J. DIFFERENTIAL GEOMETRY
Vol. **44** (1996) 634-665

THE RADIUS RIGIDITY THEOREM FOR MANIFOLDS OF POSITIVE CURVATURE

FREDERICK WILHELM

Abstract

Recall that the radius of a compact metric space $(X, dist)$ is given by $rad\ X = \min_{x\in X}\max_{y\in X} dist(x,y)$. In this paper we generalize Berger's $\frac{1}{4}$-pinched rigidity theorem and show that a closed, simply connected, Riemannian manifold with sectional curvature ≥ 1 and radius $\geq \frac{\pi}{2}$ is either homeomorphic to the sphere or isometric to a compact rank-one symmetric space.

The classical sphere theorem states that a complete, simply connected Riemannian n-manifold with positive, strictly 1/4-pinched sectional curvature is homeomorphic to S^n ([1], [16], and [21]). The weakly 1/4-pinched case is covered by

Berger's Rigidity Theorem ([2]). *Let M be a complete, simply connected Riemannian n-manifold with sectional curvature, $1 \leq sec\ M \leq 4$. Then either*

(i) *M is homeomorphic to S^n, or*

(ii) *M is isometric to a compact rank one symmetric space.*

The hypotheses of Berger's Theorem imply (with a lot of work) that the injectivity radius of M satisfies $inj\ M \geq \frac{\pi}{2}$ ([6] or [17]). The diameter therefore, also satisfies $diam\ M \geq \pi/2$, and the class of complete Riemannian manifolds with

$$(*) \qquad sec \geq 1 \text{ and } diam \geq \pi/2$$

Received January 21, 1995, and, in revised form, May 3, 1995. Supported in part by a National Science Foundation Postdoctoral Fellowship.

contains Berger's class. The former class is in fact, much vaster, since it contains, for example, metrics with arbitrarily small volume (see [3] and Example 2.4 in [11]).

On the other hand, the set of smooth manifolds admitting metrics satisfying $(*)$ is nearly the same as for Berger's class. Indeed, in [8] Gromoll and Grove extended Berger's Rigidity Theorem and the Diameter Sphere Theorem ([13]) in proving the

Diameter Rigidity Theorem. *Let M be a complete, simply connected Riemannian n-manifold with sectional curvature $sec\ M \geq 1$ and diameter $diam\ M \geq \pi/2$. Then either*

(i) *M is homeomorphic to S^n,*

(ii) *M is isometric to a compact rank one symmetric space, or*

(iii) *M has the cohomology algebra of the Cayley Plane, CaP^2.*

An open question regarding this theorem is whether the possibility (iii) can be removed from the conclusion. This seems to be a very difficult problem; however, there is a natural hypothesis that falls between those of the two rigidity theorems. Observe that the hypothesis $inj\ M \geq \pi/2$ (which is satisfied by Berger's class) implies that given *any* point $x \in M$, there is a point $y \in M$ so that $dist(x, y) \geq \pi/2$. This later condition can be expressed succinctly in terms of a well known metric invariant called the radius.

Definition 1 (Radius). Let $(X, dist)$ be a compact metric space. The *radius* of X is given by,

$$rad\ X = \min_{x \in X} \max_{y \in X} dist(x, y).$$

(The concept of radius was invented in [26]. The name *radius* was first used in [12].)

Clearly $inj\ M \geq \pi/2 \Rightarrow rad\ M \geq \pi/2 \Rightarrow diam\ M \geq \pi/2$, suggesting the following generalization of Berger's Rigidity Theorem.

Radius Rigidity Theorem. *Let M be a complete, simply connected Riemannian n-manifold with sectional curvature $sec\ M \geq 1$ and radius $rad\ M \geq \pi/2$. Then either*

(i) *M is homeomorphic to S^n, or*

(ii) *M is isometric to a compact rank one symmetric space.*

A crucial step in the proof of the Diameter Rigidity Theorem is to show that if M is not homeomorphic to S^n, then there are certain points x whose unit tangent sphere is mapped via $v \mapsto \exp_x \frac{\pi}{2}v$ onto the cut locus of x, and that this map is a Riemannian submersion with connected fibers. Since the unit tangent sphere is isometric to the unit sphere $S^n \subset \mathbb{R}^{n+1}$, the classification theorem from [9] can be invoked. It states that up to isometric equivalence the only Riemannian submersions of Euclidean spheres (with connected fibers) are the Hopf fibrations, except possibly for fibrations of the 15-sphere by homotopy 7-spheres. It was shown in [8] that if the exception could be removed from the submersion theorem in [9], then (iii) can be removed from the statement of the Diameter Rigidity Theorem (see Remark 4.4 in [8]). Although we have not been able to remove the exception from the submersion classification, we have proved the following results.

Main Lemma 2. *Let $S^n(r)$ denote $\{v \in \mathbb{R}^{n+1} \mid \|v\| = r\}$. Let $\Pi : S^{15}(1) \longrightarrow V$ be a Riemannian submersion with connected, 7-dimensional fibers, and let G be the set of points $v \in V$ such that $\Pi^{-1}(v)$ is totally geodesic. Then G is either empty, or a totally geodesic and isometrically embedded copy of $S^l(\frac{1}{2})$ for some $0 \leq l \leq 8$.*

Given a Riemannian submersion $\Pi : S^{15}(1) \longrightarrow V$, the points v such that $\Pi^{-1}(v)$ is totally geodesic will be called the *good points* of V.

Theorem 3. *Let $\Pi : S^7 \hookrightarrow S^{15}(1) \longrightarrow V$ be a Riemannian submersion. If there is a 4-dimensional set of good points, $G_V \subset V$ and a totally geodesic $S^7 \subset \Pi^{-1}(G_V)$ so that $\Pi|_{S^7} : S^7 \longrightarrow G_V$ is isometrically equivalent to the quaternionic Hopf fibration, then Π is isometrically equivalent to the Hopf fibration $S^7(\frac{1}{2}) \hookrightarrow S^{15}(1) \xrightarrow{\Pi_h} S^8(\frac{1}{2})$.*

The Riemannian manifolds with

$$(**)\quad \textit{sec } M \geq 1,\ \textit{diam } M \geq \frac{\pi}{2}, \text{ and nontrivial fundamental group}$$

were completely classified in [8]. Naturally, the class with *sec* $M \geq 1$, *rad* $M \geq \frac{\pi}{2}$, and nontrivial fundamental group is contained in $(**)$. It is not difficult to prove that this containment is proper.

Theorem 4. *Let M be a closed, Riemannian n-manifold with sectional curvature sec $M \geq 1$, radius rad $M \geq \frac{\pi}{2}$, and nontrivial fundamental group Γ. Then either:*

(i) *The universal cover $\tilde{M}$ of M is isometric to $S^n(1)$, and every orbit of the action of Γ is contained in a proper invariant totally* *geodesic subsphere, or*

(ii) *For some $d \geq 2$, M is isometric to the $\mathbb{Z}_2$-quotient of CP^{2d-1} given by the involution*

$$[z_1, z_2, \dots, z_{2d}] \mapsto [\bar{z}_{d+1}, \dots, \bar{z}_{2d}, -\bar{z}_1, \dots, -\bar{z}_d]$$

in homogeneous coordinates on CP^{2d-1}.

Moreover, all such spaces have sec $M \geq 1$ and rad $M \geq \frac{\pi}{2}$.

Recall ([27]) that a representation $\rho : \Gamma \longrightarrow O(n+1)$ is called fixed point free if and only if $S^n(1)/\rho(\Gamma)$ is a space form.

The actions of the groups in (i) are necessarily reducible; however, it is not immediately apparent (at least to the author) exactly which (reducible) space forms satisfy the conclusion of (i). As a partial answer we will prove

Theorem 5. *Let $\rho : \Gamma \longrightarrow O(n+1)$ be a fixed point free representation that decomposes as a direct sum*

$$\rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k$$

of $k \geq 2$ irreducible representations. Then the folloing hold:

(i) *A necessary condition for $S^n(1)/\rho(\Gamma)$ to have radius $= \frac{\pi}{2}$ is that ρ_i be equivalent to ρ_j for some $i \neq j$.*

(ii) *In case Γ is abelian, (i) is also a sufficient condition.*

(iii) *If Γ is not abelian and $\tilde{\rho} : \Gamma \longrightarrow O(d)$ is a fixed point free, irreducible representation, then rad $\frac{S^{2d-1}(1)}{(\tilde{\rho}\oplus\tilde{\rho})(\Gamma)} < \frac{\pi}{2}$.*

(iv) *If rad $S^n(1)/\rho(\Gamma) = \frac{\pi}{2}$ and $\sigma : \Gamma \longrightarrow O(d)$ is another fixed point free representation of Γ, then rad $\frac{S^{n+d}(1)}{(\rho\oplus\sigma)(\Gamma)} = \frac{\pi}{2}$.*

(v) *There is a k_0 (depending on Γ) such that if $k \geq k_0$, then rad $S^n(1)/\rho(\Gamma) = \frac{\pi}{2}$.*

Given a smooth manifold M, the tangent and unit tangent bundles of M will be denoted by TM and SM respectively. The unit tangent sphere at a point x will be called S_x. If $V \subset M$ is a smooth submanifold, then the normal bundle of V in M will be denoted by NV. When there is no possibility of confusion we denote $S^n(1)$ by S^n.

Given $v \in SM$ we let c_v denote the unique geodesic such that $\dot{c}_v(0) = v$. All geodesics will be parametrized by arc length on $[0, \cdot]$ unless otherwise indicated. The symbol ab will be used to denote a minimal geodesic between a and b, and the distance between a and b will be denoted by either $dist(a, b)$ or $|ab|$.

For simplicity we abbreviate compact rank-one symmetric space as CROSS.

The remainder of the paper is divided into five sections. The first two sections contain the proof of the main lemma and a review of certain material from [8]. The Radius Rigidity Theorem is proved in section 3 modulo Theorem 3. whose proof is given in section 4. Finally, section 5 contains the proofs of Theorems 4. and 5..

Remark. Several months after the preprint corresponding to this paper was written, Dimitri Alekseevsky informed the author that he had completed the classification of closed manifolds with sectional curvature ≥ 1, radius $\geq \frac{\pi}{2}$, and nontrivial fundamental group.

