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SOLITON EQUATIONS AND DIFFERENTIAL GEOMETRY

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1. Introduction

In this paper we study certain symplectic, Lie theoretic, and differential geometric properties of soliton equations.

The equation for harmonic maps from the Lorentz space $R^{1,1}$ to a symmetric space, and the equation for isometric immersions of space forms into space forms have many of the same properties as soliton equations—for example, they have Lax pairs and Bäcklund transformations—and two of the main goals of this paper are to find Hamiltonian formulations for these equations and to see how they fit into the general theory of soliton equations. As a by-product, we also find many new n-dimensional soliton systems.

It is well-known that most finite-dimensional, completely integrable, Hamiltonian systems can be obtained by applying the Adler-Kostant-Symes Theorem (AKS) to some Lie algebra $\mathcal{G}$ equipped with an ad-invariant, non-degenerate bi-linear form, and a decomposition $\mathcal{G} = \mathcal{K} + \mathcal{N}$. The symplectic manifold is some co-adjoint N-orbit $M \subset \mathcal{K}^\perp \simeq \mathcal{N}^*$ and the equation is the Hamiltonian equation of $f \mid M$, where $f : \mathcal{G} \to R$ is some suitable Ad-invariant function. For example, Kostant obtained the generalized Toda lattice ([23]) by applying the AKS theorem to $\mathcal{G} = \mathcal{K} + \mathcal{N}$ such that the corresponding G/K is a non-compact, symmetric space of split type (i.e., the rank of G/K is equal to the rank of G), and Adler and Van Moerbeke obtained the Euler-Arnold equation and Moser's geodesic flow on the ellipsoid ([6]) by applying the AKS theorem to $\mathcal{G} =$ the loop algebra of a simple Lie algebra.

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Many soliton equations can also be obtained from the AKS theorem. For example, Adler ([5]) showed that the KdV equation can be obtained by taking the Lie algebra of pseudo differential operators as $\mathcal{G}$; Drinfeld and Sokolov ([15]) obtained the Gelfand-Dikii equations and associated to each Kac-Moody algebra a hierarchy of soliton equations, and Flaschka, Newell and Ratiu ([17], [18]) obtained many remarkable properties of the soliton equations by taking $\mathcal{G}$ to be the algebra of loops on the loop algebra of $sl(n, C)$.

Another technique for generating soliton equations is the inverse scattering method, and in particular Beals and Coifman used this to obtain a hierarchy of evolution equations that will be of interest to us. To describe their equations, we first set up some notation. Given an inner product space V, we use $\mathcal{S}(V)$ to denote the space of smooth maps from R to V that lie in the Schwartz class. Let $\mathcal{U}$ be a semi-simple Lie algebra, $\mathcal{T}$ a fixed maximal abelian subalgebra of $\mathcal{U}$, $\mathcal{T}^{\perp}$ the orthogonal complement of $\mathcal{T}$ in $\mathcal{U}$ with respect to the negative of the Killing form of $\mathcal{U}$, and $a \in \mathcal{T}$ a fixed regular element. The following results are proved by Ablowitz, Kaup, Newell and Segur ([3], [4]) for $\mathcal{U} = sl(2)$, and by Beals and Coifman [7] and Sattinger [33] for arbitrary semi-simple $\mathcal{U}$:

(i) There exists a pair of compatible symplectic structures w^a and w_0 on $\mathcal{S}(\mathcal{T}^{\perp})$, which is a Hamiltonian pair in the sense of Magri [25] and Gel'fand and Dorfman [20], where $\omega^a(v_1, v_2) = (-\operatorname{ad}^{-1}(a)(v_1), v_2)$ and w_0 is defined by an integral-differential operator.

(ii) There exist a family of commuting Hamiltonians $\{F_{b.n} \mid b \in \mathcal{T}, n \in N\}$ on $\mathcal{S}(\mathcal{T}^{\perp})$ and a family of polynomial differential operators $\{Q_{b,n}(u) \mid b \in \mathcal{T}, n \in N\}$ on $\mathcal{S}(\mathcal{T}^{\perp})$ such that the Hamiltonian equation for $F_{b,n}$ is $u_t = [Q_{b,n}(u), a]$ with respect to w^a and is $u_t = [Q_{b,n+1}(u), a]$ with respect to w_0. ($u_t = [Q_{b,n+1}(u), a]$ will be called the *n-th flow equation associated to* U defined by a, b.)

For example, the second flow equation associated to $SU(2)$ defined by a, a (with $a = \operatorname{diag}(i, -i)$) is the non-linear Schrödinger equation (NLS), and the first flow associated to $SU(n)$ defined by a base a, b of $\mathcal{T}$ is the n-wave equation.

In this paper we show that the above results arise naturally from applying the AKS theorem to the Lie algebra of loops in an affine Kac-Moody algebra. Moreover, we have the following:

(1)

$$F_{b,n}(u) = -\frac{1}{n}\int_{-\infty}^{\infty} (Q_{b,n+1}(u), a)\, dx. \tag{1.1}$$

When $\mathcal{U} = su(2)$ this formula agrees with the one in [28].

(2) $(\mathcal{S}(\mathcal{T}^\perp), w^a)$ is a coadjoint orbit, and $(\mathcal{S}(\mathcal{T}^\perp), w_0)$ is a symplectic submanifold of some coadjoint orbit.

(3) Let U/K be a symmetric space of split type, and $\mathcal{U} = \mathcal{K} + \mathcal{P}$ the Cartan decomposition. Let $\mathcal{T}$ be a maximal abelian subalgebra of $\mathcal{U}$ that is contained in $\mathcal{P}$. So $\mathcal{K} \subset \mathcal{T}^\perp$. Let $i : \mathcal{S}(\mathcal{K}) \to \mathcal{S}(\mathcal{T}^\perp)$ denote the inclusion map. Then

   (i) $i^* w^a = 0$, and $i^* w_0$ is non-degenerate on $\mathcal{S}(\mathcal{K})$,

   (ii) $F_{b,2n} = 0$ on $\mathcal{S}(\mathcal{K})$,

   (iii) the Hamiltonian equation for the restriction of $F_{b,2k-1}$ to $\mathcal{S}(\mathcal{K})$ with respect to $i^*(w_0)$ is $u_t = [Q_{b,2k}(u), a]$.

It is known that the SGE is the equation for harmonic maps from $R^{1,1}$ to $SU(2)/SO(2)$ (cf.[40]). But SGE is also the -1-flow equation associated to the symmetric space $SU(2)/SO(2)$ ([28]). We generalize this to arbitrary symmetric space. In fact, given $a, b \in \mathcal{T}$, the -1-flow equation for a symmetric space U/K is $u_t = [a, g^{-1}bg]$ for $u \in \mathcal{S}(\mathcal{K})$, where $g : R \to K$ is the solution for $g^{-1}g_x = u$ and $\lim_{x\to-\infty} g(x) = e$. Moreover, we have:

(1) The -1-flow equation associated to U/K is Hamiltonian with respect to w_0.

(2) The -1-flow commutes with all the odd flows associated to U/K.

(3) Solutions of the -1-flow equation associated to U/K give rise to harmonic maps from $R^{1,1}$ to U/K.

Next we associate to each rank-n symmetric space U/K an n-dimensional first order system for maps from R^n to $\mathcal{P} \cap \mathcal{T}^\perp$. Given a basis $\{a_1, \ldots, a_n\}$, the following system of $n(n-1)/2$ first order equations for $v : R^n \to \mathcal{P} \cap \mathcal{T}^\perp$:

$$[a_i, v_{x_j}] - [a_j, v_{x_i}] = [[a_i, v], [a_j, v]], \qquad 1 \le i < j \le n, \tag{1.2}$$

is called the n-dimensional first order system associated to U/K. The relation of this system to the first flow equation associated to U/K is that for fixed $i \neq j$, the restriction of equation (1.2) to the $x_i x_j$ plane is the first flow equation associated to U/K defined by a_i, a_j on the symplectic manifold $(\mathcal{S}(\mathcal{T}^\perp), \omega_0)$ with $u = [a_i, v]$.

Using work of Beals and Coifman ([8]) on the inverse scattering for linear system, we prove that the n-dimensional system (1.2) can be solved globally for generic initial data on the line.

Now let $N^n(c)$ denote the space form of constant sectional curvature c. We prove that the equation for isometric immersion of $N^n(c)$ into $N^{2n}(c)$ with flat normal bundle and linearly independent curvature normals is exactly the n-dimensional system (1.2) associated to the symmetric space

$$M_c(n) = \begin{cases} SO(2n,1)/S(O(n)\times O(n,1)), & \text{if } c=-1,\\ SO(2n)/S(O(n)\times O(n)), & \text{if } c=0,\\ SO(2n+1)/S(O(n)\times O(n+1)), & \text{if } c=1. \end{cases}$$

In particular, this implies that:

(1) The system of equations for isometric immersions of $N^n(c)$ into $N^{2n}(c)$ is obtained by putting all the first flow equations associated to the symmetric space $M_c(n)$ together.

(2) To solve this system of n variables, it suffices to solve the first flow equations of two variables.

The literature in soliton theory is enormous, and we will only refer here to papers we use directly. There are many excellent survey articles and books; for example [2], [9], [28], [16], [29], where the reader can find more complete bibliographies.

The relations among Bäcklund transformations, Poisson loop group actions, and the inverse scattering for the j-th flow equation ($j = -1, 1, 2, ..$) will be studied in forthcoming joint papers with K. Uhlenbeck.

This paper is organized as follows: In section 2 we review the work of Beals, Coifman and Sattinger on evolution equations and inverse scattering and prove the formula (1.1). In section 3 we apply the AKS Theorem to the loop algebras of affine Kac-Moody algebras to obtain the Beals-Coifman evolution equations. In section 4 we study a sequence of compatible symplectic structures ω_r on $\mathcal{S}(\mathcal{T}^\perp)$. In section 5 we construct the -1-flow equation and study its relation to harmonic maps. In section 6 we associate to each rank-n symmetric space an n-dimensional system of first order partial differential equations and give some examples. Finally, in section 7 we explain the relation between the n-dimensional system and the equations for isometric immersions of a space form into a space form.

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2. First order linear systems and non-linear evolution equations

In this section, we state some of the results proved by Beals and Coifman ([7], [8], [9]) and Sattinger [33] concerning the relation between nonlinear evolution equations and spectral problems for first order linear systems.

Let $\mathcal{U}, \mathcal{T}$ be as in section 1, $(\,,)$ an Ad-invariant bilinear form on $\mathcal{U}$, and $\mathcal{T}^\perp$ the orthogonal complement of $\mathcal{T}$ in $\mathcal{U}$ with respect to $(\,,)$. Let

$$\langle u_1, u_2\rangle = \int_{-\infty}^{\infty} (u_1(x), u_2(x))dx$$

denote the L^2-inner product on $\mathcal{S}(\mathcal{T}^\perp)$, Given a functional $f : \mathcal{S}(\mathcal{T}^\perp) \to R$, let $\nabla f : \mathcal{S}(\mathcal{T}^\perp) \to \mathcal{S}(\mathcal{T}^\perp)$ denote the gradient of f with respect to the L^2-inner product on $\mathcal{S}(\mathcal{T}^\perp)$, i.e., $df_u(v) = \langle \nabla f(u), v\rangle$. Let $a \in \mathcal{T}$ be a regular element. Then $\mathrm{ad}(a) : \mathcal{T}^\perp \to \mathcal{T}^\perp$ is an isomorphism. So the two-form ω defined by

$$\omega_u(v_1, v_2) = \langle -\,\mathrm{ad}(a)^{-1}(v_1), v_2\rangle$$

is a symplectic structure on $\mathcal{S}(\mathcal{T}^\perp)$, and the corresponding Poisson structure for functions on $\mathcal{S}(\mathcal{T}^\perp)$ is given by $\{f_1, f_2\} = \langle [\nabla f_1, a], \nabla f_2\rangle$.

Given $u \in \mathcal{S}(\mathcal{T}^\perp)$, consider the following first order system for $\psi : R \to U$ with asymptotic condition at $-\infty$:

$$\psi_x(x,\lambda) = \psi(x,\lambda)(a\lambda + u(x)), \qquad \lim_{x\to-\infty} \exp(-a\lambda x)\psi(x,\lambda) = e, \tag{2.1}$$

where $\psi_x = \frac{\partial\psi}{\partial x}$, and $e \in U$ denotes the identity. Let

$$m(x,\lambda) = \exp(-a\lambda x)\psi(x,\lambda).$$

Then the equation (2.1) written in terms of m is

$$m_x(x,\lambda) = \lambda[m,a] + mu, \qquad \lim_{x\to-\infty} m(x,\lambda) = e. \tag{2.2}$$

Let $\mathcal{U} = su(n)$, $a = \mathrm{diag}(c_1, \ldots, c_n)$ a regular element in $su(n)$, (i.e., all the c_i's are distinct), and $\mathcal{T}$ be the space of diagonal matrices in $su(n)$. Set

$$\Gamma = \{\lambda \in C \mid \mathrm{Re}((c_j - c_k)\lambda) = 0, \quad \text{for some } j \neq k\}.$$

2.1 Theorem ([7]). *Given $u \in \mathcal{S}(\mathcal{T}^\perp)$, we have:*

(i) There exists a discrete set $D \subset C\backslash\Gamma$ such that for all $\lambda \in C\backslash(\Gamma\cup D)$ the system (2.2) has a unique solution m, $m(x,\lambda)$ is meromorphic in $\lambda \in C\setminus\Gamma$ and holomorphic in $C\setminus(\Gamma\cup D)$.

