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DIFFERENTIAL TOPOLOGICAL RESTRICTIONS CURVATURE AND SYMMETRY

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A basic question one asks in Riemannian geometry is: how are geometric properties of a manifold reflected in its topology? An analogous question in transformation groups is: what topological restrictions are forced on a manifold by the existence of an effective action of a large group? In this work, we consider a combination of these two problems, namely:

Classify positively curved manifolds with large isometry groups.

One measurement for the size of a transformation group, $G \times M \to M$, is the dimension of its orbit space, M/G, also called the *cohomogeneity* of the action. This dimension is clearly constrained by the dimension of the fixed point set, M^G, of G in M. In fact, $dim(M/G) \geq dim(M^G) + 1$ for any non-trivial action. In light of this we define the *fixed point cohomogeneity* of an action by

$$\text{(0.1)} \qquad cohom\,fix(M,G) = dim(M/G) - dim(M^G) \geq 1,$$

that is, as the codimension of M^G in M/G. Note that if $M^G = \emptyset$, then, by convention, $cohomfix(M,G) = cohom(M,G) + 1$. Thus, (M,G) has minimal fixed point cohomogeneity one, if either M is homogeneous, or G acts transitively on a normal sphere to some component of M^G. In the latter case we say that M is *fixed point homogeneous.*

Recall that simply-connected homogeneous manifolds of positive sectional curvature have been classified in Berger [?][1], Aloff, Wallach [?], [?], and Berard-Bergery [?]. As one of our main results, we provide a

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[1]Cf. also B. Wilking, The normal homogeneous space $(SU(3) \times SO(2))/U(2)$ has positive sectional curvature, Proc. Amer. Math. Soc., to appear.

complete classification of fixed point homogeneous manifolds of postive sectional curvature (cf. Theorem 2.8). As a special case, we obtain:

Theorem A. *Any simply-connected, fixed point homogeneous manifold of positive sectional curvature is diffeomorphic to either S^n, CP^m, HP^k or CaP^2.*

Another measurement for the size of $G \times M \to M$ is the dimension, $dim(G)$ of G relative to $dim(M)$. From a dual point of view, G is large if $dim(M)$ is small relative to G. This viewpoint is related to representation theory. In a sense, the most basic linear representations of a compact Lie group, G, are those of lowest dimension. Thus motivated, we may also interpret the above problem in the following manner:

> *For a given compact Lie group, G, classify the low-dimensional positively curved manifolds, M, on which G can act (almost) effectively by isometries.*

Recall that any connected, compact Lie group, G is finitely covered by a group $\tilde{G} = T^k \times G_1 \times ... \times G_l$, where each $G_i, i = 1, ..., l$, is simple. Our classification of positively curved manifolds with maximal symmetry rank in [?] can be viewed as an answer to the above problem when $G = T^k$ is abelian. In this paper, we consider the remaining building blocks, i.e., the simple Lie groups and, in particular, the classical ones.

If for each compact Lie group, G, we set

$$rep_0^+(G) = min\{n | G acts \text{ (almost) effectively by isometries on some } M^n \text{ with } sec(M^n) > 0\}, \tag{0.2}$$

then another main result of this paper can be stated as (cf. Theorems 3.7, 3.9, 3.11, 3.12, 3.13):

Theorem B. *Let G be a connected, compact, simple Lie group other than E_6, E_7, or E_8. Then*

(i) $rep_0^+(G) = min\{dim(G/H) | H \subset G \text{ closed subgroup }\}$,

(ii) *any positively curved (almost) G-manifold, M, with $dim(M) \leq 2rep_0^+(G) - \epsilon(G)$ is diffeomorphic to a positively curved homogeneous manifold, where $\epsilon(G)$ is a small number depending on G.*

The key reduction used in the proof of this result is that any $G \times M \to M$ satisfying the assumptions of Theorem B is either homogeneous, of cohomogeneity one or of fixed point cohomogeneity one. In an analagous curvature free setting we refer to the work initiated by W.-Y. Hsiang in [**?**] (see also [**?**]).

Recall that, except for the examples due to Eschenburg [**?**], [**?**] and Bazaikin [**?**], all known positively curved manifolds are homogeneous (up to diffeomorphism). Thus, Theorem B provides another motivation for the systematic work initiated here. Indeed, for most, if not for all groups G, the conclusion in (ii) will almost certainly fail when $dim(M)$ is sufficiently large. It is quite likely that methods as developed in this paper, when applied to the lowest dimensional manifolds, M, where the theorem fails, will yield enough structure on M so as to propose potentially new examples of manifolds with positive curvature. However, we will refrain from pursuing this issue here. The following are simple consequences of Theorem B.

Corollary C. *Let G/H be a homogeneous space of positive curvature. If M is a positively curved manifold with $dim(M) = dim(G/H)$ on which G acts (almost) effectively by isometries. Then M is diffeomorphic to a positively curved homogeneous manifold (which is not necessarily G/H).*

In this generality, the conclusion of Corollary C fails if the symmetry group G of M is replaced by the smaller group H. However, for the rank-one symmetric spaces, we have the following result.

Corollary D. *Let G/H be a compact rank-one symmetric space (CROSS). If M is a positively curved manifold on which H acts (almost) effectively by isometries, and $dim(M) = dim(G/H) \geq 16$, then M is diffeomorphic to a CROSS.*

In both of these corollaries, the conclusion holds for manifolds with dimension larger than $dim(G/H)$. However, Corollary D fails in dimension $7 = dim(Sp(2)/Sp(1))$, namely each Aloff-Wallach example $W^7 = SU(3)/T_{k,l}$ admits an $Sp(1)$ action but is not a CROSS. All in all, one might thus be tempted to phrase the main results of this paper as follows: any potentially new example of a manifold of positive curvature must have significantly smaller symmetry group than those of the known examples.

The point of departure for our investigations is to analyse transformation groups $G \times M \to M$ directly via the geometry of their orbit spaces $X = M/G$. These spaces form a particularly beautiful subclass of the so-called Alexandrov spaces, and our work is, to a large extent, propelled by the recent progress in this area. Of particular importance to us is the fact that if M has positive curvature, then so does X. This becomes especially restrictive if X has non-empty boundary, since in that case X is contractible by the Cheeger-Gromoll-Meyer Soul theorem adapted to Alexandrov spaces (cf. [?]). Other restrictions are obtained via Alexandrov-Toponogov type angle comparisions when applied to triangles in X with vertices at singular points.

We arrive at our results when these geometric methods, together with critical point theory for distance functions (cf. e.g. [?]), are combined with known results from Lie theory and representation theory. For general facts about representation theory, we refer the reader to [?]. All claims about dimensions of representations and inclusions among Lie groups follow easily from the theory in [?]. For facts about subgroups of exceptional Lie groups, we refer to [?]. Finally, we occasionally need to compute the normalizer of a subgroup in some specific examples. For general methods as to how to do this, we refer for example to [?], in particular to paragraph 3. We wish to thank Wu-Yi Hsiang and W. Ziller for numerous illuminating conversations in which they shared their views and expertise on the latter subjects.

1. Alexandrov geometry of orbit spaces

Throughout this paper M will denote a complete, connected Riemannian n-manifold, and G a compact Lie group which acts (almost) effectively on M by isometries. The orbit space $X = M/G$ is equipped with the orbital distance metric from M.

Although we are primarily interested in positively curved manifolds, the natural setting for our methods applies to manifolds, M, whose sectional curvature is bounded from below, i.e., $sec(M) \geq k$. It is well known that there are many geometrically equivalent formulations of the condition $sec(M) \geq k$, some of which involve distances only (cf. e.g. [?], [?] and [?]). It is therefore easy to see, that, in this distance comparision sense, X is curved from below as well, in fact, $curv(X) \geq k$. Thus, X is an example of a so-called Alexandrov space, and

$$dim(X) = cohom(M, G) \tag{1.1}$$

by definition of the *cohomogeneity* of the action $G \times M \to M$.

The local and infinitesimal structure of general Alexandrov spaces is tied to spaces of directions (cf. [?], [?]). In the case of orbit spaces $X = M/G$, these are described as follows. For $p \in M$, we denote its orbit in M by $G(p)$, and when viewed as a point in X by $\bar{p}$. The *space of directions*, $S_{\bar{p}}X$ at $\bar{p} \in X$, consists exclusively of geodesic directions. Moreover,

$$S_{\bar{p}}X = S_p^{\perp}/G_p, \tag{1.2}$$

where $S_p^{\perp}$ is the unit normal sphere to $G(p)$ at p, and $G_p = \{g \in G : gp = p\}$ is the *isotropy group* of p. Note that $\bar{p}$ is a *euclidean point* of X, i.e., $S_{\bar{p}}X = S_1^{m-1}$, where S_1^{m-1} is the unit $(m-1)$-sphere, $m = dim(X)$, if and only if $G(p)$ is a principal orbit in M. We denote the set of such points by M_e, and call it the *regular part* of M. Correspondingly, $M_s = M - M_e$ is called the *singular part* of M.

As a first application of comparision theory, we show how $curv(X) \geq k$ imposes restrictions on the singular set M_s, via $X_s = M_s/G$. For simplicity, we confine ourselves to the case where $k \geq 0$, since otherwise the diameter of X must be invoked.

Extent Lemma 1.3. *For any choice of $(q+1)$ distinct points $\bar{p_0}, ..., \bar{p}_q \in X = M/G$ one has*

$$\frac{1}{q+1} \Sigma_{i=0}^{q} xt_q S_{\bar{p}_i} X \underset{(=)}{>} \frac{\pi}{3}$$

whenever $curv(X) \underset{(=)}{>} 0$.

