J. DIFFERENTIAL GEOMETRY
48 (1998) 225-274

ASYMPTOTIC BEHAVIOR OF ANISOTROPIC CURVE FLOWS

XI-PING ZHU

Abstract

In this paper the asymptotic behavior of a closed embedded curve evolving by an anisotropic parabolic equation $\gamma_t = (\Phi(T)k + \Psi(T))N$ on a Riemannian surface is studied. It was proven by Oaks [18] that whenever the evolving curve develops a singularity, it shrinks to a point. We further show that its dilations converge to a Minkowski isoperimetrix associated Φ in C^∞ topology.

0. Introduction

This paper is concerned with the evolution of curves on a Riemannian surface M whose normal velocity is a given function of its position and its tangent direction, as well as its curvature. A particular case is the curve shortening problem where the normal velocity and geodesic curvature coincide. This case has been studied in great detail in a series of paper by Gage, Hamilton and Grayson etc. For a curve embedded in R^2, it is shown by Gage [8], Gage and Hamilton [10] and Grayson [12] that the embedded planar curve becomes convex before its curvature can blow up,and then shrinks to a point with round limiting shape, and with C^∞ convergence. For a curve embedded in a Riemanian surface, Grayson [13] proved the evolving curve either shrinks to a point in finite time, or exists for infinite time.

In a sequence of two papers ([3],[4]) Angenent developed a theory of an arbitrary uniformly parabolic equation

$$\frac{\partial \gamma}{\partial t} = V(T, k)N \tag{0.1}$$

Received October 28, 1996. The author was partially supported by CNRS-K.C.Wong Foundation and NSF of China.

for a curve $\gamma(t,\cdot)$ on a surface M. T and N denote the unit tangent and normal vectors to $\gamma(t,\cdot)$, and k the curvature of $\gamma(t,\cdot)$. Under an additional symmetry assumption, $V(-T,-k) = -V(T,k)$, he proved that whenever a curve $\gamma(t,\cdot)$ evolving by (0.1) develops a singularity it either loses at least one self-intersection, or else its total curvature drops at least π. Oaks [18] improved the Angenent's augument and showed that the latter case never occurs. As a consequence, if at the start the curve is embedded, the evolving curve either shrinks to a point in finite time or exists for infinite time. So they extended the Grayson's theorem from the curve shortening flow to the general flow (0.1).

Naturally the further development leads to investigating the formation of the singularities. In this paper, we study the formation of the singularities for the following evolution problem

$$(0.2)\qquad \begin{cases} \frac{\partial\gamma}{\partial t} = (\Phi(T)k + \Psi(T))N, \\ \\ \gamma(0,\cdot) \text{ is a smooth curve embedded on } M, \end{cases}$$

where Φ, Ψ are bounded smooth functions satisfying $\Phi \geq \lambda$ for some positive constant λ, and the symmetry condition $\Phi(-T) = \Phi(T), \Psi(-T) = -\Psi(T)$. (0.2) is general enough to unify a handful of specific evolutions which have recently received attention such as, the curve shortening flow, the flow by curvature in relative geometries ([9]), and the models for phase transitions ([5],[6],[14]). In fact, even in the simplest case of the curve shortening flow for an embedded curve, it has already been raised as a conjecture by Grayson [12] that the limiting shape of a singularity would be a circle.

Recently, in the joint work with Chou [7], we have gotten the formation of the singularities of (0.2) in the special case $M = R^2$. The arguments in that paper [7] which are inspired by the Grayson's work [12], are heavily dependent on the translation invariance of R^2. In the present paper, we give a different approach to the flow (0.2) on an arbitrary Riemannian surface M. We prove the limit shape of (0.2) in which it develops a singularity must be the Minkowski isoperimetrix associated Φ (see Definition 4.1). In the particular case of the curve shortening flow this confirms the above Grayson's conjecture.

Our arguments are inspired by the work of Hamilton [15]. In Section 2, we give two isoperimetric estimates for an evolving embedded curve of (0.2). By a rescaling argument in Section 3, we show that the curve becomes convex before it degenerates to a point. The main theorem is stated and proved in Section 4.

We would like to express our appreciation to S.Angenent, K.S.Chou and P.L.Lions for many helpful discussions and comments. This paper was completed while the author was visiting CEREMADE,University of Paris-Dauphine.

1. Preliminaries

Let M be a smooth Riemannian surface with bounded scale curvature, and denote its unit tangent bundle by $S^1(M)$. A regular closed curve in M is, by definition, an equivalence class of C^∞ immersion of the circle S^1 into M; two such immersions which differ only by an orientation-preserving reparametrization will be considered to be the same regular closed curve. For simplicity, throughout this paper, we call a regular closed curve to be a curve. Given a curve $\gamma(u) : S^1 \to M$, we write T and N for its unit tangent and unit normal vectors respectively, and write k for its geodesic curvature.

Given a C^1 family of curves $\gamma(t,\cdot) : S^1 \to M$ one can decompose the time derivative $\gamma_t(t,u)$ as

$$\gamma_t(t,u) = v'' T + v^\perp N.$$

The second component $v^\perp$ is independent of the chosen parametrization of each $\gamma(t,\cdot)$; it is the normal velocity of the family of curves.

This paper is concerned with the following initial value problem. Given a smooth embedded curve γ_0, a C^1 family of smooth curves $\gamma(t,u) : [0, t_{\max}) \times S^1 \to M$ satisfies

$$v^\perp = \Phi(T)k + \Psi(T), \tag{1.1}$$

and whose initial value $\gamma(0,\cdot) = \gamma_0$. Here Φ, Ψ satisfy the following conditions:

(H$_1$) $\Phi, \Psi : S^1(M) \to R$ are smooth bounded functions;

(H$_2$) $\lambda \le \Phi(T) \le \lambda^{-1}$ for all $T \in S^1(M)$, where λ is a positive constant;

(H$_3$) $\Phi(-T) = \Phi(T), \Psi(-T) = -\Psi(T)$, for all $T \in S^1(M)$.

The basic results in [3],[4] ensure that (1.1) has a unique maximal solution $\gamma(t,\cdot)$ defined in $[0, t_{\max}), 0 < t_{\max} \le \infty$ such that if $t_{\max} <$

$+\infty$, the curvature k is unbounded as $t \to t_{\max}$, moreover the solution $\gamma(t,\cdot)$ remains embedded on $[0, t_{\max})$. And by [18], whenever $t_{\max} < +\infty$, the evolving curve $\gamma(t,\cdot)$ of (1.1) shrinks to a point as $t \to t_{\max}$.

To analyse the formation of singularities, without loss of generality, we may always assume that $t_{\max} < +\infty$, and the whole family $\gamma(t,\cdot), t \in [0, t_{\max})$ is contained in a small neighborhood of the shrinking point on M. Then from the Theorem 3.1 and its proof in Oaks' paper [18], the initial value problem (1.1) is equivalent to the following problem on the plane R^2:

$$(1.2) \qquad \begin{cases} v^{\perp} = \Phi(x,y,\theta)k + \Psi(x,y,\theta) \\ \gamma(0,\cdot) = \gamma_0, \quad \text{a smooth curve embedded on } R^2 \end{cases}$$

where Φ, Ψ satisfy the following conditions :

$(\mathbf{H}_1)'$ $\Phi, \Psi : R^2 \times S^1 \to R$ are smooth bounded functions;

$(\mathbf{H}_2)'$ $\lambda \le \Phi(x,y,\theta) \le \lambda^{-1}$ for all $(x,y,\theta) \in R^2 \times S^1$, and λ is some positive constant;

$(\mathbf{H}_3)'$ $\Phi(x,y,\theta+\pi) = \Phi(x,y,\theta), \Psi(x,y,\theta+\pi) = -\Psi(x,y,\theta)$, for all $(x,y,\theta) \in R^2 \times S^1$.

Here we denote by (x,y) the position vector of the curve $\gamma(t,\cdot)$, and θ the angle between the tangent vector and the x-axis. Throughout this paper we shall assume that $\gamma(t,\cdot)$ is oriented in couterclockwise direction.

Let $\gamma(t,\cdot) : [0,t_{\max}) \times S^1 \to R^2$ be the maximal solution of (1.2). One can choose a parametrization $\gamma(t,u) : [0,t_{\max}) \times S^1 \to R^2$ of $\gamma(t,\cdot)$ whose time derivative $\frac{\partial\gamma}{\partial t}$ is always orthogonal to the curve $\gamma(t,\cdot)$. For this particular parametrization (1.2) is written as

$$(1.3) \qquad \begin{cases} \frac{\partial \gamma}{\partial t} = (\Phi(x(t,u),y(t,u),\theta(t,u))k(t,u) \\ \qquad\qquad + \Psi(x(t,u),y(t,u),\theta(t,u)))N \\ \\ \gamma(0,u) = \gamma_0(u), \quad t \in (0,t_{\max}), u \in S^1. \end{cases}$$

One may also parametrize each curve $\gamma(t,\cdot)$ by the arclength parameter s. Denote $L(t)$ to be the length of $\gamma(t,\cdot)$ and $A(t)$ to be the area enclosed by $\gamma(t,\cdot)$. By a standard computation, one has the following evolution equations:

$$\left.\frac{\partial k}{\partial t}\right|_u = \frac{\partial^2}{\partial s^2}(\Phi(x,y,\theta)k+\Psi(x,y,\theta))+k^2(\Phi(x,y,\theta)k+\Psi(x,y,\theta)) \tag{1.4}$$

(c.f.(2.9) in Gurtin's book [14]),

$$\frac{dL}{dt} = -\int_{\gamma(t,\cdot)} k(\Phi(x,y,\theta)k+\Psi(x,y,\theta))ds, \tag{1.5}$$

$$\frac{dA}{dt} = -\int_{\gamma(t,\cdot)} (\Phi(x,y,\theta)k+\Psi(x,y,\theta))ds, \tag{1.6}$$

(c.f.(2.26) in [14]),

and

$$\left.\frac{\partial \theta}{\partial t}\right|_u = -\frac{\partial}{\partial s}(\Phi(x,y,\theta)k+\Psi(x,y,\theta)) \tag{1.7}$$

(c.f. (2.9) in [14]),

where $\left.\frac{\partial}{\partial t}\right|_u$ means the partial derivative with u fixed.

2. Isoperimetric estimates

Consider γ to be any embedded smooth closed curve in R^2. Let Γ be any curve dividing the region D enclosed by γ into two regions D_1 and D_2 with areas A_1 and A_2, where $A_1 + A_2 = A$ is the area of D. Denote L to be the length of Γ. Define the ratio

$$G(\Gamma) = L^2\left(\frac{1}{A_1}+\frac{1}{A_2}\right) \tag{2.1}$$

and let

$$\bar{G} = \inf_{\Gamma} G(\Gamma) \tag{2.2}$$

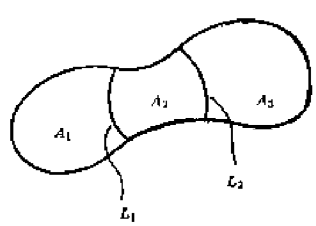
be the least possible value of $G(\Gamma)$ for all curves Γ.

In [15], Hamilton defined the ratio (2.1) for the restricted class where Γ are straight line segments,and proved the corresponding $\bar{G}$ increases under the curve shortening flow (i.e., $\Phi \equiv 1, \Psi \equiv 0$) whenever $\bar{G} \leq \pi$. In this section we shall generalize the Hamilton's isoperimetric estimate to the flow (1.2). More precisely, we shall show a lower positive bound for the ratio (2.2) under the flow(1.2). The anisotropic nature of the flow (1.2) forces us to study the ratio (2.1) on the full class of curves.

Let p be any fixed point on the boundary γ. Draw a small circle with p as its center. Consider Γ to be the arc of the circle enclosed by γ. It is easy to see that the ratio $G(\Gamma)$ can come as close to π as we wish by taking the radius of the circle small enough. So the infimum $\bar{G}$ is always not larger than π. Since our purpose is to get a lower positive bound for $\bar{G}$, it suffices to consider curves γ which satisfy $\bar{G} < \pi$. We begin with a lemma which is somewhat known in the works of Hamilton [15] and our proof is just a slight adaptation.

Lemma 2.1. *If $\bar{G} < \pi$, the infimum $\bar{G}$ is attained by a unipiece smooth curve Γ of constant curvature perpendicular to γ.*

Proof. Exactly as in Section B.1 of [15], for any given division of area $A = A_1 + A_2$, there will be a shortest curve (or collection of curves) effecting this division, and the curve (or each component curve) has constant curvature and is perpendicular to the boundary, moreover the number of components is finite. We claim that there is a component of the curve forming a new division of D such that the corresponding ratio G is smaller. Suppose, for example, that Γ has two components Γ_1 and Γ_2 of lengths L_1 and L_2, dividing A into regions of area $A_1 + A_2 + A_3 = A$ as shown



We will have

$$\begin{aligned}(L_1+L_2)^2 &\left(\frac{1}{A_2}+\frac{1}{A_1+A_3}\right)\\ &> \min\left\{L_1^2\left(\frac{1}{A_1}+\frac{1}{A_2+A_3}\right), L_2^2\left(\frac{1}{A_1+A_2}+\frac{1}{A_3}\right)\right\}.\end{aligned}$$

In fact, if not,we get

$$\frac{A_1(A_2+A_3)}{A_2(A_1+A_3)} \le \frac{L_1^2}{(L_1+L_2)^2} \quad \text{and} \quad \frac{(A_1+A_2)A_3}{A_2(A_1+A_3)} \le \frac{L_2^2}{(L_1+L_2)^2}.$$

Adding these inequalities gives

$$1+\frac{2A_1A_3}{A_2(A_1+A_3)} \le 1-\frac{2L_1L_2}{(L_1+L_2)^2}.$$

This is absurd. Then

$$\begin{aligned}\overline{G} &= \inf\{G(\Gamma) \mid && \text{for all curves } \Gamma\ \}\\ &= \inf\{G(\Gamma) \mid && \Gamma \text{ is a unipiece smooth curve}\\ & && \text{of constant curvature perpendicular to } \gamma.\}\end{aligned}$$

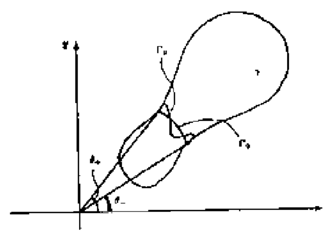
Choose a sequence of unipiece smooth curves Γ_n of constant curvature perpendicular to γ such that

$$G(\Gamma_n) \to \bar{G} \quad \text{as} \quad n \to +\infty.$$

From the assumption $\bar{G} < \pi$, we know that the constant curvature k_n of Γ_n must be uniformly bounded from above. Then it follows that there is a subsequence of Γ_n converges to a unipiece smooth curve Γ of constant curvature perpendicular to γ at two ends such that Γ achieves the infimum $\bar{G}$. Moreover the curve Γ cannot meet γ at some point in its interior. Othewise, we can divide Γ into two segments and repeat the above argument to conclude that there is a new division of D such that the corresponding ratio G is smaller, which contradicts that Γ attains the infimum $\bar{G}$. q.e.d.

Let Γ_μ be any smooth one-parameter family of curves each dividing D into two regions with the corresponding areas $A_1(\Gamma_\mu)$ and $A_2(\Gamma_\mu)$, where μ belongs to an open interval containing 0. We assume that Γ_0 is the minimizer obtained in Lemma 2.1. Now we need to compute the first and second variation of the length $L(\Gamma_\mu)$ of the curve Γ_μ and the areas $A_1(\Gamma_\mu)$ and $A_2(\Gamma_\mu)$. By Lemma 2.1, Γ_0 has a constant curvature, denoted by k_0.

First we consider the case $k_0 \neq 0$. Using polar coordinates,we may assume that Γ_μ is given by the graph of $r = r(\theta, \mu)$ between $\theta_- = \theta_-(\mu)$ and $\theta_+ = \theta_+(\mu)$, and $\Gamma_0 = \left\{(r, \theta) \,\middle|\, r = \frac{1}{|k_0|}, \theta \in [\theta_-, \theta_+]\right\}$.



Since the function $r(\theta, 0)$ is a constant and Γ_0 is perpendicular to γ, it follows that $\mu = 0$,

$$\frac{\partial r}{\partial \theta} = 0, \qquad \frac{\partial^2 r}{\partial \theta^2} = 0,$$

and

$$\frac{\partial \theta_+}{\partial \mu} = 0, \qquad \frac{\partial \theta_-}{\partial \mu} = 0.$$

The boundary γ has curvature k_+ at $\theta = \theta_+$ and k_- at $\theta = \theta_-$ which can be computed as the curvatures of the graph of

$$\theta = \theta_+(\mu) \quad \text{and} \quad r = r(\theta_+(\mu), \mu)$$

for k_+ and the same for k_-, except with a possible sign change. By a direct computation, one has

$$\left(r\frac{d^2\theta}{d\mu^2}\right)_+ = -k_+\left(\frac{\partial r}{\partial \mu}\right)_+^2$$

and

$$\left(r\frac{d^2\theta}{d\mu^2}\right)_- = k_-\left(\frac{\partial r}{\partial \mu}\right)_-^2$$

at $\mu = 0$. Here and in the following subscripts $+$ or $-$ denote the value of the function at $\theta = \theta_+$ or $\theta = \theta_-$.