(ii) For generic $u \in \mathcal{S}(\mathcal{T}^\perp)$, the set D is finite and m has simple poles on D, and there exists a map $S : (\Gamma\setminus\{0\})\cup D \to U$ such that

$$\lim_{\lambda\to\lambda_0^+} m(x,\lambda) = e^{a\lambda_0 x} S(\lambda_0) e^{-a\lambda_0 x} \lim_{\lambda\to\lambda_0^-} m(x,\lambda)$$

for $\lambda_0 \in \Gamma\setminus\{0\}$, and for $\lambda_j \in D$

$$(I - (\lambda-\lambda_j)^{-1} e^{a\lambda_j x} S(\lambda_j) e^{-a\lambda_j x}) m(x,\lambda)$$

has a removable singularity at $\lambda = \lambda_j$. The scattering map $u \mapsto S$ is injective on an dense open subset of $\mathcal{S}(\mathcal{T}^\perp)$.

2.2 Theorem ([7], [8]). *Given $b \in \mathcal{T}$, let*

$$Q_b(x,\lambda) = m(x,\lambda)^{-1} b m(x,\lambda) = \psi(x,\lambda)^{-1} b \psi(x,\lambda).$$

Then $(Q_b)_x + [a\lambda + u, Q_b] = 0$. Moreover, Q_b has the asymptotic expansion: $Q_b(x,\lambda) \sim \sum_{j=0}^{\infty} Q_{b,j}(x)\lambda^{-j}$, as $\lambda \mapsto \infty$ such that

1. *(i)] $Q_{b,0} = b$,*

(ii) $(Q_{b,j})_x + [u, Q_{b,j}] = [Q_{b,j+1}, a]$ with $\lim_{x\to-\infty} Q_{b,j} = 0$ for all $j > 0$,

(iii) $u(x,t)$ satisfies equation

$$u_t = (Q_{b,j}(u))_x + [u, Q_{b,j}(u)] = [Q_{b,j+1}(u), a]$$

if and only if its scattering data $S(\lambda,t)$ satisfies the linear equation

$$S_t(\lambda,t) = \lambda^j [S(\lambda,t), b].$$

Henceforth, we will call $u_t = [Q_{b,j+1}(u), a]$ *the* j*-th flow equation associated to* U *defined by* a, b.

2.3 Theorem ([8],[33],[10]). *With the same notation as in Theorem 2.2, we have*

(i) $Q_{b,j}(u)$ *is a polynomial differential operator in* u,

(ii) $u \mapsto Q^{\perp}_{b,j}(u)$ *is a gradient vector field on* $\mathcal{S}(\mathcal{T}^{\perp})$*, where* $x^{\perp}$ *denotes the orthogonal projection of* x *onto* $\mathcal{T}^{\perp}$,

(iii) the hierarchy of flows

$$u_t = [Q_{b,j}(u), a], \qquad j \geq 1 \tag{2.3}$$

are commuting Hamiltonians with respect to the symplectic structure ω *on* $\mathcal{S}(\mathcal{T}^{\perp})$ *defined by* $\omega(v_1, v_2) = \langle -\operatorname{ad}(a)^{-1}(v_1), v_2\rangle$,

(iv) u *is a solution of* $u_t = [Q_{b,j+1}(u), a]$ *if and only if the following linear system is solvable for all* λ

$$\begin{cases} \psi_x &= \psi(a\lambda + u) \\ \psi_t &= \psi(b\lambda^j + Q_{b,1}(u)\lambda^{j-1} + Q_{b,2}(u)\lambda^{j-2} + \cdots + Q_{b,j}(u)). \end{cases}$$

Theorem 2.2 implies that the evolution equation (2.3) is linearized by the scattering map. Moreover, Beals and Sattinger proved in [10] that equation (2.3) is completely integrable by finding the action-angle variables.

The Hamiltonian function $F_{b,n} : \mathcal{S}(\mathcal{T}^{\perp}) \to R$ corresponding to the $(n-1)$-th flow equation $u_t = [Q_{b,n}(u), a]$ is

$$F_{b,n}(u) = \int_0^1 \langle Q_{b,n}(tu)^{\perp}, u\rangle dt = \int_0^1 \langle Q_{b,n}(tu), u\rangle dt,$$

because $\nabla F_{b,n}(u) = Q_{b,n}(u)^{\perp}$. It is proved in [33] that $m(\infty) = \lim_{x\to\infty} m$ is diagonal and

$$\sum_{n\geq 1} F_{b,n}(u)\lambda^n = (b, \log(m(\infty)) = \int_{-\infty}^{\infty} (b, \frac{d}{dx}\log m(x,\lambda))dx.$$

When $\mathcal{U} = sl(2)$ and $a = \operatorname{diag}(i, -i)$, there is a simple formula in terms of Q's (cf. [28]):

$$F_{a,n}(u) = \frac{2}{n}\int_{-\infty}^{\infty} h_{n+1}(u)dx, \qquad \text{where } Q_{a,n} = \begin{pmatrix} ih_n & e_n \\ f_n & -ih_n \end{pmatrix}.$$

The following Proposition generalizes this formula to arbitrary $\mathcal{U}$.

2.4 Proposition. *The Hamiltonian functional $F_{b,n} : \mathcal{S}(\mathcal{T}^{\perp}) \to R$ is*

$$F_{b,n}(u) = -\frac{1}{n}\int_{-\infty}^{\infty}(Q_{b,n+1}(u), a)dx. \tag{2.4}$$

Proof. We write $Q = Q_b$. First we claim that for all $v \in \mathcal{S}(\mathcal{T}^{\perp})$ we have

$$(v, \frac{dQ}{d\lambda}) = (dQ_u(v), a). \tag{2.5}$$

To see this, we note that $m_x = \lambda[m, a] + mu$. So

$$\left(\frac{dm}{d\lambda}\right)_x = [m, a] + \lambda\left[\frac{dm}{d\lambda}, a\right] + \frac{dm}{d\lambda}u.$$

Let $\xi = m^{-1}\frac{dm}{d\lambda}$. Then a direct computation gives

$$\xi_x + [a\lambda + u, \xi] = a - m^{-1}am.$$

Let $Q = Q_b = m^{-1}bm$, $U = a\lambda + u$ and $\eta = m^{-1}dm_u(v)$. Then

$$\begin{aligned} dQ(a) &= \frac{d}{d\lambda}Q = [Q, \xi], \\ dQ(v) &= [Q, \eta], \\ \eta_x + [U, \eta] &= v. \end{aligned}$$

Therefore

$$\begin{aligned} (dQ(a), v) - (dQ(v), a) &= ([Q, \xi], v) - ([Q, \eta], a) \\ &= (Q, [\xi, v] - [\eta, a]). \end{aligned} \tag{2.6}$$

Another direct computation implies that

$$[\xi, \eta]_x = [[\xi, \eta], U] + [\xi, v] - [\eta, a] - [m^{-1}am, \eta]. \tag{2.7}$$

Substituting equation (2.7) into equation (2.6), we get

$$\begin{aligned} &(dQ(a), v) - (dQ(v), a) \\ &\quad = (Q, [\xi, \eta]_x + [U, [\xi, \eta]] + [m^{-1}am, \eta]) \\ &\quad = (Q, [\xi, \eta])\,|_{-\infty}^{\infty} - (Q_x + [U, Q], [\xi, \eta]) + (Q, [m^{-1}am, \eta]). \end{aligned}$$

Now the first term is zero because $\lim_{x\to\infty} m$ being diagonal implies that $\xi(\infty)$ and $\eta(\infty)$ are diagonal so the bracket is zero. The second term is zero because $Q_x + [U, Q] = 0$. The third term

$$\begin{aligned}(Q, [m^{-1}am, \eta]) &= (m^{-1}bm, [m^{-1}am, \eta]) = ([m^{-1}bm, m^{-1}am], \eta)\\ &= (m^{-1}[b, a]m, \eta)\end{aligned}$$

is zero because $[a, b] = 0$. This proves equality (2.5).

But

$$\frac{dQ}{d\lambda} \sim -\sum_{n=1}^{\infty} nQ_n(u)\lambda^{-(n+1)}.$$

Comparing the coefficients of $\lambda^{-(n+1)}$ in (2.5), we get $\nabla F_{b,n+1} = -nQ_n^{\perp}$.

q.e.d.

2.5 Example ([28]). Let $\mathcal{U} = su(2)$ and $a = \mathrm{diag}(i, -i)$. Then $u \in \mathcal{S}(\mathcal{T}^{\perp})$ is of the form $u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$. So $\mathcal{S}(\mathcal{T}^{\perp})$ is naturally isomorphic to the space of Schwarz functions $q : R \to C$. Under this identification, $\omega_a(q_1, q_2) = \frac{i}{2}\langle q_1, q_2\rangle$, and the first and second flow equations are:

$$\begin{aligned} q_t &= q_x, \\ q_t &= \frac{i}{2}(q_{xx} + 2\,|\,q\,|^2 q). \end{aligned}$$

Note that the second flow equation is the NLS equation.

2.6 Example ([8]). Let $\mathcal{U} = su(n)$, and

$$a = \mathrm{diag}(a_1, \ldots, a_n), b = \mathrm{diag}(b_1, \ldots, b_n).$$

If a is regular, then the first flow equation $u_t = [Q_{b,2}(u), a]$ for $u = (u_{ij}) : R^2 \to \mathcal{T}^{\perp}$ is

$$(u_{ij})_t = \frac{b_i - b_j}{a_i - a_j}(u_{ij})_x + \sum_k \left(\frac{b_k - b_j}{a_k - a_j} - \frac{b_i - b_k}{a_i - a_k}\right)u_{ik}u_{kj}, \qquad i \neq j.$$

When $n = 3$, this is the three-wave equation studied in [26].

3. Adler-Kostant-Symes Theory

In this section, we show that $\mathcal{S}(\mathcal{T}^{\perp})$ is a coadjoint orbit, the symplectic form defined in section 2 is the orbit symplectic form, and the n-th flow equation (2.3) arises naturally when we apply the Adler-Kostant-Symes theorem to the loop algebras of the affine Kac-Moody algebras.

A two-form ω on M is called a *weak symplectic form* if ω is closed and the induced map from TM to T^*M defined by $v \mapsto \omega(\cdot, v)$ is injective (cf. [14]); and (M, ω) will be called a weak symplectic manifold. If M is a Riemannian manifold and there exists $A \in C^\infty(TM, TM)$ such that:

(i) $A_x : TM_x \to TM_x$ is an injective, skew adjoint operator for all $x \in M$,

(ii) $\omega_x(v_1, v_2) = \langle A_x(v_1), v_2\rangle$ is closed, where $\langle , \rangle$ is the inner product on TM_x defined by the metric,

then ω is a weak symplectic form, and the Hamiltonian vector fields X_f are defined for functions $f : M \to R$ whose gradient at $x \in M$ lies in the image of A_x. In fact,

$$X_f(x) = (A_x)^{-1}(\nabla f(x)).$$

The Poisson bracket for two such functions f_1, f_2 is given by

$$\{f_1, f_2\}(x) = \langle A_x^{-1}(\nabla f_1(x)), \nabla f_2(x)\rangle,$$

and $\omega(X_{f_1}, X_{f_2}) = \{f_1, f_2\}$. Note that when M is of finite dimension, a weak symplectic form is symplectic, but when M is of infinite dimension, this is not the case in general.

Let $M \subset \mathcal{G}^*$ be a co-adjoint orbit of G. Then the weak orbit symplectic structure ω on M is defined by

$$\omega_\ell(x(\ell), y(\ell)) = \ell([x, y]), \quad x, y \in \mathcal{G}, \ell \in M,$$

where $x(\ell)(z) = -\ell([x, z])$ is the vector field induced by the co-adjoint action.

Let $\mathcal{G}$ be a Lie algebra with a non-degenerate, Ad-invariant form $(,)$, $\mathcal{K}, \mathcal{N}$ subalgebras of $\mathcal{G}$, and $\mathcal{G} = \mathcal{K} + \mathcal{N}$ as direct sum of vector spaces. Let

$$\mathcal{K}^\perp = \{x \in \mathcal{G} \mid (x, \mathcal{K}) = 0\},$$
$$\mathcal{N}^\perp = \{x \in \mathcal{G} \mid (x, \mathcal{N}) = 0\}.$$

Then $\mathcal{K}^\perp$ and $\mathcal{N}^\perp$ can be identified as linear subspaces of the dual $\mathcal{N}^*$ and $\mathcal{K}^*$ via $(,)$ respectively. Under this identification, the coadjoint action of N on $\mathcal{K}^\perp$ is given by

$$g \cdot x = \pi_{\mathcal{K}^\perp}(gxg^{-1}), \quad \text{for } g \in N, x \in \mathcal{K}^\perp,$$

and the infinitesimal action is

$$\xi(x) = \pi_{\mathcal{K}^\perp}([\xi, x]), \quad \text{for } \xi \in \mathcal{N},\ x \in \mathcal{K}^\perp.$$

We need the following slight generalization of the Adler-Kostant-Symes theorem ([6],[23]).

