J. DIFFERENTIAL GEOMETRY
48 (1998) 557-585

BUSEMANN FUNCTIONS ON MANIFOLDS WITH LOWER BOUNDS ON RICCI CURVATURE AND MINIMAL VOLUME GROWTH

CHRISTINA SORMANI

Complete noncompact Riemannian manifolds with lower bounds on Ricci curvature have often been studied by means of an analysis of their Busemann functions. Such manifolds contain rays, and given a ray, γ, its associated Busemann function is defined

$$b_\gamma(x) = \lim_{R\to\infty} R - d(x, \gamma(R)).$$

Cheeger and Gromoll used these functions to prove that a manifold with nonnegative Ricci curvature that contains a line splits isometrically [7]. S.T. Yau used them to prove that complete noncompact manifolds with nonnegative Ricci curvature have at least linear volume growth [14]. Cheeger, Gromov and Taylor proved that manifolds with quadratically decaying lower bounds on Ricci curvature have a specific lower bound on volume growth [8] using the Relative Volume Comparison Theorem [2], [11].

The main results in this paper concern manifolds with

$$Ric_x \geq \frac{(n-1)(1/4 - v^2)}{b^2(x)} \tag{0.1}$$

outside a compact set, where $v \in [0, \frac{n+1}{2(n-1)})$. This includes manifolds with nonnegative Ricci curvaure.

Received October 31, 1996, and, in revised form, February 11, 1997 and November 11, 1997. This material is based upon work supported under a National Science Foundation Graduate Fellowship.

In the first section of this paper, we prove a useful volume comparison theorem for subsets, $\Omega(K)$, covered by the flows of the gradient field of a Busemann function. See Definition 3 and Theorem 5. In particular, we prove that in a manifold with nonnegative Ricci curvature outside a compact set, the (n-1)-Hausdorff volume of $\Omega(K) \cap b^{-1}(r)$ is a nondecreasing function of r. We then employ Theorem 5, to obtain a more precise lower bound for the volume growth theorem of Cheeger-Gromov-Taylor. That is, we show that if a manifold satisfies (0.1), then

$$\liminf_{R\to\infty} \frac{Vol(B_p(R))}{R^{(1/2-v)(n-1)+1}} = C > 0 \tag{0.2}$$

for a precise value of C. See Theorem 18.

In the second section, we apply Theorem 5 to examine manifolds with *minimal volume growth*, manifolds satisfying (0.1) such that

$$\limsup_{R\to\infty} \frac{Vol(B_p(R))}{R^{(1/2-v)(n-1)+1}} = V_0 < \infty. \tag{0.3}$$

In particular, we examine manifolds with nonnegative Ricci curvature and linear volume growth. We prove that manifolds with globally nonnegative Ricci curvature that satisfy (0.1) for $v \in (0, 1/2]$ outside a compact set and have minimal volume growth as defined in (0.3), have proper Busemann functions. Furthermore, the level sets of any given Busemann function on these manifolds have at most linear diameter growth. See Theorem 19.

In [13], we will employ this theorem to prove an almost rigidity theorem about such manifolds, namely that these manifolds are asymptotically close to certain warped product manifolds in the Gromov-Hausdorff topology. In that paper we will also improve the statement of Theorem 19, proving that the diameter of the level sets grows at most sublinearly.

In the last section of this paper we provide nontrivial examples of manifolds with nonnegative Ricci curvature and linear volume growth. Example 26 demonstrates that as r approaches infinity, $diam(b^{-1}(r))$ may approach infinity. Example 27 demonstrates that $b^{-1}(r)$ need not approach a unique Gromov-Hausdorff limit as r approaches infinity even if the diameter is uniformly bounded.

The author would like to express her thanks to Jeff Cheeger for his advise and the Courant Institute of Mathematical Sciences for its support during her years as a graduate student. She would also like to thank Shing-Tung Yau for suggesting further applications of Theorem 19 and William Minicozzi for his assistance during the revision process.

1. Busemann functions and volume

All manifolds in this section are complete noncompact manifolds, M^n, with a fixed ray, γ, and its associated Busemann function, $b = b_\gamma$. All geodesics are parametrized by arclength. Note that the Busemann function is a Lipschitz function and $|\bigtriangledown b_\gamma| = 1$ almost everywhere. For a thorough description of Busemann functions and rays, see [5]. We begin with some definitions.

Definition 1. A ray, γ_x, emanating from x is called a *Busemann ray associated with* γ if it is the limit of a sequence of minimal geodesics, σ_i, from x to $\gamma(R_i)$ in the following sense,

$$\gamma_x'(0) = \lim_{R_i \to \infty} \sigma_i'(0). \tag{0.4}$$

We parametrize γ_x by arclength such that $\gamma_x(b(x)) = x$.

It is easy to verify that at least one Busemann ray exists at each point. However, a Busemann ray, γ_x, is unique if and only if $\bigtriangledown b$ exists at the point, x, in which case $\bigtriangledown b = \gamma_x'$ at x. So the Busemann rays are the integral curves of $\bigtriangledown b$. Note that if a point y is on a Busemann ray, γ_x, then the Busemann ray at y, γ_y is unique. In fact γ_y is the segment of γ_x emanating from y, [5].

Lemma 2. *Suppose $\bigtriangledown b$ exists at p_i which converge to p. If $\bigtriangledown b$ exists at p, then*

$$\lim_{i \to \infty} \bigtriangledown b_{p_i} = \bigtriangledown b_p.$$

Note that this lemma does not hold for arbitrary Lipschitz functions.

Proof. Let $p_{i'}$ be any subsequence of p_i such that $\bigtriangledown b_{p_{i'}}$ converges. Then the Busemann rays $\gamma_{p_{i'}}$ converge to a Busemann ray emanating from p. Now, $\bigtriangledown b_p$ exists, so there is only one such Busemann ray. Thus, $\bigtriangledown b_{p_{i'}} \to \bigtriangledown b_p$. q.e.d.

Definition 3. Let K be a compact set contained in M^n. Then

$$\Omega(K) \;=\; \{x : \exists z \in K \; \exists t \geq b(z) \; \exists \gamma_z \textit{ such that } x = \gamma_z(t)\}. \tag{0.5}$$

Furthermore, let $\Omega_R(K) = \Omega(K) \cap b^{-1}(R)$.

Lemma 4. *The set, $\Omega(K)$, is closed and $\Omega_R(K)$ is compact.*

Proof. Suppose $y_i \in \Omega(K)$ and $y_i \to y$. Then there exist $\bar{y}_i \in K$ and Busemann rays from each $\bar{y}_i$ to the y_i. By the compactness of K a subsequence of these rays must converge to a ray also emanating from K. This limit ray must be a Busemann ray which passes through y. Thus $y \in \Omega(K)$. q.e.d.

Note in particular that $\Omega(K)$ is a Borel set. Thus it is both Lebesgue measurable and Hausdorff measurable. Furthermore its n-dimensional Hausdorff and Lebesgue measures are equal [10, 2.10.35]. We denote the Lebesgue measure of a set Y by $Vol(Y)$. We can now state our volume comparison theorem.

Theorem 5. *Let M^n be a Riemannian manifold with a given ray γ and its associated Busemann function, $b = b_\gamma$. Fix*

$$r_0 \le r_1 < r_2 \le r_3 < r_4.$$

Let $Ric_x \ge \frac{(n-1)(\frac{1}{4}-v^2)}{b(x)^2}$ for $x \in b^{-1}([r_0,\infty))$, where $v \in (0, \frac{n+1}{2(n-1)}]$. Let $p = (\frac{1}{2} - v)(n-1)+1$.

If $K \subset b^{-1}((-\infty, r_1])$ is a compact set, then

$$(0.6)\quad Vol(\Omega(K)\cap b^{-1}([r_3,r_4])) \ \ge\ \frac{r_4^p - r_3^p}{r_2^p - r_1^p} Vol(\Omega(K)\cap b^{-1}([r_1,r_2])).$$

If $p = 0$, we replace r^p by $Ln(r)$.

Before proving this theorem we will prove a series of lemmas about a complete noncompact manifold, M^n, with no assumptions on its Ricci curvature. We begin by relating the Busemann function to a distance function.

Lemma 6. *Fix $r_1 < R$ and $K \subset b^{-1}((-\infty, r_1))$. Then*

$$(0.7)\qquad d(x, b^{-1}(R)) = R - b(x) \qquad \forall x \in b^{-1}((-\infty, R]).$$

Furthermore, if $z \in \Omega(K) \cap b^{-1}([r_1, R])$, then there is only one point, $y \in b^{-1}(R)$, such that $d(z,y) = d(z, b^{-1}(R))$. Thus $y = \gamma_x(R)$ and

$$(0.8)\qquad d(z, \Omega_R(K)) = R - b(z) \qquad \forall z \in \Omega(K)\cap b^{-1}([r_1, R]).$$

Proof. Given a point $x \in b^{-1}((-\infty, R])$, we have

$$d(x, b^{-1}(R)) \le d(x, \gamma_x(R)) \le R - b(x).$$

On the other hand, the triangle inequality implies that

$$\begin{aligned} R - b(x) &= \lim_{s\to\infty} R - s + d(x, \gamma(s)) \\ &= \lim_{s\to\infty} d(\gamma(s), \gamma(R)) + d(x, \gamma(s)) \geq d(x, \gamma(R)). \end{aligned} \tag{0.9}$$

Thus (0.7) holds.

Let $z \in \Omega(K) \cap b^{-1}([r_1, R])$ and $y \in b^{-1}(R)$ such that $d(z, y) = d(z, b^{-1}(R))$. Let σ be a minimal geodesic from z to y. Note that $f(t) = b(\sigma(t))$ is a Lipschitz function of t with Lipschitz constant less than or equal to 1. So $f'(t) \leq 1$ almost everywhere, and since

$$d(z, y) = b(y) - b(z) = \int_0^{d(z,y)} f'(t)\, dt \leq d(z, y),$$

we know $f'(t) = 1$ almost everywhere on $[0, d(y, z)]$. Thus, integrating again, we have $b(\sigma(t)) = b(z) + t$.

