal{B}^D(\tilde{x}, \tilde{y})$ of intersection pattern D, let $\widetilde{W}^D_\epsilon(f, \mathbf{H}) \subset \widetilde{W}_\epsilon(f, \mathbf{H})$ be the local uniformizers of the neighborhoods

$$W^D_\epsilon(f, \mathbf{H}) = \widetilde{W}^D_\epsilon(f, \mathbf{H})/\Gamma_f \subset W_\epsilon(f, \mathbf{H}) = \widetilde{W}_\epsilon(f, \mathbf{H})/\Gamma_f$$

of $[f]$ in $\mathcal{B}^D(\tilde{x},\tilde{y}) \subset \overline{\mathcal{B}}(\tilde{x},\tilde{y})$. We defined in Sec.3.1 a finite dimensional vector bundle R over $\widetilde{W}_\epsilon(f,\mathbf{H})$ and proved that the local moduli space $\mathcal{M}^D_{R,\epsilon}(J,H;\tilde{x},\tilde{y})$ of stable maps in $\widetilde{W}^D_\epsilon(f,\mathbf{H})$ that satisfy the weaker equation $\bar{\partial}_{J,H} f \in R$ is a smooth manifold. Now we extend this to $\widetilde{W}_\epsilon(f,\mathbf{H})$.

Proposition 3.2. *For a gluing parameter (t,τ) with $|(t,\tau)| = \delta$, when ϵ,δ are small enough, there is a gluing map*

$$T_{(t,\tau)} : \widetilde{\mathcal{M}}^D_{R,\epsilon}(J,H;\tilde{x},\tilde{y}) \to \widetilde{W}^{(t,\tau)}_\epsilon(f,\mathbf{H}),$$

which is a smooth homeomorphism of $\widetilde{\mathcal{M}}^D_{R,\epsilon}(J,H;\tilde{x},\tilde{y})$ with its $T_{(t,\tau)}$-image, such that $\bar{\partial}_{J,H} g \in R(g)$ for any

$$g \in \widetilde{\mathcal{M}}^{(t,\tau)}_{R;\epsilon}(J,H;\tilde{x},\tilde{y}) = T_{(t,\tau)}(\widetilde{\mathcal{M}}^D_{R,\epsilon}(J,H;\tilde{x},\tilde{y})).$$

Moreover, if $h \in \widetilde{W}^{(t,\tau)}_\epsilon(f,\mathbf{H})$ is a solution with $\bar{\partial}_{J,H} h \in R(h)$, then h is in the image of $T_{(t,\tau)}$.

Let

$$\widetilde{\mathcal{M}}^{\bar{D}_1}_{R,\epsilon}(J,H;\tilde{x},\tilde{y}) = \bigcup_{(t,\tau)\in D',\ D\le D'\le D_1} \widetilde{\mathcal{M}}^{(t,\tau)}_{R;\epsilon}(J,H;\tilde{x},\tilde{y})$$

be the union of all $T_{(t,\tau)}$-image parameterized by the gluing parameter $(t,\tau) \in \Lambda_{\bar{D}_1}$, where $\Lambda_{\bar{D}_1} = \{(t,\tau)\,|\,(t,\tau)\in D',\, D \le D' \le D_1\}$. Then $\widetilde{\mathcal{M}}^{\bar{D}_1}_{R,\epsilon}(J,H;\tilde{x},\tilde{y})$ is a "cornered" smooth manifold of dimension

$$r + \mu(\tilde{y}) - \mu(\tilde{x}) - 1 - \sum 2n_t - \sum n_\tau$$

with the induced "cornered" smooth structure from $\mathcal{M}^D_{R,\epsilon}(J,H;\tilde{x},\tilde{y}) \times \Lambda_{\bar{D}_1}$ via $T = \{T_{(t,\tau)}\}$, where $\mu(\tilde{x})$ is the Cauchy-Zehnder index of $\tilde{x}$, n_t and n_τ are the numbers of zero components of the parameter t and τ for a generic $(t,\tau) \in \Lambda_{\bar{D}_1}$, and $r = dim R$.

Proof. We start with the existence of $T_{(t,\tau)}$. Recall that we have used $N_{(y,v)}$ to denote the asymptotic kernel $L_{(y,v)}$, the collection of which forms a trivial bundle over $\Lambda_\delta = \{(y,v)\,|\,|(y,v)| < \delta\}$. Let $\widetilde{N}_{(y,v)}$ be the kernel of $L_{(y,v)} \oplus M_{(y,v)}$. It is clear that we can identify the above bundle with the bundle of the collection of $\widetilde{N}_{(y,v)}$ over Λ_δ. Now $\widetilde{N}_{(\alpha,\theta)}$ is the tangent space of $\widetilde{\mathcal{M}}^{(\alpha,\theta)}_{R,\epsilon}(J,H;\tilde{x},\tilde{y})$ at $f_{(\alpha,\theta)}$, hence can be thought as a coordinate chart of it. If we can prove that $\widetilde{N}_{(y,v)}$, which can be identified with $\widetilde{N}_{(\alpha,\theta)}$, can also serves as a coordinate chart for the space

$$\widetilde{\mathcal{M}}^{(y,v)}_{R,\epsilon} = \{\, g \,|\, g \in \widetilde{W}^{(y,v)}_{\epsilon}(f, \mathbf{H}),\, \bar{\partial}_{J,H} g \in R(g)\},$$

we clearly obtain a gluing map

$$T_{(y,v)} : \widetilde{\mathcal{M}}^{(\alpha,\theta)}_{R,\epsilon}(J, H; \tilde{x}, \tilde{y}) \to \widetilde{\mathcal{M}}^{(y,v)}_{R,\epsilon}.$$

Then we simply define

$$T_{(t,\tau)} = \{\, T_{(\alpha,\theta,t,\tau)} \,|\, (\alpha,\theta) \in D\}.$$

To define $T_{(y,v)}$, consider

$$\begin{aligned} &F_{(y,v)}(f) \oplus M_{(y,v)} : \\ &\qquad V^{(y,v)}_{\epsilon} \oplus R(f_{(y,v)}) \subset L^p_k(f^*_{(y,v)}TV, h) \oplus R(f_{(y,v)}) \\ &\qquad\qquad\qquad\qquad \to L^p_{k-1}(\wedge^{0,1}(f^*_{(y,v)}TV)) \end{aligned}$$

given by: $(\xi, \gamma) \mapsto F_{(y,v)}(\xi) + \gamma$. Since the map is linear on the factor R, we have the following Taylor expansion:

$$\begin{aligned} (F_{(y,v)} \oplus M_{(y,v)})(\xi,\gamma) \;&= \bar{\partial}_{J,H}(f_{(y,v)}) \\ &\quad + \; (L_{(y,v)} \oplus M_{(y,v)})(\xi,\gamma) + N_{(y,v)}(\xi). \end{aligned}$$

Now by proposition 3.1 and Lemma 3.7, we have

$$\begin{aligned} \|G_{(y,v)}N_{(y,v)}(\xi) - G_{(y,v)}N_{(y,v)}(\eta)\|_{\chi;k,p} \;&\le C\|N(\xi) - N(\eta)\|_{\chi;k-1,p} \\ &\le C \cdot C_1 \|\xi - \eta\|_{\chi;k,p}\{\|\xi\|_{\chi;k,p} \\ &\qquad + \; \|\eta\|_{\chi;k,p}\}. \end{aligned}$$

Let $\delta_1 = \frac{1}{8}c_1c_2$. Then by Lemma 3.6,

$$\|G_{(y,v)}(\bar{\partial}_{J,H} f_{(y,v)})\|_{\chi;k,p} \le C\|\bar{\partial}_{J,H} f_{(y,v)}\|_{\chi;k-1,p} < \frac{\delta_1}{2},$$

when (y, v) is small enough. Applying Picard method to the above situation, we get the solvability of the equation $\bar{\partial}_{J,H} g = R(g)$ in a δ_1-neighborhood of $f_{(y,v)}$ with solution set parametrized by a δ_1-ball of $\widetilde{N}_{(y,v)}$. This establish the existence of $T_{(t,\tau)}$.

The “uniqueness” part of the Proposition directly follows from the corresponding part of the Picard method.

To compute the dimension of $\widetilde{\mathcal{M}}^{\bar{D}_1}_{R;\epsilon}(J,H;\tilde{x},\tilde{y})$, note that the virtual dimension of $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$ is $\mu(\tilde{y})-\mu(\tilde{x})-1$, and the appearance of each bubble component will reduce the dimension by 2 and each principal component by 1. The conclusion follows by the induction. q.e.d.

In above, we only proved the surjectivity of map T in terms of the weighted (k,p)-norm. But for later application we need a version in terms of L^∞-norm for stable (J,H)-map.

Corollary 3.3.
If $g\in \widetilde{W}^{(y,v)}_{\epsilon}(f,\mathbf{H})$ *with* $\bar{\partial}_{J,H}g=0$ *and*

$$\|g-f_{(y,v)}\|_\infty<\min\{\frac{1}{2CC_1},\frac{\delta_1}{4}\}$$

with (y,v) *small enough, then* g *is in the image of* $T_{(y,v)}$.

Proof. Since $\bar{\partial}_{J,H}(g)=0$, we have

$$0=F_{(y,v)}(\tilde{g})=\bar{\partial}_{J,H}(f_{(y,v)})+L_{(y,v)}(\tilde{g})+N(\tilde{g}),$$

where $\tilde{g}$ is the coordinate of g in $L^p_k(f^*_{(y,v)}TV,h)$. Hence

$$0=G_{(y,v)}(\bar{\partial}_{J,H}(f_{(y,v)}))+\tilde{g}_{\widetilde{N}^*_{(y,v)}}+G_{(y,v)}N(\tilde{g}),$$

where $\tilde{g}_{\widetilde{N}^*_{(y,v)}}$ is the orthogonal projection of $\tilde{g}$ to the orthogonal complement $\widetilde{N}^*_{(y,v)}$ of the kernel $\widetilde{N}_{(y,v)}$ of $L_{(y,v)}$.

Therefore

$$\begin{aligned}
&\|\tilde{g}_{\widetilde{N}^*_{(y,v)}}\|_{\chi;k,p}\\
&\quad\le C(\|\bar{\partial}_{J,H}(f_{(y,v)})\|_{\chi;k-1,p}+\|N(\tilde{g})\|_{\chi;k-1,p})\\
&\quad\le C\|\bar{\partial}_{J,H}(f_{(y,v)})\|_{\chi;k-1,p}+CC_1\|\tilde{g}\|_\infty\cdot\|\tilde{g}\|_{\chi;k,p}\\
&\quad\le C\|\bar{\partial}_{J,H}(f_{(y,v)})\|_{\chi;k-1,p}+\frac{1}{2}\|\tilde{g}\|_{\chi;k,p},
\end{aligned}$$

when $\|(y,v)\|$ is small enough.

It is easy to show that $\|\tilde{g}_{\widetilde{N}}\|_{\chi;k,p}$ can be controlled by $\|\tilde{g}\|_\infty$ uniformly with respect to (y,v). This implies that

$$\|\tilde{g}\|_{\chi;k,p}\le C\|\bar{\partial}_{J,H}(f_{(y,v)})\|_{\chi;k-1,p}+\frac{1}{2}\|\tilde{g}\|_{\chi;k,p}+\frac{\delta_1}{4},$$

when $((y,v))$ is small enough. Therefore,

$$\|\tilde{g}\|_{\chi;k,p} \leq 2C\|\bar{\partial}_{J,H}(f_{(y,v)})\|_{\chi;k-1,p} + \frac{\delta_1}{2} \leq \delta_1.$$

q.e.d.

Now

$$\begin{aligned}&\widetilde{\mathcal{M}}^{\bar{D}_1}_{R,\epsilon}(J,H;\tilde{x},\tilde{y})\\ &\qquad\cong \{\,(\xi, -F_{D_1}(\xi))\,|\,\xi \in \cup L^p_k(f^*_{(y,v)}TV, h),\, F_{D_1}(\xi) \in R\},\end{aligned}$$

where the union is taken over $(y,v) \in D'$ and $D \leq D' \leq D_1$. Consider its projection to the factor R. From Smale-Sard theorem and the fact that there are only finite number of intersection patterns between D and D_1, we have

Lemma 3.9. *For a generic choice $\nu \in R$, the moduli space*

$$\widetilde{\mathcal{M}}^{\bar{D}_1,\nu}_{\epsilon}(J,H;\tilde{x},\tilde{y}) = \{\,g\,|\,g \in \widetilde{\mathcal{M}}^{\bar{D}_1}_{R,\epsilon}(J,H;\tilde{x},\tilde{y}),\, F_{D_1}(\tilde{g}) = \nu\}$$

of stable (J,H,ν)-maps is a "cornered" smooth manifold with the correct dimension $\mu(\tilde{y}) - \mu(\tilde{x}) - 1 - \sum n(D_1)$, where $n(D_1) = 2n_t + n_\tau$ for a "generic" $(t,\tau) \in D_1$. Furthermore, the transversality can be achieved for all D' with $D \leq D' \leq D_1$ simultaneously.

4. Relative virtual moduli cycle

In this subsection, we will globalize the construction of the local moduli space $\widetilde{\mathcal{M}}^{\nu_i}(J,H;\tilde{x},\tilde{y})$ of stable (J,H,ν)-maps described in previous section to get a compact moduli space $\mathcal{M}^{\nu}(J,H;\tilde{x},\tilde{y})$ with a boundary of right dimension. In fact, our construction yields a relative virtual moduli **Q**-cycle, which will play a crucial role in the construction of Floer homology in the next section. Different methods of constructing such a virtual moduli cycle in absolute case have been developed in [13] and [8] in the setting of Gromov-Witten classes during writing of this paper.

As the first step of the globalization process, we need to formulate the compactification theorem for moduli space $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$ of stable (J,H)-maps, which was stated before for smooth curves with cuspidal curves as their limits in [9] and [4], [5]. To this end, we introduce the weak C^∞-topology for $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$, which was used before by Gromov and Floer for cuspidal curves.