3.1 Theorem. *Let $\mathcal{G} = \mathcal{K} + \mathcal{N}$ be as above, $M \subset \mathcal{K}^\perp$ a coadjoint N-orbit equipped with the natural weak orbit symplectic structure ω, and $V_i : M \to \mathcal{G}$ vector fields satisfying the condition $[V_i(x), x] = 0$ for all $x \in M$. Let $X_i(x)$ denote the vector field on M defined by $X_i(x) = \pi_{\mathcal{N}}(V_i(x))(x)$. Then*

(i) $X_i(x) = [x, \pi_{\mathcal{K}}(V_i(x))]$,

(ii) $\omega(X_1, X_2) = 0$,

(iii) if X_1, X_2 are Hamiltonian vector fields then X_1 and X_2 commute.

Proof. Since $[V_i(x), x] = 0$, we have $[\pi_{\mathcal{K}}(V_i(x)), x] + [\pi_{\mathcal{N}}(V_i(x), x] = 0$. Since $[\mathcal{K}, \mathcal{K}^\perp] \subset \mathcal{K}^\perp$, we obtain

$$\begin{aligned}
X_i(x) &= \pi_{\mathcal{K}^\perp}([\pi_{\mathcal{N}}(V_i(x)), x]) = \pi_{\mathcal{K}^\perp}([x, \pi_{\mathcal{K}}(V_i(x))] = [x, \pi_{\mathcal{K}}(V_i(x))],\\
\omega_x(X_1, X_2) &= \omega_x(\pi_{\mathcal{N}}(V_1(x))(x), \pi_{\mathcal{N}}(V_2(x))(x))\\
&= (x, [\pi_{\mathcal{N}}(V_1(x)), \pi_{\mathcal{N}}(V_2(x))]) = ([x, \pi_{\mathcal{N}}(V_1(x))], \pi_{\mathcal{N}}(V_2(x)))\\
&= ([\pi_{\mathcal{K}}(V_1(x), x], \pi_{\mathcal{N}}(V_2(x))) = (\pi_{\mathcal{K}}(V_1(x)), [x, \pi_{\mathcal{N}}(V_2(x))])\\
&= (\pi_{\mathcal{K}}(V_1(x)), [\pi_{\mathcal{K}}(V_2(x)), x]) = ([\pi_{\mathcal{K}}(V_1(x)), \pi_{\mathcal{K}}(V_2(x))], x),
\end{aligned}$$

which is zero because $x \in \mathcal{K}^\perp$. This proves (i) and (ii), and (iii) is a consequence of (ii). q.e.d.

Next we apply Theorem 3.1 to the loop algebra of a Kac-Moody algebra. First we recall some definitions and basic facts about Kac-Moody algebras (c.f. [22], [31]). Note that $\mathcal{S}(\mathcal{U})$ is a Lie algebra with bracket defined by $[u, v](x) = [u(x), v(x)]$. Let ρ be the 2-cocycle on $\mathcal{S}(\mathcal{U})$ defined by

$$\rho(u, v) = \int_{-\infty}^{\infty} (u_x(x), v(x))\, dx,$$

where $u_x = du/dx$. The affine algebra of type 1 based on $\mathcal{U}$ is

$$\hat{\mathcal{U}} = \mathcal{S}(\mathcal{U}) + Rc + Rd_x$$

with the bracket operation defined by

$$[u,v]^\wedge = [u,v] + \rho(u,v)c, \quad [d_x, u]^\wedge = u_x, \quad [c,u]^\wedge = [c,d_x]^\wedge = 0.$$

The bilinear form on $\hat{\mathcal{U}}$ defined by

$$\langle u_1 + r_1 c + s_1 d_x, u_2 + r_2 c + s_2 d_x\rangle = r_1 s_2 + r_2 s_1 + \int_{-\infty}^{\infty} (u(x), v(x))\, dx \tag{3.1}$$

is non-degenerate, ad-invariant, and has index 1. The Adjoint action on $\hat{\mathcal{U}}$ is given by

$$\begin{aligned} \mathrm{Ad}(g)(u) &= gug^{-1} + \langle gug^{-1}, g_x g^{-1}\rangle\, c, \\ \mathrm{Ad}(g)(d_x) &= -g_x g^{-1} + d_x - \frac{1}{2}\langle g_x g^{-1}, g_x g^{-1}\rangle\, c. \end{aligned}$$

In particular,

$$\mathrm{Ad}(g)(d_x + u) = d_x + gug^{-1} - g_x g^{-1} + \langle (gug^{-1} - \frac{1}{2} g_x g^{-1}), g_x g^{-1}\rangle c.$$

For $a \in \mathcal{U}$, we will also use a to denote the constant map with value a.

Now consider the Lie algebra

$$\mathcal{L} = L(\hat{\mathcal{U}}) = \{u(\lambda) = \sum_{n \le n_0} u_n \lambda^n \quad \text{for some } n_0 < \infty \mid u_n \in \hat{\mathcal{U}}\}$$

with Lie bracket defined pointwise by

$$[u,v]^\wedge(\lambda) = [u(\lambda), v(\lambda)]^\wedge.$$

For each integer r, we let $\langle\,,\rangle_r$ denote the Ad-invariant, non-degenerate bilinear form on $L(\hat{\mathcal{U}})$ defined by

$$\langle u, v\rangle_r = \sum_{n+m=r} \langle u_n, v_m\rangle, \tag{3.2}$$

where $u = \sum_n u_n \lambda^n$, $v = \sum_m v_m \lambda^m \in \mathcal{L}$ and $\langle\,,\rangle$ is the bi-linear form on $\hat{\mathcal{U}}$ defined by formula (3.1) above. For $k_1 < k_2$, we let

$$\mathcal{L}_{k_1,k_2} = \{u \in \mathcal{L} \mid u = \sum_{k_1 \le n \le k_2} u_n \lambda^n\}.$$

3.2 Proposition. *Let $\mathcal{L} = L(\hat{\mathcal{U}})$ be as above and equipped with the Ad-invariant bi-linear form $\langle,\rangle_{-1}$ defined by formula (3.2). Let $\mathcal{K} = \mathcal{L}_{0,\infty}$ and $\mathcal{N} = \mathcal{L}_{-\infty,-1}$. Then the following hold:*

(i) $\mathcal{K}$ *and* $\mathcal{N}$ *are subalgebras of* $\mathcal{L}$*, and* $\mathcal{L} = \mathcal{K} + \mathcal{N}$ *as a direct sum of vector spaces.*

(ii) Let $a \in \mathcal{T}$ *be a regular element of* $\mathcal{U}$*, identify* $a\lambda + d_x + u$ *as a linear functional* $\eta \mapsto \langle a\lambda + d_x + u, \eta\rangle_{-1}$ *on* $\mathcal{N}$*,* M *the coadjoint* N*-orbit at* $a\lambda + d_x$*, and* w *the orbit symplectic form on* M*. Then the map* $i : \mathcal{S}(\mathcal{T}^{\perp}) \to M$ *defined by* $i(u) = a\lambda + d_x + u$ *is an isomorphism.*

(iii) $w_{-1} = i^*(w)$ *is a symplectic form on* $\mathcal{S}(\mathcal{T}^{\perp})$*, and*

$$(\omega_{-1})_u(v_1, v_2) = \langle J_a^{-1}(v_1), v_2\rangle,$$

where $J_a : \mathcal{T}^{\perp} \to \mathcal{T}^{\perp}$ *is the isomorphism defined by* $J_a(v) = -\operatorname{ad}(a)(v) = [v, a]$.

Proof. (i) is obvious. For (ii), let $g = e + g_1\lambda^{-1} + g_2\lambda^{-2} + \cdots \quad \in N$. Then $g^{-1} = e + h_1\lambda^{-1} + h_2\lambda^{-2} + \cdots$. Equating the coefficients of λ^j in $gg^{-1} = e$, we get $h_1 = -g_1,\ \ h_2 = g_1^2 - g_2,\ \ldots$. Using the formula for Adjoint action of $\hat{\mathcal{U}}$ and the fact that $\langle (g_1)_x, a\rangle = 0$, we obtain

$$\begin{aligned} g \cdot (a\lambda + d_x) &= \pi_{\mathcal{K}}(\mathrm{Ad}(g)(a\lambda + d_x)) \\ &= a\lambda + d_x + [a, g_1]. \end{aligned}$$

Since a is regular, the isotropy subalgebra $\mathcal{U}_a = \mathcal{T}$ and $[a, \mathcal{U}] = \mathcal{T}^{\perp}$. This proves (ii), and (iii) follows from the definition of the orbit symplectic structure. q.e.d.

In order to apply Theorem 3.1 to M, we need to find vector fields $Q : M \to \mathcal{L}$ so that

$$[a\lambda + d_x + u, Q]^{\wedge} = (Q_x + [u, Q] + \lambda[a, Q]) + \rho(u_x, Q)c = 0. \tag{3.3}$$

This means that we need to find Q such that

$$\begin{aligned} Q_x + [u, Q] + \lambda[a, Q] = 0, \\ \langle u_x, Q\rangle = 0. \end{aligned} \tag{3.4}$$

Let $Q = m^{-1}bm = \psi^{-1}b\psi$ be as in Theorem 2.2. Then Q satisfies the first equation of (3.4). Next we claim $\langle u_x, Q\rangle = 0$. To prove this claim, it suffices to prove

$$\langle u_x, Q_k(u)\rangle = 0, \quad \text{for all} \ \ k \geq 0. \tag{3.5}$$

To see this, we use $(Q_j)_x + [u, Q_j] = [Q_{j+1}, a]$ and the ad-invariance of $\langle , \rangle$. Then a direction computation gives

$$\begin{aligned}\langle u_x, Q_k(u)\rangle &= -\langle u, (Q_k)_x\rangle = -\langle u, [Q_k, u] + [Q_{k+1}, a]\rangle = -\langle u, [Q_{k+1}, a]\rangle \\ &= -\langle [u, Q_{k+1}], a\rangle = \langle (Q_{k+1})_x - [Q_{k+2}, a], a\rangle = 0,\end{aligned}$$

which proves our claim.

If Q is a solution to (3.3), then $\lambda^j Q(u)$ is also a solution for any $j \geq 0$. Applying Theorem 3.1 to $\lambda^j Q$, we obtain the following evolution equation:

$$\begin{aligned}&(a\lambda + d_x + u)_t \\ &= [a\lambda + d_x + u, \quad \pi_{\mathcal{K}}(\lambda^j Q_b(u))]^\wedge \\ &= [a\lambda + d_x + u, \quad Q_{b,0}(u)\lambda^j + Q_{b,1}\lambda^{j-1} \cdots + Q_{b,j}(u)]^\wedge. \\ &= (Q_{b,j})_x + [u, Q_{b,j}] + \sum_{k=0}^{j-1} \lambda^{j-k}([a, Q_{b,k+1}] + (Q_{b,k})_x + [u, Q_{b,k}]) \\ &= (Q_{b,j})_x + [u, Q_{b,j}] = [Q_{b,j+1}(u), a],\end{aligned}$$

which is the j-th flow equation $u_t = [Q_{b,j+1}(u), a]$. This also proves that

$$(a\lambda + u)dx + (Q_{b,0}\lambda^j + Q_{b,1}\lambda^{j-1} + \cdots + Q_{b,j})dt$$

is flat for all λ if and only if u is a solution of the j-th flow equation.

4. A sequence of weak symplectic structures

In this section, we study a sequence of weak symplectic structures ω_r on the subspace $\mathcal{S}_r(\mathcal{T}^\perp)$ of $\mathcal{S}(\mathcal{T}^\perp)$ defined by

$$(\omega_r)_u(v_1, v_2) = \langle (J_r)_u^{-1}(v_1), v_2\rangle$$

and having the following properties:

(1) $J_{-1} = J_a = -\operatorname{ad}(a)$, and $J_r = J_a(J_a^{-1}J_0)^{r+1}$.

(2) Each ω_r is induced from a weak symplectic form of some coadjoint orbit.

(3) The hierarchy of flows (2.3) are commuting Hamiltonians with respect to the weak symplectic structure ω_r.

(4) The $(n-1)$-th flow equation satisfies the Lenard-Magri relation:

$$\begin{aligned} u_t &= [Q_{b,n}(u), a] = J_{-1}(\nabla F_{b,n}(u)) \\ &= J_0(\nabla F_{b,n-1}) = J_1(\nabla F_{b,n-2}) = \cdots = J_{n-2}(\nabla F_{b,1}(u)). \end{aligned}$$

Let $P_u : \mathcal{S}_u(\mathcal{T}^\perp) \to \mathcal{S}(\mathcal{T}^\perp)$ be the operator defined by

$$P_u(v) = v_x + [u, v]^\perp - [u, \int_{-\infty}^x [u(y), v(y)]^\mathcal{T} dy], \tag{4.1}$$

where $\mathcal{S}_u(\mathcal{T}^\perp) = \{v \in \mathcal{S}(\mathcal{T}^\perp) / \int_{-\infty}^\infty [u(y), v(y)]^\mathcal{T} dy = 0\}$. We claim that P_u is injective. To see this, let

$$\tilde{v}(x) = v(x) - \int_{-\infty}^x [u(y), v(y)]^\mathcal{T} dy.$$

Then $P_u(v) = (\tilde{v})_x + [u, \tilde{v}]$. If $P_u(v) = 0$, then $(\tilde{v})_x + [u, \tilde{v}] = 0$. This implies that $\tilde{v}(x)$ is conjugate to $\tilde{v}(0)$ for all x. But $\lim_{x\to-\infty} \tilde{v}(x) = 0$. So $v = 0$ and P_u is injective.

