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HARMONIC FUNCTIONS WITH POLYNOMIAL GROWTH

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0. Introduction

Twenty years ago Yau, [56], generalized the classical Liouville theorem of complex analysis to open manifolds with nonnegative Ricci curvature. Specifically, he proved that a positive harmonic function on such a manifold must be constant. This theorem of Yau was considerably generalized by Cheng-Yau (see [15]) by means of a gradient estimate which implies the Harnack inequality. As a consequence of this gradient estimate (see [13]), one has that on such a manifold even a harmonic function of sublinear growth must be constant. In order to study further the analytic properties of these manifolds one would like to restrict the class of functions to be considered as much as possible while minimizing loss of information (cf. [22], [26]). From the results of Cheng and Yau, it follows that a natural candidate is the class of harmonic functions of polynomial growth (note that they must be of at least linear growth). In fact, in his study of these functions, Yau was motivated to make the following conjecture (see [58], [59], and [60]; see also the excellent survey article by Peter Li, [37]):

Conjecture 0.1. (Yau). For an open manifold with nonnegative Ricci curvature the space of harmonic functions with polynomial growth of a fixed rate is finite dimensional.

We recall the definition of polynomial growth.

Definition 0.2. For an open (complete noncompact) manifold, M^n, given a point $p \in M$ let r be the distance from p. Define $\mathcal{H}_d(M)$ to be

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the linear space of harmonic functions with order of growth at most d. This means that $u \in \mathcal{H}_d$ if u is harmonic and there exists some $C > 0$ so that $|u| \leq C(1 + r^d)$.

The main result of this paper is the following.

Theorem 0.3. *Conjecture 0.1 is true if M has Euclidean volume growth.*

M^n is said to have Euclidean volume growth if there exists $p \in M$ and a positive constant V such that $\mathrm{Vol}(B_r(p)) \geq \mathrm{V} r^n$ for all $r > 0$. Note that by the Bishop volume comparison theorem (see [3]) we have that $\mathrm{Vol}(B_r(p)) \leq \mathrm{V}_0^n(1) r^n$ for $r > 0$. Here as in the rest of this paper $\mathrm{V}_\Lambda^n(r)$ denotes the volume of the geodesic ball of radius r in the n-dimensional space form of constant sectional curvature Λ.

We show Theorem 0.3 by giving an explicit bound on the dimension of $\mathcal{H}_d(M)$ depending only on n and d.

From the new results given by the investigation initiated by the first author in [17], [18], and [19], and later on further developed by the first author jointly with Cheeger in [6], [7], and [8], and finally the joint work of the first author with Cheeger and Tian in [11], we have a good understanding of the geometry of spaces with Ricci curvature bounded from below.

For the present paper, it is particularly important that it was shown in [6] that every tangent cone at infinity of a manifold satisfying the assumptions of Theorem 0.3 is a metric cone. For an open manifold M^n with nonnegative Ricci curvature, we say that a metric space, M_∞, is a tangent cone at infinity of M if it is a Gromov-Hausdorff limit of a sequence of rescaled manifolds $(M, p, r_j^{-2} g)$, where $r_j \to \infty$. Recall that by Gromov's compactness theorem, [28], any sequence, $r_i \to \infty$, has a subsequence, $r_j \to \infty$, such that the rescaled manifolds $(M, p, r_j^{-2} g)$ converge in the pointed Gromov-Hausdorff topology to a length space, M_∞.

Examples of Perelman ([49]; see also [7] for further examples) show that M_∞ is not unique in general even if M has Euclidean volume growth and quadratic curvature decay (cf. [19] and [12]).

We also note that examples of Perelman (see [50]) most likely can be modified to give examples of manifolds with nonnegative Ricci curvature, Euclidean volume growth and infinite topological type.

It is a classical result that the space of harmonic functions of polynomial growth on Euclidean space is spanned by the spherical harmonics.

Recall that the spherical harmonics are the homogeneous polynomials whose restriction to every sphere centered at the origin is an eigenfunction of the spherical Laplacian. We will observe in Section 1 that this is a general property of metric cones. That is, the harmonic functions of polynomial growth on a metric cone with smooth cross-section can be written as a linear combination of harmonic functions which separate variables (into the radial and cross-sectional directions). Further, they are homogeneous in the radial direction; it follows that the restriction to the cross-section gives an eigenfunction, where the eigenvalue depends on the dimension and the order of growth. We will show that asymptotically this picture still holds in the general case of nonnegative Ricci curvature and Euclidean volume growth (cf. Theorem 4.60). That is, on many sufficiently large annuli, harmonic functions of polynomial growth will almost separate variables and be approximately homogeneous in the radial direction.

It seems worth pointing out some of the difficulties that arise in the general case of nonnegative Ricci curvature and Euclidean volume growth compared with the model case of a cone. Here we will only indicate three such. The first is the low regularity of the cross-section of tangent cones at infinity (cf. [7]). The second is that the frequency function (see Section 2 for the definition of the frequency function) is no longer monotone in the general case; see Section 11 and [26]. Thirdly, the frequency function is not known to be bounded; see [26] for further discussion of this.

Simple examples show (see Section 11) that there exist manifolds with nonnegative Ricci curvature which admit no nontrivial harmonic function with polynomial growth; in fact, we can take such a manifold to have positive sectional curvature. However, to our knowledge no such example exists with nonnegative Ricci curvature and Euclidean volume growth; see [26] for further discussion of this.

Important contributions on this Conjecture of Yau and related problems have been made by Donnelly-Fefferman, Kasue, Li, Li-Tam, Wang, and Wu (see [27], [31], [32], [36], [37], [38], [41], [42], [54], and [55]). In related work, F.-H. Lin has studied asymptotically conical elliptic operators.

The organization of this paper is as follows:

Section 1 is concerned with the description of harmonic functions with polynomial growth on cones and serves to illustrate the methods that we will employ in the general case.

In Section 2 for later use we introduce an important tool which is a generalization of Almgren's frequency function.

A lower bound for the frequency of a harmonic function on good annuli is given in Section 3.

We study in Section 4 the monotonicity properties of the frequency function for harmonic functions on manifolds with nonnegative Ricci curvature and Euclidean volume growth. We also study the asymptotic homogeneity properties of harmonic functions with polynomial growth on these manifolds.

Section 5 deals with orthogonality properties of harmonic functions on these manifolds.

We get in Section 6 an explicit upper bound for the number of orthonormal functions with bounded gradient on a compact manifold with Ricci curvature bounded from below and diameter bounded from above.

Section 7 contains the proofs of some elementary results for functions of one variable with bounded growth, that will be used later on.

Given a set of independent harmonic functions with polynomial growth on a manifold with nonnegative Ricci curvature and Euclidean volume growth, we show in Section 8, how to produce large annuli and a set of independent harmonic functions with good properties. This together with the results of Section 7 allows us to convert the (global) polynomial growth condition to information on a definite scale (local).

With the aid of the results of Sections 7 and 8 we obtain, in Section 9, a technical result that will be needed in the inductive step of the proof of Theorem 0.3.

Using the results of the previous sections, Theorem 0.3 is proved in Section 10, by giving a bound on the dimension of the space of harmonic functions with bounded growth (and suitable independence properties) on any sufficiently large annulus in an open manifold with nonnegative Ricci curvature and Euclidean volume growth.

Section 11 furnishes various examples that illustrate the difficulties in the Euclidean volume growth setting compared with the model case of a cone; see [26] for further discussion of this.

In the appendix, we will collect some consequences of the first variation of energy that we need for this paper.

Finally, we point out that in a joint paper with Cheeger (see [10]) we study the case of linear growth harmonic functions.

Throughout this paper, if N is a closed manifold, we take the convention that a function g is an eigenfunction with eigenvalue λ if $\Delta g + \lambda g = 0$. With this convention, Δ is a negative operator but we will say that the eigenvalues are positive.

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The results of this paper were announced in [21].

Some time after the submission of this paper, in part by further developing the ideas presented here, we solved the general case of the conjecture of Yau, [23]. We wish to point out that in this paper in addition to showing the finite dimensionality of $\mathcal{H}_d$, we also describe the asymptotic structure of the harmonic functions with polynomial growth (like the almost separation of variables). This finer description is in part due to the asymptotic cone structure of the manifolds considered here. It should also be pointed out that the bounds for the dimension of the space $\mathcal{H}_d$ given in this paper depend exponentially on d. In a subsequent paper to [23], we gave polynomial bounds sharp in the order of growth, see [24].

1. Harmonic functions with polynomial growth on cones

In this section N^{n-1} will be a closed smooth $(n-1)$-dimensional manifold. The study of function theory on the Euclidean cone on N is meant to illustrate the methods that we will employ in the proof of Theorem 0.3. Note however that the results of this section will not be used in the proof of Theorem 0.3.

We will often further assume that $\text{Ric}_N \geq (n-2)$. This condition is equivalent to the Euclidean cone $C(N) = (0,\infty) \times_r N^{n-1}$ having nonnegative Ricci curvature. In this section $u(r,\theta)$ is a smooth function on the Euclidean cone $C(N)$ which may be extended continuously to the vertex. On such a cone, the Laplacian can be written as

$$\Delta_{C(N)} u = \frac{\partial^2}{\partial r^2} u + \frac{(n-1)}{r} \frac{\partial}{\partial r} u + \frac{1}{r^2} \Delta_N u(r,\cdot). \tag{1.1}$$

In general, we will say that a function is homogeneous of degree p if it is of the form $u(r,\theta) = r^p g(\theta)$.

First, we claim that if $u(r,\theta) = f(r) g(\theta)$ is harmonic, then $f(r) = r^p$ for some $p \geq 0$ and g is an eigenfunction of N. To see this note that

(1.1) becomes

$$\Delta_{C(N)} u = f'' g + (n-1)\frac{f'}{r} g + \frac{f}{r^2}\Delta_N g. \tag{1.2}$$

From this we have that

$$\Delta_N g + \lambda g = 0. \tag{1.3}$$

Note also that if $\mathrm{Ric}_{N^{n-1}} \geq (n-2)$ then $\lambda = 0$ or $\lambda \geq (n-1)$. Substituting (1.3) in (1.2) gives

$$\begin{aligned} 0 = & \; f'' g + (n-1)\frac{f'}{r} g - \frac{f}{r^2}\lambda g \\ = & \left(f'' + (n-1)\frac{f'}{r} - \lambda \frac{f}{r^2}\right) g. \end{aligned} \tag{1.4}$$

Therefore

$$f'' + (n-1)\frac{f'}{r} - \lambda\frac{f}{r^2} = 0. \tag{1.5}$$

We easily see that if $p(p-1) + (n-1)p - \lambda = p^2 + (n-2)p - \lambda = 0$, i.e.,

$$p = \frac{1}{2}\left[-(n-2) + \sqrt{(n-2)^2 + 4\lambda}\right] , \tag{1.6}$$

then $f(r) = r^p$ is a solution of (1.5). Note that we take only the non-negative solution p because the negative solution has a pole singularity at the vertex. Further, we see that if we require that $p \leq d$ then $\lambda \leq d(d+n-2)$, and therefore

$$\lambda = p(p+n-2)\,. \tag{1.7}$$

Collecting the previous calculations, we have the following elementary lemma.

Lemma 1.8. *A function $u(r,\theta) = f(r)g(\theta)$ on $C(N)$ is harmonic if and only if*

$$\Delta_N g + \lambda g = 0 \tag{1.9}$$

and

$$f(r) = r^p\,, \tag{1.10}$$

where $\lambda = p(p+n-2)$.

Let $E_\lambda(N)$ denote the linear space spanned by the eigenfunctions of N with eigenvalues less than or equal to λ. Further, we let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ denote the distinct eigenvalues of N and let $p_j \geq 0$ be determined by $\lambda_j = p_j(p_j + n - 2)$.

The following theorem is well known (see [5]).

Theorem 1.11. *(Harmonic functions on a cone). If u is a harmonic function on $C(N)$, then*

$$u(r,\theta) = \sum_j a_j r^{p_j} g_j(\theta) \,, \tag{1.12}$$

where the a_j are constants. Furthermore, u has polynomial growth if and only if this is a finite sum.

Proof. We may assume that $u(0) = 0$. By the spectral theorem applied to N, we may write

$$u(1,\theta) = \sum_j a_j g_j(\theta) \,. \tag{1.13}$$

Consider the harmonic function

$$v(r,\theta) = u(r,\theta) - \sum_j a_j r^{p_j} g_j(\theta) \,. \tag{1.14}$$

Note that v vanishes on ∂B_1 and at the vertex; by the maximum principle, v vanishes identically. The second claim follows easily from the first. q.e.d.

We will now obtain a second proof of Theorem 1.10 that is closer to the proof of Theorem 0.3.

If u is a Lipschitz function on $C(N)$ then we set

$$D(r) = r^{2-n} \int_{B_r(p)} |\nabla u|^2 \,, \tag{1.15}$$

$$I(r) = r^{1-n} \int_{\partial B_r(p)} u^2 \,, \tag{1.16}$$

$$F(r) = r^{3-n} \int_{\partial B_r(p)} \left| \frac{\partial u}{\partial r} \right|^2 \,, \tag{1.17}$$

and finally the frequency (cf. [1] and Remark 2.16)

$$U(r) = \frac{r \int_{B_r(p)} |\nabla u|^2}{\int_{\partial B_r(p)} u^2} = \frac{D(r)}{I(r)}. \tag{1.18}$$

Lemma 1.19. *If u is harmonic then U is monotone nondecreasing.*

Proof. To show this note that

$$(\log U)'(r) = \frac{D'(r)}{D(r)} - \frac{I'(r)}{I(r)}. \tag{1.20}$$

Further, from the first variation of energy, i.e., Proposition A.23, we have that

$$\begin{aligned} D'(r) =& r^{2-n} \int_{\partial B_r(p)} |\nabla u|^2 + \frac{2-n}{r} D(r) \\ =& 2r^{2-n} \int_{\partial B_r(p)} \left| \frac{\partial u}{\partial r} \right|^2 \\ =& 2\frac{F(r)}{r}\,, \end{aligned} \tag{1.21}$$

and, since $\Delta u^2 = 2|\nabla u|^2$,

$$\begin{aligned} I'(r) &= r^{1-n} \int_{\partial B_r(p)} \langle \nabla u^2, \nabla r \rangle + r^{1-n} \int_{\partial B_r(p)} u^2 \Delta r + \frac{1-n}{r} I(r) \\ &= r^{1-n} \int_{B_r(p)} \Delta u^2 = 2r^{1-n} \int_{B_r(p)} |\nabla u|^2 = 2\frac{D(r)}{r}\,. \end{aligned} \tag{1.22}$$

Therefore

$$(\log U)'(r) = \frac{D'(r)}{D(r)} - 2\frac{D(r)}{rI(r)} \tag{1.23}$$

$$= 2 \left(\frac{r^{2-n} \int_{\partial B_r(p)} |\frac{\partial u}{\partial r}|^2}{D(r)} - \frac{D(r)}{rI(r)} \right) \tag{1.24}$$

$$\begin{aligned} &= \frac{2r^{3-2n}}{D(r)I(r)} \left(\int_{\partial B_r(p)} \left| \frac{\partial u}{\partial r} \right|^2 \int_{\partial B_r(p)} u^2 \right. \\ &\qquad \left. - \left(\int_{\partial B_r(p)} u \frac{\partial u}{\partial r} \right)^2 \right). \end{aligned} \tag{1.25}$$

From the Cauchy-Schwarz inequality we get that

$$(\log U)' \geq 0. \tag{1.26}$$

q.e.d.

We will later see (Theorem 1.66 and Lemma 1.26) that in the case of a cone many U are in fact constant.

Next we have the following:

Lemma 1.27. *If u is harmonic and U is constant, then $u(r,\theta) = f(r)g(\theta)$. Conversely, if $u(r,\theta) = f(r)g(\theta)$ is harmonic, then $U \equiv p$, $f(r) = r^p$ and $\Delta_N g + p(p+n-2)g = 0$.*

Proof. Since U is constant, then by the equality in the Cauchy-Schwarz inequality, see (1.23), we have

$$\frac{\partial u}{\partial r} = h(r)u. \tag{1.28}$$

Integrating (1.28) shows that $u(r,\theta) = f(r)g(\theta)$. The lemma now follows from Lemma 1.7 and an easy computation. q.e.d.

Lemma 1.29. *If $u \in \mathcal{H}_d(C(N))$ then $U \leq d$.*

Proof. Equation (1.22) is equivalent to

$$(\log I(r))' = \frac{2U(r)}{r}. \tag{1.30}$$

Integrating equation (1.30) yields

$$I(r) = \exp\left(\int_s^r \frac{2U(t)}{t}\,dt\right) I(s). \tag{1.31}$$

Since U is monotone nondecreasing, we see that U must be bounded by d. q.e.d.

Definition 1.32. (Order at infinity). If u is harmonic, then we define the order at infinity of u, $\text{ord}_\infty(u)$, by

$$\text{ord}_\infty(u) = \lim_{r\to\infty} U(r). \tag{1.33}$$

Note that this limit exists since U is monotone nondecreasing by Lemma 1.18. When u has polynomial growth, then Lemma 1.28 shows that $\text{ord}_\infty(u)$ is finite. Likewise, the monotonicity of U allows us to make the following definition.

Definition 1.34. (Order at the vertex). If u is harmonic, then we define the order at the vertex of u, $\mathrm{ord}_0(u)$, by

$$\mathrm{ord}_0(u) = \lim_{r \to 0} U(r). \tag{1.35}$$

Lemma 1.36. *If u and v are harmonic functions, then*

$$ord_\infty(u+v) \leq \max\{ord_\infty(u), ord_\infty(v)\}\,. \tag{1.37}$$

Proof. By the Cauchy-Schwarz inequality, we have that

$$\begin{aligned} \log(I_{u+v}) &\leq \log(2I_u + 2I_v) \\ &\leq \log(4) + \max\{\log(I_u), \log(I_v)\}\,. \end{aligned} \tag{1.38}$$

Further, from (1.30) it follows that

$$(\log I(r))' = \frac{2U(r)}{r} \tag{1.39}$$

for any harmonic function. If $\mathrm{ord}_\infty(u+v) > \max\{\mathrm{ord}_\infty(u), \mathrm{ord}_\infty(v)\}$, then there exist an $R > 0$ and an $\epsilon > 0$ such that for any $r > R$,

$$(\log I_{u+v}(r))' > \max\{(\log I_u(r))', (\log I_v(r))'\} + \frac{\epsilon}{r}\,. \tag{1.40}$$

Since $\frac{1}{r}$ is not integrable, this would contradict the inequality in (1.38); therefore, (1.37) follows. q.e.d.

Lemma 1.41. *Suppose that u and v are harmonic functions on $C(N)$. If in addition $v(r,\theta) = r^p g(\theta)$, then*

$$r^{1-n} \int_{\partial B_r} uv = r^{2p} \int_{\partial B_1} uv\,. \tag{1.42}$$

Proof. Using Green's formula, we get that

$$\begin{aligned} \frac{d}{dr}\left(r^{1-n}\int_{\partial B_r} uv\right) =& r^{1-n}\int_{\partial B_r} \frac{\partial}{\partial r}(uv) \\ =& r^{1-n}\int_{\partial B_r} v\frac{\partial u}{\partial r} + r^{1-n}\int_{\partial B_r} u\frac{\partial v}{\partial r} && (1.43) \\ =& 2r^{1-n}\int_{\partial B_r} u\frac{\partial v}{\partial r}\,. && (1.44) \end{aligned}$$

From the homogeneity of v, we have that $\frac{\partial v}{\partial r} = \frac{p}{r} v$. Substituting this into (1.44),

$$\frac{d}{dr}\left(r^{1-n}\int_{\partial B_r} uv\right) = \frac{2p}{r}\left(r^{1-n}\int_{\partial B_r} uv\right) . \tag{1.45}$$

Integrating (1.45) yields (1.42) and the lemma follows. q.e.d.

Definition 1.46. We say that two harmonic functions, u and v, on $C(N)$ are orthogonal if

$$\int_{\partial B_1} uv = 0 . \tag{1.47}$$

Note that by Lemma 1.40, if v is homogeneous, and u and v are orthogonal in the sense of Definition 1.45, then

$$\int_{\partial B_r} uv = 0 \tag{1.48}$$

for all $r > 0$. Also note that from the maximum principle it follows that the left side of (1.47) defines an inner product on the space of harmonic functions on $C(N)$.