Proof. Join each pair of points from $\{\bar{p_0}, ..., \bar{p}_q\}$ by a segment in X, and add up all angles between pairs of segments with common endpoints. This is carried out in two different ways: (i) takes the sum for each triangle and then add up over all triangles; (ii) takes the sum at each point and then add up over all points. Thus

$$\binom{q}{2} \Sigma_{i=0}^{q} xt_q S_{\bar{p}_i} X \geq \Sigma angles \geq \pi \binom{q+1}{3},$$

where (i) and $curv(X) \geq 0$ have been used for the right-hand inequality, and (ii) together with the definition of the q-extent (the maximal average distance between q points, cf. [?]) have been used in the left-hand inequality. q.e.d.

In this form, (1.3) is a powerful simple generalization of one of the key ideas applied in [?]. Note that $S_{\bar{p}}$ and hence $xt_q S_{\bar{p}}$ is smaller the more singular $\bar{p}$ is. Thus (1.3) yields quantitative restrictions on the number of singular orbits of various types when $sec(M) \geq 0$. For example, there can be at most two points $\bar{p_0}$, $\bar{p_1} \in X$ with $diam S_{\bar{p_i}} \leq \frac{\pi}{3}$ if $curv(X) > 0$. In the slightly more restrictive case in which $diam(S_{\bar{p_i}}) \leq \frac{\pi}{4}$, M can be described as follows:

Equivariant Sphere Theorem 1.4. *Let M be a closed manifold with $sec(M) > 0$ on which G acts (almost) effectively by isometries. Suppose $p_0, p_1 \in M$ are points such that $diam S_{\bar{p_i}} \leq \frac{\pi}{4}$, $i = 0, 1$. Then M can be exhibited as*

$$M = D(G(p_o)) \bigcup_E D(G(p_1)),$$

where $D(G(p_i))$, $i = 0, 1$ are tubular neighborhoods of the p_i-orbits and $E = \partial D(G(p_0)) = \partial D(G(p_1))$. In particular, M is homeomorphic to a sphere if $G(p_i) = p_i$, i.e., if p_i, $i = 0, 1$ are isolated fixed points for G and $diam S_{\bar{p_i}} \leq \frac{\pi}{4}$.

Proof. Let $p \in M - (G(p_0) \bigcup G(p_1))$ be chosen arbitrarily. Since $curv(X) > 0$, it follows from the assumption that $\angle(c_0, c_1) > \frac{\pi}{2}$ for any segment c_i from $\bar{p}$ to $\bar{p_i}$, $i = 0, 1$. In M this means that p is a regular point for the distance functions, $dist(G(p_i), \cdot)$, $i = 0, 1$, and the claim follows from the isotopy lemma (cf. e.g. [**?**]). q.e.d.

Note that only $curv(X) > 0$, not $sec(M) > 0$, is used in the proof. In particular, the structure of any closed manifold of cohomogeneity one with finite fundamental group is recovered in (1.4).

Even when the singularities of $X = M/G$ are too mild for (1.3) to apply (e.g. when $diam S_{\bar{p_i}} > \frac{\pi}{2}$ and thus π), they often yield interesting restrictions in a different way. The most remarkable one of these arises when X has non-empty boundary. Here $\bar{p} \in \partial X \subset X_s$ by definition, if $\partial S_{\bar{p}} X \neq \emptyset$. This inductive definition is anchored by the simple fact that the only compact 1-dimensional orbit spaces (Alexandrov Spaces) are closed intervals and circles.

Now suppose that $\partial X \neq \emptyset$ and $curv(X) > 0$. Then the Soul theorem adapted to Alexandrov spaces by Perelman [**?**] asserts that:

(1.5) $\quad dist(\partial X, \cdot) : X \to R$ is *strictly concave.*

In particular, this tells us the following:

(1.6) there is a unique point $\bar{p_1} \in X$ at maximal distance from ∂X, (1.7) for any $\bar{p} \in X - (\partial X \bigcup \{\bar{p_1}\})$ and segments c_0, c_1 from $\bar{p}$ to ∂X, and $\bar{p_1}$, respectively, one has $\angle(c_0, c_1) > \frac{\pi}{2}$,

(1.8) X is contractible.

If $\pi : M \to M/G = X$ is the quotient map, we let $M_\partial \subset M_s$ denote the subset defined by $M_\partial = \pi^{-1}\partial(X)$. Moreover, for any subset A of M (or of X), and any $r > 0$ we use $D(A, r)$ to denote the closed r-neighborhood of A. Correspondingly, $S(A, r)$ is the set of points at distance r to A and $B(A, r) = D(A, r) - S(A, r)$.

As a fairly straightforward consequence of (1.2), (1.6) and (1.7) combined with critical point arguments for $dist(M_\partial, \cdot)$ and $dist(G(p_1), \cdot)$ (via $dist(\partial X, \cdot)$ and $dist(\bar{p_1}, \cdot\,)$) (cf. e.g. [**?**] or [**?**]) one derives the following basic:

Soul Lemma 1.9. *Suppose M is a closed manifold with $sec(M) > 0$, on which a compact Lie group G acts (almost) effectively by isometries, such that $\partial(M/G) \neq \emptyset$. Then,*

(i) there is a unique orbit, $G(p_1) \subset M$ at maximal distance from $M_\partial \subset M$,

(ii) for any $p \in M - (M_\partial \bigcup G(p_1))$, the intersections $M_\partial \bigcap M^{G_p}$ and $G(p_1) \bigcap M^{G_p}$ are nonempty,

(iii) $M \simeq D(M_\partial, \epsilon) \bigcup_E D(G(p_1))$, where $E = \partial D(G(p_1)) \simeq S(M_\partial, \epsilon)$,

(iv) M_∂/G is homeomorphic to $S^{\perp}_{p_1}/G_{p_1}$.

Remark 1.10. The key point in (1.9) is that $curv(X) > 0$, not $sec(M) > 0$. If we have only $curv(X) \geq 0$, we can apply similar arguments with somewhat weaker conclusions, since $dist(\partial X, \cdot)$ is now only concave, rather than strictly concave. Hereafter, we will refer to the orbit, $G(p_1)$, in (1.9) as the "soul"-orbit of G.

Another context in which orbit spaces with non-empty boundary play a significant role is in the following result from [**?**] (for related result cf. [**?**]).

Fixed point Lemma 1.11. *Let M be a positively curved almost-effective G-manifold with G connected and with principal isotropy subgroup H. If the connected component, H_0, of H is a maximal connected subgroup of G, and $\partial((G/H_0)/H_0) \neq \emptyset$, then either $M^G \neq \emptyset$, or else G acts transitively on M.*

Remark 1.12. The condition $\partial((G/H_0)/H_0) \neq \emptyset$ occurs quite frequently: for example (G, H_0) a symmetric pair will satisfy this condition, as will often (G, H_0), where H_0 is maximal. Note however that this is not true for example with $(Sp(2), Sp(1))$, where $Sp(1)$ is maximal.

In Section 3 we will also need some basic facts about closed manifolds of cohomogeneity one which we recall here for convenience. If $dim(M/G) = 1$, then M/G is either a circle or an interval. In the first case, all orbits are principal and $\pi : M \to X = M/G$ is a fibration. Since we are interested in positively curved manifolds here, only the second case can arise by the Bonnet-Myers theorem. All interior points of the interval correspond to the principal orbits, $E = G/H$, and the endpoints of the interval correspond to two exceptional orbits $B_i = G/K_i$, $i = 0, 1$. In terms of this data, M is exhibited as the union of tubular neighborhoods $DB_i \to B_i, i = 0, 1$, with common boundary $\partial DB_0 \simeq \partial DB_1 \simeq E$. In particular, $\pi_i : E = G/H \to G/K_i = B_i$, $i = 0, 1$, are bundles with sphere fibers $K_i/H = S^{l_i}$.

Conversely, given

$$\begin{array}{ccccc} & & K_0 & & \\ & \subset & & \subset & \\ H & & & & G, \quad K_i/H = S^{l_i}, \quad i = 0, 1, \\ & \subset & & \subset & \\ & & K_1 & & \end{array} \tag{1.13}$$

we can reconstruct a cohomogeneity one G-manifold as

$$M = (G \times_{K_0} D^{(l_0+1)}) \bigcup_{G/H} (G \times_{K_1} D^{(l_1+1)}). \tag{1.14}$$

Note that given the isomorphism classes of bundles $DB_i \to B_i$, different possibilities for M can arise via different glueing maps $\partial DB_0 \simeq \partial DB_1$. Such glueing maps are G-equivariant, and are determined by an element $n \in N(H)$ (cf. (2.6)). In the description (1.13) above, this simply corresponds to replacing only one of the K_i's by its conjugate nK_in^{-1}. We further note that under the assumption that the manifold in question is simply-connected, H is connected, when $K_i/H \neq S^1$, $i = 0, 1$. The case of finite extensions H of H_0 and (possibly of) K_i, $i = 0, 1$ in G is possible only if one of the K_i/H is a circle. Before we confine our investigation to specific groups, we state one more useful general fact.

Synge (type) Lemma 1.15. *Let M be a positively curved manifold and V and W two non-intersecting, totally geodesic submanifolds of M. Then $dim(V) + dim(W) < dim(M)$.*

In our context, the submanifolds V and W in (1.15) will arise as fixed point sets for transformation groups $K \subset G$. Although we will not use it here, we remark that (1.15) holds for orbits spaces as well, and even general Alexandrov spaces (cf. [**?**]).