4.1 Remark. Given $v \in \mathcal{S}(\mathcal{T}^\perp)$, if there exists $\tilde{v} \in \mathcal{S}(\mathcal{U})$ such that $\tilde{v}^\perp = v$, $\lim_{x\to-\infty} \tilde{v}(x) = 0$ and $(\tilde{v})_x + [u, \tilde{v}] \in \mathcal{S}(\mathcal{T}^\perp)$, then $P_u(v) = \tilde{v}_x + [u, \tilde{v}]$.

4.2 Proposition. *Let $\mathcal{L} = L(\hat{\mathcal{U}})$, $\mathcal{K} = \mathcal{L}_{0,\infty}$, $\mathcal{N} = \mathcal{L}_{-\infty,-1}$ be as in Proposition 3.2, and $a \in \mathcal{T}$ a regular element. Let $\mathcal{L}$ be equipped with the bi-linear form $\langle , \rangle_{-r}$ with $r > 0$. Identify $a\lambda + d_x + u$ as a linear functional on $\mathcal{N}$ defined by $\eta \mapsto \langle a\lambda + d_x + u, \eta\rangle_{-r}$. Let $\tilde{M}_{-r}$ be the coadjoint N-orbit through $a\lambda + d_x$, $\tilde{w}_{-r}$ the orbit symplectic form on $\tilde{M}_{-r}$ and $\mathcal{S}_r(\mathcal{T}^\perp)$ the space of $u \in \mathcal{S}(\mathcal{T}^\perp)$ such that $a\lambda + d_x + u \in \tilde{M}_{-r}$. Then*

(i) $a\lambda + d_x + u \in \tilde{M}_{-r}$ for all $u \in \mathcal{S}_r(\mathcal{T}^\perp)$, so $i : \mathcal{S}(\mathcal{T}^\perp) \to \tilde{M}_{-r}$ defined by $u \mapsto a\lambda + d_x + u$ is an embedding,

(ii) $w_{-r} = i^*(\tilde{w}_{-r})$ is a weak symmetric form on $\mathcal{S}_r(\mathcal{T}^\perp)$ given by $(w_{-r})_u(v_1, v_2) = \langle (J_{-r})_u^{-1}(v_1), v_2\rangle$, where $(J_{-r})_u = J_a(J_a^{-1}P_u)^{-r+1}$, $J_a(v) = [v, a]$ and P_u is defined by formula (4.1).

Proof. For $\xi = \sum_{j\geq 1} \xi_j \lambda^{-j} \in \mathcal{N}$, the vector field induced by the coadjoint N-action is given by

$$\begin{aligned} \xi(a\lambda + d_x + u) &= \pi_{\mathcal{K}^\perp}([\xi, a\lambda + d_x + u]^\wedge) \\ &= [\xi_1, a] + \sum_{j=1}^{r-1} \{-(\xi_j)_x - [u, \xi_j] + [\xi_{j+1}, a]\}\lambda^{-j}. \end{aligned}$$

So $v_1 = \xi(a\lambda + d_x + u)$ if and only if

$$\begin{aligned} &v_1 = [\xi_1, a], \\ &-(\xi_j)_x - [u, \xi_j] + [\xi_{j+1}, a] = 0, \quad 1 \leq j \leq r-1, \end{aligned}$$

which is equivalent to

$$\xi_1 = J_a^{-1}(v_1) \in \mathcal{S}_u, \quad J_a^{-1} P_u(\xi_j^\perp) = \xi_{j+1}^\perp, \quad \text{for } 1 \leq j \leq r-1. \tag{4.3}$$

In particular,

$$\xi_r = (J_a^{-1} P_u)^{r-1} J_a^{-1}(v_1) = (J_{-r})_u^{-1}(v_1).$$

Let $M_{-r} = (a\lambda + d_x + \mathcal{S}(\mathcal{T}^\perp)) \cap \tilde{M}_{-r}$. Given any v_1 in $\mathcal{S}(\mathcal{T}^\perp)$ we can find $\xi_1, \ldots, \xi_r$ satisfy equation (4.3), and it follows that $T(M_{-r})_{a\lambda+d+u} = \mathcal{S}(\mathcal{T}^\perp)$. This implies that $M_{-r} = a\lambda + d_x + \mathcal{S}(\mathcal{T}^\perp)$, and hence proves (i).

If $\eta = \sum_{j\geq 1} \eta_j \lambda^{-j} \in \mathcal{N}$ and $\eta(a\lambda + d_x + u) = v_2 \in T(M_{-r})_u$, then

$$\begin{aligned} &v_2 = [\eta_1, a], \\ &-(\eta_j)_x - [u, \eta_j] + [\eta_{j+1}, a] = 0, \quad 1 \leq j \leq r-1. \end{aligned}$$

Next we compute the induced weak orbit symplectic form w_r directly

$$\begin{aligned} (w_{-r})_u(v_1, v_2) &= \langle [\xi, \eta]^\wedge, a\lambda + d_x + u \rangle_{-r} \\ &= \sum_{i=0}^{r-1} \langle [\xi_{r-i}, \eta_{i+1}], a \rangle + \sum_{i=1}^{r-1} \langle [\xi_{r-i}, \eta_i], u \rangle + \rho(\xi_{r-i}, \eta_i), \\ &= \langle [\xi_r, \eta_1], a \rangle + \sum_{i=1}^{r-1} \langle \xi_{r-i}, [\eta_{i+1}, a] + [\eta_i, u] - (\eta_i)_x \rangle \\ &= \langle \xi_r, [\eta_1, a] \rangle = \langle \xi_r, v_2 \rangle = \langle (J_{-r})_u^{-1}(v_1), v_2 \rangle. \end{aligned}$$

This proves (ii). q.e.d.

Similarly, we get

4.3 Proposition. *Let $\mathcal{L}$, $\mathcal{K}$, $\mathcal{N}$ be as in Proposition 4.2, and $a \in \mathcal{T}$ a regular element. Let $\mathcal{L}$ be equipped with the bilinear form $\langle , \rangle_r$ with $r > 0$. Identify $a\lambda + d_x + u$ as a linear functional $\eta \mapsto \langle a\lambda + d_x + u, \eta \rangle_r$ on $\mathcal{K}$. Let $\tilde{M}_r$ denote the coadjoint K-orbit through $a\lambda + d_x$, $\mathcal{S}_r(\mathcal{T}^\perp)$ the space of $u \in \mathcal{S}(\mathcal{T}^\perp)$ such that $a\lambda + d_x + u \in \tilde{M}_r$ and $\tilde{w}_r$ the orbit symplectic structure on $\tilde{M}_r$. Then* *(i)* $i_r : \mathcal{S}_r(\mathcal{T}^\perp) \to \tilde{M}_r$ *defined by* $i(u) = a\lambda + d_x + u$ *is an embedding,*

(ii) $w_r = i^*(\tilde{w}_r)$ *is a weak symplectic form on* $\mathcal{S}_r(\mathcal{T}^\perp)$ *and* $(w_r)_u(v_1, v_2) = \langle (J_r)_u^{-1}(v_1), v_2\rangle$, *where* $(J_r)_u = J_a(J_a^{-1}P_u)^{r+1}$.

4.4 Proposition. *Let* $\mathcal{L} = L(\hat{\mathcal{U}})$ *be equipped with the bi-linear form* $\langle,\rangle_0$, $\mathcal{K} = \mathcal{L}_{1,\infty}$ *and* $\mathcal{N} = \mathcal{L}_{-\infty,0}$. *Identify* $d_x + u$ *as a linear functional on* $\mathcal{N}$ *mapping* η *to* $\langle d_x + u, \eta\rangle_0$. *Let* $\tilde{M}_0$ *be the coadjoint* N*-orbit at* d_x, $\mathcal{S}_0(\mathcal{T}^\perp)$ *the space of* $u \in \mathcal{S}(\mathcal{T}^\perp)$ *such that* $d_x + u \in \tilde{M}_0$, *and* $\tilde{w}_0$ *the orbit symplectic form on* $\tilde{M}_0$. *Then the following hold:*

(i) $d_x + \mathcal{S}_0(\mathcal{T}^\perp) \subset \tilde{M}_0$.

(ii) Let $i : \mathcal{S}_0(\mathcal{T}^\perp) \to \tilde{M}_0$ *denote the embedding defined by* $i(u) = d_x + u$, *and* $w_0 = i^*(\tilde{w}_0)$; *then* w_0 *is a weak symplectic form on* $\mathcal{S}_0(\mathcal{T}^\perp)$ *and* $w_0(v_1, v_2) = \langle P_u^{-1}(v_1), v_2\rangle$.

The following Proposition follows from the fact that $Q_{b,j}(u) \in \mathcal{S}(\mathcal{U})$ together with the recursive formula for Q_j's (Theorem 2.2) and Remark 4.1.

4.5 Proposition. *Let* $u \in \mathcal{S}(\mathcal{T}^\perp)$, *and* $Q_{b,n}(u)$ *as in Theorem 2.2. Then*

(i) $P_u(Q_{b,n}^\perp(u)) = [Q_{b,n+1}(u), a]$,

(ii) $J_r(\nabla F_{b,j}(u)) = [Q_{b,j+r+1}(u), a]$, *if* $j + r \geq 0$,

(iii) $[Q_{b,n}(u), a] = J_0(\nabla F_{b,n-1}) = J_1(\nabla F_{b,n-2}) = \cdots = J_{n-2}(\nabla F_{b,1})$, *if* $n \geq 1$.

4.6 Proposition. *Let* $F_{b,n}(u)$ *be as in Proposition 2.4, and* $r \geq -1$. *Then*

$$\{F_{b,n} \mid b \in \mathcal{T}, n \geq 1\}$$

is a family of commuting Hamiltonians with respect to ω_r, *and the Hamiltonian equation for* $F_{b,n}(u)$ *is* $u_t = [Q_{b,n+r+1}(u), a]$.

Proof. Given $m, n \geq 1$ and $b_1, b_2 \in \mathcal{T}$, we have

$$\begin{aligned} 0 =& \{F_{b_1,m}, F_{b_2,n}\}_{-1} = \langle [Q_{b_1,m}(u), a], Q_{b_2,n}(u)\rangle \\ =& \langle a, [Q_{b_2,n}, Q_{b_1,m}]\rangle = 0. \end{aligned} \tag{4.4}$$

But

$$\begin{aligned}\{F_{b_1,m}, F_{b_2,n}\}_r &= \langle J_r(\nabla F_{b_1,m}), \nabla F_{b_2,n}\rangle \\ &= \langle [Q_{b_1,n+r+1}, a], Q_{b_2,m}\rangle = \langle [Q_{b_2,m}, Q_{b_1,n+r+1}], a\rangle \\ &= 0. \qquad \text{q.e.d.}\end{aligned}$$

The above constructions of weak symplectic structures work exactly the same way when a is chosen to be any fixed element of $\mathcal{U}$, and $\mathcal{T}$ is the centralizer of a in $\mathcal{U}$, i.e., $\mathcal{T} = \{x \in \mathcal{U} \mid [a, x] = 0\}$.

4.7 Remark. Using the Lie algebra $\mathcal{L}$ of loops in an affine Kac-Moody algebra, Reyman and Semenov-Tian-Shansky ([32]) prove that there exist two compatible Poisson structures on the space $\mathcal{S}(\mathcal{U})$ defined by

$$\{F, G\}_0(u) = \langle (\nabla F(u))_x + [u, \nabla F], \nabla G\rangle, \qquad \{F, G\}_a = \langle ([\nabla F, a], \nabla G\rangle.$$

Note that the phase space $\mathcal{S}(\mathcal{T}^\perp)$ we are working on is a subspace of $\mathcal{S}(\mathcal{U})$, and $(\mathcal{S}_0(\mathcal{T}^\perp), w^a)$ is a symplectic leaf of $(\mathcal{S}(\mathcal{U}), \{,\}_a)$, but $(\mathcal{S}(\mathcal{T}^\perp), w_0)$ is not a Poisson submanifold of $\mathcal{S}(\mathcal{U})$.

The theorem below follows by a direct computation.

