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ON POISSON ACTIONS OF COMPACT LIE GROUPS ON SYMPLECTIC MANIFOLDS

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Abstract

Let $G_{\mathcal{P}}$ be a compact simple Poisson-Lie group equipped with a Poisson structure $\mathcal{P}$, and (M, ω) be a symplectic manifold. Assume that M carries a Poisson action of $G_{\mathcal{P}}$, and there is an equivariant moment map in the sense of Lu and Weinstein which maps M to the dual Poisson-Lie group $G^*_{\mathcal{P}}$, $\mathbf{m} : M \to G^*_{\mathcal{P}}$. We prove that M always possesses another symplectic form $\tilde{\omega}$ so that the G-action preserves $\tilde{\omega}$ and there is a new moment map $\mu = e^{-1} \circ \mathbf{m} : M \to \mathcal{G}^*$. Here e is a universal (independent of M) invertible equivariant map $e : \mathcal{G}^* \to G^*_{\mathcal{P}}$. We suggest new short proofs of the convexity theorem for the Poisson-Lie moment map, the Poisson reduction theorem and the Ginzburg-Weinstein theorem on the isomorphism of $\mathcal{G}^*$ and $G^*_{\mathcal{P}}$ as Poisson spaces.

The main goal of this paper is to compare Hamiltonian and Poisson actions of compact simple Lie groups on symplectic manifolds. We prove that one can always exchange the Poisson action to a Hamiltonian one by an appropriate change of the symplectic structure. This trick reduces many questions concerning Poisson actions to their well known counterparts from the theory of Hamiltonian G-actions. In particular, we suggest new simple proofs of the convexity theorem for the Poisson-Lie moment map [5], Poisson reduction theorem [10] and the Ginzburg-Weinstein theorem [7]. The results of this paper were announced in [2].

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Compact Poisson-Lie groups

Definition 1. Let G be a simple connected simply connected compact Lie group and $\mathcal{P}$ be a Poisson bracket on G. This pair defines a Poisson-Lie group if the multiplication map $G \times G \to G$ is a Poisson map.

Up to a scalar factor Poisson-Lie structures on G are in one to one correspondence with Manin triples $(d, \mathcal{G}, \mathcal{G}^*)$.

Definition 2. A triple of Lie algebras $(d, \mathcal{G}, \mathcal{G}^*)$ is called a Manin triple if d has an invariant nondegenerate bilinear form k, and $\mathcal{G}$ and $\mathcal{G}^*$ are maximal isotropic subalgebras of d which together span d:

$$k(\mathcal{G}, \mathcal{G}) = k(\mathcal{G}^*, \mathcal{G}^*) = 0. \tag{1}$$

The algebra d is also called a Drinfeld double of $\mathcal{G}$. For $\mathcal{G}$ being a compact simple Lie algebra the double d of $\mathcal{G}$ coincides with its complexification $\mathcal{G}^{\mathbb{C}}$ considered as an algebra over real numbers. The scalar product k is given by the imaginary part of the Killing form K on $\mathcal{G}^{\mathbb{C}}$:

$$k(a, b) = Im\ K(a, b). \tag{2}$$

Up to isomorphism, the isotropic subalgebras $\mathcal{G}^*_u \subset \mathcal{G}^{\mathbb{C}}$ are classified by real valued antisymmetric bilinear forms on the Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$ [9]. For each such a form u there is a splitting

$$d = \mathcal{G}^{\mathbb{C}} = \mathcal{G} + \mathcal{G}^*_u, \tag{3}$$

where the dual Lie algebra $\mathcal{G}^*_u$ is defined as a semi-direct sum of two subalgebras

$$\mathcal{G}^*_u = \mathbf{n} + \mathcal{H}^*_u. \tag{4}$$

Here $\mathbf{n}$ is the maximal nilpotent subalgebra in $\mathcal{G}^{\mathbb{C}}$. We can always assume that it is generated by all positive root vectors of $\mathcal{G}^{\mathbb{C}}$. The other subspace $\mathcal{H}^*_u \subset \mathcal{H}^{\mathbb{C}}$ is defined as follows:

$$\mathcal{H}^*_u = \{i(a + iu(a)), a \in \mathcal{H}\}, \tag{5}$$

where $u(a)$ is the image of a under the map $\mathcal{H} \to \mathcal{H}^*$ corresponding to the form u , composed with the isomorphism $\mathcal{H}^* \to \mathcal{H}$ corresponding to the form k. Antisymmetry of u implies