Lemma 1.49. *Suppose that u is harmonic on $C(N)$ with $ord_\infty(u) = d < \infty$ and that u is orthogonal to the homogeneous harmonic functions whose growth is less than d. Then for $r > s > 0$, we have*

$$D(r) \geq \left(\frac{r}{s}\right)^{2d} D(s) . \tag{1.50}$$

Proof. Let λ be given by (1.7), that is,

$$\lambda = d(d+n-2) . \tag{1.51}$$

By the orthogonality assumption and Lemma 1.40, we get the following scale-invariant Poincaré inequality for the cross-section ∂B_r:

$$\frac{\int_{\partial B_r} |\nabla^T u|^2}{\int_{\partial B_r} u^2} \geq \frac{\lambda}{r^2} , \tag{1.52}$$

where $\nabla^T u$ is the tangential gradient. Note that

$$|\nabla u|^2 = |\nabla^T u|^2 + \left|\frac{\partial u}{\partial r}\right|^2 . \tag{1.53}$$

Using (1.53), we can rewrite (1.52) as

$$rD'(r) - (2-n)D(r) - F(r) \geq 2\lambda I(r)\,. \tag{1.54}$$

From the first variation of energy (see equation (1.21)) it follows that

$$D'(r) = 2\frac{F(r)}{r}\,. \tag{1.55}$$

Eliminating $F(r)$ in (1.54) and using (1.55), we have

$$D'(r) - \frac{2(2-n)D(r)}{r} \geq \frac{2\lambda I(r)}{r}\,. \tag{1.56}$$

Dividing (1.56) through by $D(r)$ and noting that $\frac{I}{D} = U^{-1} \geq d^{-1}$, give

$$\frac{D'(r)}{D(r)} - \frac{2(2-n)}{r} \geq \frac{2\lambda}{rU(r)} \geq \frac{2\lambda}{d\,r}\,. \tag{1.57}$$

Substituting (1.51) for λ in (1.57), combining the $\frac{1}{r}$ terms, and rewriting the first term as a logarithmic derivative, we obtain

$$(\log D(r))' \geq \frac{2d}{r}\,. \tag{1.58}$$

Integrating (1.58) yields (1.50). q.e.d.

Lemma 1.59. *If u is harmonic, $u(0) = 0$, and $ord_0(u) = 0$, then u is identically zero.*

Proof. We may assume that u is not constant; this implies that $I(r)$ is positive for every $r > 0$. By (1.31) we get

$$I(r) \leq 2^{2d} I\left(\frac{r}{2}\right)\,, \tag{1.60}$$

where $d = U(1)$ and we take $r \leq 1$. From the scale-invariant Poincaré inequality, we have that

$$\lambda_1 \int_{\partial B_r} u^2 \leq r^2 \int_{\partial B_r} |\nabla^T u|^2 \leq r^2 \int_{\partial B_r} |\nabla u|^2\,, \tag{1.61}$$

where λ_1 is the first eigenvalue of the Laplacian on N, and $\nabla^T u$ is the tangential gradient of u. Using (1.60) and the monotonicity of I, and integrating (1.61) from $\frac{r}{2}$ to r we are led to

$$\begin{aligned} 2^{-2d}\lambda_1 r I(r) \leq & \lambda_1 r I\left(\frac{r}{2}\right) \leq 2\lambda_1 \int_{\frac{r}{2}}^{r} I(t)dt \\ \leq & 2rD(r)\,. \end{aligned} \tag{1.62}$$

This shows that $U(r) \geq \lambda_1 2^{-2d}$ and the lemma follows. q.e.d.

Corollary 1.63. *Suppose that u is harmonic on $C(N)$ with $ord_\infty(u) = d < \infty$ and that u is orthogonal to the homogeneous harmonic functions whose growth is less than d. Then u is homogeneous.*

Proof. By Lemma 1.58, we may assume that $\mathrm{ord}_0(u) > 0$; let $c = \mathrm{ord}_0(u)$. By the definition and monotonicity of U,

$$cD(r) \leq I(r) \leq dD(r)\,. \tag{1.64}$$

Therefore, by Lemma 1.48, for $r > s > 0$, we have

$$I(r) \geq \frac{c}{d}\left(\frac{r}{s}\right)^{2d} I(s)\,. \tag{1.65}$$

Setting $r = 1$ and taking $s < 1$, we see that

$$I(s) \leq \frac{c}{d} s^{2d} I(1)\,. \tag{1.66}$$

By equation (1.31), (1.66) implies that $\mathrm{ord}_0(u) = d$. Since U is monotone, we conclude that U is constant. The corollary now follows from Lemma 1.26. q.e.d.

We are now ready to give a second proof of Theorem 1.10.

Theorem 1.67. *(Harmonic functions with polynomial growth on cones; second version). If N^{n-1} is a closed $(n-1)$-manifold, then*

$$\dim(\mathcal{H}_d(C(N))) = \dim(E_{d(d+n-2)}(N)). \tag{1.68}$$

In fact, if $u \in \mathcal{H}_d(M)$ then

$$u(r,\theta) = \sum_{p_j \leq d} r^{p_j} g_j(\theta)\,, \tag{1.69}$$

where g_j is an eigenfunction with eigenvalue λ_j.

Proof. The inequality "$\geq$" in (1.68) follows from Lemma 1.7. We will show the reverse inequality, i.e., "$\leq$", by induction on j. For $j = 0$ we have by Lemma 1.28 that $\mathrm{ord}_\infty(u) = 0$; by the monotonicity of U, and Lemma 1.18, u must be constant. Assume now that the theorem is true for p_j and will show that it is true for p_{j+1}. Given $u \in \mathcal{H}_{p_{j+1}}$, by the inductive hypothesis and Lemma 1.35 we may assume that $u = u' + u''$ where $u' \in \mathcal{H}_{p_{j+1}}$, $u'' = \sum_{p_k \leq p_j} r^{p_k} g_k(\theta)$, and $\int_{\partial B_1} u'v = 0$ for all $v \in \mathcal{H}_{p_j}$. We have therefore, by Lemma 1.40, that $\int_{\partial B_r} u'v = 0$ for all r and for all $v \in \mathcal{H}_{p_j}$. By Corollary 1.62, we conclude that u' is homogeneous; the theorem follows. q.e.d.

Remark 1.70. For $\text{Ric}_N \geq (n-2)$ then the case $d = 0$ in Theorem 1.10 or Theorem 1.66 is essentially a special case of the Liouville theorem of Yau, [56]; the cases $0 < d < 1$ follow from the gradient estimate of Cheng-Yau, [15], and is in this case equivalent to $\lambda_1 \geq (n-1)$ (Lichnerowicz's theorem, [45]).

Example 1.71. In many cases where $\text{Ric}_N \geq (n-2)$ and N is diffeomorphic to $\mathbf{S}^{n-1}$, it is possible to round off the metric on a cone while preserving the condition that the Ricci curvature is nonnegative. In fact the change in the metric can often be done by a compactly supported change in the warping function. As a consequence of Theorem 1.66 and Proposition 11.5 we see that for such a perturbation $\dim(\mathcal{H}_d) = \dim(E_{d(d+n-2)}(N))$.

2. Tools to study the growth of harmonic functions on manifolds

From now on, unless explicitly stated otherwise, let M^n be an n-dimensional open manifold with nonnegative Ricci curvature. Set

$$\text{V}_M = \lim_{r\to\infty} \frac{\text{Vol}(B_r(p))}{r^n}\,; \tag{2.1}$$

note that by the volume comparison theorem this limit exist (in fact the quantity in (2.1) is nonincreasing) and is independent of the point p. We will also assume that M has Euclidean volume growth, that is, $\text{V}_M > 0$. Fix a point $p \in M$ and let G denote the global Green's function on M with singularity at p. It is well known that G exists in this setting (see for instance [52]).

For ease of exposition, we will henceforth restrict our attention to the case of $n \geq 3$. The case $n = 2$ was done earlier by Li-Tam, [42] (in fact, for surfaces with finite total curvature). For another proof in the case $n = 2$ using nodal sets see Donnelly-Fefferman, [27].

Set

$$b = \left(\frac{\text{V}_M}{\text{V}_0^n(1)} G\right)^{\frac{1}{2-n}}\ . \tag{2.2}$$

When M is $\mathbf{R}^n$, the function b defined in (2.2) is just the distance function to p. When studying the global analytic properties of M, the function b is the proper replacement for the distance function (cf. Proposition 2.21; see also [17]-[19], [6]).

With this choice of b we have

$$\nabla b = \frac{\mathrm{V}_M}{(2-n)\mathrm{V}_0^n(1)} b^{n-1} \nabla G \,, \tag{2.3}$$

$$\Delta b = (n-1)\frac{|\nabla b|^2}{b} \,, \tag{2.4}$$

and

$$\Delta b^2 = 2n|\nabla b|^2 \,. \tag{2.5}$$

We define the following quantities

$$I(r) = r^{1-n} \int_{b=r} u^2 |\nabla b| \,, \tag{2.6}$$

$$D(r) = r^{2-n} \int_{b \leq r} |\nabla u|^2 \,, \tag{2.7}$$

$$F(r) = r^{3-n} \int_{b=r} \left|\frac{\partial u}{\partial n}\right|^2 |\nabla b| \,, \tag{2.8}$$

and finally the frequency function (cf. [1] and Remark 2.16) by

$$U(r) = \frac{D(r)}{I(r)} \,. \tag{2.9}$$

Observe that if $r \leq s$, then

$$D(r) \leq \left(\frac{r}{s}\right)^{2-n} D(s) \,. \tag{2.10}$$

Remark 2.11. S.Y. Cheng, [14], showed that locally the critical sets (sets where the function is constant and its gradient vanishes) of any harmonic function are of codimension two on any smooth manifold (see also Hardt-Simon, [30]; their results are valid for low regularity elliptic equations). Since the critical sets of b coincide with those of G (which is harmonic), it is easy to see that these calculations are valid on all level sets of b (and not just at regular values).

Differentiating (2.6) gives

$$I'(r) = 2\frac{D(r)}{r} \tag{2.12}$$

and therefore

$$(\log I(r))' = \frac{2U(r)}{r}\,. \tag{2.13}$$

From (2.13) we have for $s > r > 0$

$$I(s) = \exp\left(2\int_r^s \frac{U(t)}{t}dt\right) I(r)\,. \tag{2.14}$$

The quantity $I_u(r)$ is a weighted average of u^2, and $I_1(r)$ is the weighted volume of the level set $b = r$. By (2.12), $I_1(r)$ is constant. From the definition of b it is easy to see that

$$I_1(r) = n\mathrm{V}_M \leq n\mathrm{V}_0^n(1)\,. \tag{2.15}$$

Remark 2.16. This generalization of the usual frequency function for harmonic functions (and harmonic maps) on Euclidean space has the advantage of being well defined globally and reflecting the global analytic and geometric properties of the open manifold. When the manifold is Euclidean space, the monotonicity of the frequency is an analytic version of the Three Circles Theorem of J. Hadamard. This type of ratio has been used by Almgren in his study of multi-valued harmonic mappings (see [1]), by Lin for the study of mappings to cones (see [46]), and by Gromov-Schoen in their work on harmonic mappings to singular spaces (see [29] and [51]).

We shall use some asymptotic estimates of the Green's function on manifolds with nonnegative Ricci curvature. For the convenience of the reader, we recall these now. Note first that it follows directly from the Laplacian comparison theorem together with the maximum principle that the Green's function, $G(x, y)$, satisfies

$$r^{2-n} \leq G(x, y), \tag{2.17}$$

where r is the distance from x to y.

Regarding an upper bound on the Green's function, we have the following estimate (see [44] and [48]):

If $n \geq 3$ and M^n is an n-dimensional manifold with $\mathrm{Ric}_M \geq 0$ and Euclidean volume growth, then there exists a constant $C \geq 1$ such that

$$r^{2-n} \leq G(x,y) \leq C r^{2-n} \,. \tag{2.18}$$

It follows from (2.18) that there exist positive constants (depending on M) C_1 and C_2 such that

$$C_1 r \leq b \leq C_2 r \,. \tag{2.19}$$

Remark 2.20. In [44], Li-Yau proved a stronger bound on the heat kernel which implies the bound on the Green's function. In fact, they got an estimate even in the case where M does not have Euclidean volume growth.

We shall need the following improvement of this estimate. This proposition shows that in the case of nonnegative Ricci curvature and Euclidean volume growth the Green's function has conical asymptotics; cf. [6].

Proposition 2.21. *([22]). If $n \geq 3$ and M^n is an n-dimensional manifold with $Ric_M \geq 0$ and Euclidean volume growth, then for each fixed $x \in M$*

$$\lim_{r(y)\to\infty} \frac{G(x,y)}{r^{2-n}} = \frac{V_0^n(1)}{V_M} \,. \tag{2.22}$$

Observe that (2.22) implies the strengthening of (2.19):

$$\lim_{r\to\infty} \frac{b}{r} = 1 \,. \tag{2.23}$$

Furthermore, from Section 4 of [6] and (2.23) given any $\delta > 0$, there exists $R = R(p,\delta) > 0$ such that for all $r > R$, we have

$$\int_{b\leq r} \left| |\nabla b|^2 - 1 \right|^2 \leq \delta \,\mathrm{Vol}(b \leq r) \tag{2.24}$$

and

$$\int_{b\leq r} \left| \mathrm{Hess}(b^2) - 2g \right|^2 \leq \delta \,\mathrm{Vol}(b \leq r) \,, \tag{2.25}$$

where g is the metric tensor on M.

In fact, all that we essentially require of G (for this section and Sections 3 and 4) is that it is harmonic on an annulus and C^0 close to a multiple of r^{2-n}, where r is the distance to the center of the annulus. It then follows from [6] that b has the properties similar to (2.24) and (2.25) (see Section 4 of [6] and cf. [22]).

We will also use the following meanvalue inequality of Li-Schoen, which for convenience we state only for the case of nonnegative Ricci curvature.

Proposition 2.26. *(Li-Schoen, [40]). Suppose that M^n is an n-dimensional manifold with $Ric_M \geq 0$ and v is a nonnegative subharmonic function on M. Then*

$$\sup_{B_r(p)} v \leq \frac{C}{Vol(B_{\frac{3}{2}r}(p))} \int_{B_{\frac{3}{2}r}(p)} v \,, \tag{2.27}$$

where $C = C(n)$.

Often, we will get natural integral bounds for harmonic functions and their gradients; the meanvalue inequality, Proposition 2.26, will allow us to get supremum bounds on a subset.

Finally, we will use that for each r, $I(r)$ defines a quadratic form on the linear space of harmonic functions. The associated bilinear form is given by

$$r^{1-n} \int_{b=r} uv|\nabla b| \,, \tag{2.28}$$

for harmonic functions u and v. Note that from the maximum principle we have that for the regular values, s, of b, (2.28) defines an inner product on the space of harmonic functions on $\{x \,|\, b(x) \leq s\}$. Clearly, this also follows from the monotonicity of I.

3. Lower bound of the frequency

In this section, we will give several versions of a lower bound for the frequency of a harmonic function. In a future paper we plan on undertaking a more careful study of this and some of its consequences.

We now define quantities analogous to those of Section 2 which are technically easier to work with. Let

$$E(r) = r^{2-n} \int_{b \leq r} |\nabla u|^2 |\nabla b|^2 \tag{3.1}$$

and

$$W(r) = \frac{E(r)}{I(r)} \,. \tag{3.2}$$

We will first show that when M has Euclidean volume growth, the quantity E is equivalent to D when the growth of D is controlled. By definition, the equivalence of D and E implies the equivalence of U and W.

Proposition 3.3. *(Equivalence of E and D). Let M^n be a manifold of nonnegative Ricci curvature and Euclidean volume growth. Fix $p \in M$. Given $\epsilon > 0$, $\Omega_0 > 1$, and $\gamma > 1$, there exists $R = R(p, \gamma, \epsilon, \Omega_0) > 0$ such that if $r > R$, $1 < \Omega \leq \Omega_0$, and u is any harmonic function on M with*

$$D(2\Omega r) \leq \gamma D(r) \,, \tag{3.4}$$

then for all $r \leq s \leq \Omega r$

$$\left| \log \frac{D(s)}{E(s)} \right| \leq \epsilon \,. \tag{3.5}$$

Proof. Note that $|1 - s| \leq \frac{\epsilon}{1+\epsilon}$ implies that

$$|\log(s)| \leq \left| \int_{\frac{1}{1+\epsilon}}^{1} \frac{1}{t} \right| \leq \epsilon \,. \tag{3.6}$$

From [6] and the asymptotics of the Green's function, Proposition 2.21 (see also the remarks following that proposition), given any $\delta > 0$, there exists $R = R(p, \delta) > 0$ such that for all $r > R$, we have

$$\left| \log \frac{b}{r} \right| \leq \delta \,, \tag{3.7}$$

and

$$\int_{b \leq r} \left| |\nabla b|^2 - 1 \right|^2 \leq \delta^2 \operatorname{Vol}(b \leq r) \,. \tag{3.8}$$

Note that (3.8) implies by the Cauchy-Schwarz inequality that

$$\int_{b \leq r} \left| |\nabla b|^2 - 1 \right| \leq \delta \operatorname{Vol}(b \leq r) \,. \tag{3.9}$$

We shall assume that δ is small enough to arrange that $\exp(2\delta) \leq \frac{4}{3}$.

By definition, we have for $s > R$,

$$
\begin{aligned}
|D(s) - E(s)| &= s^{2-n} \left| \int_{b \leq s} |\nabla u|^2 (1 - |\nabla b|^2) \right| \\
&\leq s^2 \sup_{b \leq s} |\nabla u|^2 \, s^{-n} \int_{b \leq s} \left| 1 - |\nabla b|^2 \right| \\
&\leq s^2 \sup_{b \leq s} |\nabla u|^2 \, \delta \mathrm{V}_0^n(1) \exp(n\delta) \, ,
\end{aligned}
\tag{3.10}
$$

where the last inequality follows from (3.7), (3.9), and the Bishop volume comparison theorem.

From the Bochner formula, $|\nabla u|^2$ is a subharmonic function. Since $\exp(2\delta) \leq \frac{4}{3}$ (3.7) and Proposition 2.26 yields that for $r \geq R$,

$$
\sup_{b \leq \Omega r} |\nabla u|^2 \leq \frac{C_1}{\mathrm{V}_M} \Omega^{-2} r^{-2} D(2\Omega r) \, ;
\tag{3.11}
$$

where $C_1 = C_1(n) > 0$; in (4.22) we do this again in more detail.

Using (3.4), (3.10), and (3.11) we obtain, for $r \leq s \leq \Omega r$,

$$
\begin{aligned}
|D(s) - E(s)| &\leq \frac{C_1}{\mathrm{V}_M} \, D(2\Omega r) \delta \mathrm{V}_0^n(1) \, \exp(n\delta) \\
&\leq \frac{C_1}{\mathrm{V}_M} \, \gamma D(r) \delta \mathrm{V}_0^n(1) \, \exp(n\delta) \\
&\leq \frac{C_1}{\mathrm{V}_M} \, \gamma D(r) \delta \mathrm{V}_0^n(1) \, \left(\frac{4}{3} \right)^{\frac{n}{2}} \, .
\end{aligned}
\tag{3.12}
$$

Finally, to finish the proof, we use the trivial bound, for s between r and Ωr,

$$
D(r) = r^{2-n} \int_{b \leq r} |\nabla u|^2 \leq r^{2-n} \int_{b \leq s} |\nabla u|^2 \leq \Omega^{n-2} D(s) \, ,
\tag{3.13}
$$

and set

$$
\delta = \min \{ \frac{1}{2} \log \frac{4}{3}, \frac{\Omega^{2-n}}{C_1 \gamma} \frac{\mathrm{V}_M}{\mathrm{V}_0^n(1)} \left(\frac{3}{4} \right)^{\frac{n}{2}} \frac{\epsilon}{1 + \epsilon} \} \, .
\tag{3.14}
$$

q.e.d.

Lemma 3.16 will illustrate some of the advantages of working with E as opposed to D; the main advantage comes from the form of the first variation formula (see the appendix).

Differentiating (3.1) gives

$$E'(r) = r^{2-n} \int_{b=r} |\nabla u|^2 |\nabla b| + (2-n) \frac{E(r)}{r}\,. \tag{3.15}$$

From this, we will see in Lemma 3.16 that $r^{-2}E(r)$ is nondecreasing. In Section 4, we will investigate other monotonicity properties of E.

Lemma 3.16. *(E grows at least quadratically). Let u be a harmonic function on a manifold M with nonnegative Ricci curvature. Then $r^{-2}E(r)$ is monotone nondecreasing.*

Proof. First we note that for any subharmonic function v,

$$J_v(r) = r^{1-n} \int_{b=r} v|\nabla b| \tag{3.17}$$

is monotone nondecreasing; this follows from

$$J_v'(r) = r^{1-n} \int_{b \leq r} \Delta v\,, \tag{3.18}$$

which uses Stokes' theorem and (2.4). In particular, the above holds for $v = |\nabla u|^2$ since this is subharmonic by the Bochner formula.

Applying the co-area formula to (3.1), we obtain

$$r^{-2}E_u(r) = r^{-n} \int_0^r s^{n-1} J_v(s)ds\,. \tag{3.19}$$

Differentiating (3.19) gives

$$\left(r^{-2}E_u\right)' = -nr^{-n-1} \int_0^r s^{n-1} J_v(s)ds + r^{-1}J_v(r)\,. \tag{3.20}$$

Integrating the first term of (3.20) by parts yields

$$\left(r^{-2}E_u\right)' = r^{-n-1} \int_0^r s^n J_v'(s)ds \geq 0\,. \tag{3.21}$$

q.e.d.

If the frequency is locally bounded from above, we will get a lower bound for the frequency. We will have two versions of this lower bound. First, Lemma 3.22 will give a crude lower bound for the frequency function and later, in Section 4, Corollary 4.40 a more refined version.