A sequence $\{[u_i]\}_{i=1}^\infty$ of (J,H)-stable maps in $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$ is convergent to a (J,H)-stable map $[u_\infty]\in\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$ if there exist representatives $u_i\in[u_i]$ and $u_\infty\in[u_\infty]$ with domains Σ_i of u_i and Σ_∞ of u_∞ such that the following hold:

(i) $\Sigma_i\to\Sigma_\infty$, as $i\to\infty$ in the sense that there exists a $\widetilde{\Sigma}_\infty\in\mathcal{FM}^I_{0,k}$ without extra marked points with $I(\widetilde{\Sigma}_\infty)=I(\Sigma_\infty)$ and local parameters $(\alpha_i,\theta_i,t_i,\tau_i)$ in a neighborhood of $\widetilde{\Sigma}_\infty$ in $\mathcal{FM}^I_{0,k}$ with $(\alpha_i,\theta_i,t_i,\tau_i)\ \to\ 0$, and a family of identification maps: $\phi_i:\widetilde{\Sigma}_i\to\Sigma_i$ and $\phi_\infty:\widetilde{\Sigma}_\infty\to\Sigma_\infty$.

(ii) For each compact set $K\subset\widetilde{\Sigma}_\infty\setminus\{\text{double points}\}\cup\{\text{ cylindrical ends}\}$, let K_i be the corresponding subset of $\widetilde{\Sigma}_i$ via gluing construction in $\overline{\mathcal{FM}}_{0,k}$, when i is large enough. Then $v_{i,K_i}=(u_i\circ\phi_i)|_K$ is C^∞-convergent to $v_{\infty,K}=u_\infty\circ\phi_\infty|_K$.

(iii) $\lim_{i\to\infty}E(u_i)=E(u_\infty)$.

With respect to this weak C^∞-topology, we have

Proposition 4.1. *The moduli space $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$ of (J,H)-stable maps connecting $\tilde{x}$ and $\tilde{y}$ equipped with the weak C^∞- topology is compact. If*

$$\{[u_i]\}_{i=1}^\infty\to u_\infty$$

in $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$, then (i) $E(u_i)\to E(u_\infty)$; (ii) $Ind(u_i)=Ind(u_\infty)$ when i is large. Moreover, $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$ is Hausdorff.

We remark that essential ingredients for proving the part of theorem concerning compactness of $\overline{\mathcal{M}}(\tilde{x},\tilde{y};J,H)$ are already contained in [18],[4],[5] and [19], where the corresponding theorems are proved for cuspidal curves. The new feature for $\overline{\mathcal{M}}(\tilde{x},\tilde{y};J,H)$ is that we need carefully keep track all markings introduced in bubbling process. Such a deleting-dropping marking procedure leads to the convergence of domains of a sequence of stable-maps to a stable curve rather than just a cuspidal curve. This, in turn, leads to Hausdorffness of $\overline{\mathcal{M}}(\tilde{x},\tilde{y};J,H)$. It is well-known that for moduli space of cuspidal maps Hausdorffness does not hold.

Proof. We only sketch a proof for the part of the theorem concerning compactness, leaving the relevant analytic detail to readers to consult the above mentioned literatures. But we will give more detail for the part concerning Hausdorffness.

For simplicity, we may assume that $\Sigma_i = S^2$ with three marked points $x_1 = 0, x_2 = 1$ and $x_3 = \infty$, which correspond to a free bubble component and already contains essential features of the general case.

Let $\{[f_i]\}_{i=1}^{\infty}$ be a sequence of stable maps of bounded energy and $f_i : \Sigma_i \to V$ be a representative of $[f_i]$. We give Σ_i the usual spherical metric. We start our adding- dropping marking process by adding x_4 and x_5 to Σ_i, in such a way that x_4 is one of the points in Σ_i such that $|df_i(x_4)| = \max_{x\in\Sigma_i} |df_i(x)|$ and x_5 corresponds to $1 \in \mathbf{C}$ under the rescaling in the usual bubbling process, designed to capture the top level bubble in bubble tree and described, for instance, in [4], [5] and [15]. We may assume that x_4 and x_5 are away from $x_i, i = 1, 2, 3$. By deleting x_3 and rename x_4 to be x_3 and x_5 to be x_4, we get a sequence $(\Sigma_i; x_1, x_2, x_3, x_4) \in \mathcal{M}_{0,4}$. Since we have assumed that bubbling do happen, $|df_i(x_3)| \to \infty$. This implies that $d(x_3, x_4) \to 0$ as $i \to \infty$. Now we have an identification $\phi_i : \tilde{\Sigma}_i \to \Sigma_i$ with $\tilde{\Sigma}_i \to \tilde{\Sigma}_\infty$ in $\overline{\mathcal{M}}_{0,4}$ as $i \to \infty$. Here the "universal" curves $\{\tilde{\Sigma}_i\}$ are obtained from $\tilde{\Sigma}_\infty$ by gluing. Each $\tilde{\Sigma}_i$, and hence Σ_i, inherits a spherical-like metric from $\tilde{\Sigma}_\infty$. The above bubbling process can also be described by such a metric change. However, in this new metric, the injective radius of Σ_i goes to zero as $i \to \infty$. In order to recapture all other bubbles, especially those intermediate ones in bubble tree, we need to switch to cylindrical metric near each x_3 of $\tilde{\Sigma}_i$. Then the usual conformal rescaling process will be applicable again. In such a way, we can get all possible top level bubbles. By a similar procedure to [4], we can also obtain all intermediate bubbles by using the cylindrical coordinates and local convergence. For those intermediate bubbles with only two double points that correspond to the two ends of $S^1 \times \mathbf{R}$ in our cylindrical coordinate, we will add a new marking on the "middle" circle of $S^1 \times \mathbf{R}$, where the middle circle divides the energy of the bubble into two equal parts. Note that here both kinds of unstable bubbles have a non-trivial energy which is bounded below by some positive constant. This implies that such an adding marking and bubbling process will stop after finite steps. We conclude that there is an adding marking procedure according to successive bubbling and local convergence such that (i) after deleting $x_3 = x_3^i$, and adding new $x_3^i, \cdots, x_k^i$ to Σ_i, we have a conformal marking preserving identification

$$\phi_i : (\tilde{\Sigma}_i, \tilde{x}_1^i, \tilde{x}_2^i, \cdots, \tilde{x}_k^i) \to (\Sigma_i; x_1^i, \cdots, x_k^i),$$

with $\tilde{\Sigma}_i \in \mathcal{M}_{0,k}$, such that, $\lim_{i\to\infty} \tilde{\Sigma}_i = \tilde{\Sigma}_\infty \in \overline{\mathcal{M}}_{0,k}$ after taking a subsequence. Therefore each $\Sigma_i \equiv \tilde{\Sigma}_i$ gets a spherical like metric induced form $\tilde{\Sigma}_\infty$ through gluing.

(ii) Let $K_m \hookrightarrow \tilde{\Sigma}_\infty \setminus D$, $m = 1, 2, \cdots$, be a sequence of compact sets with $K_m \subset \overset{\circ}{K}_{m+1}$, and $\tilde{\Sigma}_\infty \setminus D = \cup_{m=1}^\infty K_m$. Here we have used D to denote the set of double points of $\tilde{\Sigma}_\infty$. Then for each fixed m, there exists an $i_{K_m} > 0$ such that $K_m \hookrightarrow \tilde{\Sigma}_i$ when $i > i_{K_m}$, by the gluing construction. Consider $\tilde{f}_{i,K_m} = f_i \circ \phi_i|_{K_m}$. We have

$$|d\tilde{f}_{i,K_m}(x)| < C = C_{K_m},$$

when $i > i_{K_m}$.

(iii) For any given ϵ, $\exists\, m(\epsilon)$ such that for all $m > m(\epsilon)$, $E(\tilde{f}_i|_{\tilde{\Sigma}_i \setminus K_m)} < \epsilon$, where i is sufficiently large.

(i) and (ii) imply that after taking subsequence, $\{\tilde{f}_i\}$ is locally convergent to $\tilde{f}_\infty : \tilde{\Sigma}_\infty \to V$. $\tilde{f}_\infty$ is almost a stable map without extra markings except that the following two cases may happen, which violate the definition of stable maps. First case is that some stable components may still contain the original marking x_1 and x_2. In this case we only need to drop the extra markings. The second case is that there may have some unstable trivial components being stabilized by x_1 and x_2. For this case, we only need to drop the corresponding marking in $\tilde{\Sigma}_i$ and contract the corresponding components in $\tilde{\Sigma}_\infty$. We leave the detail to readers to verify that such a deleting marking process will still keep (i)-(iii) above and we get a stable map limit $\tilde{f}_\infty$ without any extra markings. It follows from (ii) and (iii) that $E(\tilde{f}_\infty) = \lim_{i\to\infty} E(f_i)$, $[\tilde{f}_\infty] = [f_i]$ in $H_2(V)$ when i is large and the image of (f_i) is C^0-convergent to image of $(\tilde{f}_\infty)$. This completes the sketch of the proof of the compactness of $\overline{\mathcal{M}}(\tilde{x}, \tilde{y}; J, H)$.

To prove the Hausdorffness, let sequence $\{f_i\}$ and $\{f_i'\}$ be two different representatives of $\{[f_i]\}$ with domains

$$(\Sigma_i, x_1, x_2, x_3) \equiv (\Sigma_i', x_1', x_2', x_3') \equiv (S^2, 0, 1, \infty).$$

There exist conformal identifications $\psi_i : \Sigma_i' \to \Sigma_i$ such that $f_i' = f_i \circ \psi_i$.

Applying the above adding-deleting marking process to both of the sequences, we get $\{\tilde{f}_i\}$ and $\{\tilde{f}_i\}$ with new domains $(\tilde{\Sigma}_i, \tilde{x}_1, \cdots, \tilde{x}_k)$ and $(\tilde{\Sigma}_i', \tilde{x}_1', \cdots, \tilde{x}_{k'}')$ in $\overline{\mathcal{M}}_{0,k}$ and $\mathcal{M}_{0,k'}$ respectively. ψ_i induces an identification $\tilde{\psi}_i : \tilde{\Sigma}_i' \to \tilde{\Sigma}_i$ such that $\tilde{f}_i' = \tilde{f}_i \circ \tilde{\psi}_i$. Note that $\tilde{\psi}_i$ does not preserve markings in general. By taking a common subsequence, we get two limit stable maps $\tilde{f}_\infty$ and $\tilde{f}'_\infty$ with domains $\tilde{\Sigma}_\infty$ and $\tilde{\Sigma}'_\infty$ in $\overline{\mathcal{M}}_{0,k}$ and $\overline{\mathcal{M}}_{0,k'}$ respectively.

We need to prove that $k = k'$ and there exists a marking preserving identification $\tilde{\psi}_\infty : \tilde{\Sigma}'_\infty \to \tilde{\Sigma}_\infty$ such that $\tilde{f}'_\infty = \tilde{f}_\infty \circ \tilde{\psi}_\infty$.

To this end, consider the image of $\tilde{f}_\infty$. As a set of V consisting of all limit points of $im(\{[f_i]\})$, it is well-defined, not depending on any particular parametrization of the domains of $\{[f_i]\}$. Let $\tilde{f}^c_\infty$ be the cuspidal map obtained from $\tilde{f}_\infty$ by shrinking all its trivial components. Let $\tilde{\Sigma}^c_\infty$ be the domain of $\tilde{f}^c_\infty$. Then $\tilde{f}^c_\infty = \bar{f}^c_\infty \circ \pi$, where $\bar{f}^c$ is a simple cuspidal map, and $\pi : \tilde{\Sigma}^c_\infty \to \overline{\Sigma}^c_\infty$ is a continuous surjective map between the two cuspidal domains, which is a holomorphic branch covering on each component of $\tilde{\Sigma}^c_\infty$. $\bar{f}^c_\infty$ gives rise to a holomorphic parametrization of $im(\tilde{f}_\infty)$, which is one to one away from finite points. It is easy to see that such a simple conformal parametrization is unique up to a conformal identification of the cuspidal domains of $\bar{f}^c_\infty$. In particular, the image $\bar{D} = \tilde{f}_\infty(D)$ of double points of $\tilde{\Sigma}_\infty$ is a well-defined finite set of V.

Let $\tilde{\Sigma}^N_\infty$ be a subset of $\tilde{\Sigma}_\infty$, which is the union of domains of all nontrivial components of $\tilde{f}_\infty$, and denote $\tilde{f}_\infty|_{\tilde{\Sigma}^N_\infty}$ by $\tilde{f}^N_\infty$. We define $\tilde{D} = (\tilde{f}^N_\infty)^{-1}(\bar{D})$. Then $\tilde{D}$ contains all double points of $\tilde{\Sigma}^N_\infty$ as a subset. For each $\epsilon > 0$, let $\tilde{N}_\epsilon$ be the ϵ-neighborhood of $\tilde{D}$ in $\tilde{\Sigma}^N_\infty$ and define $\tilde{K}_\epsilon = \tilde{\Sigma}^N_\infty \setminus \tilde{N}_\epsilon$. Then for i large enough, the compact set $\tilde{K}_\epsilon$ is also contained in $\tilde{\Sigma}_i$ through gluing. We will also use $\tilde{K}^i_\epsilon$ to denote the $\tilde{K}_\epsilon$ in $\tilde{\Sigma}_i$. Next we consider ϵ-neighborhood $\bar{N}_\epsilon$ of $\bar{D}$ in V, and define compact subset $\tilde{C}_\epsilon = \tilde{\Sigma}_\infty \setminus \tilde{f}^{-1}_\infty(\bar{N}_\epsilon)$ and $\tilde{C}^i_\epsilon = \tilde{\Sigma}_i \setminus \tilde{f}^{-1}_i(\bar{N}_\epsilon)$ in $\tilde{\Sigma}_\infty$ and $\tilde{\Sigma}_i$ respectively. The following two facts concerning $\tilde{K}_\epsilon$ and $\tilde{C}_\epsilon$ are crucial for the proof of Hausdorffness:

(I) $\tilde{K}_{\epsilon_2} \subset \tilde{C}_{\epsilon_1}, \quad \tilde{K}^i_{\epsilon_2} \subset \tilde{C}^i_{\epsilon_1}$;

(II) $\tilde{C}_{\epsilon_2} \subset \tilde{K}_{\epsilon_1}, \quad \tilde{C}^i_{\epsilon_2} \subset \tilde{K}^i_{\epsilon_1}$;

where $\epsilon_1 << \epsilon_2$ and i is large enough.