Since $z \in \Omega(K) \cap b^{-1}([r_1, R])$ and $K \subset b^{-1}((-\infty, r_1))$, $\bigtriangledown b_z$ exists and the Busemann ray, $\gamma_z(t)$, is unique. By the definition of the gradient,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |b(\sigma(t)) - b(z) - g(\bigtriangledown b_z, \sigma'(0))t| < \varepsilon |t|, \ \ \forall t > \delta.$$

Substituting $b(\sigma(t)) = b(z) + t$ and dividing by $|t|$, we get,

$$\forall \varepsilon > 0, \qquad |1 - g(\bigtriangledown b_z, \sigma'(0))| < \varepsilon.$$

Thus we see that $g(\bigtriangledown b_z, \sigma'(0)) = 1$. So $\sigma(t) = \gamma_z(t - b(z))$ and $y = \gamma_z(R)$. q.e.d.

In order to prove Theorem 5, we define some star-shaped sets and finite unions of such sets, which can be studied by means of the Relative Volume Comparison Theorem [2], [11], as discussed in [6]. Since $\Omega(K)$ does not naturally contain any such sets, we define the following sets, S_{δ, r_1, r_2}, which contain star-shaped sets about points in $b^{-1}(R) \cap \Omega(K)$.

Definition 7. Given any $\delta > 0$ and any $r_1 < r_2 < R$, let

$$S_{\delta, r_1, r_2} = S_{\delta, r_1, r_2}(\Omega_R(K))$$

be the set of points x with $d(x, \Omega_R(K)) \in [R - r_2, R - r_1]$ such that there exists a minimal geodesic σ from $\Omega_R(K)$ to x with

$$L(\sigma) = d(x, \Omega_R(K)) \ \textit{and} \ g(\sigma'(0), -\bigtriangledown b) \geq 1 - \delta.$$

It is easy to verify that S_{δ,r_1,r_2} is a compact set by applying Lemma 4.

Lemma 8. *Fix $R > r_2 > r_1$, $\delta > 0$ and K a compact subset of $b^{-1}((-\infty, r_1))$. Then*

$$\bigcap_{\delta>0} S_{\delta,r_1,r_2}(\Omega_R(K)) = \Omega(K) \cap b^{-1}([r_1, r_2]).$$

Proof. Given any $x \in \Omega(K) \cap b^{-1}([r_1, r_2])$, a Busemann ray, $\gamma_x([b(x), R])$ is a minimal geodesic from x to $\Omega_R(K)$ such that $g(-\gamma_x'(R), -\nabla b) = 1$. By Lemma 6, we have

$$L(\gamma_x([b(x), R])) = R - b(x) = d(x, \Omega_R(K)).$$

Thus $x \in S_{\delta,r_1,r_2}$.

On the other hand, let $x \in S_{\delta,r_1,r_2}$ for all $\delta > 0$. We know that $\forall\delta > 0$, $\exists\sigma_\delta$, a minimal geodesic from $\sigma_\delta(0) \in \Omega_R(K)$ to x, such that

$$L(\sigma_\delta) = d(x, \Omega_R(K)) \in [R - r_2, R - r_1] \text{ and } g(\sigma_\delta'(0), -\nabla b) \geq 1 - \delta.$$

By the compactness of $\Omega_R(K)$, as δ approaches 0, a subsequence, σ_δ, must converge to a minimal geodesic, σ_0, from $\Omega_R(K)$ to x where

$$L = L(\sigma_0) = d(x, \Omega_R(K)) \in [R - r_2, R - r_1]. \tag{0.10}$$

Since $\sigma_0(0) \in \Omega_R(K)$, ∇b exists there, and by Lemma 2,

$$g(\sigma_0'(0), -\nabla b) = 1. \tag{0.11}$$

Now $\sigma_0(0)$ must be on a Busemann ray, γ_z, where $z \in K \subset b^{-1}((-\infty, r_1))$. By (0.11), $\sigma_0'(0) = -\gamma_z'(R)$, and, by (0.10), $\sigma(t) = \gamma_z(R - t)$ for $t \in [0, R - r_1]$. Thus $x = \gamma_z(R - L) \in\subset \Omega(K) \cap b^{-1}([r_1, r_2])$. q.e.d.

The following corollary of Lemma 8 is an immediate consequence of the Monotone Convergence Theorem for $\delta \to 0$ combined with the facts that S_{δ,r_1,r_2} is compact and for all $\delta < \delta'$,

$$S_{\delta,r_1,r_2}(\Omega_R(K)) \subset S_{\delta',r_1,r_2}(\Omega_R(K)). \tag{0.12}$$

Corollary 9. *Fix $R > r_2 > r_1$. If $K \subset b^{-1}((-\infty, r_1))$ is compact, then*

$$\lim_{\delta\to 0} Vol(S_{\delta,r_1,r_2}(\Omega_R(K))) = Vol(\Omega(K) \cap b^{-1}([r_1, r_2])).$$

Recall that in Definition 7, we say that $x \in S_{\delta,r_1,r_2}$ if there exists a minimal geodesic σ_0, from $\Omega_R(K))$ to x such that $\sigma_0'(0)$ is close to $-\bigtriangledown b$. However, it is possible that there are other minimal geodesics, σ, of the same length such that $\sigma_0'(0)$ is not close to $\bigtriangledown b$. This would cause us trouble when we try to approximate S_{δ,r_1,r_2} by unions of starshaped sets if not for the following lemma.

Lemma 10. *Let $K \subset b^{-1}((-\infty, r_1])$ and $R > r_3 > r_2 > r_1$. Given any $\delta > 0$, there exists*

$$h_1(\delta) = h_1(\delta, r_1, r_2, r_3, R, K, M^n) < \delta$$

such that for any $x \in S_{h_1(\delta),r_2,r_3}(\Omega_R(K))$, every minimal geodesic, σ, from $\sigma(0) \in \Omega_R(K))$ to x such that $d(x, \sigma(0)) = d(x, \Omega_R(K))$, satisfies

$$g(\sigma'(0), -\bigtriangledown b) \geq 1 - \delta.$$

Proof. Suppose there exists δ such that no such $h_1(\delta)$ exists. Thus there exist $\delta_i \to 0$, $x_i \in S_{\delta_i,r_2,r_3} \in S_{\pi,r_2,r_3}$, and σ_i minimal from $\sigma_i(0) \in \Omega_R(K)$ to x_i such that

$$L_i = d(x_i, \sigma_i(0)) = d(x_i, \Omega_R(K)), \tag{0.13}$$

but

$$g(\sigma_i'(0), -\bigtriangledown b) < 1 - \delta. \tag{0.14}$$

Since $x_i \in S_{\delta_i,r_2,r_3}$, there exist γ_i minimal from $\gamma_i(0) \in \Omega_R(K)$ to x_i such that

$$d(x_i, \gamma_i(0)) = d(x_i, \Omega_R(K)) = L_i \in [R - r_3, R - r_2],$$

and

$$g(\gamma_i'(0), -\bigtriangledown b) \geq 1 - \delta_i. \tag{0.15}$$

By the compactness of S_{π,r_2,r_3}, there exists a subsequence such that all the points and geodesics converge, $x_i \to x_\infty$, γ_i to γ_∞ and σ_i to σ_∞. Note that

$$L_i \to L = d(x_\infty, \gamma_\infty(0)) = d(x_\infty, \Omega_R(K)) \in [R - r_3, R - r_2]$$

and that $\gamma_\infty(0) \subset \Omega_R(K)$. Thus there exists $z \in K \subset b^{-1}((-\infty, r_1])$ such that $\gamma_\infty(0) = \gamma_z(R)$, and $\gamma_z(R - t)$ is minimal from $t = R$ to $t = R - r_1$.

Furthermore, using the facts that $L > 0$, $\gamma_\infty(0) \in \Omega_R(K)$, and (0.15), and applying Lemma 2 we have

$$g(\gamma'_\infty(0), -\nabla b) \geq 1.$$

Thus $\gamma_\infty(t) = \gamma_z(R-t)$ and $x_\infty = \gamma_\infty(L) = \gamma_z(R-L)$. Since $\gamma_z(R-t)$ is minimal from $t = 0$ to $t = R - r_1$ and $L \leq R - r_2 < R - r_1$, we know that x_∞ cannot be a cut point of $\gamma_\infty(0)$. Thus not only does x_∞ have a unique closest point in $\Omega_R(K)$ by Lemma 6 and not only must that point be $\gamma_\infty(0)$, there must also be only one geodesic joining x_∞ to $\gamma_\infty(0)$.

On the other hand, by the limit of (0.13),

$$d(x_\infty, \sigma_\infty(0)) = d(x_\infty, \Omega_R(K)),$$

so σ_∞ must be γ_∞. However, the limit of (0.14) and Lemma 2 imply that

$$g(\sigma'_\infty(0), -\nabla b) \leq 1 - \delta,$$

which is a contradiction. q.e.d.

We now define some star-shaped sets about points in $\Omega_R(K)$ upon which we can employ the volume comparison theorem. The following construction is similar to that contained in [6, Appendix on Volume Growth]. We begin by employing the fact that $\Omega_R(K)$ is compact.

Definition 11. Given any $\varepsilon > 0$, there exists a set

$$X_\varepsilon = \{p_1, ...p_{N_\varepsilon}\} \subset \Omega_R(K)$$

such that the tubular neighborhood, $T_\varepsilon(X_\varepsilon)$, contains $\Omega_R(K)$.

Definition 12. Given X_ε and $p \in X_\varepsilon$. We define the *star-shaped set*:

$$V_{p,\varepsilon} = \{x : d(x,p) < d(x,q) \ \ \forall q \in X_\varepsilon \textit{ such that } q \neq p\}.$$

Now we need to focus on points near $\Omega(K)$.