Lemma 3.22. *(Lower bound of the frequency; crude version). Let M be an open manifold with nonnegative Ricci curvature and Euclidean volume growth and let $p \in M$ be fixed. Given $\Omega \geq 2$, there exist $C = C(n) > 0$ and $R = R(p) > 0$ such that for any harmonic function u with $u(p) = 0$, we have for $r > R$,*

$$I(r) \leq C\Omega^{-2} D(\Omega r)\,. \tag{3.23}$$

Furthermore, if $U(s) \leq d$ for $r \leq s \leq \Omega r$, we get a lower bound for $U(\Omega r)$; that is,

$$\frac{\Omega^{2-2d}}{C} \leq U(\Omega r)\,. \tag{3.24}$$

Proof. By (2.23), we can choose $R = R(p) > 0$ such that for $r > R$

$$\left| 2 \log \frac{b}{r} \right| \leq \log \frac{4}{3}\,, \tag{3.25}$$

which implies that the set $\{b \leq r\}$ is contained in a ball of radius $\frac{2}{\sqrt{3}} r$. As in (3.11), by Proposition 2.26, we have (since $\Omega > 2$)

$$\sup_{b \leq r} |\nabla u|^2 \leq \frac{C_1}{\mathrm{V}_M} \Omega^{-2} r^{-2} D(\Omega r)\,, \tag{3.26}$$

where $C_1 = C_1(n) > 0$.

By integrating equation (3.26) along geodesics starting at p and using the fact that $u(p) = 0$, we get

$$\sup_{b \leq r} |u|^2 \leq \frac{4}{3} \frac{C_1}{\mathrm{V}_M} \Omega^{-2} D(\Omega r)\,. \tag{3.27}$$

The claim (3.23) now follows from the weighted volume bound for the level set, (2.15), with $C = \frac{4}{3} n C_1$.

If $U \leq d$, then by (2.14),

$$I(\Omega r) \leq \Omega^{2d} I(r) \leq C\, \Omega^{2d-2} D(\Omega r)\,, \tag{3.28}$$

and the second claim follows. q.e.d.

We are now prepared to give a uniform lower bound for the maximum of the frequency on arbitrary annuli outside a compact set. In contrast to Lemma 3.22 the importance of this result is that it does not require any control on the function.

Corollary 3.29. *(Uniform lower bound of the maximum of the frequency). Let M be an open manifold with nonnegative Ricci curvature and Euclidean volume growth and let $p \in M$ be fixed. There exist $C_L = C_L(n) > 0$ and $R = R(p) > 0$ such that for any harmonic function u with $u(p) = 0$, we have for $r > R$,*

$$\max_{r \leq s \leq 2r} U(s) \geq C_L \,. \tag{3.30}$$

Moreover, given $\epsilon > 0$, there exists $\Omega_L = \Omega_L(n, \epsilon) \geq 2$ such that for any harmonic function u with $u(p) = 0$, we have for $r > R$,

$$\max_{r \leq s \leq \Omega_L r} U(s) \geq (1 - \epsilon) \,. \tag{3.31}$$

Proof. Suppose that d is a uniform upper bound for the frequency; we will show that d cannot be too small. We apply Lemma 3.22 with $\Omega = 2$ to get an $R = R(p) > 0$ and a $C = C(n) > 0$ such that for $r > R$, if $U(s) \leq d$ for s between r and $2r$, then

$$U(2r) \geq \frac{2^{2-2d}}{C} \,. \tag{3.32}$$

Hence, we have

$$d \geq \frac{2^{2-2d}}{C} > 0 \,. \tag{3.33}$$

This implies that $d \geq C_L = C_L(n) > 0$.

Moreover, with the same $R = R(p)$, for any $\Omega \geq 2$ and the same $C = C(n) > 0$, if $r > R$ and $U(s) \leq (1 - \epsilon)$ for s between r and Ωr, then

$$1 > U(\Omega r) \geq \frac{1}{C} \Omega^{2\epsilon} \,. \tag{3.34}$$

This is not possible for $\Omega \geq \Omega_L = \Omega_L(n, \epsilon) = \max\{C^{\frac{1}{2\epsilon}}, 2\}$. q.e.d.

Remark 3.35. We will see in Section 4 that there are many large annuli on which the frequency function is almost monotone. As a result, Corollary 3.29 will imply a lower bound for the frequency function on the outer parts of these annuli. See Section 4 for a precise version, and [24] for other results in this direction.

We will now show how to get control of the growth of D just from a bound on the growth of I (cf. Theorem 4.60). In later sections, this will allow us to work with bounds only on the growth of I.

Proposition 3.36. *(Bounding the growth of D by the growth of I). Let M be an n-dimensional manifold with nonnegative Ricci curvature and Euclidean volume growth, and let $p \in M$ be fixed. There exists $R = R(p) > 0$ such that for $r > R$ and any $\Omega > 1$, if u is any harmonic function on M with $u(p) = 0$ such that*

$$I(2\Omega r) \leq C_1 I\left(\frac{r}{2}\right) , \tag{3.37}$$

then

$$D(\Omega r) \leq C_2\, C_1 D(r) , \tag{3.38}$$

where $C_2 = C_2(n)$.

Proof. By Lemma 3.22 we get an $R = R(p) > 0$ and a $K = K(n) > 0$ such that for $r > R$,

$$I\left(\frac{r}{2}\right) \leq K\, D(r) . \tag{3.39}$$

From (2.12), we have

$$2 \int_{\Omega r}^{2\Omega r} \frac{D(s)}{s} = I(2\Omega r) - I(\Omega r) \leq I(2\Omega r) , \tag{3.40}$$

and hence

$$2 \int_{\Omega r}^{2\Omega r} s^{n-2}\, D(s) \leq (2\Omega r)^{n-1}\, I(2\Omega r) . \tag{3.41}$$

By definition, (2.7), $s^{n-2} D(s)$ is monotone nondecreasing, and therefore (3.41) yields

$$2\,(\Omega r)^{n-1}\, D(\Omega r) \leq 2^{n-1}\, (\Omega r)^{n-1}\, I(2\Omega r) . \tag{3.42}$$

Dividing through by $2(\Omega r)^{n-1}$ gives

$$D(\Omega r) \leq 2^{n-2} I(2\Omega r) . \tag{3.43}$$

Combining (3.37), (3.39) and (3.43), we obtain (3.38) with $C_2 = 2^{n-2} K$.
q.e.d.

4. Almost monotonicity of the frequency and almost separation of variables

In this section we will show that when M has Euclidean volume growth the frequency function behaves much like it did in the cone case. In particular, we will first show that the frequency function is almost monotone (Proposition 4.11). This will allow us to show that harmonic functions with polynomial growth come close to separating variables on infinitely many large annuli (Theorem 4.60).

Differentiating $\log W$, we get

$$(\log W(r))' = \frac{E'(r)}{E(r)} - \frac{I'(r)}{I(r)}, \tag{4.1}$$

which together with (2.13), that is, with $(\log I)' = 2r^{-1}U$, implies

$$(\log W(r))' = \frac{E'(r)}{E(r)} - \frac{2D(r)}{rI(r)}\,. \tag{4.2}$$

From the first variation of energy (Proposition A.23) it follows that (3.15) is equivalent to

$$\begin{aligned} E'(r) =& 2r^{2-n}\int_{b=r}\left|\frac{\partial u}{\partial n}\right|^2 |\nabla b| + \frac{r^{1-n}}{2}\int_{b\leq r}|\nabla u|^2\Delta b^2 \\ &- r^{1-n}\int_{b\leq r}\operatorname{Hess}(b^2)(\nabla u,\nabla u) + (2-n)\frac{E(r)}{r} \\ =& 2\frac{F(r)}{r} + \frac{2}{r}E(r) - r^{1-n}\int_{b\leq r}\operatorname{Hess}(b^2)(\nabla u,\nabla u)\,, \end{aligned} \tag{4.3}$$

where the second equality follows from (2.5) and (2.8).

Substituting (4.3) for $E'(r)$ into (4.2), we get

$$\begin{aligned} (\log W(r))' =& \frac{2}{r} + \frac{2F(r)}{rE(r)} \\ &- \frac{r^{1-n}\int_{b\leq r}\operatorname{Hess}(b^2)(\nabla u,\nabla u)}{E(r)} - \frac{2D(r)}{rI(r)}\,. \end{aligned} \tag{4.4}$$

Grouping terms, we rewrite (4.4) as

$$\begin{aligned} (\log W(r))' =& \left[\frac{2}{r} - \frac{r^{1-n}\int_{b\leq r}\operatorname{Hess}(b^2)(\nabla u,\nabla u)}{E(r)}\right] \\ &+ \left[\frac{2F(r)}{rE(r)} - \frac{2D(r)}{rI(r)}\right]\,. \end{aligned} \tag{4.5}$$

By the divergence theorem, we can express $D(r)$ as the boundary integral

$$D(r) = r^{2-n} \int_{b=r} u \frac{\partial u}{\partial n} \, . \tag{4.6}$$

Applying the Cauchy-Schwarz inequality to (4.6), we get

$$\begin{aligned} D(r)^2 &\leq \left(r^{1-n} \int_{b=r} u^2 |\nabla b| \right) \left(r^{3-n} \int_{b=r} \left| \frac{\partial u}{\partial n} \right|^2 |\nabla b|^{-1} \right) \\ &= I(r) \, r^{3-n} \int_{b=r} \left| \frac{\partial u}{\partial n} \right|^2 |\nabla b|^{-1} \, , \end{aligned} \tag{4.7}$$

which is equivalent to

$$\left(\frac{2r^{2-n} \int_{b=r} \left| \frac{\partial u}{\partial n} \right|^2 |\nabla b|^{-1}}{D(r)} - \frac{2D(r)}{rI(r)} \right) \geq 0 \, . \tag{4.8}$$

In view of (4.8), we now rewrite (4.5) as

$$\begin{aligned} (\log W(r))' = &\left[\frac{2r^{2-n} \int_{b=r} \left| \frac{\partial u}{\partial n} \right|^2 |\nabla b|^{-1}}{D(r)} - \frac{2D(r)}{rI(r)} \right] \\ &+ \left[\frac{2}{r} - \frac{r^{1-n} \int_{b \leq r} \text{Hess}(b^2)(\nabla u, \nabla u)}{E(r)} \right] \\ &+ \left[\frac{2F(r)}{rE(r)} - \frac{2r^{2-n} \int_{b=r} \left| \frac{\partial u}{\partial n} \right|^2 |\nabla b|^{-1}}{D(r)} \right] \, . \end{aligned} \tag{4.9}$$

Define the first term in brackets to be

$$\left[(\log W(r))' \right]^{\text{ess}} = \left[\frac{2r^{2-n} \int_{b=r} \left| \frac{\partial u}{\partial n} \right|^2 |\nabla b|^{-1}}{D(r)} - \frac{2D(r)}{rI(r)} \right] \, ; \tag{4.10}$$

note that by (4.8), $[(\log W)']^{\text{ess}}$ is nonnegative. To this point we have not used any assumption on M other than the existence of a global Green's function. We will show that the remaining terms are small on many large annuli if M has nonnegative Ricci curvature and Euclidean volume growth, and u has polynomial growth.

Examples of [26] (see Section 11) show that the frequency function is no longer monotone under the assumptions of Theorem 0.3. However, we will now prove almost monotonicity of the frequency function on many large annuli.

Proposition 4.11. *(Almost monotonicity of W). Let M^n be an n-dimensional manifold with nonnegative Ricci curvature and Euclidean volume growth. Fix $p \in M$. Given positive constants γ, ϵ, and $\Omega_0 > 1$, there exists $R = R(p, \gamma, \epsilon, \Omega_0) > 0$ such that if $1 < \Omega \leq \Omega_0$, $r \geq R$, and u is any harmonic function on M with*

$$D(2\Omega r) \leq \gamma D(r)\,, \tag{4.12}$$

then

$$\int_r^{\Omega r} \min\{(\log W)'(t), 0\} dt > -\epsilon\,. \tag{4.13}$$

In fact we will show that

$$\int_r^{\Omega r} |(\log W)'(t) - \left[(\log W)'(t)\right]^{ess} |dt < \epsilon\,. \tag{4.14}$$

Proof. Since the first term in (4.9) is nonnegative, (4.13) follows from (4.14). Therefore, it suffices to bound the integrals of the second and third lines in (4.9).

We now recall some analytic facts. From the asymptotics of the Green's function and Section 4 of [6], given any $\delta > 0$, there exists $R_1 = R_1(p, \delta) > 0$ such that for all $r > R_1$, we have

$$\left|\log \frac{b}{r}\right| \leq \delta\,, \tag{4.15}$$

$$\int_{b \leq r} \left||\nabla b|^2 - 1\right|^2 \leq \delta^2 \operatorname{Vol}(b \leq r)\,, \tag{4.16}$$

and

$$\int_{b \leq r} \left|\operatorname{Hess}(b^2) - 2g\right|^2 \leq \delta^2 \operatorname{Vol}(b \leq r)\,, \tag{4.17}$$

where g is the metric tensor on M.

If we consider only $\delta > 0$ such that

$$\exp(2\delta) \leq \frac{4}{3}\,, \tag{4.18}$$

then (4.15) implies that for $s > R_1$,

$$\begin{aligned} \{b \leq s\} &\subset B_{s\sqrt{\frac{4}{3}}}(p)\,, \\ B_{s\sqrt{3}}(p) &\subset \{b \leq 2s\}\,. \end{aligned} \tag{4.19}$$

From Proposition 3.3, we get an $R_2 = R_2(p, \gamma, \delta, \Omega_0) > 0$ such that D and E are equivalent; that is, for $r > R_2$ and s between r and Ωr, we have

$$\left| \log \frac{D(s)}{E(s)} \right| \leq \delta\,. \tag{4.20}$$

We set $R = \max\{R_1, R_2\}$.

Note also that Lemma 3.16 together with (4.20) implies that D is almost monotone; that is, for s between r and Ωr,

$$\begin{aligned} D(r) \leq & E(r)\exp(\delta) \leq E(s)\exp(\delta) \\ \leq & D(s)\exp(2\delta)\,. \end{aligned} \tag{4.21}$$

By the Bochner formula $|\nabla u|^2$ is subharmonic. Therefore Proposition 2.26 and (4.19) yield that for $r > R_1$, $|\nabla u|^2$ is bounded by

$$\begin{aligned} \sup_{b \leq \Omega r} |\nabla u|^2 &\leq \sup_{B_{\Omega r\sqrt{\frac{4}{3}}}(p)} |\nabla u|^2 \\ &\leq C_1 \operatorname{Vol}(B_{\Omega r\sqrt{3}}(p))^{-1} \int_{B_{\Omega r\sqrt{3}}(p)} |\nabla u|^2 \\ &\leq C_1 \mathrm{V}_M{}^{-1} \left(\Omega r\sqrt{3}\right)^{-n} \int_{b \leq 2\Omega r} |\nabla u|^2 \\ &= C\,\Omega^{-2}\, r^{-2} D(2\Omega r)\,, \end{aligned} \tag{4.22}$$

where $C_1 = C_1(n) > 0$ comes from Proposition 2.26 and

$$C = C_1 \mathrm{V}_M^{-1} (\sqrt{3})^{-n} 2^{n-2}.$$

Bounding the normal derivative by the full gradient, and using the weighted area bound for the level sets, (2.15), we see that (4.22) gives a bound for F. For s between r and Ωr,

$$F(s) = s^{3-n} \int_{b=s} \left|\frac{\partial u}{\partial n}\right|^2 |\nabla b| \leq C n \mathrm{V}_M\, D(2\Omega r)\,. \tag{4.23}$$

We now bound the second line of (4.9) by

$$\begin{aligned} &\left| \frac{2}{s} - \frac{s^{1-n} \int_{b \leq s} \mathrm{Hess}(b^2)(\nabla u, \nabla u)}{E(s)} \right| \\ &\leq \frac{1}{s} \left[\frac{s^2 \sup_{b \leq s} |\nabla u|^2}{E(s)} \right] \left[s^{-n} \int_{b \leq s} |\mathrm{Hess}(b^2) - 2g| \right] \\ &\quad + \frac{2}{s} \left[\frac{E(s) - D(s)}{E(s)} \right] . \end{aligned} \tag{4.24}$$

From (4.20) we get for $s > R$ a bound on the above second term. We will now bound the first term in the second line of (4.24). From the monotonicity of E (see (4.21)) and (4.22), we have for s between r and Ωr,

$$\frac{s^2 \sup_{b \leq s} |\nabla u|^2}{E(s)} \leq C \frac{D(2\Omega r)}{E(r)} \leq C\gamma \exp(\delta) , \tag{4.25}$$

where the second inequality follows from (4.20) and the hypothesis (4.12). We use the estimate (4.17) together with (4.15), the Cauchy-Schwarz inequality, and the Bishop volume comparison theorem to bound

$$s^{-n} \int_{b \leq s} |\mathrm{Hess}(b^2) - 2g| \leq \delta \mathrm{V}_0^n(1) \, \exp(n\delta) . \tag{4.26}$$

Putting it all together, we get a bound on the second line of (4.9), for $r > R$ and s between r and Ωr,

$$\begin{aligned} &\left| \frac{2}{s} - \frac{s^{1-n} \int_{b \leq s} \mathrm{Hess}(b^2)(\nabla u, \nabla u)}{E(s)} \right| \\ &\leq \frac{1}{s} \left[C\gamma \exp(\delta) \right] \left[\delta \mathrm{V}_0^n(1) \, \exp(n\delta) \right] \\ &\quad + \frac{2}{s} \left[\exp(\delta) - 1 \right] . \end{aligned} \tag{4.27}$$

Integrating (4.27) yields

$$\begin{aligned} &\int_r^{\Omega r} \left| \frac{2}{s} - \frac{s^{1-n} \int_{b \leq s} \mathrm{Hess}(b^2)(\nabla u, \nabla u)}{E(s)} \right| \\ &\leq \left[C\gamma \exp(\delta) \right] \left[\delta \mathrm{V}_0^n(1) \, \exp(n\delta) \right] \log \Omega_0 \\ &\quad + 2 \left[\exp(\delta) - 1 \right] \log \Omega_0 \\ &\leq C \left(\frac{4}{3} \right)^{\frac{n+1}{2}} \mathrm{V}_0^n(1) \, \gamma \delta \log \Omega_0 + 2 \left[\exp(\delta) - 1 \right] \log \Omega_0 . \end{aligned} \tag{4.28}$$

It remains to bound the third line of (4.9); this must be done in an integral sense. We have

$$
\begin{aligned}
&\left[\frac{2F(s)}{sE(s)} - \frac{2s^{2-n}\int_{b=s}\left|\frac{\partial u}{\partial n}\right|^2 |\nabla b|^{-1}}{D(s)}\right] \\
&\quad = \frac{2s^{2-n}\int_{b=s}\left|\frac{\partial u}{\partial n}\right|^2 (|\nabla b| - |\nabla b|^{-1})}{D(s)} \\
&\quad\quad + \frac{2}{s}\left[\frac{D(s)-E(s)}{E(s)}\right]\frac{F(s)}{D(s)}\,.
\end{aligned}
\tag{4.29}
$$

Using (4.23) and (4.20), and then (4.21) and (4.12), we get

$$
\begin{aligned}
\left|\frac{D(s)-E(s)}{E(s)}\right|\frac{F(s)}{D(s)} &\leq (\exp(\delta)-1)C\gamma\exp(2\delta) \\
&\leq (\exp(\delta)-1)C\gamma\frac{4}{3}\,.
\end{aligned}
\tag{4.30}
$$

A similar application of (4.22), and then (4.21) and (4.12) yields

$$
\begin{aligned}
&\frac{s^{2-n}\int_{b=s}\left|\frac{\partial u}{\partial n}\right|^2 \left||\nabla b| - |\nabla b|^{-1}\right|}{D(s)} \\
&\leq C\gamma s^{-n}\int_{b=s}\left||\nabla b| - |\nabla b|^{-1}\right| ds\,\exp(2\delta) \\
&\leq \frac{4}{3}C\gamma s^{-n}\int_{b=s}\left||\nabla b| - |\nabla b|^{-1}\right| ds\,.
\end{aligned}
\tag{4.31}
$$

By the co-area formula, we see that

$$
\begin{aligned}
\int_r^{\Omega r} s^{-n}\int_{b=s}\left||\nabla b| - |\nabla b|^{-1}\right|\,ds &= \int_{\{r\leq b\leq \Omega r\}} b^{-n}\left||\nabla b|^2 - 1\right| \\
&\leq r^{-n}\int_{\{b\leq \Omega r\}}\left||\nabla b|^2 - 1\right| \\
&\leq \delta\frac{\mathrm{Vol}(\{b\leq \Omega r\})}{r^n} \\
&\leq \delta \mathrm{V}_0^n(1)\,\Omega^n\exp(n\delta) \\
&\leq \delta \mathrm{V}_0^n(1)\,\Omega^n\left(\frac{4}{3}\right)^{\frac{n}{2}}\,,
\end{aligned}
\tag{4.32}
$$

where the third to last inequality follows from (4.16), and the second to last from (4.15) and the Bishop volume comparison theorem.