Exactly as in [15], a straightforward calculation at $\mu = 0$ gives the following results.

Lemma 2.2. *At* $\mu = 0$,

$$\begin{aligned}
\frac{dL}{d\mu} &= \int_{\theta_-}^{\theta_+} v d\theta, \\
\frac{d^2L}{d\mu^2} &= \int_{\theta_-}^{\theta_+} z d\theta + |k_0| \int_{\theta_-}^{\theta_+} \left(\frac{dv}{d\theta}\right)^2 d\theta - (k_+ v_+^2 + k_- v_-^2), \\
\frac{dA_1}{d\mu} &= \frac{1}{|k_0|} \int_{\theta_-}^{\theta_+} v d\theta, \\
\frac{dA_2}{d\mu} &= -\frac{1}{|k_0|} \int_{\theta_-}^{\theta_+} v d\theta, \\
\frac{d^2A_1}{d\mu^2} &= \frac{1}{|k_0|} \int_{\theta_-}^{\theta_+} z d\theta + \int_{\theta_-}^{\theta_+} v^2 d\theta, \\
\frac{d^2A_2}{d\mu^2} &= -\frac{1}{|k_0|} \int_{\theta_-}^{\theta_+} z d\theta - \int_{\theta_-}^{\theta_+} v^2 d\theta,
\end{aligned}$$

where $v = \left.\frac{\partial r}{\partial \mu}\right|_{\mu=0}$, $z = \left.\frac{\partial^2 r}{\partial \mu^2}\right|_{\mu=0}$, *and* $A_1(\Gamma_\mu)$ *denotes the area on the origin side of* Γ_μ. q.e.d.

Now we write down the conditions that $G(\Gamma)$ attains its minimum at Γ_0. Consider

$$\ln G = 2 \ln L - \ln A_1 - \ln A_2 + \ln A,$$

where $A = A_1 + A_2$ is independent of μ. Then at $\mu = 0$,

$$\begin{aligned}
0 &= \frac{d}{d\mu} \ln G \\
&= \frac{2}{L} \int_{\theta_-}^{\theta_+} v d\theta - \frac{1}{A_1 \cdot |k_0|} \int_{\theta_-}^{\theta_+} v d\theta + \frac{1}{A_2 \cdot |k_0|} \int_{\theta_-}^{\theta_+} v d\theta \\
&= \left(\frac{2}{L} - \frac{1}{A_1 \cdot |k_0|} + \frac{1}{A_2 \cdot |k_0|}\right) \int_{\theta_-}^{\theta_+} v d\theta,
\end{aligned}$$

and, since v is arbitrary,

$$\frac{2}{L} - \frac{1}{A_1 \cdot |k_0|} + \frac{1}{A_2 \cdot |k_0|} = 0, \tag{2.4}$$

and

$$
\begin{aligned}
0 \leq\ & \frac{d^2}{d\mu^2}\ln G \\
=\ & \frac{2}{L}\frac{d^2L}{d\mu^2} - \frac{2}{L^2}\left(\frac{dL}{d\mu}\right)^2 - \frac{1}{A_1}\frac{d^2A_1}{d\mu^2} + \frac{1}{A_1^2}\left(\frac{dA_1}{d\mu}\right)^2 \\
& -\frac{1}{A_2}\frac{d^2A_2}{d\mu^2} + \frac{1}{A_2^2}\left(\frac{dA_2}{d\mu}\right)^2 \\
=\ & \frac{2}{L}\left[\int_{\theta_-}^{\theta_+} z\,d\theta + |k_0|\int_{\theta_-}^{\theta_+}\left(\frac{dv}{d\theta}\right)^2 d\theta - (k_+v_+^2 + k_-v_-^2)\right] \\
& -\frac{2}{L^2}\left(\int_{y_-(\mu)}^{y_+(\mu)} v\,dy\right)^2 \\
& -\frac{1}{A_1}\left[\frac{1}{|k_0|}\int_{y_-(\mu)}^{y_+(\mu)} z\,dy + \int_{y_-(\mu)}^{y_+(\mu)} v^2\,dy\right] \\
& +\frac{1}{A_1^2}\left(\frac{1}{|k_0|}\int_{y_-(\mu)}^{y_+(\mu)} v\,dy\right)^2 \\
& +\frac{1}{A_2}\left[\frac{1}{|k_0|}\int_{y_-(\mu)}^{y_+(\mu)} z\,dy + \int_{y_-(\mu)}^{y_+(\mu)} v^2\,dy\right] \\
& +\frac{1}{A_2^2}\left(\frac{1}{|k_0|}\int_{y_-(\mu)}^{y_+(\mu)} v\,dy\right)^2 \\
=\ & \left(\frac{2}{L} - \frac{1}{A_1\cdot|k_0|} + \frac{1}{A_2\cdot|k_0|}\right)\int_{y_-(\mu)}^{y_+(\mu)} z\,dy \\
& +\left[\frac{2|k_0|}{L}\int_{y_-(\mu)}^{y_+(\mu)}\left(\frac{dv}{d\theta}\right)^2 dy\right. \\
& \qquad \left. -\frac{1}{A_1}\int_{y_-(\mu)}^{y_+(\mu)} v^2\,dy + \frac{1}{A_2}\int_{y_-(\mu)}^{y_+(\mu)} v^2\,dy\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2}{L}(k_+v_+^2+k_-v_-^2)\\
&+\left[\frac{1}{|k_0|^2\cdot A_1^2}+\frac{1}{|k_0|^2\cdot A_2^2}-\frac{2}{L^2}\right]\left(\int_{y_-(\mu)}^{y_+(\mu)} v\,dy\right)^2,
\end{aligned}
$$

by (2.4),

$$
\begin{aligned}
= \quad & \frac{2|k_0|}{L}\int_{y_-(\mu)}^{y_+(\mu)}\left[\left(\frac{dv}{d\theta}\right)^2-v^2\right]dy-\frac{2}{L}(k_+v_+^2+k_-v_-^2)\\
&+\frac{1}{2}\left(\frac{1}{A_1\cdot|k_0|}+\frac{1}{A_2\cdot|k_0|}\right)^2\left(\int_{y_-(\mu)}^{y_+(\mu)} v\,dy\right)^2.
\end{aligned}
$$

By choosing $v=\sqrt{\Phi\left(\frac{1}{|k_0|}\cos\theta,\frac{1}{|k_0|}\sin\theta,\theta\right)}, \theta\in[\theta_-,\theta_+]$, we have

$$
\frac{dv}{d\theta}=\frac{1}{2\sqrt{\Phi}}\left[\Phi_x\cdot\frac{1}{|k_0|}(-\sin\theta)+\Phi_y\cdot\frac{1}{|k_0|}\cos\theta+\Phi_\theta\right],
$$

then

$$
\begin{aligned}
\left(\frac{dv}{d\theta}\right)^2 \quad \le \quad & \frac{1}{4\lambda}(|\Phi_x|^2+|\Phi_y|^2+|\Phi_\theta|^2)\left(\frac{1}{k_0^2}+1\right)\\
\le \quad & C_1\left(\frac{1}{k_0^2}+1\right)
\end{aligned}
$$

for some positive constant C_1 depending only on Φ.

And by noticing $|k_0|\cdot L=\theta_+-\theta_-$, we deduce, by (2.4),

$$
\begin{aligned}
\frac{2|k_0|}{L}\int_{y_-(\mu)}^{y_+(\mu)}\left(\frac{dv}{d\theta}\right)^2 dy \quad \le \quad & C_1\frac{2|k_0|}{L}\cdot\left(\frac{1}{k_0^2}+1\right)\cdot(L\cdot|k_0|)\\
\le \quad & C_1\left(1+\left(\frac{2|k_0|}{L}\right)^2\cdot L^2\right)\\
= \quad & C_1\left(1+L^2\left(\frac{1}{A_1}-\frac{1}{A_2}\right)^2\right).
\end{aligned}
$$

Thus

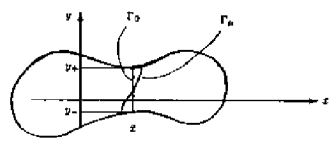
$$
\begin{aligned}
0 \quad \le \quad & C_1\left[1+L^2\left(\frac{1}{A_1}-\frac{1}{A_2}\right)^2\right]\\
&-\frac{2}{L}(k_+\Phi(x_+,y_+,\theta_+)+k_-\Phi(x_-,y_-,\theta_-))\\
&+\frac{1}{2\lambda}\left(\frac{1}{A_1}+\frac{1}{A_2}\right)^2\cdot L^2,
\end{aligned}
$$

this is,

$$
\tag{2.5}
\begin{aligned}
&\frac{2}{L}(k_+\Phi(x_+, y_+, \theta_+) + k_-\Phi(x_-, y_-, \theta_-)) \\
&\qquad \leq C_2 + C_2 L^2 \left(\frac{1}{A_1} + \frac{1}{A_2}\right)^2,
\end{aligned}
$$

where C_2 is some positive constant depending only on Φ. Here (x_+, y_+) and (x_-, y_-) are the corresponding position vectors of the boundary points of Γ_0.

Next we consider the case $k_0 = 0$. We may assume that Γ_0 lies on the vertical line over some $\bar{x}$. Consider the one-parameter family of curves Γ_μ given by the graph of $x = x(y, \mu)$ between $y = y_-(\mu)$ and $y = y_+(\mu)$. Set $y_- = y_-(0)$ and $y_+ = y_+(0)$.



By an argument similar to that in the above case, at $\mu = 0$,

$$
\begin{aligned}
\frac{\partial x}{\partial y} &= 0, \quad \frac{\partial^2 x}{\partial y^2} = 0, \\
\frac{dy_+}{d\mu} &= 0, \quad \frac{dy_-}{d\mu} = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\frac{d^2 y_+}{d\mu^2}\right)_+ &= -k_+ \left(\frac{\partial x}{\partial \mu}\right)_+^2, \\
\left(\frac{d^2 y_-}{d\mu^2}\right)_- &= k_- \left(\frac{\partial x}{\partial \mu}\right)_-^2,
\end{aligned}
$$

here and in the following, subscripts $+$ or $-$ denote the value of the function at $y = y_+$ or $y = y_-$.

Similarly, denote

$$v = \left.\frac{\partial x}{\partial \mu}\right|_{\mu=0}, \quad z = \left.\frac{\partial^2 x}{\partial \mu^2}\right|_{\mu=0}.$$

The length of Γ_μ is

$$L(\Gamma_\mu) = \int_{\theta_-}^{\theta_+} \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2} d\theta.$$

Then

$$\begin{aligned}
\left.\frac{dL}{d\mu}\right|_{\mu=0} &= \left.\int_{y_-(\mu)}^{y_+(\mu)} \frac{\left(\frac{\partial x}{\partial y}\right) \cdot \frac{\partial^2 x}{\partial y \partial \mu}}{\sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2}} dy\right|_{\mu=0} \\
&\quad + \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)_+^2} \cdot \left.\frac{dy_+}{d\mu}\right|_{\mu=0} \\
&\quad - \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)_-^2} \cdot \left.\frac{dy_-}{d\mu}\right|_{\mu=0} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\left.\frac{d^2 L}{d\mu^2}\right|_{\mu=0} &= \int_{y_-}^{y_+} \left(\frac{dv}{dy}\right)^2 dy + \left(\frac{d^2 y_+}{d\mu^2}\right)_+ - \left(\frac{d^2 y_-}{d\mu^2}\right)_- \\
&= \int_{y_-}^{y_+} \left(\frac{dv}{dy}\right)^2 dy - (k_+ v_+^2 + k_- v_-^2).
\end{aligned}$$

The area on the left-hand side of Γ_μ is

$$A_1(\Gamma_\mu) = A_1(\Gamma_0) + \int_0^\mu \int_{y_-(\tau)}^{y_+(\tau)} \frac{\partial x}{\partial \mu} dy d\tau,$$

by noticing $dxdy = \frac{\partial x}{\partial \mu} dy d\mu$. Thus

$$\begin{aligned}
\left.\frac{dA_1}{d\mu}\right|_{\mu=0} &= \left.\int_{y_-(\mu)}^{y_+(\mu)} \frac{\partial x}{\partial \mu} dy\right|_{\mu=0} \\
&= \int_{y_-}^{y_+} v dy,
\end{aligned}$$

and

$$\left.\frac{d^2A_1}{d\mu^2}\right|_{\mu=0} = \int_{y_-}^{y_+} zdy.$$

Summing up, we get the following.

Lemma 2.3. *At $\mu = 0$,*

$$\begin{aligned}
&\frac{dL}{d\mu} = 0, \qquad \frac{d^2L}{d\mu^2} = \int_{y_-}^{y_+}\left(\frac{dv}{dy}\right)^2 dy - (k_+v_+^2 + k_-v_-^2),\\
&\frac{dA_1}{d\mu} = \int_{y_-}^{y_+} vdy, \qquad \frac{d^2A_1}{d\mu^2} = \int_{y_-}^{y_+} zdy\\
&\frac{dA_2}{d\mu} = -\int_{y_-}^{y_+} vdy, \qquad \frac{d^2A_2}{d\mu^2} = -\int_{y_-}^{y_+} zdy.
\end{aligned}$$

q.e.d.

As in the preceding case, we want to write down the conditions that $G(\Gamma)$ attains its minimum at Γ_0. By computation, at $\mu = 0$

$$\begin{aligned}
0 &= \frac{d}{d\mu}\ln G = \frac{2}{L}\frac{dL}{d\mu} - \frac{1}{A_1}\frac{dA_1}{d\mu} - \frac{1}{A_2}\frac{dA_2}{d\mu}\\
&= -\left(\frac{1}{A_1} - \frac{1}{A_2}\right)\int_{y_-}^{y_+} vdy,
\end{aligned}$$

by the arbitrary of v, we have

$$A_1 = A_2. \tag{2.6}$$

and

$$
\begin{aligned}
0 &\le \frac{d^2}{d\mu^2}\ln G \\
&= \frac{2}{L}\frac{d^2L}{d\mu^2} - \frac{2}{L^2}\left(\frac{dL}{d\mu}\right)^2 - \frac{1}{A_1}\frac{d^2A_1}{d\mu^2} \\
&\quad + \frac{1}{A_1^2}\left(\frac{dA_1}{d\mu}\right)^2 - \frac{1}{A_2}\frac{d^2A_2}{d\mu^2} + \frac{1}{A_2^2}\left(\frac{dA_2}{d\mu}\right)^2 \\
&= \frac{2}{L}\left[\int_{y_-}^{y_+}\left(\frac{dv}{dy}\right)^2 dy - (k_+v_+^2 + k_-v_-^2)\right] - \frac{1}{A_1}\int_{y_-}^{y_+} z dy \\
&\quad + \frac{1}{A_1^2}\left(\int_{y_-}^{y_+} v dy\right)^2 + \frac{1}{A_2}\int_{y_-}^{y_+} z dy + \frac{1}{A_2^2}\left(-\int_{y_-}^{y_+} v dy\right)^2 \\
&= \frac{2}{L}\int_{y_-}^{y_+}\left(\frac{dv}{dy}\right)^2 dy - \frac{2}{L}(k_+v_+^2 + k_-v_-^2) \\
&\quad + \left(\frac{1}{A_1^2} + \frac{1}{A_2^2}\right)\left(\int_{y_-}^{y_+} v dy\right)^2 .
\end{aligned}
$$

By choosing $v = \sqrt{\Phi(\bar{x}, y, 0)}, y \in [y_-, y_+]$ and noticing $y_+ - y_- = L$, we deduce

$$
\frac{2}{L}(k_+\Phi(\bar{x}, y_+, 0) + k_-\Phi(\bar{x}, y_-, 0)) \le C_3 + C_3L^2\left(\frac{1}{A_1} + \frac{1}{A_2}\right)^2, \tag{2.7}
$$

where C_3 is some positive constant depending only on Φ.

Now we can state the isoperimetric estimate. Let the boundary curve γ be evolving by (1.2) and define the corresponding ratio (2.1). The corresponding minimum $\bar{G}$ is a positive function on $[0, t_{\max})$.

Theorem 2.4. *Assume the conditions $(H_1)', (H_2)'$ and $(H_3)'$, and let $\gamma(t,\cdot) : [0, t_{\max}) \times S^1 \to R^2$ be an evolving smooth embedded curve of (1.2). If $t_{\max} < \infty$, then there exists a positive constant δ such that*

$$
\bar{G}(t) \ge \delta > 0
$$

for all $t \in [0, t_{\max})$.