We point out that the utility of the methods developed in this section is amplified by the obvious fact that they also apply to all subgroups of a given transformation group. Since we are primarily interested in large groups, this will play a significant role as we proceed.

2. Fixed point homogeneous manifolds

In this section we will classify (up to equivariant diffeomorphism) positively curved, fixed point homogeneous manifolds, that is, manifolds, M, for which the fixed point cohomogeneity

$$cohom\, fix(M, G) = dim(M/G) - dim(M^G) \tag{2.1}$$

is minimal, i.e., equal to 1.

We need only consider the case in which $M^G \neq \emptyset$. If B_0 is a component of M^G with maximal dimension, i.e., $dim(B_0) = dim(M^G)$, then clearly the codimension of B_0 in $X = M/G$ is one more than the cohomogeneity of any normal sphere to B_0 under the induced G-action. Thus, if $cohomfix(M, G) = 1$, we see that G acts transitively on the normal spheres to B_0. In particular, B_0 is a component of ∂X. Moreover, for $\epsilon > 0$ sufficiently small, the ϵ-neighborhood of B_0 in X is a smooth manifold with boundary B_0, and all orbits in $B(B_0, \epsilon) - B_0$ are principal. As a special case of the Structure Lemma (1.9), we derive the following (cf. also (1.4)):

Structure Theorem 2.2. *Let M be a positively manifold with an (almost) effective, isometric G-action of fixed point cohomogeneity one and $M^G \neq \emptyset$. If B_0 is a component of M^G with maximal dimension, then the following hold:*

(i) There is a unique orbit, $B_1 = G(p_1) \simeq G/G_{p_1}$, at maximal distance to B_0 (the "soul" orbit).

(ii) All orbits in $M - (B_0 \bigcup B_1)$ are principal and diffeomorphic to $S^k \simeq G/H$, the normal sphere to B_0.

(iii) There is a G-equivariant decomposition of M, as

$$M = DB_0 \bigcup_E DB_1,$$

where DB_0, DB_1 are the normal disc bundles of B_0, B_1, respectively, in M, with common boundary E when viewed as tubular neighborhoods.

(iv) All G_{p_1}-orbits in the normal sphere S^l to B_1 at p_1 are principal and diffeomorphic to G_{p_1}/H. Moreover, B_0 is diffeomorphic to S^l/G_{p_1}.

We leave the details of the proof to the reader and point out only that if $dim B_0 > 0$ then $B_0 = \partial X$. However, if M^G is finite, and hence B_0 is a point, then X is an interval and B_0 is one of the boundary points. The other boundary point is either another fixed point for G (in fact, the only other one), or else the orbit at maximal distance from B_0 (cf. (1.4)). In either case, (2.2) holds as stated.

Note that implicitly in (2.2), we have exhibited two spherical fiber bundles:

$$K/H \to S^l \to S^l/K \simeq B_0, \tag{2.3}$$

$$K/H \to S^k \simeq G/H \to G/K \simeq B_1, \tag{2.4}$$

where $K = G_{p_1}$ is the isotropy group of the soul orbit. This already imposes severe topological restrictions due to Browder [**?**], from which it is possible to deduce that, at least cohomologically, any M as in (2.2) looks like a finite quotient of a rank-one symmetric space. Utilizing the restrictions on the pair (G, H) expressed in (2.4), we will in fact obtain such a description (Theorem (2.8)) up to (equivariant) diffeomorphism. To acheive this, we need the following:

Uniqueness Lemma 2.5. *Let M and $\hat{M}$ be two (Riemannian) G-manifolds with structure as in (2.2), i.e., there exist components $B_0 \subset M^G, \hat{B}_0 \subset \hat{M}^G$ and orbits $B_1 \subset M, \hat{B}_1 \subset \hat{M}$, such that (ii)-(iv) of (2.2) hold. If, in addition, the bundles $DB_1 \to B_1$ and $D\hat{B}_1 \to \hat{B}_1$ are G-equivalent, then M and $\hat{M}$ are G-diffeomorphic.*

Proof. We will show that under the assumptions above, any G-equivariant bundle isomorphism $f : DB_1 \to D\hat{B}_1$ extends to a G-equivariant diffeomorphism from M to $\hat{M}$. To do this, we will extend the restriction $h = f| : E \to \hat{E}$ to a G-equivariant bundle map $g : DB_0 \to D\hat{B}_0$. Namely, let g be the unique radial extension of h. Then it clearly follows that, $F = g \bigcup_h f : DB_0 \bigcup_E DB_1 \to D\hat{B}_0 \bigcup_{\hat{E}} D\hat{B}_1$ is a G-equivariant homeomorphism, $F| : M - B_0 \to \hat{M} - \hat{B}_0$ is a diffeomorphism, and so is $F| : B_0 \to \hat{B}_0$ (in fact $g|B_0 \simeq h/G$). To check that $g : DB_0 \to D\hat{B}_0$ is a diffeomorphism, it therefore suffices to see that it is linear on each fiber. Since isometries of the standard sphere $S^k = G/H$ are restrictions of linear maps of $R^{k+1} \supset D^{k+1}$, the desired linearity is an immediate consequence of the following useful fact:

Sublemma 2.6. *Let G be a connected, compact Lie group, and H a closed subgroup. Then for any G-equivariant map $F : G/H \to G/H$, there is an $n \in N(H) \subset G$ such that*

$$F(gH) = gnH.$$

Moreover, F is an isometry for any homogeneous metric on G/H which is induced from an $Ad(N(H))$-invariant metric on G.

Proof. Set $F(H) = nH$. Then $F(gH) = gnH$ for all $g \in G$, by equivariance. However, this is only well-defined if $nHn^{-1} = H$. The following simple calculation:

$$\begin{aligned} dist(F(g_1H), F(g_2H) &= dist(g_1nH, g_2nH) \\ &= dist(n^{-1}g_1^{-1}g_2nH, H) \\ &= dist(g_1^{-1}g_2H, nHn^{-1}) \\ &= dist(g_2H, g_1H) \end{aligned}$$

proves the isometry claim. q.e.d.

In order to fully exploit the Structure Theorem (2.2) and the Uniqueness Lemma (2.5), we shall now use the restrictions imposed on G by the requirement that $G/H = S^k$ (cf. also (2.4)). In fact, using the classification of groups that can act transitively on spheres (cf. [**?**], [**?**], [**?**] and [**?**]), we can assume, (by possibly replacing G by a subgroup) that the pair (G, H) is one of the following:

$$(2.7)\quad \begin{cases} (a_{k+1})(G,H) = (SO(k+1), SO(k)) & (k \geq 1), \\ (b_{m+1})(G,H) = (SU(m+1), SU(m)) & (k = 2m+1 \geq 3), \\ (c_{m+1})(G,H) = (Sp(m+1), Sp(m)) & (k = 4m+3 \geq 7), \\ (d)(G,H) = (G_2, SU(3)) & (k = 6), \\ (e)(G,H) = (Spin(7), G_2) & (k = 7), \\ (f)(G,H) = (Spin(9), Spin(7)) & (k = 15). \end{cases}$$

The strategy is now to assume that M is a fixed point homogeneous, positively curved G-manifold, where G is one of the groups in (2.7), and H is the corresponding principal isotropy subgroup. In each case, we determine all potential "soul"-isotropy groups K, such that $H \subset K \subset G$ satisfies (2.4). For those K which cannot be excluded on the basis of the Structure Theorem (2.2), we find an explicit model $\hat{M}$ with the same slice representation at the soul orbit and then apply the Uniqueness Lemma (2.5).

We know that if $cohomfix(M,G) = 1$, $M^G \neq \emptyset$ and G is one of the groups listed in (2.7), then $codim(M^G) = n, 2n, 4n, 7, 8,$ or 16 corresponding to the cases (a_n), (b_n), (c_n), (d), (e), or (f), respectively. We have used this fact in the formulation of our first main result.

Classification Theorem 2.8. *Let M be a closed, connected, fixed point homogeneous Riemannnian manifold. Then M supports an effective and isometric G-action, where G is one of the groups $SO(n), SU(n)$, $Sp(n), G_2, Spin(7)$, or $Spin(9)$ and $codim M^G = n, 2n, 4n, 7, 8,$ or 16, respectively. If moreover, $sec(M) > 0$, then M is G-equivariantly diffeomorphic to one of the following:*

(a_n) *S^m, RP^m $(m \geq n)$, or in addition, when $n = 2$, S^m/Z_q $(q \geq 3)$ or CP^m;*

(b_n) *S^m, S^m/Z_q $(m \geq 2n)$ or CP^m $(m \geq n)$, or in addition, when $n = 2$, S^m/Γ $(\Gamma \subset SU(2), (m \geq 5))$, CP^m/Z_2 $(m$ odd$)$ or HP^m;*

(c_n) *S^m, S^m/Γ $(\Gamma \subset Sp(1),\ m \geq 4n)$, CP^m $(m \geq 2n)$, CP^m/Z_2 $(m > 2n$ odd$)$ or HP^m $(m \geq n)$;*

(d) *S^m, or RP^m $(m \geq 7)$;*

(e) *S^m or RP^m $(m \geq 8)$; or* (*f*) S^m, RP^m ($m \geq 16$) *or* CaP^2,

where G *in case* (a_n) *is* $SO(n)$, *etc. as in (2.7).*

Proof. First note that the least restrictive cases are $(a_2), (b_2) = (c_1)$ and (f). By abuse of formalism, this is because $(a_k) \Rightarrow (a_{k+1})$, $(b_l) \Rightarrow (b_{l+1})$, $(c_m) \Rightarrow (c_{m+1})$, and $(b_3) \Rightarrow (d) \Rightarrow (e)$ by standard representation theory. Since (a_2) was proven in [**?**], we will discuss only the cases (b_2) and (f) here, and leave the remaining more restrictive cases to the reader.