Combining (4.30), (4.31), and (4.32), we get an integral bound for (4.29),

$$\int_r^{\Omega r}\left[\frac{2F(s)}{sE(s)}-\frac{2s^{2-n}\int_{b=s}\left|\frac{\partial u}{\partial n}\right|^2|\nabla b|^{-1}}{D(s)}\right] \tag{4.33}$$
$$\leq \frac{8}{3}C\gamma(\exp(\delta)-1)\log\Omega_0 + \left(\frac{4}{3}\right)^{\frac{n+2}{2}} C\gamma \mathrm{V}_0^n(1)\Omega_0^n\,\delta\,.$$

To control the four terms from (4.28) and (4.33), we first choose

$$\delta_1=\min\left\{\frac{1}{2}\log\frac{4}{3},\epsilon\left[4C\left(\frac{4}{3}\right)^{\frac{n+1}{2}}\mathrm{V}_0^n(1)\,\gamma\log\Omega_0\right]^{-1},\right.$$
$$\left.\left(\frac{3}{4}\right)^{\frac{n+2}{2}}\epsilon\,[C\gamma\mathrm{V}_0^n(1)\Omega_0^n]^{-1}\right\}. \tag{4.34}$$

Next, notice that for $0<s\leq\frac{1}{2}\log\frac{4}{3}$,

$$\exp s-1=\int_0^s\exp t\,dt\leq\left(\frac{4}{3}\right)^{\frac{1}{2}}s\,; \tag{4.35}$$

therefore, choose δ_2 by taking

$$\delta_2=\min\left\{\frac{1}{2}\log\frac{4}{3},\epsilon\left[\frac{32}{3}\log\Omega_0\right]^{-1},\epsilon\left[\frac{128}{9}C\gamma\log\Omega_0\right]^{-1}\right\}. \tag{4.36}$$

Taking $\delta=\min\{\delta_1,\delta_2\}$, each of the four terms from (4.28) and (4.33) is bounded by $\frac{\epsilon}{4}$, and the proposition now follows. q.e.d.

Corollary 4.37. *Let M^n be as in Proposition 4.11 and let $p\in M$ be fixed. Given a positive constant ϵ and $1<\Omega_0$, there exists $R=R(p,\gamma,\epsilon,\Omega_0)>0$ such that if $1<\Omega<\Omega_0$, $r\geq R$, and u is any harmonic function on M satisfying*

$$D(2\Omega r)\leq\gamma D(r)\,, \tag{4.38}$$

then for all $r<s<t<\Omega r$ we have that

$$I(t)\leq\left(\frac{t}{s}\right)^{2(1+\epsilon)d}I(s)\,, \tag{4.39}$$

where $d=W(\Omega r)$.

Proof. This follows from Proposition 4.11 together with (2.13) and the equivalence of $E(r)$ and $D(r)$ (Proposition 3.3). q.e.d.

In light of Proposition 4.11, the lower bounds for the maximum of U from Section 3 can now be used to derive pointwise lower bounds for the frequency on many annuli.

Corollary 4.40. *(Uniform lower bound of the frequency). Let M be a manifold with nonnegative Ricci curvature and Euclidean volume growth, and let $p \in M$ be fixed. Given $\frac{1}{3} > \epsilon > 0$, we let $\Omega_L = \Omega_L(n, \epsilon)$ be given by Corollary 3.29. Given $\gamma > 0$ and $\Omega_0 > \Omega_L$, there exists $R = R(p, \gamma, \epsilon, \Omega_0) > 0$ such that if $r > R$, $\Omega_L < \Omega < \Omega_0$, and u is any harmonic function on M with $u(p) = 0$ and*

$$D(2\Omega^2 r) \leq \gamma D(r)\,, \tag{4.41}$$

then for s between Ωr and $\Omega^2 r$,

$$(1 - 3\epsilon) \leq U(s)\,. \tag{4.42}$$

Proof. From Corollary 3.29, there exists a $R_1 = R_1(p) > 0$ such that for $r > R_1$,

$$(1 - \epsilon) \leq \max_{r \leq s \leq \Omega_L r} U(s)\,. \tag{4.43}$$

Given $\delta > 0$, Proposition 4.11 and Proposition 3.3 yield an $R_2 = R_2(p, \gamma, \delta, \Omega_0^2) > 0$ such that for $r > R_2$,

$$\int_r^{\Omega^2 r} \min\{(\log W)'(t), 0\} dt > -\delta\,, \tag{4.44}$$

and for s between r and $\Omega^2 r$,

$$\left| \log \frac{U(s)}{W(s)} \right| \leq \delta\,. \tag{4.45}$$

Combining (4.44) and (4.45) we get for $r \leq s_1 \leq s_2 \leq \Omega^2 r$

$$U(s_1) \leq \mathrm{e}^{\delta} W(s_1) \leq \mathrm{e}^{2\delta} W(s_2) \leq \mathrm{e}^{3\delta} U(s_2)\,. \tag{4.46}$$

The corollary now follows from (4.43) and (4.46) by choosing

$$\delta = \frac{1}{3} \log \frac{1 - \epsilon}{1 - 3\epsilon}\,, \tag{4.47}$$

and $R = \max\{R_1, R_2\}$. q.e.d.

Definition 4.48. (Almost separation of variables). Suppose that M^n is an open manifold, $p \in M$, $\epsilon > 0$, $\{r \le b \le \Omega r\}$ is an annulus, and u is a function on $\{r \le b \le \Omega r\}$. We say that u ϵ-almost separates variables on the annulus $\{r \le b \le \Omega r\}$ if there exists a function $h : \mathbf{R} \to \mathbf{R}$ such that for any $r \le s_1 < s_2 \le \Omega r$,

$$\int_{\{s_1 \le b \le s_2\}} b^{-n} \left(b \, \frac{\partial u}{\partial n} - h(b) \, u \, |\nabla b| \right)^2 < \epsilon \, I_u(s_2) \, . \tag{4.49}$$

The next goal is to show that harmonic functions with polynomial growth in fact almost separate variables on many large annuli. As in the conical case, we will show that a harmonic function almost separates variables by analyzing almost equality in the Cauchy-Schwarz inequality which implied the positivity of $[(\log W)']^{\text{ess}}$. This term is small because $[(\log W)']$ is small and almost equal to $[(\log W)']^{\text{ess}}$ by Proposition 4.11.

We need some preliminary results which now follow.

Proposition 4.50. *(U almost constant implies u almost separates variables). Let M^n have nonnegative Ricci curvature and Euclidean volume growth, and let $p \in M$ be fixed. Given $\epsilon, d_0, \gamma > 0$, and $\Omega_0 > 1$, there exists $\delta = \delta(d_0, \epsilon) > 0$ such that we have the following: there exists $R = R(p, \gamma, d_0, \epsilon, \Omega_0) > 0$ such that if $1 < \Omega < \Omega_0$, $r > R$, and u is any harmonic function on M satisfying*

$$D(2\Omega r) \le \gamma D(r) \, , \tag{4.51}$$

$$\max_{r \le s \le \Omega r} U(s) \le d_0 \, , \tag{4.52}$$

and

$$\left| \log \left(\frac{U(\Omega r)}{U(r)} \right) \right| < \frac{\delta}{2} \, , \tag{4.53}$$

then u ϵ-almost separates variables on the annulus $\{r \le b \le \Omega r\}$ in the sense of Definition 4.48. In fact, we can take $h(s) = U(s)$ and choose $\delta > 0$ such that

$$\delta \exp(\delta) \le \frac{\epsilon}{d_0} \, . \tag{4.54}$$

Proof. Under the hypotheses, for each $\delta > 0$ there exists $R_\delta > 0$ such that for $R_\delta < r \le s \le \Omega r$, $U(s)$ and $W(s)$ are almost equal and

$W(s)$ is almost monotone. That is, given $\delta > 0$, Proposition 3.3 and Proposition 4.11 guarantee the existence of an $R_\delta = R_\delta(p, \gamma, \delta, \Omega_0) > 0$ such that for $R_\delta < r \leq s \leq \Omega r$,

$$\left|\log \frac{D(s)}{E(s)}\right| < \frac{\delta}{2}\,, \tag{4.55}$$

and

$$\int_r^{\Omega r} \left|(\log W)'(t) - \left[(\log W)'(s)\right]^{\text{ess}}\right| dt < \frac{\delta}{2}\,. \tag{4.56}$$

Note that (4.55) is equivalent to having for $r \leq s \leq \Omega r$

$$\left|\log \frac{U(s)}{W(s)}\right| < \frac{\delta}{2}\,. \tag{4.57}$$

Further observe that (4.55) implies by Lemma 3.16 that if $s \leq s_2$

$$D(s) < \exp\left(\frac{\delta}{2}\right) E(s) \leq \exp\left(\frac{\delta}{2}\right) E(s_2) < \exp(\delta)\, D(s_2)\,. \tag{4.58}$$

Therefore, by the co-area formula, we have for $r \leq s_1 < s_2 \leq \Omega r$,

$$\begin{aligned}
&\int_{\{s_1 \leq b \leq s_2\}} b^{-n} \left(b \frac{\partial u}{\partial n} - U(b)\, u\, |\nabla b|\right)^2 \\
&\quad = \int_{s_1}^{s_2} s^{-n} \int_{b=s} \left(s \frac{\partial u}{\partial n} |\nabla b|^{-\frac{1}{2}} - U(s)\, u\, |\nabla b|^{\frac{1}{2}}\right)^2 \\
&\quad = \int_{s_1}^{s_2} \left[s^{2-n} \int_{b=s} \left|\frac{\partial u}{\partial n}\right|^2 |\nabla b|^{-1} - 2 s^{1-n} U(s) \int_{b=s} u \frac{\partial u}{\partial n} \right. \\
&\qquad\qquad \left. + U^2(s)\, s^{-n} \int_{b=s} u^2 |\nabla b|\right] \\
&\quad = \frac{1}{2} \int_{s_1}^{s_2} D(s) \left[(\log W)'(s)\right]^{\text{ess}} \\
&\quad < \frac{1}{2} \exp(\delta)\ D(s_2) \int_r^{\Omega r} \left[(\log W)'(s)\right]^{\text{ess}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\exp(\delta)\ D(s_2)\int_r^{\Omega r}(\log W)'(s) \\
&\quad + \frac{1}{2}\exp(\delta)\ D(s_2)\int_r^{\Omega r}|(\log W)'(s) - \left[(\log W)'(s)\right]^{\text{ess}}| \\
&< \frac{1}{2}\exp(\delta)\ D(s_2)\left[\log\left(\frac{W(\Omega r)}{W(r)}\right) + \frac{\delta}{2}\right] \qquad (4.59) \\
&< \frac{1}{2}\exp(\delta)\ D(s_2)\left[\log\left(\frac{U(\Omega r)}{U(r)}\right) + \frac{3}{2}\delta\right] \\
&< \exp(\delta)\ U(s_2)\, I(s_2)\,\delta \\
&\leq \delta\ \exp(\delta)\ d_0\, I(s_2)\,.
\end{aligned}
$$

The claim now follows by choosing δ by (4.54) and then taking $R = R_\delta$. q.e.d.

As an application of the techniques developed, we give now an asymptotic description of harmonic functions with polynomial growth on manifolds with nonnegative Ricci curvature and Euclidean volume growth. Namely, we show that a harmonic function with polynomial growth on such a manifold almost separates variables on an infinite sequence of large annuli. By improving the proof, we will in Section 10 give a generalization of this for a set of independent harmonic functions. This generalization will be a key step in the proof of Theorem 0.3.

Since it is the generalization of Theorem 4.60 given in Section 10, and not Theorem 4.60 itself, that we need in the proof of Theorem 0.3 the reader can choose to skip Theorem 4.60.

Theorem 4.60. *(Asymptotic description of harmonic functions with polynomial growth). Let M^n be as in Proposition 4.11, and $u \in \mathcal{H}_d(M)$. Given $\Omega > 2$ and $\epsilon > 0$, there exists a sequence $r_j \to \infty$ such that u ϵ-almost separates variables on the annulus $A_{r_j,\Omega r_j}$.*

Proof. We can assume that $u(p) = 0$. By the Cheng-Yau gradient estimate, $|\nabla u|$ grows polynomially of order at most $d-1$, and $d \geq 1$ if u is nonconstant. Combining this with the Bishop volume comparison theorem, we see that $D(r)$ grows polynomially of order at most $2d$.

Choose an $\epsilon_0 > 0$ such that

$$\epsilon_0 \exp \epsilon_0 < \frac{\epsilon}{12d}\,. \qquad (4.61)$$

Let $\Omega_L = \Omega_L(n, \frac{1}{3}) \geq 2$ and $R_1 = R_1(p) > 0$ be given by Corollary 3.29. Then for any $r > R_1$, the maximum of U on the interval $[r, \Omega_L r]$

is at least $\frac{2}{3}$. Set $\Omega_0 = \max\{\Omega, \Omega_L\}$. Choose $m > 2n$ and $\delta > 0$ so that

$$\begin{aligned} \delta &< \frac{\epsilon_0}{4} \,, \\ \exp 2\delta &< \frac{4}{3} \,, \end{aligned} \tag{4.62}$$

and

$$\frac{\log 8d}{m-2} < \frac{\epsilon_0}{4} \,. \tag{4.63}$$

Since D grows polynomially of order at most $2d$ and

$$2d < \frac{5md}{2m+1} \,, \tag{4.64}$$

there is a sequence $r_i \to \infty$ such that

$$D(\Omega_0^{2m+1} r_i) < \Omega_0^{5dm} D(r_i) \,, \tag{4.65}$$

(cf. Corollary 7.9).

By Proposition 4.11 and Proposition 3.3, we get an

$$R_2 = R_2(p, \Omega_0^{5dm}, \delta, \Omega_0^{2m+1}) > 0$$

such that for $r_i > R_2$,

$$\int_{r_i}^{\Omega_0^{2m} r_i} \min\{(\log W)'(t), 0\} dt > -\delta \,, \tag{4.66}$$

and for s between r_i and $\Omega_0^{2m} r_i$,

$$\left| \log \frac{D(s)}{E(s)} \right| \leq \delta \,. \tag{4.67}$$

Set $R = \max\{R_1, R_2\}$.

From (2.10), we have

$$D(\Omega_0^{2m} r_i) \leq \Omega_0^{5dm+n-2}\, D(r_i) \,. \tag{4.68}$$

By the definition of U and (2.13),

$$\begin{aligned} \int_{r_i}^{\Omega_0^{2m} r_i} \frac{2U(s)}{s} ds =& \log \left(\frac{D(\Omega_0^{2m} r_i)}{D(r_i)} \right) \\ & - \log \left(\frac{U(\Omega_0^{2m} r_i)}{U(r_i)} \right) \\ \leq & (5dm + n - 2) \log \Omega_0 + 3\delta \\ \leq & 6dm \log(\Omega_0) \,, \end{aligned} \tag{4.69}$$

where the inequality uses (4.62), (4.63), and (4.68). Since U is nonnegative, the bound (4.69) implies that for some s_i between $\Omega_0^m r_i$ and $\Omega_0^{2m} r_i$ we have

$$U(s_i) \leq 3d\,. \tag{4.70}$$

By (4.67) and the δ-almost monotonicity of W on the interval $[r_i, \Omega_0^{2m} r_i]$, (4.66), the bound (4.70) implies that for $r_i > R_2$, and s between r_i and $\Omega_0^m r_i$,

$$W(s) \leq 3d \exp(2\delta) \leq 4d\,, \tag{4.71}$$

where the last inequality follows from (4.62). By the δ-almost monotonicity of W on the interval $[r_i, \Omega_0^{2m} r_i]$, (4.66), the choice of Ω_0, and (4.67), we have for $r > R$ and s between $\Omega_0 r_i$ and $\Omega_0^m r_i$,

$$\frac{1}{2} \leq \frac{2}{3} \exp(-2\delta) \leq W(s)\,, \tag{4.72}$$

where again the last inequality follows from (4.62). Combining (4.71) and (4.72) yields that for s between $\Omega_0 r_i$ and $\Omega_0^m r_i$,

$$\frac{1}{2} \leq W(s) \leq 4d\,. \tag{4.73}$$

Again by the δ-almost monotonicity of W on the interval $[r_i, \Omega_0^{2m} r_i]$, (4.66), together with (4.73), we see that for $r_i > R$, there exists an integer $2 \leq k_i < m$ such that

$$\log\left(\frac{W(\Omega_0^{k_i} r_i)}{W(\Omega_0^{k_i-1} r_i)}\right) \leq 2\delta + \frac{\log(8\,d)}{m-2} < \frac{\epsilon_0}{2}\,. \tag{4.74}$$

By (4.67) and (4.74), we conclude that

$$\log\left(\frac{U(\Omega_0^{k_i} r_i)}{U(\Omega_0^{k_i-1} r_i)}\right) < \epsilon_0\,. \tag{4.75}$$

From (4.67) and (4.73) it is seen that for s between $\Omega_0 r_i$ and $\Omega_0^m r_i$,

$$\frac{3}{8} < U(s) < 5d\,, \tag{4.76}$$

which implies

$$\log\left(\frac{D(\Omega_0^{k_i+1}r_i)}{D(\Omega_0^{k_i-1}r_i)}\right) = \log\left(\frac{U(\Omega_0^{k_i+1}r_i)}{U(\Omega_0^{k_i-1}r_i)}\right) + \int_{\Omega_0^{k_i-1}r_i}^{\Omega_0^{k_i+1}r_i} \frac{2U(s)}{s} ds < \log(14\,d) + 20\,d\,\log(\Omega_0)\,. \tag{4.77}$$

We can now apply Proposition 4.50 to get an

$$R_3 = R_3(p, 14d\Omega_0^{20d}, 5d, \epsilon, \Omega_0^2) > 0$$

such that for $r_i > \max\{R, R_3\}$, u ϵ-almost separates variables on the annulus $\{\Omega_0^{k_i-1}r_i \leq b \leq \Omega_0^{k_i}r_i\}$. Finally, we note that when u ϵ-almost separates variables it also does so on all subannuli, and the claim follows since $\Omega \leq \Omega_0$. q.e.d.

5. Preserving almost orthogonality

In this section, we will use the previous work on almost separation of variables to show how to preserve the almost orthogonality condition for harmonic functions on an annulus. The importance of the results of this section is that it will allow us to show that two harmonic functions u and v have a definite separation at $b = r$

provided that:

(1) they have a definite separation at $b = \Omega r$,

(2) the growth of u and v from $b = r$ to $b = \Omega r$ has a definite bound,

(3) we have good control of v between $b = r$ and $b = \Omega r$. (In fact v needs to be very close to separating variables on the annulus $\{r \leq b \leq \Omega r\}$.)

We continue to take M^n to be an n-dimensional manifold with nonnegative Ricci curvature and Euclidean volume growth.

Proposition 5.1. *(Almost preserving orthogonality). Fix $p \in M$, $\Omega > 1$, and suppose that u and v are harmonic functions on the annulus $\{r \leq b \leq \Omega r\}$, v δ-almost separates variables on $\{r \leq b \leq \Omega r\}$, $r < s_2 \leq \Omega r$, and*

$$s_2{}^{1-n} \int_{b=s_2} uv|\nabla b| = 0\,. \tag{5.2}$$

Then for $r \leq s_1 < s_2$,

$$\left| {s_1}^{1-n} \int_{b=s_1} uv|\nabla b| \right|^2 < 8\,\delta \left(\frac{s_2}{s_1}\right)^{4d+2} I_u(s_2)\, I_v(s_2)\,, \tag{5.3}$$

where $d = \max\{h_v(s) \,|\, s_1 \leq s \leq s_2\}$, and h_v is as in Definition 4.48.

Proof. By differentiation, we get

$$\begin{aligned} \frac{d}{ds}\left[s^{1-n} \int_{b=s} uv|\nabla b|\right] =&(1-n)s^{-n} \int_{b=s} uv|\nabla b| \\ &+ s^{1-n} \int_{b=s} \left(v\frac{\partial u}{\partial s} + u\frac{\partial v}{\partial s}\right) |\nabla b| \\ &+ s^{1-n} \int_{b=s} uv \frac{\Delta b}{|\nabla b|}\,; \end{aligned} \tag{5.4}$$

using equation (2.4), we have

$$\begin{aligned} \frac{d}{ds}\left[s^{1-n} \int_{b=s} uv|\nabla b|\right] =&s^{1-n} \int_{b=s} \left(v\frac{\partial u}{\partial s} + u\frac{\partial v}{\partial s}\right) |\nabla b| \\ =&2s^{1-n} \int_{b=s} u\frac{\partial v}{\partial n}\,, \end{aligned} \tag{5.5}$$

where the second equality follows from Green's formula together with the assumption that u and v are harmonic. Define $\mathrm{err}(s)$ by

$$s\frac{d}{ds}\left[s^{1-n} \int_{b=s} uv|\nabla b|\right] = 2\,h_v(s)\left[s^{1-n} \int_{b=s} uv|\nabla b|\right] + \mathrm{err}(s)\,. \tag{5.6}$$

By (5.5) and the definition (5.6), we get

$$|\mathrm{err}(s)| \leq 2s^{1-n} \left| s\int_{b=s} u\frac{\partial v}{\partial n} - h_v(s) \int_{b=s} uv|\nabla b| \right|. \tag{5.7}$$

It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} &\left| s\int_{b=s} u\frac{\partial v}{\partial n} - h_v(s) \int_{b=s} uv|\nabla b| \right|^2 \\ &\qquad \leq \left(\int_{b=s} |u| \left| s\frac{\partial v}{\partial n} - h_v(s)\, v|\nabla b| \right| \right)^2 \\ &\qquad \leq \int_{b=s} u^2|\nabla b| \int_{b=s} \left(s\frac{\partial v}{\partial n} - h_v(s)\, v|\nabla b| \right)^2 \frac{1}{|\nabla b|} \\ &\qquad = s^{n-1} I_u(s) \int_{b=s} \left(s\frac{\partial v}{\partial n} - h_v(s)\, v|\nabla b| \right)^2 \frac{1}{|\nabla b|}\,. \end{aligned} \tag{5.8}$$

Combining equations (5.7) and (5.8) gives

$$s^{-1}|\mathrm{err}(s)|^2 \leq \; 4I_u(s)\, s^{-n} \int_{b=s} \left(s\frac{\partial v}{\partial n} - h_v(s)\, v|\nabla b| \right)^2 \frac{1}{|\nabla b|}\,. \tag{5.9}$$

Integrating equation (5.9), by the co-area formula and the monotonicity of I (specifically $I_u(s) \leq I_u(t)$ for $s \leq t$), for $r \leq s_1 < t \leq \Omega r$, we have

$$\begin{aligned} \int_{s_1}^{t} s^{-1}|\mathrm{err}(s)|^2 ds &\leq 4\, I_u(t) \int_{\{s_1 \leq b \leq t\}} b^{-n} \left(b\frac{\partial v}{\partial n} - h_v(b)\, v|\nabla b| \right)^2 \\ &\leq 4\delta I_u(t) I_v(t)\,. \end{aligned} \tag{5.10}$$

Here the second inequality follows since v δ-almost separates variables on the annulus $\{r \leq b \leq \Omega r\}$.