$$K(a, u(b)) + K(u(a), b) = 0. \tag{6}$$

Let us denote a Poisson structure corresponding to the Manin triple (3) by $\mathcal{P}_u$. Rescaling this Poisson bracket by a real factor t we get a family parametrized by pairs (t, u):

$$\mathcal{P}_{(t,u)} = t\mathcal{P}_{u/t}. \tag{7}$$

This family provides a complete classification of Poisson structures on compact simple Lie groups (up to isomorphisms). The Poisson bracket $\mathcal{P}_{(t,u)}$ behaves smoothly at the point $t = 0$. However, in the main part of this paper we always assume that $t \neq 0$ and treat the case $t = 0$ in Appendix.

Let us remark that the Lie algebras $\mathcal{G}$ and $\mathcal{G}^*$ enter the picture in a symmetric way. This means that the connected simply connected group G^*_u corresponding to the Lie algebra $\mathcal{G}^*_u$ also carries a Poisson-Lie structure defined by the Manin triple.

In our example the group G^*_u is a semi-direct product of the maximal nilpotent group $\mathbf{N}$ in $G^{\mathbb{C}}$ and the subgroup H^*_u of the complexification of the Cartan torus

$$H^*_u = \{exp(a), a \in \mathcal{H}^*_u\}. \tag{8}$$

In particular, for $G = SU(N)$ and $u = 0$, the group H^*_0 is formed by diagonal matrices of unit determinant with real positive eigenvalues. The elements of G^*_u may be visualized by the embedding into $G^{\mathbb{C}}$:

$$G^*_u = \{N\ exp\{i(a + iu(a))\}, N \in \mathbf{N}, a \in \mathcal{H}\}. \tag{9}$$

Let $a \to \bar{a}$ be an anti-involution of $G^{\mathbb{C}}$ which singles out the compact form. It is convenient to introduce a map

$$f : a \to a\bar{a}; \tag{10}$$

which maps G^*_u into a certain subspace P of $G^{\mathbb{C}}$,

$$P = \{exp\{ia\}, a \in \mathcal{G}\}. \tag{11}$$

Observe that though the dual group G^*_u depends on the choice of u, the target of the map f is always the same space P.

There is another way to characterize P:

$$P = \{x \in G^{\mathbb{C}}, \bar{x} = x\}. \tag{12}$$

The bar operation being an *anti*-involution, P is not a group. Using the fact that any element of P may be brought to the maximal torus by conjugation by some element of G, the Iwasawa decomposition and the uniqueness of a positive square root of a positive real number one easily proves that the map f is in fact invertible. Let us define the following map $e_{(t,u)}$ from $\mathcal{G}^*$ to G_u^*:

$$e_{(t,u)} = f^{-1} \circ j \ , \ j = E \circ K = exp\{2it \ \cdot\} \circ K. \tag{13}$$

Here K stands for the Killing form which converts $\mathcal{G}^*$ to $\mathcal{G}$, the exponential map E with additional i maps $\mathcal{G}$ to P and the last map f^{-1} identifies P with G_u^*. Let a be an element of $\mathcal{G}^*$ and $A = e_{(t,u)}(a)$. Then the definition (13) implies

$$\mathbf{A} \equiv A\bar{A} = j(a) = exp\{2itK(a)\}. \tag{14}$$

Both spaces $\mathcal{G}^*$ and G_u^* carry natural actions of the group G. The dual space to the Lie algebra carries the coadjoint action Ad^*:

$$K(Ad^*(g)a) = gK(a)g^{-1}. \tag{15}$$

The G-action on the group G_u^* is defined by using a somewhat generalized version of the Iwasawa decomposition:

$$g \cdot A = A^g \cdot g'. \tag{16}$$

This is an equality in $G^{\mathbb{C}}$. On the right-hand side $g' \in G$ and $A^g \in G_u^*$. Existence and uniqueness of A^g and g' are ensured by the corresponding properties of the Iwasawa decomposition. For historical reasons this action of G on G_u^* is called *dressing action* [13]. To make notation closer to the case of $\mathcal{G}^*$ we sometimes denote

$$A^g = AD^*(g)A. \tag{17}$$

Observe that

$$\mathbf{A}^g = A^g \bar{A}^g = gA\bar{A}g^{-1} = g\mathbf{A}g^{-1}. \tag{18}$$

This simple observation proves the following lemma.