Case (b_2)**.** Let B_0 be a component of $M^{SU(2)}$ with $codim B_0 = 4$, and B_1 the corresponding soul orbit. $SU(2)$ acts freely on $M - (B_0 \bigcup B_1)$, and the structure of M is determined by the slice representation of the isotropy group $K = G_{p_1}$ at $p_1 \in B_1$, according to (2.2) and (2.5). For K, there are the following possibilities:

(i) $K = SU(2)$,

(ii) $K = T^1 = S^1$, (ii)′ $K = N(T^1)$ $(N(T^1)/T^1 \simeq Z_2)$,

(iii) $K = \{1\}$, (iii)′ $K = \Gamma$ (Γ finite).

Subcase (i). $B_1 = \{p_1\} \subset M^{SU(2)}$, and $SU(2)$ acts freely on the tangent sphere S^l to M at p_1. Consequently, $l = 4m - 1$, the action of $SU(2) = Sp(1)$ on $T_{p_1}M \simeq R^{4m} \simeq H^m$ is the Hopf action (cf. [**?**, Sec. 5]), and $B_0 \simeq S^l/SU(2) = HP^{m-1}$. Now take $\hat{M} = HP^m$ with the obvious $SU(2) = Sp(1)$-action fixing $\hat{B}_1 = HP^{m-1}$ and the point $\hat{p_1} = \hat{B}_1$ at maximal distance from HP^{m-1}. By the Uniqueness Lemma (2.5), M is $SU(2)$-diffeomorphic to HP^m.

Subcase (ii). $B_1 = SU(2)/S^1 \simeq CP^1$ and $K = S^1$ acts freely on the normal sphere S^l to B_1 in M at p_1. In particular, $l = 2m - 3$ and $S^{2m-3} \to S^{2m-3}/S^1 = CP^{m-2} \simeq B_0$ is the Hopf map. By the slice theorem, the normal bundle $DB_1 \subset V_1 \to B_1$ to B_1 in M is isomorphic to

$$SU(2) \times_{S^1} R^{2m-2} \to SU(2)/S^1.$$

Now take $\hat{M} = CP^m$ with the natural action of $U(m+1)$. The $SU(2)$-action on $\hat{M}$ given via the standard inclusion $SU(2) \subset U(m+1)$ obviously fixes $\hat{B}_0 = CP^{m-2}$ and acts canonically on $\hat{B}_1 = CP^1$ at maximal distance from CP^{m-2}. By the slice theorem, the normal bundle to $\hat{B}_1$ in $\hat{M}$ is $SU(2)$-equivalent to the normal bundle of B_1 in M. Hence by (2.5), M and CP^m are $SU(2)$-diffeomorphic.

Subcase (iii). $B_1 \simeq SU(2)$, and $SU(2)$ acts freely on $M - B_0$. Since $K = \{1\}$, $B_0 \simeq S^l$, the normal sphere to B_1 in M at p_1. Again by the slice theorem, the normal bundle $DB_1 \subset V_1 \to B_1$ of B_1 in M is trivial. Now take $\hat{M} = S^l * SU(2) = S^{l+4}$ with the obvious $SU(2)$-action fixing $\hat{B}_0 = S^l$ and acting by left multiplication on $\hat{B}_1 = SU(2) \simeq S^3$ at maximal distance from S^l in S^{l+4}. By (2.5), M is $SU(2)$-diffeomorphic to S^{l+4}.

Subcase (ii)′. If $B_1 = SU(2)/N(T^1) \simeq CP^1/Z_2$, we argue as in (ii) that $l = 2m-3$ and $B_0 \simeq S^{2m-3}/N(T^1) = CP^{m-2}/Z_2$. This, however, is only possible if $m-2$ is odd. In that case, there is indeed an action of $SU(2)$ on $\hat{M} \simeq CP^m/Z_2$ which models M in the sense of (2.5). In fact, if $\tau : CP^m \to CP^m$ is the involution defining CP^m/Z_2, then the $SU(2)$- action on CP^m described above takes τ-orbits to τ-orbits (in homogeneous coordinates,

$$\tau([z_1, ..., z_{2n}; z_{2n+1}, z_{2n+2}]) = [\overline{z_{n+1}}, ..., \overline{z_{2n}}, -\overline{z_1}, ..., -\overline{z_n}; \overline{z_{2n+2}}, -\overline{z_{2n+1}}]$$

if $m = 2n+1$).

Subcase (iii)′. If $B_1 = SU(2)/\Gamma$, the finite subgroup

$$K = \Gamma \subset SU(2)$$

acts freely on the normal sphere S^l to B_1 in M at p_1, and $B_0 \simeq S^l/\Gamma$. The normal bundle to B_1 in M is isomorphic to

$$SU(2) \times_\Gamma R^{l+1} \to SU(2)/\Gamma.$$

by the slice theorem. Now consider $\hat{M} = S^l * SU(2)/\Gamma \simeq S^{l+4}/\Gamma$ where Γ acts on S^l as above and on $SU(2)$ by right translations. The $SU(2)$-action on $S^l * SU(2)$ described in (iii) induces an action on $\hat{M}$ with $\hat{B}_0 = S^l/\Gamma$. Since the normal bundles of B_1 in M and of $\hat{B}_1$ in $\hat{M}$ are isomorphic, we are done by (2.5).

Case (f). Let B_0 be a component of $M^{Spin(9)}$ with $codim(B_0) = 16$, and $B_1 = Spin(9)(p_1)$ be the corresponding soul orbit. Since the principal isotropy subgroup is $H = Spin(7)$, and H is necessarily embedded in $Spin(8) \subset Spin(9)$ via the spin representation (ref?), there are only the following possibilities for potential soul isotropy subgroups K:

(i) $K = Spin(9)$

(ii) $K = Spin(8)$;

(ii)′ $K = N(Spin(8))$ $(N(Spin(8)/Spin(8) = Z_2)$

(iii) $K = Spin(7)$;

(iii)′ $K = N(Spin(7)) \subset Spin(8)$ $(N(Spin(7))/Spin(7) = Z_2)$

Subcase (i). $B_1 = \{p_1\} \subset M^{Spin(9)}$, and all orbits of the $Spin(9)$-action on the tangent sphere $S^l \subset T_{p_1}M$ are principal and diffeomorphic to S^{15}. Since there is no proper fibration of a sphere with S^{15} as fiber (cf. e.g. [**?**]), we conclude that $l = 15$ and $B_0 \simeq Spin(9)/Spin(7) = S^{15}$. Taking $\hat{M} = S^{16}$, the suspension of $Spin(9)/Spin(7) = S^{15}$, we see via (2.5) that M is $Spin(9)$-equivalent to S^{16}.

Subcase (ii). $B_1 \simeq Spin(9)/Spin(8) = S^8$, and $Spin(8)$ acts on the normal sphere S^l to B_1 at p_1, such that all orbits are principal and diffeomorphic to $Spin(8)/Spin(7) = S^7$. Moreover, $B_0 \simeq S^l/Spin(8)$. Thus either $l = 7$ and $B_0 = \{p_0\}$, or $l = 15$ (cf. e.g. [**?**]). However, as shown in [**?**, p.236], there is no fibration of S^{15} with S^7 fibers, all of which are also orbits of a group action on S^{15}. Hence $l = 7$ and the normal sphere bundle $E \to B_1$ is $Spin(9)$-equivalent to the Hopf fibration

$$\begin{aligned} S^7 =& Spin(8)/Spin(7) \to Spin(9)/Spin(7) \\ =& S^{15} \to S^8 = Spin(9)/Spin(8). \end{aligned}$$

The same picture is apparent for the sub-action of $Spin(9) \subset F_4$ on $\hat{M} = CaP^2$. Therefore M is $Spin(9)$-equivalent to CaP^2 by (2.5).

Subcase (iii). $B_1 = Spin(9)/Spin(7) = S^{15}$, and all orbits in $M - B_0$ are diffeomorphic to S^{15} and principal. In particular, $B_0 \simeq S^l$, where S^l is the normal sphere to B_1 in M at p_1. Furthermore, the normal bundle to B_1 in M is $Spin(9)$-isomorphic to the trivial bundle,

$$Spin(9)/Spin(7) \times R^{l+1} \to Spin(9)/Spin(7),$$

where the action on R^{l+1} is trivial. Pick $\hat{M} = S^l * Spin(9)/Spin(7) = S^{l+16}$ with the obvious $Spin(9)$-action fixing $\hat{B}_0 = S^l$ and acting canonically on $Spin(9)/Spin(7) = S^{15}$. Via (2.5), we see that M is $Spin(9)$-diffeomorphic to S^{l+16}.

Subcase (ii)′. If $B_1 = Spin(9)/N(Spin(8)) \simeq S^8/Z_2$, we see, as in subcase (ii) above, that all $N(Spin(8))$-orbits in the normal sphere S^l

to B_1 at p_0 must be diffeomorphic to $N(Spin(8))/Spin(7) = S^7 \coprod S^7$. This excludes $l = 7$, and we exclude $l = 15$ as in subcase (ii). Thus $K = N(Spin(8))$ cannot occur as a soul isotropy subgroup.