If we now write

$$g(s) = s^{1-n} \int_{b=s} uv|\nabla b|\,, \tag{5.11}$$

we see that by assumption

$$g(s_2) = 0\,, \tag{5.12}$$

and that (5.6) implies

$$s|g'(s)| \leq 2d\,|g(s)| + |\mathrm{err}(s)|\,. \tag{5.13}$$

It remains to get an upper bound for $|g(s_1)|$. For ease of exposition, we set

$$a^2 = 4\delta I_u(s_2) I_v(s_2)\,. \tag{5.14}$$

If $|g(s_1)| \leq a$, then we are done. Suppose therefore that $|g(s_1)| > a$, and let s_3 be the smallest $s > s_1$ such that $|g(s)| = a$ (such an $s_3 \leq s_2$ must exist since $g(s_2) = 0$). Replacing g with $-g$ if necessary, we have for $s_1 \leq s \leq s_3$,

$$g(s) \geq a\,. \tag{5.15}$$

From (5.13) and (5.15) it follows that for $s_1 \leq s \leq s_3$,

$$s\left|(\log g)'\right| \leq 2d + \frac{|\mathrm{err}(s)|}{a}\,. \tag{5.16}$$

Now integrating (5.16) leads to

$$\log g(s_1) - \log g(s_3) \leq \int_{s_1}^{s_3} \left(\frac{2d}{s} + \frac{|\mathrm{err}(s)|}{as} \right) ds \,. \tag{5.17}$$

By absorbing inequality $2xy \leq \lambda^2 x^2 + \lambda^{-2} y^2$ we get

$$2\frac{|\mathrm{err}(s)|}{a} \leq \frac{1}{2}\left(\frac{|\mathrm{err}(s)|}{a} \right)^2 + 2 \,. \tag{5.18}$$

Using (5.18) and substituting $g(s_3) = a$, from (5.17) we obtain

$$\begin{aligned} \log g(s_1) \leq & \log a + (2d+1) \log \frac{s_3}{s_1} \\ & + \frac{1}{4} a^{-2} \int_{s_1}^{s_3} s^{-1} |\mathrm{err}(s)|^2 ds \,. \end{aligned} \tag{5.19}$$

Combining (5.19) with the estimate (5.10) gives

$$\begin{aligned} \log g(s_1) \leq & \log a + (2d+1) \log \frac{s_3}{s_1} \\ & + \frac{1}{4} a^{-2} 4 \delta I_u(s_2) I_v(s_2) \\ = & \log a + (2d+1) \log \frac{s_3}{s_1} + \frac{1}{4} \,. \end{aligned}$$

Exponentiating (5.19) yields

$$g(s_1) \leq \left(\frac{s_3}{s_1} \right)^{2d+1} a \exp \frac{1}{4} \,, \tag{5.20}$$

and the result follows since $s_3 \leq s_2$ and $\exp \frac{1}{2} < 2$. q.e.d.

For the applications it is crucial that the δ in Proposition 5.1 is chosen small compared with Ω and the growth of I_u and I_v from $b = r$ to $b = \Omega r$. Namely given this then Proposition 5.1 implies that u and v are almost orthogonal at $b = r$ in the following sense.

Definition 5.22. (Almost orthogonality). Let u and v be harmonic functions defined in a neighborhood of $b = s$. Given $\epsilon > 0$, we say that u and v are ϵ-almost orthogonal at s if

$$s^{1-n} \left| \int_{b=s} uv |\nabla b| \right| < \epsilon \, I_u^{\frac{1}{2}}(s) \, I_v^{\frac{1}{2}}(s) \,. \tag{5.23}$$

Noting that any linear functional on a Hilbert space is determined by its kernel and its action on any element orthogonal to the kernel, we see that Proposition 5.1 has the following simple corollary.

Corollary 5.24. *(Almost preserving the inner product). If u and v are harmonic functions and v δ-almost separates variables on $\{r \leq b \leq \Omega r\}$, then for $r \leq s_1 < s_2 \leq \Omega r$*

$$\left| {s_2}^{1-n} \int_{b=s_2} uv|\nabla b| - \exp\left(2 \int_{s_1}^{s_2} \frac{h(s)}{s}\, ds\right) {s_1}^{1-n} \int_{b=s_1} uv|\nabla b| \right|^2$$

$$< 32\, \delta \left(\frac{s_2}{s_1}\right)^{6d+2} I_u(s_2)\, I_v(s_2)\,, \tag{5.25}$$

where

$$d = \max_{s_1 \leq s \leq s_2} h_v(s)\,. \tag{5.26}$$

Proof. Orthogonally decompose u into

$$u = u_1 + av\,, \tag{5.27}$$

where

$${s_2}^{1-n} \int_{b=s_2} u_1 v |\nabla b| = 0\,. \tag{5.28}$$

We apply Proposition 5.1 to u_1, and use the fact that v δ-almost separates variables to control the remainder. q.e.d.

We note that in the applications we will use Proposition 5.1 and not Corollary 5.24.

6. Bounding the number of almost orthonormal Lipschitz functions

In this section we will bound the dimension of the space of almost L^2-orthonormal functions with a given Lipschitz bound under very general conditions.

Definition 6.1. Given (X, d) a compact metric space, define $\mathcal{L}(X)$ to be the set of Lipschitz functions on X. We set

$$\mathcal{L}_k(X) = \{u \in \mathcal{L}(X)\,|\, \mathrm{Lip}(u) \leq k\}.$$

Definition 6.2. (η-almost orthonormal functions). Let (X, μ) be a measure space with a probability measure, μ, and suppose that $f_1, \ldots, f_m$ are L^2 functions on X. We say that the f_i are η-almost orthonormal if

$$\int_X f_i^2 = 1\,, \tag{6.3}$$

and for $i \neq j$

$$\left|\int_X f_i f_j\right| < \eta\,. \tag{6.4}$$

In the next proposition, we think of r as the scaling factor and D_0 and k as the constants.

Proposition 6.5. *Let (X, d, μ) be a compact metric space with a probability measure, μ, and diam$(X) \leq D_0 r$. Given $k > 0$, there exist at most $\mathcal{N} - 1$ $\frac{1}{2}$-almost orthonormal functions in $\mathcal{L}_{kr^{-1}}(X)$, where $\mathcal{N} = \mathcal{N}(D_0, k, \nu)$ and ν is the maximal number of disjoint balls of radius $\frac{r}{40k}$.*

Proof. Let $f_1, \ldots, f_m$ be such functions. We let $B_1, \ldots, B_\nu$ be a maximal disjoint covering of X by balls of radius $\frac{r}{40k}$; $x_1, \ldots, x_\nu$ denote the centers of the balls. It follows from maximality that double the balls covers X. We now partition X into ν (disjoint) subsets $S_1, \ldots, S_\nu$, where $B_i \subset S_i$ and S_i is contained in twice B_i.

Let (P, μ') denote the set of points $\{x_j\}$ with probability measure μ', where $\mu'(x_j) = \mu(S_j)$. We can therefore identify functions on P with functions on X which are constant on each S_j.

Since the average of each f_i^2 is one and we have bounds on the Lipschitz constant and diameter,

$$\sup_X |f_i| \leq kD_0 + 1\,. \tag{6.6}$$

Let Λ denote the set $\{\frac{s}{10} \,|\, s \in \mathbf{Z}\,, |s| \leq 10(kD_0 + 1)\}$. We will now construct an injective map $\mathcal{M}$ from the orthonormal set of functions to the set of maps from P (the points $\{x_i\}$) to Λ: let $\mathcal{M}(f_i)(x_j) \in \Lambda$ be any closest point of Λ to $f_i(x_j)$ (there are at most two possibilities). By construction, for all $y \in S_j$,

$$\begin{aligned} |f_i(y) - \mathcal{M}(f_i)(x_j)| &\leq |f_i(y) - f_i(x_j)| + |f_i(x_j) - \mathcal{M}(f_i)(x_j)| \\ &\leq \frac{1}{20} + \frac{1}{20} = \frac{1}{10}\,, \end{aligned} \tag{6.7}$$

and hence

$$\left(\int_X |f_i - \mathcal{M}(f_i)|^2\right)^{\frac{1}{2}} \leq \frac{1}{10}\,. \tag{6.8}$$

By the triangle inequality together with (6.8), we get for $i \neq j$,

$$\begin{aligned}\left(\int_X |f_i - f_j|^2\right)^{\frac{1}{2}} &- \left(\int_X |\mathcal{M}(f_i) - \mathcal{M}(f_j)|^2\right)^{\frac{1}{2}} \\ &\leq \left(\int_X |f_i - \mathcal{M}(f_i)|^2\right)^{\frac{1}{2}} + \left(\int_X |f_j - \mathcal{M}(f_j)|^2\right)^{\frac{1}{2}} \\ &\leq \frac{1}{5}\,.\end{aligned} \tag{6.9}$$

Furthermore, since the f_i are $\frac{1}{2}$-almost orthonormal, we have

$$\int_X |f_i|^2 = 1\,, \tag{6.10}$$

and for $i \neq j$,

$$\left|\int_X f_i f_j\right| < \frac{1}{2}\,. \tag{6.11}$$

Consequently, for $i \neq j$,

$$1 < \left(\int_X |f_i - f_j|^2\right)^{\frac{1}{2}}\,. \tag{6.12}$$

Combining (6.9) and (6.12) yields that for $i \neq j$,

$$0 < \frac{4}{5} < \left(\int_X |\mathcal{M}(f_i) - \mathcal{M}(f_j)|^2\right)^{\frac{1}{2}}\,. \tag{6.13}$$

Hence, $\mathcal{M}$ is injective. The proposition follows by counting the cardinality of the set of maps between two finite point sets (in fact, $\mathcal{N} \leq (20(kD_0+1)+1)^{\nu}$). q.e.d.

Remark 6.14. (Divergence of eigenvalues). Given a gradient bound $C(\lambda)$ for all eigenfunctions with eigenvalues at most λ, that is

$$\sup_X |\nabla u| \leq C(\lambda) \sup_X |u|\,, \tag{6.15}$$

as is the case in the Cheng-Li-Yau gradient estimate (see [15], [39], and [43]), Proposition 6.5 gives a definite rate of divergence of the eigenvalues (compare [43] and Weyl's asymptotic formula). In applications $X \equiv \{b = r\}$ and the functions will be the restriction of harmonic functions on M with bounded growth. Moreover the restriction of the harmonic functions will be approximately eigenfunctions with eigenvalues given in terms of the frequency. Due to the fact that these functions are restrictions of harmonic functions on M we have a gradient bound already on M. This bound is given in terms of the frequency. For this reason we need not deal with spectral properties of X.

7. Growth properties of functions of one variable

In this section, we will prove some elementary results for functions of a single variable with polynomial growth.

The first two results (Lemma 7.1 and Corollary 7.9) show the existence of infinitely many annuli with bounded growth.

The basic idea is that for any set of $2k$ functions with polynomial growth of degree at most d, we can find a subset of k functions and infinitely many annuli for which the degree of growth from the inner radius to the outer radius of each of the functions in the subset is at most $2d$.

We will think of this elementary fact as a weak version of a uniform Harnack inequality for a set of functions with polynomial growth.

This simple idea of restricting attention to a large subset in order to make the constants independent of the number of functions in the set will be used over and over again.

In the next section, we will produce functions of one variable with the properties of the functions of this section.

The main results of this section are Corollary 7.9 and Corollary 7.21. Whereas Corollary 7.9 will be used to start the proof of Theorem 0.3 (see Corollary 8.14), and Corollary 7.21 in the inductive step in the proof of Theorem 0.3; see Section 9.

Lemma 7.1. *Suppose that $f_1, \ldots, f_l$ are positive nondecreasing functions on $(0, \infty)$ such that for some $d\,, K > 0$ and all i,*

$$f_i(r) \leq K(r^d + 1)\,. \tag{7.2}$$

For all $\Omega > 1$, $k \leq l$, and any $C > \Omega^{\frac{ld}{l-k+1}}$, there exist k of these functions $f_{\alpha_1}, \ldots, f_{\alpha_k}$ and infinitely many integers, $m \geq 1$, such that *for* $i = 1, \dots, k$*,*

$$f_{\alpha_i}(\Omega^{m+1}) \leq C f_{\alpha_i}(\Omega^m)\,. \tag{7.3}$$

Proof. We will show that there are infinitely many m such that there is some rank k subset of $\{f_i\}$, where the subset could vary with m, satisfying (7.3). This will suffice to prove the lemma; since there are only finitely many rank k subsets of the l functions, one of these rank k subsets must have been repeated infinitely often.

Set

$$g(x) = \prod_{i=1}^{l} f_i(x)\,; \tag{7.4}$$

note that

$$g(r) \leq K^l (r^d + 1)^l\,, \tag{7.5}$$

and g is a positive nondecreasing function. Assume that there are only finitely many such m and let $m_0 - 1$ be the largest. Then for all $j \geq 1$ we have

$$g(\Omega^{m_0+j}) > C^{l-k+1} g(\Omega^{m_0+j-1})\,. \tag{7.6}$$

Iterating this gives

$$g(\Omega^{m_0+j}) > C^{j(l-k+1)} g(\Omega^{m_0})\,. \tag{7.7}$$

From the upper bound on g, equation (7.5), we have for all $j > m_0$ that

$$\tilde{c}\left(\Omega^j\right)^{dl} \geq C^{j(l-k+1)} g(\Omega^{m_0})\,, \tag{7.8}$$

where $\tilde{c} = \tilde{c}(l, m_0, \Omega, K)$. Since $C > \Omega^{\frac{ld}{l-k+1}}$ and $g(\Omega^{m_0}) > 0$ this is impossible, yielding a contradiction. q.e.d.

Corollary 7.9. *(Weak version of a uniform Harnack inequality for a set of functions with polynomial growth). Suppose that* $f_1, \dots, f_{2k}$ *are positive nondecreasing functions on* $(0, \infty)$ *such that for some* $d\,, K > 0$ *and all* i*,*

$$f_i(r) \leq K(r^d + 1)\,. \tag{7.10}$$

For all $\Omega > 1$*, there exist* k *functions* $f_{\alpha_1}, \dots, f_{\alpha_k}$ *and infinitely many integers,* $m \geq 1$*, such that for* $i = 1, \dots, k$*,*

$$f_{\alpha_i}(\Omega^{m+1}) \leq \Omega^{2d} f_{\alpha_i}(\Omega^m)\,. \tag{7.11}$$

Proof. This is an immediate consequence of Lemma 7.1 with $l = 2k$.
q.e.d.

Remark 7.12. That the upper bound in Corollary 7.9 for the degree of growth of $f_{\alpha_1}, \cdots, f_{\alpha_k}$ from Ω^m to Ω^{m+1} can be made independent of k, Ω, and K is crucial for the applications.

In the proof of Theorem 0.3, we will use Corollary 7.9 to get an initial annulus on which we have some growth control (see Corollary 8.14). Henceforth, we will work on an annulus where we have this control on the growth, and then produce subannuli where we have even better control of the growth. This better control is needed in the proof; see Remark 7.25.

Lemma 7.13. *Given $\Omega > 1$, suppose that f is a positive nondecreasing function on $[r, \Omega^m r]$ such that for some $d_0 > 0$,*

$$f(\Omega^m r) \le \Omega^{d_0 m} f(r)\,. \tag{7.14}$$

Then for $d = d_0 \frac{m}{m-1}$, there exists some j with $0 \le j \le m-2$ such that

$$f(\Omega^{j+1} r) \le \Omega^d f(\Omega^j r) \tag{7.15}$$

and

$$f(\Omega^{j+2} r) \le \Omega^{2d} f(\Omega^j r)\,. \tag{7.16}$$

Proof. Suppose that the lemma is false; then for every j, we have either

$$f(\Omega^{j+1} r) > \Omega^d f(\Omega^j r) \tag{7.17}$$

or

$$f(\Omega^{j+2} r) > \Omega^{2d} f(\Omega^j r)\,. \tag{7.18}$$

In particular, either for $k = 1$ or for $k = 2$, we must have that

$$f(\Omega^k r) > \Omega^{kd} f(r)\,. \tag{7.19}$$

Continuing inductively, by (7.17) and (7.18) we get (7.19) for $k = m$ or for $k = m - 1$. Hence the monotonicity of f, together with (7.19) implies that

$$f(\Omega^m r) > \Omega^{(m-1)d} f(r)\,. \tag{7.20}$$

By the assumption (7.14), and the definition of d, (7.20) yields the desired contradiction. q.e.d.

Lemma 7.13 has the following easy corollary.

Corollary 7.21. *(Double growth condition). Given $\Omega > 1$, suppose that $f_1, \dots, f_{km}$ are positive nondecreasing functions on $[r, \Omega^m r]$ such that for some $d_0 > 0$, and all $i = 1, \dots, km$,*

$$f_i(\Omega^m r) \leq \Omega^{d_0 m} f_i(r)\,. \tag{7.22}$$

Then for $d = d_0 \frac{m}{m-1}$, there exist k functions $f_{\alpha_1}, \dots, f_{\alpha_k}$ and some j with $0 \leq j \leq m-2$ such that for $i = 1, \dots, k$,

$$f_{\alpha_i}(\Omega^{j+1} r) \leq \Omega^d f_{\alpha_i}(\Omega^j r) \tag{7.23}$$

and

$$f_{\alpha_i}(\Omega^{j+2} r) \leq \Omega^{2d} f_{\alpha_i}(\Omega^j r)\,. \tag{7.24}$$

Proof. Applying Lemma 7.13 to the functions f_i, for each i we get a j_i such that f_i satisfies (7.23) and (7.24) with $j = j_i$. Since each j_i must lie in the set $0, \dots, (m-2)$ and there are km of them, at least k of j_i must be equal. q.e.d.

Remark 7.25. In the application (Proposition 9.1) of Corollary 7.21 we will only use (7.24) for f_{α_1}. In contrast (7.23) will be used for all f_{α_i} with $i > 1$. The reason for this is that in the inductive step of Theorem 0.3 we will need to find an annulus and a subset $\{f_{\alpha_i}\}$ of $\{f_i\}$ such that these f_{α_i} have almost the same degree of growth on this annulus as on a certain larger annulus, and u_{α_1} has bounded frequency (the bound must be uniform in terms of the polynomial rate of growth of u_{α_1}). To achieve this, we apply Corollary 7.21 to find a pair of annuli one contained in the other and so that we have controlled growth on both annuli for f_{α_1}. The bounded growth on the larger annulus and the almost monotonicity of the frequency then imply the desired frequency bound for u_{α_1} on the interior annulus; see Section 9 for further details.

8. Constructing independent harmonic functions with good properties from given ones

In this section, given a linearly independent set of functions in $\mathcal{H}_d$ we will construct functions of one variable which reflect the growth and independence properties of this set. In particular, here we shall establish that these functions of one variable satisfy the conditions of Section 7.

The results of this section rely heavily on the properties of harmonic functions on manifolds with nonnegative Ricci curvature; we use in particular that I_u is monotone nondecreasing for *all* harmonic functions.

In Section 10, we will use these results to show that given linearly independent harmonic functions with polynomial growth we can produce annuli and harmonic functions on these annuli which are separated and have controlled growth.

We begin with two definitions. In the first definition we construct the functions whose growth properties will be studied.

Definition 8.1. ($w_{i,r}$ and f_i). Suppose that $u_1, \ldots, u_k$ are linearly independent harmonic functions. For each $r > 0$ we will now define an orthogonal basis $w_{i,r}$ with respect to the inner product

$$r^{1-n} \int_{b=r} uv|\nabla b| \,, \tag{8.2}$$

and functions f_i. Set $w_{1,r} = w_1 = u_1$ and $f_1(r) = I_{u_1}(r)$. Define $w_{i,r}$ by requiring it to be orthogonal to u_j for $j < i$ with respect to the inner product (8.2) and so that on $\{b = r\}$ we have

$$u_i = \sum_{j=1}^{i-1} \lambda_{ji}(r) u_j + w_{i,r} \,. \tag{8.3}$$

Set

$$f_i(r) = r^{1-n} \int_{b=r} w_{i,r}^2 |\nabla b| \,. \tag{8.4}$$

Definition 8.5. (Barrier). We will say that a function f is a (left) barrier for a function g at r if $f(r) = g(r)$ and for $s < r$, $f(s) \leq g(s)$.

We will use the barrier property to conclude that the growth of g from s to r is not larger than the growth of f from s to r (cf. Remark 8.16).

In the next proposition, we will establish some key properties of the functions f_i from Definition 8.1.