Proof. Fix any time $t_0 \in (0, t_{\max})$, and let Γ be the minimizer of $\bar{G}$ for $\gamma(t_0, \cdot)$. For any t near t_0, we choose Γ_t to be the segment of the fixed circle (or the fixed straight line) containing Γ such that the segment Γ_t divides the enclosed region D of $\gamma(t,\cdot)$ into two parts, D_1 and D_2. Since Γ_t is fixed in its interior, the time derivative of the length of Γ_t, at $t=t_0$, is just the sum of the negative normal velocity of $\gamma(t,\cdot)$ at two ends of Γ_t. By noting the orientation of $\gamma(t,\cdot)$, the length L of Γ_t evolves, at $t=t_0$,by

$$\begin{aligned}\frac{dL}{dt}=&-\big(\Phi(x_-,y_-,\theta_-)k_-+\Psi(x_-,y_-,\theta_-)\\&+\Phi(x_+,y_+,\theta+\pi)k_++\Psi(x_+,y_+,\theta+\pi)\big),\end{aligned}$$

where (x_+,y_+) and (x_-,y_-) are the position vectors of the ends of $\Gamma_t,\theta_+,\theta_-$ are the corresponding angles in (2.5) and (2.7), and k_+,k_- are the corresponding curvatures of $\gamma(t,\cdot)$ at the ends of Γ_t.

The curve Γ_t divides the boundary curve $\gamma(t,\cdot)$ into two parts, $\gamma_1(t,\cdot)$ and $\gamma_2(t,\cdot)$, where $\gamma_i(t,\cdot)$ and Γ_t enclose the region $D_i, i=1,2$.

Applying the generalized formula of (1.6) for closed piecewise-smooth evolving curve (c.f. (2.49) in [14]) yields

$$\frac{dA_1}{dt}=-\int_{\gamma_1(t,\cdot)}(\Phi(\gamma_1,\theta)k+\Psi(\gamma_1,\theta))ds,$$

and

$$\frac{dA_2}{dt}=-\int_{\gamma_2(t,\cdot)}(\Phi(\gamma_2,\theta)k+\Psi(\gamma_2,\theta))ds,$$

by noticing the normal velocity of Γ_t is zero.

Let (x_0,y_0) be the coordinates of an arbitrary fixed point on $\gamma(t_0,\cdot)$. By the assumption $(\mathrm{H}_3)'$ and a direct computation, at $t=t_0$, we find

$$\begin{aligned}&\frac{d}{dt}\ln G\\&=\frac{2}{L}\frac{dL}{dt}-\frac{1}{A_1}\frac{dA_1}{dt}-\frac{1}{A_2}\frac{dA_2}{dt}+\frac{1}{A}\frac{dA}{dt}\\&=-\frac{2}{L}[\Phi(x_+,y_+,\theta_+)k_++\Phi(x_-,y_-,\theta_-)k_-\\&\qquad-\Psi(x_+,y_+,\theta_+)+\Psi(x_-,y_-,\theta_-)]\\&\quad+\frac{1}{A_1}\int_{\gamma_1}(\Phi k+\Psi)ds+\frac{1}{A_2}\int_{\gamma_2}(\Phi k+\Psi)ds\\&\quad-\frac{1}{A_1+A_2}\int_{\gamma}(\Phi k+\Psi)ds\end{aligned}$$

$$
\begin{aligned}
(2.8) \qquad & \geq -\frac{2}{L}[\Phi(x_+, y_+, \theta_+)k_+ + \Phi(x_-, y_-, \theta_-)k_- \\
& \quad - \Psi(x_+, y_+, \theta_+) + \Psi(x_-, y_-, \theta_-)] \\
& \quad + \frac{A_2^2}{A_1 A_2 (A_1 + A_2)} \int_{\gamma_1} \Phi(x_0, y_0, \theta) k ds \\
& \quad + \frac{A_1^2}{A_1 A_2 (A_1 + A_2)} \int_{\gamma_2} \Phi(x_0, y_0, \theta) k ds \\
& \quad - \frac{A_1^2 + A_2^2}{A_1 A_2 (A_1 + A_2)} \left[\int_{\gamma} |\Phi(x, y, \theta) - \Phi(x_0, y_0, \theta)| \cdot |k| ds \right. \\
& \qquad \qquad \left. + \max_{\substack{(x,y)\in R^2 \\ \theta \in S^1}} |\Psi| \cdot L(\gamma) \right],
\end{aligned}
$$

where $L(\gamma)$ is the length of the curve $\gamma(t_0, \cdot)$.

By using the total curvature bound of Angenent [3], we know

$$
\begin{aligned}
& \int_{\gamma} |\Phi(x, y, \theta) - \Phi(x_0, y_0, \theta)| \cdot |k| ds \\
& \qquad \leq \max_{\substack{(x,y)\in \gamma \\ \theta \in S^1}} |\Phi(x, y, \theta) - \Phi(x_0, y_0, \theta)| \cdot \int_{\gamma} |k| ds \\
& \qquad \leq C_4 \cdot \max_{\substack{(x,y)\in \gamma \\ \theta \in S^1}} |\Phi(x, y, \theta) - \Phi(x_0, y_0, \theta)|,
\end{aligned}
$$

where the constant C_4 depends only on Φ, Ψ and the initial curve γ_0.

Without loss of generality, we may assume that $\theta_- \leq 0 \leq \theta_+$. Then

$$
\begin{aligned}
& \int_{\gamma_1} \Phi(x_0, y_0, \theta) k ds \\
& \quad = \int_{\theta_+ + \pi}^{\theta_- + 2\pi} \Phi(x_0, y_0, \theta) d\theta \\
& \quad = \int_{\theta_+}^{\theta_- + \pi} \Phi(x_0, y_0, \theta) d\theta \\
& \quad = \left(\int_0^{\pi} - \int_0^{\theta_+} - \int_{\theta_- + \pi}^{\pi} \right) \Phi(x_0, y_0, \theta) d\theta \\
& \quad \geq \frac{1}{2} \int_0^{2\pi} \Phi(x_0, y_0, \theta) d\theta - \left(\max_{\theta \in S^1} \Phi(x_0, y_0, \theta) \right) \cdot (\theta_+ - \theta_-)
\end{aligned}
$$

and

$$\int_{\gamma_2} \Phi(x_0, y_0, \theta) k ds$$
$$= \left(\int_{\pi}^{2\pi} + \int_{0}^{\theta_+} + \int_{\theta_- + \pi}^{\pi} \right) \Phi(x_0, y_0, \theta) d\theta$$
$$\geq \frac{1}{2} \int_0^{2\pi} \Phi(x_0, y_0, \theta) d\theta - \left(\max_{\theta \in S^1} \Phi(x_0, y_0, \theta) \right) \cdot (\theta_+ - \theta_-),$$

here we have used the symmetry assumption $(\mathrm{H}_3)'$.

Thus (2.8) can be written as, at $t = t_0$,

$$\begin{aligned}
\frac{d}{dt} \ln G \quad \geq \quad & -\frac{2}{L} \Big[\Phi(x_+, y_+, \theta_+) k_+ + \Phi(x_-, y_-, \theta_-) k_- \\
& \qquad - \Psi(x_+, y_+, \theta_+) + \Psi(x_-, y_-, \theta_-) \Big] \\
& + \frac{1}{2} \frac{A_1^2 + A_2^2}{A_1 A_2 (A_1 + A_2)} \int_0^{2\pi} \Phi(x_0, y_0, \theta) d\theta \\
& - \frac{A_1^2 + A_2^2}{A_1 A_2 (A_1 + A_2)} (\theta_+ - \theta_-) \cdot \max_{\theta \in S^1} \Phi(x_0, y_0, \theta) \\
& - \frac{A_1^2 + A_2^2}{A_1 A_2 (A_1 + A_2)} \Bigg[C_4 \cdot \max_{\substack{(x,y) \in \gamma \\ \theta \in S^1}} |\Phi(x, y, \theta) \\
& \qquad\qquad - \Phi(x_0, y_0, \theta)| + \max_{\substack{(x,y) \in R^2 \\ \theta \in S^1}} |\Psi| \cdot L(\gamma) \Bigg],
\end{aligned}$$

By the mean value theorem, it follows that, at $t = t_0$,

$$\begin{aligned}
\frac{d}{dt} \ln G \geq & -\frac{2}{L} (\Phi(x_+, y_+, \theta_+) k_+ \\
& + \Phi(x_-, y_-, \theta_-) k_-) - C_5 \left(1 + \frac{(\theta_+ - \theta_-)}{L} \right) \\
& + \frac{1}{2} \frac{A_1^2 + A_2^2}{A_1 A_2 (A_1 + A_2)} \int_0^{2\pi} \Phi(x_0, y_0, \theta) d\theta \\
& - C_5 \frac{A_1^2 + A_2^2}{A_1 A_2 (A_1 + A_2)} [(\theta_+ - \theta_-) + L(\gamma)]
\end{aligned} \tag{2.9}$$

for some positive constant C_5 depending only on Φ, Ψ and the initial curve γ_0.

At $t=t_0$, by applying (2.4)-(2.7) and the fact that $\theta_+ - \theta_- = |k_0|\cdot L$, we get

$$\begin{aligned}
&\frac{d}{dt}\ln G \\
&\quad \geq -(C_2+C_3)-(C_2+C_3)L^2\left(\frac{1}{A_1}+\frac{1}{A_2}\right)^2 \\
&\qquad - C_5\left[1+\frac{L}{2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)\right] \\
&\qquad + \frac{1}{2}\frac{A_1^2+A_2^2}{A_1A_2(A_1+A_2)}\left[\int_0^{2\pi}\Phi(x_0,y_0,\theta)d\theta\right. \\
&\qquad\qquad\qquad \left. -C_5L^2\left(\frac{1}{A_1}-\frac{1}{A_2}\right)-2C_5L(\gamma)\right] \\
&\quad \geq -(C_2+C_3+C_5)\left[1+\left(\frac{1}{A_1}+\frac{1}{A_2}\right)(\bar{G}+L(\gamma))\right] \\
&\qquad + \frac{1}{2}\frac{A_1^2+A_2^2}{A_1A_2(A_1+A_2)}\Big[2\pi\lambda-2C_5(\bar{G}+L(\gamma))\Big].
\end{aligned}$$

Since

$$\frac{A_1^2+A_2^2}{A_1A_2(A_1+A_2)} \geq \frac{1}{2}\frac{(A_1+A_2)^2}{A_1A_2(A_1+A_2)} = \frac{1}{2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right),$$

and

$$\frac{1}{A_1}+\frac{1}{A_2} \geq \frac{1}{A_1+A_2} \geq \frac{4\pi}{L^2(\gamma)} \quad \text{(the isoperimetric inequality)},$$

we deduce that, at $t=t_0$,

$$\begin{aligned}
\frac{d}{dt}\ln G \quad \geq \quad & \left[\frac{\pi^2\lambda}{L^2(\gamma)}-(C_2+C_3+C_5)\right] \\
& +\frac{1}{4}\left(\frac{1}{A_1}+\frac{1}{A_2}\right)[\pi\lambda-8(C_2+C_3+C_5)\cdot(\bar{G}+L(\gamma))],
\end{aligned}$$

provided $\bar{G}+L(\gamma)\leq 2\pi\lambda/2C_5$.

Thus there exists a positive constant μ depending only on Φ,Ψ and the initial curve γ_0 such that if $L(\gamma(t_0,\cdot))\leq\mu$ and $\bar{G}(t_0)\leq\mu$, then at $t=t_0$,

$$\frac{d}{dt}\ln G > 0. \tag{2.10}$$

Now we are in a position to end the whole proof. Without loss of generality, we may assume that the length of each $\gamma(t,\cdot)(t\in[0,t_{\max}))$ is less than μ. We claim that

$$\bar{G}(t)\geq\frac{1}{2}\min\{\bar{G}(0),\mu\}$$

for all $t\in[0,t_{\max})$. Suppose not, we pick a time $\bar{t}\in(0,t_{\max})$ such that $\bar{G}(\bar{t})=\frac{1}{2}\{\bar{G}(0),\mu\}$ and $\bar{G}(t)>\frac{1}{2}\min\{\bar{G}(0),\mu\}$ for $t\in[0,\bar{t})$. By (2.10), for any t sufficiently near $\bar{t}$ and less than $\bar{t}$,

$$G(\Gamma_t)<G(\Gamma_{\bar{t}})=\bar{G}(\bar{t}),$$

where we have chosen $t_0=\bar{t}$ and Γ_t is the segment constructed at the beginning of the proof. Therefore by definition,

$$\bar{G}(t)\leq G(\Gamma_t)<\bar{G}(\bar{t})=\frac{1}{2}\min\{\bar{G}(0),\mu\}.$$

This contradicts the choice of the time $\bar{t}$. q.e.d.

In the next section, we will use a rescaling argument to conclude the evolving curve $\gamma(t,\cdot)$ is eventually convex. The above isoperimetric estimate should be crucial. But, after rescaling, we cannot distinguish the interior or the exterior of the limit curve. So we need to establish a similar isoperimetric estimate for the exterior region of the curve $\gamma(t,\cdot)$.

Consider γ to be any embedded closed curve in the unit disk $\{(x,y)\in R^2|x^2+y^2\leq1\}$ of R^2. Denote by B_4 the disk $\{(x,y)\in R^2|x^2+y^2\leq16\}$ and by ∂B_4 its boundary. Let D be the region between ∂B_4 and γ. Set Γ be any curve (or collection of curves) dividing D into two regions D_1 and D_2 with the areas A_1 and A_2, where $A_1+A_2=A$ is the area of D.

Denote L to be the length of Γ. Similarly, we define the ratio $G'(\Gamma)$ by (2.1) and the least possible value $\bar{G}'$ by (2.2). The same proof of Lamma 2.1 gives the following result.

Lemma 2.1′. *If $\bar{G}'<\pi$, then there is a curve Γ which attains the infimum $\bar{G}'$ and satisfies :*

(i) *either Γ is a one-component curve which starts and ends on the same curve ∂B_4 or γ, or else, Γ is a two-components curve such that each components connects the curves ∂B_4 and γ.*

(ii) *Γ lies entirely in the interior of D except at its ends ;*

(iii) Γ *has constant curvature and is perpendicular to the boundary of* D. *q.e.d.*

Since the curve γ stays in the unit disk, it is easy to see that there is an absolute positive constant c_0 such that if the infimum $\bar{G}' \le c_0$, then the minimizer Γ obtained by the above lemma must be a one-component curve which starts and ends on the curve γ. The main purpose of this section is to provide a lower positive bound for the isoperimetric ratio. Thus, in the following, we can only consider the case that the minimizer Γ is perpendicular to the curve γ at its ends.

Denote the constant curvature of Γ by k_0. Let us keep the sign convention for the curvature of γ as before. By repeating the same computation as the derivation of (2.4)-(2.7), we get the same necessary conditions for Γ to be a minimizer of $\bar{G}'$. More precisely, we have

$$\frac{2|k_0|}{L} - \frac{1}{A_1} + \frac{1}{A_2} = 0, \tag{2.11}$$

$$\begin{aligned} &-\frac{2}{L}(k_+\Phi(x_+, y_+, \theta_+) + k_-\Phi(x_-, y_-, \theta_-)) \\ &\qquad \le C_6 + C_6 L^2 \left(\frac{1}{A_1} + \frac{1}{A_2}\right)^2, \end{aligned} \tag{2.12}$$

where C_6 is some positive constant depending only on Φ, (x_+, y_+) and (x_-, y_-) are the corresponding position vectors, θ_+, θ_- are the corresponding angles as before, and k_+, k_- are the corresponding curvature (by keeping the sign convention as before) of the curve γ at the boundary points of Γ.

Because the curve γ lies in the unit disk, and the circle (or straight) segment Γ lies entirely in the region D, it is obvious that the area A_1, A_2 cannot be equal. Thus we know that $k_0 \neq 0$, and the region D_1 (with the smaller area A_1) must be enclosed by Γ and a subarc γ_1 of γ. Moreover, by establishing polar coordinates, we may assume that $\Gamma = \{(r, \theta) | r = 1/|k_0|, \theta \in [\theta_-, \theta_+]\}$ and the region D_1 is on the origin side of Γ.

Now we state the isoperimetric estimate for the exterior region of the flow (1.2). Without loss of generality, we may assume that the evolving curve $\gamma(t, \cdot)$ of (1.2) is always in the unit disk of R^2. Define the ratio G'

by (2.1) for a curve Γ dividing the exterior region of $\gamma(t,\cdot)$ in the disk B_4. The corresponding minimum $\bar{G}'$ is a positive function on $[0,t_{\max})$.

Theorem 2.4′. *Assume the condition $(H_1)'$,$(H_2)'$ and $(H_3)'$, and let $\gamma(t,\cdot):[0,t_{\max})\times S^1\to R^2$ be an evolving smooth embedded curve of (1.2) in the unit disk of R^2. If $t_{\max}<+\infty$, then there exists a positve constant δ' such that*

$$\bar{G}'(t)\geq\delta'>0$$

for all $t\in[0,t_{\max})$.

Proof. From the proof of Theorem 2.4, we only need to show that there exists a positive constant μ depending only on Φ,Ψ and the initial curve γ_0 such that whenever a time $t_0\in(0,t_{\max})$ satisfies $L(\gamma(t_0,\cdot))\leq\mu$ and $\bar{G}'(t_0)\leq\mu$, then

$$\bar{G}'(t)<\bar{G}'(t_0)$$

for some t sufficiently near and less then t_0.