Lemma 1. *The map $e_{(t,u)}$ intertwines coadjoint and dressing actions of G on $\mathcal{G}^*$ and G_u^*:*

$$AD^*(g)e_{(t,u)}(a) = e_{(t,u)}(Ad^*(g)a). \tag{19}$$

The map $e_{(t,u)}$ has been introduced in [5]. We shall discuss some new properties of this map in the next sections.

The Moment map in the sense of Lu and Weinstein

Let us recall the definitions of the moment map for Hamiltonian and Poisson group actions on symplectic manifolds.

Let M be a symplectic manifold equipped with an action $\mathcal{A}$ of a compact Lie group G:

$$\mathcal{A} : G \times M \to M, \ \mathcal{A}(g, x) = x^g. \tag{20}$$

To each element $\alpha \in \mathcal{G}$ one can associate a vector field v_α on M:

$$v_\alpha = \mathcal{A}_*(\alpha). \tag{21}$$

On the left-hand side we consider α as an element of the Lie algebra $\mathcal{G}$ whereas on the right-hand side we treat it as a right invariant vector field on G.

It is convenient to introduce a universal vector field v taking values in the space $\mathcal{G}^*$ so that

$$v_\alpha =< v, \alpha > . \tag{22}$$

Definition 3. The action $\mathcal{A}$ is called symplectic if it preserves the Poisson structure on M:

$$\mathcal{A}_*(\mathcal{P}_M) = \mathcal{P}_M. \tag{23}$$

Here the Poisson tensor $\mathcal{P}$ is the inverse of the matrix of the symplectic form ω on M.

Definition 4. The action $\mathcal{A}$ is called Hamiltonian if there is a Hamiltonian μ_α for each vector field v_α:

$$\omega(\cdot, v_\alpha) = d\mu_\alpha \ , \ \mathcal{P}_M(\cdot, d\mu_\alpha) = v_\alpha. \tag{24}$$

The family of Hamiltonians μ_α can be assembled into the moment map $\mu : M \to \mathcal{G}^*$ such that

$$\mu_\alpha =< \mu, \alpha > . \tag{25}$$

For further generalizations to Poisson-Lie groups we rewrite the definition of the moment map in the following form.

Definition 5. The map $\mu : M \to \mathcal{G}^*$ is called a moment map if it satisfies the following property:

$$\omega(\cdot, v) = \mu^*(da). \tag{26}$$

Here $a \in \mathcal{G}^*$, and da is the natural linear 1-form on $\mathcal{G}^*$ with values in $\mathcal{G}^*$.

Existence of a moment map ensures the invariance of the symplectic form with respect to the G-action.

We are specifically interested in symplectic manifolds equipped with the G-action and an equivariant moment map.

Definition 6. A moment map μ is said to be equivariant if

$$Ad^*(g)\mu(x) = \mu(x^g). \tag{27}$$

Let $(G, \mathcal{P}_G)$ be a compact Poisson-Lie group, the Poisson structure $\mathcal{P}_G$ being one of the standard list parametrized by pairs (t, u) (see the previous section).

Definition 7. The action of $\mathcal{A} : G \times M \to M$ is called a Poisson action if it preserves the Poisson structure in the following sense:

$$\mathcal{A}_*(\mathcal{P}_G + \mathcal{P}_M) = \mathcal{P}_M. \tag{28}$$

Notice the difference with the standard definition (23). If M is equipped with a Poisson action of G, the symplectic structure on M *is not* invariant with respect to the G-action.

A Poisson counterpart of the notion of the moment map has been defined in [10].

Definition 8. Let G be a compact Poisson-Lie group equipped with a Poisson structure $\mathcal{P}_{(t,u)}$. Let $\mathcal{A} : G \times M \to M$ be a Poisson action of G on the symplectic manifold M. The map $\mathbf{m} : M \to G^*_u$ is called a moment map in the sense of Lu and Weinstein if

$$\omega(\cdot, v) = \frac{1}{t}\mathbf{m}^*(dAA^{-1}), \tag{29}$$

where dAA^{-1} is a right-invariant Maurer-Cartan form on G^*_u.

The equivariance condition for the Poisson moment map $\mathbf{m}$ looks as follows:

$$AD^*(g)(\mathbf{m}(x)) = \mathbf{m}(x^g). \tag{30}$$

Comparing Hamiltonian and Poisson actions

Here we formulate and prove the main result of the paper.