Subcase (iii)′. If $B_1 = Spin(9)/N(Spin(7)) = S^{15}/Z_2 = RP^{15}$, an argument, as in subcases (iii) and (ii)′, above shows that M is $Spin(9)$-diffeomorphic to $S^{l+16}/Z_2 = RP^{l+16}$. q.e.d.

Theorem A in the introduction is now an immediate corollary of Theorem 2.8.

3. Low-dimensional non-linear representations

The classification of positively curved manifolds with *maximal symmetry rank* [**?**], can also be viewed as a classification of the lowest-dimensional manifolds of positive curvature on which a given torus, T^k, can act (almost) effectively by isometries.

The principal issue in this section is to analyse the same question for the compact, connected simple Lie groups. Since we allow actions to be almost-effective, it suffices to consider simply-connected groups. Explicitly, the groups we are considering are: $Sp(n)(n \geq 2))$, $SU(n)(n \geq 2)$, $Spin(n)(n \geq 7)$, together with the exceptional groups G_2, F_4, E_6, E_7 and E_8.

Based on (0.2) in the introduction, we define inductively

$$(3.1) \qquad \begin{aligned} rep^+_{i+1}(G) = min\{n > rep^+_i(G) | G acts \text{ (almost) effectively} \\ \text{by isometries on some } M^n \text{ with } sec(M) > 0\} \end{aligned}$$

If we restrict our attention to *irreducible linear representations*, i.e., $M^n = S^n_1$, we use the notation $rep^S_0(G) < rep^S_1(G) < \cdots$, and it is obvious that $rep^+_0(G) \leq rep^S_0(G)$ for any compact Lie group G.

Since $Sp(n+1)/Sp(n)Sp(1) = HP^n$, $SU(n+1)/S(U(n)U(1)) = CP^n$, $Spin(n+1)/Spin(n) = S^n$, $G_2/SU(3) = S^6$, $F_4/Spin(9) = CaP^2$ all have positive curvature and each one is of the form G/H where $dim(H) < dim(G)$ is maximal, we read off the following simple fact:

Proposition 3.2. *If G is one of the simply connected simple groups other than E_6, E_7 or E_8, then*

$$rep^+_0(G) = dimG/H,$$

where $H \subset G$ is a proper subgroup of maximal dimension.

In contrast, it is well known that E_6, E_7 and E_8 cannot act transitively on a positively curved manifold. Hence from (1.11) we conclude that if G is one of these groups and M is positively curved manifold with $dim(M) = rep_0^+(G)$, on which G acts (almost) effectively, then the connected component H_0 of the principal isotropy subgroup H cannot be a maximal connected subgroup. In particular, we conclude

Proposition 3.3. *If G is one of E_6, E_7 or E_8, then*

$$rep_0^+(G) \geq dim(G/H) + 1,$$

where H is a proper subgroup of G with second lowest codimension.

In particular, this tells us that $rep_0^+(E_6) \geq 33$, $rep_0^+(E_7) \geq 56$ and $rep_0^+(E_8) \geq 115$. Further, we know that the lowest dimensional linear representations of these exceptional groups occur in complex dimensions 27, 56 and real dimension 248 respectively [?]. Thus $rep_0^+(E_6) \leq 52$, $rep_0^+(E_7) \leq 110$ and $rep_0^+(E_8) \leq 247$.

In the remaining part of this section we will classify low-dimensional positively curved manifolds on which the simple groups other than E_6, E_7 or E_8 can act (almost) effectively by isometries. We first observe that for these groups the lowest-dimensional irreducible linear representations yield transitive actions on the corresponding spheres. Using this fact together with the Fixed Point Lemma (1.11) we obtain:

Fixed Point Corollary 3.4. *Let G be a simply-connected, simple Lie group other than E_6, E_7 or E_8, and M a positively curved manifold on which G acts (almost) effectively by isometries. If the principal isotropy group H has maximal connected component H_0 and*

$$dim(M) \leq min\{2rep_0^S(G) + 1, rep_1^S(G)\},$$

then cohomfix(M,G)=1.

Another general situation in which fixed point homogeneous manifolds arise naturally occurs because of the simple fact that the principal isotropy group $H = G_p$ acts trivially on the normal space to the principal orbit $G(p) \simeq G/H$ at p.

Principal Isotropy Lemma 3.5. *Let M be a G-manifold with principal isotropy subgroup H, and isotropy representation*

$$H \times T_H G/H \to T_H G/H.$$

If $S_H \subset T_H G/H$ denotes the unit sphere and $cohom\, fix(S_H, K) = 1$ for some subgroup $K \subset H$, then $cohom\, fix(M, K) = 1$.

The point of this simple fact is that it typically applies to large subgroups H of a simple group G (other than E_6, E_7 or E_8).

When applying either (3.4) or (3.5), the cohomogeneity of $G \times M \to M$ is irrelevant. If, however, the principal isotropy subgroup $H \subset G$ is such that neither (3.4), nor (3.5), can be utilized, then we resort to other restrictions imposed by the assumption that

$$dim(M) = dim(M/G) + dim(G/H) \tag{3.6}$$

is relatively small. In particular, we consider only actions where the principal isotropy subgroup H has fairly small codimension in G. This in turn restricts all the possible isotropy subgroups and enhances the chances for using (1.3). So far, however, we have only been able to make systematic use of this approach when, in addition, $cohom(M, G) = dim(M/G) \leq 1$.

Theorem 3.7 (Symplectic Groups). *Let M be a simply-connected, closed manifold with $sec(M) > 0$. If $Sp(n+1), n \geq 1$, acts (almost) effectively by isometries on M and*

$$dim(M) \leq C(n) = \begin{cases} 8n - 3 = 2rep_0^+(Sp(n+1)), & n \geq 2 \\ 8 & n = 1, \end{cases}$$

then $dim(M) \geq 4n = rep_0^+(Sp(n+1)$, and M is diffeomorphic to one of either a sphere, a complex or quaternionic projective space, the flag-manifold $Sp(3)/(Sp(1))^3$ or the real homology sphere $Sp(2)/SU(2)$.

Proof. Let H denote the principal isotropy subgroup of

$$G = Sp(n+1)$$

acting on M. From (3.6) and $dim(M) \leq C(n)$ we have

$$dim(H) \geq dim(Sp(n+1)) - C(n). \tag{3.8}$$

An analysis of the possible connected subgroups $H_0 \subset Sp(n+1)$ satisfying (3.8) yields the following list:

(a)	$H_0 = Sp(n)Sp(1)$	$n \geq 1,$
(b)	$H_0 = Sp(n)U(1)$	$n \geq 1,$
(c)	$H_0 = Sp(n)$	$n \geq 1,$
(d)	$H_0 = Sp(n-1)Sp(2)$	$n \geq 2,$
(e)	$H_0 = Sp(n-1)(Sp(1))^2$	$n \geq 2,$
(f)	$H_0 = U(n+1)$	$n \leq 3,$
(g)	$H_0 = SU(n+1)$	$n \leq 3.$

In the case (g), we note that for $n \geq 2$ the action must be transitive, and since neither space obtained is of positive curvature, these cases do not occur. Note that for $n = 1$, case (g) coincides with case (c) (as $Sp(1) = SU(2)$) and we will treat it later. In the cases (d) and (f), we can use Corollary (3.4), and hence (2.8) since (G, H_0) is a symmetric pair and $rep_0^S(Sp(n+1)) = 4n+3$, $rep_1^S(Sp(n+1) = (n+1)(2(n+1)-1)-1, n \geq 2$ by standard representation theory. For case (f) $(n = 1)$, we note that the two lowest-dimensional irreducible linear representations of $Sp(2)$ occur in dimensions 5 and 8, but both are transitive on the corresponding spheres. Thus, $cohomfix(M, Sp(2)) = 1$ with $H_0 = U(2) \subset Sp(2)$.

Moreover, except for $Sp(1)$ in case (c) $(n = 1)$, all of these groups admit only one embedding in $Sp(n+1)$, up to conjugation. Aside from the standard embedding $Sp(1) = Sp(1) \times \{1\} \subset Sp(1) \times Sp(1) \subset Sp(2)$, we can embed $Sp(1)$ via the diagonal $Sp(1) = \Delta(Sp(1) \times Sp(1)) \subset Sp(2)$ and as a maximal subgroup $Sp(1) \subset Sp(2)$. The diagonal embedding can also be viewed as $Sp(1) = SU(2) \subset U(2) \subset Sp(2)$.

In the first three cases, in which the embedding is standard, i.e., (a) and (b) for all n, and (c) for $n \geq 2$, we apply the Principal Isotropy Lemma (3.5) to the subgroup $K = Sp(n) \subset H_0 \subset H$, and then appeal to the Classification Theorem (2.8).

In order to complete case (c), it remains to consider the case where the embedding of H_0 is not standard. The only possible dimensions for M are then 7 or 8. If $dim(M) = 7$, the $Sp(2)$-action is transitive and $M = Sp(2)/Sp(1)$ is the Berger homology sphere [?]. If $dim(M) = 8$, the $Sp(2)$-action is of cohomogeneity one. In particular, the only case consistent with (1.13) is that in which the embedding of H_0 is diagonal, and we will rule out this case. Note that the only possible subgroups K_0, K_1 between $Sp(1) \simeq SU(2)$ and $Sp(2)$ satisfying the conditions of (1.13) are $Sp(1)^2$ and $U(2)$.