Since the curve $\gamma(t,\cdot)$ always lies in the unit disk, there is an absolute positive constant c_0 such that if $\bar{G}'(t)\leq c_0$, then the minimizer Γ obtained by Lemma 2.1′ must be a one-component curve which starts and ends on $\gamma(t,\cdot)$. Without loss of generality, we may assume that the positive constant μ (to be chosen later) is less than c_0. Let t_0 be such a time in $(0,t_{\max})$ with $L(\gamma(t_0,\cdot))\leq\mu$ and $\bar{G}'(t_0)\leq\mu$. Denote by Γ_{t_0} the minimizer of $\bar{G}'(t_0)$. Then Γ_{t_0} must be perpendicular to the curve $\gamma(t_0,\cdot)$ at its ends and have constant curvature $k_0\neq0$.

The same as before, for any t near t_0, we choose Γ_t to be the segment of the fixed circle containing Γ_{t_0} which divides the region D (the region between ∂B_4 and $\gamma(t,\cdot)$) into two parts, D_1 and D_2. From the evolution equation (1.3), the length L of Γ_t evolves, at $t=t_0$, by

$$\begin{aligned}\frac{dL}{dt}=&\Phi(x_+,y_+,\theta_+)k_++\Psi(x_+,y_+,\theta_+)\\&+\Phi(x_-,y_-,\theta_-+\pi)k_-+\Psi(x_-,y_-,\theta_-+\pi),\end{aligned}$$

where (x_+,y_+) and (x_-,y_-) are the position vectors of the ends of $\Gamma_t,\theta_+,\theta_-$ are the corresponding angles in (2.12), and k_+,k_- are the corresponding curvatures (by keeping the sign convention as before) of $\gamma(t,\cdot)$ at the ends of Γ_t.

The curve Γ_t divides the curve $\gamma(t,\cdot)$ into two parts $\gamma_1(t,\cdot)$ and $\gamma_2(t,\cdot)$, where $\gamma_1(t,\cdot)$ and Γ_t enclose the region D_1 with the area A_1 (the smaller one). For the same reason as before, we have

$$\frac{dA_1}{dt}=\int_{\gamma_1(t,\cdot)}(\Phi k+\Psi)ds$$

and

$$\frac{dA_2}{dt} = \int_{\gamma_2(t,\cdot)} (\Phi k + \Psi) ds$$

by noticing our sign convention for the curvature of $\gamma(t, \cdot)$.

Fix any point (x_0, y_0) on $\gamma(t_0, \cdot)$. By the assumption $(\mathrm{H}_3)'$ and a direct computation, at $t = t_0$,

$$\begin{aligned}
\frac{d}{dt} \ln G' =& \frac{2}{L}\frac{dL}{dt} - \frac{1}{A_1}\frac{dA_1}{dt} \\
& - \frac{1}{A_2}\frac{dA_2}{dt} + \frac{1}{A_1 + A_2}\frac{d(A_1 + A_2)}{dt} \\
\geq & \frac{2}{L}\Big[\Phi(x_+, y_+, \theta_+)k_+ + \Phi(x_-, y_-, \theta_-)k_- \\
& \qquad + \Psi(x_+, y_+, \theta_+) - \Psi(x_-, y_-, \theta_-)\Big] \\
& - \frac{1}{A_1}\int_{\gamma_1} \Phi(x_0, y_0, \theta) k ds \\
& - \frac{1}{A_1}\left[\int_{\gamma_1} |\Phi(x, y, \theta) - \Phi(x_0, y_0, \theta)| \cdot |k| ds \right. \\
& \qquad \left. + \max_{\substack{(x,y)\in R^2 \\ \theta \in S^1}} |\Psi| \cdot L(\gamma)\right] \\
& - \frac{1}{A_2}\int_{\gamma_2} (\Phi k + \Psi) ds + \frac{1}{A_1 + A_2}\int_{\gamma} (\Phi k + \Psi) ds,
\end{aligned} \tag{2.13}$$

where $L(\Gamma)$ is the length of the curve $\gamma(t_0, \cdot)$.

By using polar coordinates as before, A_1 is the area of the region D_1 on the origin side of Γ_t. Combining with the sign convention on curvature, we have

$$\begin{aligned}
\int_{\gamma_1} \Phi(x_0, y_0, \theta) k ds =& \int_{\theta_- + \pi}^{\theta_+} \Phi(x_0, y_0, \theta) d\theta \\
=& \int_{\theta_- + \pi}^{\theta_+ + \pi} \Phi(x_0, y_0, \theta) d\theta \\
& + \int_{\theta_+ + \pi}^{\theta_+} \Phi(x_0, y_0, \theta) d\theta \\
\leq & \frac{-1}{2}\int_0^{2\pi} \Phi(x_0, y_0, \theta) d\theta \\
& + \max_{\theta \in S^1} \Phi(x_0, y_0, \theta) \cdot (\theta_+ - \theta_-),
\end{aligned} \tag{2.14}$$

where we have used the symmetry condition $(\mathrm{H}_3)'$.

Because the curve $\gamma(t,\cdot)$ lies in the unit disk, and the circle segment Γ_t lies in B_4, one easily knows that the area A_2 is not less than $\frac{1}{4}\pi(4)^2$. Thus by using the total curvature bound of Angenent [3] and the mean value theorem, it follows from (2.13) and (2.14) that, at $t=t_0$,

$$\begin{aligned} \frac{d}{dt}\ln G' \geq & \frac{2}{L}[\Phi(x_+,y_+,\theta_+)k_+ + \Phi(x_-,y_-,\theta_-)k_-] \\ & + \frac{1}{2A_1}\int_0^{2\pi}\Phi(x_0,y_0,\theta)d\theta - C_7[1+(\theta_+-\theta_-)L^{-1} \\ & + (\theta_+-\theta_-)A_1^{-1} + L(\gamma) + L(\gamma)\cdot A_1^{-1}], \end{aligned} \tag{2.15}$$

for some positive constant C_7 depending only on Φ,Ψ and the initial curve γ_0.

Further by applying (2.11),(2.12) and the fact that $\theta_+-\theta_- = |k_0|\cdot L$, we get, at $t=t_0$,

$$\begin{aligned} \frac{d}{dt}\ln G' \quad \geq \quad & -C_6 - C_6L^2\left(\frac{1}{A_1}+\frac{1}{A_2}\right)^2 + \frac{1}{2A_1}\int_0^{2\pi}\Phi(x_0,y_0,\theta)d\theta \\ & -C_7\left[1+\frac{L}{2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)+\frac{1}{2}\frac{L^2}{A_1}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)\right. \\ & \left. +\frac{1}{A_1}L(\gamma)+L(\gamma)\right] \\ \geq \quad & \left[\frac{2\pi}{L^2(\gamma)}\lambda - C_6 - C_7 - C_7L(\gamma)\right] \\ & +\frac{1}{2A_1}\left[\pi\lambda - 4(C_6+C_7)(\bar{G}+L(\gamma))\right], \end{aligned}$$

where we have used the assumption $(\mathrm{H}_1)'$ and the isoperimetric inequality. Therefore there exists a sufficiently small positive constant μ, depending only on Φ,Ψ and the initial curve γ_0, such that if $L(\gamma(t_0,\cdot))\leq\mu$ and $\bar{G}'(t_0)\leq\mu$, then at $t=t_0$,

$$\frac{d}{dt}\ln G' > 0.$$

So for t sufficiently near t_0 and less than t_0, we have

$$\bar{G}'(t) < \bar{G}'(t_0).$$

This completes the proof. q.e.d.

3. Eventual convexity

In this section we shall show the flow always evolves into a convex one before it shrinks to a point. We begin with a standard rescaling argument as in the proof of Theorem 9.1 in [3]. Let

$$\gamma(t,\cdot) : [0, t_{\max}) \times S^1 \to R^2$$

be an evolving curve of (1.3). Our assumption that $t_{\max} < +\infty$ implies that the curvature $k(t,\cdot)$ is unbounded as $t \to t_{\max}$. Thus we can find a sequence of points $(t_n, u_n) \in (0, t_{\max}) \times S^1$ such that $t_n \nearrow t_{\max}$ and

$$|k(t,u)| \leq |k(t_n, u_n)|, 0 < t \leq t_n, u \in S^1$$

holds for $n = 1, 2, \ldots$.

Denote (x_n, y_n) to be the position vector of the point $\gamma(t_n, u_n)$. Put $\varepsilon_n = |k(t_n, u_n)|^{-1}$, and define a new, rescaled version of the evolving curve $\gamma(t,\cdot)$ by

$$\gamma_n(t,\cdot) = \phi_n(\gamma(t_n + \varepsilon_n^2 t, \cdot)) : [-t_n/\varepsilon_n^2, (t_{\max} - t_n)/\varepsilon_n^2) \times S^1 \to R^2,$$

where $\phi_n(x,y) = (x - x_n, y - y_n)/\varepsilon_n$, for $(x,y) \in R^2$. Then the family of curves $\gamma_n(t,\cdot)$ satisfies

$$\frac{\partial \gamma_n}{\partial t} = (\Phi(x_n + \varepsilon_n x, y_n + \varepsilon_n y, \theta) k_n + \varepsilon_n \Psi(x_n + \varepsilon_n x, y_n + \varepsilon_n y, \theta)) N, \tag{3.1}$$

where the curvature k_n of $\gamma_n(t,\cdot)$ satisfies $|k_n(t,u)| \leq 1$ for all $u \in S^1$ and $t \in (-t_n/\varepsilon_n^2, 0]$.

Introduce an arclength parametrization

$$\gamma_n(t,s) : [-t_n/\varepsilon_n^2, (t_{\max} - t)/\varepsilon_n^2) \times R \to R^2$$

of the family of curves $\gamma_n(t,\cdot)$. This implies that, for

$$-t_n/\varepsilon_n^2 \leq t < (t_{\max} - t)/\varepsilon_n^2,$$

$\gamma_n(t,\cdot)$ is an $L(\gamma_n(t,\cdot))$ periodic function of $s \in R$. By the same arguments as in the proof of Theorem 7.1 in [3], it follows from the uniformly boundedness of k_n that the curvature k_n is uniformly Hölder continuous. The normal velocities must therefore also be uniformly Hölder continuous. Further one can get the uniformly boundedness for the derivatives of k_n on compact sets of $(-\infty, 0] \times R$. Thus one can extract a subsequence of $\gamma_n(t,\cdot)$ which converges to a family of complete curves

$\gamma_\infty : (-\infty, 0] \times R \to R^2$ in C^3 topology on every compact subset of $(-\infty, 0] \times R$. Moreover, the limit family $\gamma_\infty(t, \cdot)$ is a classical solution of

$$\frac{\partial \gamma_\infty}{\partial t} = \Phi(\bar{x}, \bar{y}, \theta) k_\infty N, \text{ on } (-\infty, 0] \times R, \tag{3.2}$$

and $|k_\infty(t, \cdot)| \leq 1$ everywhere with equality somewhere at $t = 0$. Here $(\bar{x}, \bar{y})$ is the position vector of the shrinking point of $\gamma(t, \cdot), k_\infty$ is the curvature of $\gamma_\infty(t, \cdot)$.

The next lemma extends the corresponding results of Altschuler [2], Angenent and Gurtin [6].

Lemma 3.1. *$\gamma_\infty(t, \cdot)$ is strictly convex.*

Proof. From Angenent's paper [4], one knows that, for any $t \in (0, t_{\max})$, the curve $\gamma(t, \cdot)$ has at most a finite number of nodes which are points on $\gamma(t, \cdot)$ with the vanishing normal velocity, and this number does not increase with time. After rescaling, the nodes of $\gamma(t, \cdot)$ are 1-1 corresponding to the nodes of $\gamma_n(t, \cdot)$. Consider the function $K(t) : [0, t_{\max}) \to R$ defined by

$$\begin{aligned} K(t) &= \int\limits_{\gamma(t,\cdot) \cap \{\Phi k + \Psi \geq 0\}} k ds - \int\limits_{\gamma(t,\cdot) \cap \{\Phi k + \Psi \leq 0\}} k ds \\ &\equiv \int\limits_{\gamma(t,\cdot)} (\operatorname{sgn}\{\Phi k + \Psi\}) k ds. \end{aligned}$$

By the assumption $(H_2)'$, it is easy to see

$$K(t) \geq -\lambda^{-1} \max_{\substack{(x,y) \in R^2 \\ \theta \in S^1}} |\Psi| \cdot \int_{\gamma(t,\cdot)} ds. \tag{3.3}$$

Noticing that the value of $K(t)$ is unchanging after rescaling, one has

$$\begin{aligned} K(t_n + \varepsilon_n^2 t) &= \int\limits_{\gamma_n(t,\cdot)} (\operatorname{sgn}\{\Phi k_n + \varepsilon_n \Psi\}) k_n ds \\ &= \operatorname{sgn}\left(k_n + \varepsilon_n \frac{\Psi}{\Phi}\right) \int\limits_{\gamma_n(t,\cdot)} \left(k_n + \varepsilon_n \frac{\Psi}{\Phi}\right) ds \\ &\quad - \operatorname{sgn}\left(k_n + \varepsilon_n \frac{\Psi}{\Phi}\right) \int\limits_{\gamma_n(t,\cdot)} \left(\varepsilon_n \frac{\Psi}{\Phi}\right) ds, \\ &\qquad \text{for } \ t \in [-t_n/\varepsilon_n^2, (t_{\max} - t_n)/\varepsilon_n^2). \end{aligned}$$

Appling (1.4), (1.5) and (1.7) to equation (3.1), we get

$$\begin{aligned}
&\frac{d}{dt}(K(t_n+\varepsilon_n^2 t)) + \frac{d}{dt}\left(\operatorname{sgn}\left(k_n+\varepsilon_n\frac{\Psi}{\Phi}\right)\int_{\gamma_n(t,\cdot)}\varepsilon_n\frac{\Psi}{\Phi}ds\right)\\
&\quad= \operatorname{sgn}\left(k_n+\varepsilon_n\frac{\Psi}{\Phi}\right)\int\limits_{\gamma_n(t,\cdot)}\frac{d}{dt}\left(\left(k_n+\varepsilon_n\frac{\Psi}{\Phi}\right)ds\right)\\
&\quad= \operatorname{sgn}\left(k_n+\varepsilon_n\frac{\Psi}{\Phi}\right)\int\limits_{\gamma_n(t,\cdot)}\frac{d}{dt}(k_n ds)\\
&\qquad+\operatorname{sgn}\left(k_n+\varepsilon_n\frac{\Psi}{\Phi}\right)\int\limits_{\gamma_n(t,\cdot)}\varepsilon_n\frac{d}{dt}\left[\left(\frac{\Psi}{\Phi}\right)ds\right],
\end{aligned}$$

where

$$\begin{aligned}
\int\limits_{\gamma_n(t,\cdot)}\frac{d}{dt}(k_n ds) &= \int\limits_{\gamma_n(t,\cdot)}\left[\frac{\partial^2}{\partial s^2}(\Phi k_n+\varepsilon_n\Psi)+k_n^2(\Phi k_n+\varepsilon_n\Psi)\right]ds\\
&\quad+\int\limits_{\gamma_n(t,\cdot)}k_n[-k_n(\Phi k_n+\varepsilon_n\Psi)]ds\\
&= \int\limits_{\gamma_n(t,\cdot)}\frac{\partial^2}{\partial s^2}(\Phi k_n+\varepsilon_n\Psi)ds,
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon_n\int\limits_{\gamma_n(t,\cdot)}\frac{d}{dt}\left[\left(\frac{\Psi}{\Phi}\right)ds\right]\\
&\quad=\varepsilon_n\int\limits_{\gamma_n(t,\cdot)}\left\langle\left(\left(\frac{\Psi}{\Phi}\right)_x,\left(\frac{\Psi}{\Phi}\right)_y\right),\varepsilon_n(\Phi k_n+\varepsilon_n\Psi)N\right\rangle ds\\
&\qquad+\varepsilon_n\int\limits_{\gamma_n(t,\cdot)}\left(\frac{\Psi}{\Phi}\right)_\theta\left(-\frac{\partial}{\partial s}(\Phi k_n+\varepsilon_n\Psi)\right)ds\\
&\qquad+\varepsilon_n\int\limits_{\gamma_n(t,\cdot)}\frac{\Psi}{\Phi}[-k_n(\Phi k_n+\varepsilon_n\Psi)]ds
\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon_n \int\limits_{\gamma_n(t,\cdot)} \left\langle \left(\left(\frac{\Psi}{\Phi}\right)_x, \left(\frac{\Psi}{\Phi}\right)_y \right), \varepsilon_n(\Phi k_n + \varepsilon_n \Psi) N \right\rangle ds \\
&\quad + \varepsilon_n \int\limits_{\gamma_n(t,\cdot)} (\Phi k_n + \varepsilon_n \Psi) \cdot \left[\left\langle \left(\left(\frac{\Psi}{\Phi}\right)_{x\theta}, \left(\frac{\Psi}{\Phi}\right)_{y\theta} \right), \varepsilon_n T \right\rangle \right. \\
&\quad \left. + \left(\frac{\Psi}{\Phi}\right)_{\theta\theta} k_n \right] ds + \varepsilon_n \int\limits_{\gamma_n(t,\cdot)} \frac{\Psi}{\Phi}[-k_n(\Phi k_n + \varepsilon_n \Psi)] ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{d}{dt}\left[K(t_n + \varepsilon_n^2 t) + \operatorname{sgn}\left(k_n + \varepsilon_n \frac{\Psi}{\Phi}\right) \int_{\gamma_n(t,\cdot)} \varepsilon_n \frac{\Psi}{\Phi} ds \right] \\
&\quad = -2 \sum_{p|(\Phi k_n + \varepsilon_n \Psi)(t,p)=0} \left| \frac{\partial}{\partial s}(\Phi k_n + \varepsilon_n \Psi) \right| \\
&\qquad + \operatorname{sgn}\left(k_n + \varepsilon_n \frac{\Psi}{\Phi}\right) \int_{\gamma_n(t,\cdot)} \varepsilon_n \frac{d}{dt}\left[\left(\frac{\Psi}{\Phi}\right) ds \right] \\
&\quad \leq -2 \sum_{p|(\Phi k_n + \varepsilon_n \Psi)(t,p)=0} \left| \frac{\partial}{\partial s}(\Phi k_n + \varepsilon_n \Psi) \right| \\
&\qquad + C_1 \varepsilon_n \int_{\gamma_n(t,\cdot)} (\Phi k_n^2 + \varepsilon_n^2) ds
\end{aligned}
$$

for some positive constant C_1 depending only on Φ and Ψ.

