MANIFOLDS ALL OF WHOSE FLATS ARE CLOSED

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Abstract

Let M be a locally irreducible Riemannian manifold such that (almost) any geodesic be contained in a compact immersed flat of dimension at least 2. The main result in this article implies that M is a locally symmetric space of the compact type. Therefore, our result gives a characterization of locally symmetric spaces of the compact type and higher rank in terms of flats.

0. Introduction

The purpose of this article is to give a higher rank rigidity characterization of compact locally symmetric spaces. Let us first begin with some motivations. Let M be a Riemannian manifold of nonpositive curvature with finite volume and $rank(M) = k$ in terms of Jacobi fields. Then any regular geodesic is contained in a k-flat, and hence the horizontal distribution, on the open set of regular vectors of TM, associated with the rank is integrable with totally geodesic leaves. (If M is not of nonpositive curvature, there seems to be no reason for expecting the integrability of the rank distribution and the rank is defined in terms of flats; cf. [12]). If M has in addition higher rank, i.e., $rank(M) \geq 2$, and irreducible universal cover, then M must be locally symmetric, as it follows from the celebrated theorem of Ballmann and Burns-Spatzier [1], [5] (see also [6], [8]). If M is not any more of nonpositive curvature, then the higher rank rigidity is not true. Namely, in [12] there is constructed a wide family of analytic (locally) irreducible compact nonsymmetric Riemannian manifolds of nonnegative curvatures and higher rank k, such that regular geodesics are contained in k-flats. (Some of these examples are homogeneous and were already known by Ernst

Heintze). In all of this examples the k-flats are not in general closed, a necessary condition for being the space locally symmetric (of the compact type). This condition turns out to be also sufficient for the local symmetry, as follows from the main results in this article (which make no curvature assumptions).

Theorem A. *Let M be a complete Riemannian manifold locally irreducible at any point in an open nonempty subset. Assume that there exists an open and dense subset Ω of TM such that for any $w \in \Omega$ the geodesic γ_w be contained in a compact (immersed) flat of dimension at least 2. Then, M is a locally symmetric space of the compact type.*

For an analytic manifold M (with irreducible universal cover) the same conclusion of the above theorem is true without assumming the density of Ω. In fact, from the Main Lemma of Section 3, it follows the existence of a nonempty locally symmetric subset of M.

Compare the above results with [9] and [7], where highly homogeneous situations are considered.

If M is a simple connected manifold such that every geodesic is contained in a compact flat, then each irreducible factor has the same property. In fact, if γ is geodesic lying in a factor, then its closure is a compact (immersed) flat which must also lie in this factor. If M is in addition analytic then by the Theorem each irreducible factor is either symmetric or a manifold all of whose geodesics are closed (see Remark 6.7). Therefore we have the following decomposition theorem.

Theorem B. *Let M be an analytic (complete) Riemannian manifold with finite fundamental group such that every geodesic is contained in a compact immersed flat. Then, each Riemannian factor of the universal cover of M is either symmetric of higher rank or a manifold all of whose geodesics are closed.*

It is still missing a general decomposition result.

Conjecture. *Let M be a (complete) Riemannian manifold such that every geodesic is contained in a compact immersed flat. Then, the universal cover $\tilde{M}$ of M is compact up to a Euclidean factor. Moreover, each irreducible factor of $\tilde{M}$ is either symmetric of higher rank or a manifold all of whose geodesics are closed.*

The above conjecture says, in particular, that for $\Omega = TM$ we can replace the local irreducibility condition in Theorem A by the weaker one of irreducibility of the universal cover.

Compact Riemannian manifolds of nonpostive curvatures and rank one is a wide family which includes locally symmetric spaces of the noncompact type. The results [1], [5] of Ballmann and Burns-Spatzier assert that for higher rank there are no examples other than the locally symmetric spaces. On the other hand, a wide and rich class of Riemannian manifolds, which contains the rank-one locally symmetric spaces of the compact type, is the class of manifolds all of whose geodesics are closed [4]. A higher rank version of this family could be the manifolds all of whose geodesics are contained in closed (i.e., compact) flats. The results in this paper might be interpreted as a higher rank rigidity in this dual context.

One of the main steps for proving Theorem A is the Regularity Lemma of Section 1 which, essentially says that the family of flats vary smoothly in appropriated subsets of TM. Roughly speaking, let $c(t)$ be a smooth curve in M and let $v(t)$ be a parallel field along $c(t)$. It is proved that, for every t, we can choose a compact flat $F(t)$, of dimension at least 2, which contains the geodesic $\gamma_v(t)$ such that $t \rightarrow F(t)$ is smooth (this is indeed much stronger than the property that the tangents $T_{c(t)}F(t)$ vary smoothly). Let now $g_t : F(0) \rightarrow F(t)$ be smooth and affine ($g_0 = id$). Then, using an argument relying on a lemma of Hermann transferred to the tangent bundle (see [7], [9], [10]) and standard results about variation of flats we prove that the covariant derivative of $\tilde{v}(t) := dg_t(v(0))$ is perpendicular to the tangent space to the k-flat $F(t)$ at $c(t)$. Hence $\langle v(t), \tilde{v}(t) \rangle$ is constant. Using this information we are able to prove that the local holonomy group does not act transitively on the unit sphere, and hence, by the Berger holonomy theorem [3], [11], M must be locally symmetric.

1. Preliminaries and basic facts

Let $(M^n, <, >)$ be a complete connected Riemannian manifold. We say that M is locally reducible at $p \in M$ if p has an open neighbourhood in M, which is a Riemannian product. This is equivalent to the fact that the local holonomy group at p does not act irreducibly on T_pM. The tangent bundle $TM \xrightarrow{\pi} M$ will be endowed with the Sasaki (Riemannian) metric $<< , >>$, which is defined by the following properties: *(i)* The horizontal distribution $\mathcal{H}$ is perpendicular to the vertical one ν. *(ii)* π is a Riemannian submersion. *(iii)* $<< , >>|_{\pi^{-1}(p)}$ coincides with the metric $< , >_p$ in T_pM, for any $p \in M$.

If $p \in T_pM$, γ_v denotes the (complete) geodesic defined by $\gamma_v'(0) = v$. A flat containing γ_v is an isometric totally geodesic immersion $i : F \to M$, where F is a complete connected flat Riemannian manifold, such that $\gamma_v(\mathbb{R})$ is contained in $i(F)$. We say that the flat is *compact* if F is compact, and is *toral* if F is homeomorphic to a torus. Since any compact flat manifold is covered by a torus, a geodesic is contained in a compact flat if and only if it is contained in a toral one. A k-flat means a flat of dimension k ($k \in \mathbb{N}$). Sometimes a flat $i : F \to M$ will be denoted by the pair (F, i) or simply by i or by F when the immersion is obvious from the context. We say that a flat $i : F \to M$ has no self-intersection at $q \in i(F)$ if $di\,(T_xF)$ does not depend on $x \in i^{-1}(q)$. In this case the uniquely defined subspace $di\,(T_xF)$ of T_qM will be called the tangent space to the flat at q.

