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CONSTANT MEAN CURVATURE SURFACES IN A HALF-SPACE OF $\mathbf{R}^3$ WITH BOUNDARY IN THE BOUNDARY OF THE HALF-SPACE

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The structure of the set of compact constant mean curvature surfaces whose boundary is a given Jordan curve in $\mathbf{R}^3$ seems far from being understood. We shall consider a simple, but interesting, situation concerning this problem: Assume that M be an embedded compact H-surface in $\mathbf{R}^3_+ = \{x_3 \geq 0\}$ with $\partial M = \Gamma \subset P = \{x_3 = 0\}$. Then, there is little known about the geometry and topology of M in terms of Γ. For example, *if* Γ *is convex, is* M *of genus zero?* When Γ is a circle, it follows from Alexandrov [1] that M is neccesarily a spherical cap or the planar disk bounded by Γ.

We first show that if $\Gamma_n \subset P$ is a sequence of embedded (perhaps nonconnected) curves converging to a point p, and $M_n \subset \mathbf{R}^3_+$ is a sequence of 1-surfaces ($H = 1$), with $\partial M_n = \Gamma_n$, then some subsequence of M_n converges to either p or the unit sphere tangent to P at p (the convergence being smooth in $\mathbf{R}^3 - p$). The same kind of result was obtained by Wente [8] when Γ_n is an arbitrary Jordan curve in $\mathbf{R}^3$ converging to a point p and M_n is an immersed topological disc bounded by Γ_n which minimizes area among disks bounding a fixed algebraic volume.

Our second result gives a partial answer to the question above. Given a convex Jordan curve $\Gamma \subset P$ and an embedded compact H-surface $M \subset \mathbf{R}^3_+$ bounded by Γ, we shall show that, for H sufficiently small (depending on Γ), M is a topological disc. Moreover, we can give a rather complete description of the shape of M, even near its boundary.

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As a consequence we obtain an upper bound, which depends on Γ but not on H, for the area of the above surfaces with positive genus.

Finally we remark that our assumptions about M hold under different geometric restrictions. For general $\Gamma \subset P$, if we denote by Ω the compact domain in P bounded by Γ, then the maximum principle shows that any constant mean curvature graph over Ω bounded by Γ, lies in $\mathbf{R}^3_+$ (or $\mathbf{R}^3_-$). When Γ is convex, there are two other conditions for embedded compact H-surfaces M with $\partial M = \Gamma$ which imply that M lies in one of the half spaces bounded by P; this holds if either M is transversal to P along Γ (see Earp, Brito, Meeks and Rosenberg [2]) or $Area(M)H^2 \leq 2\pi$ (see Lopez and Montiel [5]).

Preliminaries

Let $P(t)$ denote the horizontal plane $\{x_3 = t\}$, and $P = P(0)$. Vertical planes will be denoted by Q. Let $a = (0,0,1)$ and let $B(p,r)$ (respectively $D(p,r)$) denote the closed Euclidean ball in $\mathbf{R}^3$ (respectively in P) centered at p, of radius r. If the ball is centered at the origin $p = 0$, we will write simply $B(r)$ (resp. $D(r)$). In this paper an H-surface M will mean an embedded compact constant mean curvature (equal to H) surface in the half-space $\mathbf{R}^3_+$ with $\partial M \subset P$. If M is minimal, then the maximum principle implies that M is a domain in P. So we assume $H \neq 0$.

We orient H-surfaces by their mean curvature vector $\vec{H}$; so if n is the unit normal vector field along M, then $\left\langle n, \vec{H} \right\rangle = H > 0$. Γ is oriented by the orientation of M and bounds a compact region Ω in P so that $M \cup \Omega$ is a 2-cycle of $\mathbf{R}^3$. The maximum principle for surfaces with nonnegative mean curvature implies that $int(M) \cap int(\Omega) = \emptyset$ and thus $M \cup \Omega$ is an embedded piecewise smooth surface without boundary. Let W denote the closure of the bounded component of $\mathbf{R}^3 - (M \cup \Omega)$. Notice that we do not assume Γ is connected.