Assume first that $K_0 = K_1 = U(2)$. We remark further that we may exclude (by means of a general argument) the case in which $K_0 = K_1 \neq G$ [?] for a cohomogeneity one manifold of positive curvature. However, for the sake of comleteness we will prove each individual case as it arises (cf. (3.7) case (e), (3.9) cases (d) and (h), (3.11) case (c) and (3.13)). Then the principal orbit $E = Sp(2)/Sp(1)$ fibers over the exceptional orbits $B_0 \simeq B_1 \simeq Sp(2)/U(2)$ with common fibers $S^1 \simeq U(2)/Sp(1)$, and M fibers over $B_0 \simeq Sp(2)/U(2)$ with fiber S^2. Moreover, $E^{SU(2)} \subset E$ consists of two disjoint circles (namely the orbit of $N(SU(2))$), each of which is a fiber over B_0 and B_1. The corresponding S^2-fibers over B_i in M are also fixed by $H = SU(2)$, and in fact they are components of $M^{SU(2)}$. The latter fact is seen via the isotropy representation of $U(2)(\supset SU(2))$ at the corresponding fixed points in $B_i, i = 0, 1$. Now fix $S^1 \subset SU(2) \subset U(2)$, and consider M^{S^1}. At each of the fixed points $p_i, q_i \in B_i$, $i = 0, 1$, for $U(2)$ acting on B_i, the isotropy representation of $U(2)$ reveals that the corresponding components of M^{S^1} are 4-dimensional. By the Synge Lemma (1.15), they must all be contained in the same component, which, however, clearly contains the above (disjoint) S^2-components of $M^{SU(2)}$, impossible again by (1.15).

From the above, we conclude that one of the K_i's is $Sp(1) \times Sp(1)$, and it is not difficult to see that the action of $Sp(2) \simeq Spin(5)$ is not effective on M as exhibited in (1.14). The corresponding effective action is by $G = SO(5)$ with principal isotropy subgroup $H = SO(3)$ embedded in the standard way. Moreover, K_0, K_1 are either $SO(4)$ or $SO(3) \times SO(2)$. If $K_0 \simeq K_1 = SO(4)$ we proceed as follows. $H = SO(3)$ fixes two disjoint circles in $E = G/H$, each of which is mapped to one circle in B_0 and in B_1. Indeed, $M^{SO(3)}$ is a torus. This is a contradiction, since it is also totally geodesic and hence positively curved.

To complete case (c), it remains to consider the cohomogeneity one action on M^8 by $G = SO(5)$, with $H = SO(3)$, $K_0 = SO(3)SO(2)$ and $K_1 = SO(4)$. First observe that the $SO(3)$-factor of K_0 fixes a totally geodesic S^2 in M (that is, two points in B_0, two circles in E and one circle in B_1). Fix $L = SO(2) \subset SO(3)$ and consider M^L. From the isotropy representations, we see that M^L is a 4-manifold. Its intersection with B_1 is a 2-sphere. Now the $SO(2)$-factor of K_0 acts on M^L preserving $M^{SO(3)}$. It has exactly four fixed points (2 in B_0 and 2 in B_1), which is impossible if M and hence M^L have positive curvature, by the Extent Lemma (1.3) (cf. [?]).

Note that we must also worry about finite extensions of H_0 and K_i, $i = 0, 1$, in this case, since K_i/H can be a circle for at least one i. In the case where $K_0 = K_1 = U(2)$, we may extend H_0 to $H_0 \times Z_k = H$. However, the argument used to exclude the case with H connected works as well in this case. In the case where $K_0 = SO(3)SO(2)$ and $K_1 = SO(4)$, any finite extension of $H_0 = SO(3)$ in G must include the corresponding finite extension of $SO(4)$, since otherwise K_1/H will not be a sphere. This leaves us with only one possibility: $H = SO(3) \times Z_2 = O(3)$ and $K_1 = O(4)$. In this case, $M = CP^4$ (cf. [**?**].

We now turn to the remaining case (e). We will show that only $n = 2$ can occur, and in that case M is either homogeneous, i.e., $M = Sp(3)/Sp(1)^3$, or it has cohomogeneity one and $M = S^{13}$. In fact, for all $n \geq 2$ in the given range of dimensions for M, M must either be homogeneous or of cohomogeneity one. The classification of homogeneous manifolds with positive curvature leaves only the flagmanifold in dimension 12 above as a possibility. On the other hand, if M has cohomogeneity one, its data is given according to (1.13) as:

$$\begin{array}{ccccc} & & K_0 = Sp(n-1)Sp(2) & & \\ & \subset & & \subset & \\ H = Sp(n-1)Sp(1)Sp(1) & & & & G = Sp(n+1) \\ & \subset & & \subset & \\ & & K_1 = Sp(n-1)Sp(2) & & \end{array}$$

where $K_i/H \simeq Sp(n-1)Sp(2)/Sp(n-1)(Sp(1))^2$ are 4-spheres for both i. Moreover, for $n \geq 3$ there is only one possible embedding of $K_0 = K_1$. For $n = 2$, however, we can also embed $K_0 \simeq K_1$ by permuting the factors. This is exactly the description of S^{13} under the representation $\wedge^2\nu - \theta$ of $Sp(3)$ (notation from [**?**]). It remains to show that $K_0 = K_1$ cannot occur when $sec(M) > 0$.

The general case $n \geq 3$ reduces to the case $n = 2$, since it is easy to see from the isotropy representations that M^{8n-3} contains a totally geodesic submanifold N^{13} of dimension 13, which is fixed by $Sp(n-2) \subset Sp(n-2)(Sp(1))^3$, and on which $Sp(3)$ acts by cohomogeneity one, with $H = (Sp(1))^3$ and $K_0 = K_1 = Sp(1)Sp(2)$. Thus, it suffices to show that the case $n = 2$ in which $K_0 = K_1$ cannot occur.

Here, we see that the principal orbit $E = Sp(3)/(Sp(1))^3$ fibers over the two exceptional orbits $B_0 \simeq B_1 \simeq Sp(3)/Sp(1)Sp(2) = HP^2$ with common fibers S^4. In particular, M^{13} fibers over HP^2 with fiber S^5. Let $p_i \in B_i$, $i = 0, 1$, be the fixed points of $K = K_0 = K_1 = Sp(1)Sp(2)$ on B_i, and $S^4 \simeq HP^1 \simeq N_i \subset B_i$ the K-orbits in B_i at maximal distance from p_i in B_i. Note that the $Sp(1)$-factor of K acts trivially on N_i, and that the $Sp(2)$-factor acts transitively on N_i with principal isotropy $Sp(1)Sp(1)$. For the $Sp(1)$-factor of K, consider $M^{Sp(1)}$. At $p_i \in B_i$, $Sp(1)$ acts freely on the tangent sphere of B_i and trivially on the normal sphere. In particular, the 5-sphere fiber of $M \to B_i$ suspended between p_0 and p_1 is totally geodesic, and a component of $M^{Sp(1)}$. Now consider the action of $Sp(1)$ at points in $N_i \subset B_i$. On the normal sphere to N_i inside B_i, the action is free. Thus the component of $M^{Sp(1)}$ containing N_i is determined by the action of $Sp(1)$ normal to B_i at N_i. From representation theory, this $Sp(1) \times S^4 \to S^4$ action is either almost-effective and factors through $SO(3)$, or it is the suspension of the standard free action on S^3. In the first scenario, we find disjoint totally geodesic submanifolds of M of dimensions 9 and 5, contradicting (1.15). In the second scenario, we find a 5-dimensional component, V^5, of $M^{Sp(1)}$ containing the B_i's. Moreover, the $Sp(2)$-factor of K acts on V^5 with cohomogeneity one, and all orbits are of principal type $S^4 \simeq B_i$. In particular, V^5 fibers over S^1, which is impossible, since V^5 has positive curvature. q.e.d.

Theorem 3.9 (Unitary Groups). *Let M be a simply-connected, closed manifold with $sec(M) > 0$. If $SU(n+1)$, $n \geq 1$ acts (almost) effectively by isometries on M and*

$$dim(M) \leq C(n) = \begin{cases} 4n-2 = 2rep_0^+(SU(n+1)), & n \geq 3, \\ 7 & n = 2, \\ 4 & n = 1, \end{cases}$$

then $dim(M) \geq 2n = rep_0^+(SU(n+1))$, and M is diffeomorphic to one of the following: a sphere, a complex projective space, the flagmanifold $SU(3)/T^2$, an Aloff-Wallach space $SU(3)/S^1_{k,l}$ or the Berger manifold $SU(5)/Sp(2)S^1$.

Proof. As in the proof of (3.7), we list all the possibilities for the connected component, H_0, of the principal isotropy subgroup, H, under the restriction $dim(M) \leq C(n)$. The list is:

(a)	$U(n) = S(U(n)U(1))$	$(n \geq 1)$,
(b)	$SU(n)$	$(n \geq 2)$,
(c)	$S(U(n-1)U(2))$	$(n \geq 3)$,
(d)	$SU(n-1)SU(2)$	$(n \geq 3)$,
(e)	$SO(n+1)$	$(2 \leq n \leq 4)$,
(f)	$Sp\left(\dfrac{n+1}{2}\right)$	$(n = 3, 5)$,
(g)	$Sp(2)S^1$	$(n = 4)$,
(h)	T^2	$(n = 2)$,
(i)	S^1	$(n = 1, 2)$,
(j)	$\{1\}$	$(n = 1)$.

In all cases, with the exception of case (i) $(n = 2)$, there is only one embedding of H_0 in $SU(n+1)$, up to conjugation. In the first two cases we apply the Classification Theorem (2.8) via (3.5). In fact, in case (a) $(n = 1)$, $(M, U(1))$ is fixed point homogeneous and so is (M, K), with $K = SU(n)$ in the remaining cases.