Proposition 8.6. *(Properties of f_i). If $u_1, \ldots, u_k \in \mathcal{H}_d(M)$ are linearly independent, then the f_i from Definition 8.1 have the following* *three properties: There exists a constant $K > 0$ (depending on the set $\{u_i\}$) such that*

$$f_i(r) \leq K(r^{2d} + 1)\,, \tag{8.7}$$

$$f_i \text{ is a positive nondecreasing function,} \tag{8.8}$$

and

$$f_i \text{ is a barrier for } I_{w_{i,r}} \text{ at } r. \tag{8.9}$$

Proof. Note first that

$$f_i(r) \leq I_{u_i}(r)\,. \tag{8.10}$$

Furthermore, for $s < r$

$$\begin{aligned} f_i(s) &= s^{1-n} \int_{b=s} \left| u_i - \sum_{j=1}^{i-1} \lambda_{ji}(s) u_j \right|^2 |\nabla b| = I_{w_{i,s}}(s) \\ &\leq s^{1-n} \int_{b=s} \left| u_i - \sum_{j=1}^{i-1} \lambda_{ji}(r) u_j \right|^2 |\nabla b| = I_{w_{i,r}}(s) \\ &\leq r^{1-n} \int_{b=r} \left| u_i - \sum_{j=1}^{i-1} \lambda_{ji}(r) u_j \right|^2 |\nabla b| = I_{w_{i,r}}(r) \\ &= f_i(r)\,, \end{aligned} \tag{8.11}$$

where the first inequality of (8.11) follows from the orthogonality of $w_{i,r}$ to u_j for $j < i$, and the second inequality from the monotonicity of I for harmonic functions (see (2.12)). Since u_i are linearly independent, by (8.11) we get (8.8).

Using (8.11), we also see that f_i is a barrier for $I_{w_{i,r}}$ at r; this shows (8.9).

Finally, we shall verify (8.7). It follows from the asymptotics of the Green's function that $\left|\frac{r}{b}\right|$ is bounded and therefore $u_i \in \mathcal{H}_d$ implies that there exists a constant $\bar{K}$ such that $|u_i(x)| \leq \bar{K}(b(x)^d + 1)$. Using the C^0 bound on u_i and the weighted volume bound for the level set $b = r$, (2.15), we get

$$\begin{aligned} f_i(r) \leq& I_{u_i}(r) \leq \bar{K}^2 (r^d + 1)^2 I_{u \equiv 1}(r) \\ \leq& 2\bar{K}^2 (r^{2d} + 1) n \mathrm{V}_0^n(1)\,. \end{aligned} \tag{8.12}$$

If we set $K = 2\bar{K}^2 n \mathrm{V}_0^n(1)$ then we obtain (8.7). q.e.d.

Although we will not use it, we note that since $\log f_i$ is a barrier for $\log I_{w_i}$, we also get that

$$(\log f_i)'(r) \geq \left(\log I_{w_{i,r}}\right)'(r) = 2\frac{U_{w_{i,r}}(r)}{r}\,. \tag{8.13}$$

The following corollary of Corollary 7.9 and the properties of the f_i will be used to get initial control of the growth in the proof of Theorem 0.3.

Corollary 8.14. *Suppose that $u_1, \dots, u_{2k} \in \mathcal{H}_d(M)$ are linearly independent. Given $\Omega > 1$, then there exist a subset $f_{\alpha_1}, \dots, f_{\alpha_k}$ and infinitely many m such that for $i = 1, \dots, k$*

$$f_{\alpha_i}(\Omega^{m+1}) \leq \Omega^{4d} f_{\alpha_i}(\Omega^m)\,. \tag{8.15}$$

Proof. This follows immediately by combining Corollary 7.9 and Proposition 8.6. q.e.d.

Remark 8.16. We will continue to work with the functions f_i. However since the way they are defined is a bit abstract, we will try to clarify their usefulness by explaining a particular consequence of Corollary 8.14. The reader should note however that this consequence will not be used later on, and rather we will need to use more of the information that the functions f_i carry.

Given a set of $2k$ linearly independent functions $\{u_i\}$ of $\mathcal{H}_d$, Corollary 8.14 allows us to find infinitely many m for which there exist k orthonormal (at $b = \Omega^{m+1}$) harmonic functions (in fact in the span of $\{u_i\}$). Further, these k functions have growth of degree at most $2d$ on the annulus between $b = \Omega^m$ and $b = \Omega^{m+1}$. Note however that for different m the set of harmonic functions with growth of degree at most $2d$ may be different. That is, from Corollary 8.14 together with (8.9) we have the following:

Under the assumptions of Corollary 8.14 we have infinitely many m such that for $i = 1, \cdots, k$,

$$I_{w_{\alpha_i,\Omega^{m+1}}}(\Omega^{m+1}) \leq \Omega^{4d} I_{w_{\alpha_i,\Omega^{m+1}}}(\Omega^m), \tag{8.17}$$

and for $1 \leq i < j \leq k$

$$\int_{b=\Omega^{m+1}} w_{\alpha_i,\Omega^{m+1}} w_{\alpha_j,\Omega^{m+1}} |\nabla b| = 0\,. \tag{8.18}$$

Note also that if we could show that these k harmonic functions were orthogonal at Ω^m (and not at Ω^{m+1}) then, after applying the gradient estimate together with the meanvalue inequality, Theorem 0.3 would follow immediately from the results of Section 6.

9. Towards the inductive step

In this section we will use the results of Sections 7 and 8 to show a result (Proposition 9.1) that will be used in the inductive step of Theorem 0.3.

Given a large annulus and a set of independent harmonic functions $\{u_i\}$ such that the corresponding functions f_i (see Section 8) grow polynomially of order at most d_0 on this annulus, we show how to get

(a) a subset α_i,
(b) a subannulus,
(c) a nonconstant harmonic function u in the span of $\{u_j \,|\, j \leq \alpha_1\}$,
(d) a constant $d > d_0$, and
(e) a larger subannulus,

such that on this subannulus (b)

(1) f_{α_i} grows polynomially of order at most d,
(2) $\frac{1}{2} \leq U_u \leq 2d$ on double the subannulus, and
(3) U_u is almost constant,

and on the larger subannulus (e)

(i) u is orthogonal to $\{u_j \,|\, j < \alpha_1\}$ at the outer radius, and
(ii) I_u grows polynomially of order at most d.

It will be important that we will be able to take d very close to d_0 and U_u very close to being constant if we are willing to go to a relatively small subset of the functions and a relatively small subannulus. In the applications, Proposition 4.50 together with (2) and (3) will allow us to conclude that this u is very close to separating variables on this subannulus. This together with (i), (ii), and Proposition 5.1 will allow us to conclude that the harmonic functions that we define inductively in this way are almost orthogonal on a subannulus.

We will now make this precise in the following proposition.

Proposition 9.1. *(Towards the inductive step of Theorem 0.3). Let $m > 5$, $\bar{m} > 9$, $\Omega > \Omega_L(n, \frac{2}{5}) \geq 2$, and $d_0 \geq 1$ be given. Here $\Omega_L(n, \frac{2}{5})$*

is given by Corollary 3.29. Set

$$d = d_0 \frac{m}{m-1} \frac{\bar{m}}{\bar{m} - 4 - \sqrt{\bar{m}}} \, . \tag{9.2}$$

There exists $R = R(p, \bar{m}, d_0, \Omega) > 0$ *such that if* $r > R$ *and* $f_1, \dots, f_{lm\bar{m}}$ *are as in Definition 8.1 where* $u_i(p) = 0$ *and*

$$f_i(\Omega^{m\bar{m}} r) \leq \Omega^{d_0 m\bar{m}} f_i(r) \tag{9.3}$$

for all i, *then we have the following:*

There exist l *functions* $f_{\alpha_1}, \dots, f_{\alpha_l}$ *and integers* h *and* j *with* $0 \leq h \leq m-2$ *and* $\bar{m}h < j < \bar{m}(h+1) - 1$ *such that for* $i = 2, \dots, l$,

$$f_{\alpha_i}(\Omega^{j+1} r) \leq \Omega^d f_{\alpha_i}(\Omega^j r) \, , \tag{9.4}$$

and setting $u = w_{\alpha_1, \Omega^{\bar{m}(h+2)} r}$, *we have for* $\Omega^j r \leq s \leq \Omega^{j+2} r$,

$$\frac{1}{2} \leq U_u(s) \leq 2d \, , \tag{9.5}$$

$$I_u(\Omega^{\bar{m}(h+2)} r) \leq \Omega^{2d\bar{m}} I_u(\Omega^{\bar{m}h} r) \, , \tag{9.6}$$

and

$$\left| \log \frac{U_u(\Omega^{j+1} r)}{U_u(\Omega^j r)} \right| \leq \frac{\log(5d)}{\sqrt{\bar{m}}} \, . \tag{9.7}$$

Proof. First, we apply Corollary 7.21 to get $l\bar{m}$ functions $f_{\beta_1}, \dots, f_{\beta_{l\bar{m}}}$ such that for some h with $0 \leq h \leq m-2$ and $i = 1, \dots, l\bar{m}$,

$$f_{\beta_i}(\Omega^{\bar{m}(h+1)} r) \leq \Omega^{d_1 \bar{m}} f_{\beta_i}(\Omega^{\bar{m}h} r) \tag{9.8}$$

and

$$f_{\beta_i}(\Omega^{\bar{m}(h+2)} r) \leq \Omega^{2d_1 \bar{m}} f_{\beta_i}(\Omega^{\bar{m}h} r) \, , \tag{9.9}$$

where $d_1 = \frac{m}{m-1} d_0$. Note that we will only use (9.8) for $i > 1$ and (9.9) only for $i = 1$. Set $u = w_{\beta_1, \Omega^{\bar{m}(h+2)} r}$ and $\alpha_1 = \beta_1$. From the barrier property, (8.9), it follows that (9.9) implies

$$I_u(\Omega^{\bar{m}(h+2)} r) \leq \Omega^{2d_1 \bar{m}} I_u(\Omega^{\bar{m}h} r) < \Omega^{2d\bar{m}} I_u(\Omega^{\bar{m}h} r) \, . \tag{9.10}$$

In particular by Proposition 3.36 (noting that $\Omega > 2$), there exists $R_0 = R_0(p) > 0$ such that for $r > R_0$,

$$D_u(\Omega^{\bar{m}(h+2)-1}r) \leq C_2 \Omega^{2d_1\bar{m}} D_u(\Omega^{\bar{m}h+1}r) < C_2 \Omega^{3d_0\bar{m}} D_u(\Omega^{\bar{m}h+1}r)\,, \tag{9.11}$$

where $C_2 = C_2(n) > 0$.

Set

$$\epsilon = \min\left\{\frac{1}{3}\log\frac{6}{5}, \frac{1}{4}\log\frac{9}{8}, \frac{\log\frac{10}{9}}{2\sqrt{\bar{m}}}\right\}\,. \tag{9.12}$$

By Proposition 4.11, Proposition 3.3, and (9.11) we can choose $R_1 = R_1(p, C_2\Omega^{3d_0\bar{m}}, \epsilon, \Omega^{2\bar{m}-2}) > R_0$ so large so that for $r > R_1$ and $\Omega^{\bar{m}h+1}r \leq s \leq \Omega^{\bar{m}(h+2)-2}r$,

$$\left|\log\frac{D_u(s)}{E_u(s)}\right| < \epsilon \tag{9.13}$$

and

$$\int_{\Omega^{\bar{m}h+1}r}^{\Omega^{\bar{m}(h+2)-2}r} \min\{(\log W_u)'(t), 0\}dt > -\epsilon\,. \tag{9.14}$$

Note that (9.13) is equivalent to that for $\Omega^{\bar{m}h+1}r \leq s \leq \Omega^{\bar{m}(h+2)-2}r$

$$\left|\log\frac{U_u(s)}{W_u(s)}\right| < \epsilon\,. \tag{9.15}$$

From (9.14) we have for $\Omega^{\bar{m}h+1}r \leq s \leq t \leq \Omega^{\bar{m}(h+2)-2}r$,

$$W_u(s) < \mathrm{e}^{\epsilon} W_u(t)\,, \tag{9.16}$$

which together with (9.15) implies that for $\Omega^{\bar{m}h+1}r \leq s \leq t \leq \Omega^{\bar{m}(h+2)-2}r$

$$U_u(s) < \mathrm{e}^{\epsilon} W_u(s) < \mathrm{e}^{2\epsilon} W_u(t) < \mathrm{e}^{3\epsilon} U_u(t)\,. \tag{9.17}$$

Since I_u is nondecreasing, from (9.10) it follows that

$$\begin{aligned} I_u(\Omega^{\bar{m}(h+2)-2}r) &\leq I_u(\Omega^{\bar{m}(h+2)}r) \\ &\leq \Omega^{2d_1\bar{m}} I_u(\Omega^{\bar{m}h}r) \\ &\leq \Omega^{2d_1\bar{m}} I_u(\Omega^{\bar{m}(h+1)}r)\,. \end{aligned} \tag{9.18}$$

By (2.14) and (9.18), there exists some s_0 with

$$\Omega^{\bar{m}(h+1)}r \leq s_0 \leq \Omega^{\bar{m}(h+2)-2}r$$

such that

$$U_u(s_0) \leq \frac{\bar{m}}{\bar{m}-2}d_1 < \frac{5}{3}d_1 \,. \tag{9.19}$$

Combining (9.17) and (9.19) we see that, for $\Omega^{\bar{m}h+1}r \leq s \leq \Omega^{\bar{m}(h+1)}r$,

$$U_u(s) < \mathrm{e}^{3\epsilon}U(s_0) < \mathrm{e}^{3\epsilon}\frac{5}{3}d_1 \leq 2d_1 \,. \tag{9.20}$$

By Corollary 3.29 we can choose $R_2 = R_2(p) > 0$ so large that if $r > R_2$ then there exists a s_1 satisfying

$$\Omega^{\bar{m}h+1}r \leq s_1 \leq \Omega_L(n, \frac{2}{5})\Omega^{\bar{m}h+1}r < \Omega^{\bar{m}h+2}r$$

with

$$\frac{3}{5} \leq U_u(s_1) \,. \tag{9.21}$$

We now set $R = \max\{R_1, R_2\}$. By (9.21) and (9.17) we have that for $r > R$ and $\Omega^{\bar{m}h+2}r \leq s \leq \Omega^{\bar{m}(h+1)}r$

$$\frac{1}{2} \leq \mathrm{e}^{-3\epsilon}\frac{3}{5} \leq \mathrm{e}^{-3\epsilon}U(s_1) < U_u(s) \,. \tag{9.22}$$

Combining (9.20) and (9.22) we get that for $\Omega^{\bar{m}h+2}r \leq s \leq \Omega^{\bar{m}(h+1)}r$,

$$\frac{1}{2} \leq U_u(s) \leq 2d_1 \,. \tag{9.23}$$

Note that (9.23) implies that for $\Omega^{\bar{m}h+2}r \leq s \leq t \leq \Omega^{\bar{m}(h+1)}r$,

$$\left|\log \frac{U_u(s)}{U_u(t)}\right| < \log(4d_1) \,. \tag{9.24}$$

Consider the $(\bar{m}-3)$ subintervals given by $\Omega^{\bar{m}h+j}r$ to $\Omega^{\bar{m}h+j+1}r$ for $j = 2, \ldots, \bar{m}-2$. From (9.15) and (9.24) it follows that there exist at least $(\bar{m}-3-\sqrt{\bar{m}})$ subintervals on which the variation is less than $\frac{\log(5d_1)}{\sqrt{\bar{m}}}$. That is,

$$
\begin{aligned}
\sum_{j=2}^{\bar{m}-2} & \left|\log \frac{W_u(\Omega^{\bar{m}h+j+1}r)}{W_u(\Omega^{\bar{m}h+j}r)}\right| \\
&\leq \sum_{j=2}^{\bar{m}-2} \log \frac{W_u(\Omega^{\bar{m}h+j+1}r)}{W_u(\Omega^{\bar{m}h+j}r)} \\
&\quad - 2\sum_{j=2}^{\bar{m}-2} \min\left\{\log \frac{W_u(\Omega^{\bar{m}h+j+1}r)}{W_u(\Omega^{\bar{m}h+j}r)}, 0\right\} \\
&\leq \log \frac{W_u(\Omega^{\bar{m}(h+1)-1}r)}{W_u(\Omega^{\bar{m}h+2}r)} \\
&\quad - 2\int_{\Omega^{\bar{m}h+2}r}^{\Omega^{\bar{m}(h+1)-1}r} \min\{(\log W_u)'(t), 0\} dt \\
&< \log \frac{U_u(\Omega^{\bar{m}(h+1)-1}r)}{U_u(\Omega^{\bar{m}h+2}r)} + 4\epsilon < \log(4d_1) + 4\epsilon \\
&\leq \log(4d_1) + \log\frac{9}{8} = \log\left(\frac{9}{2}d_1\right) .
\end{aligned} \tag{9.25}
$$

By (9.15) and (9.25), there exist at least $(\bar{m}-3-\sqrt{\bar{m}})$ many j between 2 and $\bar{m}-2$ such that

$$
\begin{aligned}
\left|\log \frac{U_u(\Omega^{h\bar{m}+j+1})}{U_u(\Omega^{h\bar{m}+j})}\right| &< \left|\log \frac{W_u(\Omega^{h\bar{m}+j+1})}{W_u(\Omega^{h\bar{m}+j})}\right| + 2\epsilon \\
&< \frac{\log(\frac{9}{2}d_1)}{\sqrt{\bar{m}}} + 2\epsilon \leq \frac{\log(\frac{9}{2}d_1)}{\sqrt{\bar{m}}} + \frac{\log\frac{10}{9}}{\sqrt{\bar{m}}} \\
&= \frac{\log(5d_1)}{\sqrt{\bar{m}}} .
\end{aligned} \tag{9.26}
$$

We will call such intervals good.

We now consider the restriction of the $\{f_{\beta_i}\}$ to the union of the good intervals. By (9.8), these restricted functions grow with exponent at most equal to $d_2 = \frac{\bar{m}}{\bar{m}-3-\sqrt{\bar{m}}}d_1$. Again applying Corollary 7.21, this time to the union of these good intervals and the restrictions of the functions, f_{β_i}, we get a j with $\bar{m}h < j < \bar{m}(h+1)-1$ such that

$$
\left|\log \frac{U_u(\Omega^{j+1}r)}{U_u(\Omega^{j}r)}\right| < \frac{\log(5d_1)}{\sqrt{\bar{m}}} \tag{9.27}
$$

and $l-1$ functions $f_{\alpha_2}, \dots, f_{\alpha_l}$ such that for $i = 2, \dots, l$,

$$f_{\alpha_i}(\Omega^{j+1} r) \leq \Omega^d f_{\alpha_i}(\Omega^j r)\,, \tag{9.28}$$

where $d = \frac{\bar{m}-3-\sqrt{\bar{m}}}{\bar{m}-4-\sqrt{\bar{m}}} d_2$. Finally, note that (9.23) gives (9.5). q.e.d.

Note that for all $s > 0$ and all $i > 1$

$$\int_{b=s} w_{\alpha_1, \Omega^{\bar{m}(h+2)} r} w_{\alpha_i, s} |\nabla b| = 0\,. \tag{9.29}$$

This follows since $w_{\alpha_1, \Omega^{\bar{m}(h+2)} r}$ lies in the linear span of $\{u_k \,|\, k \leq \alpha_1\}$ and $w_{\alpha_i, s}$ is orthogonal to u_k $(k \leq \alpha_1)$ at $b = s$ for $i > 1$. Note also that R_0, R_1, R_2 (and hence R) are independent of l and also of the particular harmonic functions.

The key for applications of Proposition 9.1 is, for given l and $\Omega > 2$, to choose m and $\bar{m}$ so large that the degree of growth, d, of the functions $f_{\alpha_2}, \cdots, f_{\alpha_l}$ from the inner radius $b = \tilde{r}$ $(= \Omega^j r)$ to the outer radius $b = \Omega \tilde{r}$ is not much larger than d_0. In fact, in the applications, the more times that we need to iterate this step, the closer that d needs to be to d_0. Further, we will choose $\bar{m}$ and R so large that U_u is almost constant (i.e., almost separate variables) on the annulus between $b = \tilde{r}$ and $b = \Omega \tilde{r}$. Here if Ω is large then u has to be even closer to separating variables, so $\bar{m}$ needs to be even larger. The reason for this is that we want to apply Proposition 5.1 to get a definite separation at $b = \tilde{r}$.

In Section 10 we will need to keep close track of these relationships.

10. Harmonic functions with polynomial growth

As before, let M be an n-dimensional Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth, and let $p \in M$ be fixed.

We are now prepared to prove the main theorem. After some preliminary remarks, the proof will consist of three steps. First, we will find annuli and a subspace of the harmonic functions with polynomial growth such that a basis for this subspace has controlled growth on these annuli. This step relies mainly on the properties of the functions f_i constructed in Section 8 and the general properties of functions of one variable with polynomial growth.

Next, we will construct a set of harmonic functions contained in this subspace which have controlled growth, almost separate variables, and are pairwise almost orthogonal on a subannulus. We accomplish this through repeated applications of Propositions 9.1, 4.50, and 5.1. In essence, this step gives an effective version of the finiteness theorem, and it is here that we strongly use the results on the frequency function (and thus the Euclidean volume growth assumption).

Finally, we will use the uniform bound on the growth and the mean-value inequality to get a Lipschitz bound for these harmonic functions on a subannulus. Proposition 6.5 gives a bound on the number of such functions, and the theorem will then follow since we can use this to control the number of functions that we started with.