1. Reflecting good points

First we review some basic facts about Riemannian submersions. Recall that if $\pi : M \longrightarrow B$ is a Riemannian submersion, then the vectors tangent to the fibers are called vertical vectors, and the vectors perpendicular to the fibers are called horizontal vectors. We denote these two subbundles of TM by $\mathcal{V}M$ and $\mathcal{H}M$ respectively.

The fundamental tensors of a submersion were defined in [20] as follows. For arbitrary vector fields E and F on M the tensor T is defined by

$$T_E F = (\nabla_{E^v} F^v)^h + (\nabla_{E^v} F^h)^v,$$

where the superscripts h and v denote the horizontal and vertical parts of the vectors in question. Note that the first summand is the second fundamental form of a fiber applied to E^v and F^v, and the second term is the shape operator of a fiber applied to E^v and F^h.

The other fundamental tensor, A, is obtained by dualizing T, that is, by switching all horizontal and vertical parts in the definition of T. Thus

$$A_E F = (\nabla_{E^h} F^h)^v + (\nabla_{E^h} F^v)^h.$$

It is shown by O'Neill in [20], that all of the sectional curvatures of M can be written in terms of A, T, ∇A, ∇T, the sectional curvatures of B, and the intrinsic sectional curvatures of the fibers. In particular,

he proves that if X and Y are orthonormal horizontal vector fields and V is a unit vertical field, then we have:
Horizontal Curvature Equation

$$K(X,Y) = K(d\pi X, d\pi Y) - 3\|A_X Y\|^2, \text{ and} \tag{6}$$

Vertizontal Curvature Equation

$$K(X,V) = \langle (\nabla_X T)_V V, X\rangle + \|A_X V\|^2 - \|T_V X\|^2. \tag{7}$$

We refer the reader to [20] for the statements and proofs of the basic facts about T and A and other basic facts and definitions about Riemannian submersions that we will use freely and without further mention.

Now we begin the proof of the main lemma. Let Π, V, and G be as in the main lemma.

Lemma 6.

(i) *If $x \in G$, then there is a unique point $a(x) \in V$ at maximal distance from x, $dist(x, a(x)) = \frac{\pi}{2}$, and $a(x)$ is also in G.*

(ii) *V is Wiedersehen at x and $a(x)$, i.e., the cut locus of x is $a(x)$, and the cut locus of $a(x)$ is x. Furthermore, the fibers of Π are invariant under the antipodal map, and every geodesic in V is periodic with period π.*

Remark. Gromoll and Grove have proven that the fibers of *any* Riemannian submersion with connected fibers of the 15-sphere are invariant under the antipodal map (even ones with $G = \emptyset$) ([10]).

Proof. First we review the notion of

Holonomy Displacement. ([15, p. 238], also [9, p. 150]). Let γ be a geodesic in V. If we consider all of the horizontal lifts of γ to S^{15}, then we obtain a family of diffeomorphisms, $\psi(\gamma)_{s,t} : \Pi^{-1}(\gamma(s)) \longrightarrow \Pi^{-1}(\gamma(t))$ given by $\psi(\gamma)_{s,t}(z) = \gamma_z(t)$, where γ_z denotes the unique horizontal lift of γ with $\gamma_z(s) = z$. If c is a horizontal geodesic we will write $\psi(c)$ for $\psi(\Pi(c))$.

Now suppose that $F_x \equiv \Pi^{-1}(x)$ is totally geodesic. Then all horizontal geodesics emanating from F_x are in a totally geodesic 7-sphere $F_{a(x)}$ at time $\pi/2$. Hence $F_{a(x)}$ is also a fiber of Π, and $\Pi(F_{a(x)})$ is the desired point $a(x)$. This proves (i).

Since every horizontal geodesic emanating from F_x reaches $F_{a(x)}$ at time $\pi/2$, every geodesic emanating from x reaches $a(x)$ at time $\pi/2$, and hence is minimal up to time $\pi/2$. Thus V is Wiedersehen at x and by symmetry at $a(x)$.

It follows that reflection in x is a homeomorphism of V. Hence reflection in F_x is an isometry of S^{15} that maps fibers to fibers. Similarly, reflection in $F_{a(x)}$ maps fibers to fibers. But the composition of the two reflections is the antipodal map, a, of S^{15}. So if the composition of reflection in x with reflection in $a(x)$ is the identity map of V, then the fibers are invariant under the antipodal map.

To establish this, let r_x, $r_{a(x)}$, r_{F_x} and $r_{F_{a(x)}}$ be the four reflections. Note that $r_{F_x} \circ r_{F_{a(x)}} \circ r_{F_x} \circ r_{F_{a(x)}} = a \circ a = id_{S^{15}}$. So $r_x \circ r_{a(x)} \circ r_x \circ r_{a(x)} = id_V$. The differential $d(r_x \circ r_{a(x)})_{a(x)}$ is therefore a linear isometry of $T_{a(x)}V$ whose square is the identity, and hence $T_{a(x)}V$ has a basis of eigenvectors for $d(r_x \circ r_{a(x)})_{a(x)}$ with corresponding eigenvalues of either 1 or -1. Suppose v is a unit eigenvector whose eigenvalue is -1. Then $-v = d(r_x \circ r_{a(x)})_{a(x)} v = d(r_x)_{a(x)} - v$. This implies that the reflection isometry r_x fixes the geodesic $c_{-v} : t \mapsto \exp_{a(x)} -tv$, which is absurd, since $c_{-v}(\frac{\pi}{2}) = x$. So the only possible eigenvalue is 1, and we can conclude that the fibers are indeed invariant under the antipodal map, which implies immediately that every geodesic in V is periodic with period π. q.e.d.

We saw in the proof above that reflection in a totally geodesic fiber is an isometry of S^{15} that preserves the fibers of Π. By using this fact over and over again, we can prove

Lemma 7. *Let $x, a(x) \in V$ be good points at maximal distance. Let $z \in V \backslash \{x, a(x)\}$ be another good point, and let $\gamma : [0, \infty) \longrightarrow V$ be the unique geodesic that passes through x, z, and then $a(x)$ so that $\gamma(0) = x$. Then $\gamma(k \cdot dist(x, z))$ is a good point for all integers k. In particular, if $dist(x, z)$ is an irrational multiple of π, then all points along γ are good.*

Given $c > 0$, recall ([8]) that a subset A of a Riemannian manifold is called totally c-convex if and only if any geodesic of length strictly less than c whose end points lie in A lies entirely in A.

Lemma 8. *G is totally $\frac{\pi}{2}$-convex.*

Proof. Let a and b be in G and suppose that $dist(a, b) < \frac{\pi}{2}$. By Lemma 6, there is a unique minimal geodesic γ between a and b. We will show that the holonomy displacement maps determined by γ are isometries and hence that all of the fibers over γ are totally geodesic.

Let a_1 and a_2 be very close points in $\Pi^{-1}(a)$, and let γ_1, and γ_2 be the unique horizontal lifts of γ with initial points a_1 and a_2 respectively. Call the end points of the γ_i's b_1 and b_2 respectively. Then the γ_i's are both perpendicular to $\Pi^{-1}(b)$ and hence to $b_1 b_2$.

Now consider the two triangles $\triangle(b_1, a_1, a_2)$ and $\triangle(b_1, b_2, a_2)$, and note that $\sphericalangle(b_1, a_1, a_2) = \sphericalangle(b_1, b_2, a_2) = \frac{\pi}{2}$ and $|a_1 b_1| = |a_2 b_2| = |ab|$. Since the two triangles also share the common side $b_1 a_2$ and all distances can be assumed to be $< \frac{\pi}{2}$, the remaining sides $a_1 a_2$ and $b_1 b_2$ must be of equal length. This shows that the holonomy displacement map of γ that takes $\Pi^{-1}(a)$ to $\Pi^{-1}(b)$ is a local isometry. Since it is also a diffeomorphism, it is a global isometry.

Now consider the mid point c of γ, and for $i = 1, 2$ let c_i the lift of c on that lies on γ_i. Then $|a_2 c_2| = |b_2 c_2|$, $|a_1 a_2| = |b_1 b_2|$, and $\sphericalangle(a_1, a_2, c_2) = \sphericalangle(c_2, b_2, b_1) = \frac{\pi}{2}$, so

$$|a_1 c_2| = |b_1 c_2|.$$

Combining this with $|a_1 c_1| = |c_1 b_1| = 2|a_1 b_1|$ yields

$$\sphericalangle(a_1, c_1, c_2) = \sphericalangle(b_1, c_1, c_2) = \frac{\pi}{2},$$

and a similar argument shows

$$\sphericalangle(a_2, c_2, c_1) = \sphericalangle(b_2, c_2, c_1) = \frac{\pi}{2}.$$

So we can repeat the argument of the proceeding paragraph and conclude that $|c_1 c_2| = |a_1 a_2| = |b_1 b_2|$, and hence that the holonomy displacement map of γ that takes $\Pi^{-1}(a)$ to $\Pi^{-1}(c)$ is an isometry.

Since we can repeat the above procedure as often as we like, the set of good points on γ must be dense. On the other hand, $T = 0$ is a closed condition, so every point on γ must be good. q.e.d.

It follows from Lemma 8 and [5] that G is a topological manifold with (possibly empty) boundary and smooth totally geodesic interior. By Lemma 7, $\partial G = \emptyset$, and by Lemma 6, G is a topological sphere. The isometry type of G is therefore determined by

Lemma 9. *If $\gamma : [0, \pi] \longrightarrow V$ is a geodesic whose image consists entirely of good points, then all radial sectional curvatures of V along γ are constant and equal to 4.*

Proof. Let $\tilde{\gamma}$ be a horizontal lift of γ to S^{15}. Let X and $\tilde{X}$ denote the tangent fields to γ and $\tilde{\gamma}$ respectively, and let V be a vertical unit field along $\tilde{\gamma}$ so that $(\nabla_{\tilde{X}} V)^v = 0$. Then, using the equation for the vertizontal curvatures, we find that $K(\tilde{X}, V)$ along $\tilde{\gamma}$ is,

$$
\begin{aligned}
(12) \qquad 1 &\equiv K(\tilde{X}, V) \\
&= \langle (\nabla_{\tilde{X}} T)_V V, \tilde{X} \rangle + \|A_{\tilde{X}} V\|^2 - \|T_V \tilde{X}\|^2 = \|A_{\tilde{X}} V\|^2.
\end{aligned}
$$

It follows from (12) that

(13)

$$\text{the map } v \mapsto A_{\tilde{X}} v \text{ from } \mathcal{V}S^{15} \text{ to } \mathcal{H}S^{15} \cap \tilde{X}^{\perp} \text{ is bijective.}$$

Combining this and (12) we can show that

$$(14) \qquad \|A_{\tilde{X}} y\| \equiv 1 \text{ for all unit vectors } y \in \mathcal{H}S^{15} \cap \tilde{X}^{\perp}.$$

Indeed, if $\langle A_{\tilde{X}} y, v \rangle$ were larger than 1 for some unit vector $v \in \mathcal{V}S^{15}$, then we would have $|\langle A_{\tilde{X}} v, y \rangle| = |\langle v, A_{\tilde{X}} y \rangle| > 1$, contrary to (12). On the other hand, by (12) and (13), $y = A_{\tilde{X}} v$ for some unit vector $v \in \mathcal{V}S^{15}$, thus $|\langle A_{\tilde{X}} y, v \rangle| = |\langle y, A_{\tilde{X}} v \rangle| = 1$, and $\|A_{\tilde{X}} y\| \geq 1$ as well.