The cases (c), (e) and (f) are all done via (3.4). We remark first that in all these cases (G, H_0) is a symmetric pair, and secondly that

$$rep_0^S(SU(n+1) = 2(n+1) - 1$$

and

$$rep_1^S(SU(n+1)) = n(n+1) - 1$$

for $n \geq 4$, and for $n = 3$,

$$rep_0^S(SU(4)) = rep_0^S(Spin(6)) = 5$$

and $rep_1^S(SU(4)) = 7$ (and both of these representations are transitive), and in case (e) $(n = 2)$, we have $rep_0^S(SU(3)) = 5$ and $rep_1^S(SU(3)) = 7$.

We now proceed to show that case (d) cannot occur. We remark first, that for dimension reasons, the action of $SU(n+1)$ must either be transitive or of cohomogeneity one. The first option is ruled out by the classification of positively curved homogeneous manifolds, and thus we assume that $cohom(M, SU(n+1)) = 1$.

Note first that for $n \geq 4$, $H_0 = H$ and the only possible groups satisfying (1.13) are:

$$
\begin{aligned}
H =& SU(n-1)SU(2) \subset S(U(n-1)U(2)) \\
=& K_0 = K_1 \subset SU(n+1) = G,
\end{aligned} \tag{3.10}
$$

and $N(H) = K = K_i$, $i = 0, 1$, $K/H = S^1$. Remark also that, as in the discussion of case (e) in (3.7), the subcases $(n \geq 5)$ reduce to the subcase $(n = 3)$. We will first rule out the subcase $(n = 4)$. Here, M fibers over $B_0 \simeq B_1 \simeq G/K \simeq G_{3,2}$ with fiber S^2, and K fixes isolated points $p_i \in B_i, i = 0, 1$. Let $A_i \subset B_i$ be the K-orbit at maximal distance from $p_i, i = 0, 1$. Then $A_i \simeq G_{2,1} \simeq CP^2$ is fixed by the $SU(2)$-factor of H. This implies that $SU(2)$ also fixes the normal bundles of B_i in M restricted to A_i, $i = 0, 1$. The resulting G-manifold is a component of $M^{SU(2)}$ and hence totally geodesic. It fibers over $A_i \simeq CP^2$ with S^2-fiber, and the $SU(3)$-factor of H acts on it by cohomogeneity 1 or 2, either of which gives us a contradiction; the first via the Principal Isotropy Lemma (3.5), and the second via the Fixed Point Lemma (1.11).

To complete case (d), we now proceed with the subcase $(n = 3)$. According to (1.13), the only possibilities for K_i, $i = 0, 1$ are $S(U(2)U(2)) \simeq Spin(4)Spin(2)$ and $Spin(5)$. And the argument in this case mirrors the argument made for case (e) $(n = 3)$ in (3.7). The only possible in this case is $M = CP^5$ where $G = SO(6)/Z_2$ (cf. [**?**]). The details are left to the reader.

In case (g), the $SU(5)$-action is either transitive or of cohomogeneity one. However, there are no subgroups K_i, between $H_0 = Sp(2)S^1$ and $SU(5)$, satisfying (1.13). Thus, $M^{13} = SU(5)/Sp(2)S^1$, the Berger example, is the only possibility here.

In case (h) as well, the $SU(3)$-action is either transitive or of cohomogeneity one. In the homogeneous case, we obtain the flagmanifold, $M^6 = SU(3)/T^2$. When the action is of cohomogeneity one, we note first that only $K_i \simeq U(2)$ satisfies (1.13). Moreover, there are only two choices for the pair (K_0, K_1): either $K_0 = K_1$ or $K_0 \neq g^{-1}K_0 g = K_1$ is embedded via a permutation of the coordinates. The latter case characterizes S^7, where $SU(3)$ acts on R^8 via the adjoint representation. We will show that the former does not occur. First note that M^7 fibers over $B_0 \simeq B_1 \simeq SU(3)/K = SU(3)/S(U(2)U(1)) = CP^2$ with S^3-fibers, and K fixes isolated points $p_i \in B_i$, $i = 0, 1$. Let $A_i \subset B_i$ be the K-orbits at maximal distance to p_i. Then $A_i \simeq U(2)/T^2 \simeq CP^2$, and there is an $S^1 \subset T^2$ which fixes all of A_i. This S^1 also acts on the normal bundle of B_i restricted to A_i, and therefore either fixes the whole normal bundle or a 1-dimensional sub-bundle. The latter is impossible, since it would yield a totally geodesic component of M^{S^1} of the form $S^2 \times S^1$. If, on the other hand, the whole normal bundle is fixed, we get a 5-dimensional component of M^{S^1}, namely the restriction of the S^3-fibration $M \to B_0$ to A_0. It is, however, also easy to see that all S^3-fibers are totally geodesic, so a contradiction in this case is reached via the Synge Lemma (1.15).

In case (i) ($n = 2$), the action of $SU(3)$ must necessarily be transitive, and hence M is an Aloff-Walach example $SU(3)/S^1_{k,l}$.

It remains to consider cases (i) and (j), ($n = 1$), where we have $SU(2)$-actions on manifolds with $dim(M) \leq 4$. If $H_0 = S^1$, we are done by (3.5). If $H_0 = \{1\}$ and $dim(M) = 4$, the $SU(2)$-action is of cohomogeneity one. The only possible groups, K_i, between $\{1\}$ and $SU(2)$ satisfying (1.13) are Z_2, S^1 or $SU(2)$. If Z_2 arises, then $\pi_1 \neq \{1\}$, and if $SU(2)$ does, then we are done by the Classification Theorem (2.8). In the case where $K_0 \simeq K_1 \simeq S^1$, $\chi(M) = 4$, which is impossible by [?].

Note that we must also worry about finite extensions here, since the principal orbit may fiber over the singular orbit with circle fiber. There are only two such $SU(2)$ actions, both of which are ineffective. The corresponding (ineffective) $SO(3)$ actions have principal isotropy subgroup Z_2 or $Z_2 \times Z_2$. In the first case $M = CP^2$ and the $SO(3)$ action is the restriction of the standard $SU(3)$ action on CP^2. In the second case $M = S^4$ and the action of $SO(3)$ on S^4 is via the representation $S^2\rho_2 - \theta$ (notation from [?]) [?].

Theorem 3.11 (Orthogonal Groups). *Let M be a simply-connected, closed Riemannian manifold with $sec(M) > 0$. If $Spin(n+1)$, $n \geq 6$ acts isometrically and (almost) effectively on M and*

$$dim(M) \leq C(n) = 2n = 2rep_0^+(Spin(n+1)),$$

then $dim(M) \geq n = rep_0^+(Spin(n+1))$, and M is diffeomorphic to a sphere, a complex projective space, or the Cayley plane.

Proof. As in the previous two theorems, we proceed to list the connected components of the possible principal isotropy subgroups, under the given dimensional restrictions. The possibilities are:

(a)	$H_0 = Spin(n)$,	$n \geq 6$,
(b)	$H_0 = Spin(n-1)S^1$,	$n \geq 6$,
(c)	$H_0 = Spin(n-1)$,	$n \geq 6$,
(d)	$H_0 = SU(4) \simeq Spin(6)$,	$n = 7$,
(e)	$H_0 = G_2$,	$n = 6$.

In all cases except (c) and (d), H_0 is a maximal connected subgroup of $G = Spin(n+1)$, and $\partial((G/H_0)/H_0) \neq \emptyset$, since (a) and (b) are symmetric pairs, and in case (e), $(G/H_0)/H_0$ is a closed interval. However, the only positively curved manifolds on which $Spin(n+1)$ can act transitively are the spheres $S^n = SO(n+1)/SO(n)$, $S^7 = Spin(7)/G_2$ and $S^{15} = Spin(9)/Spin(7)$. Suppose then that $Spin(n+1)$ does not act transitively on M. Then by the Fixed Point Lemma (1.11), $M^{Spin(n+1)} \neq \emptyset$. The action of $Spin(n+1)$ at the normal space to a point in $M^{Spin(n+1)}$ yields a representation of dimension less than or equal to $2n$ by the assumption on $dim(M)$. For $n \neq 6, 8, 9$, or 11 the two lowest linear representations of $Spin(n+1)$ are in dimensions $n+1$ and $\frac{n(n+1)}{2}$, and Corollary 3.4 together with (2.8) yields the desired result. The three lowest dimensional representations for $Spin(7)$ are of dimensions $7, 8, 21$; for $Spin(9)$: $9, 16, 36$; for $Spin(10)$: $10, 16, 45$; and for $Spin(12)$: $12, 64, 66$. Since the two lowest-dimensional representations for $Spin(7)$, as well as for $Spin(9)$, yield transitive actions on the corresponding spheres, the Fixed Point Lemma (1.11), together with (2.8), still suffices without further considerations. In the case of $Spin(10)$, the 16-dimensional representation (which is not transitive on S^{15}) might occur when $16 \leq dim(M) \leq 18$. However, since the principal isotropy subgroup for this action on S^{15} is neither $Spin(9)$, nor $Spin(8)S^1$ (known from representation theory, or can be seen via the Fixed Point Lemma applied to $Spin(10) \times S^{15} \to S^{15}$), this case does not arise.