Note that $\tilde{C}^i_\epsilon$ behaves well under the identification map $\tilde{\psi}_i$; that is, if we run through all the above constructions for $\{\tilde{f}'_i\}$ and get $\tilde{K}'_\epsilon$, $\tilde{C}'_\epsilon$ respectively, then $\tilde{C}^i_\epsilon = \tilde{\psi}_i(\tilde{C}'_\epsilon)$. Combining this with (I) and (II) above, we conclude the following:

(III) $\tilde{K}_{\epsilon_2} \subset \tilde{\psi}_i(\tilde{K}'_{\epsilon'}) \subset \tilde{K}_{\epsilon_1}$ in Σ_i if $\epsilon_1 << \epsilon' << \epsilon_2$ and i is large enough.

This implies that each component of $\tilde{K}'_{\epsilon'}$ is contained in one and only one component of $\tilde{K}_\epsilon$ for some $\epsilon << \epsilon'$ under $\tilde{\psi}_i$, where i is large enough.

Note that those boundary components of $\tilde{K}_\epsilon$ near points of $\tilde{D}\setminus D$ are contractible in $\tilde{\Sigma}_i$. Therefore, we can determine if a component of $\tilde{K}_\epsilon$ lies in an unstable component of $\tilde{\Sigma}_\infty$ by counting how many non-contractible boundary components of it are in $\tilde{\Sigma}_i$. Since both $\tilde{\Sigma}_\infty$ and $\tilde{\Sigma}'_\infty$ have no extra markings, each component of $\tilde{K}_\epsilon$ ($\tilde{K}'_{\epsilon'}$) that lies in some unstable component of $\tilde{\Sigma}_\infty$ ($\tilde{\Sigma}'_\infty$) will contain one or two markings when ϵ (ϵ') is small. Combining this with (III) we conclude that $\tilde{\psi}_i$ maps each component of $\tilde{K}'_{\epsilon'}$ to a component $\tilde{K}_\epsilon$, preserving the number of ends (non-trivial boundary components). In particular, the markings $\tilde{x}'_{j'}s$ of $\tilde{\Sigma}'_i$, $j=1,\cdots,k'$, map into $\tilde{\Sigma}_i$ under $\tilde{\psi}_i$ such that they all stay in a compact set $K_\epsilon \hookrightarrow \tilde{\Sigma}_i$ for all large i, and that the number of $\tilde{\psi}_i(\tilde{x}'_j)$ contained in each component of K_ϵ is the same as the number of $\tilde{x}_j$ in the same component.

In the case that some components of $\tilde{K}'_{\epsilon'}$ contain two markings, say, $\tilde{x}'_j$, and $\tilde{x}'_k$, the distance between $\tilde{\psi}_i(\tilde{x}'_j)$ and $\tilde{\psi}_i(\tilde{x}'_k)$ in the corresponding component of $\tilde{K}_\epsilon$ is bounded below.

This implies that $k=k'$ and that

$$\tilde{\psi}_i : (\tilde{\Sigma}'_i; \tilde{x}'_1, \cdots, \tilde{x}'_k) \to (\tilde{\Sigma}_i; \tilde{\psi}_i(\tilde{x}'_1), \cdots, \tilde{\psi}_i(\tilde{x}'_k))$$

induces a conformal identification

$$\tilde{\psi}_\infty : (\tilde{\Sigma}'_\infty; \tilde{x}'_1, \cdots, \tilde{x}'_k) \to (\tilde{\Sigma}_\infty; \tilde{\psi}_\infty(\tilde{x}'_1) \cdots, \tilde{\psi}_\infty(\tilde{x}'_k)).$$

By letting ϵ and ϵ' go to zero and using the relation of $\tilde{K}'_{\epsilon'}$ and $\tilde{K}_\epsilon$ under $\tilde{\psi}_i$, we also conclude that as maps: $\tilde{f}'_\infty = \tilde{f}_\infty \circ \tilde{\psi}_\infty$ when restricted to non-trivial components of $\tilde{\Sigma}'_\infty$. Since $\tilde{\psi}_\infty$ also sends trivial components of $\tilde{\Sigma}'_\infty$ to trivial ones of $\tilde{\Sigma}_\infty$, we have $\tilde{f}'_\infty = \tilde{f}_\infty \circ \tilde{\psi}_\infty$. Note that domain of $\tilde{f}_\infty$ here is $(\tilde{\Sigma}_\infty; \tilde{\psi}_\infty(\tilde{x}'_1), \cdots, \tilde{\psi}_\infty(\tilde{x}'_k))$ as a stable curve with marked points. Since there are no extra markings, there exists an automorphism λ of $\tilde{\Sigma}_\infty$ such that $\tilde{\lambda}(\tilde{x}_i) = \tilde{\psi}_\infty(\tilde{x}'_i)$ for $i=1,\cdots,k$. This implies that $[\tilde{f}'_\infty] = [\tilde{f}_\infty]$.

A similar argument shows that not only any two adding-deleting marking procedure coming from bubbling as above gives rise to an equivalent limit map, but also any other adding-deleting marking process appeared in the definition of weak limit will lead to equivalent limit map. We leave to readers to carry out the detail of this similar argument. q.e.d.

If all Σ_i remain in the same topological type, from elliptic regularity and decay estimate along its cylindrical ends of stable (J,H)-map detailed in [5] it follows that this weak C^∞-topology is the same as our L^p_k-topology on $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$.

To compare the stronger L^p_k-topology on $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$ induced from $\overline{\mathcal{B}}(\tilde{x},\tilde{y})$ with the weak C^∞-topology here, we may assume only bubble components appear when topological types change in the weak limit, since the relevant result concerning principal components has been already established by using Floer's gluing process for those components. Now observe that the definition of $[u_i] \to [u_\infty]$ in the sense of weak C^∞-topology implies that $v_{i,K} \to v_{\infty,K}$ for any compact subset $K \subset \widetilde{\Sigma}^0_\infty$. Now $E(u_i) = E(v_i)$ and the energy identity in Floer-Gromov compactness theory above implies that when K is large enough, for any given $\delta > 0$, $E(v_i|_{\widetilde{\Sigma}_i \setminus K_i}) < \delta$. It follows from the monotonicity of minimal surface that the image of $v_i|_{\widetilde{\Sigma}_i \setminus K_i}$ is contained in a prescribed small δ-neighborhood of $u_\infty(d)$, where d is the set of double points. This together with the construction of pre-gluing $v_{(\alpha_i,\theta_i,t_i,\tau_i)}$ of v_∞ implies that the C^0-distance of v_i and $v_{(\alpha_i,\theta_i,t_i,\tau_i)}$ is less than any given $\epsilon > 0$, when i is large enough.

As we did before for the stable (J,H)-map f, here we can also construct the local hypersurfaces that are transversal to v_∞ at its free parameter. We still use $\mathbf{H}$ to denote the collection of those hypersurfaces. In general, v_i may not be in $\widetilde{W}_\epsilon(v_\infty,\mathbf{H})$ because v_i may not send its marked points $x = \{x_j\}$ into $\mathbf{H}$, but rather send its points $\tilde{x} = \{\tilde{x}^i_j\}$ into $\mathbf{H}$, with each $\tilde{x}^i_j \in D_{\delta_i}(x_j)$ for some δ_i depending on i. From the construction of $\overline{\mathcal{FM}}_{0,k}$, it follows that there exists another set of parameter $(\alpha'_i,\theta'_i,t'_i,\tau'_i)$, which is "close" to $(\alpha_i,\theta_i,t_i,\tau_i)$, "parametrize" $(\Sigma_{(\alpha_i,\theta_i,t_i,\tau_i)}, \tilde{x}^i_1,\cdots,\tilde{x}^i_k)$, i.e., there exists an identification

$$\psi_i : (\Sigma_{(\alpha'_i,\theta'_i,t'_i,\tau'_i)}, x_1,\cdots,x_k) \to (\Sigma_{(\alpha_i,\theta_i,t_i,\tau_i)}, \tilde{x}^i_1,\cdots,\tilde{x}^i_k).$$

Let $v'_i = v_i \circ \psi_i$.

It follows from the proof of Lemma 2.2 that $\delta_i \to 0$. Therefore, $(\alpha'_i,\theta'_i,t'_i,\tau'_i) \to 0$ also as $i \to \infty$, and $v'_{i,K} \to v_{\infty.K}$ for any component K. Clearly, $v'_i \in \widetilde{W}_\epsilon(v_\infty,\mathbf{H})$ and $[v'_i] = [v_i]$. It is easy to see that the above proof of that v_i and $v_{(\alpha_i,\theta_i,t_i,\tau_i)}$ are C^0-close also implies the same conclusion for v'_i and $v_{(\alpha_i,\theta_i,t_i,\tau_i)}$. By Corollary 3.3, when i is large enough, each $[v'_i]$ is in the image of gluing map T. This proves

Lemma 4.1. *The two topologies are equivalent on* $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$.

Because of this, $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$ is also compact with respect to the strong L^p_k-topology. Now consider the covering

$$\bigcup_{[f]\in\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})} W_{\epsilon_f}(f,\mathbf{H}_f)$$

of $\overline{\mathcal{M}}(J,H;\tilde{x}.\tilde{y})$, where

$$W_{\epsilon_f}(f;\mathbf{H}_f)=\widetilde{W}_{\epsilon_f}(f;\mathbf{H}_f)/\Gamma_f$$

is the image of the local uniformizer $\widetilde{W}_{\epsilon_f}(f;\mathbf{H}_f)$ in $\overline{\mathcal{B}}(\tilde{x},\tilde{y})$ under the natural projection, and we have used subscript f in ϵ_f and $\mathbf{H}_f$ to indicate the dependence on f. From the compactness theorem and Lemma (4.1) it follows that there exists a finite set $\{f_i; 1\le i\le m\}$ such that $\{W_{\epsilon_f}(f_i,\mathbf{H}_f); 1\le i\le m\}$ already form a covering of $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$. We will use $\widetilde{W}_i$ and W_i to denote $\widetilde{W}_{\epsilon_i}(f_i,\mathbf{H}_i)$ and $W_{\epsilon_i}(f_i,\mathbf{H}_i)$ respectively.

Let $W=\cup_{i=1}^m W_i$. There is an orbifold bundle $\mathcal{L}=\cup_{i=1}^m\mathcal{L}_i$ over it. Recall that the fiber $(\mathcal{L}_i)_{[g]}$ for $[g]\in W_i$ consists of the equivalent classes of sections $\cup_{g\in[g]}L^p_{k-1}(\wedge^{0,1}(g^*TV))$ with the obvious equivalence relations via pull-back action of reparametrization of domains. The isotropy group Γ_i of f_i, which acts on $\widetilde{W}_i$, has a lifting action on $\widetilde{\mathcal{L}}_i$ as bundle isomorphisms.

To describe the orbifold bundle structure here in detail, we need to review the standard definitions of orbifold and orbifold bundle. Let W be a Hausdorff topological space and U be an open set in W. A local ($C^\infty-$) uniformizing system $\{\widetilde{U},\Gamma,\pi\}$ for U with uniformizer $\widetilde{U}$ is defined as follows. $\widetilde{U}$ is a connected open subset of some Banach space , Γ is a finite group of effective C^∞-automorphisms of $\widetilde{U}$, and π is a C^∞-map from $\widetilde{U}$ to U such that for any $\phi\in\Gamma$, $\pi\circ\phi=\pi$ and the quotient map $\bar{\pi}:\widetilde{U}/\Gamma\to U$ is a homeomorphism. If $\{\widetilde{U}_1,\Gamma_1,\pi_1\}$ and $\{\widetilde{U}_2,\Gamma_2,\pi_2\}$ are two uniformizing system of U_1 and U_2 respectively, an injective C^∞-map λ between them is an open embedding of $\widetilde{U}_1$ into $\widetilde{U}_2$ and an injective group homomorphism from Γ_1 into Γ_2 such that λ commutes with the projections π_i, i=1,2 and that $\lambda\circ\phi=\lambda(\phi)\circ\lambda$ for any $\phi\in\Gamma_1$. If λ is invertible, it is an equivalence of the two system. In particular, each $\phi\in\Gamma$ induces an equivalence of $\{\widetilde{U},\Gamma,\pi\}$ onto itself. We say that W is a ($C^\infty-$) orbifold if there exists a family $\mathcal{U}$ of local uniformizing system for open subsets of W, called defining family of W, such that the following hold:

(i) W is covered by $\cup_{\widetilde{U}\in\mathcal{U}}\pi^U(\widetilde{U})$.

(ii) If $g \in U_1 \cap U_2$ with $U_i = \pi^{U_i}(\widetilde{U}_i)$ being $\mathcal{U}$-uniformized , there exists a $\mathcal{U}$- uniformized open set $U_3 \subset U_1 \cap U_2$ such that $g \in U_3$.

(iii) If $(\widetilde{U}_1, \Gamma_1, \pi_1)$ and $(\widetilde{U}_2, \Gamma_2, \pi_2)$ are two local uniformizing systems in $\mathcal{U}$ such that $U_1 \subset U_2$, then there exists an injective map $\lambda_{12} : \widetilde{U}_1 \to \widetilde{U}_2$.

A ($C^\infty-$) orbifold bundle $\mathcal{L}$ over W is another orbifold together with a continuous projection $\bar{p} : \mathcal{L} \to W$ satisfying the following condition:

(i) Each open set $E = \bar{p}^{-1}(U)$ for some $\mathcal{U}$-uniformized open set U in W is uniformized with respect to the uniformizing system $(\widetilde{E}, \Gamma^E, \pi^E)$ in such a way that there exists a Banach bundle structure $p : \widetilde{E} \to \widetilde{U}$, where $\widetilde{U}$ is the uniformizer of the uniformizing system $(\widetilde{U}, \Gamma^U, \pi^U)$ of U. Moreover, we require that $\Gamma^E = \Gamma^U = \Gamma$ as an abstract group, and the action of Γ^E on $\widetilde{E}$ is by the bundle maps which are lifted actions of the corresponding ones on $\widetilde{U}$, and that the induced quotient map of p is just $\bar{p}$. We will call the system $(\widetilde{E}, \widetilde{U}, \Gamma, p, \pi^E, \pi^U)$ a local uniformizing system for the orbifold bundle $\mathcal{L} \to W$.