Definition 13. Given X_ε and $p \in X_\varepsilon$. We define the *star-shaped wedge set*:

$$U_{p,\delta} = \{x : \exists \textit{ a min geod, } \sigma, \textit{ from } p \textit{ to } x \textit{ s.t. } g(\sigma'(0), -\nabla(b)) \geq 1 - \delta\}.$$

Furthermore, let

$$U_{\varepsilon,\delta} = \bigcup_{i=1}^{N_\varepsilon} (U_{p_i,\delta} \cap V_{p_i,\varepsilon}).$$

We will use the following notation for closed tubular neighborhoods of a set, X:

$$T_{a,b}(X) = \{y : a \le d(y, X) \le b\}. \tag{0.16}$$

In the proof of Theorem 5, we will be able to estimate the following ratio using the Relative Volume Comparison Theorem and the starlike qualities of the wedge sets $U_{p_i,\delta}$. That is, for any $s_1 < s_2 \le s_3 < s_4 \le R$ we will bound

$$\frac{Vol(T_{R-s_2,R-s_1}(X_{\varepsilon_i}) \cap U_{\varepsilon,\delta})}{Vol(T_{R-s_4,R-s_3}(X_{\varepsilon_i}) \cap U_{\varepsilon,\delta})} \tag{0.17}$$

from below. To do so we need to relate the distance of points in $U_{p,\delta}$ from p to the lower bound on Ricci curvature, which depends on the Busemann function.

Lemma 14. *Fix $r_1 < R$ and the compact set, $K \subset b^{-1}((-\infty, r_1])$.*

For all $\varepsilon > 0$, there exists $h_2(\varepsilon, r_1, R, K, M) > 0$ such that, for all $p \in \Omega_R(K)$ and all $\delta < h_2$ we have

$$d(\gamma_x(R), p) < \varepsilon \qquad \forall x \in U_{p,\delta} \cap B_p(R - r_1 - \varepsilon). \tag{0.18}$$

Thus

$$\begin{aligned} b(x) \in [R - d(x,p), R - d(x,p) + \varepsilon] \\ \forall x \in U_{p,\delta} \cap B_p(R - r_1 - \varepsilon). \end{aligned} \tag{0.19}$$

Proof. Given any $x, p \in M^n$, we have

$$R - d(x, \gamma(R)) \ge R - d(p, \gamma(R)) - d(x,p),$$

so taking $R \to \infty$ yields

$$b(x) = b(p) - d(x,p). \tag{0.20}$$

Since $b(x) = R - d(x, \gamma_x(R))$, we need only prove (0.18) to obtain (0.19).

Assume that (0.18) is not true. Then there exist $\varepsilon > 0$ and $\delta_i \to 0$, such that $x_i \in U_{p,\delta_i} \cap B_p(R - r_1 - \varepsilon)$, but

$$d(\gamma_{x_i}(R), p) \ge \varepsilon. \tag{0.21}$$

Thus, by the definition of U_{p,δ_i}, there exist σ_i from p to x_i, such that

$$g(\sigma_i'(0), -\bigtriangledown b) \ge 1 - \delta_i \text{ and } L_i = L(\sigma_i) \le R - r_1 - \varepsilon.$$

Note that by (0.20),

$$b(\sigma_i(t)) \geq R - t \geq r_1 + \varepsilon \qquad \forall t \in [0, L_i]. \tag{0.22}$$

Since $\bar{B}_p(R - r_1 - \varepsilon)$ is compact, a subsequence x_i converge to x_∞ and σ_i to σ_∞. If $L_i \to 0$, then $x = p$, $\gamma_{x_\infty}(R) = x = p$, and we have contradicted (0.21) for i sufficiently large. Thus $L_i \to L > 0$ and $\sigma_i'(0) \to \sigma'(0)$. Since $p \in \Omega_R(K)$, we can apply Lemma 2 to get

$$g(\sigma'(0), -\bigtriangledown b) = \lim_{i\to\infty}(\sigma_i'(0), -\bigtriangledown b) \geq \lim_{i\to\infty}(1 - \delta_i) = 1.$$

Hence $\sigma'(0) = -\bigtriangledown b$.

Since $p \in \Omega_R(K)$ there exists $z \in K \subset b^{-1}((-\infty, r_1])$ such that $p = \gamma_z(R)$. Thus $\sigma(t) = \gamma_z(R - t)$ for all $t \in [0, \min\{R - r_1, L\}]$. Thus $x = \gamma_z(R - L)$ and, by (0.22), $b(x) \geq r_1 + \varepsilon$ while $b(z) \leq r_1$. Thus γ_z is the unique Busemann ray through x; so it agrees with γ_x and $\gamma_x(R) = p$. This contradicts (0.21). q.e.d.

In Lemmas 15 and 16 we will relate the starshaped sets to S_{δ,r_1,r_2} and $\Omega(K) \cap b^{-1}([r_1, r_2])$. To do so, we must deal the fact that the wedge sets, $U_{\varepsilon,\delta}$, do not contain $\Omega(K) \cap b^{-1}[r_1, R]$ due to the gaps between the points in X_ε and the restrictions caused by δ. We must choose ε small relative to δ and r_2 to avoid this problem.

Lemma 15. *Fix $r_1 < r_2 < R$ and $\delta > 0$. Let $\varepsilon_0 > 0$. $K \subset b^{-1}(-\infty, r_1))$. There exists $h_3(M, \delta, r_1, r_2, R\} > 0$ sufficiently small that for all $\varepsilon < min\{\varepsilon_0, h_3\}$,*

$$\Omega(K) \cap b^{-1}([r_1, r_2]) \subset T_{R-r_2-\varepsilon_0, R-r_1+\varepsilon_0}(X_\varepsilon) \cap \bar{U}_{\varepsilon,\delta}.$$

where $\bar{U}$ is the closure of U.

So $Vol(\Omega(K) \cap b^{-1}([r_1, r_2])) \leq Vol(T_{a,b}(X_\varepsilon) \cap U_{\varepsilon,\delta})$.

Proof. First of all it follows from Lemma 6 that for any $\varepsilon < \varepsilon_0$, we have

$$\Omega(K) \cap b^{-1}([r_1, r_2]) \subset T_{R-r_2-\varepsilon_0, R-r_1+\varepsilon_0}(X_\varepsilon).$$

Thus if no such $h_3 > 0$ exists, there exist $\varepsilon_i \to 0$ and x_i in $\Omega(K) \cap b^{-1}([r_1, r_2])$ but not in $\bar{U}_{\varepsilon_i,\delta}$. Let q_i be any point in X_{ε_i} such that $d(x_i, q_i) = d(x_i, X_{\varepsilon_i})$. Then $x_i \in \bar{V}_{q_i,\varepsilon_i}$, so $x_i \notin \bar{U}_{q_i,\delta} = U_{q_i,\delta}$. So any minimal geodesic, σ_i, from q_i to x_i, satisfies

$$g(-\bigtriangledown b, \sigma_i'(0)) < 1 - \delta. \tag{0.23}$$

Since X_{ε_i} is a subset of the compact set, $\Omega_R(K)$, there exists a subsequence such that $q_i \to q_\infty \in \Omega_R(K)$, $\sigma_i \to \sigma_\infty$,

$$L(\sigma_i) \to L(\sigma_\infty) \geq R - r_2 > 0$$

and

$$x_i \to x_\infty \in \Omega(K) \cap b^{-1}([r_1, r_2]).$$

Since $\bigtriangledown b$ is defined at q_∞, we can take the limit of (0.23) using Lemma 2, to obtain

$$g(-\bigtriangledown b, \sigma'_\infty(0)) \leq 1 - \delta. \tag{0.24}$$

Furthermore, $d(x_\infty, q_\infty) = \lim_{i\to\infty} d(x_i, X_{\varepsilon_i}) = d(x_\infty, \Omega_R(K))$. By Lemma 6, $q_\infty = \gamma_\infty(R)$, and since $K \subset b^{-1}((-\infty, r_1))$, there exists a unique minimal geodesic γ_{x_∞} from q_∞ to x_∞. Thus $\sigma_\infty(t) = \gamma_{x_\infty}(R-t)$ for $t \leq R - r_2$, which contradicts (0.24).

Note that the volume inequality follows because

$$Vol(T_{a,b}(X_\varepsilon) \cap \bar{U}_{\varepsilon,\delta}) = Vol(T_{a,b}(X_\varepsilon) \cap U_{\varepsilon,\delta}).$$

q.e.d.

Recall Definition 7 and Lemma 10 regarding $S_{\delta,r_2,r_3}(\Omega_R(K))$.