By using (1.5) to equation (3.1), we know

$$
\begin{aligned}
\varepsilon_n \int_{\gamma_n(t,\cdot)} \Phi k_n^2 ds &= \varepsilon_n \int_{\gamma_n(t,\cdot)} k_n(\Phi k_n + \varepsilon_n \Psi) ds - \varepsilon_n^2 \int\limits_{\gamma_n(t,\cdot)} k_n \Psi ds \\
&= -\varepsilon_n \frac{d}{dt}(L(\gamma_n(t,\cdot))) - \varepsilon_n^2 \int\limits_{\gamma_n(t,\cdot)} k_n \Psi ds \\
&\leq -\varepsilon_n \frac{d}{dt}(L(\gamma_n(t,\cdot))) \\
&\quad + \varepsilon_n \left[\frac{1}{2} \int\limits_{\gamma_n(t,\cdot)} \Phi k_n^2 ds + \frac{1}{2} \int\limits_{\gamma_n(t,\cdot)} \frac{\varepsilon_n^2 \Psi^2}{\Phi} ds \right].
\end{aligned}
$$

Thus

$$\varepsilon_n \int\limits_{\gamma_n(t,\cdot)} \Phi k_n^2 \leq -2\varepsilon_n \frac{d}{dt}(L(\gamma_n(t,\cdot))) + C_2\varepsilon_n^3 \int\limits_{\gamma_n(t,\cdot)} ds$$

for some positive constant C_2 depending only on Φ and Ψ.

So

$$\begin{aligned}\frac{d}{dt}&\left[K(t_n+\varepsilon_n^2 t) + \operatorname{sgn}\left(k_n + \varepsilon_n \frac{\Psi}{\Phi}\right) \int\limits_{\gamma_n(t,\cdot)} \varepsilon_n \frac{\Psi}{\Phi} ds\right] \\ &\leq -2 \sum_{p|(\Phi k_n+\varepsilon_n\Psi)(t,p)=0} \left|\frac{\partial}{\partial s}(\Phi k_n + \varepsilon_n \Psi)\right| \\ &\quad - 2C_1\varepsilon_n \frac{d}{dt}(L(\gamma_n(t,\cdot))) \\ &\quad + C_1(C_2+1)\varepsilon_n^3 \int\limits_{\gamma_n(t,\cdot)} ds, \\ &\qquad \text{for } t \in \left(-t_n/\varepsilon_n^2, (t_{\max}-t_n)/\varepsilon_n^2\right). \end{aligned} \tag{3.4}$$

Define a function $F : [0, t_{\max}) \to R$ by

$$\begin{aligned} F(t) &= K(t) + \int\limits_{\gamma(t,\cdot)} \operatorname{sgn}(\Phi k + \Psi) \cdot \frac{\Psi}{\Phi} ds + 2C_1 \int\limits_{\gamma(t,\cdot)} ds \\ &\quad - C_1(C_2+1) \int_0^t \left(\int\limits_{\gamma(\sigma,\cdot)} ds \right) d\sigma. \end{aligned}$$

Combining (3.3), it is obvious that $F(t)$ has a lower bound on $[0, t_{\max})$, and (3.4) can be written as, for $t \in (-t_n/\varepsilon_n^2, (t_{\max}-t_n)/\varepsilon_n^2)$,

$$\frac{d}{dt}(F(t_n+\varepsilon_n^2 t)) \leq -2 \sum_{p|(\Phi k_n+\varepsilon_n\Psi)(t,p)=0} \left|\frac{\partial}{\partial s}(\Phi k_n + \varepsilon_n\Psi)\right|.$$

Therefore $F(t)$ is a nonincreasing function on $[0, t_{\max})$. Integrating the above inequality from $-t_n/\varepsilon_n$ to 0, it follows that

$$\begin{aligned} &F(t_n) - F(t_n - \varepsilon_n t_n) \\ &\quad \leq -2 \int_{-t_n/\varepsilon_n}^0 \sum_{p|(\Phi k_n+\varepsilon_n\Psi)(t,p)=0} \left|\frac{\partial}{\partial s}(\Phi k_n + \varepsilon_n \Psi)\right| dt. \end{aligned} \tag{3.5}$$

whose both sides tend to 0 as $n \to +\infty$.

Since the total curvature of the evolving curve $\gamma(t,\cdot)(t \in [0, t_{\max}))$ is bounded (c.f. [3]), the limit $\gamma_\infty(t,\cdot)$ must also have uniform boundedness for the total curvature. If $\gamma_\infty(t,\cdot)$ is compact, then the Angenent's results [4] for (3.2) yields that the number of inflection points of $\gamma_\infty(t,\cdot)$ is finite, does not increase with time, and drops whenever the curve has a degenerate inflection point. But (3.5) tells us

$$\int_{-\infty}^{0} \sum_{p|\Phi k_\infty(t,p)=0} \left|\frac{\partial}{\partial s}(\Phi k_\infty)\right| dt = \int_{-\infty}^{0} \sum_{p|k_\infty(t,p)=0} \Phi \left|\frac{\partial k_\infty}{\partial s}\right| dt = 0,$$

i.e., any inflection point for the (compact, by assumption) limit curve $\gamma_\infty(t,\cdot)$ must be degenerate. So the curvature k_∞ of $\gamma_\infty(t,\cdot)$ has only one sign.

If $\gamma_\infty(t,\cdot)$ is not compact, we have to be a little more subtle. Since the number of nodes for the evolving curve $\gamma(t,\cdot)$ is uniformly bounded (c.f. [4]), for each t, we can divide the limit $\gamma_\infty(t,\cdot)$ into three pieces such that $k_\infty(t,\cdot)$ cannot change sign from "+" to "−" on either of the two noncompact pieces. Then it follows from strong maximum principle that the curve $\gamma_\infty(t,\cdot)$ hasn't any inflection point outside some suitable large compact subarc of $\gamma_\infty(t,\cdot)$. Thus by using the Sturmian theorem (c.f.[4]), we know that the number of inflection points of $\gamma_\infty(t,\cdot)$ is finite, does not increase with time, and drops whenever the curve has a degenerate inflection point. Also (3.5) tells us that any inflection point for the limit $\gamma_\infty(t,\cdot)$ must be degenerate, and then the curvature k_∞ of $\gamma_\infty(t,\cdot)$ has only one sign.

Hence we may as well take $k_\infty > 0$, and $\gamma_\infty(t,\cdot)$ must be strictly convex. q.e.d.

The next lemma is crucial in this section.

Lemma 3.2. *$\gamma_\infty(t,\cdot)$ is compact for every $t \in (-\infty, 0]$.*

Proof. Since the evolving curve $\gamma(t,\cdot)$ remains embedded, the rescaled curves $\gamma_n(t,\cdot)$ also remain embedded, so the limit $\gamma_\infty(t,\cdot)$ must be a family of embedded convex curves. Thus for each t, there are only two posibilities: $\gamma_\infty(t,\cdot)$ is either a compact convex curve, or an unbounded convex curve with its total curvature $\leq \pi$.

Now we want to exclude the second posibility.

Suppose that $\gamma_\infty(t,\cdot)$ is an unbounded convex curve with its total curvature $\leq \pi$. For any fixed t, we can choose a Cartesian coordinate system (x, y) such that the curve $\gamma_\infty(t,\cdot)$ is the graph of a nonnegative strictly convex function $y = y(x)$, with $y(0) = 0$ defined on some interval (a, b). By the proof of Theorem 9.1 in Angenent's paper [3], it actually implies that the total curvature of $\gamma_\infty(t, \cdot)$ must be exactly π. This says

$$\lim_{x \to a} y'(x) = -\infty \quad \text{and} \quad \lim_{x \to b} y'(x) = +\infty$$

(here a, b may be infinity).

It follows directly that

$$\lim_{x \to a} \frac{y(x)}{|x|} = +\infty \quad \text{and} \quad \lim_{x \to b} \frac{y(x)}{|x|} = +\infty$$

Thus, for any arbitrarily small positive constant ε, there is a positive constant x_ε such that

$$y(x) \geq \frac{1}{\varepsilon} x_\varepsilon \quad \text{where } \ x \in (a, b) \ \text{ and satisfies } \ |x| \geq x_\varepsilon.$$

Consider the straight segment Γ to be the intersection of the horizontal line $l_\varepsilon = \left\{(x, y) \,\middle|\, x \in R, y = \frac{1}{\varepsilon} x_\varepsilon \right\}$ with the upper domain enclosed by $y = y(x)$. It is clear that the length L of Γ is not greater than $2x_\varepsilon$. But by convexity, the area enclosed by l_ε, and the graph of $y = y(x)$ is not less than $\frac{1}{2} L \left(\frac{x_\varepsilon}{\varepsilon}\right)$. Then the ratio $G(\Gamma)$ defined by (2.1) satisfies

$$G(\Gamma) \leq L^2 \cdot \frac{1}{\frac{1}{2} L \left(\frac{x_\varepsilon}{\varepsilon}\right)} \leq 4\varepsilon.$$

Since $\gamma_\infty(t, \cdot)$ is to be the limit, the original curve comes arbitrarily close to it after translating, rotating, and dilating, all of which do not affect the ratio (2.1). If the upper domain enclosed by $y = y(x)$ is corresponding to the limit of the interior domain of a series of the original curves $\gamma(t_n, \cdot)$, we must have $\bar{G}$ of $\gamma(t, \cdot)$ which can be arbitrarily small for some $t = t_n$, contradicting to Theorem 2.4. Also, if the upper domain enclosed by $y = y(x)$ is corresponding to the limit of the exterior domain of a series of the original curves $\gamma(t_n, \cdot)$, we must have $\bar{G}'$ of $\gamma(t, \cdot)$ (without loss of generality, we may assume that all $\gamma(t, \cdot)$ lie in the unit disk of R^2), which can be arbitrarily small for some $t = t_n$, contradicting to Theorem 2.4$'$.

Therefore $\gamma_\infty(t, \cdot)$ must be a compact convex curve. q.e.d.

Remark 3.3. If the function $\Phi(x, y, \theta)$ is independent of the last variable θ, then Huisken's monotonicity principle [16] and the classification of Abresch and Langer [1] show that $\gamma_\infty(t, \cdot)$ is a circle, which is of course just what we want. But for anisotropic cases, we must work harder.

Theorem 3.4. *Assume the conditions $(H_1)'$,$(H_2)'$ and $(H_3)'$, and let $\gamma(t,\cdot):[0,t_{\max})\times S^1\to R^2$ be an evolving embedded curve of (1.2). If $t_{\max}<+\infty$, then there is a $t_0<t_{\max}$ such that, for $t>t_0, \gamma(t,\cdot)$ is strictly convex.*

Proof. Let $\gamma_n(t,\cdot)$ be the rescaled sequence constructed at the beginning of this section, and let $\gamma_\infty(t,\cdot)$ be the limit. We may assume $\gamma_n(t,\cdot)$, itself, converges to $\gamma_\infty(t,\cdot)$ in C^3 topology on every compact set of $(-\infty,0]\times R$.

Lemmas 3.1 and 3.2 imply that there exist a postive number n_0 and a positive constant δ such that, for $n\geq n_0$, the curvature of the curve $\gamma_n(0,\cdot)$ satisfies

$$k_n(0,\cdot)\geq\delta>0, \quad \text{on } \gamma_n(0,\cdot).$$

So the curvature of the curve $\gamma(t_n,\cdot)$ satisfies

$$k_n(t_n,u)\geq\delta/\varepsilon_n, \quad \text{for all } u\in S^1 \text{ and } n\geq n_0. \tag{3.6}$$

Now we compute the evolution of $\Phi(x,y,\theta)$. By (1.3),(1.4) and (1.7), we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}(\Phi(x,y,\theta)) &= \Phi_x\frac{\partial x}{\partial t}+\Phi_y\frac{\partial y}{\partial t}+\Phi_\theta\frac{\partial\theta}{\partial t}\\
&= [(\Phi_x,\Phi_y)\cdot N](\Phi k+\Psi)-\Phi_\theta\frac{\partial}{\partial s}(\Phi k+\Psi)\\
&= [(\Phi_x,\Phi_y)\cdot N](\Phi k+\Psi)-\Phi_\theta\frac{\partial}{\partial s}(\Phi k)\\
&\quad -\Phi_\theta[(\Psi_x,\Psi_y)\cdot T+\Psi_\theta k],\\
\frac{\partial}{\partial s}(\Psi(x,y,\theta)) &= [(\Psi_x,\Psi_y)\cdot T]+\Psi_\theta k,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial s^2}(\Psi(x,y,\theta)) &= [(\Psi_x,\Psi_y)\cdot N]k+T\cdot\left[\begin{pmatrix}\Psi_{xx} & \Psi_{xy}\\ \Psi_{xy} & \Psi_{yy}\end{pmatrix}\cdot T\right]\\
&\quad +[(\Psi_{x\theta},\Psi_{y\theta})\cdot T]k+\frac{\Psi_\theta}{\Phi}\cdot\frac{\partial}{\partial s}(\Phi k)\\
&\quad +\left[\left(\left(\frac{\Psi_\theta}{\Phi}\right)_x,\left(\frac{\Psi_\theta}{\Phi}\right)_y\right)\cdot T\right]\Phi k+\left(\frac{\Psi_\theta}{\Phi}\right)_\theta\Phi k^2,
\end{aligned}$$

so

$$\begin{aligned}\frac{\partial}{\partial t}(\Phi k) &= k\frac{\partial \Phi}{\partial t} + \Phi\frac{\partial k}{\partial t}\\ &= k\Big\{[(\Phi_x, \Phi_y)\cdot N](\Phi k + \Psi) - \Phi_\theta \frac{\partial}{\partial s}(\Phi k)\\ &\qquad -\Phi_\theta[(\Psi_x, \Psi_y)\cdot T + \Psi_\theta k]\Big\}\\ &\quad +\Phi\left\{\frac{\partial^2}{\partial s^2}(\Phi k) + k^2(\Phi k + \Psi)\right.\\ &\qquad +[(\Psi_x, \Psi_y)\cdot N]k + T\cdot\left[\begin{pmatrix}\Psi_{xx} & \Psi_{xy}\\ \Psi_{xy} & \Psi_{yy}\end{pmatrix} T\right]\\ &\qquad +[(\Psi_{x\theta}, \Psi_{y\theta})\cdot T]k + \frac{\Psi_\theta}{\Phi}\cdot\frac{\partial}{\partial s}(\Phi k)\\ &\qquad \left. + \left[\left(\left(\frac{\Psi_\theta}{\Phi}\right)_x, \left(\frac{\Psi_\theta}{\Phi}\right)_y\right)\cdot T\right]\Phi k + \left(\frac{\Psi_\theta}{\Phi}\right)_\theta \Phi k^2\right\},\end{aligned}$$

this is

$$\begin{aligned}\frac{\partial}{\partial t}(\Phi k) =& \Phi\frac{\partial^2}{\partial s^2}(\Phi k) + \left(\frac{\Psi_\theta}{\Phi} - k\Phi_\theta\right)\frac{\partial}{\partial s}(\Phi k)\\ &+ \Phi k^3 + A_1 k^2 + A_2 k + A_3,\end{aligned} \tag{3.7}$$

hereA_1, A_2, and A_3 are some bounded smooth functions.

Denote $(\Phi k)_{\min}(t)$ to be the minimum of the function $\Phi(x, y, \theta)k$ on $\gamma(t, \cdot)$. Then (3.7) implies, (by $(\mathrm{H}_2)'$),

$$\frac{d}{dt}(\Phi k)_{\min} \geq \lambda^2((\Phi k)_{\min})^3 - C_1[(\Phi k)^2_{\min} + (\Phi k)_{\min} + 1] \tag{3.8}$$

for all $t \in (0, t_{\max})$. Here C_1 is a positive constant. By combining (3.6) and (3.8), we deduce that there is a sufficient large number $n_1(\geq n_0)$ such that, for $t \in [t_{n_1}, t_{\max})$,

$$(\Phi k)_{\min}(t) \geq (\Phi k)_{\min}(t_{n_1}) > 0. \tag{3.9}$$

Therefore the curve $\gamma(t, \cdot)$ is strictly convex for each $t \in [t_{n_1}, t_{\max})$.

q.e.d.