Let Y be a unit field along γ that is perpendicular to X, and let $\tilde{Y}$ denote the horizontal lift of Y. Then it follows from (14) that

$$K(X, Y) = K(\tilde{X}, \tilde{Y}) + 3\|A_{\tilde{X}} \tilde{Y}\|^2 = 1 + 3 * 1 = 4.$$

q.e.d.

2. Review of the Diameter Rigidity Theorem

If M satisfies the hypotheses of the Radius Rigidity Theorem, then M also satisfies the hypotheses of the Diameter Rigidity Theorem, so the only way M can fail to satisfy the conclusion of the Radius Rigidity Theorem is if it has the cohomology algebra of CaP^2. We assume throughout sections 2 and 3 that $sec\ M \geq 1$, $Rad\ M \geq \pi/2$, $\pi_1(M) = \{e\}$, and $H^*(M) \cong H^*(CaP^2)$, and we attempt to show that M is isometric to CaP^2.

By the Diameter Sphere Theorem ([13]), *diam* $M = \pi/2$. We would like to focus on this property for awhile; so let $\hat{M}$ be a Riemannian n-manifold with $sec\ \hat{M} \geq 1$, $\pi_1(\hat{M}) \cong \{e\}$, and *diam* $\hat{M} = \pi/2$, that is not homeomorphic to S^n. Many basic aspects of the geometry of $\hat{M}$ can be described in terms of so called *dual sets* ([8]). (Cf. also [23], [24], and [25].)

Definition 10 (Dual Sets). For any subset $B \subset \hat{M}$, the dual set of B is

$$B' = \{x \in \hat{M} \mid dist(x, B) = \pi/2\}.$$

The following properties of dual sets were observed in [8] (cf. also [23], [24], and [25]):

(i) B' is totally π-convex.

(ii) $B \subset B''$.

(iii) If $A \subset B$, then $A' \supset B'$.

(iv) $B' = B'''$.

It follows from (i) and [5] that B is a topological manifold with (possibly empty) boundary and smooth, totally geodesic interior.

If we start with a set B so that $B' \neq \emptyset$ and set $A = B'$, then $A = (A')'$. Thus

$$A = \{x \in \hat{M} \mid dist(x, A') = \pi/2\} \text{ and } A' = \{x \in \hat{M} \mid dist(x, A) = \pi/2\},$$

and A and A' are called a *dual pair.*

The proof in [8] proceeds from this point to use comparison theory and other geometric and topological tools to argue that the geometry of $\hat{M}$ is more and more like the geometry of a CROSS. For example, it is shown that $\partial A = \partial A' = \emptyset$, that cutlocus$(A) = A'$ and cutlocus$(A') = A$, and that for any $p \in A$ the map $\Pi_p : UNA_p \longrightarrow A'$ from the unit normal sphere to A at p to A' given by $\Pi_p(u) = \exp(\frac{\pi}{2}u)$ is a Riemannian submersion with connected fibers. This allows them to apply the classification theorem in [9] and conclude that Π_p is isometrically equivalent to a Hopf fibration (except possibly if the fibers are 7-dimensional). The proof is completed with further comparison arguments. The exception to the conclusion is accounted for by the fact that the classification in [9] is not quite complete. It leaves open the possibility of nonstandard Riemannian submersions of the 15-sphere by homotopy 7-spheres. On the other hand, it is possible to prove that this is the only obstruction.

Proposition 11. *If $\hat{M}$ has a dual pair (A, A') such that one of the submersions Π_p is isometrically equivalent to a Hopf fibration, then $\hat{M}$ is isometric to a CROSS.*

In the course of proving Proposition 11 we will use

Lemma 12. *The group of holonomy displacement maps for the Hopf fibration $\Pi_h : S^{15} \longrightarrow S^8$ that take a fiber to itself contains a subgroup that is isomorphic to $SO(8)$.*

Proof of Lemma 12. Let F and G be any fibers. Let r_F denote reflection in F, and let γ be a minimal geodesic between $\Pi_h(F)$ and $\Pi_h(G)$. Then the two maps r_F and $\psi(\gamma)_{0,2dist(\Pi_h(F),\Pi_h(G))}$ from G to $r_F(G)$ coincide.

By the proof of Lemma 6, r_F is a symmetry of Π_h, and covers the isometry, $r_{\Pi(F)}$, given by reflection of $S^8(\frac{1}{2})$ in $\Pi(F)$. We claim that the set of reflections in points of S^8 generates $SO(9)$. To see this, let $\{e_1, e_2, \dots, e_9\}$ be an orthonormal basis for $\mathbb{R}^9$ and observe that if $v, w \in Span(e_1, e_2) \cap S^8$, then $r_v \circ r_w$ is the identity on $Span\{e_3, \dots, e_9\}$, and by choosing v and w appropriately, $r_v \circ r_w$, can be any preassigned orientation preserving isometry of $Span\{e_1, e_2\}$. Therefore the group generated by reflections in points of S^8 can act as any preassigned rotation on any great circle while fixing the orthogonal complement of the circle, and therefore is $SO(9)$.

It was shown in [14] that the group of symmetries of Π_h is isomorphic to $Spin(9)$, acting as follows:

(i) Any orientation preserving isometry of $S^8(\frac{1}{2})$ lifts to a symmetry g of Π_h, and

(ii) Given any such g there is exactly one other symmetry, $antipodal-map \circ g$, which induces the same map on S^8.

The $antipodalmap : F \longrightarrow F$, can also be viewed as the restriction $r_{A(F)}|_F$ where $A(F)$ is the fiber such that $dist(F, A(F)) = \frac{\pi}{2}$. Therefore the restriction to F of any symmetry of Π_h which preserves F can be realized as a holonomy displacement map for Π_h. On the other hand, Proposition 7.10 in [14] asserts that the group of symmetries of Π_h that preserve F contains a subgroup isomorphic to $SO(8)$. Therefore the group of of holonomy displacement maps for Π_h that take F to itself contains a subgroup isomorphic to $SO(8)$. q.e.d.

Proof of Proposition 11.

By the Diameter Rigidity Theorem, we may assume that $H^*(\hat{M}) \cong H^*(CaP^2)$. Say $p \in A$, and Π_p is isometrically equivalent to a Hopf fibration. Then A' is isometric to a CROSS, $P^m(K)$.

First suppose that $A = \{p\}$. Choose $p_0 \in \mathbb{C}aP^2$ and an equivalence $\iota : S_p \longrightarrow S_{p_0}$ between Π_p and Π_{p_0}. Extend ι to a linear isometry $\iota : T\hat{M}_p \longrightarrow T\mathbb{C}aP^2_{p_0}$. Then the map $f = \exp_{p_0} \circ \iota \circ \exp_{p_0}^{-1} : \hat{M} \backslash \{p\}' \longrightarrow$

$\mathbb{C}aP^2\backslash\{p_0\}'$ extends to a well defined homeomorphism $f : \hat{M} \longrightarrow CaP^2$. Moreover, f is an isometry, provided $\{x\}'$ is isometric to $S^8(\frac{1}{2})$ for all $x \in A'$. Indeed, knowing that $\{x\}'$ is isometric to $S^8(\frac{1}{2})$ would tell us that $df|_{T\{x\}'}$ is an isometry (whose image is $T\{f(x)\}'$). On the other hand it follows from the proofs of 3.1 and 3.6 in [8] that $df|_{N\{x\}'}$ is an isometry whose image is $N\{f(x)\}'$.

Since A' is isometric to $S^8(\frac{1}{2})$, $\{x\}' \cap A'$ is a singleton and $\{x\}'' = \{x\}$ for all $x \in A'$. Therefore $(\{x\}, \{x\}')$ is a dual pair, and it is enough to show that the Riemannian submersion $\Pi_x : S_x \longrightarrow \{x\}'$ is standard or equivalently (by the main theorem in [22]) that the fibers of Π_x are totally geodesic.

Set $a(x) = \{x\}' \cap A'$. Then $\Pi_x^{-1}(a(x)) = SA'_x$ and $\Pi_x^{-1}(p) = S\{a(x)\}'_x$ are both totally geodesic. It will follow that all of the fibers of Π_x are totally geodesic if we can show that all of the holonomy displacement maps for horizontal geodesics in S_x emanating from SA'_x are isometries.

If c is such a geodesic, and v is a unit vertical vector at $c(0)$, then $d\psi(c)_{0,t}(v) = J(t)$ where J is the Jacobi field along c with initial conditions

$$J(0) = v \text{ and } J'(0) = A_{\dot{c}(0)}v + T_v\dot{c}(0) \tag{18}$$

(cf [9, p. 150]). Since $c(0)$ is in a totally geodesic fiber, the initial conditions for J simplify to

$$J(0) = v \text{ and } J'(0) = A_{\dot{c}(0)}v. \tag{19}$$

Therefore $\langle J'(0), J(0)\rangle = 0$, so

$$J(t) = \cos t P_t(J(0)) + \sin t P_t(J'(0)), \tag{20}$$

where for $z \in T(S\hat{M}_x)_{c(0)}$, $P_t(z)$ is the parallel image of z along c at $c(t)$. It will follow that all of the $\psi(c)_{s,t}$'s are isometries provided we can show that $\|A_{\dot{c}(0)}v\| \equiv 1$ for all unit vertical vectors v at $c(0)$.