Theorem 1. *Let (M, ω) be a symplectic manifold which carries an action $\mathcal{A}$ of a compact Poisson-Lie group G equipped with a Poisson bracket $\mathcal{P}_{(t,u)}$. Assume that there exists an equivariant moment map $\mathbf{m} : M \to G^*$. Then one can define another symplectic form $\tilde{\omega}$ on M with the following properties:*

1) $\tilde{\omega}$ is preserved by $\mathcal{A}$;

2) $\tilde{\omega}$ belongs to the same cohomology class as ω;

3) the map $\mu = e_{(t.u)}^{-1} \circ \mathbf{m}$ provides an equivariant moment map for the G-action $\mathcal{A}$ with respect to the symplectic structure $\tilde{\omega}$.

The main technical tool for proving this theorem is provided by the following lemma.

Lemma 2. *There exists a 2-form $\Omega_{(t,u)}$ on $\mathcal{G}^*$ such that the following two properties are fulfilled:*

1) The form $\Omega_{(t,u)}$ is closed, i.e., $d\Omega_{(t,u)} = 0$.

2) $\Omega_{(t,u)}(\cdot, v) = \frac{1}{t} e^*_{(t,u)} dAA^{-1} - da$.

Here v is the universal vector field corresponding to the coadjoint action of G on $\mathcal{G}^*$, $a \in g^*$ and $A = e_{(t,u)}(a) \in G^*_u$.

Proof of Lemma. It is convenient to introduce a special notation for $\alpha = K(a) \in \mathcal{G}$. Let us consider the following 2-form on $\mathcal{G}^*$:

$$\begin{aligned} \Omega_{(t,u)} = & \frac{1}{4it}\{K^* \sum_{k=2}^{\infty} \frac{(2it)^k}{k!} K(ad^{k-2}(\alpha) d\alpha \wedge d\alpha) \\ & + e^*_{(t,u)} K(A^{-1} dA \wedge d\bar{A}\bar{A}^{-1})\}. \end{aligned} \tag{31}$$

We claim that it satisfies both conditions of Lemma 2.

It is convenient to split $\Omega_{(t,u)}$ into two pieces:

$$\Omega_{(t,u)} = \omega_1 + \omega_2, \tag{32}$$

where

$$\begin{aligned} \omega_1 = & \frac{1}{4it} K^* \sum_{k=2}^{\infty} \frac{(2it)^k}{k!} K(ad^{k-2}(\alpha) d\alpha \wedge d\alpha), \\ \omega_2 = & \frac{1}{4it} e^*_{(t,u)} K(A^{-1} dA \wedge d\bar{A}\bar{A}^{-1}). \end{aligned} \tag{33}$$

1) A direct calculation shows

$$
\begin{aligned}
d\omega_2 &= \frac{1}{4it} e^*_{(t,u)} d\{K(A^{-1}dA \wedge d\bar{A}\bar{A}^{-1})\} \\
&= -\frac{1}{4it} e^*_{(t,u)} \{K((A^{-1}dA)^2 \wedge d\bar{A}\bar{A}^{-1}) \\
&\qquad + K((A^{-1}dA) \wedge (d\bar{A}\bar{A}^{-1})^2)\} \qquad (34) \\
&= -\frac{1}{12it} j^* K(d\mathbf{A}\mathbf{A}^{-1} \wedge (d\mathbf{A}\mathbf{A}^{-1})^2).
\end{aligned}
$$

Let us recall that $\mathbf{A} = A\bar{A} = j(a)$.

Using equation

$$
d\mathbf{A}\mathbf{A}^{-1} = (E^{-1})^* \left(\frac{e^{2it\lambda} - 1}{\lambda}\right)_{\lambda = ad(a)} d\alpha \qquad (35)
$$

one can easily show that

$$
\begin{aligned}
d\omega_1 = {} & d\left\{\frac{1}{4it} K^* \sum_{k=2}^{\infty} \frac{(2it)^k}{k!} K(ad^{k-2}(\alpha) d\alpha \wedge d\alpha)\right\} \qquad (36) \\
= {} & \frac{1}{12it} j^* K(d\mathbf{A}\mathbf{A}^{-1} \wedge (d\mathbf{A}\mathbf{A}^{-1})^2).
\end{aligned}
$$

Together (34) and (36) imply the first statement of the lemma.