We say that M is a *small H-surface* if $M \subset B(p,r)$ for some $p \in \mathbf{R}^3$ and $r < 1/H$, otherwise we will say that M is a *large H-surface.* It follows from the maximum principle that, for small H-surfaces M, one has $M \subset \bigcap_\alpha B_\alpha$, where B_α denotes the family of balls $B(q,\rho)$ such that $q \in \mathbf{R}^3$, $\rho \leq 1/H$ and $\partial M \subset B(q,\rho)$.

One of our main tools will be the Alexandrov reflection technique. This idea was first introduced in [1] to show that the sphere is the unique compact H-surface (without boundary) embedded in $\mathbf{R}^3$. Now we sketch the main steps of this method, adapted to our situation. A detailed version of it can be found, for instance, in [6]. Let Q be a vertical plane disjoint from M, and b a nonzero vector normal to Q. Suppose that one parallel translates Q in the direction of b until the new plane Q_1 touches M for the first time. Then when one translates Q_1 slightly further (always in the direction of b), to a position Q_2, the closure of the part of M swept out, M_2, is a graph over a part of Q_2, and the reflection, M_2^*, of M_2 in the plane Q_2 is contained in W. This is clear if Q_1 touches M only at the interior points. To see that the same is true in the case that some boundary point of M lies in Q_1, we only need to observe that, thanks to the boundary maximum principle for mean convex surfaces, the angle (on the W side) between M and Ω along Γ is everywhere positive (notice that this angle may be equal to π, since we are not assuming that M is transverse to P). Denote by Γ_2 the part of Γ left behind the plane Q_2, and by Γ_2^* its symmetric image by Q_2. Thus $\Gamma_2^* \subset \Omega$.

Now one continues parallel displacement of Q_2 until the last position Q_3 such that, following the notation above, the reflection M_3^* of M_3 in the plane Q_3, is contained in W. Then M_3 is a graph over a domain in Q_3 and, using the (interior and boundary version of the) maximum principle for H-surfaces, one proves that Q_3 is also the last position such that $\Gamma_3^* \subset \Omega$. So Q_3 depends only on Γ. Moreover we obtain that, either M is symmetric whith respect to Q_3 or $int(M_3^*)$ does not meet M. Notice that Γ_3 is necessarily nonempty and that, if Γ is a convex curve, then the part of M outside $\Omega \times [0,\infty[$ is a graph over a part of $\Gamma \times [0,\infty[$; in particular, each component has genus zero.

Now consider a horizontal plane $P(t)$ and put $M(t) = M \cap \{x_3 \geq t\}$. If we assume $P(t)$ is very high, then the above reflection argument works when one moves $P(t)$ down, if we change W to $W' = W \cup (\Omega \times]-\infty, 0])$. In this case, if $P(t_1)$ is the highest plane touching M, then $M \cap P(t_1)$ contains only interior points of M. Let $P(t_3)$ be the lowest horizontal plane in $\mathbf{R}^3_+$ such that the reflected image $M(t_3)^*$ of $M(t_3)$ in $P(t_3)$ is contained in W'. Then $M(t_3)$ is a vertical graph over a part of P, and either $t_3 = 0$, in which case M is a graph over Ω, or t_3 occurs exactly at the first time that the reflected surface touches Γ. In the latter case, either M is a spherical cap or $int(M(t_3)^*)$ meets M only at points of Γ and always transversally. Remark that, in particular, the highest point of M must lie in the cylinder $int(\Omega) \times]0,\infty[$. Moreover, it follows from the height estimate below that $t_3 < 1/H$ and, so the height of M is at most $2/H$.

Finally note that, if $M \subset \Omega \times [0, \infty[$, or more generally, if $\langle n(p), a \rangle \leq 0$ for all $p \in \Gamma$, then doing Alexandrov reflexion with horizontal planes we get that M is a graph over Ω.

Estimates of height, area and curvature for H-surface graphs

Serrin [7] observed that one has height estimates for H-graphs with zero boundary values. Suppose $M \subset \mathbf{R}^3_+$ is an H-graph over a compact domain $\Omega \subset P$ with $\partial M = \partial \Omega$. Then we can see that the highest point of M is below height $1/H$ as follows: let $\varphi = Hx_3 + n_3$. Thus, if K denotes the Gauss curvature of M, then

$$\Delta \varphi = (H(2H) - (4H^2 - 2K))n_3 = -2(H^2 - K)n_3 \geq 0,$$

since our choice of n makes $n_3 \leq 0$. From $\varphi = n_3 \leq 0$ on ∂M, it follows that $\varphi \leq 0$ on M. So $Hx_3 \leq -n_3 \leq 1$, and $x_3 \leq 1/H$, as desired.