(ii) If $U_1 \subset U_2$, there exists an injective map $\lambda = (\lambda^E, \lambda^U)$ between the local uniformizing systems $(\widetilde{E}_1, \widetilde{U}_1, \Gamma_1, p_1, \pi^{E_1}, \pi^{U_1})$ and $(\widetilde{E}_2, \widetilde{U}_2, \Gamma_2, p_2, \pi^{E_2}, \pi^{U_2})$ in the obvious sense. More precisely, we have

$$\begin{array}{ccc} \widetilde{E}_1 & \xrightarrow{\lambda^E} & \widetilde{E}_2 \\ \Big\downarrow p_1 & & \Big\downarrow p_2 \\ \widetilde{U}_1 & \xrightarrow{\lambda^U} & \widetilde{U}_2 \end{array}$$

such that λ^E and λ^U are the injective maps of local uniformizing systems of $\widetilde{E}_i$'s and $\widetilde{U}_i$'s, i=1,2, and that λ^E is a bundle map. In particular for any $u \in \widetilde{U}_1$,

$$\lambda^E|_{(\widetilde{E}_1)_u} : (\widetilde{E}_1)_u \to (\widetilde{E}_2)_{\lambda^U(u)}$$

is an isomorphism.

Let W be an orbifold and $\bar{u} \in W$ be in the image of some local uniformizing system $(\widetilde{U}, \Gamma_U, \pi_U)$. Clearly all isotropy groups Γ_u, $u \in (\pi_U)^{-1}(\bar{u})$ are conjugate to each other. We define order of $\bar{u}$ and u to be the order of Γ_u. It follows from the definition of orbifolds that the order of $\bar{u}$ is well-defined, independent of any particular choice of local uniformizing systems.

For the purpose of this paper we need to extend the notion of smooth orbifold to the case of partially smooth orbifold. A Hausdorff topological space W is said to be a partially smooth (Banach) manifold if it is a stratified Banach manifold and is a partially smooth orbifold if it can be covered by a local uniformizing system $\mathcal{U}$ such that each uniformizer in $\mathcal{U}$ is an open set of some partially smooth manifold. Here we require that all maps and group actions involved in the definition of partially smooth orbifold are not only continuous but also smooth when restricted to each strata. All other notions we introduced above for smooth orbifolds can be easily extended to the partially smooth case.

In the rest of this paper, we will simply use orbifold and orbifold bundle to refer partially smooth ones. We now come back to the orbifold

$$W = \cup_{i=1}^m W_{\epsilon_i}(f_i; \mathbf{H}) = \cup_{i=1}^m W_i$$

and the orbifold bundle $\mathcal{L} \to W$ mentioned before in this section. We have

Lemma 4.2. *W is an orbifold and $\mathcal{L} \to W$ is an orbifold bundle over W.*

Proof. $\mathcal{L} \to W$ is covered by $\{\mathcal{L}_i \to W_i\}_{i=1}^m$, which is uniformized by $\widetilde{\mathcal{L}}_i \to \widetilde{W}_i$. We only need to prove that for any $\bar{u} \in W_i \cap W_j$ there exists an open neighborhood U of $\bar{u}$ in $W_i \cap W_j$ such that the induced uniformizations of $E = \mathcal{L}|_U \to U$ from the above two uniformizations are equivalent.

To this end, we describe the induced uniformizations near $\bar{u}$ first. Let $\pi_i^{-1}(\bar{u}) = \{u_{i,k}\}$, $k = 1, \cdots, K_i$. Then all isotropy group $\Gamma_{i,k} = \Gamma_{u_{i,k}}$ are conjugate to each other and $K_i = \#(\Gamma_i/\Gamma_{i,k})$. If we choose an open neighborhood U_i of $\bar{u}$ in W_i small enough so that $\pi_i^{-1}(U_i)$ can be decomposed as a disjoint union of its K_i components $\widetilde{U}_{i,k}$, $k = 1, \cdots, K_i$, with $u_{i,k} \in \widetilde{U}_{i,k}$, then $\Gamma_{i,k}$ acts on $\widetilde{U}_{i,k}$ so that $\widetilde{U}_{i,k}/\Gamma_{i,k} = U_i$. Therefore we get K_i equivalent local uniformizing systems $(\widetilde{U}_{i,k}, \Gamma_{i,k}, \pi_{i,k})$ for U_i. Similarly we get K_j local uniformizing systems $(\widetilde{U}_{j,k}, \Gamma_{j,k}, \pi_{j,k})$ for $U_j \subset W_j$ with $\bar{u} \in U_j$. We may assume that $U_i = U_j = U$. We need to prove, for example, for $k = 1$, that the two local uniformizing systems $(\widetilde{U}_{i,1}, \Gamma_{i,1}, \pi_{i,1})$ and $(\widetilde{U}_{j,1}, \Gamma_{j,1}, \pi_{j,1})$ are equivalent. Since $u_{i,1}$ and $u_{j,1}$ are merely two different parametrizations of the same stable map $\bar{u}$, they have the same intersection pattern, say D. Let $\widetilde{U}_{i,1}^D$ and $\widetilde{U}_{j,1}^D$ be the corresponding strata in $\widetilde{U}_{i,1}$ and $\widetilde{U}_{j,1}$. We define an equivalence $\lambda^D = \lambda^D_{\widetilde{U}_{i,1}, \widetilde{U}_{j,1}}$ between $\widetilde{U}_{i,1}^D$ and $\widetilde{U}_{j,1}^D$ as follows. Let $\Sigma = \Sigma_0$ be the domain of $u_{i,1}$ and $u_{j,1}$ with marked points $x = \{x_i\}$, $i = 1, \cdots, k$, and $\mathbf{H}_i$ and $\mathbf{H}_j$ be the corresponding collections of hypersurfaces in V

used to define the pre-slicing $\widetilde{W}_i$ and $\widetilde{W}_j$. Choose an automorphism $\phi_{i,j}:\Sigma\to\Sigma$ such that $u_{j,1}=u_{i,1}\circ\phi_{i,j}$. Recall that any element $g\in\widetilde{U}^D_{i,1}\subset\widetilde{W}^D_i$ implies that $g(x)\in\mathbf{H}_i$.

Now $g\circ\phi_{i,j}$ is close to $u_{j,1}=u_{i,1}\circ\phi_{i,j}$ if g is close to $u_{i,1}$. Therefore there exists a unique x_g in a small neighborhood of x such that $g\circ\phi_{i,j}(x_g)\in\mathbf{H}_j$. Let $\phi_g:\Sigma\to\Sigma$ be the automorphism sending x to x_g. We define $\lambda^D(g)=g\circ\phi_{i,j}=\phi_g$. Clearly λ^D maps $\widetilde{U}^D_{i,1}$ to $\widetilde{U}^D_{j,1}$ and commutes with projections π_i and π_j. By a similar procedure for extending group actions of Γ_i from $\widetilde{W}^D_i$ to $\widetilde{W}_i$, we can easily extend λ^D to $\lambda:\widetilde{U}_{i,1}\to\widetilde{U}_{j,1}$ which still commutes with π_i and π_j. This proves that W has an orbifold structure. The proof for $\mathcal{L}\to W$ being an orbifold bundle is similar. We omit it here. q.e.d.

Remark 4.1. Let D_T be the top strata whose domain Σ_{D_T} is $S^1\times\mathbf{R}^1$, and D_B be the strata whose domain Σ_{D_B} consists of "broken" cylinders of at least two elements. If W^{D_T} and W^{D_B} are the corresponding strata of W, then from the construction of our slicing it follows that $W^{D_T}\cup W^{D_B}$ is contained in $(W)^s$, where $(W)^s$ is the set of smooth (i.e., order 1) points of W. Let S be the singular set of W which consists of all points of order greater than 1. Then the domains of any its elements have at least one bubble component.

As we know from Lemma 2.7 that the $\overline{\partial}_{J,H}$- operator gives rise to a Γ_i- equivariant stratawise smooth section of $\widetilde{\mathcal{L}}_i\to\widetilde{W}_i$, $i=1,\cdots,m$, and hence descends to a well-defined section of the orbifold bundle $\mathcal{L}_i\to W_i$. These local sections can be pasted together to yield a well-defined global section of the bundle $\mathcal{L}\to W$. As before we still use $\overline{\partial}_{J,H}$ to denote this section. As we showed in last section that on each $\widetilde{W}_i$, we can use a "generic" ν_i-perturbation to alter the $\overline{\partial}_{J,H}$-section such that

$$\widetilde{\mathcal{M}}^{\nu_i}_{R_i}=(\overline{\partial}_{J,H}+\nu_i)^{-1}(0)$$

has a boundary of the correct dimension. The question here is how to globalize this. Note that in order to achieve transversality in each $\widetilde{W}_i$, it is necessary to use non-equivariant perturbation $\nu_i:\widetilde{W}_i\to\widetilde{\mathcal{L}}_i$, which is a multivalued section of $\mathcal{L}_i\to W_i$. If $W_i\cap W_j\neq\emptyset$, we have to know how to transform $\nu_i|_{\pi_i^{-1}(W_i\cap W_j)}$ into a section (or sections) of the bundle $\widetilde{\mathcal{L}}_j\to\widetilde{W}_j$ restricting to $\pi_j^{-1}(W_i\cap W_j)$. It follows from the definition of orbifolds that for any $\bar{g}\in W_i\cap W_j$, there exists a neighborhood $U=U(\bar{g})$ of $\bar{g}$ with $U\subset W_i\cap W_j$ and two equivalent uniformizer $\widetilde{U}_i$ and $\widetilde{U}_j$ of U with $\widetilde{U}_i\subset\widetilde{W}_i$ and $\widetilde{U}_j\subset\widetilde{W}_j$. Let $\lambda^U_{ij}:\widetilde{U}_i\to\widetilde{U}_j$ be a equivalence between them. Then it has a lifting $\lambda_{ij}^{\mathcal{L}}: \widetilde{\mathcal{L}}_i \to \widetilde{\mathcal{L}}_j$ such that $\lambda_{ij} = (\lambda_{ij}^{\mathcal{L}}, \lambda_{ij}^{U})$ together give a equivalence of the two local orbifold bundles. This obviously induces a transform of $\nu_i|_{\widetilde{U}_i}$ to $\widetilde{U}_j$. However this construction is purely local and λ_{ij} is not canonical due to the automorphisms of Γ_{u_i} and Γ_{u_j}. These difficulties are also the reasons that one can not find a global uniformization for a orbifold bundle in general. However it is still possible to find some weaker induced structure of our orbifold bundle, which serves well as a suitable substitute of a "global uniformizer" of $\mathcal{L} \to W$.

We already sketched how to construct such a "global uniformizer" in the introduction of this paper. The rest of this section is devoted to the details of this construction and its related relative virtual moduli cycle.

We start with constructing $V_{i_1,\cdots,i_n}$ mentioned in the introduction, where the indices $i_1, \cdots, i_n$ corresponding to all possible indices of non-empty multi-intersections $W_{i_1} \cap \cdots W_{i_n}$. We will use $\mathcal{N}$ to denote the collection of all such indices. We define the length of the index $(i_1, \cdots, i_n)$ to be n and let $\mathcal{N}_n \subset \mathcal{N}$ be the set of indices of length n. We will also use short notation I to denote $(i_1, \cdots, i_n)$.

Lemma 4.3. *There exists an open covering* $\{V_{i_1,\cdots,i_n}\}$, $(i_1, \cdots, i_n) \in \mathcal{N}$ *of* $\overline{\mathcal{M}}(J, H; \tilde{x}, \tilde{y})$ *such that*

(i) $V_{i_1,\cdots,i_n} \subset W_{i_1} \cap W_{i_2} \cap \cdots \cap W_{i_n}$, *for all* $(i_1, \cdots, i_n) \in \mathcal{N}$;

(ii) $Cl(V_{I_1}) \cap Cl(V_{I_2}) = \emptyset$ *if the length* $l(I_1) = l(I_2)$, *and* $I_1 \neq I_2$.

Proof. We may assume that there exist open sets $W_i^1 \subset\subset W_i$, $i = 1, \cdots, m$ such that $\{W_i^1, i = 1, \cdots, m\}$ already forms a covering of $\overline{\mathcal{M}}(J, H; \tilde{x}, \tilde{y})$. For each fixed i we can find pairs of open sets $W_i^j \subset\subset U_i^j$, $j = 1, \cdots, m$ such that

$$W_i^1 \subset\subset U_i^1 \subset\subset W_i^2 \subset\subset U_i^2 \cdots \subset\subset W_i^m = W_i.$$

Now define

$$V_{i_1,\cdots,i_n} = W_{i_1}^n \cap W_{i_2}^n \cdots \cap W_{i_n}^n \setminus (\cup_{J \in \mathcal{N}_{n+1}} Cl(U_{j_1}^n) \cap Cl(U_{j_2}^n) \cdots \cap Cl(U_{j_{n+1}}^n)$$

where $J = (j_1, \cdots, j_{n+1})$.

Clearly the family $\{V_{i_1,\cdots,i_n}, (i_1, \cdots, i_n) \in \mathcal{N}\}$ so constructed satisfies the condition in the lemma. q.e.d.