Remark 3.5. In a recently joint paper [7] with Chou, we proved the above theorem for the case where Φ and Ψ are independent of the first two variables, x and y. The arguments in that paper [7] are inspired by the ideas of Grayson [12], which are heavily depending on the translation invariance of the flow.

4. Asymptotic behavior

Let $\gamma(t,\cdot):[0,t_{\max})\times S^1\to R^2$ be an evolving embedded curve of (1.2) with $t_{\max}<+\infty$. We have proved that $\gamma(t,\cdot)$ will become convex after some time and have obtained the isoperimetric estimates. Now we prepare to state and prove the asymptotic shape of the shrinking curve.

For any fixed $(x_0,y_0)\in R^2, \Phi(x_0,y_0,\theta)$ is a function on S^1 satisfying the symmetry condition $(\mathrm{H}_3)'$. From the works of Gage [9] and Gage & Li [11], one can write $\Phi(x_0,y_0,\theta)$ as

$$\Phi(x_0,y_0,\theta)=\bar{h}(\theta)/\bar{k}(\theta), \tag{4.1}$$

where $\bar{h},\bar{k}$ are the support function and curvature of some smooth, symmetric, strictly convex curve $\mathcal{J}$, and $\mathcal{J}$ is uniquely determined up to a dilation.

Definition 4.1. We call the curve $\mathcal{J}$ a Minkowski isoperimetrix associated $\Phi(x_0,y_0,\theta)$.

The definition is a generalization of the concept of circle. In fact, if $\Phi(x_0,y_0,\theta)$ is a positive constant, the corresponding Minkowski isoperimetrix is just a circle.

Main Theorem. *Assume the conditions $(H_1)'$,$(H_2)'$ and $(H_3)'$, and let $\gamma(t,\cdot):[0,t_{\max})\times S^1\to R^2$ be an evolving embedded curve of (1.2) with $t_{\max}<+\infty$. Then as $t\to t_{\max}$, the rescaled evolving curve $(t_{\max}-t)^{-1/2}(\gamma(t,\cdot)-(\bar{x},\bar{y}))$ converges to a Minkowski isoperimetrix associated $\Phi(\bar{x},\bar{y},\theta)$ in C^∞ topology, where $(\bar{x},\bar{y})$ is the position vector of the shrinking point of $\gamma(t,\cdot)$.*

Corollary 4.2. *Let $\gamma(t,\cdot):[0,t_{\max})\times S^1\to M$ be an evolving embedded curve of the curve shortening problem. If $t_{\max}<+\infty$, then $\gamma(t,\cdot)$ shrinks to a point with round limiting shape, and with C^∞ convergence.*

Proof. From the computation in the proof of Theorem 3.1 in Oaks' paper [18], one knows the corresponding $\Phi(x_0,y_0,\theta)$ for the curve shortening problem is independent of the last variable θ. So the corresponding Minkowski isopermetrix must be a circle. q.e.d.

Proof of Main Theorem. By Theorem 3.4, we may assume that $\gamma(t,\cdot)$ is strictly convex for all $t\in[0,t_{\max})$. Thus one can parametrize the curve as (t,θ).

Write

$$\gamma=\gamma(t,\theta):[0,t_{\max})\times[0,2\pi]\to R^2,$$

$$k = k(t,\theta) : [0, t_{\max}) \times [0, 2\pi] \to R^+.$$

The evolution equation (1.4) can be written as

$$(4.2) \qquad \begin{aligned} \frac{\partial k}{\partial t} &= k^2[\Phi(x(t,\theta), y(t,\theta), \theta)k + \Psi(x(t,\theta), y(t,\theta), \theta)]_{\theta\theta} \\ &\quad + k^2[\Phi(x(t,\theta), y(t,\theta), \theta)k + \Psi(x(t,\theta), y(t,\theta), \theta)] \end{aligned}$$

(c.f. (2.20) in [14]), where $\frac{\partial k}{\partial t}$ is the derivative of k with respect to t holding θ fixed.

The support function, $h(t,\theta) = \langle \gamma(t,\theta), -N \rangle$ is the distance from the origin to the tangent line of the convex curve $\gamma(t,\cdot)$. The evolution equation for the support function is

$$(4.3) \qquad \frac{\partial h}{\partial t} = -[\Phi(x(t,\theta), y(t,\theta), \theta)k + \Psi(x(t,\theta), y(t,\theta), \theta)].$$

In fact

$$\left.\frac{\partial \gamma(t, \theta(t,u))}{\partial t}\right|_u = \left.\frac{\partial \gamma}{\partial t}\right|_\theta + \frac{\partial \gamma}{\partial \theta} \cdot \left.\frac{\partial \theta}{\partial t}\right|_u .$$

Then

$$\begin{aligned} \left.\frac{\partial h}{\partial t}\right|_\theta &= \left\langle \left.\frac{\partial \gamma}{\partial t}\right|_\theta, -N \right\rangle \quad \text{(by noting } -N = (\sin\theta, -\cos\theta)) \\ &= \left\langle \left.\frac{\partial \gamma}{\partial t}\right|_u, -N \right\rangle \\ &= -[\Phi(x(t,\theta), y(t,\theta), \theta)k + \Psi(x(t,\theta), y(t,\theta), \theta)]. \end{aligned}$$

From the convexity of $\gamma(t,\cdot)$ and the isoperimetric estimate in Theorem 2.4, it follows that there exists a positive constant C such that

$$(4.4) \qquad D(t)/r_{\text{in}}(t) \le C, \qquad \text{for all } t \in [0, t_{\max}),$$

where $D(t)$ is the diameter of $\gamma(t,\cdot)$, and r_{in} is the inradius of $\gamma(t,\cdot)$.

Rescale the curve $\gamma(t,\cdot)$ by

$$\widetilde{\gamma}(t,\theta) = \frac{1}{\sqrt{2(t_{\max} - t)}}(\gamma(t,\theta) - (\bar{x}, \bar{y})).$$

Then the corresponding rescaled curvatuve

$$\widetilde{k}(t,\theta) = \sqrt{2(t_{\max} - t)} \cdot k(t,\theta),$$

and the enclosed area of $\widetilde{\gamma}(t,\cdot)$ (by (1.6)),

$$
\begin{aligned}
\widetilde{A}(t) =& \frac{1}{2(t_{\max}-t)}A(t)\\
=& \frac{1}{2(t_{\max}-t)}\int_t^{t_{\max}}\left(\int_{\gamma(\tau,\cdot)}(\Phi k+\Psi)ds\right)d\tau\\
=& \frac{1}{2}\int_0^{2\pi}\Phi(\bar{x},\bar{y},\theta)d\theta\\
&+\frac{1}{2(t_{\max}-t)}\int_t^{t_{\max}}\left[\int_{\gamma(\tau,\cdot)}(\Phi(x,y,\theta)-\Phi(\bar{x},\bar{y},\theta))\times kds\right.\\
&\qquad\left.+\int_{\gamma(\tau,\cdot)}\Psi(x,y,\theta)ds\right]d\tau\\
\to& \frac{1}{2}\int_0^{2\pi}\Phi(\bar{x},\bar{y},\theta)d\theta, \quad \text{as } t\to t_{\max},
\end{aligned}
\tag{4.5}
$$

by noticing,

$$
\begin{aligned}
&\left|\int_{\gamma(\tau,\cdot)}(\Phi(x,y,\theta)-\Phi(\bar{x},\bar{y},\theta))\times kds\right|\\
&\quad =\leq \max_{\substack{(x,y)\in\gamma(\tau,\cdot)\\ \theta\in[0,2\pi]}}|\Phi(x,y,\theta)-\Phi(\bar{x},\bar{y},\theta)|\cdot\int_{\gamma(\tau,\cdot)}kds\\
&\quad = \max_{\substack{(x,y)\in\gamma(\tau,\cdot)\\ \theta\in[0,2\pi]}}|\Phi(x,y,\theta)-\Phi(\bar{x},\bar{y},\theta)|\cdot 2\pi,
\end{aligned}
$$

$$
\left|\int_{\gamma(\tau,\cdot)}\Psi(x,y,\theta)ds\right|\leq \max_{\substack{(x,y)\in\gamma(\tau,\cdot)\\ \theta\in[0,2\pi]}}|\Psi(x,y,\theta)|\cdot\int_{\gamma(\tau,\cdot)}ds
$$

and the fact that $\gamma(t,\cdot)$ shrinks to the point $(\bar{x},\bar{y})$.

So by combining (4.4) and (4.5),there are some positive constants δ and Λ such that

$$
0<\delta\leq\widetilde{r}_{\text{in}}(t)\leq\widetilde{D}(t)\leq\Lambda<+\infty \tag{4.6}
$$

for all $t\in[0,t_{\max})$. Here $\widetilde{D}(t),\widetilde{r}_{\text{in}}(t)$ are the diameter, the inradius of $\widetilde{\gamma}(t,\cdot)$ respectively.

From (3.6) and (3.9), as well as $(\mathrm{H}_1)'$, without loss of generality, one can assume that

$$\begin{cases} \frac{1}{2}\Phi(x(t,\theta), y(t,\theta), \theta)k(t,\theta) \geq |\Psi(x(t,\theta), y(t,\theta), \theta)|, \\ k(t,\theta) \geq 1 \end{cases} \tag{4.7}$$

hold for all $t \in [0, t_{\max}), \theta \in [0, 2\pi]$.

The remainder proof is divided into three steps.

Step 1. Estimate the upper bound of the rescaled curvature $\widetilde{k}$.

Let $\bar{t}$ be any fixed time in $(0, t_{\max})$, and let $B(x_0, y_0; r_{\text{in}}(\bar{t}))$ be a disk with center at (x_0, y_0), radius $r_{\text{in}}(\bar{t})$, and contained in the region enclosed by $\gamma(\bar{t}, \cdot)$. By equations (1.3) and (4.7), the disk $B(x_0, y_0; r_{\text{in}}(\bar{t}))$ is also contained in all regions enclosed by the curves $\gamma(t, \cdot)$ with $t \in [0, \bar{t})$.

In this step, without loss of generality, we may choose the coordinate such that (x_0, y_0) to be the origin of R^2.

We use a trick due to Chou [19]. Consider a positive function $\phi(t,\theta)$ on $[0, \bar{t}] \times [0, 2\pi]$,

$$\phi(t,\theta) = \frac{-h_t(t,\theta)}{h(\theta,t) - \frac{1}{2}r_{\text{in}}(\bar{t})},$$

which is well defined by noting

$$h(\theta,t) - \frac{1}{2}r_{\text{in}}(\bar{t}) \geq \frac{1}{2}r_{\text{in}}(\bar{t}) \tag{4.8}$$

for all $t \in [0, \bar{t}], \theta \in [0, 2\pi]$.

Suppose that $\phi(t,\theta)$ attains its maximum at $t = t_0$ and $\theta = \theta_0$. We distinguish two cases : $t_0 > 0$ and $t_0 = 0$.

If $t_0 > 0$, then at (t_0, θ_0),

$$0 = \frac{\partial\phi}{\partial\theta} = \frac{-h_{\theta t}}{h - \frac{1}{2}r_{\text{in}}(\bar{t})} + \frac{h_t h_\theta}{\left(h - \frac{1}{2}r_{\text{in}}(\bar{t})\right)^2}, \tag{4.9}$$

$$0 \leq \frac{\partial\phi}{\partial t} = \frac{-h_{tt}}{h - \frac{1}{2}r_{\text{in}}(\bar{t})} + \frac{(h_t)^2}{\left(h - \frac{1}{2}r_{\text{in}}(\bar{t})\right)^2}, \tag{4.10}$$

$$
\begin{aligned}
0 \geq & \frac{\partial^2 \phi}{\partial \theta^2} \\
= & \frac{-(h_{\theta\theta})_t}{h - \frac{1}{2} r_{\text{in}}(\bar{t})} + \frac{2 h_{\theta t} h_\theta}{\left(h - \frac{1}{2} r_{\text{in}}(\bar{t})\right)^2} \\
& - \frac{2 h_t h_\theta^2}{\left(h - \frac{1}{2} r_{\text{in}}(\bar{t})\right)^3} + \frac{h_t h_{\theta\theta}}{\left(h - \frac{1}{2} r_{\text{in}}(\bar{t})\right)^2} \\
= & \frac{-(h_{\theta\theta})_t}{h - \frac{1}{2} r_{\text{in}}(\bar{t})} + \frac{h_t h_{\theta\theta}}{\left(h - \frac{1}{2} r_{\text{in}}(\bar{t})\right)^2}.
\end{aligned} \tag{4.11}
$$

By combining (4.3) and the relation $k^{-1} = h_{\theta\theta} + h$, as well as above (4.9)-(4.11),it follows

$$
\begin{aligned}
& \frac{(h_t)^2}{\left(h - \frac{1}{2} r_{\text{in}}(\bar{t})\right)^2} \\
& \quad \geq \frac{h_{tt}}{h - \frac{1}{2} r_{\text{in}}(\bar{t})} \\
& \quad = \frac{1}{h - \frac{1}{2} r_{\text{in}}(\bar{t})} \left[\Phi k^2 ((h_{\theta\theta})_t + h_t) - \left(\Phi_x \frac{dx}{dt} + \Phi_y \frac{dy}{dt} \right) k \right. \\
& \qquad \qquad \left. - \left(\Psi_x \frac{dx}{dt} + \Psi_y \frac{dy}{dt} \right) \right] \\
& \quad \geq \frac{\Phi k^2}{h - \frac{1}{2} r_{\text{in}}(\bar{t})} \left[\frac{h_{\theta\theta} + h - \frac{1}{2} r_{\text{in}}(\bar{t})}{h - \frac{1}{2} r_{\text{in}}(\bar{t})} \right] h_t \\
& \qquad - \frac{1}{h - \frac{1}{2} r_{\text{in}}(\bar{t})} \left[\left(\Phi_x \frac{dx}{dt} + \Phi_y \frac{dy}{dt} \right) k + \left(\Psi_x \frac{dx}{dt} + \Psi_y \frac{dy}{dt} \right) \right] \\
& \quad = \frac{1}{2} r_{\text{in}}(\bar{t}) \cdot \frac{\Phi k^2}{h - \frac{1}{2} r_{\text{in}}(\bar{t})} \phi - \frac{\Phi k}{h - \frac{1}{2} r_{\text{in}}(\bar{t})} \phi \\
& \qquad - \frac{1}{h - \frac{1}{2} r_{\text{in}}(\bar{t})} \left[\left(\Phi_x \frac{dx}{dt} + \Phi_y \frac{dy}{dt} \right) k + \left(\Psi_x \frac{dx}{dt} + \Psi_y \frac{dy}{dt} \right) \right] .
\end{aligned} \tag{4.12}
$$

Since $(x, y) = h_\theta T - hN$, it is easy to know

$$
\begin{aligned}
\left(\frac{dx}{dt}, \frac{dy}{dt} \right) &= h_{\theta t} T - h_t N \\
&= \frac{h_t h_\theta}{h - \frac{1}{2} r_{\text{in}}(\bar{t})} T - h_t N
\end{aligned}
$$

by (4.9).

Thus (4.12) can be written as

$$\begin{aligned}
&\phi^2 + \frac{\Phi k}{h - \frac{1}{2} r_{\text{in}}(\bar{t})}\phi \\
&\quad \geq \frac{1}{2} r_{\text{in}}(\bar{t}) \cdot \frac{\Phi k^2}{h - \frac{1}{2} r_{\text{in}}(\bar{t})}\phi \\
&\quad - \frac{1}{h - \frac{1}{2} r_{\text{in}}(\bar{t})}\left[(\Phi_x, \Phi_y) \cdot T(-\phi h_\theta) - (\Phi_x, \Phi_y) \cdot N h_t\right] k \\
&\quad - \frac{1}{h - \frac{1}{2} r_{\text{in}}(\bar{t})}\left[(\Psi_x, \Psi_y) \cdot T(-\phi h_\theta) - (\Psi_x, \Psi_y) \cdot N h_t\right],
\end{aligned}$$

by (4.7),

$$\begin{aligned}
\phi^2 + 2\phi^2 \quad \geq \quad & \frac{1}{4} r_{\text{in}}(\bar{t}) k \phi^2 - 2|h_\theta| \cdot \Phi^{-1} \cdot |(\Phi_x, \Phi_y) \cdot T| \cdot \phi^2 \\
& -2\Phi^{-1} \cdot |(\Phi_x, \Phi_y) \cdot N| \cdot \phi^2 \left(h - \frac{1}{2} r_{\text{in}}(\bar{t})\right) \\
& -2|h_\theta| \cdot \Phi^{-1} \cdot |(\Psi_x, \Psi_y) \cdot T| \cdot \phi^2 \\
& -2\Phi^{-1} \cdot |(\Psi_x, \Psi_y) \cdot N| \cdot \phi^2 \left(h - \frac{1}{2} r_{\text{in}}(\bar{t})\right).
\end{aligned}$$

So

$$C_1 \phi^2 \geq \frac{1}{4} r_{\text{in}}(\bar{t}) k \phi^2,$$

i.e.,

$$k(t_0, \theta_0) \leq 4C_1 / r_{\text{in}}(\bar{t}),$$

for some positive constant C_1 depending only on Φ, Ψ and the diameter of the initial curve.