Proof. (Theorem 0.3). Fix $\Omega > \max\{\Omega_L(n, \frac{2}{5}), 4\}$. Here $\Omega_L(n, \frac{2}{5})$ is given by Corollary 3.29. Set

$$k^2 = \left(\frac{4}{3}\right)^{\frac{n}{2}} 16\tilde{C} n\, d\, 4^{32d}, \tag{10.1}$$

where $\tilde{C} = \tilde{C}(n) > 0$ is the constant occuring in the meanvalue inequality, Proposition 2.26.

By (2.23) we can choose $R_0 = R_0(p) > 0$ so large that for $r > R_0$,

$$\left|\log \frac{b}{r}\right| < \log \frac{2}{\sqrt{3}}\,. \tag{10.2}$$

If $r > R_0$ then

$$X \equiv \{b = \frac{\Omega}{4} r\} \subset A_{\frac{\Omega\sqrt{3}}{8} r, \frac{\Omega}{2\sqrt{3}} r}(p)\,. \tag{10.3}$$

The relative volume comparison theorem, [3], [28], together with (10.3) imply that there exists an integer $\nu = \nu(n, k)$ such that there exist at most ν disjoint balls of radius $\frac{\Omega r}{40k}$ with centers contained in X. We think of X as a metric space with distance function given by the restriction of the Riemannian distance on M. Let μ be the probability measure on X given by

$$\mu(A) = \frac{(\Omega r)^{1-n}}{4^{1-n} n \mathrm{V}_M} \int_{A \cap \{b = \frac{\Omega}{4} r\}} |\nabla b|\,. \tag{10.4}$$

Note that the normalization in (10.4) comes from the fact that $I_1 \equiv n\mathrm{V}_M$, that is (2.15). Applying Proposition 6.5 to X with the probability measure μ, we get a constant $\mathcal{N} = \mathcal{N}(k, n)$ such that for $\eta = \frac{1}{2}$ any set of η-almost orthonormal functions with Lipschitz bound $\frac{k}{\Omega r}$ on X has at most $\mathcal{N} - 1$ elements.

We will show that if $\dim \mathcal{H}_d \geq C$, where $C = C(\mathcal{N})$, then for all $R > 0$ we can find an $r > R$ and $\mathcal{N}$ η-almost orthonormal functions with Lipschitz bound $\frac{k}{\Omega r}$ on X. This contradiction yields the result.

Choose integers $m > 5$ and $\bar{m} > 9$ so large that

$$\left(\frac{m}{m-1}\right)^{\mathcal{N}} \left(\frac{\bar{m}}{\bar{m} - 4 - \sqrt{\bar{m}}}\right)^{\mathcal{N}} \leq 2\,, \tag{10.5}$$

and

$$\exp\left(2\,\frac{\log(40d)}{\sqrt{\bar{m}}}\right) \leq 2\,. \tag{10.6}$$

Further, let $(\bar{m}_i)_{i=1,\cdots,\mathcal{N}}$ satisfy $\bar{m}_i \geq \bar{m}$, and set

$$\epsilon_i = 16d\,\frac{\log(40d)}{\sqrt{\bar{m}_i}}\,. \tag{10.7}$$

Observe that (10.6) implies

$$\frac{\epsilon_i}{8d}\exp\frac{\epsilon_i}{8d} \leq \frac{\epsilon_i}{4d}\,. \tag{10.8}$$

To simplify notation, define inductively $\mathcal{N}_{\mathcal{N}}$, $\mathcal{N}_{\mathcal{N}-1}$, $\mathcal{N}_{\mathcal{N}-2}$, $\cdots$, $\mathcal{N}_1$, $\mathcal{N}_0$, by

$$\begin{aligned} \mathcal{N}_{\mathcal{N}} &= 0\,, \\ \mathcal{N}_i &= (\mathcal{N}_{i+1} + 1)\,\bar{m}_{i+1} m\,. \end{aligned} \tag{10.9}$$

Further, for $i = 0, \ldots, \mathcal{N} - 1$, set

$$\begin{aligned} \mathcal{M}_i &= m^{\mathcal{N}-i}\Pi_{j=i+1}^{\mathcal{N}}\bar{m}_j\,, \\ \mathcal{M}_{\mathcal{N}} &= 1\,, \end{aligned} \tag{10.10}$$

and

$$\Omega_i = \Omega^{\mathcal{M}_i}\,. \tag{10.11}$$

Finally, for $i > 1$ let

$$\eta_i^2 = 8\epsilon_{i-1}\Omega_i^{208\,d\,\bar{m}_i + 4\bar{m}_i}\,. \tag{10.12}$$

Note that $\Omega_i^{208\,d\,\bar{m}_i + 4\bar{m}_i}$ depend not on $\bar{m}_{i-1}$ but only on $\bar{m}_j$ for $j \geq i$. On the other hand ϵ_{i-1} depends only on $\bar{m}_{i-1}$. These two facts will allow us to choose $\bar{m}_{i-1}$ large so that $\eta_i \leq \eta$, where $\eta \leq \frac{1}{2}$. That is, choose $\bar{m}_{\mathcal{N}}, \bar{m}_{\mathcal{N}-1}, \dots, \bar{m}_1$ inductively so large that $\eta_i \leq \eta$. Observe that this implies that if $\Omega_i^{\bar{m}_i}$ is large then ϵ_{i-1} must be small. The numbers $\bar{m}_i$ are now fixed, as are the quantities $\mathcal{N}_i, \mathcal{M}_i, \Omega_i, \epsilon_i, \eta_i$ which are defined from the $\bar{m}_i$.

We will show that we can take $C(\mathcal{N}) = 2\mathcal{N}_0 + 1$. To see this, suppose that $u_0 \equiv 1, u_1, \dots, u_{2\mathcal{N}_0} \in \mathcal{H}_d(M)$ are linearly independent. We may assume that $u_i(p) = 0$ for all $i > 0$. Given this set of harmonic functions, we will now proceed, for all $R > 0$, to construct an $r > R$ and a set, $\{v_i\}$, of η-almost orthogonal harmonic functions on the annulus $\{r \leq b \leq \Omega r\}$. In addition each v_i will ϵ_i-almost separate variables on an annulus, $\{r_i \leq b \leq \Omega_i r_i\}$, containing the annulus, $\{r \leq b \leq \Omega r\}$. In fact, for any j with $1 \leq j \leq \mathcal{N}$ the functions v_i with $i \leq j$ will be pairwise η_j-almost orthogonal on the larger annulus $\{r_j \leq b \leq \Omega_j r_j\} \supset \{r \leq b \leq \Omega r\}$. Note that in order to show that v_i and v_j are η_j-almost orthogonal (with $\eta_j \leq \eta$) on $\{r_j \leq b \leq \Omega_j r_j\}$ if $i < j$ we will need v_i to ϵ_i-almost separate variables on $\{r_i \leq b \leq \Omega_i r_i\} \supset \{r_j \leq b \leq \Omega_j r_j\}$. Since Ω_{i-1} is much larger than Ω_{j-1} if $i < j$ we will need v_i to be much closer to separating variables than v_j; cf. Section 5. Note also that for different r the v_i may be different. It will be possible to do the following construction for infinitely many annuli; however, in the end we need only do our construction on a (single) sufficiently large annulus.

As in Definition 8.1, we set

$$f_1 = I_{u_1}, \tag{10.13}$$

$$w_{1,r} = u_1, \tag{10.14}$$

and let $f_2, \dots, f_{2\mathcal{N}_0}$ and $w_{2,r}, \dots, w_{2\mathcal{N}_0,r}$ be as in that definition (with respect to $u_1, \dots, u_{2\mathcal{N}_0}$). These f_i and $w_{i,r}$ will be fixed from now on.

The first step of the proof consists of finding a sequence of annuli where a subset of the functions, $f_1, \dots, f_{2\mathcal{N}_0}$, has controlled growth. To get the initial control, we apply Corollary 8.14 to get a subset $f_{\alpha_1}, \dots, f_{\alpha_{\mathcal{N}_0}}$ and infinitely many integers j_0 such that for $i = 1, \dots, \mathcal{N}_0$,

$$f_{\alpha_i}(\Omega_0^{j_0+1}) \leq \Omega_0^{4d} f_{\alpha_i}(\Omega_0^{j_0}) \, . \tag{10.15}$$

Fix such a j_0, and set $r_0 = \Omega_0^{j_0}$.

The next step is, for any such j_0 which is sufficiently large, to inductively construct an independent set of harmonic functions which are pairwise η-almost orthogonal and whose growth is controlled. At the i-th stage of the induction, we will be working with $\mathcal{N}_i$ functions and i independent harmonic functions on an annulus $\{r_i \leq b \leq \Omega_i r_i\} \subset \{r_{i-1} \leq b \leq \Omega_{i-1} r_{i-1}\} \subset \cdots \subset \{r_1 \leq b \leq \Omega_1 r_1\} \subset \{r_0 \leq b \leq \Omega_0 r_0\}$. These $\mathcal{N}_i$ functions will grow at most like $2d_i$ on these annuli, where for $i = 1, \ldots, \mathcal{N}$,

$$d_i = 2d \left(\frac{m}{m-1}\right)^i \Pi_{j=1}^i \left(\frac{\bar{m}_j}{\bar{m}_j - 4 - \sqrt{\bar{m}_j}}\right) . \tag{10.16}$$

Note that from the choice of m and $\bar{m}$ (see (10.5)), we have

$$d_1 < d_2 < \cdots < d_{\mathcal{N}} \leq 4d \, . \tag{10.17}$$

Applying Proposition 9.1, we get a $\bar{R}_1 = \bar{R}_1(p, \bar{m}_1, 8d, \Omega_1)$ such that if j_0 is large enough to ensure that

$$r_0 = \Omega_0^{j_0} > \bar{R}_1 \, , \tag{10.18}$$

then there exist $\mathcal{N}_1 + 1$ functions $f_{\beta_1}, \ldots, f_{\beta_{\mathcal{N}_1+1}}$ and integers h_1 and j_1 with $0 \leq h_1 \leq m-2$ and $\bar{m}_1 h_1 < j_1 < \bar{m}_1(h_1+1) - 1$ such that for $i = 2, \ldots, \mathcal{N}_1 + 1$,

$$f_{\beta_i}(\Omega_1^{j_1+1} r_0) \leq \Omega_1^{2d_1} f_{\beta_i}(\Omega_1^{j_1} r_0) \, , \tag{10.19}$$

for $\Omega_1^{j_1} r_0 \leq s \leq \Omega_1^{j_1+2} r_0$

$$\frac{1}{2} \leq U_{v_1}(s) \leq 4d_1 \leq 16d \, , \tag{10.20}$$

and

$$\begin{aligned} \left| \log \frac{U_{v_1}(\Omega_1^{j_1+1} r_0)}{U_{v_1}(\Omega_1^{j_1} r_0)} \right| &\leq \frac{\log(10 d_1)}{\sqrt{\bar{m}_1}} \\ &\leq \frac{\log(40d)}{\sqrt{\bar{m}_1}} = \frac{\epsilon_1}{16d} \, . \end{aligned} \tag{10.21}$$

Here $v_1 = w_{\beta_1, \Omega_1^{\bar{m}_1(h_1+2)} r_0}$. Set $r_1 = \Omega_1^{j_1} r_0$. Note that

$$\Omega_1^2 r_1 < \Omega_1^{\bar{m}_1(h_1+2)} r_0 \leq \Omega_0 r_0.$$

The frequency bounds (10.20) combined with (2.9) and (2.14) give the following bound for the growth of D_{v_1},

$$
\begin{aligned}
D_{v_1}(\Omega_1^2 r_1) =& U_{v_1}(\Omega_1^2 r_1) I_{v_1}(\Omega_1^2 r_1) \\
\leq& 16d I_{v_1}(\Omega_1^2 r_1) \leq 16d\Omega_1^{64d} I_{v_1}(r_1) \\
\leq& 32d\Omega_1^{64d} D_{v_1}(r_1)\,.
\end{aligned} \tag{10.22}
$$

Now by Proposition 4.50 we have that (10.20), (10.21), and (10.22) together with (10.8) give the existence of an

$$R_1 = R_1(p, 32d\Omega_1^{64d}, 16d, \epsilon_1, \Omega_1^2) > R_0$$

such that if

$$r_0 = \Omega_0^{j_0} > R_1\,, \tag{10.23}$$

then v_1 ϵ_1-almost separates variables on the annulus $\{r_1 \leq b \leq \Omega_1 r_1\}$.

We proceed inductively. Again by Proposition 9.1, we get a $\bar{R}_2 = \bar{R}_2(p, \bar{m}_2, 8d, \Omega_2)$ (in fact, $\bar{R}_2 = \bar{R}_1$ will do) such that if j_0 is large enough to ensure that

$$r_0 = \Omega_0^{j_0} > \bar{R}_2\,, \tag{10.24}$$

then there exist $\mathcal{N}_2 + 1$ functions $f_{\gamma_1}, \dots, f_{\gamma_{\mathcal{N}_2+1}}$, where

$$f_{\gamma_i} \in \{f_{\beta_2}, \cdots, f_{\beta_{\mathcal{N}_1+1}}\},$$

and integers h_2 and j_2 with $0 \leq h_2 \leq m-2$ and

$$\bar{m}_2 h_2 < j_2 < \bar{m}_2(h_2+1) - 1$$

such that for $i = 2, \dots, \mathcal{N}_2 + 1$,

$$f_{\gamma_i}(\Omega_2^{j_2+1} r_1) \leq \Omega_2^{2d_2} f_{\gamma_i}(\Omega_2^{j_2} r_1)\,, \tag{10.25}$$

for $\Omega_2^{j_2} r_1 \leq s \leq \Omega_2^{j_2+2} r_1$

$$\frac{1}{2} \leq U_{v_2}(s) \leq 4d_2 \leq 16d\,, \tag{10.26}$$

$$
\begin{aligned}
I_{v_2}(\Omega^{\bar{m}_2(h_2+2)} r) \leq& \Omega^{4d_2\bar{m}_2} I_{v_2}(\Omega^{\bar{m}_2 h_2} r) \\
\leq& \Omega^{16d\bar{m}_2} I_{v_2}(\Omega^{\bar{m}_2 h_2} r)\,,
\end{aligned} \tag{10.27}
$$

and

$$\left|\log \frac{U_{v_2}(\Omega_2^{j_2+1}r_1)}{U_{v_2}(\Omega_2^{j_2}r_1)}\right| \leq \frac{\log(10d_2)}{\sqrt{\bar{m}_2}} \leq \frac{\log(40d)}{\sqrt{\bar{m}_2}} = \frac{\epsilon_2}{16d}\,. \tag{10.28}$$

Here $v_2 = w_{\gamma_1,\Omega_2^{\bar{m}_2(h_2+2)}r_1}$. Set $r_2 = \Omega_2^{j_2}r_1$. Note that

$$\begin{aligned} r_1 &< \Omega_2^{\bar{m}_2 h_2}r_1 < r_2 < \Omega_2 r_2 < \Omega_2^2 r_2 \\ &< \Omega_2^{\bar{m}_2(h_2+2)}r_1 \leq \Omega_1 r_1\,. \end{aligned} \tag{10.29}$$

Using Proposition 5.1 we will now show that v_1 and v_2 are η_2-almost orthogonal on the annulus

$$\{\Omega_2^{\bar{m}_2 h_2}r_1 \leq b \leq \Omega_2^{\bar{m}_2(h_2+2)}r_1\} \supset \{r_2 \leq b \leq \Omega_2 r_2\}.$$

By definition, v_1 is a linear combination of $u_1, \cdots, u_{\beta_1}$ and at $b = \Omega_2^{\bar{m}_2(h_2+2)}r_1$, v_2 is orthogonal to all u_i with $i < \gamma_1$; therefore

$$\int_{b=\Omega_2^{\bar{m}_2(h_2+2)}r_1} v_1 v_2 |\nabla b| = 0\,. \tag{10.30}$$

Note also that by (2.14) and (10.20) we have

$$I_{v_1}(\Omega_2^{\bar{m}_2(h_2+2)}r_1) \leq \Omega_2^{\bar{m}_2 64d} I_{v_1}(\Omega_2^{\bar{m}_2 h_2}r_1)\,. \tag{10.31}$$

Since v_1 ϵ_1-almost separates variables on the annulus, from Proposition 5.1, (10.27), (10.30), (10.31) it follows that

$$\{r_1 \leq b \leq \Omega_1 r_1\} \supset \{\Omega_2^{\bar{m}_2 h_2}r_1 \leq b \leq \Omega_2^{\bar{m}_2(h_2+2)}r_1\},$$

and by (10.20), we get for $\Omega_2^{\bar{m}_2 h_2}r_1 \leq s \leq \Omega_2^{\bar{m}_2(h_2+2)}r_1$

$$\begin{aligned} &\left|s^{1-n}\int_{b=s} v_1 v_2 |\nabla b|\right|^2 \\ &\quad < 8\epsilon_1\, \Omega_2^{(64d+2)2\bar{m}_2} I_{v_1}(\Omega_2^{\bar{m}_2(h_2+2)}r_1) I_{v_2}(\Omega_2^{\bar{m}_2(h_2+2)}r_1) \\ &\quad \leq 8\epsilon_1\, \Omega_2^{(64d+2)2\bar{m}_2}\, \Omega_2^{64d\bar{m}_2} I_{v_1}(\Omega_2^{\bar{m}_2 h_2}r_1)\, \Omega_2^{16d\bar{m}_2} I_{v_2}(\Omega_2^{\bar{m}_2 h_2}r_1) \\ &\quad = \eta_2^2 I_{v_1}(\Omega_2^{\bar{m}_2 h_2}r_1) I_{v_2}(\Omega_2^{\bar{m}_2 h_2}r_1) \leq \eta_2^2 I_{v_1}(s) I_{v_2}(s)\,. \end{aligned} \tag{10.32}$$

Here the last inequality follows from the monotonicity of I; that is, for $\Omega_2^{\bar{m}_2 h_2} r_1 \leq s$ we have $I_{v_i}(\Omega_2^{\bar{m}_2 h_2} r_1) \leq I_{v_i}(s)$. This proves that v_1 and v_2 are η_2-almost orthogonal on all level sets in the annulus

$$\{\Omega_2^{\bar{m}_2 h_2} r_1 \leq b \leq \Omega_2^{\bar{m}_2 (h_2+2)} r_1\} \supset \{r_2 \leq b \leq \Omega_2 r_2\}.$$

The frequency bounds (10.26) combined with (2.9) and (2.14) give the following bound for the growth of D_{v_2},

$$D_{v_2}(\Omega_2^2 r_2) = U_{v_2}(\Omega_2^2 r_2) I_{v_2}(\Omega_2^2 r_2) \tag{10.33}$$

$$\leq 16d I_{v_2}(\Omega_2^2 r_2) \leq 16d \Omega_2^{64d} I_{v_2}(r_2) \tag{10.34}$$

$$\leq 32d \Omega_2^{64d} D_{v_2}(r_2)\,.$$

Now, as above, by Proposition 4.50 we have that (10.26), (10.28), and (10.33) together with (10.8) yield the existence of an

$$R_2 = R_2(p, 32d\Omega_2^{64d}, 16d, \epsilon_2, \Omega_2^2) > R_0$$

(in fact $R_2 = R_1$ will do) such that if

$$r_0 = \Omega_0^{j_0} > R_2\,, \tag{10.35}$$

then v_2 ϵ_2-almost separates variables on the annulus $\{r_2 \leq b \leq \Omega_2 r_2\}$.

For each j_0 satisfying (10.15), (10.18), and (10.23), after $\mathcal{N}$ stages we are left with $\mathcal{N}$ linearly independent harmonic functions $v_1, \dots, v_{\mathcal{N}}$ on the annulus $\{r \leq b \leq \Omega r\}$, where

$$r = r_{\mathcal{N}} = \Omega_{\mathcal{N}}^{j_{\mathcal{N}}} \cdots \Omega_0^{j_0}\,. \tag{10.36}$$

Note that $[r_1, \Omega_1 r_1] \supset [r_2, \Omega_2 r_2] \supset \cdots \supset [r_{\mathcal{N}}, \Omega_{\mathcal{N}} r_{\mathcal{N}}] = [r, \Omega r]$.

On the annulus, $\{r \leq b \leq \Omega r\}$, these harmonic functions, $\{v_i\}$, must:

(a) have U bounded by $16d$,

(b) ϵ_i-almost separate variables,

and, most importantly,

(c) be pairwise η-almost orthogonal at all level sets.

The last step is to get a gradient bound, and hence a Lipschitz bound, on X for these functions. This will allow us to apply the results of Section 6 to deduce a contradiction.