Let

$$W_{i_1,\cdots,i_n} = W_{i_1} \cap \cdots \cap W_{i_n}, \quad \mathcal{L}_{i_1,\cdots,i_n} = \mathcal{L}|_{W_{i_1,\cdots,i_n}}.$$

There are n uniformizing systems

$$(\widetilde{\mathcal{L}}_{i_1,\cdots,\widehat{i_k},\cdots,i_n}, \widetilde{W}_{i_1,\cdots,\widehat{i_k},\cdots,i_n}, \pi^{\mathcal{L}}_{i_1,\cdots,\widehat{i_k},\cdots,i_n}, \pi^{W}_{i_1,\cdots,\widehat{i_k},\cdots,i_n})$$

of

$$(\mathcal{L}_{i_1,\cdots,i_n}, W_{i_1,\cdots,i_n}), \quad k = 1, \cdots, n,$$

with covering group Γ_{i_k}, induced from

$$(\pi^{\mathcal{L}}_{i_k}, \pi^{W}_{i_k}) : (\widetilde{\mathcal{L}}_{i_k}, \widetilde{W}_{i_k}) \to (\mathcal{L}_{i_k}, W_{i_k}),$$

where

$$\widetilde{W}_{i_1,\cdots,\widehat{i_k},\cdots,i_n} = (\pi^{W}_{i_k})^{-1}(W_{i_1,\cdots,i_n})$$

and

$$\widetilde{\mathcal{L}}_{i_1,\cdots,\widehat{i_k},\cdots,i_n} = (\widetilde{\mathcal{L}}_{i_1,\cdots,i_n})|_{\widetilde{W}_{i_1,\cdots,\widehat{i_k},\cdots,i_n}}.$$

We want to construct the pull-back of these morphisms, denoted by

$$(\pi^{\mathcal{L}}_{i_1,\cdots,i_n}, \pi^{W}_{i_1,\cdots,i_n}) : (\widetilde{\mathcal{L}}^{\Gamma_{i_1,\cdots,i_n}}_{i_1,\cdots,i_n}, \widetilde{W}^{\Gamma_{i_1,\cdots,i_n}}_{i_1,\cdots,i_n}) \to (\mathcal{L}_{i_1,\cdots,i_n}, W_{i_1,\cdots,i_n})$$

with covering group

$$\Gamma_{i_1,\cdots,i_n} = \Gamma_{i_1} \times \cdots \times \Gamma_{i_n}.$$

We start to define

$$\begin{aligned} \widetilde{W}^{\Gamma_I}_I &= \widetilde{W}^{\Gamma_{i_1,\cdots,i_n}}_{i_1\cdots,i_n} \\ &= \left\{ u \mid u \in \prod_{k=1}^{n} W_{i_k}, \begin{array}{lcr} \pi^W_{i_k}(u_k) & \in W_{i_1,\cdots,i_n}, & \\ \pi^W_{i_k}(u_k) & = & \pi^W_{i_l}(u_l) \end{array} \right\}, \end{aligned}$$

where $u = (u_1, \cdots, u_n)$ with $u_k \in \widetilde{W}_{i_k}$ and $\Gamma_I = \Gamma_{i_1,\cdots,i_n}$. Then $\pi^W_{i_1,\cdots,i_n}$ is the composition of $\prod_{k=1}^n \pi_{i_k}$ restricting to $\widetilde{W}^{\Gamma_I}_I$ with $\triangle_n^{-1}$ of the inverse of n-fold diagonal. If $J = (j_1, \cdots, j_m) \subseteq I = (i_1, \cdots, i_n)$, there exists an obvious projection map

$$(\pi^W)^I_J : \widetilde{W}^{\Gamma_I}_I \to \widetilde{W}^{\Gamma_J}_J$$

induced from the corresponding projection $\prod_{i_k \in I} \widetilde{W}_{i_k}$ to $\prod_{j_l \in J} \widetilde{W}_{j_l}$ such that $\pi^W_J \circ (\pi^W)^I_J = E^I_J \circ \pi^W_I$, where E^I_J is the inclusion $W_I \hookrightarrow W_J$.

All the above constructions can be directly extended to bundle case and we get a system of bundles $\{p_I : \widetilde{\mathcal{L}}^{\Gamma_I}_I \to \widetilde{W}^{\Gamma_I}_I\}$, $I \in \mathcal{N}$.

Note that for any fixed I with $l(I) > 1$, $\widetilde{W}_I^{\Gamma_I}$ is not a (partially) smooth manifold in general but rather a (partially) smooth variety, i.e., locally it is a finite union of (partially) smooth manifolds. In fact for $u \in \widetilde{W}_I^{\Gamma_I}$ with $u = (u_1, \cdots, u_n)$, $\bar{u} = \pi_{i_k}(u_k)$, there exists an open neighborhood U of $\bar{u}$ in W_I, such that for the inverse image $\widetilde{U}_k = \pi_{i_k}^{-1}(U)$ in $\widetilde{W}_{i_k}$, there exist $(n-1)$ equivalence maps $\lambda_k : \widetilde{U}_1 \to \widetilde{U}_k$, $k = 2, \cdots, n$. Composing with the actions of automorphism group Γ_{u_k} of $\widetilde{U}_k$, we get $\prod_{i=2}^n |\Gamma_{u_i}|$ equivalence maps:

$$\phi_k \lambda_k : \widetilde{U}_1 \to \widetilde{U}_k, \quad k = 2, \cdots, n, \quad \phi_k \in \Gamma_{u_k}.$$

Clearly $u = (u_1, \cdots, u_n) \in \prod_{k=1}^n U_k$ is contained in $\widetilde{W}_I^{\Gamma_I}$ if and only if $u_k = \phi_k \lambda_k(u_1)$ for some $\phi_k \in \Gamma_{u_k}$. Similar results hold for bundles. Therefore, locally the bundle decomposes into its $|\Gamma_{u_i}|^{n-1}$ irreducible components, each being a vector bundle.

We summarize up the above discussion in the following lemma.

Lemma 4.4. *There exists a pull-back*

$$\pi_I : (\widetilde{\mathcal{L}}_I^{\Gamma_I}, \widetilde{W}_I^{\Gamma_I}) \to (\mathcal{L}_I, W_I)$$

of the n uniformizing systems

$$\pi_{i_1, \cdots, \widehat{i_k}, \cdots, i_n} : (\widetilde{\mathcal{L}}_{i_1, \cdots, \widehat{i_k}, \cdots, i_n}, \widetilde{W}_{i_1, \cdots, \widehat{i_k}, \cdots, i_n}) \to (\mathcal{L}_I, W_I)$$

in the category of (partially) smooth variety such that the automorphism group of π_I is $\Gamma_I = \Gamma_{i_1} \times \Gamma_{i_2} \cdots \times \Gamma_{i_n}$ and the induced quotient map

$$\overline{\pi}_I : (\widetilde{\mathcal{L}}_I^{\Gamma_I}/\Gamma_I, \widetilde{W}_I^{\Gamma_I}/\Gamma_I) \to (\mathcal{L}_I, W_I)$$

is a homeomorphism. The inverse image $\pi_I^{-1}((\mathcal{L}_I)^s, (W_I)^s)$ of smooth points is a pair of smooth manifolds and the restriction of π_I to $\pi_I^{-1}((\mathcal{L}_I)^s, (W_I)^s)$ is a $|\Gamma_I|$-fold covering of $((\mathcal{L}_I)^s, (W_I)^s)$. For any $J \subset I$, there exists a projection

$$\pi_J^I : (\widetilde{\mathcal{L}}_I^{\Gamma_I}, \widetilde{W}_I^{\Gamma_I}) \to (\widetilde{\mathcal{L}}_J^{\Gamma_J}, \widetilde{W}_J^{\Gamma_J}),$$

whose generic fiber contains $|\prod_{j_l \in I \setminus J} \Gamma_{j_l}|$ points. Moreover, we have $\pi_J \circ \pi_J^I = \overline{E}_J^I \circ \pi_I$ for each $I \in \mathcal{N}$.

Now we define

$$\widetilde{V}_I = (\pi_I)^{-1}(V_I), \quad \widetilde{E}_I = (\pi_I)^{-1}(\mathcal{L}|_{V_I}).$$

Then the bundle $(\widetilde{E}_I, \widetilde{V}_I)$ are still a pair of smooth varieties, and for any $J \subset I$ the projection π_J^I still can be defined when restricted to $(\pi_J^I)^{-1}(\widetilde{E}_J, \widetilde{V}_J) \cap (\widetilde{E}_I, \widetilde{V}_I)$. Since locally $\widetilde{V}_I$ is a finite union of its smooth component and $\widetilde{E}_I$ decomposes into vector bundles over these local components, a local section of the bundle can be defined as a union of single-valued sections over those local components. They agree to each other over smooth points. As the local sections so defined are functorial with respect to restriction, a section of the bundle can be defined by patching these local sections together. We will say a section $S_I : \widetilde{V}_I \to \widetilde{E}_I$ is smooth if locally S_I restricted to any of those smooth components is smooth. For a smooth section S_I, we say S_I is transversal to zero section if locally S_I restricted to any of the smooth components of $\widetilde{V}_I$ is transversal to zero section .

Now let $(\widetilde{E}, \widetilde{V})$ be the collection $\{(\widetilde{E}_I, \widetilde{V}_I), \pi_J^I; J \subset I \in \mathcal{N}\}$ of the system of bundles together with their morphisms. We define a global section $S = \{S_I; I \in \mathcal{N}\}$ of such a system by requiring the obvious compatibility condition:

$$(\pi_J^I)^* S_J = S_I (\pi_J^I)^{-1} \widetilde{V}_J$$

over smooth points. Note that from now on, it is to be understood that pull-back of sections is only defined over smooth points. S is said to be transversal to zero section if each S_I is.

Now the section $\overline{\partial}_{J,H} : W \to \mathcal{L}$ gives rise to a global section of the bundle system $(\widetilde{E}, \widetilde{V})$ in an obvious way. Our goal now is to perturb $\overline{\partial}_{J,H}$ to get a global transversal section. To this end, we need to know how an element $\nu_i \in R_i$ can be interpreted as a global section of $(\widetilde{E}, \widetilde{V})$ first. By multiplying with some Γ_i-equivariant cut-off function β_i, we may assume that the support of each element ν_i is contained in $\widetilde{W}_i^1 = \pi_i^{-1}(W_i^1)$ and that $\{U_i^0; i = 1, \cdots, m\}$ already forms a covering of $\overline{\mathcal{M}}(J, H; \tilde{x}, \tilde{y})$, where $\widetilde{U}_i^0 = \{u \mid u \in \widetilde{W}_i, \beta_i(u) > 0\}$ and $U_i^0 = \pi_i(\widetilde{U}_i^0)$. Here we need to assume that the index p in L_k^p-norm which we used before to define $\widetilde{W}_i$ is even in order to be able to construct β_i. Now since each ν_i vanishes near the boundary of $\widetilde{W}_i$, we may consider it as a global multi-valued section $\bar{\nu}_i$ of $\mathcal{L} \to W$ supported in $U_i^0 \subset\subset W_i^1$.

Lemma 4.5. *Each $\nu_i \in R_i$ gives rise to a global section $\tilde{\nu}_i = \{(\tilde{\nu}_i; I \in \mathcal{N}\}$ of the system $(\widetilde{E}, \widetilde{V})$.*

Proof. Let $I \in \mathcal{N}$ with $i \notin I$ and consider V_I.

Recall that if $I = \{i_1, \cdots, i_n\}$, then

$$V_I = W^n_{i_1} \cap W^n_{i_2} \cdots \cap W^n_{i_n} \setminus \cup_{J \in \mathcal{N}_{n+1}} Cl(U^n_{j_1}) \cdots \cap Cl(U^n_{j_{n+1}})$$

with $J = (j_1, \cdots, j_{n+1})$. Since $i \notin I$,

$$\begin{aligned} V_I &\subseteq W^n_{i_1} \cap \cdots \cap W^n_{i_n} \setminus W^n_{i_1} \cap \cdots \cap W^n_{i_n} \cap Cl(W^1_i) \\ &\subseteq W \setminus Cl(W^1_i). \end{aligned}$$

Therefore, the intersection $Cl(U^0_i) \cap Cl(V_I) = \emptyset$. Hence $\tilde{\nu}_i|_{V_I} \equiv 0$ for any $I \in \mathcal{N}$ with $i \notin I$. We define $(\tilde{\nu}_i)_I \equiv 0$ if $i \notin I$.

Now assume that $i \in I$.

When $l(I) = 1$, $I = \{i\}$, $\widetilde{V}_I = \widetilde{V}_i$ and $(\tilde{\nu}_i)_I$ is just $\nu_i : \widetilde{W}_i \to \widetilde{\mathcal{L}}_i$ restricted to $\widetilde{V}_i$.

If we denote $\{i\}$ by I_i, then for any I with $n = l(I) > 1$, we have

$$\pi^I_{I_i} : (\widetilde{\mathcal{L}}^{\Gamma_I}_I, \widetilde{W}^{\Gamma_I}_I) \to (\widetilde{\mathcal{L}}_I, \widetilde{W}_I) \subset (\widetilde{\mathcal{L}}_i, \widetilde{W}_i).$$

Therefore, $(\pi^I_{I_i})^*(\tilde{\nu}_i)_{I_i}$ after the obvious extension gives rise to a section of $\widetilde{E}_I \to \widetilde{V}_I$, denoted by $(\tilde{\nu}_i)_I$. Clearly the section $(\tilde{\nu}_i)_I$, $I \in \mathcal{N}$ so constructed are compatible to each other and yields a well-defined global section $\tilde{\nu}_i = \{(\tilde{\nu}_i)_I, I \in \mathcal{N}\}$ of the system $(\widetilde{E}, \widetilde{V})$. q.e.d.

Let $\Gamma(\widetilde{E}, \widetilde{V})$ be the space of global (smooth) sections of $(\widetilde{E}, \widetilde{V})$. The correspondence $\nu_i \to \tilde{\nu}_i$ is a linear map of the vector space R_i into $\Gamma(\widetilde{E}, \widetilde{V})$. Define $R = \oplus^m_{i=1} R_i$. Then the above maps induce a linear map $R \to \Gamma(\widetilde{E}, \widetilde{V})$. Consider the system

$$(\widetilde{E} \times Z_\delta = \pi^*_1 \widetilde{E}, \widetilde{V} \times Z_\delta) = \{(\widetilde{E}_I \times Z_\delta = \pi^*_1 \widetilde{E}_I, \widetilde{V}_I \times Z_\delta); I \in \mathcal{N}\}$$

of bundles, where Z_δ is a δ-neighborhood of zero of R, and π_1 is the projection to the first factor of $\widetilde{E} \times Z_\delta$. There is a well-defined global section $\overline{\partial}_{J,H} + e$ defined as follows,

$$(\overline{\partial}_{J,H} + e)(u_I, \nu) = \overline{\partial}_{J,H} u_I + e((\tilde{\nu})_I, u_I)$$

for any $(u_I, \nu) \in \widetilde{V}_I \times Z_\delta$, where $e : \Gamma(\widetilde{V}, \widetilde{E}) \times \widetilde{V} \to \widetilde{E}$ is the evaluation map.

Theorem 4.1. *$\overline{\partial}_{J,H} + e$ is a smooth section of $(\pi^*_1 \widetilde{E}, \widetilde{V} \times Z_\delta)$, which is transversal to zero section. It follows that when δ is small enough for a generic choice of the perturbation term $\nu \in Z_\delta$ the section $\overline{\partial}_{J,H} + \tilde{\nu}$:*

$\widetilde{V} \to \widetilde{E}$ is transversal to zero section and that the family of perturbed moduli spaces

$$\widetilde{\mathcal{M}}^{\nu} = \{\widetilde{\mathcal{M}}^{\nu}_I = (\overline{\partial}_{J,H} + \tilde{\nu}_I)^{-1}(0); I \in \mathcal{N}\}$$

is compatible in the sense that

$$\pi^I_J(\widetilde{\mathcal{M}}^{\nu}_I) = \widetilde{\mathcal{M}}^{\nu}_J \cap (Im\pi^I_J), \quad J \subset I.$$

Moreover, the image $\mathcal{M}^{\nu}$ of $\widetilde{\mathcal{M}}^{\nu}$ in W is compact with boundary components of "right" dimension.

