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INTERIORS OF COMPACT CONTRACTIBLE N-MANIFOLDS ARE HYPERBOLIC ($N \geq 5$)

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Abstract

The interior of every compact contractible PL n-manifold ($n \geq 5$) supports a complete geodesic metric of strictly negative curvature. This provides a new family of simple examples illustrating the negative answer to a question of M. Gromov which asks whether metrically convex geodesic spaces which are topological manifolds must be homeomorphic to Euclidean spaces. The first examples verifying the negative answer to this question were given by M. Davis and T. Januszkiewicz [11].

0. Introduction

One goal of Riemannian geometry is to use local information about a manifold to make conclusions about its global structure. A prime example is the classical Cartan-Hadamard Theorem which guarantees that every complete simply connected Riemannian manifold with non-positive sectional curvature at each point is diffeomorphic to Euclidean space. The success of Riemannian geometry has inspired generalizations of its definitions and methods to wider classes of spaces. One effort, initiated by A. D. Aleksandrov (see [1], [2] and [3]) in the 1950's, and returned to prominence by M. Gromov in the 1980's, uses properties of triangles to extend the notion of curvature, $\kappa(X)$, at a point x, to "geodesic spaces". These are metric spaces in which (as in complete Riemannian manifolds), the distance between two points can always be realized by a geodesic arc between them. A result of this theory which illustrates the extent to which it generalizes Riemannian geometry is the following version of the Cartan-Hadamard Theorem. (See [13] and [14]).

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Theorem 0.1 (Cartan-Hadamard-Alexandrov). *Let $(X d)$ be a complete geodesic space and suppose that $\kappa(x) \leq 0$ for all $x \in X$. Then X is metrically convex and hence, contractible.*

This theorem will be discussed in more detail below.

The question of whether there is a full generalization of the Cartan-Hadamard Theorem for geodesic spaces was posed by M. Gromov who in [13] asked:

Question 0.2. If X is a metrically convex geodesic space which is a topological n-manifold, must X be homeomorphic to $\mathbb{R}^n$?

A negative answer to Gromov's question was recently given by Davis and Januszkiewicz in [11], where a method is described for constructing counterexamples in dimensions ≥ 5. These examples are the universal covers of manifolds produced by a complicated "hyperbolization" process applied to a non-combinatorial triangulation of S^n. In this note we add a large collection of simple examples to the list of "exotic" metrically convex n-manifolds by proving:

Main Theorem. *The interior of every compact contractible PL n-manifold $(n \geq 5)$ supports a complete geodesic metric of strictly negative curvature.*

Note. The "PL" hypothesis is unnecessary except possibly when $n = 5$. Indeed, if C^n is a compact contractible n-manifold, then its Kirby-Siebenmann invariant, which lies in $H^4(C^n; \mathbb{Z}^2)$ vanishes. Consequently, the results of [16], which apply to manifolds with boundary of dimension ≥ 6, imply that C^n admits a PL structure when $n > 5$. However, there may exist non-triangulable compact contractible 5-manifolds. Indeed, if there is a non-triangulable homology 4-sphere Σ, the existence of which is not precluded by presently known results, then the cone on Σ can be resolved (by [8] or [17]) to obtain a non-triangulable compact contractible 5-manifold. So the "PL" hypothesis is possibly non-redundant when $n = 5$.

The proof of the Main Theorem employs a mixture of geometry and topology – most notably geometric constructions by V. N. Berestovskii [6], and topological results from [4] which utilize a manifold recognition theorem of R. D. Edwards [12]. Paper [4] provides a simple picture of a compact contractible manifold which makes it possible to define explicitly a metric on its interior.

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N. Berestovskii and the referee for comments that led to a change in the final form of the proof of the Main Theorem. We will discuss this change further in Section 5.

1. Definitions

Here we define and comment on various notions of curvature in metric spaces. The reader is cautioned that these definitions are not all standard. There are many instances in the literature where the same term has been given different meanings, other instances where a single concept goes by several different names, and still more cases where different but (for the most part) equivalent definitions have evolved for the same core idea. Because of this, the terms and definitions we have chosen sometimes differ from those used in the original sources.

Throughout this paper all metric spaces are complete and locally compact. An isometric map from an interval into a metric space (X, ρ) is called a *geodesic arc*. A *triangle* in X consists of three points (called *vertices*) together with three geodesic arcs (called edges) connecting them. We say that (X, ρ) is a *geodesic space* if every pair of points in X can be connected by a geodesic arc. If $Y \subset X$ and if every geodesic arc in X between points in Y is contained in Y, then we call Y a *strongly geodesic subspace* of X; if this property holds locally, then Y is called a *locally strongly geodesic subspace*. We say that (X, ρ) is *metrically convex* if for any two geodesic arcs $\alpha : [a, b] \to X$ and $\gamma : [c, d] \to X$, the map $\Phi : [a, b] \times [c, d] \to \mathbb{R}$ defined by $\Phi(s, t) = \rho(\alpha(s), \gamma(t))$ is convex (i.e., $\Phi((1-\lambda)p + \lambda q) \leq (1-\lambda)\Phi(p) + \lambda\Phi(q)$ for all $p, q \in [a, b] \times [c, d]$ and $0 \leq \lambda \leq 1$.) If this property holds locally, then (X, ρ) is said to be *locally metrically convex*. Note that metrical convexity implies that the geodesic arc joining two points is unique up to reparametrization of the domain by translation.

For each $K \in \mathbb{R}$ and each positive integer n, let $M^n(K)$ denote the (unique up to isometry) complete simply-connected Riemannian n-manifold of constant sectional curvature K, and let ρ_K denote the path length distance function on $M^n(K)$. For example, $M^2(-1)$ is the hyperbolic plane, $M^2(0)$ is the Euclidean plane $\mathbb{R}^2$, and $M^2(1)$ is the unit sphere S^2 in $\mathbb{R}^3$ with the usual path length metric.

If T is a triangle in a geodesic space (X, ρ) and $K \in \mathbb{R}$, then a *comparison triangle* for T in $M^2(K)$ is a triangle in $M^2(K)$ with edges of the same length as the corresponding edges of T. It is easily seen that for any $K \in \mathbb{R}$, every triangle of perimeter $< 2\pi/\sqrt{K}$ (where we define $2\pi/\sqrt{K} = \infty$ if $K \leq 0$) in a geodesic space has a comparison triangle in $M^2(K)$. Moreover, it is a standard fact that a comparison triangle in $M^2(K)$ is unique up to isometry of $M^2(K)$.

Let $K \in \mathbb{R}$ and let T be a triangle in the geodesic space (X, ρ) with vertices A, B and C and perimeter $< 2\pi/\sqrt{K}$. We say that T *satisfies* CAT(K) if for any $P \in \{A, B, C\}$ and any $Q \in T$, $\rho(P, Q) \leq \rho_K(P', Q')$ where P' and Q' are the corresponding points on a comparison triangle in $M^2(K)$. We say that X *satisfies* CAT(K) if every triangle in X with perimeter $< 2\pi/\sqrt{K}$ satisfies CAT(K). If a point x of X has a neighborhood in which every triangle with perimeter $< 2\pi/\sqrt{K}$ satisfies CAT(K), we say that X *satisfies* CAT(K) *at* x, and write $\kappa(x) \leq K$. It $\kappa(x) \leq K$ for each $x \in X$, we say that X *satisfies* CAT(K) locally, we write $\kappa(X) \leq K$ and we also say that X has curvature $\leq K$. If $\kappa(X) \leq K < 0$, we say that X *has strictly negative curvature* or that X *is hyperbolic.*

Remark. Our curvature criterion (the CAT(K) inequality) differs from Aleksandrov's original criterion, which we denote by CAT$^A(K)$. Roughly speaking, a triangle in a geodesic space satisfies CAT$^A(K)$ if the sum of its angle measures is less than the sum of the angles measures of a comparison triangle in $M^2(K)$. Of course, one must define an appropriate notion of angle measure in a geodesic space before applying this criterion. A development of this strategy is found in [3]. A similar condition, also credited to Aleksandrov and referred to as "Criterion A" in [20] again uses a type of angle measure in a geodesic space as its curvature criterion.

Yet another curvature criterion, this one similar to the CAT(K) inequality, will be denoted CAT$^*(K)$. A triangle T in a geodesic space satisfies $CAT^*(K)$ if for any two points P and Q on T, $\rho(P, Q) \leq \rho_K(P', Q')$ where P' and Q' are the corresponding points on a comparison triangle in $M^2(K)$.

Using any of the above definitions, one may define the curvature of a geodesic space to be $\leq K$ at a point x provided x has a neighborhood in which the chosen criterion is satisfied by all triangles with perimeter less that $2\pi/\sqrt{K}$ contained in that neighborhood. To see that these competing definitions lead to equivalent results, consult Theorem 4 and Remark 8 of [20] and Theorem 3.2 and Remark 5.4 of [3].

2. Outline of the proof of the Main Theorem

The Main Theorem places a negative curvature metric on the interior of every compact contractible PL manifold of dimension ≥ 5. Here we outline the construction to motivate our later considerations.

Let $\mathcal{O}(W)$ denote the open cone of the topological space W. (A precise definition is given in Section 5.) Given a compact contractible n-manifold $C^n (n \geq 5)$, the main result of [4] allows us to represent $\text{int}(C^n)$ as the union of three pieces: two open cones $\mathcal{O}(Q_0)$ and $\mathcal{O}(Q_1)$, and the product of an open cone $\mathcal{O}(\Sigma)$ with $[0,1]$. (See Figures 2 and 3 in Section 6.) Here, Q_0, Q_1 and Σ are simplicial complexes and Σ is identified with a subcomplex Σ_i of Q_i for $i = 0, 1$. (In fact, Q_i is a compact $(n-1)$-manifold and Σ_i is its boundary.) Moreover, as subsets of $\text{int}(C^n), \mathcal{O}(Q_0)$ and $\mathcal{O}(Q_1)$ are disjoint, and for $i = 0$ or $1, \mathcal{O}(Q_i)$ intersects $\mathcal{O}(\Sigma) \times [0,1]$ in the set $\mathcal{O}(\Sigma_i) = \mathcal{O}(\Sigma) \times \{i\}$.

Let $K < 0$. We will impose a $\text{CAT}(K)$ structure on $\text{int}(C^n)$ by putting $\text{CAT}(K)$ structures on the three pieces $\mathcal{O}(Q_0), \mathcal{O}(Q_1)$ and $\mathcal{O}(\Sigma) \times [0,1]$ so that for $i = 0$ or $1, \mathcal{O}(\Sigma_i)$ and $\mathcal{O}(\Sigma) \times \{i\}$ are isometric strongly geodesic subsets of $\mathcal{O}(Q_i)$ and $\mathcal{O}(\Sigma) \times [0,1]$, respectively. Then the union of the $\text{CAT}(K)$ structures on the three pieces yields a $\text{CAT}(K)$ structure on $\text{int}(C^n)$. The construction of $\text{CAT}(K)$ metrics on the three pieces is described in Section 5, and exploits techniques developed by Berestovskii in [6] and extended in [3]. These techniques first allow us to put $\text{CAT}(1)$ structures on the simplicial complexes Q_0, Q_1 and Σ so that for $i = 0$ or 1, Σ_i is a strongly geodesic subset of Q_i which is isometric to Σ. The techniques then allow us to place $\text{CAT}(K)$ structures on the open cones $\mathcal{O}(Q_0), \mathcal{O}(Q_1)$ and $\mathcal{O}(\Sigma)$ so that for $i = 0$ or 1, $\mathcal{O}(\Sigma_i)$ is a strongly geodesic subspace of $\mathcal{O}(Q_i)$ which is isometric to $\mathcal{O}(\Sigma)$. It then remains to put a $\text{CAT}(K)$ structure on $\mathcal{O}(\Sigma) \times [0,1]$ in which $\mathcal{O}(\Sigma) \times \{0\}$ and $\mathcal{O}(\Sigma) \times \{1\}$ are strongly geodesic subspaces isometric to $\mathcal{O}(\Sigma)$. This is accomplished via Lemmas 5.4 and 5.5. The first of these lemmas shows how to impose a $\text{CAT}(K)$ structure on $X \times \mathbb{R}$, given a $\text{CAT}(K)$ structure on X; and the second lemma shows that if, in addition, X is an open cone, then the $\text{CAT}(K)$ structure on $X \times \mathbb{R}$ can be chosen so that each level $X \times \{t\}$ is a strongly geodesic subspace isometric to X. These lemmas clearly solve the remaining problem of putting the appropriate $\text{CAT}(K)$ structure on $\mathcal{O}(\Sigma) \times [0,1]$, finishing the argument.