2) To evaluate the form $\Omega_{(t,u)}$ on the universal vector field v we notice that

$$
da(v_\epsilon) = -K(ad(\alpha)\epsilon) \qquad (37)
$$

for any $\epsilon \in \mathcal{G}$. Taking into account (35) we infer

$$
\omega_1(\cdot, v_\epsilon) = \frac{1}{4it} j^* K(d\mathbf{A}\mathbf{A}^{-1} + \mathbf{A}^{-1}d\mathbf{A}, \epsilon) - < da, \epsilon > . \qquad (38)
$$

Another straightforward computation leads to

(39)

$$
\omega_2(\cdot, v_\epsilon) = \frac{1}{4it} e^*_{(t,u)} K(A^{-1}dA - d\bar{A}\bar{A}^{-1}, A^{-1}\epsilon A - \bar{A}\epsilon\bar{A}^{-1}).
$$

Combining the last two equations we conclude

(40)

$$
\Omega_{(t,u)}(\cdot, v_\epsilon) = \frac{1}{2it} e^*_{(t,u)} K(dAA^{-1} + \bar{A}^{-1}d\bar{A}, \epsilon) - < da, \epsilon > .
$$

Taking into account the definition (2) of the nondegenerate scalar product on $\mathcal{G}^C$ one can rewrite this formula as

$$\Omega_{(t,u)}(\cdot, v) = \frac{1}{t} e^*_{(t,u)} dAA^{-1} - da. \tag{41}$$

This observation completes the proof of Lemma 2.

Remark. One can guess the expression (31) for the 2-form $\Omega_{(t,u)}$ comparing Kirillov symplectic forms on the coadjoint orbits to the symplectic forms on the orbits of dressing transformations computed in [4], [3].

Proof of Theorem. By the assumptions of the theorem the manifold M is equipped with two maps $\mathbf{m} : M \to G^*_u$ and $\mu : M \to \mathcal{G}^*$, where $\mathbf{m}$ is the moment map in the sense of Lu and Weinstein and $\mu = e^{-1}_{(t,u)} \circ \mathbf{m}$. Let us define a 2-form $\tilde{\omega}$ on M by the formula

$$\tilde{\omega} = \omega - \mu^* \Omega_{(t,u)}. \tag{42}$$

In fact, the form $\tilde{\omega}$ provides the new symplectic structure on M which we are looking for.

First, observe that $\tilde{\omega}$ is a closed 2-form on M:

$$d\tilde{\omega} = d\omega - \mu^* d\Omega_{(t,u)} = 0. \tag{43}$$

Moreover, $\tilde{\omega}$ belongs to the same cohomology class as ω. Indeed, $\Omega_{(t,u)}$ is a closed 2-form on the linear space $\mathcal{G}^*$. Hence, it is exact, and its pull-back $\mu^*\Omega_{(t,u)}$ is also an exact form.

Let us evaluate $\tilde{\omega}$ on the universal vector field v:

$$\begin{aligned} \tilde{\omega}(\cdot, v) &= \omega(\cdot, v) - \mu^*\Omega_{(t,u)}(\cdot, v) \\ &= \frac{1}{t}\mathbf{m}^*(dAA^{-1}) - \mu^*(\frac{1}{t} e^*_{(t,u)}(dAA^{-1}) - da) \\ &= \mu^*(da). \end{aligned} \tag{44}$$

In particular, this implies that $\tilde{\omega}$ is G-invariant:

$$\mathcal{L}_v \tilde{\omega} = (di_v + i_v d)\tilde{\omega} = d\mu^*(da) = 0. \tag{45}$$

So, if $\tilde{\omega}$ defines a symplectic structure on M, it is G-invariant and possesses an equivariant moment map $\mu : M \to \mathcal{G}^*$.

The last point is to check the nondegeneracy of $\tilde{\omega}$. Assume that at some point $x \in M$ the form $\tilde{\omega}$ is degenerate. This means that there exists a nonvanishing vector ξ so that

$$\tilde{\omega}_x(\cdot, \xi) = 0, \tag{46}$$

which implies

$$\omega_x(\cdot, \xi) = \mathbf{m}^*(e^{-1}_{(t,u)})^* \Omega_{(t,u)}(\cdot, \mathbf{m}_* \xi) \equiv \mathbf{m}^* \eta. \tag{47}$$

The right-hand side is a pull-back of a certain 1-form η on G_u^* along the map $\mathbf{m}$. Any such a form η can be represented as

$$\eta = < dAA^{-1}, \zeta > \tag{48}$$

with some $\zeta \in \mathcal{G}$. Now consider a vector

$$\tilde{\xi} = \xi - \frac{1}{t} v_\zeta \tag{49}$$

at the point $x \in M$. It is easy to see that the form ω annihilates this vector:

$$\omega_x(\cdot, \xi - \frac{1}{t} v_\zeta) = \eta - t\frac{1}{t} < dAA^{-1}, \zeta >= 0. \tag{50}$$

This means that the form ω is also degenerate at x which contradicts the assumptions of the theorem. So, $\tilde{\omega}$ defines a symplectic structure on M. This completes the proof of Theorem 1.