Let $M_t : T\hat{M}_x \longrightarrow T\hat{M}_x$ denote the map given by multiplication by t, and let $S(x,r) = \{y \in \hat{M} \mid dist(x,y) = \frac{\pi}{2}\}$. The proof in [8] that Π_x is a Riemannian submersion also shows that the map $\Pi_r = \exp_x \circ M_{\frac{\pi}{2r}} \circ \exp_x^{-1} : S(x,r) \longrightarrow \{x\}'$ is a Riemannian submersion if we multiply the intrinsic metric on $S(x,r)$ by $\frac{1}{\sin r}$. Given a vector $y \in S\{x\}'_{a(x)}$ and a vector $v \in SA'_{a(x)}$. Consider the hinge (c_y, c_v). The hinge angle is $\frac{\pi}{2}$ and $dist(c_v(\frac{\pi}{2}), c_y(t)) \equiv \frac{\pi}{2}$ for all t. Therefore (as pointed out in [8]) (c_y, c_v) is spanned by a totally geodesic surface $S_{y,v}$ of constant curvature 1.

Let Y denote the tangent field of c_y. For a given point $c_y(s)$ along c_y, let γ_s be the minimal geodesic in $S_{y,v}$ from x to $c_y(s)$. Then the Jacobi field, J, along γ_s with $J(0) = 0$ and $J(\frac{\pi}{2}) = Y(s)$ is $J(t) = \sin t P_t(Y(s))$ where $P_t : T\hat{M}_{\gamma_s(\frac{\pi}{2})} \longrightarrow T\hat{M}_{\gamma_s(t)}$ is parallel transport along γ_s. It follows from the proof of 3.6 in [8] that $J(t)$ is horizontal for Π_t, and it is clear that $d\Pi_t(J(t)) = Y(s)$. So $J(t)$ is the horizontal lift of $Y(s)$ with respect Π_t at $\gamma_s(t)$. Allowing v to vary over all vectors in $SA'_{a(x)}$ we see how to find all horizontal lifts of Y for all of the Π_r's.

Let $z \in S\{x\}'_{a(x)}$ be perpendicular to y and let $\tilde{Y}$ (resp. $\tilde{Z}$) be the vector field whose value at a point in $q \in S(x,r) \cap \Pi_r^{-1}(c_y)$ (resp. a point in $S(x,r) \cap \Pi_r^{-1}(c_z)$) is the horizontal lift of Y (resp Z) with respect to Π_r.

Let ∂_r be the unit radial field emanating from x. Then using the fact that $\tilde{Y}$ and ∂_r (resp. $\tilde{Z}$ and ∂_r) span a totally geodesic surface of constant curvature 1 it can be shown that

$$\nabla_{\tilde{Y}} \partial_r = \cot r \tilde{Y} \text{ and } \nabla_{\tilde{Z}} \partial_r = \cot r \tilde{Z}.$$

Thus if we extend $\tilde{Y}$ and $\tilde{Z}$ to vector fields everywhere tangent to the metric spheres about x, we find that

$$\langle \nabla_{\tilde{Y}} \tilde{Z}, \partial_r \rangle|_{A'} = -\langle \tilde{Z}, \nabla_{\tilde{Y}} \partial_r \rangle|_{A'} = 0.$$

Therefore using the Gauss equation and the O'Neil's equation for horizontal curvatures we get

$$\begin{aligned} sec_{\hat{M}}(\tilde{Y}, \tilde{Z})|_{A' \cap S(x,r)} &= sec_{S(x,r)}(\tilde{Y}, \tilde{Z}) - cot^2 r \\ &= \frac{1}{\sin^2 r} sec_{\frac{1}{\sin r} S(x,r)}(\tilde{Y}, \tilde{Z}) - \cot^2 r \\ &= \frac{1}{\sin^2 r}(sec_{\{x\}'}(y,z) - 3\|A^r_{\tilde{Y}} \tilde{Z}\|^2 - \cos^2 r) \\ &= \frac{1}{\sin^2 r}(1 + 3\|A^0_{\tilde{Y}_0}, \tilde{Z}_0\|^2 - 3\|A^r_{\tilde{Y}} \tilde{Z}\|^2 - \cos^2 r) \\ &= \frac{\sin^2 r + 3(\|A^0_{\tilde{Y}_0} \tilde{Z}_0\|^2 - \|A^r_{\tilde{Y}} \tilde{Z}\|^2)}{\sin^2 r}, \end{aligned} \tag{21}$$

where we have adopted the conventions that A^0 is the A-tensor of Π_x, A^r is the A-tensor of Π_r, $\tilde{Y}_0 = d(M_{\frac{1}{r}} \circ \exp_x^{-1})\tilde{Y}$, $\tilde{Z}_0 = d(M_{\frac{1}{r}} \circ \exp_x^{-1})\tilde{Z}$, and $\frac{1}{\sin r}S(x,r)$ is the space obtained from $S(x,r)$ by scaling the intrinsic metric by a factor of $\frac{1}{\sin r}$.

Note that

$$(22) \qquad (A^r_{\tilde{Y}}\tilde{Z})_0 = \frac{1}{2}([\tilde{Y},\tilde{Z}]^v)_0 = \frac{1}{2}([\tilde{Y}_0,\tilde{Z}_0]^v) = (A^0_{\tilde{Y}_0}\tilde{Z}_0).$$

The second equality is a consequence of the fact that $d(M_{\frac{1}{r}} \circ \exp_x^{-1})$ maps horizontal vectors to horizontal vectors, and vertical vectors to vertical vectors.

It follows from (22) that the difference between the two A-tensor terms on the right-hand side of (21) is due entirely to the dilation of the map $dM_{\frac{1}{r}} \circ \exp_p^{-1} X$ on vertical vectors. (The map is an isometry on horizontal vectors.) The vertical vectors along A' are tangent to A', which is isometric to $S^8(\frac{1}{2})$; so the dilation (with respect to the metric on $S(x,r)$ scaled by $\frac{1}{\sin r}$) of the vertical vectors is $\frac{1}{\cos r}$. Therefore the right-hand side of (21) is

$$\frac{(\sin^2 r + 3\|A^0_{Y_0}Z_0\|^2(1-\cos^2 r)}{\sin^2 r} = 1 + 3\|A^0_{Y_0}Z_0\|^2.$$

In particular, the sectional curvature, $sec_M(\tilde{Y},\tilde{Z})$, is independent of the base point. There was nothing special in the above argument about the fixed point $x \in A'$; so it shows that the sectional curvature of planes normal to A' is invariant under parallel transport along broken geodesics in A'.

The normal holonomy of A' was described in [8] in the following manner. Let $u \in NA'_q$ be a normal vector, and let γ in A' be a geodesic with $\gamma(0) = q$. Then arguing as on page 645 we find a totally geodesic surface, $S_{u,\gamma}$, of constant curvature 1 which spans the hinge (c_u, γ). The parallel image of u at $\gamma(t)$ is the initial tangent vector of the minimal segment $\gamma(t)p_u$ from $\gamma(t)$ to p along $S_{u,\gamma}$. The set of negatives of final tangent vectors,

$$\{-\dot{\gamma(t)p_u}(\frac{\pi}{2}) \mid t \in \mathbb{R}\},$$

of these geodesics determines a geodesic $\tilde{\gamma}$ in S_p, and from 3.6 in [8] we see that $\tilde{\gamma}$ is a horizontal lift of γ. Thus the map $u \mapsto -\dot{c}_u(\frac{\pi}{2})$ between UNA' and S_p is an equivalence of S^7-bundles over A', which is equivariant with respect to the two types of holonomy maps determined by γ (the $\psi(\gamma)$'s in S_p and the parallel transport in NA'). But by Lemma 12 the group of holonomy displacement maps for Π_h that take a fiber to itself contains a subgroup that is isomorphic to $SO(8)$. It follows that the normal sectional curvatures of A' are constant.

Returning to our fixed point $x \in A'$ and applying the Horizontal Curvature Equation we see that there is some constant c so that

$$\|A^0_y z\| \equiv c$$

for all orthonormal, horizontal vectors, y, z, along $\Pi_x^{-1}(a(x))$. Arguing as in the proof of Lemma 9 we see that

$$\|A^0_y v\| \equiv c$$

for all unit horizontal y along $\Pi_x^{-1}(a(x))$ and all unit vertical v along $\Pi_x^{-1}(a(x))$.

Applying equations (19) and (20) we get that the differentials of any unit vertical v along $\Pi_x^{-1}(a(x))$ under the holonomy displacement maps for a horizontal geodesic c_y emanating from the foot point of v are given by the Jacobi field

$$J_{v,y}(t) = \cos t P_t(v) + \sin t P_t(A^0_y v)$$

along c_γ. Substituting in the value $t = \frac{\pi}{2}$ we get $\|J_{v,y}(\frac{\pi}{2})\| \equiv c$ for all possible choices of v and y. This implies that the map $\psi(c_y)_{0,\frac{\pi}{2}}$ is a dilation by a factor of c. On the other hand, $\Pi_x^{-1}(a(x)) = SA'_x$ and $\psi(c_y)_{0,\frac{\pi}{2}}(SA'_x) = S\{a(x)\}'_x$ are both isometric to S^7, so $c = 1$, and it follows that the $\psi(c_y)_{s,t}$'s are isometries for all horizontal geodesics, c_y, emanating from $\Pi_x^{-1}(a(x))$. Therefore Π_x is standard, and we have dispensed with the case $A =$ *point*.

Now consider the case $A \neq \{p\}$. Since A' is isometric to $P^m(K)$, the dual set B (in A') of any singleton $\{x\} \subset A'$ is isometric to $P^{m-1}(K)$, and the double dual of $\{x\}$ (in $\hat{M}$) is again $\{x\}$. It follows from the convexity properties of A' that the fibers of the submersion $SA'_x \longrightarrow B$ are also fibers of the submersion $\Pi_x : S\hat{M}_x \longrightarrow \{x\}'$, and the dimension of these fibers is < 7 because $dim\ A' < 16$.

Therefore the submersion Π_x is equivalent to a Hopf fibration (with 1- or 3-dimensional fibers), and by [8, p. 236] N is isometric to a CROSS.

q.e.d.

We now restrict our attention to the possibly exceptional manifold M.

Proposition 13.

(i) *The set of dual pairs covers M.*

(ii) *Every dual pair consists of a singleton and a set that is homeomorphic to* S^8.