We claim that M *satisfies area and curvature estimates (depending only on H) on compact subsets of int*$(\mathbf{R}^3_+)$. To see this, let $\varepsilon > 0$ and $M(\varepsilon)$ denote the part of M above $P(\varepsilon)$. Assume $M(\varepsilon)$ is not empty. Since φ is subharmonic and $\varphi = H\varepsilon/4 + n_3 \leq H\varepsilon/4$ on $\partial M(\frac{\varepsilon}{4})$, we have that, on $M(\frac{\varepsilon}{2})$,

$$n_3 \leq H\varepsilon/4 - Hx_3 \leq H\varepsilon/4 - H\varepsilon/2 \leq -H\varepsilon/4.$$

M is the graph of a certain function $u \in C^\infty(\Omega)$. Consider the compact subdomain $\Omega(\varepsilon) = \{x \in \Omega / u(x) \geq \varepsilon\}$. The above estimate of n_3 yields an estimate of the gradient of u in $\Omega(\frac{\varepsilon}{2})$, which , in turn, we can use (in an obvious way) to show that the (Euclidean) distance between $\Omega(\varepsilon)$ and $\partial\Omega(\frac{\varepsilon}{2})$ is larger than some positive constant δ depending only on H and ε. Thus, for each $p \in \Omega(\varepsilon)$ we have that $D(p, \delta) \subset \Omega(\frac{\varepsilon}{2})$ and we control u and $|\nabla u|$ on $D(p, \delta)$. Then standard results in elliptic equations (see [3], Theorems 13.1 and 6.2) show that we have $C^{2,\alpha}$ estimates for u on $\Omega(\varepsilon)$ depending on H and ε. Hence the claim follows directly.

The main results

Theorem 1. *Let $M_n \subset \mathbf{R}^3_+$ be H-surfaces with $H = 1$, and $\Gamma_n = \partial M_n \subset D(r_n) = \{(x_1, x_2, 0)/x_1^2 + x_2^2 \leq r_n^2\}$, with r_n a sequence converging to zero. Then there is a subsequence of M_n, which converges either to the origin $0 \in \mathbf{R}^3$ or to the sphere $S \subset \mathbf{R}^3_+$ of radius one tangent to P at 0. In the first case the surfaces converge as subsets and in the second one the convergence is smooth (any C^k) on compact subsets of $\mathbf{R}^3 - 0$.*

Proof. From the height estimates and the Alexandrov reflection technique it follows that all the M_n are contained in a fixed compact subset of $\mathbf{R}^3$. Let $r > 0$ and let Q be a vertical plane outside this compact subset. For n large, $\partial M_n \subset D(r)$ so, using the Alexandrov method, one can parallelly translate Q until it meets $\partial D(r)$, and the part of M_n swept out by Q is a graph over a part of Q. Therefore one has uniform area and curvature estimates for this part of M_n. Alexandrov reflection with horizontal planes gives that the part of each M_n above $P(1)$ is a vertical graph, so one has uniform area and curvature estimates for the $M_n(1+\delta)$, $\delta > 0$.

Standard compactness techniques yield a subsequence (which we also call) M_n that converges on compact subsets of $\mathbf{R}^3 - I$, where $I = 0 \times [0, 1]$; see for instance [9]. The limit is either empty, or a surface M of mean curvature one, properly embedded in $\mathbf{R}^3 - I$ (embeddedness follows because the part of M contained in each one of the half-spaces $\{\alpha x_1 + \beta x_2 > 0\}$, $\alpha^2 + \beta^2 = 1$, and $\{x_3 > 1\}$ is a graph). If the limit is empty, then for n large, M_n is uniformly close to I. Thus M_n is a small 1-surface and, as $\Gamma_n \subset B(r_n)$, it follows that $M_n \subset B(r_n)$. So, M_n converges to 0.