Remark. It is easy to see that we can exchange the roles of Hamiltonian and Poisson actions in Theorem 1 . Moreover, we can directly compare Poisson actions with different values of parameters t and u.

Corollaries for Poisson actions

Here we give new short proofs of several results on the actions of Poisson-Lie groups on symplectic manifolds.

Recently Flaschka and Ratiu [5] proved the following convexity theorem for the moment map in the sense of Lu and Weinstein (see also [8], [11]).

Corollary 1. *Let M be a compact symplectic manifold which carries a Poisson action $\mathcal{A}$ of the compact group G equipped with the Poisson structure $\mathcal{P}_{(t,u)}$. Assume that there exists an equivariant moment map $\mathbf{m} : M \to G_u^*$, and define the map $\mu = e_{(t,u)}^{-1} \circ \mathbf{m}$. Then the intersection of $\mu(M)$ with the positive Weyl chamber W_+*

$$\mu_+(M) = \mu(M) \cap W_+ \tag{51}$$

is a convex polytop.

Proof. As we know, the map μ provides a Hamiltonian equivariant moment map for some symplectic structure on M. Convexity property for the map $\mathbf{m}$ as stated above coincides with the standard convexity for the Hamiltonian moment map μ [1], [6].

The technique of Hamiltonian reduction has been generalized to Poisson actions by Lu [10]. Here we need some new notation and definitions to formulate a statement.

Definition 9. The value $\gamma \in G_u^*$ is called a regular value of the moment map $\mathbf{m} : M \to G_u^*$ if some quotient of G over a discrete (possibly trivial) subgroup F of the center of G acts freely on the preimage $\mathbf{m}^{-1}(\mathcal{O}_\gamma)$of the dressing orbit $\mathcal{O}_\gamma = AD^*(G)\gamma$.

It is convenient to introduce a special notation for the canonical projection

$$\pi : M \to M/G \tag{52}$$

to the quotient space M/G and for the embedding of $\mathbf{m}^{-1}(\gamma)$ into M:

$$i_\gamma : \mathbf{m}^{-1}(\gamma) \to M. \tag{53}$$

Corollary 2. *Let M be a symplectic manifold which carries a Poisson action $\mathcal{A}$ of the compact group G equipped with the Poisson structure $\mathcal{P}_{(t,u)}$. Assume that there exists an equivariant moment map $\mathbf{m} : M \to G_u^*$. Choose some $\gamma \in G_u^*$ which is a regular value of the moment map. Then $M_\gamma = \pi(\mathbf{m}^{-1}(\gamma)$ is a symplectic manifold with symplectic structure ω_γ defined via*

$$\pi^* \omega_\gamma|_{\mathbf{m}^{-1}(\gamma)} = i_\gamma^* \omega. \tag{54}$$

Proof. Let us switch to the symplectic structure $\tilde{\omega}$ on M and let $c = e_{(t,u)}^{-1}(\gamma)$. The map $e_{(t,u)}$ being equivariant, the space M_γ coincides with the reduced space obtained by the Hamiltonian reduction over the value c of the moment map μ. In fact, symplectic structures of the Hamiltonian and Poisson reduced spaces also coincide, since

$$i_\gamma^*(\omega - \tilde{\omega}) = i_\gamma^* \mu^* \Omega_{(t,u)} = 0. \tag{55}$$

The latter is true because the embedding i_γ chooses the point in $c \in \mathcal{G}^*$ and the pull-back of the 2-form $\Omega_{(t,u)}$ to this point vanishes for dimensional reasons.

By now we compared (M,ω) and $(M,\tilde{\omega})$ as symplectic G-spaces. It is clear that they do not coincide in this category as the G-action preserves $\tilde{\omega}$ and changes ω. However, it possible that (M,ω) and $(M,\tilde{\omega})$ are isomorphic as symplectic spaces (now we disregard the G-action). This is indeed the case, the isomorphism between (M,ω) and $(M,\tilde{\omega})$ is called Ginzburg-Weinstein isomorphism [7].