(iii) *If* (p, V) *and* (q, W) *are distinct dual pairs, then* $V \cap W$ *is a point.*

Proof. (i) is an immediate consequence of properties (ii) and (iv) of dual sets and the fact that $rad\ M = \pi/2$.

To prove (ii) first note that if (p, V) is a dual pair and V is not 8-dimensional, then the Riemannian submersion $\Pi_p : SM_p \longrightarrow V$ is isometrically equivalent to a Hopf fibration, and M is isometric to some CROSS other than CaP^2. If V is 8-dimensional, then the fibers of Π_p are homotopy 7-spheres (see Theorem 5.1 in [4]). It follows from the long exact homotopy sequence of the fibration Π_p that V is a homotopy 8-sphere, and hence a topological 8-sphere. Finally, if there is a dual pair (A, A') so that neither A nor A' is a point, then for all $p \in A$, $1 \leq dim\ UNA_p \leq 14$, and the submersion Π_p is equivalent to a Hopf fibration.

To prove (iii) observe that since $sec\ M > 0$ and $dim\ V + dim\ W = dim\ M$, a Synge Theorem type of argument shows that $V \cap W \neq \emptyset$ (see [7] and also Proposition 1.4 in [8]). Next observe that $V \cap W = \{p, q\}'$, so $(V \cap W, (V \cap W)')$ is a dual pair. By (ii), one of these dual sets is a point. Since $p, q \in (V \cap W)'$, we conclude that $(V \cap W)$ is a point.

q.e.d.

If (p, V) is a dual pair, then we will (optimistically) refer to V as a *Cayley line*. This name is partially justified by the fact that once we have proven that M is isometric to CaP^2 we will know that all of these V's are isometric to CaP^1.

3. Intersecting cayley lines

In this section we prove the Radius Rigidity Theorem modulo Theorem 3.

If (p, V) is a dual pair, then we have seen that it is enough to show that the submersion $\Pi_p : S_p \longrightarrow V$ is isometrically equivalent to the Hopf fibration $S^7 \hookrightarrow S^{15} \longrightarrow S^8$. This holds if its fibers are totally geodesic (see [22]). Roughly speaking, the strategy of our proof is to find dual pairs (p, V) so that Π_p contains more and more totally geodesic fibers. Our method for finding totally geodesic fibers will be to find more and more "good points" in M.

Definition 14 (Good Point). If (p, V) is a dual pair, then we shall call a point $x \in V$ good if and only if $\Pi_p^{-1}(x)$ is totally geodesic.

A point $m \in M$ will be called good if and only if m is a good point of W for some Cayley line $W \subset M$. The sets of good points in V and M will be denoted by G_V and G_M respectively.

The fact that G_M is rather large is a consequence of Proposition 13 and the next result.

Proposition 15. *Let (p, V) be a dual pair.*

(i) *A point $x \in V$ is good if and only if there is a Cayley line W so that $V \cap W = \{x\}$.*

(ii) *G_M is closed. In fact, $m \in G_M$ if and only if there are points $x, y \in M$ so that $dist(x, m) = dist(m, y) = dist(x, y) = \frac{\pi}{2}$.*

Remark. Gromoll and Grove were also aware of the fact that $G_M \neq \emptyset$ ([10]).

Proof. If there is a dual pair (z, W) so that $W \cap V = \{x\}$, then p and z are distinct points in $\{x\}'$, so $\{x\}'$ is a Cayley line and hence intersects V at a single point y. Since p and x are distinct points of $\{y\}'$, $\{y\}'$ is a Cayley line. It follows that the set of minimal geodesics from p to x is contained in $\{y\}'$. Thus $\Pi_p^{-1}(x)$ is contained in the unit tangent sphere $S\{y\}'_p$ to $\{y\}'$ at p. But since both of these sets are homotopy 7-spheres they must coincide. Since $S\{y\}'_p$ is totally geodesic in SM_p, $\Pi_p^{-1}(x)$ is as well. This proves the "if" part of (i).

On the other hand, if $x \in G_V$, then by Lemma 6 there is a unique point $a(x) \in V$ so that $dist(x, a(x)) = \frac{\pi}{2}$. Since $x, p \in \{a(x)\}'$, $\{a(x)\}'$ is a Cayley line. By Proposition 13, $x = \{a(x)\}' \cap V$. This proves the "only if" part of (i). Since $dist(x, a(x)) = dist(a(x), p) = dist(p, x) = \frac{\pi}{2}$, it also proves the "only if" part of (ii).

To prove the "if" part of (ii) note that $x, y \in \{m\}'$, $m, y \in \{x\}'$, and $m, x \in \{y\}'$. So $\{m\}'$, $\{x\}'$, and $\{y\}'$ are all Cayley lines, and m, for example, is good since $\{m\} = \{x\}' \cap \{y\}'$. q.e.d.

The Radius Rigidity Theorem would follow if we could show that there is a Cayley line V so that every point in V is good. In part, we will do this by finding Cayley lines with good points in sets that are isometric to spheres of constant curvature 4 of progressively higher and higher dimension.

Since each point in M lies on at least one Cayley line, we can certainly find six distinct lines $W, V_1, V_2, \ldots, V_5$. It could be that the V_i's intersect W at only 1 or 2 points, but if this is the case, then at least three of them (say V_1, V_2, and V_3) intersect W at the same point, x. Then $\{x\}'$ is a Cayley line and the points $\{x\}' \cap V_1$, $\{x\}' \cap V_2$, and

$\{x\}' \cap V_3$ must all be distinct. By the main lemma $dim\ G_{\{x\}'} \geq 1$. On the other hand if the V_i's intersect W in three or more points, then it follows that $dim G_W \geq 1$. So we can find a Cayley line V such that $dim\ G_V \geq 1$.

To prove the Radius Rigidity Theorem we argue by contradiction. From the main lemma and Propositions 13 and 15 it follows that the set of good points in each Cayley line is isometric to $S^k(\frac{1}{2})$ for some $0 \leq k \leq 8$. Let V be a Cayley line whose set of good points has maximal dimension, d. We have seen that $d \geq 1$, and if the Radius Rigidity Theorem were false, then by Proposition 11 we would know $d \leq 7$. Consider the configuration C consisting of all Cayley lines of the following types:

type 1 V,

type 2 All the Cayley lines between the good points of V and V',

type 3 For each W of type 2 we also include all of the Cayley lines between each of the good points of W and W' that are neither of type 1 nor of type 2.

We point out that if W is of type 2, then W' is a good point of V, and if U is a line of type 3 between a good point of W and W', then U' is a good point of W.

Suppose we could find a Cayley line Z that is not included in the configuration above. Then either,

$$Z \cap V \notin G_V \text{ or}$$
$$Z \cap V \in G_V.$$

But neither of these is possible. The first can not occur because G_V consists of all of the good points of V. On the other hand, if $Z \cap V \in G_V$, then $(Z \cap V)'$ is a line of type 2, and Z is a Cayley line between $(Z \cap V)'$ and $Z \cap V$, implying that Z is of type 1, 2, or 3, a contradiction. Therefore,

$$C \text{ contains all of the lines of } M. \tag{26}$$

Next we prove

Lemma 16. *We may assume that dim $G_U = d$ for every line U in the configuration.*

Proof. We prove this in a step by step manner.

We know that there is at least one line of type 3, since otherwise the configuration would only be 8-dimensional in a neighborhood of a bad point of V and hence would not cover M. Since all of the lines of type 2 intersect at V', they must intersect at distinct points of each line of type 3. Thus $dim\ G_U \geq 1$ for all lines of type 3, and if U is a line of type 3, then there are infinitely many lines between U and $U' \in W_0$, where W_0 is a line of type 2. Since all of these lines intersect at U' they must intersect each line of type 2 (other than W_0) at infinitely many places. Therefore $dim\ G_W \geq 1$ for all lines $W \neq W_0$ of type 2. Since the set of all good points in M is closed, $dim\ G_{W_0} \geq 1$ as well.

For each point $v \in G_V$, the set $L_v \equiv \{$lines U in the configuration $\mid v \in U\}$ can be topologized by declaring that it is homeomorphic to $G_{\{v\}'}$. We will show that for each $v \in G_V$,

$$L_v^d \equiv \{\text{lines } U \text{ in } L_v \mid dim\ G_U = d\}$$

is both closed and open. Since $V \in L_v^d$ and $\cup_{v \in G_v} L_v = C$, it will follow that $dim\ G_U = d$ for every line in the configuration. Let $\{U_i\}$ be a sequence in L_v^d which is converging to a line U in L_v. Then by passing to a subsequence if necessary, we may assume that $\{G_{U_i}\}$ converges (in the classical Hausdorff topology) to some subset G of U (cf Theorem 4.2 in [18]). By the main lemma, G is isometric to $S^d(\frac{1}{2})$, and by Proposition 15, $G \subset G_U$. In fact $G = G_U$ by the maximality of d. So L_v^d is closed. To see that it is open, let U in L_v^d and let $W \in L_v$ be close to U. Consider the set $L(U, U')$ of lines between U and U'. Each $u \in G_U$ is on exactly one line $Z_u \in L(U, U')$, and the map $G_U \longrightarrow G_W$ given by $u \longmapsto Z_u \cap W$ preserves distances up to small additive error. It follows from the main lemma and the maximality of d that if W was originally chosen to be sufficiently close to U, then G_W is isometric to $S^d(\frac{1}{2})$. q.e.d.

Consider the following subset of TM:

$$TC|_V(\pi/2) = \{v \in TM|_{G_V} \mid \|v\| \leq \pi/2 \text{ and } v \text{ is tangent to a line in the configuration}\}.$$

If the Radius Rigidity Theorem is false, then $\exp : TC|_V(\frac{\pi}{2}) \longrightarrow M$ is a surjective Lipshitz map, $M \backslash G_M$ and $\exp|^{-1}_{TC|_V(\frac{\pi}{2})}(M \backslash G_M)$ are open, dense sets, and $\exp : \exp|^{-1}_{TC|_V(\frac{\pi}{2})}(M \backslash G_M) \longrightarrow M \backslash G_M$ is a smooth bijection. Indeed, exp is surjective since the configuration has to cover M, and exp has unique preimages on $M \backslash G_M$. The set $M \backslash G_M$ is open and dense, since G_M consists of points of the form U' where U is a line in the configuration, and the points of this form all lie in proper subspheres of lines of type 2. $\exp|^{-1}_{TC|_V(\frac{\pi}{2})}(M\backslash G_M)$ is open and dense for similar reasons.