In an earlier version of this paper, Lemmas 5.4 and 5.5 were only conjectured, and a more ad hoc method was used to put a $\text{CAT}(K)$

structure on $\mathcal{O}(\Sigma) \times [0,1]$. In particular, it was noted that results of Berestovskii impose a CAT(K) structure on $\mathcal{O}(\mathcal{S}(\Sigma))$ where $\mathcal{S}(\Sigma)$ denotes the suspension of Σ; and it was observed that $\mathcal{O}(\Sigma)\times[0,1]$ embeds naturally in $\mathcal{O}(\mathcal{S}(\Sigma))$ so that for each $t \in [0,1], \mathcal{O}(\Sigma) \times \{t\}$ embeds onto a strongly geodesic subspace which is isometric to $\mathcal{O}(\Sigma)$. In response to the referees encouragement and a communication from Berestovskii, we found proofs of these lemmas and substituted them for the ad hoc argument.

3. Elementary properties of the spaces $M^n(K)$

Here we record some simple properties of the spaces $M^n(K)$ which we will use below.

If $K \neq 0$ and $\varepsilon = K/|K|$, then $M^n(K)$ and $M^n(\varepsilon)$ are closely related by the following observation. If M is a Riemannian manifold and $c > 0$, then multiplying M's Riemannian metric by $1/c$ has the effect of multiplying M's sectional curvature operator by c. This is easily verified directly from the definitions of the curvature and sectional curvature operators. Consequently, the identity map from M with the original Riemannian metric to M with $1/c$ times the original metric is an angle preserving (i.e., conformal) diffeomorphism which multiplies distance by $1/\sqrt{c}$. So if $K \neq 0, \varepsilon = K/|K|$ and $k = \sqrt{|K|}$, then we can regard $M^n(\varepsilon)$ and $M^n(K)$ as having the same underlying manifold and the same angle measures; and if two points are at distance d in $M^n(\varepsilon)$, then they are at distance d/k in $M^n(K)$.

Let $K < 0$, set $k = \sqrt{|K|}$, and let T be a triangle in $M^n(K)$ with sides of lengths a, b and c and angles of measures α, β and γ where side a is opposite angle α, side b is opposite angle β, and side c is opposite angle γ. If $K = -1$, then $M^n(K)$ is hyperbolic n-space and the hyperbolic sine and cosine laws are:

$$\frac{\sin(\alpha)}{\sinh(a)} = \frac{\sin(\beta)}{\sinh(b)} = \frac{\sin(\gamma)}{\sinh(c)}$$

and

$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma).$$

(See [9, p. 238].) In general, if $K < 0$, then when viewed in $M^n(-1)$, T has the same angle measures α, β and γ, and has sides of length ka, kb and kc. So the hyperbolic sine and cosine laws yield the equations:

$$\frac{\sin(\alpha)}{\sinh(ka)} = \frac{\sin(\beta)}{\sinh(kb)} = \frac{\sin(\gamma)}{\sinh(kc)}$$

and

$$\cosh(kc) = \cosh(ka)\cosh(kb) - \sinh(ka)\sinh(kb)\cos(\gamma).$$

These equations may be regarded as the sine and cosine laws for $M^n(K)$.

Next we introduce rectangular coordinates on $M^2(K)$ when $K < 0$. We describe two inequivalent ways to do this, and we find transformation formulas relating the two. Let $K < 0$ and set $k = \sqrt{|K|}$. Choose a point $\mathbb{O} \in M^2(K)$, and choose geodesic lines ξ and η in $M^2(K)$ which intersect orthogonally at $\mathbb{O}$. Think of $\mathbb{O}$ as the origin and ξ and η as the X- and Y-axes. Choose isometrics $x \mapsto A_x : \mathbb{R} \to \xi$ and $y \mapsto B^y : \mathbb{R} \to \eta$ such that $A_0 = B^0 = \mathbb{O}$. For each $x \in \mathbb{R}$, let η_x denote the geodesic line in $M^2(K)$ through A_x orthogonal to ξ. Also for each $y \in \mathbb{R}$, let ξ^y denote the geodesic line in $M^2(K)$ through B^y orthogonal to η. Then $\eta_0 = \eta$ and $\xi^0 = \xi$, and both $\{\eta_x : x \in \mathbb{R}\}$ and $\{\xi^y : y \in \mathbb{R}\}$ fiber $M^2(K)$. For each $x \in \mathbb{R}$, let $y \mapsto A_x^y : \mathbb{R} \to \eta_x$ be the unique isometry such that $A_x^0 = A_x$ and A_x^1 and B^1 lie in the same component of $M^2(K) - \xi$. For each $y \in \mathbb{R}$, let $x \mapsto B_x^y : \mathbb{R} \to \xi^y$ be the unique isometry such that $B_0^y = B^y$ and B_1^y and A_1 lie in the same component of $M^2(K) - \eta$. Then $M^2(K) = \{A_x^y : x, y \in \mathbb{R}\} = \{B_x^y : x, y \in \mathbb{R}\}$. and we regard the functions $(x, y) \mapsto A_x^y : \mathbb{R} \times \mathbb{R} \to M^2(K)$ and $(x, y) \mapsto B_x^y : \mathbb{R} \times \mathbb{R} \to M^2(K)$ as two ways to assign rectangular coordinates to the points of $M^2(K)$. Since in general $A_x^y \neq B_x^y$ for $x, y \in \mathbb{R}$, these two ways are inequivalent.

Let $M^2(K)^+$ denote the "right half space" of $M^2(K)$; i.e., set $M^2(K)^+ = \{A_s^t : s \geq 0 \text{ and } t \in \mathbb{R}\} = \{B_s^t : s \geq 0 \text{ and } t \in \mathbb{R}\}$.

Consider a point P in $M^2(K)$. Then there are rectangular coordinates (x', y') and $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $A_{x'}^{y'} = B_x^y = P$. (See Figure 1.) We assert that (x', y') and (x, y) are related by the following transformation formulas:

$$(1) \qquad \left\{ \begin{array}{lcl} x & = & \frac{1}{k}\sinh^{-1}(\sinh(kx')\cosh(ky')) \\ y & = & \frac{1}{k}\tanh^{-1}\left(\frac{\tanh(ky')}{\cosh(kx')}\right) \\ x' & = & \frac{1}{k}\tanh^{-1}\left(\frac{\tanh(kx)}{\cosh(ky)}\right) \\ y' & = & \frac{1}{k}\sinh^{-1}(\sinh(ky)\cosh(kx)) \end{array} \right\}$$

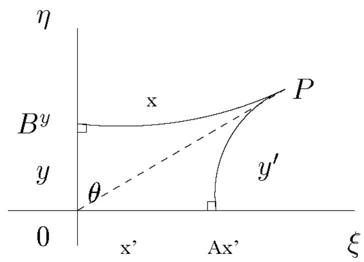


FIGURE 1

To prove these formulas, set $r = \rho_K(\mathbb{O}, P)$ and let θ denote the angle $B^y\mathbb{O}P$. (See Figure 1.) We apply the hyperbolic sine and cosine laws in the triangles $\mathbb{O}A_{x'}P$ and $\mathbb{O}B^yP$ to obtain the equations

$$\frac{\sinh(kx)}{\sin\theta} = \frac{\sinh(kr)}{1} = \frac{\sinh(ky')}{\sin\left(\frac{\pi}{2}-\theta\right)}, \tag{2a}$$

$$\cosh(kr)\cosh(kx)\cosh(ky), \tag{2b}$$

$$\cosh(kr)\cosh(kx')\cosh(ky'), \tag{2c}$$

$$\cosh(kx) = \cosh(ky)\cosh(kr) - \sinh(ky)\sinh(kr)\cos\theta, \tag{2d}$$

$$\cosh(ky') = \cosh(kx')\cosh(kr) - \sinh(kx')\sinh(kr)\cos\left(\frac{\pi}{2}-\theta\right). \tag{2e}$$

Equations (2a) imply

$$\sinh(kx) = \sinh(kr)\cos\left(\frac{\pi}{2}-\theta\right), \tag{3a}$$

$$\sinh(ky') = \sinh(kr)\cos\theta. \tag{3b}$$

Equations (2b) and (2c) imply

$$\cosh(kx)\cosh(ky) = \cosh(kx')\cosh(ky'). \tag{4}$$

Substituting (2c) and (3a) in (2e) yields

$$\cosh(ky') = \cosh^2(kx')\cosh(ky') - \sinh(kx')\sinh(kx).$$

Solving this equation for $\sinh(kx)$ and using the identity $\cosh^2(kx') - \sinh^2(kx') = 1$ gives us

$$\sinh(kx) = \sinh(kx-)\cosh(ky'). \tag{5}$$

Similarly, substituting (2b) and (3b) in (2d) and solving for $\sinh(ky')$ yield

$$\sinh(ky') = \sinh(ky)\cosh(kx) \tag{6}$$

Using equations (6) and (4), we obtain

$$\tanh(ky) = \frac{\sinh(ky)}{\cosh(ky)} = \frac{\sinh(ky')}{\cosh(kx)\cosh(ky)} = \frac{\sinh(ky')}{\cosh(kx')\cosh(ky')}.$$

Hence,

$$\tanh(ky) = \frac{\tanh(ky')}{\cosh(kx')}. \tag{7}$$

A similar application of equations (6) and (4) gives

$$\tanh(kx') = \frac{\tanh(kx)}{\cosh(ky)}. \tag{8}$$

The transformation formulas (1) now follow from equations (5), (6), (7) and (8).

4. Curvature, metric convexity and contractibility

In this section we briefly discuss some connections between curvature, metric convexity and contractibility. This will allow us to outline a proof of the Cartan-Hadamard-Aleksandrov Theorem, and to see the link between this result and Question 0.2.

Let $K \leq 0$ and suppose X is a simply connected geodesic space such that $\kappa(X) \leq K$. Then X satisfies CAT(K) by Theorems 7 and 13 of [5]. It follows that X is metrically convex by Proposition 29 of [20]. It is then easy to prove the contractibility of X. Fix a point $x_0 \in X$ and simply "slide" any other point of X toward x_0 along the (unique) geodesic arc joining the two points. The metric convexity of X guarantees that this process is well defined and continuous.

We conclude that if X is simply connected and $\kappa(X) \leq 0$, then X is contractible. This is the Cartan-Hadamard-Aleksandrov Theorem.

We also see that for $K \leq 0$, a contractible manifold of curvature $\leq K$ which is not homeomorphic to $\mathbb{R}^n$ provides a negative answer to Gromov's question which satisfies CAT(K).

5. Open cones and products

Here we describe methods for putting geometric structures on open cones and products of open cones with intervals. Such spaces are crucial to the proof of the Main Theorem because, as was explained earlier, the interior of every compact contractible manifold can be assembled from such pieces.