Observation. It is standard to show that if $i : F \to M$ is a flat containing the geodesic γ_v, then there exists a geodesic $\tilde{\gamma}$ in F such that $\gamma_v = i \circ \tilde{\gamma}$.

We shall need the following lemma, whose proof will be given in Section 6 .

Regularity Lemma. *Let $\tilde{\mathcal{O}}$ be an open subset of TM with the property that every geodesic with initial condition in $\tilde{\mathcal{O}}$ be contained in a compact flat of dimension at least 2. Then, there exist an open nonempty subset $\mathcal{O}$ of $\tilde{\mathcal{O}}$ and a choice of toral k-flats $i_v : F_v \to M$, $k \geq 2$, containing the geodesic γ_v, $v \in \mathcal{O}$ with the following properties:*

i) *The flat (F_v, i_v) has no self-intersection at $\pi(v)$, for any $v \in \mathcal{O}$.*

ii) *$\mathcal{F}$ defines a C^∞ distribution on $\mathcal{O}$, where $\mathcal{F}_v$ is the unique subspace of the horizontal subspace $\mathcal{H}_v$ such that $d\pi\,(\mathcal{F}_v)$ coincides with the tangent space to the flat at $\pi(v)$.*

In all what follows in the paper, unless otherwise stated, $(M, <>)$ will denote a Riemannian manifold, and $\tilde{\mathcal{O}}$ will be an open subset of TM such that any geodesic in M with initial condition in $\tilde{\mathcal{O}}$ be contained in a compact flat. We will follow the notation of the Regularity Lemma and consider the open and nonempty (connected) subset $\mathcal{O}$ of $\tilde{\mathcal{O}}$ and the C^∞ distribution $\mathcal{F}$ on $\mathcal{O}$ (i.e., $\mathcal{F}$ is a linear subbundle of $T\mathcal{O} \to \mathcal{O}$).

Remark 1.1. Let $\tilde{f} : \mathcal{F} \to \mathcal{O} \times M$ be defined by $\tilde{f}(u) = (\tilde{\pi}(u), f(u))$, where $\tilde{\pi}$ is the projection to its base of the vector bundle $\mathcal{F}$ over $\mathcal{O}$, and $f(u) = exp_{\pi(\tilde{\pi}(u))}(d\pi(u))$. It is easy to check that $\tilde{f}$

is an immersion. Observe that f restricted to any fibre of $\mathcal{F} \xrightarrow{\tilde{\pi}} \mathcal{O}$ is a totally geodesic isometric (flat) immersion, i.e., a flat. Identifying the tangent space at the origin to a fibre $\mathcal{F}_v = (\tilde{\pi})^{-1}(v)$ with $\mathcal{F}_v$ one has that $df(\mathcal{F}_v) = d\pi(\mathcal{F}_v)$. The above defined maps $\tilde{f}$ and f will be used in the next sections.

If $v \in \mathcal{O}$, let $i_v : F_v \to M$ be the toral k-flat containing the geodesic γ_v given by the Regularity Lemma. The k-flat (F_v, i_v) has no self-intersection at $\pi(v)$, and its tangent space at this point is $d\pi(\mathcal{F}_v)$. Thus (F_v, i_v) determines uniquely a subgroup G (which depends on v) of the group of translations $\tau(\mathcal{F}_v)$ of $\mathcal{F}_v$. Namely, G is uniquely determined by the following property: there exists a diffeomorphism $h : \mathcal{F}_v/G \to F_v$ such that $i \circ h \circ pr = exp_{\pi(v)} \circ d\pi$, where pr is the projection from $\mathcal{F}_v$ onto $\mathcal{F}_v/G$. So, from now on, the toral k-flat (F_v, i_v) will be regarded as $\mathcal{F}_v/G$ where G is a subgroup of $\tau(\mathcal{F}_v)$. Any such a G admits always a basis $b = (b_1, \cdots, b_k)$ over $\mathbb{Z}$ (briefly, a $\mathbb{Z}$-basis). Notice that if $D \in \mathbb{Z}^{k \times k}$ with $det(D) \neq 0$, and $\tilde{b} = b.D$, then $\tilde{b}_1 \cdots, \tilde{b}_k$ generates a subgroup $\tilde{G}$ of G $\mathcal{F}_v/\tilde{G}$ is also a toral k-flat containing γ_v, and $\tilde{b}$ is a $\mathbb{Z}$-basis of $\tilde{G}$. Observe that $\|b_i\|$ is greater than or equal to the injectivity radius $r_{\pi(v)}$ of M at $\pi(v)$, since $\{ t \mapsto exp_{\pi(v)} t d\pi(b_i) \}$ is a closed geodesic through $\pi(v)$ $(i = 1, \cdots, k)$. Moreover, $\|b_i\| \geq \delta_v := sup\{r_x : x \in exp \circ d\pi(\mathcal{F}_v)\}$. Observe that if $w \in G$ is of minimal length (identifying vectors with translations), then there always exists a $\mathbb{Z}$-basis b' of G such that $b'_1 = w$. In fact this is a consequence of the following well-known result: any subgroup of $\mathbb{Z}^k$ admits a $\mathbb{Z}$-basis of the form $(d_1 v_1, \cdots, d_j v_j)$, where $d_1, \cdots, d_j \in \mathbb{Z}$ $(j \leq k)$, and $(v_1, \cdots, v_k)$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^k$. In particular, one obtains that $vol(\mathcal{F}_v/G) \geq 2^{-k} vol(B_{\delta_v})$, where vol denotes volume, and B_{δ_v} is the Euclidean k-dimensional open ball of radius δ_v. Note that $vol(\mathcal{F}_v/G)$ coincides with $vol(b) := (det(\langle b_i, b_j \rangle))^{\frac{1}{2}}$, (i.e., the volume determined by $b_1, \cdots, b_k$), for any $\mathbb{Z}$-basis b of G. Before finishing this section let us define $B(\mathcal{F})$ to be the principal bundle over $\mathcal{O}$ of basis of the vector bundle $\mathcal{F} \to \mathcal{O}$. Remark that any $\mathbb{Z}$-basis of G may be regarded as an element of $B(\mathcal{F})$. The norm of any $b \in B(\mathcal{F})$ is defined by

$$\|b\| = \|(\langle b_i, b_j \rangle)\|_E,$$

where $\| \ \|_E$ is the usual Euclidean norm of matrices.

2. Smooth choice of flats

We keep the same notation and assumptions of Section 1. Let W be an arbitrary open subset of $\mathcal{O}$ and let W_i for $i \in \mathbb{N}$ be the subset of W which is defined by the following property: $v \in W$ belongs to W_i if there exists a toral k-flat $\mathcal{F}_v/G$ such that G admits a $\mathbb{Z}$-basis b with $\|b\| \leq i$. Observe that when it is clear from the context, we identify the elements of $\mathcal{F}$ with the corresponding associated translations.