The fact that exp is surjective and Lipshitz yields a contradiction in case $d \le 3$ since it implies that $dim_{Haus}\ M \le dim_{Haus}\ TC|_V(\pi/2) \le 3+3+8 = 14$.

The case $5 \le d \le 7$ is also easy to eliminate since in this case $dim TC|_V(\frac{\pi}{2}) \ge 5+5+8 > 16 = dim\ M$. So it is impossible for $\exp|_{TC|_V(\frac{\pi}{2})}$ to be a smooth bijection from the open dense set $\exp|^{-1}_{TC|_V(\frac{\pi}{2})}(M\backslash G_M)$ to the open dense set $M\backslash G_M$.

The case $d = 4$ is also not possible, but it is much harder to rule out. First we prove

Proposition 17. *If $d = 4$, then G_M is a totally geodesic submanifold of M that is isometric to HP^2 with its canonical metric with $1 \le sec\ HP^2 \le 4$.*

Proof. For any line U in C we can let U play the role of V and define a configuration C_U consisting of lines of type $1_U, 2_U$, and 3_U in a way analogous to what we did on page 649. Of course assertion (26) is valid for each C_U, and for each such configuration C_U, $G_M = \cup_{W \text{ a line of type } 2_U} G_W$, since otherwise there would be a line not included in C_U.

Now let u and w be two points in G_M. Since w must lie on a line of type $2_{\{u\}'}$, there is a Cayley line Z containing u and w. Since G_Z is isometric to $S^4(\frac{1}{2})$, we can find a geodesic in G_Z between u and w. Using Lemma 6 and the fact that Z is totally π-convex we see that if $dist(u, w) < \frac{\pi}{2}$, then the geodesic constructed above is the unique minimal geodesic in M between u and w. This shows that G_M is totally $\frac{\pi}{2}$-convex and hence, by [5], a topological manifold with boundary and smooth, totally geodesic interior. But the above construction also indicates that every geodesic in G_M can be indefinitely prolonged (to a geodesic in G_M). Therefore $\partial G_M = \emptyset$. Thus G_M with its intrinsic metric is a Riemannian manifold with sectional curvature ≥ 1 and diameter $= \frac{\pi}{2}$. The proposition follows by analyzing the structure of the dual sets in G_M and applying the classification theorem in [8]. q.e.d.

It follows that the restriction of $\Pi_{V'}$ to $(SG_M)_{V'}$ is a Riemannian submersion that is isometrically equivalent to the quaternionic Hopf fibration $S^3(1) \hookrightarrow S^7(1) \longrightarrow S^4(\frac{1}{2})$. By combining this with Theorem 3, we see that the case $d = 4$ is also impossible.

4. Comparing submersions

In this section we prove Theorem 3. Although the argument is very technical, there is an over all strategy: to show, via boot strapping, that Π is more and more similar to the Hopf fibration $S^7(\frac{1}{2}) \hookrightarrow S^{15}(1) \xrightarrow{\Pi_h} S^8(\frac{1}{2})$. This similarity will manifest itself in various forms: common values of the two submersions on larger and larger subsets of S^{15}, common values of the A and T tensors of the two submersions on larger and larger subsets of $TS^{15} \oplus TS^{15}$, and common values of the differentials of the various holonomy displacement maps associated to the two submersions on more and more vectors.

With two notable exceptions, the arguments are elementary. The exceptions are in the proof of Assertion 1 (p. 654) and the statement of Assertion 2 (p. 655) where we appeal to the classification in [9] to conclude that certain Riemannian submersions $S^7 \longrightarrow S^4$ are standard.

Proof. Fix an embedding of $\mathbb{H}$ into $\mathbb{C}a$. Let $S^7_{\mathbb{H}}$ denote the subset of S^{15} all of whose coordinates are quaternion, $O(S^7_{\mathbb{H}})$ the opposite 7-sphere, $\Pi_h : S^{15} \longrightarrow S^8$ the Hopf fibration defined as in [14], S^7_a the given copy of $S^7 \subset \Pi^{-1}(G_V)$, and S^7_b the opposite 7-sphere in S^{15}.

Assertion 1. $image(\Pi|_{S^7_b}) = G_V$, $\Pi|_{S^7_b} : S^7_b \longrightarrow G_V$ is a Riemannian submersion, and we may assume without loss of generality that

- $S^7_a = S^7_{\mathbb{H}}$,
- $S^7_b = O(S^7_{\mathbb{H}})$,
- $\Pi^{-1}(G_V) = \Pi_h^{-1}(\Pi_h(S^7_{\mathbb{H}}))$, and
- $\Pi|_{\Pi^{-1}(G_V)} = \Pi_h|_{\Pi_h^{-1}(\Pi_h(S^7_{\mathbb{H}}))}$.

Proof of Assertion 1. Let $a : G_V \longrightarrow G_V$ be the antipodal map. Given $v \in G_V$, let $O_v(\Pi|_{S^7_a}^{-1}(v))$ denote the totally geodesic 3-sphere in $\Pi^{-1}(v)$ which is at distance $\frac{\pi}{2}$ from $\Pi|_{S^7_a}^{-1}(v)$, and let $O_{a(v)}(\Pi|_{S^7_a}^{-1}(a(v)))$ be the totally geodesic 3-sphere in $\Pi^{-1}(a(v))$ which is at distance $\frac{\pi}{2}$ from $\Pi|_{S^7_a}^{-1}(a(v))$. Then

$$S^7_b = O_v(\Pi|_{S^7_a}^{-1}(v)) * O_{a(v)}(\Pi|_{S^7_a}^{-1}(a(v))), \tag{29}$$

i.e., S^7_b is the join (by geodesics in S^{15}) of $O_v(\Pi|_{S^7_a}^{-1}(v))$ and $O_{a(v)}(\Pi|_{S^7_a}^{-1}(a(v)))$. Indeed, let x be any point in

$$O_v(\Pi|_{S^7_a}^{-1}(v)) * O_{a(v)}(\Pi|_{S^7_a}^{-1}(a(v))),$$

and let y be any point in $\Pi|_{S_a^7}^{-1}(v)$. Then x lies on a geodesic of length $\frac{\pi}{2}$ which passes through $O_v(\Pi|_{S_a^7}^{-1}(v))$ and $O_{a(v)}(\Pi|_{S_a^7}^{-1}(a(v)))$. Let ν be the point on this geodesic which is also in $O_v(\Pi|_{S_a^7}^{-1}(v))$. Then $dist(\nu, y) = \frac{\pi}{2}$ and $\sphericalangle(\nu y, \nu x) = \frac{\pi}{2}$. Therefore $dist(x, y) = \frac{\pi}{2}$. A similar argument shows that $dist(x, z) = \frac{\pi}{2}$ for points $z \in \Pi|_{S_a^7}^{-1}(a(v))$. Finally if z is an arbitrary point in S_a^7, then z lies on a geodesic γ of length $\frac{\pi}{2}$ with end points in $\Pi|_{S_a^7}^{-1}(v)$ and $\Pi|_{S_a^7}^{-1}(a(v))$. Since the end points of γ are both at distance $\frac{\pi}{2}$ from x, all of γ (and z in particular) is at distance $\frac{\pi}{2}$ from x. Therefore $O_v(\Pi|_{S_a^7}^{-1}(v)) * O_{a(v)}(\Pi|_{S_a^7}^{-1}(a(v))) \subset S_b^7$, but both are 7-spheres, so the other containment must hold as well.

It follows from (18) that the holonomy displacement maps given by the geodesics in G_V are isometries (cf 3.3 in [15]). They must also preserve S_a^7. Thus if $c : [0, \cdot) \longrightarrow G_V$ is a geodesic with $c(0) = v$, then using (29) we see that the $\psi(c)$'s must preserve S_b^7. It follows from this and (29) that $\Pi(S_b^7) = G_V$. Furthermore, $\Pi|_{S_b^7} : S_b^7 \longrightarrow G_V$ is a Riemannian submersion along $O_v(\Pi|_{S_a^7}^{-1}(v))$. But (29) is valid for all $v \in G_V$, so $\Pi|_{S_b^7}$ is a Riemannian submersion everywhere. By the classification theorem in [9], $\Pi|_{S_b^7}$ is equivalent to the quaternionic Hopf fibration.

Let

$$(S_b^7, \Pi) \xrightarrow{g} (S_a^7, \Pi) \xrightarrow{f} (S_{\mathbb{H}}^7, \Pi_h) \xrightarrow{E} (O(S_{\mathbb{H}}^7), \Pi_h)$$

be equivalences of submersions. Then the direct sum of f and $E \circ f \circ g$ is an isometry of S^{15} whose restriction to $\Pi^{-1}(G_V)$ is an equivalence of the fibrations $\Pi|_{\Pi^{-1}(G_V)}$ and $\Pi_h|_{\Pi_h^{-1}(\Pi_h(S_{\mathbb{H}}^7))}$ that maps S_a^7 to $S_{\mathbb{H}}^7$ and S_b^7 to $O(S_{\mathbb{H}}^7)$. So Assertion 1 is proven.

For the remainder of the proof we make the assumptions allowed by Assertion 1, but we continue to use the old notation (S_a^7, S_b^7 ect.) for the sets associated to Π.

Fix some $v \in G_V$, and let $S_{a,b}^7 = \Pi|_{S_a^7}^{-1}(v) * \Pi|_{S_b^7}^{-1}(a(v))$.

Assertion 2. The restriction of Π to $S_{a,b}^7$ is a Riemannian submersion onto its image with 3-dimensional fibers, and hence (by [9]) is isometrically equivalent to the quaternionic Hopf fibration.

It will follow from the proof of Assertion 2 that $\Pi_h|_{S_{a,b}^7}$ is also a Riemannian submersion. Thus proving Assertion 2 is consistent with the overall strategy of showing that Π and Π_h are more and more a like.

Proof of Assertion 2. We will denote an arbitrary horizontal geodesic from $\Pi|_{S^7_a}^{-1}(v)$ to $\Pi|_{S^7_b}^{-1}(a(v))$ by $\gamma_{a,b}$, and an arbitrary horizontal geodesic from $\Pi|_{S^7_a}^{-1}(v)$ to $\Pi|_{S^7_a}^{-1}(a(v))$ by $\gamma_{a,a}$.