First we state a fundamental theorem of Berestovskii which puts a CAT(1) structure on every finite simplicial complex. This result is the ultimate source of all geometry imposed on spaces in this paper. Because it limits us to triangulated spaces, it also accounts for the "PL" hypothesis in the Main Theorem. Indeed, if a result comparable to Berestovskii's were known for all compact topological manifolds (including the non-triangulable ones), then the Main Theorem without the "PL" hypothesis would follow by a trivial modification of the present proof.

Berestovskii's theorem even imposes CAT(1) structures on *non-connected* simplicial complexes. Since such objects cannot possibly be geodesic spaces, we require a notion which generalizes CAT(K) to non-connected spaces. To this end, for $K > 0$, define a metric space (W, d) to be a *K-domain* if it satisfies the following:

(a) if $d(w, w') < \pi/\sqrt{K}$, then w and w' can be joined by a geodesic in W,

(b) triangles in W with perimeter less than $2\pi/\sqrt{K}$ satisfy CAT(K).

Note that a K-domain need not be connected, and, thus, may not be a geodesic space.

If Γ is a simplicial complex, let $|\Gamma|$ denote its underlying polyhedron. By a *K domain metric* on a simplicial complex Γ we mean a metric d on $|\Gamma|$ such that for every subcomplex Δ of Γ (including $\Delta = \Gamma$), the restriction of d to $|\Delta|$ makes $|\Delta|$ into a K domain. In [6], Berestovskii showed that each finite dimensional simplex (regarded as the simplicial complex determined by its faces) admits a 1-domain metric. (See Lemma 2 of [6].) Since every finite simplicial complex Γ can be embedded in a simplex σ of sufficiently high dimension so that Γ and all its subcomplexes are subcomplexes of σ, we have the following version of Berestovskii's theorem.

Lemma 5.1. *Every finite simplicial complex* Γ *admits a 1-domain metric. (Hence, the polyhedron underlying every subcomplex of* Γ *becomes a 1-domain under this metric.)*

As we mentioned above, we could remove the "PL" hypothesis from the statement of the Main Theorem if we knew a version of Lemma 5.1 for compact topological manifolds. In particular, it would suffice to establish the following assertion. Given a (possibly non-triangulable) compact topological n-manifold W without boundary and a compact $(n-1)$-dimensional submanifold V of W without boundary such that V separates W and V is collared in W (i.e., there is a topological embedding of $V \times \mathbb{R}$ into W which sends $V\{0\}$ onto V), then there is a metric d on W which makes W a 1-domain and such that the restriction of d to V makes V a 1-domain.

Since open cones are contractible, it is consistent with Theorem 0.1 that they admit $\mathrm{CAT}(K)$ metrics for $K \leq 0$. Moreover, since an open cone has such a simple structure, one can hope to define a $\mathrm{CAT}(K)$ metric on it by an explicit formula. Indeed, one of the virtues of 1-domains is that the open cone over a 1-domain admits an explicitly defined $\mathrm{CAT}(K)$ metric. The formula for this metric is based on the cosine law for $M^n(K)$. The idea for defining a metric on a cone via a cosine law originates in [6] and is more fully elaborated in [3]. We will outline the essential points.

If W is a topological space, the *open cone* over W is the quotient space $\mathcal{O}(W) = (W \times [0,\infty))/(W \times \{0\})$. The *vertex* of $\mathcal{O}(W)$ is the point of $\mathcal{O}(W)$ which is the image of $W \times \{0\}$ under the quotient map $W \times [0,\infty) \to \mathcal{O}(W)$. The space W is called the *base* of $\mathcal{O}(W)$. For $(w,r) \in W \times [0,\infty)$, we let rw denote the point of $\mathcal{O}(W)$ which is the image of (w,r) under the quotient map $W \times [0,\infty) \to \mathcal{O}(W)$. Thus, for each $w \in W$, $0w$ denotes the vertex of $\mathcal{O}(W)$.

Let (W,d) be a metric space. Define the metric θ on W by the formula $\theta(w,w') = \min\{d(w,w'),\pi\}$. Then θ is equivalent to d. Let $K < 0$, and set $k = \sqrt{|K|}$. Define the K *cosine law metric* on $\mathcal{O}(W)$ to be the function $\sigma_K : \mathcal{O}(W) \times \mathcal{O}(W) \to [0, \infty)$ defined by the formula

$$\sigma_K(r_1 w_1, r_2 w_2) = (1/k)\cosh^{-1}\Big(\cosh(kr_1)\cosh(kr_2) - \sinh(kr_1)\sinh(kr_2)\cos(\theta(w_1, w_2))\Big).$$

Clearly, this formula is motivated by the cosine law in $M^n(K)$. In fact, $M^n(K)$ is isometric to $(\mathcal{O}(S^{n-1}), \sigma_K)$.

Lemma 5.2 ([3, p.17]). *Let (W, d) be a metric space, $K < 0$, and set $k = \sqrt{|K|}$. Then the K cosine law metric σ_K is indeed a metric on $\mathcal{O}(W)$. σ_K is a complete metric on $\mathcal{O}(W)$ if and only if d is a complete metric on W. Furthermore, $(\mathcal{O}(W), \sigma_K)$ is a geodesic space satisfying* $\mathrm{CAT}(K)$ *if and only if (W, d) is a 1-domain.*

Corollary 5.3. *Let d be a 1-domain metric on a finite simplicial complex Γ. Let $K < 0$ and let σ_K be the K cosine law metric on $\mathcal{O}(|\Gamma|)$. Let Δ be any subcomplex of Γ. Then σ_K restricts to the K cosine law metric on $\mathcal{O}(|\Delta|)$, and $\mathcal{O}(|\Delta|)$ is a strongly geodesic subspace of $\mathcal{O}(|\Gamma|)$.*

Proof. It is obvious from the formula for σ_K that σ_K restricts to the K cosine law metric on $\mathcal{O}(|\Delta|)$. To prove that $\mathcal{O}(|\Delta|)$ is a strongly geodesic subspace of $\mathcal{O}(|\Gamma|)$ first note that $\mathcal{O}(|\Delta|)$ with the restricted metric is itself a geodesic space. Since $\mathcal{O}(|\Gamma|)$ is CAT(K), it is metrically convex. (See Proposition 29 of [20].) Hence, geodesics in $\mathcal{O}(|\Gamma|)$ between points of $\mathcal{O}(|\Delta|)$ are unique. It follows that $\mathcal{O}(|\Delta|)$ is a strongly geodesic subspace of $\mathcal{O}(|\Gamma|)$ q.e.d.

As stated above, we find it necessary to put metrics of negative curvature not only on open cones, but also on the products of certain open cones with the interval $[0, 1]$. Moreover,we need to do this in such away that the "0-level" and the "1-level" are strongly geodesic subspaces, each isometric to the original open cone. This task splits naturally into two steps, the first of which is interesting in its own right. In the first step, Lemma 5.4, we show how to put a negatively curved metric on $X \times \mathbb{R}$ given a negatively curved metric on X. (The "warped product" construction of [7] accomplishes a similar objective for negatively curved Riemannian manfolds by unrelated methods.) Second, in Lemma 5.5, under the additional hypothesis that X is an open cone, we modify the metric on $X \times \mathbb{R}$ so that each level $X \times \{t\}$ is a totally geodesic subspace isometric to X. We remark that we do not know how to make the levels $X \times \{t\}$ totally geodesic without the additional hypothesis that X is an open cone. Indeed, we conjecture that, with no assumptions on X

beyond negative curvature, it is impossible to put a negatively curved metric on $X \times \mathbb{R}$ so that the levels are totally geodesic.

Lemma 5.4. *Suppose* (X, σ) *is a metric space and* $K < 0$ *such that* (X, σ) *satisfies* $\mathrm{CAT}(K)$. *Then there is a metric* τ *on* $X \times \mathbb{R}$ *with the following properties:*

a) If σ *is a complete metric on* X*, then* τ *is a complete metric on* $X \times \mathbb{R}$.

b) $(X \times \mathbb{R}, \tau)$ *is a geodesic space satisfying* $\mathrm{CAT}(K)$.

c) $x \mapsto (x, 0) : (X, \sigma) \to (X \times \mathbb{R}, \tau)$ *is an isometric embedding onto a strongly geodesic subspace.*

d) For each $x \in X, t \mapsto (x, t) : \mathbb{R} \to (X \times \mathbb{R}, \tau)$ *is an isometric embedding onto a strongly geodesic subspace.*

e) If Y *is a strongly geodesic subspace of* X*, then* $Y \times \mathbb{R}$ *is a strongly geodesic subspace of* $X \times \mathbb{R}$.

f) If Y *is a strongly geodesic subspace of* X*, then the restriction of* τ *to* $Y \times \mathbb{R}$ *is completely determined by the restriction of* a *to* Y.

Proof. First we give a geometric description of how to compute τ. Then we give an explicit formula. Let $(x, s), (y, t) \in X \times \mathbb{R}$. $\tau((x, s), (y, t))$ is evaluated by the following procedure. Construct a geodesic quadrilateral $PQQ'P'$ in $M^2(K)$ such that PQ is perpendicular to PP' and QQ', $\rho_K(P, Q) = \sigma(X, Y), \rho_K(P, P') = |s|, \rho_K(Q, Q') = |t|$, and Q and Q' are on the same (opposite) side of PP' if s and t have the same (opposite) sign. (See Figure 2.) (This description determines $PQQ'P'$ uniquely up to isometry in $M^2(K)$.) Call $PQQ'P'$ a *reference quadrilateral* for $(x, s), (y, t)$. Then set $\tau((x, s), (y, t)) = \rho_K(P', Q')$.

Let $k = \sqrt{|K|}$. We now verify that r is determined by the following formula:

$$\begin{aligned}&\tau((x, s), (y, t)) \\ &\quad = (1/k) \cosh^{-1} \Big(\cosh(ks) \cosh(kt) \cosh(k\sigma(x, y)) - \sinh(ks) \sinh(kt) \Big)\end{aligned}$$

for $(x, s), (y, t) \in X \times \mathbb{R}$. Abbreviate $\sigma(x, y)$ to σ and $\tau((x, s), (y, t))$ to τ , set $\delta = \rho_K(P, Q')$, and let θ denote the angle at P in the triangle

$PP'Q'$. Then from the hyperbolic cosine law in the triangle $PP'Q'$, we obtain

$$\cosh(k\tau) = \cosh(k|s|)\cosh(k\delta) - \sinh(k|s|)\sinh(k\delta)\cos(\theta).$$

From the hyperbolic cosine and sine laws in the triangle PQQ', we obtain

$$\cosh(k\delta) = \cosh(k|t|)\cosh(k\sigma)$$

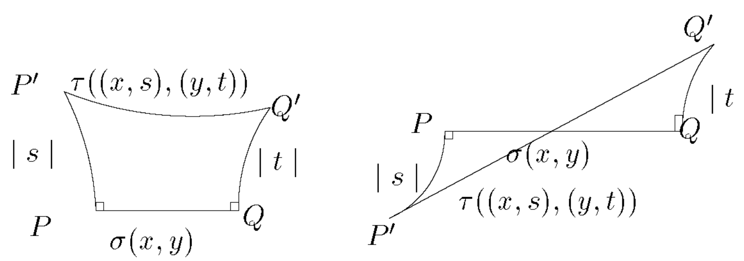
and

$$\frac{\sinh(k\delta)}{1} = \frac{\sinh(k|t|)}{\sin(\pm((\pi/2) - \theta))},$$

the sign depending on whether s and t have the same or opposite sign. Thus,

$$\sinh(k\delta)\cos(\theta) = \pm\sinh(k|t|).$$

Substituting the expressions for $\cosh(k\delta)$ and $\sinh(k\delta)\cos(\theta)$ in the equation for $\cosh(k\tau)$, removing absolute value operators, applying $\cosh^{-1}$ and dividing by k on both sides of the equation yields the desired formula.