Lemma 2.1. *Let $\{v_n\}_{n \in \mathbb{N}}$ be a sequence in W_i converging to $v \in W$, and let b^n for $n \in \mathbb{N}$ be a $\mathbb{Z}$-basis of G_n with $\|b^n\| \leq i$, where $\mathcal{F}_{v_n}/G_n$ is a toral k-flat containing the geodesic γ_{v_n}. Then the sequence $\{b^n\}_{n \in \mathbb{N}}$ in $B(\mathcal{F})$ has a subsequence converging to a $\mathbb{Z}$-basis b (with $\|b\| \leq i$) of the covering group G of some toral k-flat $\mathcal{F}_v/G$ which contains the geodesic γ_v. In particular, W_i is closed in W.*

Proof. Let $\varepsilon > 0$ be such that $exp_{\pi(v)}$ is a diffeomorphism from the Euclidean open ball of radius 2ε of $T_{\pi(v)}M$ onto the ball $B_{2\varepsilon}(\pi(v))$ of M. If $q \in B_\varepsilon(\pi(v))$, then $r_q \geq \varepsilon$ where r_q is the injectivity radius at q. If $V = \pi^{-1}(B_\varepsilon(\pi(v))) \cap \mathcal{O}$, $w \in V$ and F is a toral k-flat containing γ_w, then $vol(F) \geq c(\varepsilon)$ where $c(\varepsilon)$ is the volume of the Euclidean k-dimensional ball of radius ε (see Section 2). Thus $vol(\mathcal{F}_{v_n}/G_n) = vol(b^n) \geq c(\varepsilon)$ for large n. This condition together with $\|b^n\| \leq i$ implies that there is a subsequence $\{b^{n_j}\}$ converging to some $b \in B(\mathcal{F})_v$. Let $m_1, \cdots, m_k \in \mathbb{Z}$. Then $a_j = m_1(b^{n_j})_1 + \cdots + m_k(b^{n_j})_k$ converges in the bundle $\mathcal{F}$ to $a = m_1 b_1 + \cdots + m_k b_k$ as $j \to \infty$. Let τ_{a_j} be the translation on $\mathcal{F}_{v_{n_j}}$ defined by a_j. Observe that $\tilde{\pi}(a_j) = v_{n_j}$, where $\tilde{\pi}$ is the projection from $\mathcal{F}$ to $\mathcal{O}$. We have that $exp_{\pi(v_{n_j})} \circ d\pi \circ \tau_{a_j} = exp_{\pi(v_{n_j})} \circ d\pi|_{\mathcal{F}_{v_{n_j}}}$ which implies that $exp_{\pi(v)} \circ d\pi \circ \tau_a = exp_{\pi(v)} \circ d\pi|_{\mathcal{F}_v}$. If G is the subgroup of translations of $\mathcal{F}_v$ generated by $b_1, \cdots, b_k$, then the k-flat $exp_{\pi(v)} \circ d\pi : \mathcal{F}_v \to M$ factorizes through $\mathcal{F}_v/G$, which is a toral k-flat containing the geodesic γ_v. It is clear that $\|b\| \leq i$ and that b is a $\mathbb{Z}$-basis of G.   q.e.d.

Let $G(v)$ for $v \in \mathcal{O}$ denote the subgroup of the group of translations $\tau(\mathcal{F}_v)$ of $\mathcal{F}_v$ generated by all the subgroups G_v of $\tau(\mathcal{F}_v)$ such that $\mathcal{F}_v/G_v$ is a toral k-flat. Since the immersion $exp_{\pi(v)} \circ d\pi : \mathcal{F}_v \to M$ factorizes through $\mathcal{F}_v/G(v)$, we easily get that $G(v)$ is a discrete (topological) subgroup of $\tau(\mathcal{F}_v)$ and hence $\mathcal{F}_v/G(v)$ is a toral k-flat such that $G(v) \supset G$, for any toral k-flat $\mathcal{F}_v/G$. The k-flat $\mathcal{F}_v/G(v)$ will be called the *minimal toral k-flat*.

Remark. If the sequence $\{b^n\}$ in Lemma 1 is replaced by a sequence

of $\mathbb{Z}$-basis of the minimal toral k-flat $G/G(v_n)$, in general it is not true that it accumulates to a $\mathbb{Z}$-basis of a minimal toral k-flat.

Keeping the notation of the beginning of this section, we have that $W = \underset{i \in \mathbb{N}}{\cup} W_i$. From Lemma 2.1 it follows that W_i is closed in W. So, using the theorem of Baire we conclude that there exists $i_0 \in \mathbb{N}$ such that the interior $(W_{i_0})^\circ$ of W_{i_0} is nonempty. Since W is an arbitrary open subset of $\mathcal{O}$, $\mathcal{O}^*$ is an open and dense subset of $\mathcal{O}$, where $\mathcal{O}^*$ is defined by the following property: $v \in \mathcal{O}$ belongs to $\mathcal{O}^*$ if there exist a neighbourhood V_v of v and $i \in \mathbb{N}$, such that $q \in V_v$ implies that there exists a toral flat $\mathcal{F}_q/G_q$ such that G_q admits a $\mathbb{Z}$-basis b with $\|b\| \leq i$.

From now on we shall regard $\mathcal{F}$ as a distribution defined only on $\mathcal{O}^*$, i.e., a vector subbundle of $T\mathcal{O}^* \to \mathcal{O}^*$.

Lemma 2.2. *Let $c : [0,1] \to \mathcal{O}^*$ be of differentiability class C^r $(r \in \mathbb{N} \cup \{0\})$. Then, c can be lifted to a curve $b : [0,1] \to B(\mathcal{F})$ of class C^r, where $b(t)$ is a $\mathbb{Z}$-basis of some toral k-flat $\mathcal{F}_{c(t)}/G_{c(t)}$ for every $t \in [0,1]$.*

Proof. From the definition of $\mathcal{O}^*$ and the compactness of $[0,1]$ we get that there is $i \in \mathbb{N}$ such that, for any $t \in [0,1]$, there exists a toral k-flat $\mathcal{F}_{c(t)}/\tilde{G}_{c(t)}$ which admits a $\mathbb{Z}$-basis $\tilde{b}(t)$ with $\|\tilde{b}(t)\| \leq i$. Let now $t_0 \in [0,1]$ be fixed, and let $K \subset B(\mathcal{F})_{c(t_0)}$ be the set of all $\mathbb{Z}$-basis b' of arbitrary toral k-flat containing $\gamma_{c(t_0)}$ with $\|b'\| \leq i$. The set K is finite (observe that any element of K can be written as $b.D$, where b is a $\mathbb{Z}$-basis of the minimal toral k-flat $G(c(t_0))$ and $D \in \mathbb{Z}^{k \times k}$). Let $\tilde{b}^1, \cdots, \tilde{b}^s$ be the different elements of K and let $V_j \subset B(\mathcal{F})$ be an open neighbourhood containing $\tilde{b}^j$ such that $V_j \cap V_{j'} = \emptyset$ if $j \neq j'(j, j' = 1, \cdots, s)$. From Lemma 2.1 we easily obtain that there exists I_{t_0}, an open connected neighbourhood at t_0, in $[0,1]$, such that $\tilde{b}(t)$ belongs to $\overset{s}{\underset{j=1}{\cup}} V_j$. Using that the covering group of any toral k-flat which contains $\gamma_{c(t_0)}$ is contained in the minimal toral k-flat $G(c(t_0))$, it is easily seen that there exist $b_{t_0} \in B(\mathcal{F})_{c(t_0)}$ and $D_1, \cdots, D_s \in \mathbb{Z}^{k \times k}$ with $\tilde{b}^j.D_j = b_{t_0}$. Define now $\bar{b}(t) = \tilde{b}(t).D_j$ if $\tilde{b}(t) \in V_j$ $(t \in I_{t_0})$. Then $\underset{t \to t_0}{lim}\bar{b}(t) = \bar{b}(t_0)$ or, equivalently (taking components), $\underset{t \to t_0}{lim}\bar{b}_i(t) = (b_{t_0})_i = \bar{b}_i(t_0), (i = 1, \cdots, k)$ where $\bar{b}_i$ is a curve from I_{t_0} into $\mathcal{F}$. We shall next show that $\bar{b}_i$ is, like c, of differentiability class C^r restricted to an appropriated neighbourhood of t_0. Let A be an open neighbourhood of $\bar{b}_i(t_0)$ in $\mathcal{F}$ such that $\tilde{f} : A \to \mathcal{O}^* \times M$ be an embedding where $\tilde{f}(u) = (\tilde{\pi}(u), exp_{\pi(\tilde{\pi}(u))}d\pi(u))$ is the map defined in Remark 1.1. Let