Now we assume the M_n converges to a surface M properly embedded in $\mathbf{R}^3 - I$. If one does Alexandrov reflection, for each $r > 0$ vertical planes can be moved up to $\partial D(r)$, and the reflected images of M by these planes lie in the domain enclosed by M (since this holds for M_n, n large). So this works up till $r = 0$ by continuity, M is a rotational surface about the vertical line through 0, and each component of M has multiplicity one. M has height at most two so it is neither a Delaunay surface nor a stack of spheres of radius one. So M is the sphere S of radius one passing through 0.

Finally we show the convergence is uniform on compact subsets of

$\mathbf{R}^3 - 0$. Given $\varepsilon > 0$, there exists $r > 0$ so that for n large,

$$M_n \cap (D(r) \times]\frac{3}{2}, \infty[) = M_n \cap (D(r) \times]2 - \varepsilon, 2 + \varepsilon[),$$

and this intersection is a graph above $D(r)$. Coming down with horizontal planes $P(t)$ from $t = 2$ to $t = 1$ we see that $M_n \cap (D(r) \times [\varepsilon, 2 - \varepsilon]) = \emptyset$. So we have uniform estimates for M_n on compact subsets of $\mathbf{R}^3 - 0$, not just on compact subsets of $\mathbf{R}^3 - I$.

Remark 1. From Theorem 1 we conclude that *given a positive integer k and $\varepsilon, \delta > 0$, there exists $r = r(k, \varepsilon, \delta) > 0$ such that any large H-surface $M \subset \mathbf{R}^3_+$ with $H = 1$ and $\partial M \subset D(r) \subset P$ satifies that $M - B(\delta)$ is the graph (with respect to the normal lines of the sphere) of a function u, defined on a domain of S, with $\|u\|_{C^k} < \varepsilon$.* In fact, if this statement were false we could construct a sequence M_n which would contradict Theorem 1.

Remark 2. Theorem 1 remains true if one assumes that the surfaces $M_n \subset \mathbf{R}^3_+$ are compact, embedded, have constant mean curvature $H = 1$ and *no neccesarily planar boundary* $\partial M_n \subset B(r_n)$, with $r_n \to 0$ as $n \to \infty$. To prove that, instead of using vertical planes, one does Alexandrov reflection with planes that are ε-tilted from the vertical, i.e., planes Q whose unit normal vector $b(\varepsilon)$ satisfies $\langle b(\varepsilon), a \rangle = \varepsilon$. Given $\varepsilon, r > 0$, one can choose r_n small enough so that Alexandrov reflection works with ε-tilted planes $Q + tb(\varepsilon)$, t coming from $-\infty$, up till the plane reaches $B(r)$. Taking $\varepsilon \to 0$ we get the assertion in Theorem 1; see [4] for more details in a related situation.

Theorem 2. *Let $\Gamma \subset P$ be a strictly convex curve. There is an $H(\Gamma) > 0$, depending only on the extreme values of the curvature of Γ, such that whenever $M \subset \mathbf{R}^3_+$ is an H-surface bounded by Γ, with $0 < H < H(\Gamma)$, then M is topologically a disk and either M is a graph over the domain Ω bounded by Γ or $N = M \cap (\Omega \times]0, \infty[)$ is a graph over Ω and $M - N$ is a graph over a subannulus of $\Gamma \times]0, \infty[$, with respect to the lines normal to $\Gamma \times]0, \infty[$. In the latter case (i.e., when M is not a graph over Ω), given any θ, $0 < \theta < \pi/2$, and H-surface M as above, one can also ensure that the angle between n and a is less than θ along Γ (so $H(\Gamma)$ will also depend on θ).*

Before proving Theorem 2, we state a lemma whose proof we will give later. We remark that the radius r of the lemma is independent of the value of H.

Lemma 3. *Let $\Gamma \subset P$ be strictly convex. Then there is an $r > 0$, depending only upon the extreme values of the curvature of Γ, such that whenever $M \subset \mathbf{R}^3_+$ is an H-surface with boundary Γ, there is a point $p \in \Omega$ (p depends on M) such that $D(p,r) \subset int(\Omega)$ and $M \cap (D(p,r)\times]0,\infty[)$ is a graph over $D(p,r)$.*

Proof of Theorem 2. Let M be an H-surface as in Theorem 2 and let us first suppose that M is small. In this case, if H is smaller than the curvature values of Γ, then $M \subset \bigcap_\alpha B_\alpha$, where B_α is the family of closed Euclidean balls of radius $1/H$ centered at points of P with $\Gamma \subset B_\alpha$. From the relation between the curvatures of Γ and B_α, it is clear that each point of Γ lies in the boundary of some B_α. In particular, $\bigcap_\alpha B_\alpha$ is contained in the solid cylinder $\Omega \times [0,\infty[$, and thus M is a graph over Ω.