First note that any geodesic $\gamma_{a,b}$ and any geodesic $\gamma_{a,a}$ with common initial point make angle $\frac{\pi}{2}$ with each other. Therefore the holonomy displacement maps for the former type of geodesics must leave $S^7_{a,b}$ invariant. This also shows that $\Pi|_{S^7_{a,b}}$ is a Riemannian submersion along $\Pi|_{S^7_{a,b}}^{-1}(v)$.

We can also show that the intersections of the fibers of Π with $S^7_{a,b}$ are either empty or diffeomorphic to S^3. This is clear for the fibers $\Pi^{-1}(v)$ and $\Pi^{-1}(a(v))$. Any point $x \in S^7_{a,b}\backslash[\Pi^{-1}(v) \cup \Pi^{-1}(a(v))]$ lies on a unique horizontal geodesic $\gamma_{a,b}$ between $\Pi|_{S^7_a}^{-1}(v)$ and $\Pi|_{S^7_b}^{-1}(a(v))$. Say $x = \gamma_{a,b}(t)$, and recall (p. 639) that $\psi(\gamma_{a,b})_{s,t}$ denotes the holonomy displacement maps for $\gamma_{a,b}$. Then we have seen that $\psi(\gamma_{a,b})_{0,t}(\Pi|_{S^7_a}^{-1}(v)) \subset S^7_{a,b}$. On the other hand, the images of all other points of $\Pi^{-1}(v)$ under $\psi(\gamma_{a,b})_{0,t}$ lie on horizontal geodesics from $\Pi^{-1}(v)$ to $\Pi^{-1}(a(v))$ that do not begin on $\Pi|_{S^7_a}^{-1}(v)$ and hence can not belong to $S^7_{a,b}$. Therefore $\Pi^{-1}(\Pi(x)) \cap S^7_{a,b} = \psi(\gamma_{a,b})_{0,t}(\Pi|_{S^7_a}^{-1}(v)) \cong S^3$.

Let $\mathcal{V}S^7_{a,b}$ denote the intersection of $\mathcal{V}S^{15}$ and $TS^7_{a,b}$. Let $H_{a,b}$ be the set of vectors in $TS^7_{a,b}$ which are perpendicular to $\mathcal{V}S^7_{a,b}$. Then the differential of Π is injective on $H_{a,b}$ and hence maps it onto $T\Pi(S^7_{a,b})$, so the restriction of Π to $S^7_{a,b}$ is a submersion. If we knew that $H_{a,b}$ were a subset of $\mathcal{H}S^{15}$, then $\Pi|_{S^7_{a,b}}$ would have to be Riemannian, and Assertion 2 would be proven.

To obtain this conclusion about $H_{a,b}$ (and also to complete the proof of Theorem 3.) we shift our efforts from comparing Π and Π_h on subsets of M to comparing the A tensors, the T tensors, and the differentials of various holonomy displacement maps associated to the two submersions.

Let $X_{a,a}$ and $X_{a,b}$ denote extensions of the tangent fields of $\gamma_{a,a}$ and $\gamma_{a,b}$ to basic horizontal fields, and for each real number t we let $X_{a,a}(t) = \dot{\gamma}_{a,a}(t)$ and $X_{a,b}(t) = \dot{\gamma}_{a,b}(t)$. It follows from the proof of Lemma 9 that

$$\|A_{X_{a,a}}v\| \equiv 1 \text{ and } \|A_{X_{a,a}}y\| \equiv 1 \tag{30}$$

for all unit $v \in \mathcal{V}S^{15}|_{\Pi^{-1}(\Pi(\gamma_{a,a}))}$ and all unit vectors

$$y \in \mathcal{H}S^{15}|_{\Pi^{-1}(\Pi(\gamma_{a,a}))}$$

which are perpendicular to $X_{a,a}$. It follows from Assertion 1 that

$$A_{X_{a,a}}v = A^h_{X_{a,a}}v, \tag{31}$$

where A^h denotes the A tensor for the Hopf fibration and v is an arbitrary vector in $\mathcal{V}S^{15}|_{\Pi^{-1}(\Pi(\gamma_{a,a}))}$.

Now suppose that v is a unit vector in $\mathcal{V}S^{15}|_{\Pi^{-1}(\Pi(\gamma_{a,a}(0)))} \cap NS^7_{a,b}$ and

$$A_{X_{a,a}}v = A^h_{X_{a,a}}v = X_{a,b}. \tag{32}$$

Then

(33)
$$\begin{aligned}\langle A_{X_{a,b}}v, X_{a,a}\rangle &= -\langle v, A_{X_{a,b}}X_{a,a}\rangle \\ &= \langle v, A_{X_{a,a}}X_{a,b}\rangle = -\langle A_{X_{a,a}}v, X_{a,b}\rangle = -1.\end{aligned}$$

So in particular

$$A_{X_{a,b}}X_{a,a} = v, \tag{34}$$

and a calculation similar to (33) shows

$$\langle A^h_{X_{a,b}}v, X_{a,a}\rangle = -1 \quad \text{and} \quad A^h_{X_{a,b}}X_{a,a} = v. \tag{33^h}$$

Therefore,

$$A_{X_{a,b}}X_{a,a} = A^h_{X_{a,b}}X_{a,a} \tag{35}$$

along $\Pi^{-1}(\Pi(\gamma_{a,b}(0)))$.

It follows from (33^h) that $A^h_{\dot\gamma_{a,b}}v = -\dot\gamma_{a,a}$. We would like to conclude that the same equation holds for A, so we have to verify that the components of $A_{\dot\gamma_{a,b}(0)}v$ in all directions perpendicular to $\dot\gamma_{a,a}(0)$ are 0.

To do this we recall that (with the opposite sign convention) one of O'Neill's "fundamental equations of a submersion" is

$$\begin{aligned}\langle R(X,V)Y,W\rangle = &-\langle(\nabla_X T)_V W, Y\rangle - \langle(\nabla_V A)_X Y, W\rangle \\ &+ \langle T_V X, T_W Y\rangle - \langle A_X V, A_Y W\rangle\end{aligned}$$

where X and Y are horizontal fields and V and W are vertical ones. Letting $X = X_{a,a}$, $Y = X_{a,b}$, setting $V = W$, and evaluating at $t = 0$

we find

$$
\begin{aligned}
0 = &- \langle \nabla_V (A_{X_{a,a}} X_{a,b}), V\rangle + \langle A_{\nabla_V X_{a,a}} X_{a,b}, V\rangle \\
&+ \langle A_{X_{a,a}} \nabla_V X_{a,b}, V\rangle - \langle A_{X_{a,a}} V, A_{X_{a,b}} V\rangle \\
= &- \langle \nabla_V (A_{X_{a,a}} X_{a,b}), V\rangle - \langle A_{X_{a,b}} (\nabla_{X_{a,a}} V)^h, V\rangle \\
&+ \langle A_{X_{a,a}} (\nabla_{X_{a,b}} V)^h, V\rangle - \langle A_{X_{a,a}} V, A_{X_{a,b}} V\rangle \\
= &- \langle \nabla_V (A_{X_{a,a}} X_{a,b}), V\rangle + \langle A_{X_{a,a}} V, A_{X_{a,b}} V\rangle \\
&- \langle A_{X_{a,b}} V, A_{X_{a,a}} V\rangle - \langle A_{X_{a,a}} V, A_{X_{a,b}} V\rangle \\
= &- \langle \nabla_V (A_{X_{a,a}} X_{a,b}), V\rangle - \langle A_{X_{a,a}} V, A_{X_{a,b}} V\rangle.
\end{aligned} \tag{36}
$$

Equation (36) is also valid with A replaced by A^h. But $\langle A^h_{X_{a,a}} V, A^h_{X_{a,b}} V\rangle = 0$ so $\langle \nabla_V (A^h_{X_{a,a}} X_{a,b}), V\rangle = 0$. Combining this with (35) we have

$$\langle \nabla_V (A_{X_{a,a}} X_{a,b}), V\rangle|_{\Pi^{-1}(\Pi(\gamma_{a,b}(0)))} = 0,$$

and therefore

$$\langle A_{X_{a,a}} V, A_{X_{a,b}} V\rangle|_{\Pi^{-1}(\Pi(\gamma_{a,b}(0)))} = 0 \tag{37}$$

for all $V \in \mathcal{V}S^{15}|_{\Pi^{-1}(\Pi(0))}$. If $v \in NS^7_{a,b}|_{\gamma_{a,b}(0)} \cap \mathcal{V}S^{15}$, then from equation (31) we have, $A_{\dot\gamma_{a,a}} v \in NS^7_{a,a}$. So it follows from equation 37 that

$$A_{\dot\gamma_{a,b}} v \in TS^7_{a,a} \tag{38}$$

for all $v \in NS^7_{a,b}|_{\gamma_{a.b}(0)} \cap \mathcal{V}S^{15}$.

To prove the formula $A_{\dot\gamma_{a,b}} v = -\dot\gamma_{a,a}$, it remains to verify that the components of $A_{\dot\gamma_{a,b}} v$ in directions $x \in \mathcal{H}S^7_a \cap (\dot\gamma_{a,a}(0))^\perp$ are 0. So we compute

$$\langle A_{\dot\gamma_{a,b}} v, x\rangle = -\langle v, A_{\dot\gamma_{a,b}} x\rangle = 0.$$

The last equality is a consequence of (34), the fact that x is perpendicular to $\dot\gamma_{a,a}(0)$, and that the restriction of $z \mapsto A_{\dot\gamma_{a,b}(0)} z$ to $\mathcal{H}S^7_a|_{\dot\gamma_{a,b}(0)}$ is an isometry onto its image. Thus we conclude that

$$A_{\dot\gamma_{a,b}(0)} v = -\dot\gamma_{a,a}, \text{ and hence that } A_{\dot\gamma_{a,b}(0)} w = A^h_{\dot\gamma_{a,b}(0)} w \tag{39}$$

for all $w \in \mathcal{V}S^{15}|_{\gamma_{a,b}(0)} \cap (TS^7_a)^\perp$.

Given a unit vector $v \in \mathcal{V}S^{15}|_{\gamma_{a,b}(0)} \cap NS^7_{a,b}$ let V be the field of images of the tangent field of $t \mapsto \exp tv$ under the $d\psi(\gamma_{a,b})_{0,t}$'s. As pointed out in (18) $V|_{\gamma_{a,b}}$ is the Jacobi field with initial conditions

$$J(0) = v \text{ and } J'(0) = A_{\dot\gamma_{a,b}(0)}v.$$

Therefore it follows from (39) that the values of V would be no different if we had defined it via the holonomy displacement maps for Π_h (rather than Π).