4.8 Theorem. *Let U/K be a symmetric space, $\sigma : \mathcal{U} \to \mathcal{U}$ the corresponding involution, and $\mathcal{U} = \mathcal{K} + \mathcal{P}$ the Cartan decomposition. Let $\mathcal{A}$ be a maximal abelian subalgebra in $\mathcal{P}$, $a \in \mathcal{A}$ a regular element with respect to the $Ad(K)$-action on $\mathcal{P}$, and $\mathcal{K} = \mathcal{K}_a + \mathcal{K}_1$ the orthogonal decomposition, where $\mathcal{K}_a = \{x \in \mathcal{K} \mid [x, a] = 0\}$. Let $Q_b(u) \sim \sum_{j=0}^{\infty} Q_{b,j}(u)\lambda^{-j}$ be as in Theorem 2.2, and $F_{b,j}$ the corresponding Hamiltonian defined by formula (2.4). If $u \in \mathcal{S}(\mathcal{K}_1)$, then the following hold:*

(1) $Q_{b,n}(u)$ is in $\mathcal{P}$ if n is even and is in $\mathcal{K}$ if n is odd,

(2) $i^*(\omega_r)$ is a weak symplectic form if r is even, and is zero if r is odd, where $i : \mathcal{S}(\mathcal{K}_1) \to \mathcal{S}(\mathcal{T}^\perp)$ is the inclusion,

(3) $F_{b,j} = 0$ on $\mathcal{S}(\mathcal{K}_1)$ if j is even,

(4) $\{F_{b,2m+1} \mid b \in \mathcal{T}, m \geq 0\}$ are commuting Hamiltonians with respect to $i^*(\omega_{2r})$ for all $r \geq 0$, and the corresponding Hamiltonian equations are $u_t = [Q_{2(m+r)+2}(u), a]$ for $m \geq 0$.

Let $\mathcal{L}^\sigma(\hat{\mathcal{U}}) = \{u \in \mathcal{L}(\hat{\mathcal{U}}) \mid \sigma(u(-\lambda)) = u(\lambda)\}$ denote the Lie subalgebra of $\mathcal{L}(\hat{\mathcal{U}})$ twisted by σ. Then $u(\lambda) = \sum_j u_j \lambda^j$ lies in $\mathcal{L}^\sigma(\hat{\mathcal{U}})$ if and only if $u_j \in \mathcal{K}$ for all even j and $u_i \in \mathcal{P}$ for odd j. Let

$$\mathcal{L}^\sigma_{0,\infty}(\hat{\mathcal{U}}) = \mathcal{L}^\sigma(\hat{\mathcal{U}}) \cap \mathcal{L}_{0,\infty}(\hat{\mathcal{U}}),$$
$$\mathcal{L}^\sigma_{-\infty,-1}(\hat{\mathcal{U}}) = \mathcal{L}^\sigma(\hat{\mathcal{U}}) \cap \mathcal{L}_{-\infty,-1}(\hat{\mathcal{U}}).$$

Then the above Theorem can also be obtained by applying Theorem 3.1 to the Lie algebra splitting $\mathcal{L}^\sigma(\hat{\mathcal{U}}) = \mathcal{L}^\sigma_{0,\infty} + \mathcal{L}^\sigma_{-\infty,-1}$ and vector fields $\lambda^{2m+1} Q_b(u)$.

4.9 Example. Suppose $\mathcal{U} = su(2)$ and $a = \mathrm{diag}(i,-i)$. Identifying $\mathcal{S}(\mathcal{T}^\perp)$ as $\mathcal{S}(C)$ via

$$\begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \mapsto q,$$

we have:

$$\begin{aligned}
(J_0)_q(r) &= r_x + 4iq \int_{-\infty}^x \mathrm{Im}(\bar{q}r), \\
(J_{-1})_q(r) &= -2ir, \\
(J_r)_q(r) &= J_{-1}(J_{-1}^{-1}(J_0)_q)^{r+1}.
\end{aligned}$$

Let $\sigma : su(2) \to su(2)$ be the involution $\sigma(X) = -X^t$. Then $\mathcal{K} = so(2)$, $\mathcal{K}_a = 0$, $\mathcal{K}_1 = \mathcal{K}$,

$$\begin{aligned}
\mathcal{S}(\mathcal{K}_1) &= \left\{ i \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix} \mid q \in \mathcal{S}(R) \right\} \simeq \mathcal{S}(R) \\
(J_0)_q(r) &= r_x, \\
(J_{-2})_q^{-1}(r) &= -\frac{1}{4} r_x - q \int_{-\infty}^x q(y) r(y) dy,
\end{aligned}$$

and the third flow equation associated to $SU(2)/SO(2) = S^2$ defined by a, which is written in terms of q, is the MKdV equation

$$q_t = \frac{1}{4}(q_{xxx} + 6q^2 q_x).$$

5. The -1-flow equation $u_t = [a, Q_{b,-1}(u)]$

The main purpose of this section is to describe the -1-flow equation.

5.1 Theorem. *Let g be the solution of $g^{-1} g_x = u$ with $\lim_{x\to-\infty} g(x) = e$, and $Q_{b,-1}(u) = g^{-1} b g$. Then:*

(i) $(Q_{b,-1})_x + [u, Q_{b,-1}] = 0$ *and* $\lim_{x\to-\infty} Q_{b,-1}(u)(x) = b$,

(ii) $P_u(Q^{\perp}_{b,-1}(u)) = [b, u]$, *where* P_u *is the operator defined by formula (4.1),*

(iii) $\{F_{b,n} \mid b \in \mathcal{T}, n \geq 1\}$ *is a family of commuting Hamiltonians with respect to* ω_{-2}*, and the corresponding Hamiltonian equations are*

$$u_t = J_{-2}(\nabla F_{b,1})(u) = [a, Q_{b,-1}(u)],$$
$$u_t = J_{-2}(\nabla F_{b,n})(u) = [Q_{b,n-1}(u), a], \quad \text{for } n \geq 2,$$

(iv) u *is a solution of* $u_t = [a, Q_{b,-1}(u)]$ *if and only if the connection* $\theta_\lambda = (a\lambda + u)dx + \lambda^{-1}Q_{b,-1}(u)dt$ *is flat for all* λ*.*

Proof. (i) is obvious. To prove (ii), let $\xi = Q_{b,-1}(u) - b$. Because $(Q_{b,-1})_x + [u, Q_{b,-1}] = 0$, we have

$$\xi_x + [u, \xi] = (Q_{b,-1})_x + [u, Q_{b,-1} - b] = -[u, b].$$

Then (ii) follows from Remark 4.1.

To prove (iii), we note that $[a, Q_{b,1}(u)] = [a, J_a^{-1}J_b(u)] = [b, u]$, so that

$$\begin{aligned}(J_{-2})_u(\nabla F_{b,1}(u)) &= J_a(J_a^{-1}P_u)^{-1}(Q_{b,1}(u)) = J_aP_u^{-1}J_a(Q_{b,1}(u)) \\ &= J_aP_u^{-1}([Q_{b,1}(u), a] = J_aP_u^{-1}([u, b]) \\ &= -J_a(Q^{\perp}_{b,-1}(u))) = [a, Q_{b,-1}(u)].\end{aligned}$$

For $n \geq 2$, Proposition 4.5 gives $J_{-2}(\nabla F_{b,n})(u) = [Q_{n-1}(u), a]$. This completes the proof of (iii). Statement (iv) follows from a direct computation. q.e.d.

The equation $u_t = [a, Q_{b,-1}(u)]$ for $u : R^2 \to \mathcal{T}^{\perp}$ is called *the* -1*-flow equation defined by* a, b *associated to* U. If $g : R^2 \to U$ is as in Theroem 5.1, then the -1-flow equation can be written as

$$(g^{-1}g_x)_t = [a, g^{-1}bg]. \tag{5.1}$$

5.2 Proposition. *Let* $F_{b,-2}(u) = \langle Q_{b,-1}(u), a\rangle$*. Then the Hamiltonian equation for* $F_{b,-2}$ *with respect to the symplectic structure* w_0 *is the* -1*-flow equation* $u_t = [Q_{b,-1}(u), a]$*.*

Proof. Let $g(u)$ be as in Theorem 5.1, and $v \in \mathcal{S}(\mathcal{T}^{\perp})$. Set $\eta = g^{-1}dg_u(v)$. It follows from a direct computation that $\eta_x + [u, \eta] = v$ and

$\eta(-\infty) = 0$. Since $v \in \mathcal{S}(\mathcal{T}^\perp)$, by Remark 4.1, $P_u(\eta^\perp) = v$. A direct computation then gives

$$\begin{aligned} d(F_{b,-2})_u(v) &= -\langle [g^{-1}bg, g^{-1}dg_u(v)], a\rangle \\ &= -\langle g^{-1}dg_u(v), [a, g^{-1}bg]\rangle = -\langle P_u^{-1}(v), [a, g^{-1}bg]\rangle \\ &= -w_0(v, [a, g^{-1}bg]\rangle. \end{aligned}$$

So the Hamiltonian field of $F_{b,-2}$ is $u \mapsto [a, g^{-1}bg] = [a, Q_{b,-1}(u)]$.

q.e.d.

Let $\mathcal{U}, \mathcal{K}, \mathcal{P}, \mathcal{A}, \mathcal{K}_a, \mathcal{K}_1$ be as in Theorem 4.8, and $b \in \mathcal{A}$. If $u \in \mathcal{S}(\mathcal{K}_1)$, then $Q_{b,-1}(u) \in \mathcal{S}(\mathcal{P})$, and the -1-flow equation associated to the symmetric space U/K with respect a, b is the equation $u_t = [a, Q_{b,-1}(u)]$ for $u : R^2 \to \mathcal{K}_1$. If U/K is of split type, then $\mathcal{K}_a = 0$ and the -1-flow equation is for $u : R^2 \to \mathcal{K}$.

5.3 Example. To write down the -1-flow associated to $SU(2)/SO(2)$, we first note that give $u = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}$ and let

$$g = \begin{pmatrix} \cos f & \sin f \\ -\sin f & \cos f \end{pmatrix}, \qquad \text{where } f(x) = \int_{-\infty}^{x} q(y)dy.$$

Then $g^{-1}g_x = u$ and $\lim_{x\to-\infty} g = e$. Let $a = \operatorname{diag}(i, -i)$, and $b = -a/4$. Then a direct computation shows that the -1-flow equation associated to $SU(2)/SO(2)$ defined by a, b, written in terms of f gives the SGE

$$2f_{xt} = \sin 2f.$$

Note that the MKdV and the SGE are commuting Hamiltonian flows with respect to the weak symplectic structures w_{-2} and w_0.

5.4 Example. Let $\mathcal{U} = su(n)$, $\sigma(X) = -X^t$, and a, b be regular diagonal matrices with pure imaginary entries in $su(n)$. Then the -1-flow equation associated to $SU(n)/SO(n)$ defined by a, b for u is

$$(g^{-1}g_x)_t = [a, g^{-1}bg],$$

where $g : R^2 \to SO(n)$ and $g^{-1}g_x = u$. The associated one-parameter family of flat connections is $\theta_\lambda = (a\lambda + g^{-1}g_x)dx + \lambda^{-1}g^{-1}bgdt$. Note that equation (5.2) is the SG-matrix equation studied in [12]. But our associated linear problem to equation (5.2) is different from the one given in [12].

In the following, we show that solutions of the -1-flow equation associated to U (symmetric space U/U_0 respectively) give rise to harmonic maps from the Lorentz space $R^{1,1}$ into Lie group U (symmetric space U/U_0 respectively).

Let $R^{1,1}$ denote the Lorentz space given by the metric $dxdt$, and

$$\mathcal{E} : \mathcal{S}(R^{1,1}, U) \to R, \quad \mathcal{E}(s) = \int_{R^2} (s^{-1}s_x, s^{-1}s_t)dxdt$$

denote the energy functional. A map $s : R^{1,1} \to U$ is *harmonic* if s is a critical point of $\mathcal{E}$. The Euler Lagrange equation is the harmonic map equation:

$$(s^{-1}s_x)_t = -(s^{-1}s_t)_x = \frac{1}{2}[s^{-1}s_x, s^{-1}s_t]. \tag{5.3}$$

Note that $\Omega(x,t) = A(x,t)dx + B(x,t)dt$ is a flat, $\mathcal{U}$-valued connection 1-form on the (x,t)-plane (i.e., $d\Omega = -\Omega \wedge \Omega$) if and only if there exists a unique $E : R^2 \to U$ such that $E^{-1}dE = \Omega$ with $E(0,0) = e$. Such E will be called the *parallel transport* of Ω. Given $\phi : R^2 \to U$, the *gauge transformation* of ϕ on the space of connections is $\phi * \Omega = \phi\Omega\phi^{-1} - d\phi\phi^{-1}$. It is easy to see that

(i) Ω is flat if and only if $\phi * \Omega$ is flat,

(ii) if E is the parallel transport of the flat connection Ω, then $\phi(0,0)^{-1}E\phi^{-1}$ is the parallel transport of $\phi * \Omega$.