If $t_0 = 0$, then at (t_0, θ_0)

$$k(t_0, \theta_0) \leq \max_{\theta \in [0, 2\pi]} k(0, \theta) \cdot D(0) / r_{\text{in}}(\bar{t}).$$

Therefore we get, for any $\theta \in [0, 2\pi]$,

$$\begin{aligned} k(\bar{t},\theta) &\le \lambda^{-1}(\Phi(x(\bar{t},\theta), y(\bar{t},\theta),\theta)k(\bar{t},\theta)) \\ &\le 2\lambda^{-1}(-h_t(\bar{t},\theta)) && (\text{by}(4.7)) \\ &= 2\lambda^{-1}\phi(\bar{t},\theta)\cdot\left(h(\bar{t},\theta)-\frac{1}{2}r_{\text{in}}(\bar{t})\right) \\ &\le 2\lambda^{-1}D(\bar{t})\cdot\phi(t_0,\theta_0) \\ &\le 4\lambda^{-1}D(\bar{t})\cdot(r_{\text{in}}(\bar{t}))^{-1}\cdot(\Phi k+\Psi)(t_0,\theta_0) && (\text{by}(4.8)) \\ &\le 8\lambda^{-2}\left(\frac{D(\bar{t})}{r_{\text{in}}(\bar{t})}\right)\cdot k(t_0,\theta_0) && (\text{by}(4.7)) \\ &\le 8\lambda^{-2}\left(\frac{D(\bar{t})}{r_{\text{in}}(\bar{t})}\right)\cdot\left[4C_1+D(0)\cdot\max_{\theta\in[0,2\pi]}k(0,\theta)\right]\cdot r_{\text{in}}^{-1}(\bar{t}). \end{aligned}$$

Thus, by (4.4) and (4.6) we have

$$\begin{aligned} \widetilde{k}(\bar{t},\theta) &\le 8\lambda^{-2}C\left[4C_1+D(0)\cdot\max_{\theta\in[0,2\pi]}k(0,\theta)\right](\widetilde{r}_{\text{in}}(\bar{t}))^{-1} \\ &\le 8\lambda^{-2}C\left[4C_1+D(0)\cdot\max_{\theta\in[0,2\pi]}k(0,\theta)\right]\cdot\delta^{-1} \end{aligned} \tag{4.13}$$

for any $\bar{t} \in [0, t_{\max})$ and $\theta \in [0, 2\pi]$.

This gives the upper bound for the rescaled curvature $\widetilde{k}$.

Step 2. Estimate the derivatives of the rescaled curvature $\widetilde{k}$.

Introduce a new time parameter

$$\tau = -\frac{1}{2}\ln\left(\frac{t_{\max}-t}{t_{\max}}\right) \in [0, +\infty).$$

The support function $\widetilde{h}(\tau,\theta)$ of $\widetilde{\gamma}(\tau,\cdot)$ satisfies

$$\begin{aligned} \frac{\partial\widetilde{h}}{\partial\tau} = &-\Big(\Phi(x(\tau,\theta), y(\tau,\theta),\theta)\widetilde{k} \\ &\quad + \sqrt{2t_{\max}}e^{-\tau}\Psi(x(\tau,\theta), y(\tau,\theta),\theta)\Big) \\ &+ \widetilde{h}. \end{aligned} \tag{4.14}$$

In fact

$$\widetilde{h} = \frac{1}{\sqrt{2(t_{\max}-t)}}(h + \langle(\bar{x},\bar{y}),N\rangle) = \frac{e^{\tau}}{\sqrt{2t_{\max}}}(h + \langle(\bar{x},\bar{y}),N\rangle),$$

and

$$\begin{aligned}
\frac{\partial \widetilde{h}}{\partial \tau} &= \frac{e^{\tau}}{\sqrt{2t_{\max}}}(h + \langle (\bar{x}, \bar{y}), N \rangle) + \frac{e^{\tau}}{\sqrt{2t_{\max}}} \frac{\partial h}{\partial t} \cdot \frac{dt}{d\tau} \\
&= \widetilde{h} + \sqrt{2t_{\max}} e^{-\tau}[-\Phi k - \Psi] \\
&= -[\Phi \widetilde{k} + \sqrt{2t_{\max}} e^{-\tau} \Psi] + \widetilde{h},
\end{aligned}$$

where we have used (4.3).

Let $\tau_1 > 0$ be any time in $(0, +\infty)$, and choose the center of a circle with radius $\widetilde{r}_{\text{in}}(\tau_1)$ enclosed by $\widetilde{\gamma}(\tau_1, \cdot)$ as the origin of R^2. Then at time τ_1, by (4.6), we have

$$\delta \leq \widetilde{r}_{\text{in}}(\tau_1) \leq \widetilde{h}(\tau_1, \theta) \leq \widetilde{D}(\tau_1) \leq \Lambda$$

for $\theta \in [0, 2\pi]$. By the boundedness of $\widetilde{k}$, from (4.14) it follows that there exists a positive constant Δ, independent of τ_1, such that

$$\frac{\delta}{2} \leq \widetilde{h}(\tau, \theta) \leq 2\Lambda \tag{4.15}$$

for $\tau \in [\tau_1, \tau_1 + \Delta], \theta \in [0, 2\pi]$.

Consider the rescaled curve $\widetilde{\gamma}(\tau, \cdot), \tau \in [\tau_1, \tau_1 + \Delta]$, parametrized as spherical graph by choosing the origin of R^2 to be center of the above enclosed circle at time τ_1,

$$\widetilde{\gamma}(\tau, \phi) = (\widetilde{r}(\tau, \phi) \cos \phi, \widetilde{r}(\tau, \phi) \sin \phi), \quad \phi \in [0, 2\pi].$$

By a standard computation, we have

$$d\widetilde{s} = \sqrt{\widetilde{r}_{\phi}^2 + \widetilde{r}^2}\, d\phi, \tag{4.16}$$

$$\widetilde{k} = \frac{-\widetilde{r}\widetilde{r}_{\phi\phi} + 2\widetilde{r}_{\phi}^2 + \widetilde{r}^2}{(\widetilde{r}_{\phi}^2 + \widetilde{r}^2)^{3/2}}, \tag{4.17}$$

where $\widetilde{s}$ is the arclength parameter of $\widetilde{\gamma}(t, \cdot)$.

Now we want to write down the evolution equation for $\widetilde{k}$ in the spherical coordinate. First we compute the normal time derivative $k^0 = \frac{\partial k(t,u)}{\partial t}\Big|_u$ (recall our original parametrization in (1.3)),

$$k^0 = \frac{\partial k}{\partial t} + k_{\phi} \cdot \phi^0,$$

where $\frac{\partial k}{\partial t} = \left.\frac{\partial k(t,\phi)}{\partial t}\right|_{\phi}$ and $\phi^0 = \left.\frac{\partial \phi(t,u)}{\partial t}\right|_{u}$.

The position vector is

$$\gamma(t,\phi) = (r(t,\phi(t,u))\cos(\phi(t,u)), r(t,\phi(t,u))\sin(\phi(t,u))) + (\bar{x},\bar{y}),$$

where $r(t,\phi) = \sqrt{2(t_{\max}-t)}\widetilde{r}(t,\phi)$.

The normal time derivative of γ is,

$$\begin{aligned} \left.\frac{\partial \gamma}{\partial t}\right|_{u} &= \phi^0(r_\phi\cos\phi - r\sin\phi, r_\phi\sin\phi + r\cos\phi) + r_t(\cos\phi,\sin\phi) \\ &= \phi^0\sqrt{r_\phi^2+r^2}T + r_t(\cos\phi,\sin\phi), \end{aligned}$$

where $r_t = \left.\frac{\partial r}{\partial t}\right|_{\phi}$, T is the unit tangent vector.

Then by (1.3),

$$\phi^0 = -\frac{r_t}{\sqrt{r_\phi^2+r^2}}T\cdot(\cos\phi,\sin\phi),$$

$$\Phi k + \Psi = r_t N\cdot(\cos\phi,\sin\phi),$$

with

$$T = \frac{1}{\sqrt{r_\phi^2+r^2}}(r_\phi\cos\phi - r\sin\phi, r_\phi\sin\phi + r\cos\phi),$$

$$N = \frac{1}{\sqrt{r_\phi^2+r^2}}(-r_\phi\sin\phi - r\cos\phi, r_\phi\cos\phi - r\sin\phi).$$

So

$$\begin{aligned} \phi^0 &= -\frac{r_t}{\sqrt{r_\phi^2+r^2}}\cdot\frac{r_\phi}{\sqrt{r_\phi^2+r^2}} \\ &= +\frac{r_\phi}{(\sqrt{r_\phi^2+r^2})^2}\cdot\frac{(\Phi k+\Psi)}{(r/\sqrt{r_\phi^2+r^2})} \\ &= +\frac{(\Phi k+\Psi)r_\phi}{r\sqrt{r_\phi^2+r^2}}, \end{aligned}$$

hence

$$k^0 = \frac{\partial k}{\partial t} + \frac{(\Phi k+\Psi)r_\phi k_\phi}{r\sqrt{r_\phi^2+r^2}}.$$

On the other hand, by (1.4), one has

$$\begin{aligned}
k^0 &= \frac{\partial^2}{\partial s^2}(\Phi k+\Psi)+k^2(\Phi k+\Psi)\\
&= \frac{1}{\sqrt{r_\phi^2+r^2}}\frac{\partial}{\partial\phi}\Big(\frac{1}{\sqrt{r_\phi^2+r^2}}\frac{\partial}{\partial\phi}(\Phi(x(t,\phi),y(t,\phi),\theta(t,\phi))k\\
&\qquad\qquad +\Psi(x(t,\phi),y(t,\phi),\theta(t,\phi)))\Big)\\
&\quad +k^2(\Phi(x(t,\phi),y(t,\phi),\theta(t,\phi))k+\Psi(x(t,\phi),y(t,\phi),\theta(t,\phi))).
\end{aligned}$$

Then, for the rescaled $\widetilde{k}=\widetilde{k}(\tau,\phi)$,

$$\begin{aligned}
\frac{\partial}{\partial\tau}\tilde{k} &= \frac{\partial}{\partial\tau}(\sqrt{2t_{\max}}e^{-\tau}k(\tau,\phi))\\
&= \sqrt{2t_{\max}}(-e^{-\tau})k(\tau,\phi)+\sqrt{2t_{\max}}e^{-\tau}\frac{\partial k}{\partial t}\cdot\frac{dt}{d\tau}\\
&= -\tilde{k}+(\sqrt{2t_{\max}}e^{-\tau})^3\frac{\partial k}{\partial t}\\
&= -\tilde{k}+(\sqrt{2t_{\max}}e^{-\tau})^3\left\{\frac{1}{\sqrt{r_\phi^2+r^2}}\frac{\partial}{\partial\phi}\left[\frac{1}{\sqrt{r_\phi^2+r^2}}\frac{\partial}{\partial\phi}(\Phi k+\Psi)\right]\right.\\
&\qquad\qquad \left.+k^2(\Phi k+\Psi)-\frac{(\Phi k+\Psi)r_\phi k_\phi}{r\sqrt{r_\phi^2+r^2}}\right\}\\
&= -\tilde{k}+\frac{1}{\sqrt{\tilde{r}_\phi^2+\tilde{r}^2}}\frac{\partial}{\partial\phi}\left[\frac{1}{\sqrt{\tilde{r}_\phi^2+\tilde{r}^2}}\frac{\partial}{\partial\phi}(\Phi\tilde{k}+\sqrt{2t_{\max}}e^{-\tau}\Psi)\right]\\
&\quad +\tilde{k}^2(\Phi\tilde{k}+\sqrt{2t_{\max}}e^{-\tau}\Psi)-\frac{(\Phi\tilde{k}+\sqrt{2t_{\max}}e^{-\tau}\Psi)\tilde{r}_\phi\tilde{k}_\phi}{\tilde{r}\sqrt{\tilde{r}_\phi^2+\tilde{r}^2}},
\end{aligned}$$

where Φ and Ψ can be written as

$$\Phi=\Phi(\sqrt{2t_{\max}}e^{-\tau}\widetilde{x}(\tau,\phi)+\bar{x},\sqrt{2t_{\max}}e^{-\tau}\widetilde{y}(\tau,\phi)+\bar{y},\theta(\tau,\phi)),$$

$$\Psi=\Psi(\sqrt{2t_{\max}}e^{-\tau}\widetilde{x}(\tau,\phi)+\bar{x},\sqrt{2t_{\max}}e^{-\tau}\widetilde{y}(\tau,\phi)+\bar{y},\theta(\tau,\phi)),$$

with $(\widetilde{x},\widetilde{y})$ to be the position vector of $\widetilde{\gamma}(\tau,\cdot)$.

This is, for $\tau \in [\tau_1, \tau_1 + \Delta]$,

$$
\begin{aligned}
\frac{\partial \tilde{k}}{\partial \tau} =& \frac{1}{\sqrt{\tilde{r}_\phi^2 + \tilde{r}^2}} \frac{\partial}{\partial \phi} \left[\frac{1}{\sqrt{\tilde{r}_\phi^2 + \tilde{r}^2}} \frac{\partial}{\partial \phi} (\Phi \tilde{k} + \sqrt{2 t_{\max}} e^{-\tau} \Psi) \right] \\
& - \frac{(\Phi \tilde{k} + \sqrt{2 t_{\max}} e^{-\tau} \Psi) \tilde{r}_\phi \tilde{k}_\phi}{\tilde{r} \sqrt{\tilde{r}_\phi^2 + \tilde{r}^2}} \\
& + [\tilde{k}(\Phi \tilde{k} + \sqrt{2 t_{\max}} e^{-\tau} \Psi) - 1] \cdot \tilde{k}.
\end{aligned} \tag{4.18}
$$

By using the boundedness of $\widetilde{k}$, (4.15),(4.17) and (4.6), we can find a positive constant C, independent of τ_1, such that

$$
\begin{gathered}
\frac{\delta}{2} \leq \widetilde{r}(\tau, \phi) \leq 2\Lambda, \\
|\widetilde{r}_\phi(\tau, \phi)| \leq C, \\
|\widetilde{r}_{\phi\phi}(\tau, \phi)| \leq C,
\end{gathered}
$$

for all $\tau \in [\tau_1, \tau_1 + \Delta]$, and $\phi \in [0, 2\pi]$.

Then we can apply the standard results of linear uniformly parabolic equations (c.f. Theorem III 10.1 in [17]) to bound the derivatives and higher derivatives of $\widetilde{k}(\tau, \phi)$ on $\left[\tau_1 + \frac{\Delta}{4}, \tau_1 + \frac{3}{4}\Delta\right] \times [0, 2\pi]$; moreover these bounds are independent of τ_1. Further, by using (4.16) and (4.17), we deduce that the derivatives and higher derivatives of $\widetilde{k}$ with respect to the arclength parameter $\widetilde{s}$ are uniformly bounded for all $\tau \in [0, +\infty)$.

Step 3. Establish the convergence.

The following argument suggested by S. Angenent simplifies much of our original one. Without loss of generality, we may assume that $\gamma(t, \cdot)$ shrinks to the origin of R^2. Consider the rescaled curves

$$
\widetilde{\gamma}(\tau, \cdot) = \frac{1}{\sqrt{2(t_{\max} - t)}} \gamma(t, \cdot)
$$

with $\tau = -\frac{1}{2} \ln \left(\frac{t_{\max} - t}{t_{\max}} \right) \in [0, +\infty)$.

By (1.3) and (4.7), it is easy to know that the origin is always enclosed by $\gamma(t, \cdot)$. And by (4.6) one sees that $\widetilde{\gamma}(\tau, \cdot)$ stay in a fixed bounded region. If necessary, we can introduce the arclength parametrization: $\widetilde{\gamma} : [0, +\infty) \times R \to R^2$ by regarding $\widetilde{\gamma}(\tau, \cdot)$ to be an $\widetilde{L}(\tau)$ periodic function of $\widetilde{s} \in R$. Also we know the above two steps that $\widetilde{\gamma}(\tau, \cdot)$ have uniformly bounded curvature $\widetilde{k}$ and uniformly bounded derivatives of $\widetilde{k}$. Thus for any sequence $\tau_n \to +\infty$, we may take a subsequence τ_{n_j} such that for $\overline{\tau} \in [-1,1]$, $\widetilde{\gamma}\left(\tau_{n_j}+\overline{\tau},\cdot\right)$converges in C^∞ topology to a family of curves $\widetilde{\gamma_\infty}\left(\overline{\tau},\cdot\right)$.

Since

$$\begin{aligned}\frac{\partial\widetilde{\gamma}}{\partial\tau} &= \frac{\partial}{\partial\tau}\left(\frac{e^\tau}{\sqrt{2t_{\max}}}\gamma\right)\\ &= \frac{e^\tau}{\sqrt{2t_{\max}}}\gamma+\frac{e^\tau}{\sqrt{2t_{\max}}}\frac{\partial\gamma}{\partial t}\cdot\frac{dt}{d\tau}\\ &= \widetilde{\gamma}+\left(\Phi\widetilde{k}+\sqrt{2t_{\max}}e^{-\tau}\Psi\right)N,\end{aligned}$$

we have that the limit $\widetilde{\gamma_\infty}\left(\overline{\tau},\cdot\right)$ evolves by

$$\frac{\partial\widetilde{\gamma_\infty}}{\partial\overline{\tau}} = \widetilde{\gamma_\infty}+\Phi\left(0,0,\theta\right)\widetilde{k_\infty}N, \tau\in\left[-1,1\right], \tag{4.19}$$

where $\widetilde{k_\infty}$ is the curvature of the limit $\widetilde{\gamma_\infty}$.