We are left then with case (c) and (d). Note however that there is an outer automorphism of $Spin(8)$ (triality) which take $SU(4)$ to $Spin(6)$ and so case (d) is contained in case (c). For $n \neq 8$, there is only one embedding of $Spin(n-1)$ in $Spin(n+1)$, and $Spin(n+1)/Spin(n-1)$ does not carry a homogeneous metric of positive curvature, so for these n, $Spin(n+1) \times M \to M$ must be of cohomogeneity one and $dim(M) = 2n$. For the non-standard embedding of $Spin(7)$ in $Spin(9)$, however, $Spin(9)/Spin(7) = S^{15}$.

Now suppose that $Spin(n+1)$ acts (almost) effectively on M^{2n} with cohomogeneity one and principal isotropy subgroup $H = H_0 = Spin(n-1)$ embedded in the standard fashion (also when $n = 8$). The possible subgroups K_0, K_1 satisfying (1.13) are then $Spin(n-1)S^1$ and $Spin(n)$. Note also that there is only one embedding of $Spin(n-1)S^1$ in $Spin(n+1)$, whereas $Spin(n)$ admits embeddings parametrized by S^1 (these, however, all yield the same manifold up to diffeomorphism). As we have seen in previous theorems, it suffices to consider only the subcases $(n = 4)$ and $(n = 5)$, and since $Spin(5) = Sp(2)$ and $Spin(6) = SU(4)$, both have already been ruled out. Thus, case (c) does not occur when $H = H_0 = Spin(n-1) \subset Spin(n+1)$ is standard. If $H/H_0 \neq \{1\}$, then $M = CP^n$ (cf. [**?**]).

Finally, we consider case (c) $(n = 8)$, where the embedding of $Spin(7)$ in $Spin(9)$ is not standard, i.e., suppose $Spin(9)$ acts on M^{16} by cohomogeneity one, with principal isotropy subgroup $H_0 = Spin(7)$ embedded via the spin-representation in $Spin(8) \subset Spin(9)$. According to (1.13) there are only two possibilities for K_i, $i = 0, 1$, corresponding to 3 possible scenarios: (i) $K_0 = K_1 = Spin(9)$, (ii) $K_0 = Spin(9)$ and $K_1 = Spin(8)$ and (iii) $K_0 \simeq K_1 = Spin(8)$. The first two cases correspond to S^{16} and CaP^2 respectively, and we show how to rule out the third case here. To do so, we consider the orbit space $M/Spin(8)$. Here $G/K_i = S^8$, and the action of $Spin(8)$ on the singular orbits fixes 2 isolated points and is transitive on the normal S^7 to both of these points, that is, $M/Spin(8)$, $(G/K_i)/Spin(8) =$I, $i = 0, 1$. $M^{Spin(8)}$ consists of four isolated points. Moreover the induced representation of $Spin(8)$ on the tangent spaces R^{16} to the fixed points in M yields a cohomogeneity one action on S^{15} with principal isotropy group G_2 and orbit space $[0, \pi/2]$. A contradiction is thus obtained via the Extent Lemma (1.3).

q.e.d.

The exceptional groups G_2 and F_4 have $SU(3)$ and $Spin(9)$ as subgroups of maximal dimension, respectively. Since $G_2/SU(3) = S^6$ and $F_4/Spin(9) = CaP^2$ both have (homogeneous) positive curvature metrics, it follows that $rep_0^+(G_2) = 6$ and $rep_0^+(F_4) = 16$. Further, it is known that $rep_1^S(G_2) = 14$ and $rep_0^S(F_4) = 25$ and $rep_1^S(F_4) = 51$. By arguments as in the previous three theorems, we derive:

Theorem 3.12. *Let M be a positively curved manifold with* $\pi_1(M) = \{1\}$. *If* G_2 *acts (almost) effectively on* M *by isometries and*

$$dim(M) \leq 11 = 2rep_0^+(G_2) - 1,$$

then $dim(M) \geq 6 = rep_0^+(G_2)$, *and* M *is diffeomorphic to a sphere.*

Theorem 3.13. *Let* M *be a simply-connected manifold with* $sec(M) > 0$, *as above. If* F_4 *acts isometrically and (almost) effectively on* M *and*

$$dim(M) \leq 25 = 2rep_0^+(F_4) - 7,$$

then $dim(M) \geq 16 = rep_0^+(F_4)$, *and* M *is diffeomorphic to a sphere, the Cayley plane, or the flagmanifold* $F_4/Spin(8)$.

The Corollaries C and D in the introduction now follow easily from Theorems (3.7), (3.9), (3.11), (3.12), and (3.13), and the classification of positively curved homogeneous manifolds.

References

[1] S. Aloff & N. L. Wallach, *An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures*, Bull. Amer. Math. Soc. **81** (1975) 93-97.

[2] Y. Bazaikin, *On one family of 13-dimensional closed Riemannian manifolds with positive curvature*, PhD thesis, to appear in Siberian Math. J.

[3] V. N. Berestovskii, *Spaces with bounded curvatures and distance geometry*, Sibirsk. Mat. Zh. **27** (1986) 11-25.

[4] L. Berard Bergery, *Les varietes Riemaniennes homogenes simplement connexes de dimension impair a courbure strictement positive*, J. Math. Pures Appl. **55** (1976) 47-68.

[5] M. Berger, *Les varietes Riemaniennes homogenes simplement connexes a courbure strictement positive*, Ann. Scuola Norm. Sup. Pisa **15** (1961) 179-246.

[6] A. Borel, *Some remarks about transformation groups on spheres and tori*, Bull. Amer. Math. Soc. **55** (1949) 580-587.

[7] ———, *Le plan projectif des octaves et les spheres comme espaces homogenes*, C.R. Acad. Sci. Paris **230** (1950) 1378-1381.

[8] W. Browder, *Higher torsion in H-spaces*, Trans. Amer. Math. Soc. **108** (1963) 353-375.

[9] Y. Burago, M. Gromov & G. Perelman, *A.D. Alexandrov's spaces with curvatures bounded from below. I*, Uspekhi Mat. Nauk., Tom 47 BbIIIyCK 2 (284), (1992) 3-51 (Russian), or Russian Math Surveys, **42** (1992) 3-51.

[10] E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Mat. Sb. **30** (1952) 349-462 (Russian), Amer. Math. Soc. Transl. **6** (1957) 111-244.

[11] J. H. Eschenberg, *New examples of manifolds with strictly positive curvature*, Invent. Math. **66** (1982) 469-480.

[12] ______, *Inhomogneous spaces of positive curvature*, Differential Geom. Appl. **2** (1992) 123-132.

[13] D. Gromoll & K. Grove, *A generalization of Berger's rigidity theorem for positively curved manifolds*, Ann. Sci. École Norm. Sup. **20** (1987) 227-239.

[14] ______, *The low-dimensional metric foliations of Euclidean spheres*, J. Differential Geom. **28** (1988) 143-156.

[15] K. Grove, *Critical point theory for distance functions*, Proc. Amer. Math. Soc. Summer Instit. Differential Geom., UCLA, 1990.

[16] K. Grove & S. Markvorsen, *New extremal problems for the Riemannian recognition problem via Alexandrov geometry*, J. Amer. Math. Soc. **8** (1995) 1-28.

[17] K. Grove & C. Searle, *Positively curved manifolds of maximal symmetry rank*, J. Pure Appl. Algebra **91** (1994) 137-142.

[18] K. Grove, et al, *Global G-manifold resolutions with applications*, in preparation.

[19] K. Grove & W. Ziller, in preparation.

[20] W.-Y. Hsiang, *On the classification of differentiable $SO(n)$ actions on simply-connected manifolds*, Amer. J. Math. **88** (1966) 137-153.

[21] W.-C. Hsiang & W.-Y. Hsiang, *Differentiable actions of compact connected classical groups. I*, Amer. J. Math. **89** (1967) 705-786.

[22] W.-Y. Hsiang & B. Kleiner, *On the topology of positively curved 4-manifolds with symmetry*, J. Differential Geom. **30** (1989) 615-621.

[23] W.-Y. Hsiang & H. B. Lawson, *Minimal submanifolds of low cohomogeneity,* J. Differential Geom. **5** (1971) 1-38.

[24] B. Kleiner, *Riemannian 4-manifolds with non-negative curvature and continuous symmetry*, PhD thesis, U.C. Berkeley, 1990.

[25] D. Montgomery & H. Samelson, *Transformation groups on spheres*, Ann. of Math. **44** (1943) 454-470.

[26] G. Perelman, *Aklexandrov's spaces with curvature bounded from below. II*, Preprint.

[27] ______, *Elementary Morse theory for Alexandrov spaces*, St. Petersburg Math. J. **5/1** (1994) 207-214.

[28] A. Petrunin, PhD thesis, University of Illinois, Urbana, 1995.

[29] F. Podestá, *A Fixed point theorem*, Preprint, 1994.

[30] J. Poncet, *Groupes de Lie compacts de transformations de l'espaces Euclidean et les spheres comme espaces homogenes*, Comment. Math. Helv. **33** (1959) 109-120.

[31] H. Samelson, *Notes on Lie algebras*, Van Nostrand Reinhold, Cincinnati, 1969.

[32] A. Wald, *Begrundung Einer koordinatenlosen differentialgeometrie der flamchen*, Ergeb. Eines Math. Kolloquiums, Vol. H.7, 1935, 24-46.

[33] N. R. Wallach, *Compact homogeneous Riemannian manifolds with strictly positive curvature*, Ann. of Math. **96** (1972) 277-295.

[34] M. Wang & W. Ziller, *On isotropy irreducible Riemannian manifolds*, Acta Math. **166** (1991) 223-261.

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