Lemma 16. *Fix $r_1 < r_2 < r_3 < R$ and $K \subset b^{-1}((-\infty, r_1])$. Given any $\delta > 0$, let $h_1(\delta) < \delta$ be the real number defined in Lemma 10.*

Then there exists $h_4 = h_4(M^n, h_1(\delta), R, r_1, r_2, r_3) > 0$ such that given any $\varepsilon_0 > 0$, for all $\varepsilon < min\{h_4, \varepsilon_0\}$, we have

$$T_{R-r_3+\varepsilon_0, R-r_2-\varepsilon_0}(X_\varepsilon) \cap U_{\varepsilon,h_1(\delta)} \subset S_{2\delta,r_2,r_3}(\Omega_R(K)).$$

Proof. First of all it follows from Definition 11, that for any $\varepsilon < \varepsilon_0$, we have

$$T_{R-r_3+\varepsilon_0, R-r_2-\varepsilon_0}(X_\varepsilon) \subset T_{R-r_3, R-r_2}(\Omega_R(K)).$$

Thus if no such $h_4 > 0$ exists, then there is a sequence ε_i approaching 0 and x_i in $T_{R-r_3+\varepsilon_0, R-r_2-\varepsilon_0}(X_{\varepsilon_i})$ such that there exist minimal geodesics, γ_i, from X_{ε_i} to x_i satisfying

$$L(\gamma_i) = d(x_i, X_{\varepsilon_i}) \text{ and } g(-\bigtriangledown b, \gamma'_i(0)) \geq 1 - h_1(\delta). \tag{0.25}$$

However, there also exist minimal geodesics, σ_i, from $\Omega_R(K)$ to x_i satisfying

$$L(\sigma_i) = d(x_i, \Omega_R(K)) \text{ and } g(-\bigtriangledown b, \sigma'_i(0)) < 1 - 2\delta. \tag{0.26}$$

Since $\Omega_R(K)$ and $T_{R-r_3+1,R-r_2-1}(\Omega_R(K))$ are compact we can take a subsequence of the i such that, $\varepsilon_i \to 0$, $x_i \to x \in T_{R-r_3,R-r_2}(\Omega_R(K))$, and the geodesics σ_i and γ_i converge to minimal geodesics σ and γ respectively. Note that $\sigma(0) \in \Omega_R(K)$ with $d(x, \sigma(0)) = d(x, \Omega_R(K))$, and the same is true for $\gamma(0)$. Thus both geodesics have positive length and, applying Lemma 2 to (0.25) and (0.26), we have

$$g(-\bigtriangledown b, \gamma'(0)) \geq 1 - h_1(\delta) \tag{0.27}$$

and

$$g(-\bigtriangledown b, \sigma'(0)) \leq 1 - 2\delta. \tag{0.28}$$

Note that (0.27) implies that $x \in S_{h_1(\delta),r_1,r_2,R,K}$.

Applying Lemma 10, this implies that for any minimal geodesic $\bar{\sigma}$ from $\Omega_R(K)$ to x such that $d(x, \bar{\sigma}(0)) = d(x, \Omega_R(K))$, we have

$$g(-\bigtriangledown b, \bar{\sigma}'(0)) \geq 1 - \delta, \tag{0.29}$$

which contradicts (0.28). q.e.d.

We now prove Theorem 5 using the Relative Volume Comparison Theorem [3], [11], with comparison manifolds from [8], on the star shaped sets $U_{p,\delta}$ by employing Lemma 14 for arbitrary, $\varepsilon_0 > 0$ and $R > 0$. Imitating [6], we thus obtain a lower bound on the ratio, (0.17). To apply Lemmas 15 and 16 we need to take ε of X_ε small relative to δ, and δ small relative to R and ε_0. Once the dependence on ε is eliminated, we can take $\delta \to 0$ and apply Corollary 9. In the last step we take $R \to \infty$, and finally $\varepsilon_0 \to 0$ to obtain the theorem.

Proof of Theorem 5. As in the hypothesis, fix $r_0 \leq r_1 < r_2 \leq r_3 < r_4$ and $K \subset b^{-1}((-\infty, r_1])$. Recall that, for all $y \in b^{-1}([r_0, \infty))$ we have

$$Ric(y) \geq \frac{(n-1)(\frac{1}{4} - v^2)}{b(y)^2}. \tag{0.30}$$

Fix $R > r_4$ and $\varepsilon_0 < \frac{1}{10}\min\{r_2 - r_1, r_4 - r_3, R - r_4\}$. By Lemma 14, there exists

$$h_2 = h_2(\varepsilon_0, r_1, R, K, M), \tag{0.31}$$

such that if $p \in \Omega_R(K)$ and $\delta < h_2$ then, for all $h < \delta$,

$$\begin{aligned} b(y) \in& [R - d(y,p), R - d(y,p) + \varepsilon_0], \\ \forall y \in& U_{p,h} \cap B_p(R - r_1 - \varepsilon_0). \end{aligned} \tag{0.32}$$

Recall the definition in Lemma 10 of

$$h_1 = h_1(\delta, r_1, r_3 - \varepsilon_0, r_4 + \varepsilon_0, R, K, M^n) < \delta. \tag{0.33}$$

Since $h_1 < \delta$, (0.32) holds for all $y \in U_{p,h_1} \cap B_p(R - r_1 - \varepsilon_0)$.

Let $\varepsilon_v = 0$ if $v \geq 1/2$, and let $\varepsilon_v = \varepsilon_0$ if $v > 1/2$. Then,

$$Ric(x) \geq \frac{(n-1)(\frac{1}{4} - v^2)}{(R - d(y, \gamma(R)) + \varepsilon_v)^2} \tag{0.34}$$

for all $x \in U_{p,h_1} \cap B_p(R - r_1 - \varepsilon_0)$.

This is the curvature bound used in [8], so we can apply the volume comparison theorem on any starlike set, $V_p \subset U_{p,h_1} \cap B_p(R - r_1 - \varepsilon_0)$. We compare V_p to the corresponding region about the origin in the warped product manifold of [8] with the metric $dt^2 + J_{R,\varepsilon}(t)^2\,\omega$ where ω is the standard metric on S^{n-1} and

$$J_{R,\varepsilon}(t) = \frac{R + \varepsilon_v}{2v}\left(-\left(\frac{R - t + \varepsilon_v}{R + \varepsilon_v}\right)^{\frac{1}{2}+v} + \left(\frac{R - t + \varepsilon_v}{R + \varepsilon_v}\right)^{\frac{1}{2}-v}\right). \tag{0.35}$$

Thus, imitating [6] to apply the volume coparison theorem to a starlike set rather than a ball, we obtain

$$\frac{Vol(Ann_p(R - r_4, R - r_3) \cap V_p)}{Vol(Ann_p(R - r_2, R - r_1 - \varepsilon_0) \cap V_p)} \geq \frac{V(r_3, r_4, R, \varepsilon_v)}{V(r_1 + \varepsilon_0, r_2, R, \varepsilon_v)}, \tag{0.36}$$

where

$$\begin{aligned} &V(s_1, s_2, R, \varepsilon_v) \\ &\quad = \int_{R-s_2}^{R-s_1} J_{R,\varepsilon_v}(t)^{n-1}\, dt \\ &\quad = \sum_{j=0}^{n-1} \frac{{}_{n-1}C_j\,(-1)^j}{p + 2vj}\left(\left(\frac{s_2 + \varepsilon_v}{R + \varepsilon_v}\right)^{p+2vj} - \left(\frac{s_1 + \varepsilon_v}{R + \varepsilon_v}\right)^{p+2vj}\right), \end{aligned} \tag{0.37}$$

and $p = (1/2 - v)(n - 1) + 1$.

Set the constants h_3 and h_4 equal to $h_3(M, h_1, r_1 + 2\varepsilon_0, r_2 - \varepsilon_0, R)$ of Lemma 15 and $h_4(M, h_1, R, r_1, r_3 - \varepsilon_0, r_4 + \varepsilon_0)$ of Lemma 16 respectively. Let

$$\varepsilon < \min\{h_3, h_4, \varepsilon_0\}. \tag{0.38}$$

Recall X_ε of Definition 11. Given $p \in X_\varepsilon$ let

$$V_p = V_{p,\varepsilon} \cap U_{p,h_1} \cap B_p(R - r_1 - \varepsilon_0).$$

Then, by the definition of $U_{\varepsilon,\delta}$, we have the following disjoint union,

$$\bigcup_{p\in X_\varepsilon} V_p = T_{R-r_1-\varepsilon_0}(X_\varepsilon) \cap U_{\varepsilon,h_1}.$$

Thus, imitating [6], we obtain

$$\begin{aligned}
&\frac{Vol(T_{R-r_4,R-r_3}(X_\varepsilon)\cap U_{\varepsilon,h_1})}{Vol(T_{R-r_2,R-r_1-\varepsilon_0}(X_\varepsilon)\cap U_{\varepsilon,h_1})}\\
&= \frac{\sum_{p\in X_\varepsilon} Vol(Ann_p(R-r_4,R-r_3)\cap V_p)}{\sum_{p\in X_\varepsilon} Vol(Ann_p(R-r_2,R-r_1-\varepsilon_0)\cap V_p)}\\
&\geq \frac{\sum_{p\in X_\varepsilon}\left(Vol(Ann_p(R-r_2,R-r_1-\varepsilon_0)\cap V_p)\frac{V(r_3,r_4,R,\varepsilon_v)}{V(r_1+\varepsilon_0,r_2,R,\varepsilon_v)}\right)}{\sum_{p\in X_\varepsilon} Vol(Ann_p(R-r_2,R-r_1-\varepsilon_0)\cap V_p)}\\
&= \frac{V(r_3,r_4,R,\varepsilon_v)}{V(r_1+\varepsilon_0,r_2,R,\varepsilon_v)}.
\end{aligned} \tag{0.39}$$

We can now apply Lemmas 15 and 16 and our choice of the constants h_1 in (0.33) and ε in (0.38) to obtain

$$\frac{Vol(S_{2\delta,r_3-\varepsilon_0,r_4+\varepsilon_0}(\Omega_R(K)))}{Vol(\Omega_R(K)\cap b^{-1}([r_1+2\varepsilon_0,r_2-\varepsilon_0]))} \geq \frac{Vol(T_{R-r_4,R-r_3}(X_\varepsilon)\cap U_{\varepsilon,h_1})}{Vol(T_{R-r_2,R-r_1-\varepsilon_0}(X_\varepsilon)\cap U_{\varepsilon,h_1})}.$$

Thus combining this inequality with (0.39) yields the following statement:

Given any $R > r_4$, any $\varepsilon_0 < \frac{1}{10}\min\{r_2-r_1, r_4-r_3, R-r_4\}$, and any $\delta < h_2(\varepsilon_0, r_1, R, K, M)$ and setting $\varepsilon_v \leq \varepsilon_0$ as above (0.34), we have

$$\frac{Vol(\,S_{2\delta,r_3-\varepsilon_0,r_4+\varepsilon_0}(\Omega_R(K))\,)}{Vol(\,\Omega(K)\cap b^{-1}([r_1+2\varepsilon_0,r_2-\varepsilon_0])\,)} \geq \frac{V(r_3,r_4,R,\varepsilon_v)}{V(r_1+\varepsilon_0,r_2,R,\varepsilon_v)}. \tag{0.40}$$

Note that (0.40) does not depend on h_1 or ε.