The following Theorem is well-known ([39]):

5.5 Theorem [39]. *If $s : R^{1,1} \to U$ is a solution of the harmonic map equation (5.3), then $\Omega_\lambda = (1-\lambda)(\frac{1}{2}s^{-1}s_x)dx + (1-\lambda^{-1})(\frac{1}{2}s^{-1}s_t)dt$ is flat for all λ. Conversely, if $\Omega_\lambda(x,t) = (1-\lambda)A(x,t)dx + (1-\lambda^{-1})B(x,t)dt$ is flat for all λ, then $s = E_{-1}$ satisfies the harmonic map equation (5.3), where E_λ is the parallel transport of Ω_λ.*

By a direct computation, we have

5.6 Corollary. *Let u be a solution to the -1-flow equation $u_t = [a, Q_{b,-1}(u)]$, and $\Phi_\lambda(x,t)$ the parallel transport of the corresponding one-parameter family of flat connections $\theta_\lambda(x,t) = (a\lambda + u(x,t))dx + \lambda^{-1}Q_{b,-1}(u)dt$. Then*

(i) $\Phi_1 * \theta_\lambda = (1-\lambda)(-\Phi_1 a\Phi_1^{-1})dx + (1-\lambda^{-1})(-\Phi_1 v\Phi_1^{-1})dt$, where $v = Q_{b,-1}(u)$,

(ii) $s = \Phi_{-1}\Phi_1^{-1}$ *is harmonic from* $R^{1,1}$ *to* U *and* $s^{-1}s_x \in U \cdot (-2a)$ *and* $s^{-1}s_t \in U \cdot (-2b)$*, where* $U \cdot x_0$ *denotes the Adjoint* U*-orbit at* x_0.

5.7 Proposition. *Let* $a \in \mathcal{T}$ *be a regular element of* $\mathcal{U}$*. If* $s : R^{1,1} \to U$ *is harmonic such that* $s^{-1}s_x \in U \cdot (-2a)$*, then there exists* $\psi : R^2 \to U$ *such that*

$$\psi * \Omega_\lambda = (a\lambda + u)dx + \lambda^{-1}vdt$$

for some u, v*, where* Ω_λ *is the one-parameter family of flat connections associated to* s *as in Theorem 5.5.*

Proof. Let $A = \frac{1}{2}s^{-1}s_x$ and $B = \frac{1}{2}s^{-1}s_t$. Then

$$\Omega_\lambda = (1-\lambda)Adx + (1-\lambda^{-1})Bdt.$$

Choose $\phi : R^2 \to U$ so that $A = -\phi a \phi^{-1}$. Let $f = -\phi^{-1}B\phi$. First we claim that the following statements are true:

(1) $f_x = -[\phi^{-1}\phi_x, f]$ and $f_x^{\mathcal{T}} = -[\phi^{-1}\phi_x, f]^{\mathcal{T}}$.

(2) $(\phi^{-1}\phi_t - f)(x,t) \in \mathcal{T}$.

Note that (1) follows from a direct computation. To prove (2), by a direct computation we get $A_t = -[A, \phi_t\phi^{-1}]$. But s harmonic implies that $A_t = [A, B]$. So we have $[A, B + \phi_t\phi^{-1}] = 0$, which implies that

$$[\phi^{-1}A\phi, \phi^{-1}B\phi + \phi^{-1}\phi_t] = [a, f - \phi^{-1}\phi_t] = 0.$$

Since a is regular, the centralizer of a is $\mathcal{T}$, and (2) follows.

Now suppose there exists $h : R^2 \to T$ such that $(\phi h)^{-1} * \Omega_\lambda$ is of the form

$$(a\lambda + u)dx + \lambda^{-1}vdt$$

for some $u(x,t) \in \mathcal{T}^\perp$ and $v(x,t) \in \mathcal{U}$. So we want

$$\begin{aligned}(\phi h)^{-1} * \Omega_\lambda &= \{h^{-1}\phi^{-1}(1-\lambda)A\phi h + h^{-1}h_x + h^{-1}\phi^{-1}\phi_x h\}dx \\ &\quad + \{h^{-1}\phi^{-1}(1-\lambda^{-1})B\phi h + h^{-1}h_t + h^{-1}\phi^{-1}\phi_t h\}dt \\ &= \{(-(1-\lambda)a + h^{-1}h_x + h^{-1}\phi^{-1}\phi_x h\}dx \\ &\quad + \{-(1-\lambda^{-1})h^{-1}fh + h^{-1}h_t + h^{-1}\phi^{-1}\phi_t h\}dt \\ &= (a\lambda + u)dx + \lambda^{-1}vdt\end{aligned}$$

for some $u : R^2 \to \mathcal{T}^\perp$ and $v : R^2 \to \mathcal{U}$. This implies that h must satisfy the following equations:

$$\begin{cases} h^{-1}h_x + (h^{-1}\phi^{-1}\phi_x h)^{\mathcal{T}} = a, \\ -h^{-1}fh + h^{-1}h_t + h^{-1}\phi^{-1}\phi_t h = 0, \end{cases}$$

or equivalently,

$$\begin{cases} h_x h^{-1} = a - (\phi^{-1}\phi_x)^{\mathcal{T}}, \\ h_t h^{-1} = f - \phi^{-1}\phi_t. \end{cases} \tag{5.4}$$

By statement (2) above, the right-hand side of the second equation has value in $\mathcal{T}$. So such h exists if and only if the integrability condition holds. Since T is abelian, the integrability condition for system (5.4) is

$$(h_x h^{-1})_t - (h_t h^{-1})_x = 0.$$

The following computation shows that system (5.4) is integrable:

$$\begin{aligned} (a - (\phi^{-1}\phi_x)^{\mathcal{T}})_t - (f - \phi^{-1}\phi_t)_x &= -(\phi^{-1}\phi_x)_t^{\mathcal{T}} - f_x^{\mathcal{T}} + (\phi^{-1}\phi_t)_x^{\mathcal{T}} \\ &= -[\phi^{-1}\phi_x, \phi^{-1}\phi_t]^{\mathcal{T}} - f_x^{\mathcal{T}}, \quad \text{by (2)} \\ &= [f, \phi^{-1}\phi_x]^{\mathcal{T}} - f_x^{\mathcal{T}}, \end{aligned}$$

which is zero by (1). q.e.d.

Assume $a, b \in \mathcal{U}$ such that a is regular and $[a, b] = 0$. The above discussion says that there is a bijective correspondence between the space of harmonic maps $s : R^{1,1} \to U$ such that $s^{-1}s_x \in U \cdot (-2a)$ and $s^{-1}s_t \in U \cdot (-2b)$ and the space of solutions of the -1-flow equation $u_t = [a, Q_{b,-1}(u)]$.

Recall that the σ-action of U on U for an involution $\sigma : U \to U$ is defined by $g \cdot x = gx\sigma(g)^{-1}$ and the orbit $U \cdot e$ is totally geodesic and is isometric to the symmetric space U/K.

5.8 Proposition. *Let σ be an involution of $\mathcal{U}$, $\mathcal{U} = \mathcal{K} + \mathcal{P}$ the Cartan decomposition, $\mathcal{T} \subset \mathcal{P}$ a maximal abelian subalgebra, $a, b \in \mathcal{T}$, and a regular with respect to the $Ad(K)$-action on $\mathcal{P}$. Suppose $u : R^2 \to \mathcal{K}_a^\perp \cap \mathcal{K}$ satisfies the condition that $\theta_\lambda = (a\lambda + u)dx + \lambda^{-1}vdt$ is flat for all λ, where $\mathcal{K}_a = \{x \in \mathcal{K} \mid [x, a] = 0\}$. Let Φ_λ is the parallel transport of θ_λ, Then*

(1) $s = \Phi_{-1}\Phi_1 : R^{1,1} \to U$ is harmonic,

(2) *the image of s lies in the orbit M of the σ-action at e; since M is totally geodesic and is isometric to the symmetric space U/K, s is harmonic from $R^{1,1}$ to the symmetric space U/K.*

Proof. Since $\sigma(\theta_\lambda) = \theta_{-\lambda}$, $\sigma(\Phi_\lambda) = \Phi_{-\lambda}$. So $\sigma(\Phi_{-1}) = \Phi_1$. In particular,

$$s = \Phi_{-1}\Phi_1^{-1} = \sigma(\Phi_1)\Phi_1^{-1} \in U(\sigma)\cdot e. \quad \text{q.e.d.}$$

6. The n-dimensional systems

In this section, we associate to each rank-n, semi-simple Lie group U (or a rank-n symmetric space U/K) an n-dimensional system for maps $u : R^n \to \mathcal{T}^\perp$ (or $u : R^n \to \mathcal{T}^\perp \cap \mathcal{P}$ respectively), which is the first flow equation when restricted to any of the two variables of R^n. Using a theorem of [7], we also show that the Cauchy problem for this n-dimensional system has global solutions for generic initial data.

Let $\mathcal{U}, \mathcal{T}, \mathcal{T}^\perp$ be as before, and n the rank of $\mathcal{U}$. Let $a_1, \ldots, a_n \in \mathcal{T}$ be a basis of $\mathcal{T}$ consisting of regular elements. For $a \in \mathcal{T}$, we let J_a and $w^a = w_{-1}$ be as in section 3. Then on the symplectic manifold $(\mathcal{S}(\mathcal{T}^\perp), w^a)$, the first flow equation defined by $a, b \in \mathcal{T}$ is

$$\begin{aligned} u_t &= [Q_{b,2}(u), a] = Q_{b,1}(u)_x + [u, Q_{b,1}(u)] \\ &= (J_a^{-1}J_b(u))_x + [u, J_a^{-1}J_b(u)]. \end{aligned} \tag{6.1}$$

Let t_1 denote the variable of $u_1 : R \to \mathcal{T}^\perp$, and let t_j ($j \neq 1$) denote the flow variable for the first flow equation given by a_1, a_j. Set $v = -J_{a_1}^{-1}(u_1)$, i.e., $u_1 = [a_1, v]$. Then

(i) $J_{a_1}^{-1}J_{a_j}(u_1) = [a_j, v]$,

(ii) the first flow equation defined by a_1, a_j written in terms of v is

$$[a_1, v_{t_j}] = [a_j, v_{t_1}] + [[a_1, v], [a_j, v]]. \tag{6.2}$$

Similarly, for $u_2 \in \mathcal{S}(\mathcal{T}^\perp)$, the first flow equation defined by a_2, a_j ($j \neq 2$) is

$$(u_2)_{t_j} = (J_{a_2}^{-1}J_{a_j}(u_2))_{t_2} + [u_2, J_{a_2}^{-1}J_{a_j}(u_2)]. \tag{6.3}$$

But if $u_2 = [a_2, v]$, then $J_{a_2}^{-1} J_{a_j}(u_2) = [a_j, v]$, and equation (6.3) becomes

$$[a_2, v_{t_j}] = [a_j, v_{t_2}] + [[a_2, v], [a_j, v]].$$

What this says is that the first flow equations on these n different symplectic manifolds

$$\{(\mathcal{S}(\mathcal{T}^\perp), w^{a_i}) \mid 1 \le i \le n\}$$

are compatible. In other words, if $v \in \mathcal{S}(R^n, \mathcal{T}^\perp)$ satisfies

$$[a_i, v_{t_j}] - [a_j, v_{t_i}] = [[a_i, v], [a_j, v]], \qquad \text{for all } \ 1 \le i \ne j \le n, \tag{6.4}$$

then the restriction of $u_i = [a_i, v]$ to the $t_i t_j$-plane satisfies the Hamiltonian equations

$$(u_i)_{t_j} = [Q_{a_j,2}(u_i), a_i], \qquad 1 \le i \ne j \le n. \tag{6.5}$$

It is obvious that equation (6.4) is the condition for the following one-parameter family of connections on R^n being flat:

$$\Theta_\lambda = \sum_{j=1}^{n} (a_j \lambda + [a_j, v]) dt_j. \tag{6.6}$$

We will call equation (6.4) the *n-dimensional system associated to the rank-n Lie group U.*

It is proved in [8] that the Cauchy problem for the two dimension system (6.2) can be solved globally by the inverse scattering method for generic initial data provided

$$\int_{-\infty}^{\infty} |\, u(x,t)\,|^2 dx$$

stays bounded for all t in the maximal forward time interval $[0, T)$. Let $\triangle$ be the root system associated to $\mathcal{U}$ with respect to $\mathcal{T}$, and $\mathcal{U} = \mathcal{T} + \sum_{\alpha\in\triangle} \mathcal{U}_\alpha$ the root space decomposition. Then

$$\begin{aligned} F_{b,1}(u) &= \int_{-\infty}^{\infty} (J_a^{-1} J_b(u), u)\, dx \\ &= \int_{-\infty}^{\infty} \sum_{\alpha\in\triangle} \frac{\alpha(b)}{\alpha(a)} \,|\, u_\alpha\,|^2 dx, \end{aligned}$$

where u_α is the $\mathcal{U}_\alpha$-component of u. We may choose regular elements a, b lying in the same Weyl chamber in $\mathcal{T}$. Then there exists $c_0 > 0$ such that $\frac{\alpha(b)}{\alpha(a)} \geq c_0$ for all $\alpha \in \triangle$. This implies that

$$F_{b,1}(u) \geq c_0 \int_{-\infty}^{\infty} |\, u(x,t)\, |^2 dx.$$

Since $F_{b,1}$ is conserved under the flow, the energy remains bounded for all $t \in [0, T)$. This proves that Cauchy problem for the n-dimensional system can be solved globally for generic initial data. We summarize our discussion in the following theorem:

6.1 Theorem. *Let $\mathcal{U}$ be a rank-n semi-simple Lie algebra, $\mathcal{T}$ a maximal abelian subalgebra of $\mathcal{U}$, and $a_1, \ldots, a_n \in \mathcal{T}$ a basis of $\mathcal{T}$ consisting of regular elements. A smooth map $v : R^n \to \mathcal{T}^\perp$ is a solution to the n-dimensional system (6.4) associated to U if and only if the one-parameter family of connections defined by formula (6.6) on R^n is flat. Moreover,*

(i) for generic $f \in \mathcal{S}(\mathcal{T}^\perp)$, the Cauchy problem for the system (6.4) with initial condition $v(t_1, 0, \ldots, 0) = f(t_1)$ has global solution on R^n,

(ii) if $v(t_1, \ldots, t_n)$ is a solution of system (6.4), then the restriction of $u_i = [a_i, v]$ to any plane parallel to the $t_i t_j$-plane satisfies the first flow equation associated to U defined by a_i, a_j on the symplectic manifold $(\mathcal{S}(\mathcal{T}^\perp), w^{a_i})$ for all $j \neq i$: $(u_i)_{t_j} = [Q_{a_j,2}(u_i), a_i] = (u_j)_{t_i} + [u_i, u_j]$.