now $\bar{I}_{t_0} \subset I_{t_0}$ be an open neighbourhood of t_0 such that $\bar{b}_i(\bar{I}_{t_0}) \subset A$, (such an $\bar{I}_{t_0}$ does exist because $\bar{b}_i$ is continuous at t_0). We have that $\tilde{f} \circ \bar{b}_i(t) = (c(t), exp_{\pi(c(t))}(d\pi\bar{b}_i(t))) = (c(t), \pi \circ c(t))$, because $\{s \mapsto exp(sd\pi\bar{b}_i(t))\}$, $s \in [0,1]$, is a closed geodesic. Therefore, $\tilde{f} \circ \bar{b}_{|\bar{I}_{t_0}}$ is of class C^r and hence $\bar{b}_{i\,|\bar{I}_{t_0}}$ is of class C^r, $i = 1, \cdots, k$. So, $\bar{b}_{|\bar{I}_{t_0}}$ is of class C^r and $\bar{b}(t)$, for every $t \in \bar{I}_{t_0}$, is a $\mathbb{Z}$-basis of the covering group of a toral k-flat containing $\gamma_{c(t)}$. Just for the sake of finishing the proof, let $t^\circ = 0 < t^1 < \cdots < t^l = 1$ be a partition such that $c_{|[t_j,t_{j+1}]}$ admits a C^r lifting $b^j(t)$ to $B(\mathcal{F})$, being $b^j(t)$ a $\mathbb{Z}$-basis of a toral k-flat containing $\gamma_{c(t)}$, $t \in [t_j, t_{j+1}]$. We will only define $b(t)$ on $[0, t_2]$; the iteration of the following method provides the desired b defined on $[0, 1]$. There exist $D_0, D_1 \in \mathbb{Z}^{k \times k}$ and a $\mathbb{Z}$-basis $b(t_1)$ of the covering group of a toral k-flat containing $\gamma_{c(t_1)}$ such that $b^\circ(t_1).D_0 = b^1(t_1).D_1 = b(t_1)$. Let $b_{|[0,t_1]} = b^0.D_0$ and $b_{|[t_1,t_2]} = b^1.D_1$. Then b satisfies the desired property. (The fact that b is of class C^r at t_1, is obtained by applying $\tilde{f}$ to each of its components).   q.e.d.

Remark 2.3.   If in the Main Lemma we do not assume that the dimension of the flats are at least 2, the same conlusion is true, as it is clear from the proof, but the dimension of the flats can be 1, i.e., the geodesics with initial condition in $\mathcal{O}$ are closed. All the results of this section are also true for $k = 1$ (in the proof we never used the dimension of the flats). In this particular case Lemma 2.2 is well known. Observe that once we can vary smoothly compact flats, they must have all the same volume, because, being the flats totally geodesic, they are minimal submanifolds. For $k = 1$ this means that all of these closed geodesics have the same length (more precisely, they admit a common period; cf. [4])

3. Affine maps between k-flats and the statement of the Main Lemma

We keep the assumptions and notation of Sections 1 and 2. Assume furthermore that M is locally irreducible at some $p = \pi(v_0)$, where $v_0 \in \mathcal{O}^*$ (locally irreducible at p means that there exists no neighbourhood of p which is a Riemannian product, or equivalently, that the local holonomy group Φ_p^{loc} acts irreducibly on T_pM). Let $\epsilon > 0$ be such that the ball $B_\epsilon(v_0)$ be contained in $\mathcal{O}^*$ (recall that TM is endowed with the Sasaki metric). If $c : [0, 1] \to M$ is a piecewise differentiable curve with

$c(0) = p = \pi(v_0) = c(1)$ and length$(c) < \epsilon$, then its horizontal lift v, with $v(0) = v_0$, is a curve in $B_\epsilon(v_0)$ because of $\|v'(t)\|^2 = |c'(t)|^2 + |\frac{D}{dt}v(t)|^2 = |c'(t)|^2$. By Lemma 2.2 there exists a lift $b(t)$ of $v(t)$ to $B(\mathcal{F})$ such that $b(t)$ is piecewise differentiable and such that $b(t)$ is a $\mathbb{Z}$-basis of the covering group G_t of a toral k-flat torus $F(t)$ which contains the geodesic $\gamma_{v(t)}$. We have, for $t \in [0,1]$, that the linear map $l_t : \mathcal{F}_{v(\circ)} \to \mathcal{F}_{v(t)}$ which maps the basis $b(0)$ into $b(t)$ projects down to an affine map $\bar{l}_t : F(0) \to F(t)$. Let now $g : [0,1] \times F(0) \to M$ be defined by $g_t(u) = f \circ \bar{l}_t(u) = exp_{c(t)} \circ d\pi (u)$; in order to save notation we do not distinguish between $f_{|\mathcal{F}_v}$ and its projection to the quotient $F(t) = \mathcal{F}_{v(t)}/G_t$. The map g is continuous, and furthermore is differentiable at those (t, w) such that c is differentiable at t. After identifying v_0 with its horizontal lifting to TTM it makes sense to assume $v(t) = dg_t(v_0)$ which will always be regarded as a piecewise differentiable curve in TM, i.e., a tangent field along $c(t)$. It is computed as follows: if $v_0 = \sum\limits_{i=1}^{k} a_i d\pi(b_i(0))(a_i \in \mathbb{R})$, then $dg_t(v_0) = \sum\limits_{i=1}^{k} a_i d\pi(b_i(t))$.