Proof. $\overline{\partial}_{J,H} + e$ is obviously smooth. From the main estimate of last section and the construction of R_i it follows that

$$(\overline{\partial}_{J,H} + e_i)|_{\widetilde{U}^0_i \times Z_\delta} : \widetilde{U}^0_i \times Z_\delta \to \pi_1^*(\widetilde{\mathcal{L}}_i|_{\widetilde{U}_i})$$

is transversal to zero section, where e_i is the same as the evaluation map, but replaced R by R_i.

Now $\{U^0_i; i = 1, \cdots, m\}$ already forms a covering of $\overline{\mathcal{M}}(J, H; \tilde{x}, \tilde{y})$. Given any $u \in \widetilde{V}_I$, let $\overline{u} = \pi^I(u) \in V_I$. Then there exists a U^0_i such that $\overline{u} \in U^0_i$. Since $U^0_i \cap V_I \neq \emptyset$, $i \in I$ as we proved before. Therefore $\overline{\partial}_{J,H} + e$ is also transversal to zero section near

$$(\pi^I_{I_i})^{-1}(\widetilde{U}^0_i \cap \widetilde{W}_{i_1,\cdots,i_n}),$$

where $I_i = \{i\}$ and $I = i_1, \cdots, i_n$ as before. This proves the transversality for $\overline{\partial}_{J,H} + e$. By the implicit function theorem applied locally to each (partially) smooth component of $\widetilde{V} = \{\widetilde{V}_I; I \in \mathcal{N}\}$ we obtain that

$$(\overline{\partial}_{J,H} + e)^{-1}(0) = \{(\overline{\partial}_{J,H} + e)^{-1}_I(0); I \in \mathcal{N}\} \subset \widetilde{V} \times Z_\delta$$

is a family of "cornered" (partially) smooth subvariety.

Let

$$\pi : (\overline{\partial}_{J,H} + e)^{-1}(0) \to Z_\delta$$

be the restriction of projection of $\widetilde{V} \times R$ to R. It is easy to see that Smale-Sard theorem is still applicable in this case. We conclude that for " generic" choice of $\nu \in R$, $\overline{\partial}_{J,H} + \nu$ is already a transversal section of $(\widetilde{E}, \widetilde{V})$.

The compatibility of the family of zero set

$$\{\widetilde{\mathcal{M}}^{\nu}_I; I \in \mathcal{N}\} = \{\overline{\partial}_{J,H} + e)^{-1}_I(0)\}$$

follows from the fact that $\overline{\partial}_{J,H}+\nu$ is a global section of $(\widetilde{E},\widetilde{V})$.

What left is to prove the compactness of $\mathcal{M}^\nu$.

Because

$$\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})\subset\cup_{i=1}^m U_i^0,$$

$\overline{\partial}_{J,H}$ never becomes zero along the boundary

$$\partial(\cup_{i=1}^m U_i^0)=Cl(\cup_{i=1}^m U_i^0)\setminus\cup_{i=1}^m U_i^0.$$

But from the construction of R, it follows that $\nu\equiv 0$ along $\partial(\cup_{i=1}^m U_i^0)$. Therefore we have $\mathcal{M}^\nu\subset\cup_{i=1}^m U_i^0\subset\subset V$. Let $\{\overline{u}_i\}_{i=1}^\infty$ be a sequence of $\mathcal{M}^\nu$. We may assume that all $\overline{u}_i$ are contained in V_I for some $I\in\mathcal{N}$. Let $u_i\in\widetilde{\mathcal{M}}_I^\nu\subset\widetilde{V}_I$ with $\overline{u}_i=\pi^I(u_i)$. We need to prove first that the section ν_I of the bundle $\widetilde{E}_I\to\widetilde{V}_I$ has a bounded L_k^p-norm. Since $\nu=\sum_{i,j}a_{ij}e_{ij}$ with $\{e_{ij};j=1,\cdots,n_i\}$ being the basis of R_i, our assumption that $\nu\in Z_\delta\subset R$ implies that all $|a_{ij}|$ are bounded. Therefore we only need to prove that all the lifting $\{e_{ij}\}_I$ over $\widetilde{V}_I$ of e_{ij} , which is defined over $\widetilde{W}_i$ originally, are still bounded. As noted before, we only need to consider the case that $i\in I$. From the construction ν_I, the boundedness of $\|\mu_I\|_{k,p}$ will follow easily if we can prove that all "coordinate changes" between $\widetilde{W}_k$'s are induced from those reparametrizations which stay inside a compact subset of $SL(2,\mathbf{C})$. To this end, we consider $W_i^c=Cl(W_i)$, $i=1,\cdots,m$ and $W_{i,j}^c=W_i^c\cap W_j^c$ with $\widetilde{W}_i^c$ and $\widetilde{W}_{i,j}^c\subset\widetilde{W}_i^c$ be the lifting of them in the uniformizer $\widetilde{W}_i^*$. Here we have assumed that the uniformizer $\widetilde{W}_i^*$ is defined over a slight larger set than $\widetilde{W}_i$.

Let $\mathcal{M}_{ij}=\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})\cap W_{i,j}^c$. Then $\mathcal{M}_{ij}$ is compact. Let

$$\{Z_k^{i,j},Z_k^{i,j}\subset\mathcal{B}(\tilde{x},\tilde{y}),k=1,\cdots,m^{ij}\}$$

be an open covering of $\mathcal{M}_{ij}$ in $\mathcal{B}(\tilde{x},\tilde{y})$ such that each component of $\pi_i^{-1}(Z_k^{i,j})$ and $\pi_j^{-1}(Z_k^{i,j})$ in $\widetilde{W}_i^c$ and $\widetilde{W}_j^c$ respectively is a uniformizer of $Z_k^{i,j}$. Now for each fixed pair of components of $\pi_i^{-1}(Z_k^{i,j})$ and $\pi_j^{-1}(Z_k^{i,j})$, the equivalence between them are induced by some automorphisms of domain which are contained in a compact subset of $\prod SL(2,\mathbf{C})$. Since there are only finite $Z_k^{i,j}$, all these coordinate changes are still induced from a compact set of $\prod SL(2,\mathbf{C})$. Now let $Z^{i,j}=\cup_k Z_k^{i,j}$ and use $(W_i\setminus\cup_{k\neq i}W_{ik}^c)\cup_{k\neq i}Z^{i,k}$ to replace W_i, $i=1,\cdots,m$. They still form an open covering of $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$ and all previous constructions work in the same way as before. All "coordinate changes" now stay in a compact set .

We need to prove that $u_i \in \widetilde{\mathcal{M}}^{\nu}_I \subset \widetilde{V}_I$ has a limit in $\prod_{i_k \in I} Cl(\widetilde{W}_{i_k})$. Since we can do this componentwisely, we may assume that $l(I) = 1$, and $I = \{1\}$. Then we have $\{u_i\}_{i=1}^{\infty} \in \widetilde{U}^0_1 \subset \widetilde{W}_1$ and

$$\nu_i = \overline{\partial}_{J,H}(u_i) \in (\widetilde{\mathcal{L}}_1)_{u_i}$$

has a uniform bounded L^p_k-norm. Note that from the construction of R it follows that ν_i vanishes near all double points of the domain of u_i. By the standard elliptic estimate, there exists a $u_\infty \in \widetilde{W}_1$ such that some subsequence of $\{u_i\}_{i=1}^{\infty}$, still denoted by $\{u_i\}$, is weakly C^∞-convergent to u_∞. Similar argument for proving the equivalence of weak C^∞ -topology and the L^p_k-topology for $\overline{\mathcal{M}}(J, H; \tilde{x}, \tilde{y})$ stated at the beginning of this section proves that the above convergence is actually also in the L^p_k-topology. q.e.d.

As we noted before, for the top strata D_T or the strata D_B of "broken" connecting orbits, $W_i^{D_T}$ and $W_i^{D_B}$ are contained in the smooth locus $W_i^s \subset W_i$. It follows that $\widetilde{V}_I^{D_T} = \pi_I^{-1}(V_I^{D_T})$ and $\widetilde{V}_I^{D_B} = \pi_I^{-1}(V_I^{D_B})$ are also smooth. For our purpose, we only need to consider those "broken" connecting orbits of only two components. We will still use D_B to refer this particular intersection pattern. Then $\widetilde{\mathcal{M}}_I^{\nu,D_T} = \widetilde{\mathcal{M}}_I^{\nu} \cap \widetilde{V}_I^{D_T}$ and $\widetilde{\mathcal{M}}^{\nu,D_B} = \widetilde{\mathcal{M}}_I^{\nu} \cap \widetilde{V}_I^{D_B}$ are smooth manifolds of dimension $\mu(\tilde{y}) - \mu(\tilde{x}) - 1$ and $\mu(\tilde{y}) - \mu(\tilde{x}) - 2$ respectively. In fact, let $\widetilde{\mathcal{M}}_I^{\mu,c} = \widetilde{\mathcal{M}}_I^{\mu,D_T} \cup \widetilde{\mathcal{M}}_I^{\mu,D_B}$. Then $\widetilde{\mathcal{M}}_I^{\nu,c}$ is a smooth manifold with boundary and its boundary is $\partial\widetilde{\mathcal{M}}_I^{\nu,c} = \widetilde{\mathcal{M}}_I^{\nu,D_B}$. Let $N_i =$ order of Γ_i, where Γ_i is the isotropy group of f_i with $\widetilde{W}_i = \widetilde{W}_{\epsilon_i}(f_i; \mathbf{H})$. Then $\widetilde{W}_i^s = \pi_i^{-1}(W_i^s) \to W_i^s$ is a N_i-folded covering. It follows that $\widetilde{V}_I^{D_T}$ (resp. $\widetilde{V}_I^{D_B}$) is a $N_I = \prod_{i \in I} N_i$-fold covering of $V_I^{D_T}$ (resp. $V_I^{D_B}$), and

$$\pi_J^I : \pi_I^{-1}(\widetilde{V}_J^{D_T}) \cap \widetilde{V}_I^{D_T} \to \pi_J^I(\pi_I^{-1}(\widetilde{V}_J^{D_T}) \cap \widetilde{V}_I^{D_T})$$

is a N_I/N_J-folded covering. Now

$$(\overline{\partial}^I_{J,H} + \nu_I) = (\pi_J^I)^*(\overline{\partial}^J_{J,H} + \nu_J).$$

It follows that $(\pi_J^I)_*(\widetilde{\mathcal{M}}_I^{\nu,D_T}) = N_I/N_J \widetilde{\mathcal{M}}_J^{\nu,D_T}$. Therefore if we consider each $\frac{1}{N_I}\widetilde{\mathcal{M}}_I^{\nu,D_T}$, $i \in \mathcal{N}$ as a rational "geometric" chain, then they can be pasted together to define a "fundamental" cycle of $\overline{\mathcal{M}}^{\nu}$. We formally denote this as

$$C(\overline{\mathcal{M}}^{\nu}) = \sum_{I \in \mathcal{N}} \frac{1}{N_I} \widetilde{\mathcal{M}}_I^{\nu,c}.$$

Here the summation is somehow abused since on the overlap of two pieces of $C(\overline{\mathcal{M}}^{\nu})$ we only count them once.

More precisely, for each $\widetilde{\mathcal{M}}_I^{\nu,c}$, $I \in \mathcal{N}$, let $\{\widetilde{K}_I^n\}_{n=1}^{\infty}$ be an increasing sequence of compact submanifolds with boundary of $\widetilde{\mathcal{M}}_I^{\nu,c}$, such that $\widetilde{\mathcal{M}}_I^{\nu,c} = \cup_n \widetilde{K}_I^n$. Fix a $\widetilde{K}_I^n$ and choose a triangulation of it. For each simplex in $\widetilde{K}_I^n$ we choose its orientation induced from that of $\widetilde{\mathcal{M}}_I^{\nu,c}$. Then we take the summation of all oriented symplex of top dimension in $\widetilde{K}_I^n$. This gives rise to a singular chain in $\widetilde{\mathcal{M}}_I^{\nu,c}$. We use $S(\widetilde{K}_I^n)$ to denote it. Let

$$\widetilde{S}(\widetilde{K}_I^n) = \frac{1}{N_I}\pi_I \circ S(\widetilde{K}_I^n)$$

be the corresponding rational singular chain in $\overline{\mathcal{M}}^{\nu}$.

Now for each $I \in \mathcal{N}$, choose a fixed $\widetilde{K}_I^n$ in $\widetilde{\mathcal{M}}_I^{\nu,c}$. By using subdivisions and the above compatibility condition of those $\widetilde{\mathcal{M}}_I^{\nu,c}$'s, we can arrange that for each simplex $c_I^n \in \widetilde{K}_I^n$, $\pi_J^I(c_I^n)$ is a simplex of $\widetilde{K}_J^n$, if $J < I$ and $\widetilde{K}_J^n \cap \pi_J^I(\widetilde{K}_I^n) \neq \emptyset$.

Now let n varies, we may also arrange that $\widetilde{K}_I^m|_{\widetilde{K}_I^n}$ is obtained from $\widetilde{K}_I^n$ by some divisions as simplicial complex if $m > n$. It follows that for each fixed n we can define a singular chain $C(\widetilde{K}^n)$ to be the summation of all those singular maps in $\widetilde{S}(\widetilde{K}_I^n)$, $I \in \mathcal{N}$. Note that on those overlaps where more than one singular maps appear we only count once. There is an obivious restriction map $r_n^m : C(\widetilde{K}^m) \to C(\widetilde{K}^n)$, $m > n$, given by the subdivision mentioned above. Therefore, we can define $C(\overline{\mathcal{M}}^{\nu})$ to be the inverse limit of $\{C(\widetilde{K}^n)\}_n$. Note that each element of $\overline{\mathcal{M}}^{\nu}$ will be covered by some $\widetilde{K}^m$.

Now we have

Theorem 4.2. *$C(\overline{\mathcal{M}}^{\nu})$ is a relative virtual moduli cycle of dimension $\mu(\tilde{y}) - \mu(\tilde{x}) - 1$ with*

$$\partial(C(\overline{\mathcal{M}}^{\nu})) = \sum_{I\in\mathcal{N}} \frac{1}{N_I}\widetilde{\mathcal{M}}^{\nu,D_B}.$$

In particular, when $\mu(\tilde{y}) - \mu(\tilde{x}) \leq 2$, we have $\widetilde{\mathcal{M}}_I^{\nu,c} = \widetilde{\mathcal{M}}_I^{\nu}$, hence $C(\overline{\mathcal{M}}^{\nu}) = \sum_{I\in\mathcal{N}} \frac{1}{N_I}\widetilde{\mathcal{M}}_I^{\nu}$.