Corollary 3. *For arbitrary values of parameters t and u, (M,ω) and $(M,\tilde{\omega})$ are isomorphic as symplectic spaces. In particular, orbits of dressing transformations are symplectomorphic to the corresponding coadjoint orbits.*

Proof. Choose some primitive $\alpha_{(t,u)}$ of the 2-form $\Omega_{(t,u)}$:

$$\Omega_{(t,u)} = d\alpha_{(t,u)}. \tag{56}$$

We would like to vary parameters t and u of the Poisson bracket of G. For simplicity we change only t. When t varies, the symplectic form $\omega = \tilde{\omega} + \mu^*\Omega_{(t,u)}$ changes as:

$$\frac{\partial}{\partial t}\omega = \mu^* \frac{\partial \Omega_{(t,u)}}{\partial t} = \mu^* d\, \frac{\partial \alpha_{(t,u)}}{\partial t}. \tag{57}$$

Denote

$$\beta_{(t,u)} = \frac{\partial \alpha_{(t,u)}}{\partial t} \tag{58}$$

and construct a vector field $V_{(t,u)}$

$$V_{(t,u)} = \mathcal{P}_M(\cdot, \mu^*\beta_{(t,u)}). \tag{59}$$

The vector field $V_{(t,u)}$ is a certain linear combination of the vector fields v_ϵ with coefficients $\mathcal{E}(\mathbf{m}(x))$ which depend only on the value of the moment map $\mathbf{m}(x)$:

$$V_{(t,u)} =< \mathcal{E}(\mathbf{m}(x)), v > \ , \ \beta_{(t,u)} =< \mathcal{E}(A), dAA^{-1} > . \tag{60}$$

The Lie derivative of the symplectic structure ω with respect to $V_{(t,u)}$ coincides with the t-derivative:

$$\mathcal{L}_{V_{(t,u)}}\omega = di_{V_{(t,u)}}\omega = d\mu^*\beta_{(t,u)} = \frac{\partial\omega}{\partial t}. \tag{61}$$

Integrating the (t-dependent) field $V_{(t,u)}$ we construct a family of Ginzburg-Weinstein isomorphisms which identify (M,ω) and $(M,\tilde{\omega})$ for different values of t. One can construct symplectomorphisms between these spaces with different values of u in a similar fashion.

Remark. Formula (60) for the vector field $V_{(t,u)}$ makes it possible to extend the Ginzburg-Weinstein isomorphism to Poisson manifolds which carry a Poisson G-action and possess an equivariant moment map $\mathbf{m} : M \to G^*_u$ in the following sense:

$$v = \frac{1}{t}\mathcal{P}_M(\cdot, \mathbf{m}^*(dAA^{-1})). \tag{62}$$

This condition implies that symplectic leaves are preserved by the G-action. Integrating the vector field (60) one can obtain a diffeomorphism $D_{(t,u)}$ of M which preserves symplectic leaves and replaces the Poisson structure $\mathcal{P}_M$ by the G-invariant Poisson structure $\tilde{\mathcal{P}}_M$. Restricted to each symplectic leaf $D_{(t,u)}$ coincides with the Ginzburg-Weinstein symplectomorphism described above. Thus the new Poisson G-space $(M, \tilde{\mathcal{P}}_M)$ possesses an equivariant moment map $\mu = e^{-1}_{(t,u)} \circ \mathbf{m}$ which arises from the equivariant moment maps on each symplectic leaf.

Let us apply this construction to the Poisson space G^*_u equipped with the Poisson structure $\mathcal{P}^*_{(t,u)}$ from the standard list. The dressing action of G preserves symplectic leaves, the moment map is equal to the identity $\mathbf{m} = id : G^*_u \to G^*_u$. The Ginzburg-Weinstein diffeomorphism $D_{(t,u)}$ endows G^*_u with a new G-invariant Poisson structure $\tilde{\mathcal{P}}^*_{(t,u)}$ and a new moment map $\mu = e^{-1}_{(t,u)} : G^*_u \to \mathcal{G}^*$. Both maps $D_{(t,u)}$ and μ are invertible Poisson maps. Thus, an invertible Poisson map $e^{-1}_{(t,u)} \circ D_{(t,u)}$ establishes a Poisson isomorphism of $(G^*_u, \mathcal{P}^*_{(t,u)})$ and $\mathcal{G}^*$ equipped with the standard Kirillov-Kostant-Sourieu bracket. In fact, we have recovered the original version of the Ginzburg-Weinstein isomorphism [7].