To get the gradient bound, observe first that the uniform bound on U_{v_i} on the interval $[r, \Omega r]$,

$$U_{v_i}(s) \leq 16d\,, \tag{10.37}$$

implies by (2.14) that

$$I_{v_i}(\Omega r) \leq 4^{32d} I_{v_i}\left(\frac{\Omega}{4} r\right) . \tag{10.38}$$

Furthermore, (10.37) yields that

$$D_{v_i}(\Omega r) \leq 16d\, I_{v_i}(\Omega r) . \tag{10.39}$$

By (10.2) together with the meanvalue inequality, Proposition 2.26, we get a constant $\tilde{C} = \tilde{C}(n) > 0$ such that for $r > R$

$$\begin{aligned} \sup_{B_{\frac{\Omega}{\sqrt{3}} r}} |\nabla v_i|^2 \leq & \quad \frac{\tilde{C}}{\mathrm{Vol}(B_{\frac{\sqrt{3}}{2}\Omega r})} \int_{B_{\frac{\sqrt{3}}{2}\Omega r}} |\nabla v_i|^2 \\ \leq & \quad \left(\frac{4}{3}\right)^{\frac{n}{2}} \frac{\tilde{C} D_{v_i}(\Omega r)}{(\Omega r)^2 \mathrm{V}_M} . \end{aligned} \tag{10.40}$$

Combining (10.38), (10.39), and (10.40) leads to the gradient estimate

$$\begin{aligned} \sup_{B_{\frac{\Omega}{\sqrt{3}} r}} |\nabla v_i|^2 &\leq \left(\frac{4}{3}\right)^{\frac{n}{2}} \frac{\tilde{C} D_{v_i}(\Omega r)}{(\Omega r)^2 \mathrm{V}_M} \\ &\leq \left(\frac{4}{3}\right)^{\frac{n}{2}} \frac{16 \tilde{C} d\, I_{v_i}(\Omega r)}{(\Omega r)^2 \mathrm{V}_M} \\ &\leq \left(\frac{4}{3}\right)^{\frac{n}{2}} \frac{16 \tilde{C} d\, 4^{32d} I_{v_i}\left(\frac{\Omega}{4} r\right)}{(\Omega r)^2 \mathrm{V}_M} . \end{aligned} \tag{10.41}$$

We now normalize the v_i to get $\mathcal{N}$ η-almost orthogonal harmonic functions $\tilde{v}_1, \ldots, \tilde{v}_{\mathcal{N}}$ on $\{r \leq b \leq \Omega r\}$ with $I_{\tilde{v}_i}(\frac{\Omega}{4} r) = n\mathrm{V}_M$ for $i = 1, \ldots, \mathcal{N}$. From (10.41) it follows that

$$\sup_{B_{\frac{\Omega}{\sqrt{3}} r}} |\nabla \tilde{v}_i|^2 \leq \left(\frac{4}{3}\right)^{\frac{n}{2}} 16 \tilde{C} n\, d\, 4^{32d} (\Omega r)^{-2} . \tag{10.42}$$

Finally note that by the triangle inequality we have that if $x, y \in \{b = \frac{\Omega}{4} r\} \subset B_{\frac{\Omega}{2\sqrt{3}} r}$ then the minimal geodesic (in M) between x and y lies entirely inside $B_{\frac{\Omega}{\sqrt{3}} r}$.

Since the $\tilde{v}_i$ are η-almost orthonormal at $b = \frac{\Omega}{4} r$ and satisfy the Lipschitz estimate which follows from the gradient bound (10.42) together with the the above application of the triangle inequality, we can apply Proposition 6.5 to obtain the theorem. q.e.d.

Remark 10.42. (Conical case). If M is $C(N)$ (as it is in Section 1), then U is monotone nondecreasing by Lemma 1.18. Given a set of independent harmonic functions $\{u_i\} \subset \mathcal{H}_d$, at $r = 1$ we can extract an orthonormal basis v_i for the space spanned by the u_i. By Lemma 1.28, the frequency of v_i ($\in \mathcal{H}_d$) is uniformly bounded by d. Integrating this out to $r = 2$ leads to a uniform bound on $I_{v_i}(2)$ and hence, given the bound on U_{v_i}, we get a uniform bound on $D_{v_i}(2)$. By the Li-Schoen meanvalue inequality, we obtain a Lipschitz bound for $r \leq 1$ for the independent functions. Proposition 6.5 now yields a bound on the dimension of $\mathcal{H}_d(C(N))$ just in terms of d and the lower bound on the Ricci curvature of N. In contrast to the results of Section 1, this bound is not sharp.

11. Examples

In contrast to the Euclidean case, it is possible for M with nonnegative Ricci curvature to admit harmonic functions with nonintegral rates of growth (cf. Example 1.70). Even if M is Ricci flat and Kähler, examples exist.

Example 11.1. (Tian-Yau, [53]). There exist Ricci flat Kähler manifolds with Euclidean volume growth which have harmonic functions with growth strictly between one and two.

We note that there are manifolds with positive sectional curvature which admit no nontrivial harmonic functions with polynomial growth. To our knowledge, no such example has been constructed with Euclidean volume growth even under the less restrictive assumption of nonnegative Ricci curvature; see [26] for more on this.

Example 11.2. There exist manifolds with nonnegative Ricci curvature (in fact, positive sectional curvature) which admit no nontrivial polynomial growth harmonic function. In fact, one may round off a metric of the form $dr^2 + r^{2\alpha} d\theta^2$, where $\alpha < 1$ and $d\theta^2$ is the standard metric on $\mathbf{S}^{n-1}$.

In this case, since $\partial B_r(p)$ is connected, if $u \in \mathcal{H}_d$, then by the maximum principle there exists $x \in \partial B_r(p)$ with $u(x) = u(p)$. Integrating along curves beginning at x, we obtain $u \equiv u(p)$ by using the gradient estimate together with the facts that u has polynomial growth and that $\operatorname{diam} \partial B_r(p)/r \to 0$. This example was observed by Kasue in [31].

Example 11.3. (Klembeck, [31], and cf. [47]). In the holomorphic case, there exists a Kähler metric on $\mathbf{C}^n$ of positive sectional curvature and quadratic curvature decay which does not admit any nonconstant holomorphic functions with polynomial growth.

The next example reveals some of the difficulties in the general case compared with the model case of a cone. It shows in particular that unlike the model case of a cone the frequency of a harmonic function on a manifold with nonnegative Ricci curvature and Euclidean volume growth may not be monotone.

Example 11.4. ([26]). There exist manifolds with nonnegative Ricci curvature, Euclidean volume growth, and quadratic curvature decay which admit harmonic functions with polynomial growth whose frequency oscillate between two different numbers.

Let us explain the idea behind Example 11.4. From Section 1 (Theorem 1.66) we know that for a cone the order of growth of $u \in \mathcal{H}_d$ (on a large annulus) is given in terms of an eigenvalue of the cross-section (see (1.6)). If we consider a manifold which on a large annulus looks roughly like an annulus in a cone centered at the vertex, then the growth of such a u will be given almost in terms of an eigenvalue of the cross-section of the cone. By changing the cross-section slowly into a different cross-section (see [7]) which is not isospectral to the original one we can change the growth of u. Oscillating back and forth between two non-isospectral cross-sections gives a harmonic function with polynomial growth on a manifold with nonnegative Ricci curvature, Euclidean volume growth, and for which the order at infinity is not well defined.

We refer to [7] for an extensive discussion of examples of manifolds with nonnegative Ricci curvature and nonuniqueness of tangent cones at infinity. Examples of manifolds with nonnegative Ricci curvature, Euclidean volume growth, quadratic curvature decay, and for which the tangent cone at infinity is not unique were first constructed by Perelman, [49].

We end this section by showing that if one makes a small pertubation (in an appropriate norm) of the metric, then $\dim \mathcal{H}_d$ remains unchanged. In fact we have the following proposition which says that any polynomial growth asymptotically harmonic function lies within a bounded distance of a harmonic function (this new harmonic function will of course be forced to have the same rate of growth).

Proposition 11.5. *Suppose that* $Ric_{M^n} \geq 0$ *and* $Vol(B_r(p)) \geq V r^n$

for some $V > 0$ (here we assume that $n \geq 3$). Suppose also that u is a smooth function on M, and u and $|\nabla u|$ have polynomial growth. If in addition

$$|\Delta u| \leq f(r) \tag{11.6}$$

for some bounded, integrable (on $\mathbf{R}^n$), and nonnegative function, f, then

$$\lim_{t\to\infty} e^{t\Delta} u(x) \tag{11.7}$$

exists for all x, and further

$$\|e^{t\Delta}u - u\|_\infty \leq C(n)\left(\int_0^\infty f(s)s^{n-1}ds + \sup_{[0,\infty)} f\right). \tag{11.8}$$

Proof. Since $\mathrm{e}^{t\Delta}u(x) = \int_M H(x,y,t)u(y)dy$, we have that

$$\begin{aligned} \frac{\partial}{\partial t}(\mathrm{e}^{t\Delta}u)(x) &= \int_M \frac{\partial}{\partial t}H(x,y,t)u(y)dy \\ &= \int_M \Delta H(x,y,t)u(y)dy. \end{aligned} \tag{11.9}$$

Integrating by parts gives

$$\begin{aligned} \int_{B_r(p)} \Delta H(x,y,t)u(y)dy = &\int_{\partial B_r(p)} \nabla_n H(x,y,t)u(y)dy \\ &- \int_{\partial B_r(p)} H(x,y,t)\nabla_n u dy \\ &+ \int_{B_r(p)} H(x,y,t)\Delta u dy. \end{aligned} \tag{11.10}$$

By the Bishop volume comparison theorem and the fact that u and $|\nabla u|$ have polynomial growth together with the fact that H and $\int_{A_{r,r+1}(p)} |\nabla H|^2$ decay exponentially we get that

$$\lim_{r\to\infty} \int_{B_r(p)} \Delta H(x,y,t)u(y)dy = \lim_{r\to\infty}\int_{B_r(p)} H(x,y,t)\Delta u dy. \tag{11.11}$$

Further,

$$\left|\int_{B_r(p)} H(x,y,t)\Delta u dy\right| \leq \int_{B_r(p)} H(x,y,t) f(r(y))dy. \tag{11.12}$$

Using the Li-Yau estimate on the heat kernel [see [44]; i.e., for any $\epsilon > 0$ there exists a constant $C > 0$ such that

$$H(x,y,t) \leq C t^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{(4+\epsilon)t}\right)], \tag{11.13}$$

we have

$$\begin{aligned}\int_M & H(x,y,t) f(r(y))dy \\ & \leq C \int_M t^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{(4+\epsilon)t}\right) f(r(y))dy\,.\end{aligned} \tag{11.14}$$

We will now deal separately with the cases $t \geq 1$ and $t < 1$. For $t < 1$, we bound the right-hand side in terms of the sup of f and the integral of the Euclidean heat kernel (again using the Bishop volume comparison theorem). That is,

$$\begin{aligned}\int_M & H(x,y,t) f(r(y))dy \\ & \leq C \sup f \int_M t^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{(4+\epsilon)t}\right) dy \\ & \leq C \sup f\,.\end{aligned} \tag{11.15}$$

For t at least one, we have that $\exp\left(-\frac{|x-y|^2}{4t}\right) \leq 1$, and

$$\int_M H(x,y,t) f(r(y))dy \leq C t^{-\frac{n}{2}} \int_M f(r(y))dy\,. \tag{11.16}$$

Now combining (11.15) and (11.16) yields the bound

$$\begin{aligned}\int_0^\infty \left|\frac{\partial}{\partial t}(\mathrm{e}^{t\Delta}u)(x)\right| \leq & \int_0^1 (C \sup f) \\ & + \int_M f(r(y))dy \int_1^\infty C t^{-\frac{n}{2}} \\ \leq & C(n) \left(\int_0^\infty f(s) s^{n-1} ds + \sup_{[0,\infty)} f\right),\end{aligned} \tag{11.17}$$

which proves (11.7). Since

$$\mathrm{e}^{t\Delta}u(x) - u(x) = \int_M H(x,y,t)(u(y)-u(x))dy,$$

(11.8) follows by the same argument. q.e.d.

Appendix A. The first variation of energy

In this appendix, we will, for the sake of completeness, collect some well-known consequences of the first variation of energy that we need for this paper.

In the following, we will take M to be a complete Riemannian manifold and u to be a smooth function on M. Given a one-parameter family ϕ_t of diffeomorphisms of M we define a one-parameter family of functions $u_t = u \circ \phi_t$. We let v denote $\frac{d\phi_t}{dt}$.

Let B be a bounded domain in M. Henceforth, we suppose that the diffeomorphisms are the identity outside of B; equivalently, we take the vector field v to have support in B. Now we define E_t to be the Dirichlet energy of u_t in B, that is,

$$E_t = \int_B |\nabla u_t|^2 \,. \tag{A.1}$$

Lemma A.2. (First Variation). *Let M and v be as above. Then the first variation of energy is given by*

$$E'(0) = 2 \int_B \nabla(v)(\nabla u, \nabla u) - \int_B |\nabla u|^2 div(v) \,. \tag{A.3}$$

In the following, we will work in normal coordinates. We can then rewrite (A.3) as

$$E'(0) = 2 \int_B v_{ij} u_i u_j - \int_B u_i^2 v_{jj} \,, \tag{A.4}$$

where additional indices refer to covariant derivatives, and the usual summation conventions are to be understood.

Proof. By definition, we have

$$E_t' = \frac{d}{dt} \int_B |\nabla u_t|^2 \,. \tag{A.5}$$

Differentiating under the integral sign gives

$$E_t' = \int_B \frac{d}{dt} |\nabla u_t|^2 = 2 \int_B \langle \nabla_{\frac{d}{dt}} (\nabla u_t) \,, \nabla u_t \rangle \,. \tag{A.6}$$

We will now calculate the integrand above in normal coordinates. By the chain rule,

$$\text{(A.7)} \qquad \left.\frac{d}{dt}\right|_{t=0} |\nabla u_t|^2 = 2\left(\dot{u}_j \phi_{0,ji} + u_j \dot{\phi}_{0,ji}\right)(u_j \phi_{0,ji}) \,.$$

By construction, we have

$$\text{(A.8)} \qquad \phi_{0,ji} = \delta_{ij} \text{ and } \dot{\phi}_{0,ji} = v_{ji} \,.$$

Combining (A.7) and (A.8) yields

$$\text{(A.9)} \qquad \left.\frac{d}{dt}\right|_{t=0} |\nabla u_t|^2 = 2\left(\dot{u}_i u_i + u_j u_i v_{ji}\right) \,.$$

Note that we can rewrite the first term above as

$$\text{(A.10)} \qquad 2\dot{u}_i u_i = \left.\frac{d}{dt}\right|_{t=0} |\nabla u|^2(\phi_t) \,.$$

Integrating equation (A.10), we get that the integral of the first term in equation (A.9) is given by

$$\text{(A.11)} \qquad \left.\frac{d}{dt}\right|_{t=0} \int_B |\nabla u|^2(\phi_t) \,,$$

which becomes, by the change of variables formula,

$$\text{(A.12)} \qquad \left.\frac{d}{dt}\right|_{t=0} \int_B |\nabla u|^2 \text{Jac}(\phi_t^{-1}) = \int_B |\nabla u|^2 \left.\frac{d}{dt}\right|_{t=0} \text{Jac}(\phi_t^{-1}) \,.$$

To first order, we have that $\phi_{t,ij} = \delta_{ij} + tv_{ij}$, therefore

$$\text{(A.13)} \qquad \begin{aligned} \text{Jac}(\phi_t) &= \det(\delta_{ij} + tv_{ij}) \\ &= 1 + t(v_{jj}) + O(t^2) \,. \end{aligned}$$

Thus

$$\text{(A.14)} \qquad \left.\frac{d}{dt}\right|_{t=0} \int_B |\nabla u|^2 \text{Jac}(\phi_t^{-1}) = \int_B |\nabla u|^2 (-\text{div}(v)) \,.$$

Putting the above all together, the lemma now follows. q.e.d.

We will now derive some general identities from the first variation formula, by making careful choices of the domain B and the variation vector field v. Let b be a Lipschitz function such that

$$\text{(A.15)} \qquad B = \{x \in M \mid b(x) \leq r\}.$$

Take $\gamma : \mathbf{R} \to \mathbf{R}$ to be a cut-off function with support in $\{|x| \leq r\}$ such that

$$\gamma(x) = 1 \text{ for } x \leq (r-\epsilon) \text{ and } \gamma \to 0 \text{ linearly otherwise.} \tag{A.16}$$

Let A_ϵ denote the region in B on which $b \geq (r-\epsilon)$. Finally, we choose the variation vector field v to be

$$v(x) = \frac{1}{2}\gamma(b(x))\nabla b^2 . \tag{A.17}$$

We will often write γ for $\gamma \circ b$.

Note that by (A.17),

$$\mathrm{div}(v) = \frac{1}{2}\gamma\Delta(b^2) + \frac{1}{2}\gamma'\langle \nabla b, \nabla b^2\rangle . \tag{A.18}$$

It follows from (A.18) that given any function u,

$$\begin{aligned} \int_B |\nabla u|^2 \mathrm{div}(v) =& \frac{1}{2}\int_B \gamma |\nabla u|^2 \Delta b^2 \\ & - \frac{1}{\epsilon}\int_{\mathrm{A}_\epsilon} |\nabla u|^2 b \, |\nabla b|^2 . \end{aligned} \tag{A.19}$$

Similarly,

$$v_{ji} = \frac{1}{2}\left[\gamma' b_i (b^2)_j + \gamma (b^2)_{ji}\right] . \tag{A.20}$$

$$\begin{aligned} 2\int_B \nabla v(\nabla u, \nabla u) = & \int_B \gamma \mathrm{Hess}(b^2)(\nabla u, \nabla u) \\ & - \frac{2}{\epsilon}\int_{\mathrm{A}_\epsilon} b\langle \nabla u, \nabla b\rangle^2 . \end{aligned} \tag{A.21}$$

Combining (A.19) and (A.21), the first variation formula (A.3) implies the following:

$$\begin{aligned} E'(0) = & -\frac{1}{2}\int_B \gamma |\nabla u|^2 \Delta b^2 \\ & + \frac{1}{\epsilon}\int_{\mathrm{A}_\epsilon} |\nabla u|^2 b \, |\nabla b|^2 \\ & + \int_B \gamma \mathrm{Hess}(b^2)(\nabla u, \nabla u) \\ & - \frac{2}{\epsilon}\int_{\mathrm{A}_\epsilon} b\langle \nabla u, \nabla b\rangle^2 . \end{aligned} \tag{A.22}$$

If we now let ϵ approach zero, we get the following proposition.

Proposition A.23. *If u is harmonic, and b and B are as above, then*

$$\begin{aligned} &\frac{1}{2}\int_B |\nabla u|^2 \Delta b^2 - \int_B \mathit{Hess}(b^2)(\nabla u, \nabla u) \\ &\quad = r\int_{\partial B} |\nabla u|^2 |\nabla b| - 2r\int_{\partial B} \left|\frac{\partial u}{\partial n}\right|^2 |\nabla b|\,, \end{aligned} \tag{A.24}$$

where $\frac{\partial u}{\partial n}$ is the normal derivative of u on ∂B. Recall that $\partial B = \{x | b(x) = r\}$.

Proof. Since u is harmonic, it is a critical point for the energy functional and $E'(0) = 0$. By (A.22), we have

$$\begin{aligned} &\frac{1}{2}\int_B \gamma |\nabla u|^2 \Delta b^2 - \int_B \gamma \mathrm{Hess}(b^2)(\nabla u, \nabla u) \\ &\quad = \frac{1}{\epsilon}\int_{\mathrm{A}_\epsilon} |\nabla u|^2 b\, |\nabla b|^2 - \frac{2}{\epsilon}\int_{\mathrm{A}_\epsilon} b\langle \nabla u, \nabla b\rangle^2 . \end{aligned} \tag{A.25}$$

As $\epsilon \to 0$, the left-hand side in (A.25) clearly approaches the left-hand side in (A.24). Furthermore, the tube A_ϵ is to first order a subdomain of the normal bundle of ∂B of width $\frac{\epsilon}{|\nabla b|}$. It follows that the right-side of (A.25) approaches the right-side of (A.24). q.e.d.

Corollary A.26. *If u is harmonic, and ρ is the distance function from a fixed point $p \in M$, then*

$$\begin{aligned} &\frac{1}{2}\int_{B_r} |\nabla u|^2 \Delta \rho^2 - \int_{B_r} \mathit{Hess}(\rho^2)(\nabla u, \nabla u) \\ &\quad = r\int_{\partial B_r} |\nabla u|^2 - 2r\int_{\partial B_r} \left|\frac{\partial u}{\partial r}\right|^2 , \end{aligned} \tag{A.27}$$

where B_r is the ball of radius r, and $\frac{\partial u}{\partial r}$ is the radial derivative of u on ∂B_r.

Remark A.28. If we take M to be $C(N)$, the cone on a compact manifold N, then $\mathrm{Hess}(\rho^2) = 2\delta_{ij}$ where ρ is the distance from the vertex. For example, $C(\mathbf{S}^{(n-1)})$ is $\mathbf{R}^n$. Therefore, Corollary A.26 implies that

$$\begin{aligned} (n-2)\int_{B_r} |\nabla u|^2 = &\, r\int_{\partial B_r} |\nabla u|^2 \\ &- 2r\int_{\partial B_r} \left|\frac{\partial u}{\partial r}\right|^2 , \end{aligned} \tag{A.29}$$

where B_r is the ball of radius r centered at the vertex of the cone. If we let $D(r)$ denote the scaled energy on the ball of radius r centered at the vertex, then

$$D'(r) = (2-n)r^{-1}D(r) + r^{2-n}\int_{\partial B_r} |\nabla u|^2 . \tag{A.30}$$

By substituting equation (A.29), equation (A.30) becomes

$$D'(r) = 2r^{2-n}\int_{\partial B_r} \left|\frac{\partial u}{\partial r}\right|^2 \geq 0. \tag{A.31}$$

Equation (A.31) is the usual monotonicity of scaled energy.

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