Corollary 4.1. *When $\mu(\tilde{y}) - \mu(\tilde{x}) = 1$,*

$$\mathcal{M}^{\nu} = \cup_{I\in\mathcal{N}}\mathcal{M}_I^{\nu}$$

is a finite set, and

$$C(\overline{\mathcal{M}}^{\nu})=\sum_{I\in\mathcal{N}}\frac{1}{N_I}\widetilde{\mathcal{M}}_I^{\nu}$$

with each $\widetilde{\mathcal{M}}_I^{\nu}$ *being finite.*

When $\mu(\tilde{y})-\mu(\tilde{x})=2$,

$$C(\overline{\mathcal{M}}^{\nu})=\sum_{I\in\mathcal{N}}\frac{1}{N_I}\widetilde{\mathcal{M}}^{\nu}$$

and

$$\partial(C(\overline{\mathcal{M}}^{\nu}))=\sum_{I\in\mathcal{N}}\frac{1}{N_I}\widetilde{\mathcal{M}}^{\nu,D_B}$$

with each $\widetilde{\mathcal{M}}_I^{\nu,D_B}$ *being a finite set. Moreover, the oriented number* $\#\partial(C(\overline{\mathcal{M}}^{\nu}))=0$.

Remark 4.2. Here we have used the fact that each of these $\mathcal{M}_I^{\nu,D}$, $I\in\mathcal{N}$ has a canonical orientation. Details related to this can be found in [5], [7] and [19].

Now each "broken" connecting orbit $u\in\widetilde{\mathcal{M}}_I^{\nu,D_B}=\widetilde{\mathcal{M}}_I^{\nu,D_B}(\tilde{x},\tilde{y})$ with $\mu(\tilde{y})-\mu(\tilde{x})=2$ has a form $u=(u_1,u_2)$ with $u_1\in\widetilde{\mathcal{M}}_{I_1}^{\nu_1}(\tilde{x},\tilde{z})$, $u_2\in\widetilde{\mathcal{M}}_{I_2}^{\nu_2}(\tilde{z},\tilde{y})$ and $\mu(\tilde{z})-\mu(\tilde{x})=1$, $\mu(\tilde{y})-\mu(\tilde{z})=1$. Therefore if we construct each

$$\widetilde{\mathcal{M}}^{\nu}(\tilde{x},\tilde{y})=\{\widetilde{\mathcal{M}}_I^{\nu}(\tilde{x},\tilde{y}),I\in\mathcal{N}(\tilde{x},\tilde{y})\}$$

inductively with respect to $\mu(\tilde{x},\tilde{y})=\mu(\tilde{y})-\mu(\tilde{x})$ starting with $\mu(\tilde{x},\tilde{y})=1$, then for $\mu(\tilde{x},\tilde{y})=2$, $\widetilde{\mathcal{M}}^{\nu,D_B}(\tilde{x},\tilde{y})$ of "broken " connecting orbits has been already constructed and is just

$$\bigcup_{\substack{\mu(\tilde{x},\tilde{z})=1\\ \mu(\tilde{z},\tilde{y})=1}}\widetilde{\mathcal{M}}^{\nu}(\tilde{x},\tilde{z})\times\widetilde{\mathcal{M}}^{\nu}(\tilde{z},\tilde{y}).$$

It is easy to see that we can extend this "product" type ν defined for strata $\widetilde{V}^{D_B}$ to $\widetilde{V}$ itself in our previous construction. Since

$$\begin{aligned}\widetilde{V}^{D_B}(\tilde{x},\tilde{y})=&\{\widetilde{V}_I^{D_B}(\tilde{x},\tilde{y});I\in\mathcal{N}(\tilde{x},\tilde{y})\}\\ =&\left\{\cup_{\substack{\mu(\tilde{x},\tilde{z})=1\\ \mu(\tilde{z},\tilde{y})=1\\ I_1\cup I_2=I}}\widetilde{V}_{I_1}^{D_T}(\tilde{x},\tilde{z})\times\widetilde{V}_{I_2}^{D_T}(\tilde{z},\tilde{y});\ \begin{matrix}I_1\in\mathcal{N}(\tilde{x},\tilde{z}),\\ I_2\in\mathcal{N}(\tilde{z},\tilde{y})\end{matrix}\right\},\end{aligned}$$

we have $N_I=\sum_{I_1+I_2=I}N_{I_1}\times N_{I_2}$. It follows that

Corollary 4.2. *When $\mu(\tilde{y})-\mu(\tilde{x})=2$,*

$$\sum_{\substack{\mu(\tilde{x},\tilde{z})=1\\ \mu(\tilde{z},\tilde{y})=1}} C(\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{z}))\times C(\overline{\mathcal{M}}^{\nu}(\tilde{z},\tilde{y}))=\partial(C(\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{y}))).$$

Therefore $\#(\sum_{\substack{\mu(\tilde{x},\tilde{z})=1\\ \mu(\tilde{z},\tilde{y})=1}} C(\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{z}))\times C(\overline{\mathcal{M}}^{\nu}(\tilde{z},\tilde{y}))$

$$=\sum_{I_1,I_2,\tilde{z}}\frac{1}{N_{I_1}N_{I_2}}\#(\widetilde{\mathcal{M}}_{I_1}^{\nu_{I_1}}(\tilde{x},\tilde{z})\times\widetilde{\mathcal{M}}_{I_2}^{\nu_{I_2}}(\tilde{z},\tilde{y}))$$

is zero.

These last two corollaries and their analogies for the s-dependent pair (J_s,H_s) are all what we need to extend Floer homology to any symplectic manifold and to prove Arnold Conjecture in general.

We remark that it is possible to formulate the construction of the virtual moduli cycles by using the desingularization of the bundle system used here.

5. Floer cohomology and Arnold conjecture

In this section we will complete our long journey of extending Floer cohomology to an arbitrary symplectic manifold without any positivity assumption on its first Chern class and proving Arnold conjecture in general. In view of the "classical" Floer cohomology our task here is quite a routine after the last two corollaries which we proved in the last section.

Recall that for any generic Hamiltonian function H, we have defined the graded $\mathbf{Q}$-space: $C^*(H)=C^*(H;\mathbf{Q})=\oplus_n C^n(H)$ as follows.

Each element $\xi\in C^n(H)$ can be written as a formal sum $\xi=\sum_{\mu(\tilde{z})=n}\xi_{\tilde{z}}\tilde{z}$, with $\xi_{\tilde{z}}\in\mathbf{Q}$ and $\tilde{z}\in\tilde{\mathbf{P}}(H)$ such that for any $c>0$,

$$\#\{\tilde{z}\,|\,\xi_{\tilde{z}}\neq 0, a_H(\tilde{z})\leq c\}<\infty.$$

$C^*(H)$, as a vector space over $\mathbf{Q}$, is of course infinite dimensional in general, but it is finite dimensional over the Novikov ring $\wedge_\omega$, which can be defined as follows (see [10] for more details). Recall that we have used Γ to denote the image of $\pi_2(V)$ in $H_2(V;\mathbf{Z})$ under the Hurewicz homomorphism modulo torsion. The symplectic form gives rise to a homomorphism $\omega:\Gamma\to\mathbf{R}$. If $\{e_i; i=1,\cdots,m\}$ be the $\mathbf{Z}$ basis of Γ, we may identify any element $A \in \Gamma$, $A = A_i e_i$ with $(A_1, \cdots, A_m)$. Choose m indeterminants $t = (t_1, \cdots, t_m)$. We define $\wedge_\omega$ to be the collection of formal sums

$$\lambda = \sum_{A \in \Gamma} \lambda_A \cdot t^A$$

with $\lambda_A \in \mathbf{Q}$ satisfying the condition that

$$\#\{A \in \Gamma \mid \lambda_A \neq 0, \omega(A) \leq c\} < \infty$$

for any $c > 0$. Here we have used t^A to denote $t_1^{A_1} \cdot t_2^{A_2} \cdots t_m^{A_m}$ with $A = (A_1, \cdots, A_m)$. $\wedge_\omega$ is a ring with the obvious multiplication

$$\lambda \cdot \mu = \sum_{A,B} \lambda_A \cdot \lambda_B t^{A+B}.$$

In our case that all coefficients $\lambda_A \in \mathbf{Q}$, the ring $\wedge_\omega$ is in fact a field. $C^*(H)$ becomes a vector space over $\wedge_\omega$ under the following scalar product:

$$\lambda \cdot \xi = \sum_{\tilde{z}} \sum_{A} \lambda_A \cdot \xi_{(-A)\#\tilde{z}} \tilde{z},$$

where $(-A)\#\tilde{z}$ is the usual connect sum. Clearly the dimension of $C^*(H)$ as a $\wedge_\omega$-space is just $\#\mathbf{P}(H)$. Note that the above scalar product does not preserve the grading of $C^*(H)$.

In order to make the grading $\mathbf{Q}$-space $C^*(H)$ into a cochain complex, we introduce a "generic" ω-compatible almost complex structure J and its associated moduli space of stable (J, H)-maps $\overline{\mathcal{M}}(J, H; \tilde{x}, \tilde{y})$ which is the stable compactification of the moduli space $\mathcal{M}^{D_T}(J, H; \tilde{x}, \tilde{y})$ of the connecting orbits between $\tilde{x}$ and $\tilde{y}$. We can associate $\mathcal{M}^{D_T}(J, H; \tilde{x}, \tilde{y})$ with a finite open covering $\mathcal{W} = \{W_i; i = 1, \cdots, m\}$ with $W_i \in \overline{\mathcal{B}}(\tilde{x}, \tilde{y})$.

Let $\mathcal{N}$ be the nerve of $\mathcal{W}$. Then we can define the compact moduli space of stable (J, H, ν)-maps

$$\mathcal{M}^\nu(\tilde{x}, \tilde{y}) = \{\mathcal{M}^\nu(\tilde{x}, \tilde{y}); I \in \mathcal{N}\}$$

and its associated relative virtual moduli cycle

$$C(\overline{\mathcal{M}}^\nu(\tilde{x}, \tilde{y})) = \sum_{I \in \mathcal{N}} \frac{1}{N_I} \widetilde{\mathcal{M}}_I^{\nu,c}(\tilde{x}, \tilde{y}).$$

In the case that $\mu(\tilde{y}) - \mu(\tilde{x}) \leq 2$, $\widetilde{\mathcal{M}}_I^{\nu,c}(\tilde{x}, \tilde{y})$ in the last expression is just $\widetilde{\mathcal{M}}_I^\nu(\tilde{x}, \tilde{y})$.

Now the coboundary operator $\delta = \delta_{J,H,\nu} : C^*(H) \to C^*(H)$ is defined by

$$\delta(\tilde{x}) = \sum_{\mu(\tilde{y})=k+1} \#(C(\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{y})) \cdot \tilde{y}$$

for any $\tilde{x} \in C^k(H)$.

It follows from Corollary 4.1 that $n(\tilde{x},\tilde{y}) = \#(C(\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{y})))$ is finite. The Corollary 4.2 implies that

$$\delta \cdot \delta(\tilde{x}) = \sum_{\mu(\tilde{z})=k+1} \sum_{\mu(\tilde{y})=k+2} n(\tilde{x},\tilde{z}) \cdot n(\tilde{z},\tilde{y})\tilde{y} = 0$$

for any $\tilde{x} \in C^k(H)$.

Therefore we define Floer cohomology $FH^*(V,\omega; J,H,\nu)$ of (V,ω) associated to (J,H,ν) to be the homology of the cochain complex $(C^*(H), \delta_{J,H,\nu})$.

Rename the "parameter" (J,H,ν) by (J_0,H_0,ν_0). Our goal now is to prove that $FH^*(V,\omega;J_0,H_0,\nu_0)$ is isomorphic to $H^*(V,\wedge_\omega) = H^*(V,\mathbf{Q}) \otimes \wedge_\omega$, and as a consequence, to prove Arnold conjecture in general.

To this end, consider a "generic" time-independent pair (J_1,H_1). When the C^2-norm of H_1 is small enough, all elements of $\mathbf{P}(H_1)$ will degenerate into non-degenerate critical points of H_1. The lifting $\tilde{\mathbf{P}}(H_1)$ of $\mathbf{P}(H_1)$ is defined as before. But any element $\tilde{z} \in \mathbf{P}(H_1)$ has a form $\tilde{z} = [z,w]$ with $w \in \Gamma$. $\mu(\tilde{z}) = Ind_{H_1}(z) - n + 2c_1(w)$, where $Ind_{H_1}(z)$ is the Morse index of the critical point z and $n = (\dim V)/2$.