Taking the limit as δ approaches 0 and applying Corollary 9, we obtain

$$\frac{Vol(\,\Omega(K)\cap b^{-1}([r_3-\varepsilon_0,r_4+\varepsilon_0])\,)}{Vol(\,\Omega(K)\cap b^{-1}([r_1+2\varepsilon_0,r_2-\varepsilon_0])\,)} \geq \frac{V(r_3,r_4,R,\varepsilon_v)}{V(r_1+\varepsilon_0,r_2,R,\varepsilon_\delta)}. \tag{0.41}$$

Since this holds for all $R \geq r_4$, taking $R \to \infty$ in (0.41) and (0.37) leads to

$$\frac{Vol(\,\Omega(K) \cap b^{-1}([r_3 - \varepsilon_0, r_4 + \varepsilon_0])\,)}{Vol(\,\Omega(K) \cap b^{-1}([r_1 + 2\varepsilon_0, r_2 - \varepsilon_0])\,)} =$$

$$= \lim_{R\to\infty} \frac{\sum_{j=0}^{n-1} \frac{{}_{n-1}C_j\,(-1)^j}{p+2vj}\Big((\frac{r_4+\varepsilon_v}{R+\varepsilon_v})^{p+2vj} - (\frac{r_3+\varepsilon_v}{R+\varepsilon_v})^{p+2vj}\Big)}{\sum_{j=0}^{n-1} \frac{{}_{n-1}C_j(-1)^j}{p+2vj}\Big((\frac{r_2+\varepsilon_v}{R+\varepsilon_v})^{p+2vj} - (\frac{r_1+\varepsilon_0+\varepsilon_v}{R+\varepsilon_v})^{p+2vj}\Big)}$$

$$= \lim_{R\to\infty} \frac{(\frac{r_4+\varepsilon_v}{R+\varepsilon_v})^p - (\frac{r_3+\varepsilon_v}{R+\varepsilon_v})^p}{(\frac{r_2+\varepsilon_v}{R+\varepsilon_v})^p - (\frac{r_1+\varepsilon_0+\varepsilon_v}{R+\varepsilon_v})^p} = \frac{(r_4+\varepsilon_v)^p - (r_3+\varepsilon_v)^p}{(r_2+\varepsilon_v)^p - (r_1+\varepsilon_v)^p}.$$

We can now take ε_0 to 0 and bring ε_v to 0 with it to complete the proof. q.e.d.

The following corollary of Theorem 5 is proven using the Coarea Formula [10, 3.2.11], which states that if a set A is Lebesgue measureable, and a real valued function f is Lipschitz, then the $(n-1)$-Hausdorff volume, $Vol_{n-1}(A \cap f^{-1}(s))$, is integrable and

$$\int_A |\bigtriangledown f|\, dL^n x = \int_{-\infty}^{\infty} Vol_{n-1}(A \cap f^{-1}(s))\, ds, \tag{0.42}$$

where $|\bigtriangledown f|$ is defined almost everywhere because f is Lipschitz. Recall that the Busemann function is also Lipschitz [5] and that $\Omega(K)$ is measurable [Lemma 4]. See [12] for details.

Corollary 17. *Let* $Ric \geq \frac{(n-1)(\frac{1}{4}-v^2)}{b(x)^2}$ *for* $x \in b^{-1}([r_0,\infty))$ *where* $v \in (0, \frac{n+1}{2(n-1)})$*. Let* $r_1 \geq r_0$ *and* $K \in b^{-1}((-\infty, r_1])$ *be a compact set. Then there exists a nondecreasing function* $V(r)$ *defined on* $r \in [r_1,\infty)$ *such that*

$$V(r) = \frac{Vol_{n-1}(\Omega(K) \cap b^{-1}(r))}{r^{(\frac{1}{2}-v)(n-1)}} \tag{0.43}$$

almost everywhere. In particular, for almost every $s_2 > s_1 \geq r_1$ *we have*

$$\frac{Vol_{n-1}(\Omega_{s_1}(K)}{ps_1^{(\frac{1}{2}-v)(n-1)}} \leq \frac{Vol(\Omega(K) \cap b^{-1}([s_1,s_2]))}{s_2^p - s_1^p} \leq \frac{Vol_{n-1}(\Omega_{s_2}(K))}{ps_2^{(\frac{1}{2}-v)(n-1)}}$$

where $p = (1/2 - v)(n-1) + 1$.

Theorem 5 and Corollary 17 are essential ingredients to the proof of the properness of the Busemann function in the next section. We now apply them to prove the following refinement of the Volume Growth Theorem of [8].

Theorem 18. *Fix x_0 in M^n and $r > r_1 > r_0 \geq 0$. Let γ be a ray from x_0. Let $Ric \geq \frac{(n-1)(\frac{1}{4}-v^2)}{b(x)^2}$ at all points x such that $b(x) \geq r_0$ where $v \in [0, \frac{n+1}{2(n-1)}]$. Let $p = (\frac{1}{2} - v)(n-1) + 1$. Then,*

$$\begin{aligned} Vol(B_{x_0}(R + r_1 - r_0)) & \\ \geq \frac{Vol_{n-1}(B_{x_0}(r_1) \cap b^{-1}(r_0))}{p r_0^{p-1}} \left(R^p - r_0^p \right). \end{aligned} \tag{0.44}$$

When $p = 0$, replace r^p by $ln(r)$.

Proof of Theorem 18. Let K be $\bar{B}_{r_2}(x_0)$. If $y \in \Omega(K) \cap b^{-1}([r_1, r])$, there exists $\bar{y} \in K$ such that y is on a geodesic ray emanating from $\bar{y}$. So $d(\bar{y}, y) = b(y) - b(\bar{y}) < r - r_0$, while $d(\bar{y}, x_0) \leq r_2$. Thus $y \in B_{r+r_2-r_1}(x_0)$. Hence,

$$Vol(\Omega(\bar{B}_{r_2}(x_0)) \cap b^{-1}([r_1, r])) \leq Vol(B_{r+r_2-r_1}(x_0)),$$

and the lemma follows from Theorem 5. q.e.d.

The power, p, is the same as in the Volume Growth Theorem of [8], but the constant is now the maximal possible constant and is achieved by warped product manifolds with a metric of the form $dr^2 + r^{2(1/2-v)} g_0$.

2. Compactness of level sets

In this section we prove the following theorem.

Theorem 19. *Let M^n have nonnegative Ricci curvature everywhere and satisfy $Ric_x \geq \frac{(n-1)(1/4-v^2)}{b^2(x)}$ on $b^{-1}(r_0, \infty)$, where $v \in (0, 1/2]$. Suppose that M^n also has minimal volume growth, so that*

$$\limsup_{R \to \infty} \frac{Vol(B_p(R))}{R^{(1/2-v)(n-1)+1}} = V_0 < \infty. \tag{0.45}$$

Then the Busemann function's level sets, $b^{-1}(r)$, are compact. Furthermore, the diameter of these levels grows at most linearly, i.e.,

$$\limsup_{R \to \infty} \frac{diam(b^{-1}(R))}{R} < \infty, \tag{0.46}$$

where the diameter is measured in the ambient manifold, M^n.

In particular, we prove that on a manifold with nonnegative Ricci curvature and linear volume growth all Busemann functions have compact level sets.

The key ingredient to the proof of this theorem is the volume growth of $\Omega(K)$ [Theorem 5 and Corollary 17]. We show that given minimal volume growth, any Busemann function has level sets with finite $(n-1)$-volume and the ratio, $Vol_{n-1}(r)/(r^{1/2-v})$, is nondecreasing and bounded above. Thus we can take a compact set, K, and a Busemann level whose intersection with K has a sufficiently large $(n-1)$-volume. This insures that "most" of the volume of the manifold is contained in $\Omega(K)$. We then show that balls of large diameter cannot fit outside of $\Omega(K)$ using the Relative Volume Comparison Theorem and the globally nonnegative Ricci curvature.

Lemma 20. *Let M^n have $Ric_x \geq \frac{(n-1)(1/4-v^2)}{b^2(x)}$ on $b^{-1}(r_0,\infty)$, where $v \in (0, \frac{n+1}{2(n-1)})$, and let M^n have minimal volume growth, (0.45).*

Then the level sets of any Busemann function, b, have finite $(n-1)$-volume. Furthermore, if $r_2 > r_1 \geq r_0$, then

(0.47)
$$\frac{Vol_{n-1}(b^{-1}(r_1))}{r_1^{p-1}} \leq \frac{p\,Vol(b^{-1}(r_1,r_2))}{r_2^p - r_1^p} \leq \frac{Vol_{n-1}(b^{-1}(r_2))}{r_2^{p-1}} \leq V_0,$$

where $p = (n-1)(1/2-v)+1$ and V_0 is defined in (0.45).

Note that this lemma, unlike Theorem 19, includes manifolds with quadratically decaying negative Ricci curvature ($v > 1/2$) outside a compact set.