Now let U/K be a rank-n symmetric space, $\sigma : \mathcal{U} \to \mathcal{U}$ the corresponding involution, $\mathcal{U} = \mathcal{K} + \mathcal{P}$ the Cartan decomposition, and $\mathcal{T} \subset \mathcal{P}$ a maximal abelian subalgebra. Let $a_1, \ldots, a_n$ be a basis for $\mathcal{T}$ consisting of regular elements with respect to the $\mathrm{Ad}(K)$-action on $\mathcal{P}$. Then a similar argument to the above will lead to a natural system for $v : R^n \to \mathcal{P} \cap \mathcal{T}^\perp$:

$$(6.7) \qquad [a_i, v_{t_j}] - [a_j, v_{t_i}] = [[a_i, v], [a_j, v]], \qquad \text{for all } \ 1 \leq i \neq j \leq n,$$

(note that this equation is different from (6.4) because $a_i \in \mathcal{T} \subset \mathcal{P}$ and $v(t) \in \mathcal{P} \cap \mathcal{T}^\perp$). System (6.7) will be called the n-*dimensional system associated to the rank-n symmetric space U/K*. To summarize, we have

6.2 Theorem. *Let U/K be a rank-n symmetric space, $\sigma : \mathcal{U} \to \mathcal{U}$ the corresponding involution, $\mathcal{U} = \mathcal{K} + \mathcal{P}$ the Cartan decomposition, and $\mathcal{T} \subset \mathcal{P}$ a maximal abelian subalgebra. Let $a_1, \ldots, a_n$ be a basis for $\mathcal{T}$ consisting of regular elements with respect to the $Ad(K)$-action on $\mathcal{P}$. Then a smooth map $v : R^n \to \mathcal{P} \cap \mathcal{T}^\perp$ is a solution of system (6.7) associated to the symmetric space U/K if and only if the following one-parameter family of $\mathcal{U}$-valued connections on R^n is flat:*

$$\Theta_\lambda = \sum_{i=1}^{n} (a_i \lambda + [a_i, v]) dt_i. \tag{6.8}$$

(Note that here $a_i \in \mathcal{T} \subset \mathcal{P}, v \in \mathcal{T}^\perp \cap \mathcal{P}$, and $[a_i, v] \in \mathcal{K}$.) Moreover,

(i) for generic $f \in \mathcal{S}(\mathcal{T}^\perp \cap \mathcal{P})$, there exists a unique $v \in C^\infty(R^n, \mathcal{T}^\perp \cap \mathcal{P})$ such that v is a solution of system (6.7) and $v(t_1, 0, \ldots, 0) = f(t_1)$,

(ii) if $v(t_1, \ldots, t_n)$ is a solution of system (6.7), then the restriction of $u_i = [a_i, v]$ to the $t_i t_j$-plane satisfies the first flow equation associated to U/K defined by a_i, a_j: $(u_i)_{t_j} = [Q_{a_j,2}(u_i), a_i] = (u_j)_{t_i} + [u_i, u_j]$ on the symplectic manifold $(\mathcal{S}(\mathcal{K} \cap (\mathcal{K}_o)^\perp), w_0)$, where $\mathcal{K}_o = \{x \in \mathcal{K} \mid [x, b] = 0, \text{for all}\, b \in \mathcal{T}\}$.

Next we give some explicit examples.

6.3 Examples.

(1) $M_{-1}(n) = SO(2n, 1)/S(O(n) \times O(n, 1))$.

The Cartan decomposition of $M_{-1}(n)$ is $\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1$, where $\mathcal{U} = so(2n, 1)$,

$$\mathcal{U}_0 = so(n) \times so(n,1) = \left\{ \begin{pmatrix} X & 0 & 0 \\ 0 & Y & \eta^t \\ 0 & \eta & 0 \end{pmatrix} \,\middle|\, X, Y \in so(n), \eta \in R^n \right\},$$

$$\mathcal{U}_1 = \left\{ \begin{pmatrix} 0 & F & \xi^t \\ -F^t & 0 & 0 \\ \xi & 0 & 0 \end{pmatrix} \,\middle|\, F \in gl(n), \xi \in R^n \right\}.$$

Let

$$\mathcal{T} = \left\{ \begin{pmatrix} 0 & -C & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in gl(2n+1) \middle| C \in gl(n) \text{ is diagonal} \right\},$$

$$a_i = \begin{pmatrix} 0 & -C_i & 0 \\ C_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{where } C_i = \operatorname{diag}(c_{i1}, c_{i2}, \ldots, c_{in}).$$

Then $\mathcal{T}$ is a maximal abelian subalgebra in $\mathcal{U}_1$. Note that $a_1, \ldots, a_n$ are regular and form a basis of $\mathcal{T}$ if and only if

$$\det((c_{ij})) \neq 0, \qquad c_{ij}^2 \neq c_{ik}^2 \quad \text{for all } j \neq k. \tag{6.9}$$

Let $gl(n)_* = \{(x_{ij}) \in gl(n) \mid x_{ii} = 0 \text{ for all } 1 \le i \le n\}$. Then

$$\mathcal{U}_1 \cap \mathcal{T}^\perp$$
$$= \left\{ \begin{pmatrix} 0 & F & \xi^t \\ -F^t & 0 & 0 \\ \xi & 0 & 0 \end{pmatrix} \in \mathcal{U}_1 \middle| \xi = (b_1, \cdots, b_n), F = (f_{ij}) \in gl(n)_* \right\}.$$

Make linear change of coordinates by setting $x_j = \sum_{i=1}^n c_{ij} t_i$. Then

$$\sum_{i=1}^n C_i dt_i = \operatorname{diag}\Big(\sum_i c_{i1} dt_i, \sum_i c_{i2} dt_i, \ldots, \sum_i c_{in} dt_i\Big)$$
$$= \operatorname{diag}(dx_1, \ldots, dx_n) = \delta.$$

So the one-parameter family of flat connections (6.8) is:

$$\Theta_\lambda = \sum_{i=1}^n (a_i \lambda + u_i) dt_i$$
$$= \lambda \begin{pmatrix} 0 & -\delta & 0 \\ \delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \delta F^t - F\delta & 0 & 0 \\ 0 & \delta F - F^t \delta & \delta \xi^t \\ 0 & \xi\delta & 0 \end{pmatrix}.$$

The system (6.7) written in terms of differential forms is:

$$\begin{cases} -F\delta + \delta F^t \ \text{ is left flat}, \\ d\omega = \omega \wedge \omega + \delta\xi^t \wedge \xi\delta, & \text{where } \omega = -\delta F + F^t \delta, \\ d\xi \wedge \delta = \xi(-F\delta + \delta F^t) \wedge \delta, & \text{where } \delta = \operatorname{diag}(dx_1, \ldots, dx_n). \end{cases} \tag{6.10}$$

(2) $M_0(n) = SO(2n)/S(O(n) \times O(n))$.

Similarly, the n-dimensional system associated to the symmetric space $M_0(n)$ is a system for $v : R^n \to \mathcal{U}_1 \cap \mathcal{T}^\perp$

$$v = \begin{pmatrix} 0 & F \\ -F^t & 0 \end{pmatrix}, \qquad F = (f_{ij}) \in gl(n)_*,$$

and the system (6.7) in x coordinates becomes

$$\begin{cases} -F\delta + \delta F^t & \text{is left flat,} \\ -\delta F + F^t\delta & \text{is right flat,} \end{cases} \tag{6.11}$$

and the corresponding one-parameter family of flat connections is

$$\Theta_\lambda = \lambda \begin{pmatrix} 0 & -\delta \\ \delta & 0 \end{pmatrix} + \begin{pmatrix} -F\delta + \delta F^t & 0 \\ 0 & \delta F - F^t\delta \end{pmatrix}.$$

(3) $M_1(n) = SO(2n+1)/S(O(n) \times O(n+1))$.

For this example, we use the following real form of $so(2n+1, C)$, which is isomorphic to $so(2n+1)$:

$$\left\{ \begin{pmatrix} X & i\eta^t \\ i\eta & 0 \end{pmatrix} \middle| X \in so(2n), \quad \eta \in R^{2n} \right\},$$

where $i = \sqrt{-1}$. Then a similar calculation as for $M_{-1}(n)$ shows that the n-dimensional system associated to the symmetric space $M_1(n)$ is given by

$$\begin{cases} -F\delta + \delta F^t & \text{is left flat,} \\ d\omega = \omega \wedge \omega - \delta\xi^t \wedge \xi\delta, & \text{where } \omega = -\delta F + F^t\delta, , \\ d\xi \wedge \delta = \xi(-F\delta + \delta F^t) \wedge \delta. & \end{cases} \tag{6.12}$$

and the corresponding one-parameter family of flat connections is

$$\Theta_\lambda = \lambda \begin{pmatrix} 0 & -\delta & 0 \\ \delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -F\delta + \delta F^t & -\delta & 0 \\ \delta & \delta F - F^t\delta & i\delta\xi^t \\ 0 & i\xi\delta & 0 \end{pmatrix}.$$

6.4 Theorem. *Let $F = (f_{ij}) : R^n \to gl(n)_*$, and $\xi = (b_1, \dots, b_n) : R^n \to R^n$ be smooth maps. Then the n-dimensional system associated to the symmetric space $M_c(n)$ is the following system for (F, ξ):*

$$\begin{cases} -F\delta + \delta F^t & \text{is left flat,} \\ d\omega = \omega \wedge \omega - c\delta\xi^t \wedge \xi\delta, & \text{where } \omega = -\delta F + F^t\delta, \\ (b_i)_{x_j} = f_{ij} b_j. \end{cases} \tag{6.13}$$

The one-parameter family of flat connections associated to this system is

$$\Theta_\lambda = \lambda \begin{pmatrix} 0 & -\delta & 0 \\ \delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \delta F^t - F\delta & 0 & 0 \\ 0 & \delta F - F^t\delta & \sqrt{-c}\,\delta\xi^t \\ 0 & \sqrt{-c}\,\xi\delta & 0 \end{pmatrix},$$

where $\delta = \mathrm{diag}(dx_1, \dots, dx_n)$.

Note that if $c \neq 0$, then the system (6.13) for (b, F) in $(x_1, \dots, x_n)$ coordinates is:

(6.14)

$$\begin{cases} (b_i)_{x_j} - f_{ij} b_j = 0, & \text{if } i \neq j, \\ (f_{ij})_{x_i} + (f_{ji})_{x_j} + \sum_k f_{ki} f_{kj} = 0, & \text{if } i \neq j, \\ (f_{ij})_{x_k} - f_{ik} f_{kj} = 0, & \text{if } i, j, k \text{ are distinct}, \\ (f_{ij})_{x_j} + (f_{ji})_{x_i} + \sum_k f_{ik} f_{jk} = -c b_i b_j, & \text{if } i \neq j, \end{cases}$$

and in $(t_1, \dots, t_n)$ coordinates is:

$$\begin{cases} (-FC_k + C_k F^t)_{t_j} - (-FC_j + C_j F^t)_{t_k} \\ \qquad = [-FC_k + C_k F^t, -FC_j + C_j F^t], \\ (-C_k F + F^t C_k)_{t_j} - (-C_j F + F^t C_j)_{t_k} \\ \qquad = -[-C_k F + F^t C_k, -C_j F + F^t C_j] \\ \qquad\quad + c(C_j \xi^t \xi C_k - C_k \xi^t \xi C_j), \\ (\xi C_k)_{t_j} - (\xi C_j)_{t_k} \\ \qquad = -\xi C_k F^t C_j + \xi C_j F^t C_k. \end{cases} \tag{6.15}$$

If $c = 0$ then the system (6.13) for F in $(x_1, \dots, x_n)$ coordinates is:

$$\begin{cases} (f_{ij})_{x_i} + (f_{ji})_{x_j} + \sum_k f_{ki} f_{kj} = 0, & \text{if } i \neq j, \\ (f_{ij})_{x_k} - f_{ik} f_{kj} = 0, & \text{if } i, j, k \text{ are distinct}, \\ (f_{ij})_{x_j} + (f_{ji})_{x_i} + \sum_k f_{ik} f_{jk} = 0, & \text{if } i \neq j, \end{cases} \tag{6.16}$$

and in $(t_1, \dots, t_n)$ coordinates is:

$$(6.17)\qquad \begin{cases} (-FC_k + C_kF^t)_{t_j} - (-FC_j + C_jF^t)_{t_k} \\ \qquad = [-FC_k + C_kF^t, -FC_j + C_jF^t], \\ (-C_kF + F^tC_k)_{t_j} - (-C_jF + F^tC_j)_{t_k} \\ \qquad = -[-C_kF + F^tC_k, -C_jF + F^tC_j]. \end{cases}$$

7. Isometric immersions of space forms into space forms

In this section we show that the n-dimensional system (6.13) associated to the symmetric space $M_n(c)$ is the equation for the isometric immersions of n-dimensional space forms of sectional curvature c into $2n$-dimensional space forms of the same curvature c with $c = -1, 0, 1$.