Remark 3.1. Under the above notation, assume that

$$v(1) \in df(\mathcal{F}_{v(0)})$$

and that it is so close to v_0 that it belongs to $\mathcal{O}^*$. Assume furthermore that the geodesic $\gamma_{v(1)}$ is dense in the k-flat $(f, \mathcal{F}_{v_0}/G_0)$, i.e., in the image of the k-flat. Then, $df(\mathcal{F}_{v(1)}) = df(\mathcal{F}_{v(0)})$, since the k-flat $(f, \mathcal{F}_{v_{(1)}}/G_1)$ contains the geodesic $\gamma_{v(1)}$ which is dense in $(f, \mathcal{F}_{v_0}/G_0)$ and this last k-flat has no self intersection at $\pi(v_0)$. Moreover,

$$(df_{|\mathcal{F}_{v(1)}})^{-1} \circ df_{|\mathcal{F}_{v(0)}} = (d\pi_{|\mathcal{F}_{v(1)}})^{-1} \circ d\pi_{|\mathcal{F}_{v(0)}} : \mathcal{F}_{v(0)} \to \mathcal{F}_{v(1)}$$

is an isometry, which induces a bijection from the set $\mathcal{G}^{v(0)}$ of covering groups of arbitrary toral k-flat containing the geodesic $\gamma_{v(0)}$, into the set $\mathcal{G}^{v(1)}$ of covering groups of arbitrary toral k-flat containing $\gamma_{v(1)}$ (recall that, in our notation, any toral k-flat containing γ_w is regarded as $\mathcal{F}_w/G$ where G is a discrete subgroup of the group of translations of $\mathcal{F}_w$). In fact, if $\mathcal{F}_{v(0)}/G$ is a toral k-flat containing $\gamma_{v(0)}$, then $f : \mathcal{F}_{v(0)}/G \to M$ is a k-flat containing also $\gamma_{v(1)}$, which corresponds to a toral k-flat $\mathcal{F}_{v(1)}/G'$.

3.2. We keep the notation of this section. If $w \in \mathcal{O}^*$, let $\mathbb{Z}_w$ be the subset of $B(\mathcal{F})_w$ which consists of all $\mathbb{Z}$-basis of covering groups

of arbitrary toral k-flat containing the geodesic γ_w. We have that $\mathbb{Z}_w$ is countable. In fact, choose b^0, a $\mathbb{Z}$-basis of the covering group of the minimal toral k-flat. Then, any element of $\mathbb{Z}_w$ can be uniquely written as $b^0 \cdot D$, where $D \in \mathbb{Z}^{k \times k}$ and $det(D) \neq 0$. Let $A_w \subset \mathbb{R}^{k \times k}$ be defined as follows: $a = (a^1, \cdots, a^k) \in \mathbb{R}^{k \times k}$ belongs to A_w if $w = a^1 d\pi(b_1) + \cdots + a^k d\pi(b_k)$ for some $b = (b_1, \cdots, b_k) \in \mathbb{Z}_w$. The set A_w is clearly countable. Let

$$\tilde{\mathbb{Z}}_w = \{(d\pi(b_1), \cdots, d\pi(b_k)) : (b_1, \cdots, b_k) \in \mathbb{Z}_w, \}.$$

The set A_w, as well as $\tilde{\mathbb{Z}}_w$, is countable. Observe, under the notation and assumptions of Remark 3.1, that $\tilde{\mathbb{Z}}_{v(0)} = \tilde{\mathbb{Z}}_{v(1)}$. This observation together with the above defined sets will be used in the proof of the following crucial lemma, which shall be proved in Section 5.

Main Lemma. *Under the assumptions and notation of this and previous sections. Let $v_0 \in \mathcal{O}^*$ such that M is locally irreducible at $p = \pi(v_0)$. Then M is locally symmetric in an open neighbourhood of p.*

4. Smooth variations of k-flats

In this section we include some standard facts about variations of k-flats. Let T^k be a flat Riemannian torus and let $\mu : \mathbb{R} \times T^k \to M$ be differentiable such that, for each $t \in \mathbb{R}, \mu_t : T^k \to M$ is a totally geodesic immersion in the sense that μ_t maps geodesics of T^k into geodesic of M (we do not assume μ_t to be an isometric immersion). If $\gamma(s)$ is an arbitrary geodesic in T^k, then $t \mapsto \mu_t \circ \gamma$ defines a variation of geodesics. Thus, $\frac{\partial \mu}{\partial t}|_{t_0}$ is a field along μ_{t_0} which is a Jacobi field along any geodesic of the k-flat $T_t^k := \mu_{t_0} : T^k \to M$. Decompose $\frac{\partial \mu}{\partial t}|_{t_0} = X_{t_0}^N + X_{t_0}^T$, where $X_{t_0}^T$ is the orthogonal projection to the tangent space of T_t^k, and $X_{t_0}^N$ is its normal component (more precisely, $X_{t_0}^T(q)$ is the projection of $\frac{\partial \mu}{\partial t}|_{t_0}(q)$ to the subspace $d\mu_{t_0}(T_q T^k)$). It is standard and well known to show that $X_{t_0}^T$ is a Jacobi field along any geodesic of $T_{t_0}^k$. Since $T_{t_0}^k$ is compact, $X_{t_0}^T$ is parallel along any geodesic and hence it is a parallel field (this is indeed Hermann's lemma [10]). Let $\gamma(s)$ be a geodesic in T^k, and consider $\mu_t(\gamma(s))$. Then $\frac{D}{\partial t} d\mu_t(\gamma'(s)) = \frac{D}{\partial t}\frac{\partial}{\partial s}\mu_t(\gamma(s)) = \frac{D}{\partial s}\frac{\partial}{\partial t}\mu_t(\gamma(s)) = \frac{D}{\partial s}X_t^T(\gamma(s)) + \frac{D}{\partial s}X_t^N(\gamma(s)) = \frac{D}{\partial s}X_t^N(\gamma(s)) \perp d\mu_t(T_{\gamma(s)}T^k)$, because T_t^k is a totally geodesic immersion and hence its shape operator is identically zero. So, $\frac{D}{dt}d\mu_t(w)$ is perpendicular to $d\mu_t(T_u T^k)$, for all $w \in T_u T^k$ (cf. [7]).