Proof. By Theorem 17, we know that for any compact set $K \subset b^{-1}((-\infty, r_1])$, we have

$$\frac{Vol_{n-1}(b^{-1}(r_1)\cap\Omega(K))}{r_1^{p-1}} \leq \frac{Vol(\Omega_{r_1,r_2}(K))}{r_2^p - r_1^p} \leq \frac{Vol(B_x(diam(K)+r_2))}{r_2^p - r_1^p}.$$

Taking the limsup as r_2 approaches infinity and using the minimal volume growth, (0.45), we get

$$\frac{Vol_{n-1}(b^{-1}(r_1)\cap\Omega(K))}{r_1^{p-1}} \leq V_0. \tag{0.48}$$

Taking a sequence of compact sets $K_i \subset K_{i+1}$ such that $\bigcup K_i = b^{-1}((-\infty, r_1])$, we get

$$Vol_{n-1}(b^{-1}(r_1)) \leq V_0\, r_1^{p-1} \quad \forall r_1 \geq r_0. \tag{0.49}$$

In particular, the volume of any level set is finite. Similarly, we can substitute this sequence K_i into the following equation

$$\frac{Vol_{n-1}(b^{-1}(r_1)\cap\Omega(K))}{r_1^{p-1}} \le \frac{Vol_{n-1}(b^{-1}(r_2)\cap\Omega(K))}{r_2^{p-1}} \le \frac{Vol_{n-1}(b^{-1}(r_2))}{r_2^{p-1}},$$

which holds for all $r_2 > r_1$ by Theorem 17. Thus $Vol_{n-1}(b^{-1}(r))$ is nondecreasing for $r \ge r_1$. We can then apply the coarea formula to complete the proof. q.e.d.

Definition 21. *Given a Busemann function, b, let*

$$V_{b,\infty} = \lim_{r\to\infty} \frac{Vol_{n-1}(b^{-1}(r))}{r^{p-1}}. \tag{0.50}$$

Proof of Theorem 19. Fix any Busemann function, b. We wish to show that all the level sets of b are compact.

Note that once a given level set $b^{-1}(r)$ is shown to be compact then, for all $a < r$, $b^{-1}(a)$ is compact as well. This is true because given any pair of points $x, y \in b^{-1}(a)$ the Busemann rays γ_x, γ_y intersect with $b^{-1}(r)$ at the points $\gamma_x(r), \gamma_y(r)$. Thus,

$$d(x,y) \le d(\gamma_x(r),\gamma_y(r)) + 2(r-a) \le diam(b^{-1}(r)) + 2(r-a).$$

This arguement only works for $a < r$. So we must show that there exist a sequence of compact levels sets, $b^{-1}(r_i)$, where $r_i \to \infty$.

By Lemma 20 and the definition of $V_{b,\infty}$, we know we can find r_1 such that $Vol(b^{-1}(r_1))/(r_1)^{p-1}$ is as close to $V_{b,\infty}$ as we wish. Furthermore, we can find a compact set $K \subset b^{-1}((-\infty, r_1])$ large enough that $Vol(b^{-1}(r_1)\cap K)$ is almost the entire volume of the level $b^{-1}(r_1)$. So we can chose r_1 such that

$$\frac{V_{b,\infty} - V_{r_1}}{V_{r_1}} \le \frac{1}{5^{n+1}}, \tag{0.51}$$

where

$$V_{r_1} = \frac{Vol_{n-1}(b^{-1}(r_1)\cap K)}{(r_1)^{p-1}}. \tag{0.52}$$

Assume there exists an increasing sequence of real numbers h_i approaching infinity and a sequence of points $p_i \in b^{-1}(r_1 + h_i)$ such that $d(p_i, \Omega(K)) = h_i$. Recall Definition 3 of $\Omega(K)$. We now proceed to prove that such a sequence cannot exist.

Fix $p = p_i$ and $h = h_i$ temporarily and consult Figure 1. Since $d(p, \Omega(K)) = h$, we know that $B_p(h) \cap \Omega(K) = \emptyset$. On the other hand $B_p(h) \subset b^{-1}(r_1, r_1 + 2h)$. Therefore,

$$Vol(B_p(h)) \leq Vol(b^{-1}(r_1, r_1 + 2h)) - Vol(\Omega_{r_1, r_1+2h}(K)), \tag{0.53}$$

where $\Omega_{r_1,r_1+2h}(K) = b^{-1}(r_1, r_1 + 2h) \cap \Omega(K)$. By Corollary 17 and Lemma 20, this implies that

$$Vol(B_p(h)) \leq V_{b,\infty}((r_1 + 2h)^p - (r_1)^p) - V_{r_1}((r_1 + 2h)^p - (r_1)^p). \tag{0.54}$$

On the other hand, the globally nonnegative Ricci curvature combined with the Relative Volume Comparison Theorem [2], [11] gives a lower bound on this volume,

$$Vol(B_p(h)) \geq \left(\frac{h}{R}\right)^n Vol(B_p(R)). \tag{0.55}$$

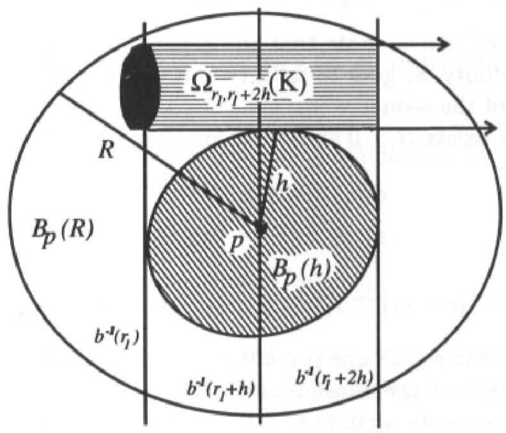

FIGURE 1. $B_p(h) \cap \Omega(K) = \emptyset$

We now choose R large enough so that $B_p(R) \supset \Omega_{r_1,r_1+2h}(K)$. To get from p to any point in $\Omega_{r_1,r_1+2h}(K)$ we travel first a distance h to reach $\Omega_{r_1,r_1+2h}(K)$, and then at most a distance $2h$ along a Busemann ray to reach K. Traveling at most $diam(K)$ we reach any point in K and then we can travel back up a Busemann ray a distance at most $2h$ to get to any point in $\Omega_{r_1,r_1+2h}(K)$. Thus we can take $R = 5h + diam(K)$ and we have

$$Vol(B_p(h)) \geq \left(\frac{h}{(5h + diam(K))}\right)^n Vol(\Omega_{r_1,r_1+2h}(K)). \tag{0.56}$$

Finally we employ Theorem 17 again to bound the volume on the right from below:

$$Vol(B_p(h)) \geq \left(\frac{h}{(5h + diam(K))}\right)^n V_{r_1}((r_1 + 2h)^p - (r_1)^p) \tag{0.57}$$

Combining our upper and lower bounds (Eqn 0.54 and 0.57), we get the following inequality

$$(V_{b,\infty} - V_{r_1})((r_1 + 2h)^p - (r_1)^p) \geq \frac{h^n V_{r_1}((r_1 + 2h)^p - (r_1)^p)}{(5h + diam(K))^n}. \tag{0.58}$$

Cancelling $((r_1 + 2h)^p - (r_1)^p)$, rearranging and substituting (0.51) we get,

$$\frac{1}{5^{n+1}} \geq \frac{(V_{b,\infty} - V_{r_1})}{V_{r_1}} \geq \left(\frac{h_i}{(5h_i + diam(K))}\right)^n \tag{0.59}$$

for all h_i defined above. Note that the dependence on p_i has vanished. Taking i to infinity, h_i goes to infinity and we get the contradiction of the existence of the sequence p_i.

Thus there exists $H > 0$ such that for all $p \in b^{-1}((H + r_1, \infty))$

$$d(p, \Omega(K)) \neq b(p) - r_1 \tag{0.60}$$

We claim that

$$b^{-1}(r + r_1) \subset T_r(\Omega_{r_1, r_1+2r}(K)) \qquad \forall r \geq H. \tag{0.61}$$

If not. there exists $r \geq H$ and $p \in b^{-1}(r_1 + r)$ such that $d(p, \Omega(K)) > r$. By the definition of the Busemann function, $b^{-1}([r_1 + r, \infty))$ is connected, so there exists a curve $C : [0, L] \to b^{-1}([r_1 + r, \infty))$ such that $C(0) = p$ and $C(L) \in \Omega(K) \cap b^{-1}(r_1 + r)$. Then

$$d(C(0), \Omega(K)) - b(C(0)) + r_1 > 0$$

while

$$d(C(L), \Omega(K)) - b(C(L)) + r_1 \leq -H < 0.$$

By the Intermediate Value Theorem, there exists $t \in (0, L)$ such that $d(C(t), \Omega(K)) - b(C(t)) + r_1 = 0$, contradicting (0.60). Thus (0.61) holds.

Fixing $x \in K$, we have

$$b^{-1}(r_1 + r) \subset B_x(3r + diam(K)) \qquad \forall r \geq H. \tag{0.62}$$

This implies that $b^{-1}(r+r_1)$ is compact for $r \geq H$ and, consequently, b is a proper function. Furthermore, the diameter of $b^{-1}(r)$ grows at most linearly. q.e.d.

Remark 22. Note that we cannot prove a similar theorem for manifolds with a negative quadratically decaying lower bound on Ricci curvature. That is, if $v > 1/2$ and we assume that $Ric(x) \geq \frac{(n-1)(1/4-v^2)}{b^2}$ for $x \in b^{-1}(r_0, \infty)$ and $Ric(x) \geq -\Lambda$ for all $x \in M^n$, the above proof does not follow. In particular, the Relative Volume Comparison Theorem in (0.55) would involve *sinh* and thus (0.59) would not produce a contradiction when taking h_i to infinity.

Recall the definition of V_0 in (0.45) and of $V_{b,\infty}$ in Definition 21.

Corollary 23. *Let M^n have Ricci $\geq \frac{(n-1)(\frac{1}{4}-v^2)}{b(x)^2} \geq 0$ for $x \in b^{-1}(r_0, \infty)$ and globally nonnegative Ricci curvature. If M^n has minimal volume growth then either*

i) M^n splits, in which case $V_{b,\infty} = V_0/2$ for all Busemann functions, b,

or

ii) $b_{min} = \inf_{x \in M^n} b(x) > -\infty$ exists, in which case $V_{b,\infty} = V_0$ for all Busemann functions, b.