Suppose now that M is a large H-surface. Let $r > 0$ and let $p \in \Omega$ be given by Lemma 2.1. Let Σ' be the unique vertical catenoid meeting P in the circle $C_0 = \partial D(p,\rho)$ where $\rho < r$ and ρ is smaller than the smallest radius of curvature of Γ (the latter condition allows us to translate C_0 horizontally in Ω so as to touch every point of Γ), and such that the angle between Σ' and P along C_0 is θ. Here the angle θ is the angle between Σ' and the noncompact component of $P - C_0$; so Σ' is a graph over this noncompact component. Let $\Sigma = \Sigma' \cap \{0 \le x_3 \le 1\}$ and let C_1 be the circle of Σ at height one. Let $V = \{v \in P/C_0 + v \subset \Omega\}$ and let $D(R)$ be a sufficiently large disk in P so that the translated disc $D(R) + a \subset \{x_3 = 1\}$ contains the circle $C_1 + v$ for all $v \in V$. Clearly V is compact and convex.

Now consider Alexandrov reflection of M with horizontal planes coming down from above M. A rescaled version of Theorem 1 will imply that if H is small enough, the part of the reflected surface which is in $D(R) \times [0,\infty[$ is uniformly near $D(R)$; it is a graph above $D(R)$ of height less than $1/2$, and $M \cap (D(R) \times [\frac{1}{2},\infty[)$ is a graph above $D(R)$ of height larger than $1/H$. We make this precise in the next paragraph.

We know the highest point q of M is in $\Omega\times]0,\infty[$, hence q is in $D(R) \times [0,\infty[$. From Remark 1, if H is small enough, the image of $M(\frac{1}{H})$ by the homothety of ratio H centered at the origin is arbitrarily near the unit upper half-sphere $S \cap \{x_3 \ge 1\}$, with respect to the C^k-distance, $k \ge 1$. Thus the component G of $M \cap (D(R) \times [0,\infty[)$, that contains q is almost flat when H is small. In particular, since M is horizontal at q, G is a graph above $D(R)$, $G \subset \{x_3 > 1/H\}$ and the total vertical oscillation of G is less than one half for H small enough (depending on R). Then when one does Alexandrov reflection with horizontal planes, the last position of the reflected image G^* of G is a graph above $D(R)$ of height less than one half (remenber that when one does reflection coming down from q, the last position occurs just the first time that a point of M reflects to a point of ∂M). Since the oscillation of G is less than one half, G^* lies below $D(R) \times \frac{1}{2}$. Moreover, as another consequence of the Alexandrov technique, we obtain that $M \cap (D(R) \times [\frac{1}{2}, \frac{1}{H}]) = \emptyset$.

Therefore $\Sigma + a$ and $C_1 + ta$, $0 \leq t \leq 1$, are contained in $int(W)$ (recall W is the compact component bounded by $M \cup \Omega$) and, as Lemma 2.1 gives $C_0 + ta \subset int(W)$ for $0 < t \leq 1$, it follows that $\Sigma \subset W$. Otherwise when one translates $\Sigma + a$ down to Σ, there would be a first point of contact of $\Sigma + ta$ with M. This contact point occurs on the W side of M, the side to which the mean curvature vector of M points. This is impossible since the point of contact is an interior point of both M and $\Sigma + ta$ and $\Sigma + ta$ is a minimal surface.

We know that the boundary component of $\Sigma + v$, for $v \in V$, at height one, is contained in $int(W)$. Hence $\Sigma + v \subset W$ for each $v \in V$ by similar reasoning as above: the family $\Sigma + tv$, $0 \leq t \leq 1$, can have no first point of interior contact with M as t goes from 0 to 1.