5. Proofs of Main Results

Proof of Main Lemma.   Let $v_0 \in \mathcal{O}^*$ such that M is locally irre-
ducible at $\pi(v_0)$. We will show that the local holonomy group of M at
$\pi(v_0)$ does not act transitively on the sphere of radius $|v_0|$ of $T_{\pi(v_0)}M$.
Then, by the Berger-Simons holonomy theorem [3], [11], M must be
locally symmetric near p. Let $\varepsilon > 0$ be such that the ball of radius
ε and center v_0 in TM is contained in $\mathcal{O}^*$ such that the local holon-
omy group of M at $\pi(v_0)$ coincides with the holonomy group of the ball
$B_\varepsilon(\pi(v_0))$ of radius ε and center $\pi(v_0)$ in M. Let $c : [0, 1] \rightarrow B_\varepsilon(\pi(v_0))$
be a piecewise differentiable curve with $c(0) = \pi(v_0) = c(1)$, such that
$length(c) < \varepsilon$. Let $v(t)$ be the parallel transport of v_0 along $c(t)$. By
Section 3 the curve $v(t)$ in $\mathcal{O}^*$ can be lifted to a piecewise smooth curve
$b(t)$ in $B(\mathcal{F})$ such that, for each t, $b(t)$ is a $\mathbb{Z}$-basis of the covering group
G_t of some toral k-flat $F(t)$ which contains the geodesic $\gamma_{v(t)}$. Following
the notation of that section we can produce a piecewise differentiable
$g : [0, 1] \times F(0) \rightarrow M$, defined by $g_t(q+G_0) = exp_{c(t)} d\pi(l_t(q)) = f \circ l_t(q)$,
where $q \in \mathcal{F}_{v(0)}$ and $l_t : \mathcal{F}_{v(0)} \rightarrow \mathcal{F}_{v(t)}$ is the linear map which maps the
basis $b(0)$ into the basis $b(t)$ (see Section 3). We have, by Section 4, that
$\frac{D}{dt} dg_t(v_0)$ is perpendicular to the k-flat $g_t : F(0) \rightarrow M$ at $0+G_0$, i.e., it is
perpendicular to $dg_t(T_{\bar{0}}F(0))$ (observe that we have regarded v_0 as a hor-
izontal vector of $T_{v_0}TM$. Namely, we identify v_0 with $\frac{d}{dt}|_0\phi_t(v_0)$ where
ϕ_t denotes the geodesic flow on TM). In particular, $\langle \frac{D}{dt} dg_t(v_0), v(t) \rangle = 0$
because $v(t)$ belongs to the tangent space of the k-flat $g_t : F(0) \rightarrow M$
(recall that $F(t)$ is a k-flat containing $\gamma_{v(t)}$). Since $\frac{D}{dt} v(t) = 0$ we get
that $\frac{d}{dt}\langle dg_t(v_0), v(t) \rangle = \langle \frac{D}{dt} dg_t(v_0), v(t) \rangle + \langle dg_t(v_0), \frac{D}{dt} v(t) \rangle = 0$. Hence
$\langle dg_1(v_0), v(1) \rangle = \langle dg_0(v_0), v_0 \rangle = |v_0|^2$. Without loss of generality we
may assume that $|v_0| = 1$. So, we have

$$(I) \qquad \qquad \langle dg_1(v_0), v(1) \rangle = 1.$$

We will come back to this equality after some considerations. As-
sume now that the local holonomy group of M at $\pi(v_0)$ does act tran-
sitively on the unit sphere of $T_{\pi(v_0)}M$. Then, any element in the unit
sphere $S^{k-1}(v_0)$ of $df(\mathcal{F}_{v_0}) = dg_0(T_{\bar{0}}F(0))$ can be achieved by the par-
allel transport along some piecewise differentiable loop at $\pi(v_0)$. There
exists ε, which may be assumed to be the same as that used in the
begining of the proof, such that the parallel transport along piecewise
differentiable loops at $\pi(v_0)$ of length less than ε contains an open neigh-
bourhood of the local holonomy group (see [7, Appendix]. Though, in
this reference, the result we need is stated for piecewise C^1 loops, by

following the proof one can easily check that it is true for piecewise differentiable ones). This implies that there exists an open neighbourhood V of v_0 in $S^{k-1}(v_0)$ such that any element of V can be achived by the parallel transport along some piecewice differentiable loop at $\pi(v_0)$ of length less than ε. We need to consider the subset $\tilde{V}$ of V defined by the condition: $v \in V$ belongs to $\tilde{V}$ if the geodesic γ_v is dense in the toral k-flat $\mathcal{F}_{v_0}/G_0$. It is standard to show that $\tilde{V}$ is of second category in the sense of Baire (i.e., $\tilde{V}$ is the complement of countable many subsets of V, any of which closures has empty interior). In fact, if γ_v, $v \in V$, is not dense in the k-flat $\mathcal{F}_{v_0}/G_0$, then γ_v can be identified with a one-parameter subgroup which lies in a proper compact subgrop of $\mathcal{F}_{v_0}/G_0$ (observe that the closure of a subgroup of a torus is a subtorus). But there are only countable many subtori of a given torus. It is now not hard to conclude that $\tilde{V}$ is the complement in V of countable many proper linear spaces (intersected with V). Namely, those corresponding to the tangent spaces to proper subtori. Comming back to the equality (I); assume that $v(1) \in \tilde{V}$. Thus, by Remark 3.1, $df(\mathcal{F}_{v_0}) = df(\mathcal{F}_{v(1)})$. If $v_0 = \sum\limits_{i=1}^{k} a_i d\pi(b_i(0))$, then $dg_1(v_0) = \sum\limits_{i=1}^{k} a_i d\pi(b_i(1))$ (see Section 3), which belongs to the countable set

$$\bar{\mathbb{Z}}_{v_0} := \{a^1 \tilde{b}_1 + \cdots + a^k \tilde{b}_k : a \in A_{v_0}, \ \tilde{b} \in \tilde{\mathbb{Z}}_{v(1)}\}$$
$$= \{a^1 \tilde{b}_1 + \cdots + a^k \tilde{b}_k : a \in A_{v_0}, \ \tilde{b} \in \tilde{\mathbb{Z}}_{v_0}\}$$

of $T_p M$ (the above equality is due to $\tilde{\mathbb{Z}}_{v(1)} = \tilde{\mathbb{Z}}_{v(0)}$; see Section 3.2). Observe that the above defined set depends only on v_0. Then, by the formula (I), $v(1)$ belongs to the union of the countable family of hyperplanes (in $df(\mathcal{F}_{v_0}) = d\pi(\mathcal{F}_{v_0})$)

$$\{x :< z, x >= 1\}_{z \in \bar{\mathbb{Z}}_{v_0}}.$$

But, since $v(1)$ is arbitrary in $\tilde{V}$, we have that

$$\tilde{V} = \bigcup_{z \in \bar{\mathbb{Z}}_{v_0}} \{x :< z, x >= 1\} \cap \tilde{V},$$

which is imposible due to the well-known theorem of Baire (recall that $\tilde{V}$ is of second cathegory. Therefore, the local holonomy group does not act transitively on the unit sphere of $T_{\pi(v_0)} M$.   q.e.d.

Proof of the Theorem. Let Ω' be any open nonempty subset of Ω. By the Main Lemma there exists a nonempty subset Ω'' of Ω' such that

Ω'' is locally either symmetric (and irreducible) or reducible at any of its points. So, in the notation of Lemma 6.6, $\Omega_1 \cup \Omega_2$ is dense in M. Moreover, from the assumptions, Ω_1 is nonempty. Thus, by Lemma 6.6, M is locally symmetric. The fact that M is of the compact type follows from the next lemma.   q.e.d.