s and t have same sign $\qquad$ s and t have opposite sign

FIGURE 2

Suppose that $PQQ'P'$ is a quadrilateral in $M^2(K)$ such that PQ is perpendicular to PP' and QQ'. Call PQ the *base*, PP' and QQ' the *sides*, and $P'Q'$ the *top* of this quadrilateral. We observe that the preceding remarks give us a formula for the length of the top of $PQQ'P'$ in terms of the lengths of its base and sides. Specifically, if we set $\sigma = \rho_K(P,Q), s = \rho_K(P,P'), t = \rho_K(O,Q')$, and $\tau = \rho_K(P',Q')$, then we have shown that

$$\tau = (1/k)\cosh^{-1}\Big(\cosh(ks)\cosh(kt)\cosh(k\sigma) - \sinh(ks)\sinh(kt)\Big).$$

We must establish that τ is a metric on $X \times \mathbb{R}$, which makes it a CAT(K) geodesic space. For this purpose it is convenient to introduce "coordinates" on $M^3(K)$. To this end let M_0 be a 2-dimensional submanifold of $M^3(K)$ that is isometric to $M^2(K)$. For each $x \in M_0$, let ζ_x denote the geodesic line in $M^3(K)$ through x orthogonal to M_0. Then $\{\zeta_x : x \in M_0\}$ fibers $M^3(K)$. (This is easily visualized in the Poincaré ball model O^3 of $M^3(K)$ by thinking of M_0 as the intersection of the XY-plane with O^3. Then the geodesics $\{\zeta_x : x \in M_0\}$ are simply the intersection of O^3 with circles that are centered in the XY-plane and orthogonal to the XY-plane and to ∂O^3.) Call one of the two components of $M^3(K) - M_0$ *positive* and the other *negative*. For each $x \in M_0$, let $t \mapsto C^t_x : \mathbb{R} \to \zeta_x$ be the unique isometry such that $C^0_x = x$ and for $t > 0, C^t_x$ lies in the positive component of $M^3(K) - M_0$. Now $(x,t) \mapsto C^t_x : M_0 \times \mathbb{R} \to M^3(K)$ is a bijection which "assigns coordinates" to the points of $M^3(K)$.

One further bit of notation: if $S \subset M_0$, set $V(S) = \cup_{x \in S} \zeta_x$. Hence, if λ is a geodesic line in M_0, then $V(\lambda)$ is an isometric copy of $M^2(K)$. (Again this is easy to see in O^3 with M_0 identified with the intersection of the XY-plane and O^3. λ can be assumed to be the intersection of the X-axis and O^3. This identifies $V(\lambda)$ with the intersection of the XZ-plane and O^3 which is clearly isometric to $M^2(K)$.) Furthermore, if α is a geodesic arc joining two points x and y of λ, then $V(\alpha)$ is a convex subset of $V(\lambda)$ and, hence, of $M^3(K)$. Indeed, $V(\alpha)$ is the intersection of two closed half-spaces of $V(\lambda)$ determined by ζ_x and ζ_y. Here, when we say that a set is "convex", we mean that whenever it contains two points, it contains the geodesic arc joining them.

We now make an observation which will be used several times below. Suppose $(x,s),(y,t) \in X \times \mathbb{R}$ and $x', y' \in M_0$ such that $\rho_K(x',y') = \sigma(x,y), x'' = C^S_{x'}$ and $y'' = C^t_{y'}$. Then $x'y'y''x''$ is a reference quadrilateral for $(x,s),(y,t)$ and, therefore, $\tau((x,s),(y,t)) = \rho_K(x'',y'')$. To justify this observation, note that if λ is the geodesic line in M_0 that passes through x' and y', then the quadrilateral $x'y'y''x''$ lies in $V(\lambda)$. Also note that $\rho_K(x',y') = \sigma(x,y), \rho_K(x',x'') = |r|, \rho_K(y',y'') = |s|$, and x'' and y'' lie on the same side of λ in $V(\lambda)$ if and only if they lie on the same side of M_0 in $M^3(K)$ if and only if r and s have the same sign.

We now verify that τ is a metric on $X \times \mathbb{R}$. Only the triangle inequality is not obvious. Let $(x,r),(y,s)$ and $(z,t) \in X \times \mathbb{R}$. Let T_0 denote the geodesic triangle with vertices x, y and z in X, and let T'_0 denote a comparison triangle with vertices x', y' and z' in M_0. Set $x'' = C^r_{x'}, y'' = C^s_{y'}$

and $z'' = C^t_{z'}$. Now, as we observed above, $x'y'y''x''$ is a reference quadrilateral for $(x,r),(y,s), x'z'z''x''$ is a reference quadrilateral for $(x,r),(z,t)$, and $z'y'y''z''$ is a reference quadrilateral for $(z,t),(x,r)$. Hence, $\tau((x,r),(y,s)) = \rho_K(x'',y''), \tau((y,s),(z,t)) = \rho_K(y'',z'')$ and $\tau((x,r),(z,t)) = \rho_K(x'',z'')$. Now it is clear that since the metric ρ_K satisfies the triangle inequality, then so does τ.

We must also verify that τ induces the product topology on $X \times \mathbb{R}$. it is clear from the formula for τ that if the sequence $\{(x_i, s_i)\}$ converges to the point (y,t) in $X \times \mathbb{R}$ with the product topology, then $\tau((x_i,s_i),(y,t)) \to 0$ as $i \to \infty$. We must prove the converse. For that purpose we exploit the identity

$$\cosh(a-b) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)$$

to rewrite the formula for τ as

$$\begin{aligned}\tau((x,s),(y,t)) = (1/k)\cosh^{-1}\big(&\cosh(ks)\cosh(kt)(\cosh(k\sigma(x,y)) - 1) \\ &+ \cosh(k(s-t))\big)\end{aligned}$$

Also recall that $\cosh(0) = 1$ and $\cosh(t) > 0$ if $t \neq 0$. It follows that if $\tau((x_i,s_i),(y,t)) \to 0$ as $i \to \infty$ then

$$\cosh(ks_i)\cosh(kt)(\cosh(k\sigma(x_i,y)) - 1) + \cosh(k(s_i - t)) \to 1.$$

Hence, $(\cosh(k\sigma(x_i,y)) - 1) \to 0$ and $\cosh(k(s_i - t)) \to 1$. This implies that $\{x_i\}$ converges to y in X and $\{s_i\}$ converges to t in $\mathbb{R}$. So $\{(x_i,s_i)\}$ converges to (y,t) in $X \times \mathbb{R}$ with the product topology.

By an argument very similar to the one just presented, it can be proved that if $\{(x_i,s_i)\}$ is a Cauchy sequence in $(X \times \mathbb{R}, T\tau)$, then $\{x_i\}$ and $\{s_j\}$ are Cauchy sequences in (X,σ) and $\mathbb{R}$, respectively. It follows that if a is a complete metric on X, then τ is a complete metric on $X \times \mathbb{R}$.

Next we argue that $(X \times \mathbb{R}, \tau)$ is a geodesic space. Let (x,s) and $(y,t) \in X \times \mathbb{R}$. Choose $x', y' \in M_0$ so that $\rho_K(x',y') = \sigma(x,y)$. Let α denote the geodesic arc in X joining x to y, let α' denote the geodesic arc in M_0 joining x' to y', and let $f : \alpha \to \alpha'$ denote the unique isometry such that $f(x) = x'$ and $f(y) = y'$. We define an isometry $g : \alpha \times \mathbb{R} \to V(\alpha')$ by $g(z,u) = C^u_{f(z)}$. Clearly g is a bijection. To prove that g is an isometry, let $(z,u),(w,v) \in \alpha \times \mathbb{R}$. Set $z' = f(z)$, $w' = f(w)$, $z'' = g(z,u)$ and $w'' = g(w,v)$. Then, as observed above, $z'w'w''z''$ is a reference quadrilateral for $(z,u),(w,v)$. Hence, $\tau((z,u),(w,v)) = \rho_K(z'',w'') =$

$\rho_K(g(z,u),g(w,v))$, proving g is an isometry. Since $V(\alpha')$ is a convex subset of $M^3(K)$, the geodesic arc γ in $M^3(K)$ joining $g(x,s)$ to $g(y,t)$ lies in $V(\alpha')$. Since g is an isometry, $g^{-1}\circ\gamma$ is a geodesic arc in $X\times\mathbb{R}$ joining (x,s) to (y,t).

To prove that $(X\times\mathbb{R},\tau)$ satisfies CAT(K), we will first establish that the geodesic joining two points of $X\times\mathbb{R}$ is *unique*. To this end let (x,r) and $(y,s)\in X\times\mathbb{R}$ and let α denote the geodesic arc in X joining x to y. According to the previous paragraph, $\alpha\times\mathbb{R}$ is isometric to a convex subset of $M^3(K)$. Since two points in a convex subset of $M^3(K)$ are joined by a unique geodesic within the convex set, we conclude that there is exactly one geodesic in $\alpha\times\mathbb{R}$ joining (x,r) to (y,s). We must eliminate the possibility of a second geodesic in $X\times\mathbb{R}$ which joins (x,r) to (y,s) but which does not lie in $\alpha\times\mathbb{R}$. For that purpose, consider a point $(z,t)\in(X-\alpha)\times\mathbb{R}$. We will prove that $\tau((x,r),(z,t))+\tau((z,t),(y,s))>\tau((x,r),(y,s))$. It will then follow that no geodesic joining (x,r) to (y,s) can pass through (z,t). Let T_0 denote the geodesic triangle with vertices x,y and z in X, and let T_0' denote a comparison triangle with vertices x',y' and z' in M_0. Then α is the edge of T_0 joining x to y. Let α' denote the edge of T_0' joining x' to y'. Since X is CAT(K), it is metrically convex (by Proposition 29 of [20]), so that points in X are joined by unique geodesics. Since $z\notin\alpha$, it follows that no geodesic joining x to y in X passes through z. Hence, $\sigma(x,z)+\sigma(z,y)>\sigma(x,y)$. Therefore,

$$\rho_K(x',z')+\rho_K(z',y')>\rho_K(x',y').$$

Consequently, $z'\notin\alpha'$. Now set $x''=C^r_{x'}, y''=C^s_{y'}$ and $z''=C^t_{z'}$. Then, as we observed above, $x'y'y''x''$ is a reference quadrilateral for $(x,r),(y,s),x'z'z''x''$ is a reference quadrilateral for $(x,r),(z,t)$, and $z'y'y''z''$ is a reference quadrilateral for $(z,t),(x,r)$. Hence,

$$\tau((x,r),(y,s))=\rho_K(x'',y''),\quad \tau((y,s),(z,t))=\rho_K(y'',z''),$$

and $\tau((x,r),(z,t))=\rho_K(x'',z'')$. Let α'' denote the geodesic arc in $M^3(K)$ which joins x'' to y''. Since $V(\alpha')$ is a convex subset of $M^3(K)$ and $x'',y''\in V(\alpha')$, we have $\alpha''\subset V(\alpha')$. Since $z''\in\zeta_{z'}$ and $z'\notin\alpha'$, it follows that $z''\notin V(\alpha')$, so that $z''\notin\alpha''$. Since points in $M^3(K)$ are joined by *unique* geodesics, $\rho_K(x'',z'')+\rho_K(z'',Yy)>\rho_K(x'',z'')$. Therefore, $\tau((x,r),(z,t))+\tau((z,t),(y,s))>\tau((x,r),(y,s))$, and we conclude that (z,t) does not lie on any geodesic in $X\times\mathbb{R}$ which joins (x,r) to (y,s). Consequently, any geodesic in $X\times\mathbb{R}$ which joins (x,r) to (y,s) must lie in $\alpha\times\mathbb{R}$ and is, therefore, unique.