Let $N^n(c)$ denote the simply connected, Riemmanian manifold of constant sectional curvature c. Then $N^n(0) = R^n$, $N^n(1)$ is the standard unit sphere S^n, and $N^n(-1) = \{x \in R^{n,1} \mid (x, y)_1 = -1\}$, where $R^{n,1}$ is the Lorentz space equipped with the bilinear form $(x, y)_1 = x_1y_1 + \dots + x_ny_n - x_{n+1}y_{n+1}$.

Let M^n be a submanifold of $N^{2n}(c)$ with constant sectional curvature c and flat normal bundle. Then by the Ricci equation, given any $p \in M$ the collection of shape operators $\{A_v \mid v \in \nu(M)_p\}$ is a commuting family of self-adjoint operators. So there exists a local orthonormal frame field $e_1, \dots, e_{2n}$ such that $e_{n+1}, \dots, e_{2n}$ are parallel normal fields and $\{e_1, \dots, e_n\}$ is a common eigenbasis for the shape operators. Let b_i^α denote the eigenvalues of A_{e_α}, i.e., $A_{e_\alpha}(e_i) = b_i^\alpha e_i$, and $v_i = \sum_{\alpha=n+1}^{2n} b_i^\alpha e_\alpha$. Then given any $v \in \nu(M)$ we have $A_v(e_i) = \langle v, v_i\rangle e_i$. It is easy to see that the v_i's are well-defined, and they are called the *curvature normals* of the submanifold M.

7.1 Theorem. *Suppose $\mathcal{O}$ is a simply connected open subset of $N^n(c)$, $X : \mathcal{O} \to N^{2n}(c)$ is an isometric immersion with flat normal bundle, and the n curvature normals are linearly independent. Let $\{e_{n+1}, \dots, e_{2n}\}$ be a parallel orthonormal normal frame field. Then the following hold:*

(i) There exist line of curvature coordinates $(x_1, \dots, x_n)$ and smooth maps $\xi = (b_1, \dots, b_n) : \mathcal{O} \to R^n$ and $A = (a_{ij}) : \mathcal{O} \to SO(n)$ such

that the two fundamental forms are of the form

$$I = \sum_{i=1}^{n} b_i^2 dx_i^2,$$
$$II = \sum_{j=1}^{n} a_{ji} b_i dx_i^2 e_{n+j}.$$

(ii) *Let* $\xi = (b_1, \ldots, b_n)$, *and* $F = (f_{ij})$, *where* $f_{ij} = \frac{(b_i)_{x_j}}{b_j}$ *if* $i \neq j$ *and* 0 *if* $i = j$. *Then* (b, F) *is a solution of the n-dimensional system (6.13) associated to the symmetric space* $M_c(n)$.

Proof. Statement (i) can be proved exactly the same way as for isometric immersions of $N^n(c)$ into $N^{2n-1}(c+1)$ (cf. [13], [27] and [38]).

To prove (ii), we use the local theory of submanifolds in space forms (cf. [30]). Set $w_i = b_i dx_i$. Then the connection 1-form is

$$w_{ij} = -f_{ij} dx_i + f_{ji} dx_j = (-\delta F + F^t \delta)_{ij}, \tag{7.1}$$

$w_{i,n+j} = a_{ji} dx_i$ and $w_{n+i,n+j} = 0$. The Codazzi equation

$$dw_{i,n+j} = \sum_{k=1}^{n} w_{ik} \wedge w_{k,n+j}$$

gives $(a_{ji})_{x_k} = f_{ik} a_{jk}$ for $i \neq k$. Since $\sum_{k=1}^{n} a_{jk}^2 = 1$, for $j \neq i$ we have

$$(a_{ji})_{x_i} = -\frac{\sum_{k \neq i} (a_{jk}^2)_{x_i}}{2a_{ji}} = -\sum_{k \neq i} \frac{a_{jk}(a_{jk})_{x_i}}{a_{ji}} = -\sum_{k \neq i} a_{jk} f_{ki}.$$

So $dA = A(-F\delta + \delta F^t)$, and hence F satisfies the first equation in system (6.13). The structure equation $dw_i = \sum_j w_{ij} \wedge w_j$ gives the third equation, and the Gauss equation $dw_{ij} = \sum_{k=1}^{n} w_{ik} \wedge w_{kj} - c\, w_i \wedge w_j$ gives the second equation of the system (6.13). q.e.d.

As a consequence of Theorem 7.1 and the fundamental theorem of submanifolds in space forms, we have

7.2 Corollary. *Let* (F, ξ) $(F = (f_{ij}), \xi = (b_1, \ldots, b_n))$ *be a solution of the system (6.14) for* $c = \pm 1$ *on* R^n. *Then there exist smooth maps* $A = (a_{ij}) : R^n \to SO(n)$ *and* $X : R^n \to N^{2n}(c)$ *such that* X *is an* *immersion, and has constant curvature c, flat normal bundle and I, II as in Theorem 7.1 on the set of all $x \in R^n$ with $b_i(x) \neq 0$ for all $1 \leq i \leq n$.*

7.3 Corollary. *Let F be a solution of the system (6.16). Then there exist a smooth map $A = (a_{ij}) : R^n \to SO(n)$ such that $A^{-1}dA = -F\delta + \delta F^t$. Moreover, given constants $\mu_1, \ldots, \mu_n$ and set $b_i = \sum_j \mu_j a_{ji} > 0$, then there exists a smooth map $X : R^n \to N^{2n}(c)$ such that X is an immersion, and has zero sectional curvature, flat normal bundle, and I, II as in Theorem 7.1 on the set of $x \in R^n$ such that $b_i(x) \neq 0$ for all $1 \leq i \leq n$.*

Theorem 7.4 below follows from Theorem 6.1 (ii), Corollary 7.2 and Corollary 7.3.

7.4 Theorem. *Fix a non-zero vector $c = (c_1, \ldots, c_n)$ such that $c_i^2 \neq c_j^2$ for all $i \neq j$. Then given generic rapidly decay smooth functions $f_{ij}^o, b_i^o : R \to R$ with $f_{ii} = 0$ for $1 \leq i, j \leq n$, there exists a smooth map $X : R^n \to N^{2n}(c)$ unique up to isometry of $N^{2n}(c)$ such that*

(i) $I = X^*(ds^2) = \sum_i b_i(x)^2 dx_i^2$, where ds^2 is the metric on $N^{2n}(c)$,

(ii) $b_i(c_1 t, \ldots, c_n t) = b_i^o(t)$ and $((b_i)_{x_i}/b_j)(c_1 t, \ldots, c_n t) = f_{ij}^o(t)$,

(iii) X is an immersion, and has constant curvature c, flat normal bundle on the set of $x \in R^n$ such that $b_i(x) \neq 0$ for all $1 \leq i \leq n$.

Next we discuss the relation between local isometric immersions of $N^n(c)$ into $N^{2n}(c)$ with flat normal bundle and local isometric immersions of $N^n(c)$ into $N^{2n-1}(c+1)$. It is proved by Cartan [13] that $N^n(c)$ cannot be locally, isometrically, immersed into $N^{2n-2}(c+1)$, but can be into $N^{2n-1}(c+1)$. Moreover, if $X : \mathcal{O} \to N^{2n-1}(c+1)$ is a local isometric immersion of $N^n(c)$, then the normal bundle is flat. Using the work of Moore [27] and Cartan [13], it is known that given any parallel normal frame field $e_{n+1}, \ldots, e_{2n-1}$ there exist a line of curvature coordinates $(x_1, \ldots, x_n)$ and a smooth map $A = (a_{ij}) : \mathcal{O} \to SO(n)$ such that the first and second fundamental forms of the immersion X are given by

$$
\begin{aligned}
I &= \sum_{i=1}^n a_{1i}^2 dx_i^2, \\
II &= \sum_{i=1, j=2}^n a_{1i} a_{ji} dx_i^2 \otimes e_{n+j-1}.
\end{aligned}
\tag{7.2}
$$

Define

$$f_{ij} = \begin{cases} \frac{(a_{1i})_{x_j}}{a_{1j}}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

and set $F = (f_{ij})$. Then the Gauss, Codazzi and Ricci equation of the immersion is a system for (A, F):

$$\begin{cases} A^{-1} dA = -F\delta + \delta F^t, \\ d\omega = \omega \wedge \omega - c\delta A^t \mu A \delta, \end{cases} \tag{7.3}$$

where $\delta = \operatorname{diag}(dx_1, \ldots, dx_n)$, $\omega = -\delta F + F^t \delta$, and $\mu = \operatorname{diag}(1, 0, \ldots, 0)$.

Conversely, if $(A, F) : R^n \to SO(n) \times gl(n)_*$ is a solution of (7.3), then it follows from the fundamental theorem of submanifolds in space forms that there is a local immersion $X : \mathcal{O} \to N^{2n-1}(c+1)$ such that the immersed submanifold has flat normal bundle, constant sectional curvature c and its first and the second fundamental forms are given by the formula (7.2). In other words, system (7.3) is the equation for isometric immersion of $N^n(c)$ into $N^{2n-1}(c+1)$. It is well-known that the equation (7.3) with $c = -1$ for isometric immersions of -1 curvature surface into R^3 is the Sine-Gordon equation, and the equation (7.3) with $c = 0$ for isometric immersions of flat surfaces in S^3 is the wave equation. So the system (7.3) is called the GSGE (generalized Sine-Gordon equation) in [37] and [38] for $c = -1$, and is called the GWE (generalized wave equation) in [36] for $c = 0$.

Let $i : N^{2n-1}(c+1) \to N^{2n}(c)$ be a standard isometric, totally umbilic embedding of $N^{2n-1}(c+1)$ (cf. Chapter 2 of [30]). If $X : \mathcal{O} \to N^{2n-1}(c+1)$ is a local isometric immersion of $N^n(c)$, then $i \circ X : \mathcal{O} \to N^{2n}(c)$ is a local isometric immersion of $N^n(c)$ with flat normal bundle and linearly independent curvature normals. Moreover, if x is the line of curvature coordinate for the immersion X, then the following hold:

(i) x is the line of curvature coordinate as in Theorem 7.1 for $i \circ X$.

(ii) If (A, F) is a solution of equation (7.3) and let $\xi = (a_{11}, \ldots, a_{1n})$. Then (F, ξ) is a solution of equation (6.13).

A direct computation also gives the following:

7.5 Proposition. *Let $A = (a_{ij}) : R^n \to SO(n)$, and $F = (f_{ij}) : R^n \to gl(n)_*$. If (A, F) is a solution of the system (7.3) and let $\xi = (a_{11}, \ldots, a_{1n})$ the first row of A, then (F, ξ) is a solution of the system*

(6.13) associated to the symmetric space $M_c(n)$. Conversely, if $(F,\xi): R^n \to gl_*(n) \times S^{n-1}$ is a solution of (6.13), then there exists a smooth map $A: R^n \to SO(n)$ such that

(i) the first row of A is ξ,

(ii) $A^{-1}dA = -F\delta + \delta F^t$,

(iii) (A,F) is a solution of the equation (7.3).

7.6 Corollary. *Let $\mathcal{O}$ be a simply connected, open subset of R^n, $A=(a_{ij}): \mathcal{O} \to SO(n)$, $\xi = (a_{11}, \ldots, a_{1n})$, and $F=(f_{ij}): \mathcal{O} \to gl(n)_*$ be smooth maps. Then the (A,F) is a solution of the equation (7.3) if and only if Θ_λ as in Theorem 6.4 is flat for all λ.*

7.7 Remark. It was proved in [1] that (A,F) is a solution of equation (7.3) if and only if

$$\frac{\lambda^{-1}}{2}\begin{pmatrix} 0 & -\delta A^t J \\ -JA\delta & 0 \end{pmatrix} + \begin{pmatrix} -\delta F + F^t\delta & 0 \\ 0 & 0 \end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix} 0 & -\delta A^t \\ -A\delta & 0 \end{pmatrix} \tag{7.4}$$

is a flat $o(n,n)$-connection on R^n for all λ, where $J = \mathrm{diag}(-1,1,\ldots,1)$. The inverse scattering associated to the flat connection (7.4) was solved in [1].

The result of this section shows that the system (7.3) for isometric immersions from $N^n(c)$ into $N^{2n-1}(c+1)$ has the standard linear problem considered in [7], and is the n-dimensional system associated to the symmetric space $M_n(c)$. Hence the restriction of system (7.3) to the t_it_j-plane $(i \neq j)$

(i) is the first flow equation associated to $M_n(c)$,

(ii) is the Hamiltonian equation for $F_{b,1}$ with respect to the symplectic structure w_0,

(iii) commutes with all the odd flows assoicated to $M_n(c)$.

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