Appendix. The case of t=0

Here we collect some details on the special family of Poisson structures $\mathcal{P}_{(0,u)}$

$$\mathcal{P}_{(0,u)} = \lim_{t\to 0} t\mathcal{P}_{u/t} \tag{63}$$

on compact Lie groups. All results obtained in the main text generalize to this special family. In fact, calculations become much easier. For this reason, we provide only the basic definitions and formulas related to the proof of Lemma 2. The proofs of Theorem 1 and of all Corollaries do not change.

For the special family of Poisson structures (63) the dual Lie algebra is a subset in the semi-direct product of the Cartan subalgebra $\mathcal{H}$ and the dual Lie algebra $\mathcal{G}_0^*$ considered as an Abelian Lie algebra:

$$\mathcal{G}^*_{(0,u)} = \{(ih+n, -u(h)), h \in \mathcal{H}, n \in \mathbf{n}^{\mathbb{C}}\}. \tag{64}$$

The $\mathcal{H}$ component acts on the $\mathcal{G}_0^*$ component by the natural coadjoint action.

The corresponding Lie group is a subgroup in the semi-direct product of the Cartan subgroup $H \subset G$ and $\mathcal{G}_0^*$ (viewed as an Abelian group with addition playing the role of the group operation):

$$G^*_{(0,u)} = \{(ih+n, exp\{-u(h)\}), h \in \mathcal{H}, n \in \mathbf{n}^{\mathbb{C}}\}. \tag{65}$$

The equivariant map $e_u : \mathcal{G}_0^* \to G^*_{(0,u)}$ is defined as

$$e_u(ih+n) = (ih+n, exp\{-u(h)\}). \tag{66}$$

The inverse map e_u^{-1} is a forgetting map which drops the second component of the pair.

It is instructive to compare Maurer-Cartan forms for the Abelian group $\mathcal{G}_0^*$:

$$a = ih + n\ ,\ da = idh + dn, \tag{67}$$

and for the group $G^*_{(0,u)}$:

$$\begin{gathered} A = (ih+n, exp\{-u(h)\}), \\ dAA^{-1} = (idh + dn - [u(dh), n], -u(dh)). \end{gathered} \tag{68}$$

Let us mention that the second component in the pair describing dAA^{-1} is disregarded in the pairing with elements of $\mathcal{G}$.

The definition of the moment map in the sense of Lu and Weinstein modifies as follows:

Definition 10. Let G be a compact Poisson-Lie group equipped with a Poisson structure $\mathcal{P}_{(0,u)}$. Let $\mathcal{A} : G \times M \to M$ be a Poisson action of G on the symplectic manifold M. The map $\mathbf{m} : M \to G^*_{(0,u)}$ is called a moment map in the sense of Lu and Weinstein if

$$\omega(\cdot, v) = \mathbf{m}^*(dAA^{-1}), \tag{69}$$

where dAA^{-1} is a right-invariant Maurer-Cartan form on $G^*_{(0,u)}$.

Lemma 2 in this situation is reformulated as:

Lemma 3. *There exists such a 2-form Ω_u on $\mathcal{G}^*$, so that the following two properties are fulfilled:*

1) The form Ω_u is closed, i.e., $d\Omega_u = 0$.

2) $\Omega_u(\cdot, v) = e_u^* dAA^{-1} - da$.

Here v is the universal vector field corresponding to the coadjoint action of G on $\mathcal{G}^*$, $a \in g^*$ and $A = e_u(a) \in G^*_{(0,u)}$.

Proof. The 2-form Ω_u which fulfils these two properties looks as

$$\Omega_u = \frac{1}{2} u(dh \wedge dh), \tag{70}$$

where h is a Cartan projection of $(ih + n) \in \mathcal{G}_0^*$.

Obviously, Ω_w is closed. Evaluating it on the universal vector field v one finds:

$$\begin{aligned} \Omega_u(\cdot, v_\alpha) &= \frac{1}{2i} K(u(dh), [\alpha, n + \bar{n}]) \\ &= -< [u(dh), n], \alpha > \\ &= < e_u^* dAA^{-1} - da, \alpha > . \end{aligned} \tag{71}$$

This completes the proof of Lemma 3.

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