Our choice of C_0 guarantees that for each $q \in \Gamma$, there is a $v \in V$ such that $C_0 + v$ is tangent to Γ at q. Hence θ is strictly larger than the outer angle that W makes with P at q. So this holds along Γ.

Since the horizontal translations of Σ, $\Sigma + tv$, $v \in V$, $0 \leq t < 1$ are all in W and $D(r) \times]0,1] \subset int(W)$ by Lemma 2.1, we know that $\Omega \times]0,1] \subset int(W)$. Also M meets $D(R) \times [\frac{1}{2}, \infty[$ in a graph above $D(R)$ of height larger than $1/H$, so $M \cap (\Omega \times]0,\infty[)$ is a graph above Ω of height larger than $1/H$. The part of M outside $\Omega \times]0,\infty[$ is a graph over a subannulus of $\Gamma \times [0,\infty[$, so M is topologically a disk and Theorem 2 is established.

Proof of Lemma 2.1. Consider doing Alexandrov reflection with horizontal planes, coming down from above M. If we can come down to P, then M is a graph above Ω and 2.1 is clear. Otherwise there is a height $t > 0$ where the reflected surface touches Γ for the first time at a point $q \in \Gamma$. So $q \times]0,\infty[$ intersects M exactly once and $q \times]0, 2t[\subset int(W)$. Also the part of M above height t is a vertical graph.

Now consider Alexandrov reflection with vertical planes Q, v normal to Q, $|v| = 1$. Suppose one can do Alexandrov reflection of M, moving the plane Q slightly beyond q, and denote by $J(v)$ the open segment in $int(\Omega)$ joining q to its reflected image by this plane. Clearly the vertical rectangle $J(v)\times]0,2t[\subset int(W)$. Suppose we could repeat this reasoning for a family of directions $F \subset \{v \in P; |v| = 1\}$, such that, for some $p \in \Omega$ and $r > 0$, we have $D(p,r) \subset \cup_{v\in F} J(v)$. Then we would have

$$D(p,r)\times]0,2t[\subset \cup_{v\in F} J(v)\times]0,2t[\subset int(W).$$

Hence $D(p,r)\times]0,\infty[$ would intersect M only at points above $P(2t)$, and this intersection would be a graph above $D(p,r)$ as desired. So we have to understand the horizontal directions v for which Alexandrov reflection goes beyond a given point $q \in \Gamma$.

First recall, that for horizontal directions v, one can always do Alexandrov reflection up till Γ. Let k be the minimum curvature of Γ and let $C \subset P$ be a circle of curvature k. So if C is tangent to Γ at q, then Γ is inside C. Take $\rho > 0$ smaller than the smallest radius of curvature of Γ. Then the tubular neighbourhood of Γ in P of radius ρ is an embedded annulus. Thus for each horizontal v, $|v| = 1$, one can do Alexandrov reflection with vertical planes orthogonal to v at least at a distance $\rho/2$ beyond the first time the plane meets Γ and, so, at least at a distance $\rho/2$ beyond the first time the vertical plane meets the circle C.

Now consider those horizontal vectors v such that Alexandrov reflection with planes orthogonal to v left behind q (this will hold for those directions in some neighborhood $F \subset \{v \in P/\,|v| = 1\}$ of the inward pointing normal to C at q). It is clear from the geometry of the circle, that $\cup_{v\in F} J(v)$ contains a disk $D(p,r)$, $r > 0$, where r depends on ρ and C (but not on $q \in \Gamma$). This completes the proof of Lemma 2.1. q.e.d.

Corollary 4. *Let $\Gamma \subset P$ be a strictly convex curve. Then there exist $V(\Gamma)$, $A(\Gamma) > 0$, depending only on the extreme values of the curvature of Γ, such that any H-surface $M \subset \mathbf{R}^3_+$ bounded by Γ which either encloses a volume* $\mathrm{Vol}(W) > V(\Gamma)$ *or verifies $Area(M) > A(\Gamma)$, is a topological disk.*

Proof. From the height estimate we have that $M \subset B(r + \frac{2}{H})$ where $r > 0$ is chosen such that $\Gamma \subset D(r)$. So W is contained in the same ball and

$$\mathrm{Vol}(W) \le c(r + 2/H)^3,$$

for a certain positive constant c. Thus if $\mathrm{Vol}(W)$ is big enough, we will have $H < H(\Gamma)$ and the result follows from Theorem 2.