We now prove that $(X \times \mathbb{R}, \tau)$ satisfies CAT(K). Let $(x,r),(y,s)$ and $(z,t) \in X \times \mathbb{R}$, and let T be the geodesic triangle with vertices $(x,r),(y,s)$ and (z,t) in $X \times \mathbb{R}$. Let T_0 be the geodesic triangle with vertices x, y and z in X, and let T_0' be a comparison triangle with vertices x', y' and z' in M_0. As before, set $x'' = C^r_{x'}, y'' = C^s_{y'}$ and $z'' = C^t_{z'}$. Then $x'y'y''x''$ is a reference quadrilateral for $(x,r),(y,s)$, $x'z'z''x''$ is a reference quadrilateral for $(x,r),(z,t)$, and $z'y'y''z''$ is a reference quadrilateral for $(z,t),(x,r)$; and $\tau((x,r),(y,s)) = \rho_K(x'',y''), \tau((y,s),(z,t)) = \rho_K(y'',z'')$ and $\tau((x,r),(z,t)) = \rho_K(x'',z'')$. Let T' denote the geodesic triangle in $M^3(K)$ with vertices x'', y'' and z''. The three points x'', y'' and z'' lie in a 2-dimensional submanifold of $M^3(K)$ which is isometric to $M^2(K)$, and this submanifold also contains the geodesic triangle T'. So T' is a comparison triangle for T. Let (w,u) be a point on the edge α of T opposite (x,r), and let w'' be the corresponding point on the edge α' of T' opposite x''. (See Figure 3.) We must prove that $\tau((x,r),(w,u)) < \rho_K(x'',w'')$. Let α_0 be the edge of T_0 opposite x, and let α_0' denote the edge of T_0' opposite x'. We previously showed that there is an isometry $g : \alpha_0 \times \mathbb{R} \to V(\alpha_0')$ such that $v \mapsto g(v,0)$ is an isometry from α_0 to $\alpha_0', g(y,0) = y', g(z,0) = z'$, and $g(p,v) = C^v_{g(p,0)}$ for $(p,v) \in \alpha_0 \times \mathbb{R}$. Set $w' = g(w,0)$; then w' is the point on α_0' which corresponds to the point w on α_0. Since X satisfies CAT(K) and T_0' is a comparison triangle for T_0, $\sigma(x,w) \le \rho_K(x',w')$. Since $V(\alpha_0')$ is a convex subset of $M^3(K)$ that contains $g(y,s) = y''$ and $g(z,t) = z''$, we have $\alpha' \subset V(\alpha_0')$. Hence, $g^{-1}(\alpha')$ is a geodesic in $X \times \mathbb{R}$ joining (y,s) to (z,t). Since such geodesics are unique, $g^{-1}(\alpha') = \alpha$. So $g(\alpha) = \alpha'$, which implies that $g(w,u) = w''$. Thus, $w'' = C^u_{w'}$, and therefore $w'' \in \zeta_{w'}$. Let λ denote the geodesic line in M_0 passing through x' and w'. Since $x'' \in \zeta_{x'}$, $x'w'w''x''$ is a quadrilateral in $V(\lambda)$ such that $x'w'$ is perpendicular to $x'x''$ and $w'w''$. Also $\rho_K(x',x'') = \rho_K(C^0_{x'}, C^r_{x'}) = r$ and $\rho_K(w',w'') = \rho_K(C^0_{w'}, C^u_{w'}) = u$. Using our formula for the length of the "top" of such a quadrilateral, we have

$$\rho_K(x'',w'') = (1/k)\cosh^{-1}\big(\cosh(kr)\cosh(ku)\cosh(k\rho_K(x',w')) \\ - \sinh(kr)\sinh(ku)\big).$$

On the other hand, our formula for the metric τ gives:

$$\tau((x,r),(w,u)) = (1/k)\cosh^{-1}\big(\cosh(kr)\cosh(ku)\cosh(k\sigma(x,w)) \\ - \sinh(kr)\sinh(ku)\big).$$

As $\sigma(x,w) \le \rho_K(x',w')$ and the hyperbolic cosine function is strictly increasing on $[0, \infty)$, we conclude that $\tau((x,r),(w,u)) \le \rho_K(x'', w'')$. Thus, $(X \times \mathbb{R}, \tau)$ satisfies $\mathrm{CAT}(K)$.

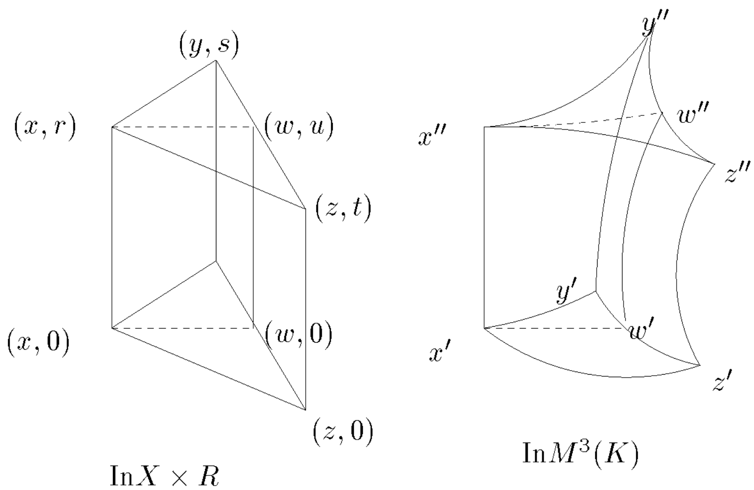


FIGURE 3

It is clear from the formula for r that the functions

$$x \mapsto (x,0) : (X, \sigma) \to (X \times \mathbb{R}, \tau)$$

and $t \mapsto (x,t) : \mathbb{R} \to X \times \mathbb{R}$ (for fixed $x \in X$) are isometric embeddings. Moreover, since the domains of these isometric embeddings are geodesic spaces, and since the geodesic joining a pair of points in $(X \times \mathbb{R}, \tau)$ is unique, the images of these isometric embeddings are strongly geodesic subspaces of $X \times \mathbb{R}$.

Suppose Y is a strongly geodesic subspace of X. Let (x,s) and $(y,t) \in Y \times \mathbb{R}$. Then x and y are joined by a unique geodesic α_0 in Y. Our earlier argument showed that there is a unique geodesic α in $X \times \mathbb{R}$ joining (x,s) to (y,t) and $\alpha \subset \alpha_0 \times \mathbb{R}$. Hence, $\alpha \subset Y \times \mathbb{R}$. It follows that $Y \times \mathbb{R}$ is a strongly geodesic subspace of $X \times \mathbb{R}$.

Finally, conclusion f) of this lemma is an immediate consequence of the formula for τ. q.e.d.

Lemma 5.5. *Let $K < 0$, and suppose X is an open cone and a is a K cosine law metric on X such that (X, σ) satisfies* $\mathrm{CAT}(K)$. *Then there is a metric τ^* on $X \times \mathbb{R}$ with the following properties.*

a) $(X \times \mathbb{R}, \tau^*)$ *is a geodesic space satisfying* $\mathrm{CAT}(K)$.

b) *If a is a complete metric on X, then τ^* is a complete metric on $X \times \mathbb{R}$.*

c) *For each $t \in \mathbb{R}, x \mapsto (x,t) : (X,\sigma) \to (X \times \mathbb{R}, \tau^*)$ is an isometric embedding onto a strongly geodesic subspace.*

d) *If v is the vertex of the open cone X, then*

$$t \mapsto (v,t) : \mathbb{R} \to (X \times \mathbb{R}, \tau^*)$$

is an isometric embedding onto a strongly geodesic subspace.

Proof. We assign $X \times \mathbb{R}$ the metric τ constructed in Lemma 5.4. We will construct a homeomorphism $h : X \times \mathbb{R} \to X \times \mathbb{R}$ with the following properties:

a) For each $t \in \mathbb{R},\ x \mapsto h(x,t) : (X,\sigma) \to (X \times \mathbb{R}, \tau)$ is an isometric embedding onto a strongly geodesic subspace of $X \times \mathbb{R}$.

b) If v is the vertex of X, then $t \mapsto h(v,t) : \mathbb{R} \to (X \times \mathbb{R}, \tau)$ is an isometric embedding onto a strongly geodesic subspace of $X \times \mathbb{R}$.

Given h, it is clear that a metric τ^* on $X \times \mathbb{R}$ which satisfies the conclusions of Lemma 5.5 is defined by the formula $\tau^*((x,s),(y,t)) = \tau(h(x,s),h(y,t))$.

First we give a geometric description of h. Then we will exhibit formulas for h and h^{-1} which make their continuity clear.

Suppose X is the open cone on the space $W : X = \mathcal{O}(W)$. For each $w \in W$, let $R_w = \{sw : s \geq 0\}$; i.e., R_w is the ray in X generated by w. Recall that the notation $M^2(K)^+ = \{A_s^t : s \geq 0 \text{ and } t \in \mathbb{R}\} = \{B_s^t : s \geq 0 \text{ and } t \in \mathbb{R}\}$ was introducted in Section 3. For each $w \in$W, define bijections $f_w : R_w \times \mathbb{R} \to M^2(K)^+$ and $g_w : R_w \times \mathbb{R} \to M^2(K)^+$ by $f_w(sw,t) = A_s^t$ and $g_w(sw,t) = B_s^t$. Then define the bijection $h : X \times \mathbb{R} \to X \times \mathbb{R}$ by $h|R_w \times \mathbb{R} = f_w^{-1} \circ g_w$ for each $w \in W$.

Here is the idea behind the definition of h. Fix $t \in \mathbb{R}$. Our aim is to make $x \mapsto (x,t) : (X,\sigma) \to (X \times \mathbb{R}, \tau)$ an isometric embedding. For a fixed $w \in W$, this goal entails that $s \mapsto (sw,t) : [0,\infty) \to R_w \times \mathbb{R}$ be an isometric embedding. From the definition of the metric τ in Lemma 5.4 it is easily seen that $f_w : (R_w \times \mathbb{R}, \tau) \to (M^2(K)^+, \rho_K)$ is an isometry. Unfortunately, $s \mapsto f_w(sw,t) = A_s^t : [0,\infty) \to M^2(K)^+$ is not an isometric embedding. (Indeed, $f_w(R_w \times \{t\})$ is not a geodesic ray in $M^2(K)^+$.) We conclude that $s \mapsto (sw,t) : [0,\infty) \to R_w \times \mathbb{R}$ is not an isometric embedding. So our aim is initially frustrated. On the other hand, $s \mapsto g_w(sw,t) = B_s^t : [0,\infty) \to M^2(K)^+$ is an isometric embedding, and $g_w(R_w \times \{t\})$ is a geodesic ray in $M^2(K)^+$. (See Figure 4.) Thus, $s \mapsto f_w^{-1} \circ g_w(sw,t) : [0,\infty) \to R_w \times \mathbb{R}$ is an isometric embedding, and $f_w^{-1} \circ g_w(R_w \times \{t\})$ is a geodesic ray in $R_w \times \mathbb{R}$. This suggests the above definition of h.