Lemma. *Let M^n ($n \geq 2$) be a complete locally symmetric Riemannian manifold whose universal cover $\tilde{M} \stackrel{pr}{\to} M$ is irreducible. Assume that there exists an open subset W of TM such that every geodesic γ_w, $w \in W$, is contained in a compact flat. Then $\tilde{M}$ is of the compact type.*

Proof. Suppose $\tilde{M}$ is a symmetric space of the noncompact type. We shall derive a contradiction. Let $v \in pr^{-1}(W)$ be a regular vector and let $G = K.A.N$ (K compact, A abelian and N nilpotent) be the Iwasawa decomposition with respect to v, where G is the connected component of the identity of the full group of the isometries of $\tilde{M}$. It is well known that $A.N$ is a subgroup of G which acts simply transitively on $\tilde{M}$. Moreover, $A.pr\,(v)$ is the maximal flat in $\tilde{M}$ which contains the geodesic γ_v, and the horosphere $N.pr\,(v)$ is perpendicular to $A.pr\,(v)$ at $pr\,(v)$. Let now $X \neq 0$ belong to the Lie algebra of N and let $\gamma_s(t) = exp\,(sX).\gamma_v(t)$. It is well known that the distance $\tilde{d}(\gamma_s(t), \gamma_v(t))$ tends to 0 as t goes to infinity, because the variation is perpendicular to the maximal flat. Then the distance in M, for fixed s, $d(pr\,(\gamma_s(t)), pr\,(\gamma_v(t)))$ tends to 0 as t goes to infinity, i.e., they are asymptotic.. We also have that, for small s any geodesic $pr\,(\gamma_s(t))$ is contained in a compact flat. Thus, all these geodesics must coincide. (In fact, let γ be a geodesic contained in a compact flat. Then, its clousure is an immerersed flat (F, i) whose image consists of all the accumulation points of γ. Let $\tilde{\gamma}$ be a geodesic asymptotic to γ which is contained in a compact flat. Then, the closure $cl(\tilde{\gamma})$ of $\tilde{\gamma}$ coincides with its points of accumulation. From the asymptotic condition it follows that $cl(\tilde{\gamma}) = i(F)$. Thus, $\tilde{\gamma}$ factorizes through (F, i) and consequently, by the asymptoticity, they both must coincide). Hence, for any small s, there must exist an element of the covering group of M which maps the geodesic $\gamma_s(t)$ into $\gamma_v(t)$. But this is impossible for small s because $\gamma_s(0)$ is close to $\gamma_v(0)$. A contradiction.   q.e.d.

Remark. The proofs can be simplified, if in Theorem A we further assume that the flats are isometric. In fact, we easily get that in the notation of Section 5 formula (I) implies that $dg_1(v_0) = v(1)$ due to the Cauchy- Schwarz equality with g_t chosen to be an isometry. This can be used to give a conceptual proof (different from that in [7]) of the local symmetry of compact k-flat homogeneous spaces [9].

6. The proof of Regularity Lemma

This section will be devoted to prove the Regularity Lemma of section 1.

Lemma 6.1. *Let M be a Riemannian manifold, let $i : F \to M$ be a compact (immersed) flat and let $p \in i(F)$. Then there exists a convex neighbourhood W of p in M such that, if $q \in i(F) \cap W$, then the minimizing geodesic segment γ in M from p to q is the image by i of some minimizing geodesic segment in F.*

The proof is standard and omitted.

From now on M will be a Riemannian manifold and $\tilde{\mathcal{O}}$ an open nonempty subset of of the tangent bundle $TM \xrightarrow{\pi} M$ with the following property: if $v \in \tilde{\mathcal{O}}$, then the geodesic γ_v is contained in a compact (immersed) flat (which without loss of generality is assumed to be a toral one; see Section 1) of dimension at least 2.

Let $\tilde{\mathcal{O}}_{j,m}^k$ for $j, k, m \in \mathbb{N}$ be the subset of $\tilde{\mathcal{O}}$ defined by the following property: $v \in \tilde{\mathcal{O}}$ belongs to $\tilde{\mathcal{O}}_{j,m}^k$ if there exists a k-dimensional toral flat $i_v : \mathbb{R}^k/\Gamma_v \to M$, containing the geodesic γ_v and such that:

i) the flat $(\mathbb{R}^k/\Gamma_v, i_v)$ has no self intersection at $\pi(v)$ and Γ_v admits a $\mathbb{Z}$-basis b with $\|b\| \leq j$,

ii) the open geodesic ball $B_{\frac{1}{m}}(\pi(v))$ satisfies the assertion of Lemma 6.1 for the flat $(\mathbb{R}^k/\Gamma_v, i_v)$ and the point $\pi(v)$.

We have the following lemma whose proof is standard and also omitted (see Section 2).

Lemma 6.2. $\tilde{\mathcal{O}}_{j,m}^k$ *is a closed subset of $\tilde{\mathcal{O}}$, for any $j, k, m \in \mathbb{N}$.*

If $\mathbb{V}$ is a Euclidean n-dimensional vector space, and S_1, S_2 are k-dimensional subspaces of $\mathbb{V}$, we want to measure how far is S_1 from S_2. It can be done in several ways. We will choose, for instance, the following way: choose a basis $v_1^i, \cdots, v_k^i$ of S_i $(i = 1, 2)$, and consider the (unoriented) line of $\Lambda^k(\mathbb{V})$ given by $l_i = \mathbb{R}.v_1^i \wedge \cdots \wedge v_k^i$. It is well known that l_i depends only on the subspace S_i and not on the chosen basis. We define the distance $d(S_1, S_2)$, between S_1 and S_2, as the angle ϕ $(0 \leq \phi \leq \frac{\pi}{2})$ between the unoriented lines l_1 and l_2 ($\Lambda^k(\mathbb{V})$ is endowed with the scalar product induced by $\mathbb{V}$).

Let $K_{j,m}^k$ for any $k, j, m \in \mathbb{N}$ be the subset of $\tilde{\mathcal{O}}$ defined by the property: $v \in \tilde{\mathcal{O}}$ belongs to $K_{j,m}^k$ if there exists a toral k-flat (F, i) containing the geodesic γ_v and satisfying the following:

i) The toral k-flat (F, i) admits a $\mathbb{Z}$-basis b with $\|b\| \leq j$.

ii) There exist $q_1, q_2 \in F$ such that $i(q_1) = i(q_2) = \pi(v)$ and $d(di(T_{q_1}F), di(T_{q_2}F)) \geq \frac{1}{m}$, where d is the distance between k-dimensional subspaces of $T_{i(v)}M$ defined above.

Lemma 6.3. *For any $j, k, m \in \mathbb{N}$, $K_{j,m}^k$ is a closed subset of $\tilde{\mathcal{O}}$. Moreover, the interior of $K_{j,m}^k$ is empty.*

Proof. The fact that $K_{j,m}^k$ is a closed subset of $\tilde{\mathcal{O}}$ is standard, so its proof is omitted. Let us then prove that its interior is empty. Let $v \in K_{j,m}^k$ and let (F, i) be a k-flat containing the geodesic γ_v which has the properties (i) and (ii) of the above paragraph. Let V be any neighbourhood of v in TM, and choose a non zero $v' \in di(TF) \cap V$ such that: **(a)** the k-flat (F, i) has no self-intersection at $\pi(v')$, **(b)** the geodesic $\gamma_{v'}$ is dense in F. By the compactness of F it is clear that such a v' does exist. From the density of the geodesic $\gamma_{v'}$ it follows that if (F', i') is a k-flat in M containing $\gamma_{v'}$, then $i'(F') = i(F)$. Hence, (F', i') has no self-intersection at $\pi(v')$. So, v' does not belong to $K_{j,m}^k$. Since V is arbitrary, the proof is finished.   q.e.d.