Proof. By Lemma 20, we know that for all r_1,

$$V_{b,\infty} \geq \lim_{r \to \infty} \frac{Vol(b^{-1}(r_1, r))}{r^p}. \tag{0.63}$$

We first suppose there is a Busemann function, b, defined on the manifold which does not have a minimum value, b_{min}. Then there exist a sequence of points $p_i \in M^n$ such that $b(p_i) = -i$. From each point p_i, there is a Busemann ray γ_i parametrized so that $b(\gamma_i(t)) = t$. Then $(\gamma_i(0), \gamma_i'(0))$ is a sequence of unit vectors in the tangent space restricted to $b^{-1}(0)$. Since $b^{-1}(0)$ is compact, a subsequence $(\gamma_j(0), \gamma_j'(0))$ converges to (p, v) where $p \in b^{-1}(0)$. Note that $exp_p(tv)$ is a line because it is a limit of rays in the positive direction and the limit of increasingly long minimal geodesics in the negative direction.

By the Splitting Theorem [7], we know that a manifold with globally nonnegative Ricci curvature which contains a line is isometric to a metric product $N^{n-1} \times \mathbb{R}$ where N^{n-1} is the level set of a Busemann function. Since N^{n-1} must therefore be compact, it is not hard to see that it must be a level set of any Busemann function, $\bar{b}$, as well. Thus

$$\frac{Vol(\bar{b}^{-1}((-s,s)))}{(s+D)^p} = \frac{2\,Vol(\bar{b}^{-1}((0,s)))}{(s+D)^p}. \tag{0.64}$$

On the other hand, for $p_0 \in b^{-1}(0)$ and $D = diam(N^{n-1})$,

$$\frac{Vol(B_{p_0}(s+D))}{(s+D)^p} \geq \frac{Vol(\bar{b}^{-1}((-s,s)))}{(s+D)^p} \geq \frac{Vol(B_{p_0}(s))}{(s+D)^p}. \tag{0.65}$$

Combining (0.64) with (0.65) and taking s to infinity, we get

$$V_0 \geq 2V_{\bar{b},\infty} \geq V_0.$$

We have proven case (i).

We now consider the alternate case, where every Busemann function, b, has a minimum value, b_{min}. Then for $p \in b^{-1}(b_{min})$, we have

$$B_p(R) \subset b^{-1}(b_{min}, b_{min}+R).$$

So

$$V_{b,\infty} \geq \lim_{R\to\infty} \frac{b^{-1}(b_{min}, b_{min}+R)}{R^p} \geq \limsup_{R\to\infty} \frac{B_p(R)}{R^p} = V_0. \tag{0.66}$$

However, by Lemma 20, $V_{b,\infty} \leq V_0$, implying case (ii). q.e.d.

3. Examples

In this section, we construct some examples of noncompact manifolds with nonnegative Ricci curvature and linear volume growth. Example 26 demonstrates that the diameters of the Busemann level sets can grow logarithmically. Example 27 demonstrates that the sequences of subsets of the form $b^{-1}(r_i, r_i+l)$ can have more than one limit as r_i approaches infinity. This is similar in concept to examples of the nonuniqueness of tangent cones at infinity [9].

Both examples are doubly warped products of the Hopf bundle, S^3, crossed with the real line. S^3 is a compact Lie group; so it has three left invariant vector fields, X, Y and Z, which are orthonormal in the standard metric on S^3. We construct metrics on $R \times S^3$ of the form,

$$g = dr^2 + u^2(r)\sigma_Z^2 + w^2(r)(\sigma_X^2 + \sigma_Y^2), \tag{0.67}$$

where σ_X, σ_Y and σ_Z are the covectors of X, Y and Z respectively.

We require that the Ricci curvature be nonnegative. If we let $T = \frac{d}{dr}$, then the following formula for the Ricci curvature's eigenvalues holds [1].

$$Ric(T,T) = \frac{-u''}{u} + 2\frac{-w''}{w}, \tag{0.68}$$

$$Ric(Z,Z) = \frac{-u''}{u} - 2\frac{u'}{u}\frac{w'}{w} + 2\frac{u^2}{w^4}, \tag{0.69}$$

$$\begin{aligned} Ric(X,X) =& Ric(Y,Y) \\ =& \frac{-w''}{w} - \left(\frac{w'}{w}\right)^2 - \frac{u'}{u}\frac{w'}{w} + 4\left(\frac{w^2 - \frac{1}{2}u^2}{w^4}\right). \end{aligned} \tag{0.70}$$

The main part of our constructions will consist of designing the ends of the manifold by choosing functions $u(r)$ and $w(r)$ where $r \geq r_0$ for some r_0. Since we need to construct complete manifolds of nonnegative Ricci curvature, we must close up the manifolds smoothly at some $r_1 < r_0$. In fact we will close up the ends, which have boundary diffeomorphic to S^3, with simply connected balls by extending u and w so that they are 0 at r_1 and the metric near r_1 is a metric of constant curvature. The following lemma justifies this extension of $u(r)$ and $w(r)$, given certain assumptions about their values and derivatives at r_0.

Lemma 24. *Given any $w_0 > 0$ there exists $\delta > 0$ and there exists $t_0 > 0$ such that if*

$$w_1 \in [0, \delta), \quad u_0 \in (w_0 - \delta, w_0], \quad u_1 \in (-\delta, 0], \tag{0.71}$$

then there exist functions $u, w : [0, t_0] \longrightarrow \mathbf{R}$ such that

$$u(t_0) = u_0, \quad u'(t_0) = u_1, \quad w(t_0) = w_0, \quad w'(t_0) = w_1, \tag{0.72}$$

and such that the metric, (0.67), is complete and has positive Ricci curvature on the ball of radius t_0 in $\mathbf{R}^4$.

Proof. Let $w(r) = u(r) = A\sin(r/A)$ for $r \in [0, t_1]$, where $A = w_0/(1 - w_1^2)^{1/2}$ and $t_1 = A\,\mathrm{Arcsin}(u_0/A)$. This metric has constant sectional curvature and is complete at $r = 0$. Note that $u'(t_1) = \sqrt{1 - (u_0/A)^2}$.

Let $t_0 = A\,\mathrm{Arcsin}(w_0/A) \leq A(\pi/2)$. For $r \in [t_1, t_0]$, let $w(r) = A\sin(r/A)$ and let

$$u(r) = \frac{a_1(r-t_1)^3}{(t_0-t_1)^2} + \frac{a_2(r-t_1)^2}{t_0-t_1} + u'(t_1)(r-t_1) + u(t_1), \tag{0.73}$$

where $a_1 = u'(t_1) + u_1$ and $a_2 = -2u'(t_1) - u_1$. Thus $u(r)$ and $u'(r)$ are continuous at t_1 and satisfy (0.72).

For δ sufficiently small, we can take $u(r)$ uniformly close on $[t_1, t_0]$ to u_0 which is close to w_0, and we can take $u'(r)$ and $u''(r)$ uniformly close to 0 on $[t_1, t_0]$. Thus, for $r \in (t_1, t_0)$, by (0.68), (0.69) and (0.70), we have $Ric(T,T)$ arbitrarily close to $1/w_0 > 0$, $Ric(Z,Z)$ arbitrarily close to $1/w_0 + 2/w_0^2 > 0$, and $Ric(X,X) = Ric(Y,Y)$ arbitrarily close to $2/w_0^2 > 0$. We may have to take δ quite small to insure that these Ricci curvatures are positive.

Since the Ricci curvature is positive on both sides of t_1 and since u and w are C^1 on $[0, t_0]$, u and w can be smoothed on an arbitrarily small neighborhood of $r^{-1}(t_1)$ to C^∞ functions preserving the positive Ricci curvature. See, for example, [1]. q.e.d.

We are constructing manifolds with linear volume growth. Thus $Vol_3(r^{-1}(t))$, which is proportional to $A(t) = u(t)w(t)^2$, must approach a constant at infinity. However, it cannot approach its asymptote too quickly if w, satisfying (0.68) with $u = A/w^2$, is to approach infinity. See [12]. So we set

$$A(r) = k - \frac{\varepsilon}{\ln r}, \tag{0.74}$$

where $k > 1$ and $\varepsilon > 0$ are constants.

The proof of the following lemma is just calculus combined with (0.69), (0.70), (0.68) and (0.74). Condition (i) alone implies $Ric(T,T) \geq 0$.

Lemma 25. *If the doubly warped product manifold $\mathbf{R}^+ \times S^3$ with the metric, (0.67), has $u(r)w(r)^2 = A(r) = k - \frac{\varepsilon}{\ln r}$ where $k > 1$ and $\varepsilon > 0$ are constants, and has $w(r) > 0$ satisfying the four conditions:*

$$(i)\quad \left|\frac{w'(r)}{w(r)}\right| \leq \sqrt{\frac{\varepsilon}{k}}\,\frac{1}{3r\ln r}, \qquad (ii)\quad 4w^2(r) - 2u^2(r) \geq w^2(r),$$

$$(iii)\quad \limsup_{r\to\infty}(w'/w)' r\ln r \leq k_2, \qquad (iv)\quad w^2(r) \leq k_3(\ln r)^{2\sqrt{\varepsilon}/(3\sqrt{k})}$$

for some constants $k_2, k_3 > 0$, then there exists $r_0 > 0$ depending on k, ε, k_2, and k_3, such that g has nonnegative Ricci curvature for $r \geq r_0$.