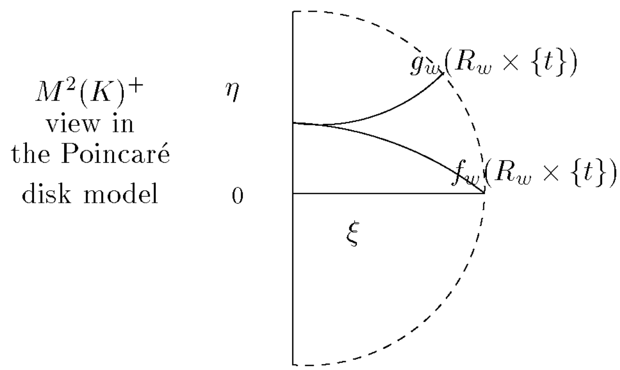


FIGURE 4

To discover a formula for h, we note the definition of h implies that $h(sw,t) = (s'w,t')$ if and only if $B_s^t = A_{s'}^{t'}$. It then follows from the transformation formulas (1) in Section 3 that

$$h(sw,t) = \left(\left(\frac{1}{k}\tanh^{-1}\left(\frac{\tanh(ks)}{\cosh(kt)}\right)\right)w, \frac{1}{k}\sinh^{-1}(\sinh(kt)\cosh(ks))\right)$$

and

$$h^{-1}(s'w,t') = \left(\left(\frac{1}{k}\sinh^{-1}(\sinh(ks')\cosh(kt'))\right)w, \frac{1}{k}\tanh^{-1}\left(\frac{\tanh(kt')}{\cosh(ks')}\right)\right).$$

It is clear from these formulas that h and h^{-1} are continuous. Thus, h is a homeomorphism.

Let $t \in \mathbb{R}$. We will now prove that $x \mapsto h(x,t) : X \to X \times \mathbb{R}$ is an isometric embedding. Let $w_1, w_2 \in W$ and $s_1, s_2 \in [0,\infty)$. We must show that

$$\tau(h(s_1w_1,t), h(s_2w_2,t)) = \sigma(s_1w_1, s_2w_2).$$

We could do this by a computation involving the formulas of τ, h and σ and some hyperbolic trigonometric identities. Instead, we choose to give a geometric argument in which we construct a reference quadrilateral for $h(s_1w_1,t), h(s_2w_2,t)$ in which the "top" has length $\sigma(s_1w_1, s_2w_2)$.

Recall that σ is a K cosine law metric on X. Hence, there is a metric d on W such that if we set $\theta = \min\{d(w_1,w_2), \pi\}$, then for any

$s_1', s_2' \in [0, \infty)$,

$$\sigma(s_1' w_1, s_2' w_2) = (1/k)\cosh^{-1}\left(\cosh(ks_1')\cosh(ks_2') - \sinh(ks_1')\sinh(ks_2')\cos(\theta)\right).$$

Let M_0 be a 2-dimensional submanifold of $M^3(K)$ that is isometric to $M^2(K)$. Choose a point Z of M_0, and let ω be the geodesic line in $M^3(K)$ passing through Z orthogonal to M_0. Choose a point $B \in \omega$ such that $\rho_K(Z, B) = |t|$. Let χ_1 and χ_2 be geodesic rays in M_0 emanating from Z so that the angle between them has measure θ. For the moment, let $i = 1$ or 2. Let H_i denote the union of all the geodesic lines in $M^3(K)$ that pass through points of χ_i and are orthogonal to M_0. Then H_i is isometric to $M^2(K)^+, \partial H_i = \omega$ and there is a unique isometry $e_i : M^2(K)^+ \to H_i$ such that $e_i(\mathbb{O}) = Z$ and $e_i(B^t) = B$. (Here we are again using the notation established in Section 3.) Thus $e_i(\xi \cap M^2(K)^+) = \chi_i$. There is a unique geodesic ray in H_i which emanates from B and is orthogonal to ω; let P_i denote the point on this ray such that $\rho_K(B, P_i) = s_i$. (Then, P_i is the point in H_i such that $\rho_K(B, P_i) = s_i$ and the geodesic joining B to P_i is orthogonal to ω.) Because of the way H_i is defined, it contains a unique geodesic line that passes through P_i and is orthogonal to M_0; let A_i denote the point where this line passes through M_0. Since $H_i \cap M_0 = \chi_i$, $A_i \in \chi_i$. (Thus, A_i is the point on χ_i such that the geodesic joining A_i to P_i is orthogonal to M_0.) Set $s_i' = \rho_K(Z, A_i)$ and $t_i' = \pm\rho_K(A_i, P_i)$ so that t and t_i' have the same sign. (See Figure 5.) It follows that $e_i \circ g_{w_i}(s_i w_i, y) = e_i(B^t_{s_i}) = P_i = e_i(A^{t_i'}_{s_i'}) = e_i \circ f_{w_i}(s_i' w_i, t_i')$. Hence, $h(s_i w_i, t) = (s_i' w_i, t_i')$.

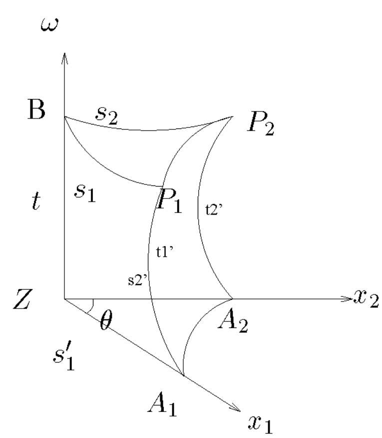


FIGURE 5

Let λ be the geodesic line in M_0 passing through A_1 and A_2, and let V be the union of all the geodesic lines in $M^3(K)$ that pass through points of λ and are orthogonal to M_0. Then V is isometric to $M^2(K)$ and contains the quadrilateral $A_1A_2P_2P_1$. We now show that $A_1A_2P_2P_1$ is a reference quadrilateral for $h(s_1w_1,t), h(s_2w_2,t)$. The geodesics A_1P_1 and A_2P_2 are perpendicular to M_0 and thus to A_1A_2. Since the angle at Z in the triangle ZA_1A_2 has measure 0, the hyperbolic cosine law implies that

$$\begin{aligned}&\rho_K(A_1,A_2)\\ &\quad= (1/k)\cosh^{-1}(\cosh(ks_1')\cosh(ks_2') - \sinh(ks_1')\sinh(ks_2')\cos(\theta)).\end{aligned}$$

Thus, $\rho_K(A_1A_2) = \sigma(s_1'w_1, s_2'w_2)$. Also, $\rho_K(A_i,P_i) = |t_i'|$ for $i = 1,2$. Furthermore, P_1 and P_2 are on the same side of M_0 as the point B and are, therefore, on the same side of A_1A_2 in V; and t_1' and t_2' have the same sign as t. We conclude that $A_1A_2P_2P_1$ is a reference quadrilateral for $(s_1'w_1,t_1'), (s_2'w_2,t_2')$ and, hence, for $h(s_1w_1,t), h(s_2w_2,t)$. Thus, by definition, $\tau(h(s_1w_1,t), h(s_2w_2,t)) = \rho_K(P_1,P_2)$.

We now compute $\rho_K(P_1,P_2)$. Let M_1 be the union of all the geodesic lines in $M^3(K)$ that pass through the point B and are orthogonal to ω. Then M_1 is isometric to $M^2(K)$ and contains the triangle BP_1P_2. We assert that the angle at B in the triangle BP_1P_2 has measure 0. To see this, consider the point B' on ω half way between Z and B, and let M' denote the 2-dimensional submanifold that is isometric to $M^2(K)$, passes through B' and is orthogonal to ω. Reflection of $M^3(K)$ through M' is an isometry that carries M_0 onto M_1, carries ω onto itself and fixes each of the geodesic rays $H_i \cap M'$. Hence, this reflection carries each H_i onto itself. Thus, it carries $\chi_i = H_i \cap M_0$ onto $H_i \cap M_1$. Since, $H_i \cap M_1$, is the geodesic ray emanating from B through P_i, it follows that the angle at B in triangle BP_1P_2 is congruent to the angle between χ_1 and χ_2, proving our assertion. Applying the hyperbolic cosine law in the triangle BP_1P_2 now yields

$$\begin{aligned}&\rho_K(P_1,P_2)\\ &\quad= (1/k)\cosh^{-1}\Big(\cosh(ks_1)\cosh(ks_2) - \sinh(ks_1)\sinh(ks_2)\cos(\theta)\Big).\end{aligned}$$

Thus, $\rho_K(P_1,P_2) = \sigma(s_1w_1, s_2w_2)$, and we conclude that

$$\tau(h(s_1w_1,t), h(s_2w_2,t)) = \sigma(s_1w_1, s_2w_2).$$

So $x \mapsto h(x,t) : X \to X \times \mathbb{R}$ is an isometric embedding.

Since X is a geodesic space and $x \mapsto h(x,t) : X \to X \times \mathbb{R}$ is an isometric embedding, $h(X \times \{t\})$ is a geodesic space. Since the geodesic joining two points of $X \times \mathbb{R}$ is unique, it follows that $h(X \times \{t\})$ is a strongly geodesic subspace of $X \times \mathbb{R}$.

Let v be the vertex of $X = \mathcal{O}(W)$. Let $w \in W$. Then $v = 0w$. For $t \in \mathbb{R}$, since $f_w(0w,t) = A_0^t = B_0^t = g_w(0w,t)$, $h(v,t) = f_w^{-1} \circ g_w(0w,t) = (0w,t) = (v,t)$, Therefore, Lemma 5.4.d implies that $t \mapsto h(v,t) : \mathbb{R} \to (X \times \mathbb{R}, \tau)$ is an isometric embedding onto a strongly geodesic subspace. q.e.d.

At this point, we report that the referee suggested a clever alternative approach to the results of this section in which Lemmas 5.2 and 5.5 are derived from Lemma 5.4 under the additional hypothesis that W is a compact polyhedron. We have not chosen the referee's approach in order to leave open the possibility of removing the "PL" hypothesis, from our Main Theorem. As we remarked earlier, if the appropriate topological manifold version of Lemma 5.1 is ever proved, then the remainder of our argument would prove the Main Theorem without the "PL" hypothesis. This feature of our argument would be lost if we were to follow the course suggested by the referee. However, because the referee's argument is quite efficient and does lead to a proof of the Main Theorem as it presently stands, we outline it briefly.

Using Lemma 5.4, the referee proves analogues of Lemmas 5.2 and 5.5 which we shall call Lemmas 5.2' and 5.5'. We leave it to the reader to verify that Lemmas 5.2' and 5.5' can replace Lemmas 5.2 and 5.5 in the proof of the Main Theorem given in Section 7.