Now we can run through the whole theory again for the time-independent pair (J_1,H_1) as we did for (J_0,H_0). However there is a difference between these two cases. Namely, there is an extra symmetry along θ-direction for the time-independent pair (J_1,H_1). Because of this, the definition for $\overline{\mathcal{FM}_{0,k}}$ of $\mathcal{F}$-stable curves here needs to be modified by requiring that the automorphism group of each principal component is $\mathbf{R}^1 \times S^1$ rather than just S^1. The rest of the theory before can also be modified to incorporate this symmetry. After such a modification, we will still get a orbifold covering $\mathcal{W} = \{W_i; i = 1, \cdots, m\}$ of $\overline{\mathcal{M}}(J,H;\tilde{x},\tilde{y})$ with nerve $\mathcal{N}$ as before. However, due to the extra θ- symmetry, the dimension of

$$\mathcal{M}^{\nu_1} = \{\mathcal{M}_I^{\nu_1}, I \in \mathcal{N}\}$$

and its associated relative virtual cycle

$$C(\overline{\mathcal{M}}^{\nu_1}) = \sum_{I \in \mathcal{N}} \frac{1}{N_1} \widetilde{\mathcal{M}}_I^{\nu_1,c}$$

is $\mu(\tilde{y})-\mu(\tilde{x})-2$ instead of $\mu(\tilde{y})-\mu(\tilde{x})-1$ as long as all the elements used to construct $\mathcal{M}^{\nu_1}$ and $C(\mathcal{M}^{\nu_1})$ are away from "Morse cell" $\overline{\mathcal{M}}_{H_1}(x,y)$. Here $\overline{\mathcal{M}}_{H_1}(x,y)$ is the moduli space of unparametried "broken" gradient lines connecting x and y which is contained in $\overline{\mathcal{M}}(J_1,H_1;\tilde{x},\tilde{y})$ as an isolated compact component, if $[w_x]=[w_y]\in H_2(V)$ with $\tilde{x}=[x,w_x]$ and $\tilde{y}=[y,w_y]$ hence $c_1(w_x)=c_1(w_y)$. Now assume that $\mu(\tilde{y})-\mu(\tilde{x})=1$. Then away from $\overline{\mathcal{M}}_{H_1}(x,y)$, $\mathcal{M}^{\nu}(J_1,H_1;\tilde{x},\tilde{y})=\emptyset$ simply for dimension reason. Therefore we only need to consider $\overline{\mathcal{M}}_{H_1}(x,y)$ with $Ind_{H_1}(y)-Ind_{H_1}(x)=1$ in this case. In fact, because each element of $\overline{\mathcal{M}}_{H_1}(x,y)$ is homotopically trivial, we also have $w_x=w_y$ in the case. Combining all these together we have

Lemma 5.1. *For a generic time-independent pair (J_1,H_1) with H_1 being a C^2- small Morse function, the Floer cochain complex $(C^*(H_1),\delta_{J_1,H_1,\nu_1})$ constructed as before is just the usual Morse cochain complex with $\wedge_\omega$ as its coefficient ring. Therefore, the Floer cohomology $FH^*(V,\omega;J_1,H_1,\nu_1)$ is isomorphic to $H^*(V,\mathbf{Q})\otimes\wedge_\omega$ of the ordinary cohomology with $\wedge_\omega$ as coefficient ring.*

In view of this, in order to calculate $FH^*(V,\omega;J_0,H_0,\nu_0)$ for the time-independent pair (J_0,H_0), we only need to prove that

$$FH^*(V,\omega;J_0,H_0,\nu_0)\cong FH^*(V,\omega;J_1,H_1,\nu_1).$$

For this purpose, we define a chain homomorphism

$$\phi_1^0:(C^*(H_0),\delta_{J_0,H_0,\nu_0})\to(C^*(H_1),\delta_{J_1,H_1,\nu_1})$$

as follows.

Let (J_s,H_s), $s\in\mathbf{R}$ be a "generic" pair of a family of s-dependent ω-compatible almost complex structure J_s and Hamiltonian functions H_s such that $(J_s,H_s)\equiv(J_0,H_0)$ when $s<0$ and $(J_s,H_s)=(J_1,H_1)$ when $s\geq 1$. Such a family of (J_s,H_s) can be viewed as a deformation between (J_0,H_0) and (J_1,H_1). Let $\tilde{z}_0\in\tilde{\mathbf{P}}(H_0)$ and $\tilde{z}_1\in\tilde{\mathbf{P}}(H_1)$. We can define the moduli space $\overline{\mathcal{M}}(J_s,H_s;\tilde{z}_0,\tilde{z}_1)$ of stable (J_s,H_s)-maps connecting $\tilde{z}_0$ and $\tilde{z}_1$ in a similar way as before, but the equation for a stable (J_s,H_s)-map $u:\Sigma\to V$ with principal component $u_p:\Sigma_p\to V$, $p=1,\cdots,L_1$, need to be modified by requiring that there exists an index $p_0\in\{1,\cdots,L_1\}$ such that (i) u_p is a stable (J_0,H_0)-map if $p<p_0$ and is a stable (J_1,H_1)-map if $p>p_0$; (ii)

$$\frac{\partial u_{p_0}}{\partial s}(s,\theta)+J_s(u_{p_0}(s,\theta))\cdot\frac{\partial u_{p_0}}{\partial\theta}(s,\theta)+\nabla H_s(u_{p_0}(s,\theta),\theta)=0.$$

As we did for (J_1, H_1), we can also extend the previous theory to include this case. But there is a difference of this case with the case of (J_0, H_0) again.

Because of the s-dependence of the equation for stable (J_s, H_s)-maps, we do not have the s-invariance for the particular principal component u_{p_0}. We have to incorporate this into all the construction before. The output of such a modified theory for (J_s, H_s) is that the dimension of the compact moduli space $\mathcal{M}^{\nu_s}(J_s, H_s; \tilde{z}_0, \tilde{z}_1)$ and its associated relative cycle $C(\overline{\mathcal{M}}^{\nu_s, c}(J_s, H_s; \tilde{z}_0, \tilde{z}_1))$ is just $\mu(\tilde{z}_0, \tilde{z}_1)$ rather than $\mu(\tilde{z}_0, \tilde{z}_1) - 1$. In particular, if the relative index $\mu(\tilde{z}_0, \tilde{z}_1) = 0$, $\mathcal{M}^{\nu_s}(J_s, H_s; \tilde{z}_0, \tilde{z}_1)$ is just a finite set and the oriented number

$$n(\tilde{z}_0, \tilde{z}_1) = \# C(\overline{\mathcal{M}}^{\nu_s}(J_s, H_s; \tilde{z}_0, \tilde{z}_1))$$

is a well-defined rational number. We define

$$\phi_1^0(\tilde{z}_0) = \sum_{\mu(\tilde{z}_1)=k} n(\tilde{z}_0, \tilde{z}_1)\tilde{z}_1$$

for $\tilde{z}_0 \in C^k(H_0)$.

Lemma 5.2.

$$\phi_1^0 : (C^*(H_0), \delta_{J_0,H_0,\nu_0}) \to (C^*(H_1), \delta_{J_1,H_1,\nu_1})$$

is a chain homomorphism.

Proof. As in the "classical" Floer cohomology, this follows from the analogue statement of Corollary 4.2 for (J_s, H_s). q.e.d.

Similarly, we can also define an "inverse" chain map ϕ_0^1 of ϕ_1^0 from

$$(C^*(H_1), \delta_{J_1,H_1,\nu_1})$$

to

$$(C^*(H_0), \delta_{J_0,H_0,\nu_0}).$$

Now we can state our last theorem.

Theorem 5.1. *ϕ_1^0 induces an isomorphism $(\phi_1^0)^*$ of the Floer cohomology*

$$FH^*(V, \omega; J_0, H_0, \nu_0)$$

and

$$FH^*(V, \omega; J_1, H_1, \nu_1).$$

Proof. The proof is identical to Floer's original proof for the "classical" Floer cohomology after replacing all his constructions involving the "classical" moduli space of (J_s, H_s)-holomorphic maps by our relative virtual moduli cycles. We will only indicate main steps involved here and leave the details to the readers.

Let $\phi_0^0 = \phi_0^1 \circ \phi_1^0$ and $\phi_1^1 = \phi_1^0 \circ \phi_0^1$. We only need to prove that $(\phi_0^0)^*$ and $(\phi_1^1)^*$ are identity maps of $FH^*(V, \omega; J_0, H_0, \nu_0)$ and $FH^*(V, \omega; J_1, H_1, \nu_1)$ respectively. Since the proof for the two cases are the same, we only deal with $\phi_0^0 = \phi_0^1 \circ \phi_1^0$. It follows from the definition of ϕ_0^1 and ϕ_1^0 that for any $\tilde{x} \in C^k(H_0)$,

$$\phi_0^0(\tilde{x}) = \sum_{\mu(\tilde{z})=k} \sum_{\mu(\tilde{y})=k} n(\tilde{x}, \tilde{y}) \cdot n(\tilde{y}, \tilde{z}) \tilde{z} \in C^k(H_0),$$

where $\tilde{y} \in \tilde{\mathbf{P}}(H_1)$ and

$$n(\tilde{x}, \tilde{y}) = \#(C(\overline{\mathcal{M}}^{\nu}(J_s, H_s; \tilde{x}, \tilde{y}))).$$

Therefore $\sum_{\mu(\tilde{y})=k} n(\tilde{x}, \tilde{y}) \cdot n(\tilde{y}, \tilde{z})$ is just all possible pair (u_1, u_2) with $u_1 \in \mathcal{M}^{\nu_s}(J_s, H_s; \tilde{x}, \tilde{y})$, $u_2 \in \mathcal{M}^{\nu_s}(J_s, H_s; \tilde{y}, \tilde{z})$ counted with sign and the fractional multiplicity in the corresponding virtual cycle. Now we introduce a new parameter $\rho \in [0, +\infty)$ and a one parameter family of s-dependent family (J_s^ρ, H_s^ρ). However, unlike (J_s, H_s) where the variation of (J_s, H_s) along s-direction is concentrated in $\{s; 0 \leq s \leq 1\}$, (J_s^ρ, H_s^ρ) varies along $\{s; -\rho - 1 \leq s \leq \rho \quad or\ \rho \leq s \leq \rho + 1\}$. More precisely, we define

$(i)\ (J_s^\rho, H_s^\rho) = (J_{s-\rho}, H_{s-\rho}) \quad when\ s > \rho$
$(ii)\ (J_s^\rho, H_s^\rho) = (J_{-s-\rho}, H_{-s-\rho}) \quad when\ s < -\rho$
$(iii)\ (J_s^\rho, H_s^\rho) = (J_1, H_1) \quad when\ -\rho < s < \rho$

For such a two parameter (ρ, s)-family, we can also develop all previous theory in this case. In particular, for ρ large enough, there is a gluing map:

$$T_\rho : \mathcal{M}^{\nu_s}(J_s, H_s; \tilde{x}, \tilde{y}) \times \mathcal{M}^{\nu_s}(J_s, H_s; \tilde{y}, \tilde{z}) \to \mathcal{M}^{\nu_s^\rho}(J_s^\rho, H_s^\rho; \tilde{x}, \tilde{z}),$$

which is a orientation preserving bijection. It follows that when ρ is large enough, ϕ_0^0 is the same as $(\phi_\rho)_0^0$ defined by

$$(\phi_\rho)_0^0(\tilde{x}) = \sum_{\mu(\tilde{y})=k} n(\tilde{x}, \tilde{y}) \tilde{y}$$

for $\tilde{x} \in C^k(H_0)$, where

$$n(\tilde{x}, \tilde{y}) = \#(C(\overline{\mathcal{M}}^{\nu_s^\rho}(J_s^\rho, H_s^\rho; \tilde{x}, \tilde{y})).$$

Now let ρ vary to $\rho = 0$ first. We get (J_s^0, H_s^0) which is (J_s, H_s) when $s \geq 0$ and (J_{-s}, H_{-s}) when $s \leq 0$. Clearly (J_s^0, H_s^0) can be deformed further into (J_0, H_0) of the original s-independent pair. Let (J_s^ρ, H_x^ρ) be the latter deformation with $-1 \leq \rho \leq 0$ and $(J_s^{-1}, H_s^{-1}) = (J_0, H_0)$. It is easy to see that $(\phi_{-1})_0^0$ is just the identity map of $C^*(H_0)$. Therefore $(\phi_0^0)^*$ will be identity map of $FH^*(V, \omega; J_0, H_0, \nu_0)$ if we can prove that the chain maps $(\phi_{\rho_1})_0^0$ and $(\phi_{\rho_2})_0^0$ are homotopic to each other. This latter statement can be proved in the same way as the "classical" case with the same kind of modification we mentioned before. q.e.d.

Corollary 5.1 *(Arnold Conjecture). For a non-degenerate time-dependent Hamiltonian function H,*

$$\#\mathbf{P}(H) \geq \sum_i b_i(V).$$

This completes the proof of the Arnold conjecture.

References

[1] V.I. Arnold, *Mathematical methods of classical mechanics (Appendix 9)*, Nauka 1974, Engl. Transl. Berlin, Heidelberg, New York; Springer, 1978.

[2] Ya. Eliashberg, *A theorem on the structure of wave fronts*, Functional Anal. Appl. **21** (1981) 65-72.

[3] C. C. Conley & E. Zehnder, *The Birkhoff-Lewis fixed point theorem and a conjecture of V. I. Arnold*, Invent. Math. **73** (1983) 33-49.

[4] A. Floer, *The unregularized gradient flow of the symplectic action*, Comm. Pure Appl. Math. **41** (1988) 775-813.

[5] ______, *Symplectic fixed points and holomorphic spheres*, Comm. Math. Phys. **120** (1989) 575-611.

[6] ______, *Proof of the Arnold conjecture for surfaces and generalizations to certain Kähler manifolds*, Duke Math. J. **53** (1986) 1-32.

[7] A. Floer & H. Hofer, *Coherent orientations for periodic orbit problems in symplectic geometry*, Math. Z. **212** (1994) 13-38.

[8] K. Fukaya & K. Ono, *Arnold conjecture and Gromov-Witten invariants*, Preprint, 1996.

[9] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985) 307-347.

[10] H. Hofer & D. A. Salamon, *Floer homology and Novikov rings*, The Floer Memorial Vol., Progr. Math. Vol. 133, 483-524.

[11] G. Liu, *Associativity of Quantum multiplication*, Comm. Math. Phys. **191** (1998) 265-282.

[12] J. Li & G. Tian, *Virtual moduli cycles and GW-invariants of algebraic varieties*, J. of Amer. Math. Soc. **11** (1998) 119-174.

[13] ———, *Virtual moduli cycles and GW-invariants of general symplectic manifolds*, Proc. 1st IP conference at Univ. Calif., Irvine, Internat. Press, 1996.

[14] R. B. Lockhard & R. C. McOwen, *Elliptic operators on noncompact manifolds*, Ann. Scuola. Norm. Sup. Pisa Cl. Sci. (4) **12** (1985) 409-446.

[15] D. McDuff, *Examples of symplectic structures*, Invent. Math. **89** (1987) 13-36.

[16] D. McDuff & D.A. Salamon, *J-holomorphic curves and quantum cohomology*, University Lecture Series, Amer. Math Soc, Providence (1994).

[17] K. Ono, *On the Arnold conjecture for weakly monotone symplectic manifolds*, Invent. Math. **119** (1995) 519-537.

[18] T.H. Parker & J.G. Wolfson, *Pseudoholomorphic maps and bubble trees*, J. Geom. Anal. **3** (1993) 63-98.

[19] Y. Ruan & G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. **42** (1995) 259-368.

[20] J. C. Sikorav, *Points fixes d'un symplectiomorphisme homologue a l'identite*, J. Differential Geom. **22** (1982) 49-79.

[21] A. Weinstein, *On extending the Conley Zehnder fixed point theorem to other manifolds*, Proc. Sympos. Pure Math. Vol. 45, Providence, R.I., Amer. Math. Soc., 1986.

[22] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. **17** (1982) 661-692.

Massachusetts Institute of Technology