Lemma 6.4. *There exist $j, k, m \in \mathbb{N}$, $k \geq 2$ such that $\tilde{\mathcal{O}}_{j,m}^k$ has nonempty interior.*

Proof. We have that each of the sets on the right-hand side of $\tilde{\mathcal{O}} = \bigcup_{j,k,m \in \mathbb{N}, k \geq 2} (\tilde{\mathcal{O}}_{j,m}^k \cup K_{j,m}^k)$ is closed. By the well known Theorem of Baire at least one of them must have nonempty interior. But this cannot be the case of any of the $K_{j,m}^k$, due to Lemma 6.3.   q.e.d.

Let v belong to the (nonempty) interior of that $\tilde{\mathcal{O}}_{j,m}^k$ given by Lemma 6.4 and let (F, i) be a toral k-flat, containing γ_v, used for the definition of $\tilde{\mathcal{O}}_{j,m}^k$. We may assume, without loss of generality, that the geodesic γ_v is the image by i of a dense geodesic in the torus F. Choose $0 < t_1 < \cdots < t_k \in \mathbb{R}$ such that: *i)* $\gamma_v(t_1), \cdots, \gamma_v(t_k)$ all belong to the open geodesic ball $B_{\frac{1}{m}}(\pi(v))$, *ii)* $dexp_{\pi(v)}^{-1}(\gamma_v(t_1)), \cdots, dexp_{\pi(v)}^{-1}(\gamma_v(t_k))$ is a basis of the tangent space to the k-flat (F, i); recall the definition of the set $\tilde{\mathcal{O}}_{j,m}^k$ and in particular that (F, i) has no self-intersection at $\pi(v)$, where exp is the exponential map of M. Let $\mathcal{O}$ be a neighbourhood of v (in the interior of $\tilde{\mathcal{O}}_{j,m}^k$) such that $v' \in \mathcal{O}$ implies that $\gamma_{v'}(t_1), \cdots, \gamma_{v'}(t_k)$ belong to the open geodesic ball $B_{\frac{1}{m}}(\pi(v')$ and that $dexp_{\pi(v')}^{-1}(\gamma_{v'}(t_1)), \cdots, dexp_{\pi(v')}^{-1}(\gamma_{v'}(t_k))$ are linearly independent. Now choose, for any $v' \in \mathcal{O}$, a toral k-flat $(F_{v'}, i_{v'})$ just as in the definition of the set $\tilde{\mathcal{O}}_{j,m}^k$. Then the tangent space to the image of the k-flat

$(F_{v'}, i_{v'})$ at $\pi(v')$ is given by the linear span $H_{v'}$ of $dexp^{-1}_{\pi(v')}(\gamma_{v'}(t_1)), \cdots,$
$dexp^{-1}_{\pi(v')}(\gamma_{v'}(t_k))$. It is clear that the horizontal lift $\mathcal{F}_{v'}$ of $H_{v'}$ to $T\mathcal{O}$
defines a C^∞ distribution. This finishes the proof of the Regularity
Lemma of Section 1.

Remark 6.5. In the above notation, the k-flats associated with
elements of $\mathcal{O}$ have $\mathbb{Z}$-basis with norm less than or equal to j. This
makes Lemma 2.1 of Section 2 unnecessary. Nevertheless, for didactic
reasons we decide to include it. Also, its proof is ilustrative for the
standard facts (whose proofs are omitted) of this section.

The following technical lemma is used in the proof of the Theorem.

Lemma 6.6. *Let M be be a connected Riemannian manifold and
let*

$\Omega_1 = \{x \in M : M$ *is locally irreducible and symmetric near* $x\}$,
$\Omega_2 = \{x \in M : M$ *is locally reducible near* $x\}$.
*Asumme that $\Omega_1 \cup \Omega_2$ is dense in M and that Ω_1 is not empty. Then
$\Omega_1 = M$, i.e., M is locally symmetric everywhere.*

Proof. Assume there is $q \notin \Omega_1$ and let $c : [0, 1] \to M$ be curve with
$c(0) \in \Omega_1$, $c(1) = q$. Let $t_0 \in (0, 1)$ be the first value of the parameter
such that $c(t_0) \notin \Omega_1$. The curvature tensor $R_{c(t_0)}$ of M at $c(t_0)$ is al-
gebraically the same as (the algebraic class of) the curvature tensor of
the (locally irreducible and locally symmetric) connected component of
Ω_1 which contains $c(0)$. It is well known, from the theory of symmetric
spaces, that the skew-symmetric endomorphisms generated by such a
curvature tensor do not leave invariant any proper subspace of $T_{c(t_0)}M$;
in fact, this is a consequence of the Ambrose-Singer holonomy theorem.
There must then exist an open connected neighbourhood V of $c(t_0)$ in
M such that $V \cap \Omega_2$ is empty. In fact, if not, there would exist a se-
quence $\{x_n\}$ in Ω_2 converging to $c(t_0)$. Because of the local reduciblility,
the (skew-symmetric endomorphisms generated by the) curvature ten-
sors R_{x_n} leave the proper subspaces invariant, let us say S_n, of $T_{x_n}M$.
It is clear that S_n acummulates to a proper subspace of $T_{c(t_0)}$ which
must be invariant under the corresponding curvature tensor. This is a
contradiction which shows the existence of such a V. Since $\Omega_1 \cup \Omega_2$ is
dense in M, Ω_1 is dense in V. The curvature tensor is parallel in Ω_1 and
therefore must be parallel in V. Hence V is locally symmetric. From
the connectedness of V and the fact that the curvature tensor $R_{c(t_0)}$
does not leave proper subspaces invariant, it follows that V is locally

irreducible. Hence $V \subset \Omega_1$ and $c(t_0) \in \Omega_1$, a contradiction which shows that $\Omega_1 = M$.   q.e.d.

Remark 6.7. Let M be a simply connected analytic Riemannian manifold such that every geodesis is contained in a compact flat. Then, as remarked in the introduction, each irreducible factor has the same property. If M_1 is such a factor, it is locally irreducible at any point due to the analyticity. Then, by Lemma 2.2 (see Remark 2.3) there exists an open nonempty subset $\mathcal{O}^*$ of TM such that every geodesic with initial condition in this subset is contained in a compact k-flat, and the k-flats vary smoothly. If $k \geq 2$, by the Main Lemma, M_1 is locally symmetric in the projection of $\mathcal{O}^*$ and hence, by the analyticity, locally symetric around any point. If $k = 1$, then every geodesic with initial condition in $\mathcal{O}^*$ is closed and they all admit a common period; let us say τ. Let Φ_t, $t \in \mathbb{R}$, be the geodesic flow in TM. Then $\Phi_\tau : TM \to TM$ is a map which coincides with the identity map in the open subset $\mathcal{O}^*$. Hence, by the analyticity, Φ_τ is the identity map. So, every geodesic in M_1 is closed with common period τ.

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