Lemma 5.2'. *If Γ is a finite simplicial complex and $K < 0$, then there is a complete* $\mathrm{CAT}(K)$ *structure on $\mathcal{O}|\Gamma|$ in which $\mathcal{O}(|\Delta|)$ is a strongly geodesic subspace for each subcomplex Δ of Γ.*

Proof. This construction is based on the observation that for each simplex a $\sigma \in \Gamma$, there is a homeomorphism h_σ identifying the pair $(\mathcal{O}(\sigma), \mathcal{O}(\partial\sigma))$ with the pair $(\mathcal{O}(\partial\sigma) \times [0,\infty), \mathcal{O}(\partial\sigma) \times \{0\})$. This identification reveals that we can use Lemma 5.4 to extend a complete $\mathrm{CAT}(K)$ metric on $\mathcal{O}(\sigma)$ to a complete $\mathrm{CAT}(K)$ metric on $\mathcal{O}(\sigma)$. Now we proceed by induction on the number of simplices in Γ. Let σ be a top dimensional simpiex of Γ. We can assume that there is a complete $\mathrm{CAT}(K)$ structure on $\mathcal{O}(|\Gamma - \{(\sigma\}|)$ in which $\mathcal{O}(|\Delta|)$ is a strongly geodesic subspace for each subcomplex Δ of $\Gamma - \{\sigma\}$. In particular, $\mathcal{O}(\partial\sigma)$ is a strongly geodesic subspace. We extend the complete $\mathrm{CAT}(K)$ structure on $\mathcal{O}(\partial\sigma)$ to a complete $\mathrm{CAT}(K)$ structure on $\mathcal{O}(\sigma)$. Since $\mathcal{O}(|\Gamma|)$

is the union of the two complete CAT(K) metric spaces $\mathcal{O}(|\Gamma - \{\sigma\}|)$ and $\mathcal{O}(\sigma)$) along the strongly geodesic subspace $\mathcal{O}(\partial\sigma)$, $\mathcal{O}(|\Gamma|)$ has a complete CAT(K) structure ([5, Corollary 5, p.192]). The same union principle implies that $\mathcal{O}(|\Delta|)$ is a strongly geodesic subspace for each subcomplex Δ of Γ. We make an additional observation which will help in the proof of Lemma 5.5': for each $\sigma \in \Gamma$, assuming that we have fixed the homeomorphism $h_\sigma : \mathcal{O}(\sigma) \to \mathcal{O}(\partial\sigma) \times [0, \infty)$ then the metric on $\mathcal{O}(\sigma)$ is completely determined by the metric on $\mathcal{O}(\partial\sigma)$ via Lemma 5.4.f. q.e.d.

Lemma 5.5'. *If Γ is a finite simplicial complex and $K < 0$, then there is a complete* CAT(K)*structure on $\mathcal{O}(|\Gamma|) \times \mathbb{R}$ such that for each subcomplex Δ of Γ, $\mathcal{O}(|\Gamma|) \times \mathbb{R}$ is a strongly geodesic subspace, and for each $t \in \mathbb{R}, \mathcal{O}(|\Delta|) \times \{t\}$ is a strongly geodesic subspace isometric to $\mathcal{O}(|\Delta|)$. (For each subcomplex Δ of Γ, $\mathcal{O}(\Delta)$ is assumed to carry the metric constructed in Lemma 5.2')*

Proof. This construction is based on the observation that for each simplex $\sigma \in \Gamma$, a homeomorphism H_σ identifying the pair $(\mathcal{O}(\sigma) \times \mathbb{R}, \mathcal{O}(\partial\sigma) \times \mathbb{R})$ with the pair

$$((\mathcal{O}(\partial\sigma) \times \mathbb{R}) \times [0, \infty), (\mathcal{O}(\partial\sigma) \times \mathbb{R}) \times \{0\})$$

is determined by the condition that for each $t \in \mathbb{R}, H_\sigma$ maps $\mathcal{O}(\sigma) \times \{t\}$ onto $(\mathcal{O}(\partial\sigma) \times \{t\}) \times [0, \infty)$ in exactly the way that h_σ maps $\mathcal{O}(\sigma)$ onto $\mathcal{O}(\partial\sigma) \times [0, \infty)$. In other words, if $x \in \mathcal{O}(\sigma), y \in \mathcal{O}(\partial\sigma)$ and $s \in [0, \infty)$ such that $h_\sigma(x) = (y, s)$, then $H_\sigma(x, t) = ((y, t), s)$. The identification H_σ allows us to use Lemma 5.4 to extend a complete CAT(K) metric on $\mathcal{O}(\partial\sigma) \times \mathbb{R}$ to a complete CAT(K) metric on $\mathcal{O}(\sigma) \times \mathbb{R}$. Moreover, for $t \in \mathbb{R}$, if $\mathcal{O}(\partial\sigma) \times \{t\}$ is a strongly geodesic subspace of $\mathcal{O}(\partial\sigma) \times \mathbb{R}$ isometric to $\mathcal{O}(\partial\sigma)$, then according to Lemma 5.4.e and f, $\mathcal{O}(\partial\sigma) \times \{t\}$ is a strongly geodesic subspace of $\mathcal{O}(\sigma) \times \mathbb{R}$ isometric to $\mathcal{O}(\sigma)$.

Again we induct on the number of simplices in Γ. We let σ be a top dimensional simplex of Γ. We can assume that there is a complete CAT(K) structure on $\mathcal{O}(|\Gamma - \{\sigma\}|) \times \mathbb{R}$ such that for each subcomplex Δ of $\Gamma - \{\sigma\}, \mathcal{O}(|\Delta|) \times \mathbb{R}$ is a strongly geodesic subspace, and for each $t \in \mathbb{R}, \mathcal{O}(|\Delta|) \times \{t\}$ is a strongly geodesic subspace isometric to $\mathcal{O}(|\Delta|)$. Thus, $\mathcal{O}(\partial\sigma) \times \mathbb{R}$ is a strongly geodesic subspace, and we can extend the complete CAT(K) structure on $\mathcal{O}(\partial\sigma) \times \mathbb{R}$ to a complete CAT(K) structure on $\mathcal{O}(\sigma) \times \mathbb{R}$. Now, as in the proof of the previous lemma, the union of the complete CAT(K) structures on $\mathcal{O}(|\Gamma - \{\sigma\}|) \times \mathbb{R}$ and $\mathcal{O}(\sigma) \times \mathbb{R}$ is a complete CAT(K) structure on $\mathcal{O}(|\Gamma|) \times \mathbb{R}$, and

$\mathcal{O}(|\Delta|) \times \mathbb{R}$ is a strongly geodesic subspace for each subcomplex Δ of Γ. Next consider a subcomplex Δ of Γ containing σ and fix $t \in \mathbb{R}$. Then $\mathcal{O}(|\Delta - \{\sigma\}|) \times \{t\}$ and $\mathcal{O}(\partial\sigma) \times \{t\}$ are strongly geodesic subspaces isometric to $\mathcal{O}(|\Delta - \{\sigma\}|)$ and $\mathcal{O}(\partial\sigma)$, respectively. It follows that $\mathcal{O}(\sigma) \times \{t\}$ is a strongly geodesic subspace isometric to $\mathcal{O}(\sigma)$. Finally, since $\mathcal{O}(|\Delta|) \times \{t\}$ is the union of $\mathcal{O}(|\Delta - \{\sigma\}|) \times \{t\}$ and $\mathcal{O}(\sigma) \times \{t\}$ along the $\mathcal{O}(\partial\sigma) \times \{t\}$, it follows that $\mathcal{O}(|\Delta|) \times \{t\}$ is a strongly geodesic subspace isometric to $\mathcal{O}(|\Delta|)$. q.e.d.

We end this section by considering the $K = 0$ case of these lemmas and the Main Theorem. If $K \leq K'$, then $M^2(K)$ satisfies CAT(K') ([3, Corollary 5.1, p.21]). It follows that if $K \leq K'$, then any space which satisfies CAT(K) also satisfies CAT(K'). Hence, our Main Theorem implies the $K = 0$ version of itself. However, one can also prove the $K = 0$ version of the Main Theorem by deriving it from $K = 0$ versions of the lemmas in this section. As it happens, the $K = 0$ versions of these lemmas are, in general, easier to prove than their $K < 0$ counterparts. This is particularly the case for Lemmas 5.4 and 5.5. Thus, the proof of the $K = 0$ version of the Main Theorem is simpler than the $K < 0$ case. Because some readers might be primarily interested in the proof of the $K = 0$ case, we now briefly describe the $K = 0$ versions of the lemmas of this section. We leave to the reader the task of assembling them into a derivation of the $K = 0$ version of the Main Theorem. This derivation is essentially the same as the $K < 0$ derivation described in Section 7.

Lemma 5.1 need not be changed.

To formulate the $K = 0$ version of Lemma 5.2, we must first define the 0 cosine law metric on an open cone. Let (W, d) be a metric space and, as before, define the metric θ on W by the formula $\theta(w, w') = \min\{d(w, w'), \pi\}$. Define the 0 *cosine law metric* on $\mathcal{O}(W)$ to be the function $\sigma_0 : \mathcal{O}(W) \times \mathcal{O}(W) \to [0, \infty)$ defined by the formula $\sigma_0(r_1 w_1, r_2 w_2) = \left(r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta(w_1, w_2))\right)^{1/2}$. (This formula is clearly motivated by the cosine law in $M^n(0) = \mathbb{R}^n$. In fact, $\mathbb{R}^n$ is isometric to $(\mathcal{O}(S^{n-1}), \sigma_0)$.) The $K = 0$ version of Lemma 5.2 simply says that σ_0 is a metric on $\mathcal{O}(W)$ which is complete if d is complete, and that $(\mathcal{O}(W), \sigma_0)$ is a geodesic space satisfying CAT(0) if and only if (W, d) is a 1-domain. The reference is the same as for the $K < 0$ version of Lemma 5.2: [3, p.17].

Replace "$K < 0$" by "$K = 0$" to obtain the statement of the $K = 0$ version of Corollary 5.3. The proof is the same as before.

In the $K = 0$ case, Lemmas 5.4 and 5.5 collapse into one proposition. The reason is that as K approaches 0 from below, the two essentially different ways of putting rectangular coordinates on $M^2(K)$ converge. As a result, the $K = 0$ analogue of Lemma 5.4 puts a CAT(0) metric on $X \times \mathbb{R}$ in which the levels $X \times \{t\}$ are strongly geodesic subspaces isometric to X. Specifically, the $K = 0$ version of Lemma 5.4 says that if (X, σ) is a metric space satisfying CAT(0), then a metric τ on $X \times \mathbb{R}$ is defined by the formula $\tau((x,s),(y,t)) = \big((\sigma(x,y))^2 + (s-t)^2\big)^{1/2}$ and has the following properties:

a) If σ is a complete metric on X, then τ is a complete metric on $X \times \mathbb{R}$.

b) $(X \times \mathbb{R}, \tau)$ is a geodesic space satisfying CAT(0).

c) For each $t \in \mathbb{R}, x \mapsto (x,t) : X \to X \times \mathbb{R}$ is an isometric embedding onto a strongly geodesic subspace.

d) For each $x \in X, t \mapsto (x,t) : \mathbb{R} \to X \times \mathbb{R}$ is an isometric embedding onto a strongly geodesic subspace.

e) If Y is a strongly geodesic subspace of X, the $Y \times \mathbb{R}$ is a strongly geodesic subspace of $X \times \mathbb{R}$.

Property b) can be proved by adapting (and simplifying) appropriate parts of the proof of Lemma 5.4 to the $K = 0$ situation. The proofs of properties a) and c) through e) are immediate.

6. Arc spines

Let C^n be a compact contractible PL n-manifold ($n \geq 5$). In [4] it is shown that there is a map $f : \partial C^n \to [0,1]$ such that the mapping cylinder of $f, \mathrm{Cyl}(f)$, is homeomorphic to C^n. For later convenience, we give $\mathrm{Cyl}(f)$ the following non-standard parametrization. $\mathrm{Cyl}(f) = ((\partial C^n \times [0,\infty] \cup [0,1])/\sim$ where for each $x \in \partial C^n, \sim$ identifies $(x,0)$ with $f(x) \in [0,1]$. The specific form of the mapping cylinder structure imposed on C^n will be a key ingredient in the proof of the Main Theorem. In order to see this structure, we briefly review the main points of [4].

First one obtains a PL embedded copy $\Sigma^{n-2} \times [0,1]$ in ∂C^n, where Σ^{n-2} is a PL homology $(n-2)$-sphere such that the inclusion $\Sigma^{n-2} \times [0,1] \to \partial C^n$ induces a π_1-epimorphism. Lemma 1 of [4] describes the construction of a *topologically* embedded $\Sigma^{n-2} \times [0,1]$ in ∂C^n. However, when $n > 5$, the construction in [4] is clearly piecewise linear; and in the case $n = 5$, [4] appeals to [10] from which it is clear that if ∂C^n

is PL (as it is here), then the construction of $\Sigma^{n-2} \times [0,1]$ can also be done in the PL category.