Remember the original curve $\gamma\left(t,\cdot\right)$ to be strictly convex, so the curvature $\widetilde{k_\infty}\left(\overline{\tau},\cdot\right)$ of the limit $\widetilde{\gamma_\infty}\left(\overline{\tau},\cdot\right)$ must be nonnegative in $[-1,1]$. In the following we show that $\widetilde{k_\infty}\left(\overline{\tau},\cdot\right)$ is in fact positive in the open interval $(-1,1)$.

Denoted

$$\gamma_\infty\left(\overline{t},\cdot\right) = \sqrt{2t_{\max}}e^{-\overline{\tau}}\widetilde{\gamma_\infty}\left(\overline{\tau},\cdot\right)$$

with $\overline{t} = t_{\max}\left(1-e^{-2\overline{\tau}}\right) \in \left[t_{\max}\left(1-e^2\right), t_{\max}\left(1-e^{-2}\right)\right]$. The curvature $k_\infty\left(\overline{t},\cdot\right)$ of $\gamma_\infty\left(\overline{t},\cdot\right)$ must be also nonnegtive. And we notice that the curve $\gamma_\infty\left(\overline{t},\cdot\right)$ evolves by

$$\begin{aligned}\frac{\partial\gamma_\infty\left(\overline{t},\cdot\right)}{\partial\overline{t}} &= \frac{d\overline{\tau}}{d\overline{t}}\cdot\frac{\partial}{\partial\overline{\tau}}\left(\sqrt{2t_{\max}}e^{-\overline{\tau}}\widetilde{\gamma_\infty}\left(\overline{\tau},\cdot\right)\right)\\ &= \frac{1}{2\left(t_{\max}-\overline{t}\right)}\left[-\sqrt{2t_{\max}}e^{-\overline{\tau}}\widetilde{\gamma_\infty}\left(\overline{\tau},\cdot\right)\right.\\ &\quad \left.+\sqrt{2t_{\max}}e^{-\overline{\tau}}\left(\widetilde{\gamma_\infty}\left(\overline{\tau},\cdot\right)+\Phi\left(0,0,\theta\right)\widetilde{k_\infty}\left(\overline{\tau},\cdot\right)N\right)\right]\\ &= \Phi\left(0,0,\theta\right)\widetilde{k_\infty}\left(\overline{\tau},\cdot\right)N\end{aligned}$$

for $\overline{t} \in \left[t_{\max}\left(1-e^2\right), t_{\max}\left(1-e^{-2}\right)\right]$.

Thus it is well-known from the strongly maximum principle that the curvature $k_\infty\left(\bar{t},\cdot\right)$ must be positive in the open interval

$$\left(t_{\max}\left(1-e^2\right),t_{\max}\left(1-e^{-2}\right)\right).$$

Therefore $\widetilde{k}_\infty\left(\overline{\tau},\cdot\right)$ must be also positive in $(-1,1)$.

Let us come back to the rescaled curves $\widetilde{\gamma}\left(\tau,\cdot\right),\tau\in[0,+\infty)$. We claim that there is a positive constant $\delta>0$ such that

$$\inf_{\tau\in[0,+\infty),\widetilde{\gamma}(\tau,\cdot)}\widetilde{k}\left(\tau,\cdot\right)\geq\delta>0. \tag{4.20}$$

In fact, if not, there are some sequences $\tau_n\to+\infty$ and

$$P_n\in\widetilde{\gamma}\left(\tau_n,\cdot\right)\subset R^2$$

such that

$$\widetilde{k}\left(\tau_n,P_n\right)\to0 \qquad \text{as } n\to+\infty.$$

From the above argument, we may take a subsequence n_j such that $\widetilde{\gamma}\left(\tau_{n_j}+\overline{\tau},\cdot\right)$ converges in C^∞ topology to a family of curves $\widetilde{\gamma_\infty}\left(\overline{\tau},\cdot\right)$ with $\overline{\tau}\in[-1,1]$ and P_{n_j} converges to some point $P\in\widetilde{\gamma_\infty}\left(0,\cdot\right)$ with $\widetilde{k_\infty}\left(0,P\right)=0$. But in the above we have shown that $\widetilde{k_\infty}\left(\overline{\tau},\cdot\right)$ is positive in (-1.1). This contradiction proves the claim (4.20).

Since the origin is always enclosed by $\gamma\left(t,0\right)$, the support function $\widetilde{h}\left(\tau,\cdot\right)$ of the rescaled curve $\widetilde{\gamma}\left(\tau,\cdot\right)$ is positive for all $\tau\in[0,+\infty)$. By (4.20) we may assume

$$\Phi\widetilde{k}+\sqrt{2t_{\max}}e^{-\tau}\Psi\geq k_0>0 \tag{4.21}$$

on $\widetilde{\gamma}\left(\tau,\cdot\right)$ for all $\tau\in[0,+\infty)$. Here k_0 is some positive constant independent of τ. We further claim that in fact $\widetilde{h}\geq\frac{k_0}{2}$ for all τ. For, if on the contray that $\widetilde{h}<\frac{k_0}{2}$ for some (τ_0,θ_0), by (4.14) and (4.21) one must have $\widetilde{h}\left(\tau_0,\theta_0+1\right)<0$, which is impossible. Hence $\widetilde{h}\geq\frac{k_0}{2}$ for all $\tau\in[0,+\infty)$.

Consider the functional

$$J(\tau) = \int_0^{2\pi} (\widetilde{h}_\theta^2 - \widetilde{h}^2 + 2\Phi \ln \widetilde{h} + 2Be^{-\tau}) d\theta, \tag{4.22}$$

where $\tau \in [0, +\infty)$, and B is a constant to be determined. We have

$$\begin{aligned}
\frac{dJ(\tau)}{d\tau} =& 2\int_0^{2\pi} \left[-(\tilde{h}_{\theta\theta} + \tilde{h})\tilde{h}_\tau + \Phi \frac{\tilde{h}_\tau}{\tilde{h}} + \left(\frac{\partial \Phi}{\partial \tau}\right) \ln \widetilde{h} - Be^{-\tau} \right] d\theta \\
=& 2\int_0^{2\pi} \left(-\frac{1}{\tilde{k}} + \frac{\Phi}{\tilde{h}} \right) \tilde{h}_\tau d\theta + 2\int_0^{2\pi} \left(\frac{\partial \Phi}{\partial \tau}\right) \ln \widetilde{h} d\theta \\
& - 2B \int_0^{2\pi} e^{-\tau} d\theta \\
=& -2\int_0^{2\pi} \frac{1}{\tilde{h}\tilde{k}} (\Phi\tilde{k} - \tilde{h})(\Phi\tilde{k} + \sqrt{2t_{\max}} e^{-\tau}\Psi - \tilde{h}) d\theta \\
& - 2\int_0^{2\pi} \sqrt{2t_{\max}} e^{-\tau} \left[\frac{\partial \Phi}{\partial x} \left(\sqrt{2t_{\max}} e^{-\tau} \widetilde{x}, \sqrt{2t_{\max}} e^{-\tau} \widetilde{y}, \theta\right) \cdot \widetilde{x} \right. \\
& \left. + \frac{\partial \Phi}{\partial y} \left(\sqrt{2t_{\max}} e^{-\tau} \widetilde{x}, \sqrt{2t_{\max}} e^{-\tau} \widetilde{y}, \theta\right) \cdot \widetilde{y} \right] \ln \widetilde{h} d\theta - 4\pi B e^{-\tau} \\
\leq& -\int_0^{2\pi} \frac{1}{\tilde{h}\tilde{k}} (\Phi\tilde{k} - \tilde{h})^2 d\theta + \int_0^{2\pi} \frac{1}{\tilde{h}\tilde{k}} (2t_{\max}) e^{-2\tau} \Psi^2 d\theta \\
& - 2\int_0^{2\pi} \sqrt{2t_{\max}} e^{-\tau} \left(\frac{\partial \Phi}{\partial x} \cdot \widetilde{x} + \frac{\partial \Phi}{\partial y} \cdot \widetilde{y} \right) \ln \widetilde{h} d\theta - 4\pi B e^{-\tau}.
\end{aligned}$$

Since $\widetilde{h} \geq \frac{k_0}{2}$ for all and the rescaled curves $\widetilde{\gamma}(\tau, \cdot)$ stay in a fixed bounded region, it is certainly that there exists a positive constant B which depends on k_0, δ and the functions Φ, Ψ such that

$$\begin{aligned}
& \int_0^{2\pi} \frac{1}{\widetilde{h}\widetilde{k}} (2t_{\max}) e^{-2\tau} \Psi^2 d\theta \\
& \quad -2\int_0^{2\pi} \sqrt{2t_{\max}} e^{-\tau} \left(\frac{\partial \Phi}{\partial x} \cdot \widetilde{x} + \frac{\partial \Phi}{\partial y} \cdot \widetilde{y} \right) \ln \widetilde{h} d\theta \\
& \quad -4\pi B e^{-\tau} \leq 0 \qquad \text{for all } \tau \in [0, +\infty).
\end{aligned}$$

So

$$\begin{aligned}\frac{dJ}{d\tau} &\le -\int_0^{2\pi}\frac{1}{\tilde{h}\tilde{k}}(\Phi\tilde{k}-\tilde{h})^2 d\theta \\ &= -\frac{1}{\Lambda}\int_{\tilde{\gamma}(\tau,\cdot)}(\Phi\tilde{k}-\tilde{h})^2 d\,\tilde{s} \\ &\le 0\end{aligned} \tag{4.23}$$

by (4.6).

By using $\widetilde{h}_{\theta\theta}+\widetilde{h}=\frac{1}{\widetilde{k}}$, we have

$$\begin{aligned}\left|\widetilde{h}_\theta\right| &\le \int_0^{2\pi}\left(\frac{1}{\widetilde{k}}+\widetilde{h}\right)d\theta \\ &\le \widetilde{L}(\tau)+2\pi\Lambda \le 3\pi\Lambda,\end{aligned}$$

where $\widetilde{L}(\tau)$ is the length of $\widetilde{\gamma}(\tau,\cdot)$. Combining the fact $\widetilde{h}\ge\frac{k_0}{2}$ for all $\tau\in[0,+\infty)$, we know that the functional $J(\tau)$ is bounded below on $[0,+\infty]$. Furthermore, from (4.23), we conclude

$$\lim_{\tau\to+\infty}\int_{\widetilde{\gamma}(\tau,\cdot)}\left(\Phi\widetilde{k}-\widetilde{h}\right)d\widetilde{s}=0. \tag{4.24}$$

Let τ_n be any sequence of times diverging to infinity. From the above,$\widetilde{\gamma}(\tau_n,\cdot)$ must contain a C^∞ convergent subsequence. And by (4.24), any limit of a convergent subsequence of $\widetilde{\gamma}(\tau_n,\cdot)$ must satisfy $\Phi(0,0,\theta)\widetilde{k}(\theta)=\widetilde{h}(\theta)$. By the symmetry assumption $(\mathrm{H}_3)'$ and the uniqueness result of Gage [9], Gage & Li [11], any two solutions of the equation $\Phi(0,0,\theta)\widetilde{k}(\theta)=\widetilde{h}(\theta)$ differ only by a dilation. But by (4.5), the enclosed area of $\widetilde{\gamma}(\tau,\cdot)$ tends to a positive constant. Then any limit of a convergent subsequence of $\widetilde{\gamma}(\tau_n,\cdot)$ must be the same solution of the equation $\Phi(0,0,\theta)\widetilde{k}(\theta)=\widetilde{h}(\theta)$. Thus $\widetilde{\gamma}(\tau,\cdot)$ converges in C^∞ topology to a Minkowski isoperimetrix associated $\Phi(0,0,\theta)$. q.e.d.

As an application of the main theorem, we would like to mention the following example of H. Matano and H. Taniyama (by oral communication).

Example 4.3. Let (x,y,z) be the Cartesian coordinate of R^3. A surface of rotation about the z-axis can be parameterized by

$$\{(x=x(u)\cos v, y=x(u)\sin v, z=z(u))|0\le u\le 1, 0\le v\le 2\pi, x(u)>0\},$$

i.e., the surface is gotten by rotating the (x,z)- plane curve

$$\{(x = x(u), z = z(u)) \,|\, 0 \leq u \leq 1\}$$

about the z-axis.

Consider the mean curvature flow

$$(4.25) \qquad \begin{cases} \frac{\partial X}{\partial t} = H\nu \\ X(\cdot, 0) \text{ is a embedded, smooth and compact} \\ \qquad \text{surface of rotation about the z-axis} \end{cases}$$

where $X(\cdot, t)$ is a one-parameter family of surfaces in R^3, and $H\nu$ is the corresponding mean curvature vector. It is clear that $X(\cdot, t)$ remains to be a surface of rotation about the z-axis on the maximal time interval $[0, T)$ for the existence for (4.25). This is, $X(\cdot, t)$ is the surface by rotating some (x,z)-plane curve

$$\gamma(\cdot, t) = \{(x = x(u, t), z = z(u, t)) \,|\, 0 \leq u \leq 1, x(u, t) > 0\}$$

about the z-axis. Thus (4.25) is equivalent to the evolution of $\gamma(\cdot, t)$ by

$$(4.26) \qquad \begin{cases} \frac{\partial \gamma}{\partial t} = \left(k + \frac{1}{x}\sin\theta\right) N \\ \gamma(\cdot, 0) \text{ is a embeded, smooth and closed} \\ \qquad \text{curve in the (x,z)-plane,} \end{cases}$$

where k is the curvature of $\gamma(\cdot, t)$, N is the normal vector and θ is the angle between the tangent vector and the x-axis. Hence if there is some positive constant δ such that $\text{dist}(z - axis, \gamma(\cdot, t)) \geq \delta$ on the maximal time interval $[0, T)$, then by the main theorem, $\gamma(\cdot, t)$ shrinks to a point as $t \nearrow T$, and $\gamma(\cdot, t)$ is asympototically a circle.

References

[1] U. Abresch & J. Langer, *The normalized curve shortening flow and homothetic solutions*, J. Differential Geom. **23** (1986) 175-196.

[2] S. Altschuler, *Singularities of the curve shrinking flow for space curves*, J. Differential Geom.**34** (1991) 491-514.

[3] S. Angenent, *Parabolic equations for curves on surfaces (I). Curver with p-integrable curvature*, Ann. of Math.**132** (1990) 451-483.

[4] ______, *Parabolic equations for curves on surfaces (II). intersections, blowup and generalized solutions*, Ann. of Math. **133** (1991) 171-215.

[5] S. Angenent & M. Gurtin, *Multiphase thermomechanics with interfacial structure 2. Evolution of an isothermal interface*, Arch. Rational Mech. Anal. **108** (1989) 323-391.

[6] ______, *Anisotropic motion of a phase interface*, J. Reine Angew. Math. **446** (1994) 1-47.

[7] K. Chou & X. Zhu, *A convexity theorem for a class of anisotropic flows for plane curves*, Preprint, 1996.

[8] M. Gage, *An isoperimetric inequality with applications to curve shortening*, Duke Math. J. **50** (1983) 1225-1229.

[9] ______, *Evolving plane curves by curvature in relative geometries*, Duke Math. J. **72** (1993) 441-466.

[10] M. Gage & R. Hamilton, *The shrinking of convex plane curves by the heat equation*, J. Differential Geom. **23** (1986) 69-96.

[11] M. Gage & Y. Li, *Evolving plane curves by curvature in relative geometries. II*, Duke Math. J. **75** (1994) 79-98.

[12] M. Grayson, *The heat equation shrinks embedded plane curves to points*, J. Differential Geom. **26** (1987) 285-314.

[13] ______, *Shortening embedded curves*, Ann. of Math. **129** (1989) 71-111.

[14] M. Gurtin, *Thermomechanics of evolving phase boundaries in the plane*, Oxford Math. monographs, Oxford University Press, 1993.

[15] R. Hamilton, *Isoperimetric estimates for the curve shrinking flow in the plane*, Modern methods in complex analysis, The Princeton conference in honor of Gunning and Kohn, (ed. T. Bloom, et al.), Ann. of Math. Stud. 137, Princeton Univ. Press, 1996.

[16] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. **31** (1990) 285-299.

[17] O. A. Ladyzhenskaya, V. A. Solonikov & N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Transl. Math. Monographs Amer. Math. Soc. **23** (1968).

[18] J. Oaks, *Singularities and self intersections of curves evolving on surfaces*, Indiana Univ. Math. J. **43** (1994) 959-981.

[19] K. Tso, *Deforming a hypersurface by its Gauss-Kronecker curvature*, Comm. Pure Appl. Math. **38** (1985) 867-882.

ZHONGSHAN UNIVERSITY, CHINA