Computing the Laplacian of $|X|^2$, X being the position vector, in $\mathbf{R}^3$ and its restriction to M, using the divergence theorem twice we obtain, the standard formulae:

$$3\,\mathrm{Vol}(W) + \int_M \langle X, n\rangle = 0,$$

$$\mathrm{Area}(M) + H\int_M \langle X, n\rangle = \frac{1}{2}\int_{\partial M} \langle X, \nu\rangle\,,$$

where ν is the outward pointing conormal vector along ∂M. Thus

$$2\mathrm{Area}(M) = 6H\,\mathrm{Vol}(M) + \int_{\partial M} \langle X, \nu\rangle\,.$$

As H is bounded above by $1/r'$, where $r' > 0$ is chosen so that $D(p, r') \subset \mathrm{int}(\Omega)$ (note that some of the balls $B(p, r') + ta$, t coming from $-\infty$, will have a first contact with the positively curved side of $int(M)$ and thus the maximum principle gives $H \leq 1/r'$) it follows that when $\mathrm{Area}(M)$ is big then $\mathrm{Vol}(W)$ is big too. So we conclude our argument using the step above. q.e.d.

Remark 3. The results of this paper extend to compact hypersurfaces with constant mean curvature embedded in the Euclidean halfspace $\mathbf{R}^{n+1}_+$ and with boundary in $P = \partial\mathbf{R}^{n+1}_+$.

To see that Theorem 1 extends, we only need to note that both the Alexandrov reflection technique and the curvature estimates for H-graphs work for any dimension.

Concerning Theorem 2, $\Gamma \subset P = \mathbf{R}^n$ will be a strictly convex compact hypersurface of P, and $H(\Gamma)$ will depend on the maximum and the minimum principal curvatures of Γ. The only change in our arguments is that, as the height of the higher dimensional vertical Catenoid Σ' is bounded, now the piece $\Sigma \subset \Sigma'$ we use in our proof will be, not $\Sigma' \cap \{0 \leq x_{n+1} \leq 1\}$, but $\Sigma = \Sigma' \cap \{0 \leq x_{n+1} \leq \varepsilon(\theta)\}$, where $\varepsilon(\theta) > 0$ is small enough to assure the compactness of Σ.

Remark 4. An interesting problem is to understand the topology and geometry of solutions to the isoperimetric problem for a convex curve $\Gamma \subset P$. More precisely, given $V \geq 0$, we know from geometric measure theory, that there exists an embedded constant mean curvature surface M with $\partial M = \Gamma$, which (together with the planar domain Ω enclosed by Γ) bounds a volume V, and minimizes area among such surfaces. What is the nature of M as V goes from 0 to infinity? When does M traverse the plane P?

Another problem we wish to mention is the following: Let Γ_1 and Γ_2 be convex curves in parallel planes. Is there a constant mean curvature surface M with boundary $\Gamma_1 \cup \Gamma_2$, M topologically an annulus?

References

[1] A. D. Alexandrov, *Uniqueness theorems for surfaces in the large I*, Vestnik Leningrad Univ. Math. **11** (1956) 5-17.

[2] R. Earp, F. Brito, W. H. Meeks III & H. Rosenberg, *Structure theorems for constant mean curvature surfaces bounded by a planar curve*, Indiana Univ. Math. J. **40** (1991) 333-343.

[3] D. Gilbarg & N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer, Berlin, 1983.

[4] N. Korevaar, R. Kusner & B. Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geom. **30** (1989) 465-503.

[5] R. Lopez & S. Montiel, *Constant mean curvature surfaces with planar boundary*, Duke Math. J., to appear.

[6] R. Schoen, *Uniqueness, symmetry and embeddedness of minimal surfaces*, J. Differential Geom. **18** (1983) 701-809.

[7] J. Serrin, *On surfaces of constant mean curvature which span a given space curve*, Math. Z. **112** (1969) 77-88.

[8] H. C. Wente, *Large solutions to the volume constrained plateau problem*, Arch. Rational Mech. Anal. **75** (1980) 59-77.

[9] B. White, *Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals*, Invent. Math. **88** (1987) 243-256.

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