By pulling in the ends of $\Sigma^{n-2} \times [0,1]$ slightly, we may assume that both $\Sigma^{n-2} \times \{0\}$ and $\Sigma^{n-2} \times \{1\}$ are bicollared. Then $\partial C^n - (\Sigma^{n-2} \times (0,1))$ is the union of two disjoint PL manifolds Q_0 and Q_1 which are homology $(n-1)$-cells with PL homeomorphic boundaries $\Sigma^{n-2} \times \{0\}$ and $\Sigma^{n-2} \times \{1\}$, respectively. The map $f : \partial C^n \to [0,1]$ sends Q_0 to 0, Q_1 to 1, and $\Sigma^{n-2} \times \{t\}$ to t for each $t \in (0,1)$. (See Figure 6.)

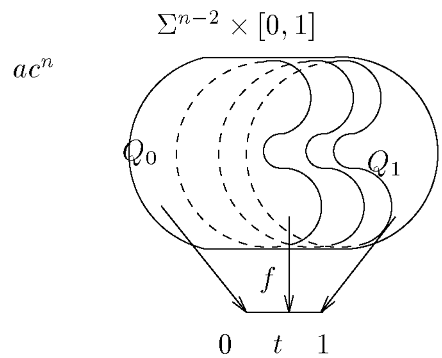


FIGURE 6

Now if $\mathcal{C}(Q_0), \mathcal{C}(Q_1)$ and $\mathcal{C}(\Sigma^{n-2})$ denote the cones $(Q_0 \times [0,\infty])/(Q_0 \times \{0\})$, $(Q_1 \times [0,\infty])/(Q_1 \times \{0\})$ and $(\Sigma^{n-2} \times [0,\infty]/(\Sigma^{n-2} \times \{0\})$, then we may view $\mathrm{Cyl}(f)$ as the adjunction space

$$\mathcal{C}(Q_0) \cup_{\omega_0} \mathcal{C}(\Sigma^{n-2}) \times [0,1] \cup_{\omega_1} \mathcal{C}(Q_1),$$

where for $i = 0$ or 1, ω_i is a PL homeomorphism from $\mathcal{C}(\partial Q_i)$ (a subset of $\mathcal{C}(Q_i)$) onto $\mathcal{C}(\Sigma^{n-2}) \times \{i\}$ which sends cone lines to cone lines. (See Figure 7.)

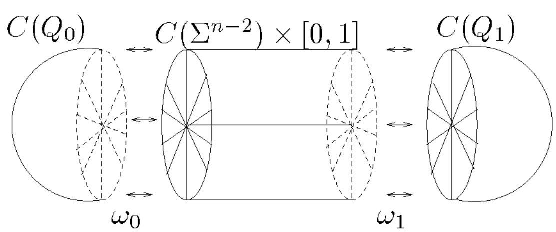


FIGURE 7

Since we may view an open cone as a subset of the corresponding cone, we may restrict ω_i to a homeomorphism $\widetilde{\omega}_i$, from $\mathcal{O}(\partial Q_i)$ onto

$\mathcal{O}(\Sigma^{n-2}) \times \{i\}$ for $i = 0$ or 1. Then $\mathrm{int}(C^n)$ may be realized as

$$\mathcal{O}(Q_0) \cup_{\widetilde{\omega}_0} \mathcal{O}(\Sigma^{n-2}) \times [0,1]) \cup_{\widetilde{\omega}_1} \mathcal{O}(Q_1).$$

7. Proof of the Main Theorem

We now prove our main theorem in the following slightly stronger form.

Theorem 7.1. *Let C^n be a compact contractible n-manifold, $n \geq 5$, and let $K < 0$. Then* $\mathrm{int}(C^n)$ *supports a metric ρ under which* $(\mathrm{int}(C^n), \rho)$ *is a complete geodesic space satisfying* $\mathrm{CAT}(K)$. *Consequently,* $\mathrm{int}(C^n)$ *supports a hyperbolic metric.*

Proof. We decompose ∂C^n into $\Sigma^{n-2} \times [0,1], Q_0$ and Q_1 as described in the preceding section. Since ∂Q_0 and ∂Q_1 are PL homeomorphic, we may choose triangulations of Q_0 and Q_1 under which ∂Q_0 and ∂Q_1 are isomorphic subcomplexes. Let $\phi : \partial Q_0 \to \partial Q_1$ be a simplicial isomorphism. Then the adjuction space $Q_0 \cup_\phi Q_1$ is a simplicial complex (homeomorphic to ∂C^n) in which Q_0, Q_1 and $\partial Q_0 = \partial Q_1$ are subcomplexes. Lemma 5.1 provides $Q_0 \cup_\phi Q_1$ with a metric d under which $Q_0 \cup_\phi Q_1$, and each of its subcomplexes is a 1-domain. Then Lemma 5.2 provides $\mathcal{O}(Q_0 \cup_\phi Q_1)$ with a complete metric under which it is a geodesic space satisfying $\mathrm{CAT}(K)$. Moreover, according to Corollary 5.3, the subcones $\mathcal{O}(Q_0), \mathcal{O}(Q_1)$ and $\mathcal{O}(\partial Q_0) = \mathcal{O}(\partial Q_1)$ are strongly geodesic subspaces of $\mathcal{O}(Q_0 \cup_\phi Q_1)$ satisfying $\mathrm{CAT}(K)$.

We use the PL homeomorphism between Σ^{n-2} and ∂Q_0 to put a metric on Σ^{n-2} that makes it a 1-domain isometric to ∂Q_0 and ∂Q_1. Then Lemma 5.2 provides a complete metric for $\mathcal{O}(\Sigma^{n-2})$ which makes it a geodesic space satisfying $\mathrm{CAT}(K)$ that is isometric to $\mathcal{O}(\partial Q_0)$ and $\mathcal{O}(\partial Q_1)$. We apply Lemma 5.5 to obtain a complete metric on $\mathcal{O}(\Sigma^{n-2}) \times \mathbb{R}$ which makes it a geodesic space satisfying $\mathrm{CAT}(K)$ in which $\mathcal{O}(\Sigma^{n-2}) \times \{t\}$ is a strongly geodesic subspace isometric to $\mathcal{O}(\Sigma^{n-2})$ for each $t \in \mathbb{R}$. It follows that $\mathcal{O}(\Sigma^{n-2}) \times [0,1]$ is a geodesic space satisfying $\mathrm{CAT}(K)$ in which $\mathcal{O}(\Sigma^{n-2}) \times \{0\}$ and $\mathcal{O}(\Sigma^{n-2}) \times \{1\}$ are strongly geodesic subspaces isometric to $\mathcal{O}(\Sigma^{n-2})$.

The preceding section decomposes $\mathrm{int}(C^n)$ into the pieces $\mathcal{O}(Q_0), \mathcal{O}(\Sigma^{n-2}) \times [0,1]$ and $\mathcal{O}(Q_1)$ where $\mathcal{O}(\partial Q_0) \subset \mathcal{O}(Q_0)$ is identified with $\mathcal{O}(\Sigma^{n-2}) \times \{0\} \subset \mathcal{O}(\Sigma^{n-2}) \times [0,1]$ and $\mathcal{O}(\partial Q_1) \subset \mathcal{O}(Q_1)$ is identified with $\mathcal{O}(\Sigma^{n-2}) \times \{1\} \subset \mathcal{O}(\Sigma^{n-2}) \times [0,1]$. Each piece is a geodesic space satisfying CAT(K). Moreover, $\mathcal{O}(\partial Q_0), \mathcal{O}(\Sigma^{n-2}) \times \{0\}$, $\mathcal{O}(\Sigma^{n-2}) \times \{1\}$ and $\mathcal{O}(\partial Q_1)$ are isometric and are strongly geodesic subspaces of $\mathcal{O}(Q_0), \mathcal{O}(\Sigma^{n-2}) \times [0,1]$ and $\mathcal{O}(Q_1)$, respectively. It follows from Corollary 5 on p.192 of [5] that the metrics on these pieces can be assembled into a metric on $\text{int}(C^n)$ which makes it a complete geodesic space satisfying CAT(K). q.e.d.

Observation. The hyperbolic metric we have constructed on $\text{int}(C^n)$ has the following curious feature: there is a geodesic arc in $\text{int}(C^n)$ which is wild. (An arc α in the interior of an n-manifold is *tame* if α has a neighborhood U such that the pair (U, α) is homeomorphic to $(\mathbb{R}^n, [0,1] \times \{(0,0,\cdots,0)\})$. α is *wild* if it is not tame.) The wild arc originates from the identification in [4] of C^n with the mapping cylinder $\text{Cyl}(f)$ of a map $f : \partial C^n \to [0,1]$. In this construction, the interval $[0,1]$ which is the target of f and which embeds naturally in $\text{Cyl}(f)$ is wild. (This is explained in the proof of Theorem 2 of [4]. There it is noted that even in the case that C^n is an n-ball, it is possible to choose f so that its target is wild in $\text{Cyl}(f)$.) Thus, there is a naturally occurring wild arc in $\text{int}(C^n)$. It remains to argue that this wild arc is a geodesic under the metric imposed on $\text{int}(C^n)$.

Under the identification of $\text{Cyl}(f)$ with

$$\mathcal{C}(Q_0) \cup_{\omega_0} \mathcal{C}(\Sigma^{n-2}) \times [0,1] \cup_{\omega_1} \mathcal{C}(Q_1)$$

explained in Section 6, it is apparent that the interval $[0,1]$ which is the target of f gets identified with $\{v\} \times [0,1]$ where v is the vertex of the cone $\mathcal{O}(\Sigma^{n-2})$. Thus, $\{v\} \times [0,1]$ is a wild arc in

$$\begin{aligned}\text{int}(&\mathcal{C}(Q_0) \cup_{\omega_0} \mathcal{C}(\Sigma^{n-2}) \times [0,1] \cup_{\omega_1} \mathcal{C}(Q_1)) \\ &= \mathcal{O}(Q_0) \cup_{\widetilde{\omega}_0} \mathcal{O}(\Sigma^{n-2}) \times [0,1] \cup_{\widetilde{\omega}_1} \mathcal{O}(Q_1).\end{aligned}$$

Now note that v is also the vertex of the open cone $\mathcal{O}(\Sigma^{n-2})$. Therefore, according to Lemma 5.5.d, $t \mapsto (v,t) : \mathbb{R} \to \mathcal{O}(\Sigma^{n-2}) \times \mathbb{R}$ is an isometric embedding. Hence, $\{v\} \times [0,1]$ is a geodesic arc in $\mathcal{O}(\Sigma^{n-2}) \times [0,1]$.

Finally, we remark that since our construction applies to the n-ball, there is a hyperbolic metric on $\mathbb{R}^n$ containing a wild geodesic arc.

8. Dimensions < 5

For $n = 1$ and $2, \mathbb{R}^n$ is the only contractible open n-manifold. Work of D. Rolfsen [18] implies that any simply connected 3-manifold supporting a geodesic metric of non-positive curvature is homeomorphic to $\mathbb{R}^3$. The following question is apparently open.

Question 8.1. Is every simply connected 4-manifold which supports a geodesic metric of non-positive (or strictly negative) curvature homeomorphic to $\mathbb{R}^4$?

More specifically, we ask:

Question 8.2. Do any (or all) interiors of compact contractible 4-manifolds not homeomorphic to B^4 support metrics of non-positive (or strictly negative) curvature?

For some partial results in dimension 4, see [19].

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