The first example is constructed by solving the ordinary differential equation,

$$\frac{w'(r)}{w(r)} = \sqrt{\frac{\varepsilon}{k}}\,\frac{1}{3r\ln r}. \tag{0.75}$$

Example 26. Given any $k > 1$ and $\varepsilon > 0$, then there exists a complete manifold, M^n, with nonnegative Ricci curvature, linear volume growth and $diam(b^{-1}(r)) = w(r)$ approaching infinity. In particular, the end is isomorphic to the doubly warped product, $[r_0, \infty) \times S^3$ with the metric, (0.67), where

$$w(r) = k^{1/3}\left(\frac{\ln r}{\ln r_0}\right)^{\frac{\sqrt{\varepsilon}}{3\sqrt{k}}} \quad \textit{and} \quad u(r) = \frac{k - \frac{\varepsilon}{\ln r}}{w(r)^2}. \tag{0.76}$$

It is easy to verify that this w obeys all four conditions of Lemma 25. It is also easy to see that w and $u = A/w$ satisfy the conditions at r_0 required by Lemma 24, thus demonstrating that M^n is complete. Note that while the diameter growth of such a manifold is unbounded it is still sublinear. In fact, in [13], we prove that all manifolds with nonnegative Ricci curvature and linear volume growth have sublinear diameter growth.

We will now construct another metric on $R^+ \times S^3$ with nonnegative Ricci curvature, linear volume growth and a bound on the diameter of the level sets but without a unique limit of the level sets. We want to find a function $w(r)$ which alternates between two values as r approaches infinity. Once again w'/w must not be integrable from 0 to infinity and is bounded as in (0.68) and $A(r)$ is defined as in (0.74). We now define w as a solution of an iteratively defined but integrable ordinary differential equation.

First note that any function, f(r), which satisfies,

$$\frac{f'(r)}{f(r)} = \sqrt{\frac{\varepsilon}{k}}\,\frac{1}{3r\ln r} \tag{0.77}$$

will increase to any value in a finite amount of time. That is, there exists a function $L_+(r, f_1, f_2)$ defined for all positive r and positive $f_2 > f_1$ such that if $f(r)$ is a solution of the above equation and $f(r) = f_1$, then $f(r + L_+) = f_2$. In fact,

$$L_+(r, f_1, f_2) = r^{(f_2/f_1)^{3\sqrt{k/\varepsilon}}} - r. \tag{0.78}$$

Note how large L_+ is relative to r for a fixed ratio f_2/f_1.

On the other hand, if we have a function, g(r), satisfying,

$$\frac{g'(r)}{g(r)} = -\sqrt{\frac{\varepsilon}{k}}\,\frac{1}{3r\ln r}, \tag{0.79}$$

then it will decrease to any positive value in a finite amount of time. There exists a function $L_-(r, g_1, g_2)$ defined for all positive r and positive $g_2 < g_1$ such that if $g(r)$ is a solution of the above equation and $g(r) = g_1$, then $g(r + L_-) = g_2$. Note that L_- also grows dramatically as a function of r.

We must design our function $w(r)$ following the specification of condition (i) of Lemma 25; so it can only grow as fast as $f(r)$ and decrease as fast as $g(r)$. We will require w to solve (0.77) long enough to reach an upper value, k, and then require it to solve (0.79) long enough to return to a lower value, $k^{2/3}$, and then return to (0.77) and so on. To smooth out the differential equation, we define $h(r)$ to be a C^∞ function such that $h = 1$ for $r \le 0$ and $h = -1$ for $r \ge \pi$ and $-1 \le h \le 1$ everywhere. Note that the solution of a C^∞ integrable ordinary differential equation is C^∞.

Let r_0 be some positive number large enough to insure that $L_+(r, k, 2k)$, $L_+(r, k^{2/3}, k)$, $L_-(r, k^{2/3}, k^{1/3})$ and $L_-(r, k, k^{2/3})$ are all larger than 2π for all $r > r_0$. This insures that nothing much can happen in the smoothing interval. We may take r_0 even larger later in order to satisfy the other Ricci curvature conditions in Lemma 25.

Let $w(r_0) = k^{1/3}$. Let $w(r)$ increase to k by satisfying (0.77) for $r \in (r_0, r_1)$ where $r_1 = r_0 + L_+(r_0, w(r_0), k)$.

On the interval between r_1 and $r_2 = r_1 + \pi$, we require that,

$$\frac{w'(r)}{w(r)} = h(r - r_1)\sqrt{\frac{\varepsilon}{k}}\,\frac{1}{3r\ln r}. \tag{0.80}$$

Thus $w(r)$ continues to be smooth and to obey condition (i). Note that our restrictions on r_0 combined with the fact that $| h(r) | \le 1$ tells us that $k^{2/3} < w(r_2) < 2k$.

Let $w(r)$ decrease to $k^{2/3}$ by satisfying (0.79) for $r \in (r_2, r_3)$ where $r_3 = r_2 + L_-(r_0, w(r_0), k)$.

On the interval between r_3 and $r_4 = r_3 + \pi$, we require that,

$$\frac{w'(r)}{w(r)} = -h(r - r_3)\sqrt{\frac{\varepsilon}{k}}\,\frac{1}{3r\ln r}. \tag{0.81}$$

Thus $w(r)$ continues to be smooth, and to obey condition (i), and $k^{1/3} < w(r_4) < k$.

At this point we continue the process by requiring that $w(r)$ increase to k by satisfying (0.77) once again, then satisfying (0.80) to turn around, satisfying (0.79) to decrease back to $k^{2/3}$ and so on. In this way we guarantee that,

$$\begin{aligned} &w(r_{4i+1}) = k, \quad w(r_{4i+3}) = k^{2/3} \\ &\textit{and} \quad k^{1/3} < w(r) < 2k \quad \forall r > r_0. \end{aligned} \tag{0.82}$$

It is not hard to verify that conditions (i)-(iv) of Lemma 25 are also satisfied for r_0 chosen sufficiently large. See [12].

Finally we close up the manifold with a four-dimensional disk, by smoothly extending $w(r)$ and $v(r)$ down to 0 at some point $r_1 < r_0$. We are able to do this because w and u have values and derivatives at r_0 which satisfy the hypothesis of Lemma 24 for r_0 chosen sufficiently large.

Example 27. Given any $k > 1$ and $\varepsilon > 0$, we can find an $r_0 > 0$ and we can construct a manifold M^n with an end which is isometric to the doubly warped product, $[r_0, \infty) \times S^3$ with the metric, (0.67), where w is a smooth function defined as above and $u(r) = A(r)/w(r)^2$. This manifold has nonnegative Ricci curvature and linear volume growth. The diameters of the level sets of the Busemann function are uniformly bounded.

The level sets alternate between two different Riemannian manifolds: $r^{-1}(r_{4i+1})$, a Hopf sphere with the metric,

$$\frac{(k - \frac{\varepsilon}{\ln r})^2}{k^4}\sigma_Z^2 + k^2(\sigma_X^2 + \sigma_Y^2) \;\approx\; \frac{1}{k^2}\sigma_Z^2 + k^2(\sigma_X^2 + \sigma_Y^2), \tag{0.83}$$

and $r^{-1}(r_{4i+3})$, a Hopf sphere with the metric,

$$\frac{(k - \frac{\varepsilon}{\ln r})^2}{k^{8/3}}\sigma_Z^2 + k^{4/3}(\sigma_X^2 + \sigma_Y^2) \;\approx\; \frac{1}{k^{2/3}}\sigma_Z^2 + k^{4/3}(\sigma_X^2 + \sigma_Y^2), \tag{0.84}$$

where the sequence r_i is defined above. Here $b^{-1}(r, r+L)$ does not approximate a unique metric in the Gromov-Hausdorff sense as r approaches infinity.

Notice that the length of the intervals between the level sets of maximum diameter and those of minimum diameter is increasing to infinity; see (0.78). So given any L we can go far enough out that a region

$b^{-1}(r, r+L)$ is close to the isometric product, $b^{-1}(r) \times (r, r+L)$, in the Gromov-Hausdorff sense. This behavior will be proven necessary in [13].

We believe that similar examples can be constructed for manifolds with quadratically decaying Ricci curvature bounds and minimal volume growth. Such examples might be constructed using the same doubly warped product of the Hopf Sphere with the positive real axis but a slightly different differential equation.

References

[1] M. Anderson. *Metrics of positive Ricci curvature with large diameter.* Manuscripta Math. **68** (1990) 405–415.

[2] R. Bishop. *A relation between volume, mean curvature and diameter,* Amer. Math. Soc. Notices **10** (1963) 364.

[3] R. Bishop & R. Crittenden. *Geometry of manifolds,* Pure Applied Math., Vol. XV. Academic Press, New York, 1964.

[4] S. Bochner. *Vector fields and Ricci curvature,* Bull. Amer. Math. Soc. **52** (1946) 776–797.

[5] H. Busemann, *The Geometry of geodesics,* Academic Press, New York, NY, 1955.

[6] J. Cheeger, *Critical points of distance functions and applications to geometry*, Geometric Topology and Recent Developments, Montecatini Terme, 1990, 1–38; Lecture Notes in Math., Vol. 1504, Springer, Berlin, 1991.

[7] J. Cheeger & D. Gromoll *The splitting theorem for manifolds of nonnegative Ricci curvature,* J. Differential Geom. **6** (1971/72) 119–128.

[8] J. Cheeger, M. Gromov & M. Taylor. *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds,* J. Differential Geom. **17** (1982) 15–53.

[9] J. Cheeger & G. Tian. *On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay,* Invent. Math. **118** (1994) 493–571.

[10] H. Federer, *Geometric measure theory*, Grundlehren Math. Wiss., Band, Vol. 153, Springer, New York, 1969

[11] M. Gromov, J. Lafontaine & P. Pansu, *Structures métriques pour les Variétés Riemanniennes*, (French) Textes Math. Vol. 1, Cedic-Nathan, Paris, 1981.

[12] C. Sormani, *The almost rigidity of manifolds with lower bounds on Ricci curvature and minimal volume growth* preprint 1998.

[13] ______, *The almost rigidity of manifolds with lower bounds on Ricci curvature and minimal volume growth*, Preprint, 1997.

[14] S. T. Yau, *Some function theoretic properties of complete Riemannian manifolds and their applications to geometry,* Indiana Univ. Math. J. **25** (1976) 659–670.

HARVARD UNIVERSITY