We also see that $\langle J'(0), J(0)\rangle = 0$ and by (39) $\|J(0)\| = \|J'(0)\| = 1$; so

$$J(t) = \cos tP_t(J(0)) + \sin tP_t(J'(0)), \tag{40}$$

where for $z \in TS^{15}_{\gamma_{a,b}(0)}$, $P_t(z)$ is the parallel image of z along $\gamma_{a,b}$ at $\gamma_{a,b}(t)$. Since $J(0)$ and $J'(0)$ are both in $NS^7_{a,b}$, we get from (40) that $V|_{\gamma_{a,b}}(t) = J(t) \in NS^7_{a,b}$ for all t.

Let $\mathcal{V}^hS^{15}$ denote the vertical bundle for the Hopf fibration. Then since v can be any vector in $N(S^7_{a,b})_{\gamma_{a,b}(0)}$ we have

$$NS^7_{a,b} \cap \mathcal{V}^hS^{15} = NS^7_{a,b} \cap \mathcal{V}S^{15}, \tag{41}$$

and the fibers of each of these bundles are 4-dimensional. Therefore

$$\mathcal{V}S^{15}|_{S^7_{a,b}} = \mathcal{V}S^7_{a,b} \oplus (NS^7_{a,b} \cap \mathcal{V}S^{15}), \tag{42}$$

and it follows that $H_{a,b}$ is horizontal. So assertion 2 is proved.

Setting $\mathcal{V}^hS^7_{a,b} = \mathcal{V}^hS^{15} \cap TS^7_{a,b}$, we also have

$$\mathcal{V}^hS^{15}|_{S^7_{a,b}} = \mathcal{V}^hS^7_{a,b} \oplus (NS^7_{a,b} \cap \mathcal{V}^hS^{15}). \tag{43}$$

Since $[X_{a,b}, V] = 0$ and $S^7_{a,b}$ is totally geodesic, (42) implies

$$T_V\dot\gamma_{a,b} = (\nabla_V X_{a,b})^v = (\nabla_{\dot\gamma_{a,b}}V)^v \in NS^7_{a,b}. \tag{44}$$

From (41), (42), and (43) it follows that the right hand-side of (44) equals $T^h_V\dot\gamma_{a,b}$, where T^h is the T tensor of Π_h. Since $T^h \equiv 0$, and $V|_{\gamma_{a,b}}(t)$ can be any vector in $NS^7_{a,b} \cap \mathcal{V}S^{15}$, we have

$$T_v\dot\gamma_{a,b} = 0 \tag{45}$$

for all $v \in NS^7_{a,b} \cap \mathcal{V}S^{15}$. Combining this with Assertion 2 we get

$$T_v \dot{\gamma}_{a,b} = 0 \tag{46}$$

for all $v \in \mathcal{V}S^{15}|_{\gamma_{a,b}}$, and hence

$$\langle T_v w, \dot{\gamma}_{a,b} \rangle = -\langle w, T_v \dot{\gamma}_{a,b} \rangle = 0$$

for all $v, w \in \mathcal{V}S^{15}|_{\gamma_{a,b}}$. Combining this and (46) with equation (7) yields

$$\|A_{\dot{\gamma}_{a,b}} v\| \equiv 1 \tag{47}$$

for all unit $v \in \mathcal{V}S^{15}|_{\gamma_{a,b}}$. Using (47) and arguing as in the proof of Lemma 9, we obtain

$$\|A_{\dot{\gamma}_{a,b}} x\| \equiv 1 \tag{48}$$

for all horizontal unit vectors x along $\gamma_{a,b}$ which are perpendicular to $\gamma_{a,b}$. Combining (48) with the equation for horizontal curvatures we conclude that all radial sectional curvatures along $\Pi(\gamma_{a,b})$ are equal to 4. Therefore if $c_{a,b}$ is any horizontal lift of $\Pi(\gamma_{a,b})$ (not necessarily contained in $S^7_{a,b}$),

$$\|A_{\dot{c}_{a,b}} v\| \equiv 1 \tag{49}$$

for all unit $v \in \mathcal{V}S^{15}|_{c_{a,b}}$.

Combining (49) with (18) shows that the images of any unit vector $v \in \mathcal{V}S^{15}_{c_{a,b}(0)}$ under the $d\psi(c_{a,b})_{0,t}$'s are given by a Jacobi field J with initial conditions that satisfy $\langle J'(0), J(0)\rangle = 0$ and $\|J(0)\| = \|J'(0)\| = 1$; so as before $J(t) = \cos tP_t(J(0)) + \sin tP_t(J'(0))$, where for $w \in TS^{15}_{c_{a,b}(0)}$, $P_t(w)$ is the parallel image of w along $c_{a,b}$ at $c_{a,b}(t)$. In particular $\|J\| \equiv 1$; so all of the $\psi_{0,t}$'s are isometries, and all of the points in $\Pi(S^7_{a,b})$ are good. By Assertion 2 and Lemma 8, $\Pi(S^7_{a,b})$ is totally geodesic and isometric to $S^4(\frac{1}{2})$. Since it intersects G_V orthogonally, it follows from the main lemma that every point in V is good. q.e.d.

5. The nonsimply connected case

Let M satisfy the hypotheses of Theorem 4. As we indicated in the introduction the classification theorem in [8] applies to M. In particular, we know that M is either isometric to a space form or the quotient of CP^{2k-1} in Theorem 4.(ii).

Suppose M is a space form, O is an orbit of the action of Γ on S^n, and $p : S^n \longrightarrow M$ is the universal covering map. Since $rad\ M = \frac{\pi}{2}$, we can find a dual pair (A, A') in M with $p(O) \in A$. Since A is totally π-convex so is $\tilde{A} = p^{-1}(A)$. Since $\partial A = \emptyset$ (2.5, 3.4, and 3.5 in [8]), $\partial \tilde{A} = \emptyset$. $\tilde{A}$ is therefore a Γ-invariant great subsphere of S^n that contains O.

On the other hand, if S^n/Γ is a space form and an orbit O of Γ is contained in a proper, invariant, totally geodesic subsphere, S^k, then $A(S^k) = \{x \in S^n \mid dist(x, S^k) = \frac{\pi}{2}\}$ is also invariant. Therefore, if every orbit of Γ is contained in a proper, invariant, great subsphere, then $dist(p(O), p(A(S^k))) = \frac{\pi}{2}$, and $rad\ S^n/\Gamma = \frac{\pi}{2}$.

To complete the proof of Theorem 4 it remains to show that the space in (4., ii) has radius $= \frac{\pi}{2}$. The orbit of an arbitrary point for the corresponding action on S^{4d-1} is

$$
\begin{aligned}
(z_1, z_2, \dots, z_{2d}) &\mapsto (\overline{z}_{d+1}, \dots, \overline{z}_{2d}, -\overline{z}_1, \dots, -\overline{z}_d) \\
&\mapsto (-z_1, \dots, -z_d, -z_{d+1}, \dots, -z_{2d}) \\
&\mapsto (-\overline{z}_{d+1}, \dots, -\overline{z}_{2d}, \overline{z}_1, \dots, \overline{z}_d) \mapsto (z_1, z_2, \dots, z_{2d}).
\end{aligned}
$$

Thus each orbit (in S^{4d-1}) is contained in an invariant geodesic that is perpendicular to the fibers of the Hopf fibration $S^1 \hookrightarrow S^{4d-1} \longrightarrow CP^{2d-1}$. It follows that each orbit in CP^{2d-1} is contained in an invariant geodesic. If $\gamma : [0, \pi] \longrightarrow CP^{2d-1}$ is an invariant geodesic, then $(image(\gamma))'$ is also invariant (and $\neq \emptyset$ since $d \geq 2$). So the radius of the quotient is $\frac{\pi}{2}$. q.e.d.

Now we focus on the proof of (5).

Proof of (i). Let $\rho : \Gamma \longrightarrow O(n+1)$ be a fixed point free representation that respects an orthogonal splitting $V_1 \oplus V_2 \oplus \cdots \oplus V_k$ of $\mathbb{R}^{n+1}$ so that $\rho|_{V_i}$ is irreducible for all i. It follows from Theorem 7.2.18 in [27] that $dim\ V_i = dim\ V_j (\equiv d)$ for all i, j. Suppose $rad\ S^n/\rho(\Gamma) = \frac{\pi}{2}$. Then using Theorem (4, i) we can first find a proper, invariant subspace that is not a direct sum of V_i's and an irreducible invariant subspace W that is distinct from all of the V_i's. The orthogonal projections $p_i : W \longrightarrow V_i$ are all ρ-equivariant, so by Schur's Lemma, they are either zero maps or isomorphisms. If they are all zero maps, then we have $W \subset (V_1 \oplus V_2 \oplus \cdots \oplus V_k)^{\perp}$, which is impossible. So at least one of the projections (say p_1) is an isomorphism. If all of the other p_i's are zero maps, then we have $W \subset (V_2 \oplus \cdots \oplus V_k)^{\perp}$, which would imply that $W = V_1$, also impossible. So at least one other projection (say p_2) is an isomorphism. Thus $\rho|_W$ is linearly (and hence orthogonally by Lemma 4.7.1 in [27]) equivalent to both $\rho|_{V_1}$ and $\rho|_{V_2}$.

Proof of (ii). By (iv) it suffices to consider the case $k = 2$. In this case the action of $\rho(\Gamma)$ on S^3 is orthogonally equivalent to a subaction $\bar{\rho}(\Gamma)$ of the Hopf action, and the Hopf fibration $S^1 \hookrightarrow S^3 \longrightarrow S^2$ induces a Riemannian submersion $S^1 \hookrightarrow S^3/\bar{\rho}(\Gamma) \longrightarrow S^2(\frac{1}{2})$. Thus $rad\ S^3/\rho(\Gamma) = rad\ S^3/\bar{\rho}(\Gamma) \geq rad\ S^2(\frac{1}{2}) = \frac{\pi}{2}$.

Proof of (iii). Suppose there are points $u, v \in S^{d-1}$ and members $g_1, g_2, \dots, g_d, g_{d+1}$ of Γ so that $\{g_1(u), g_2(u), \dots, g_d(u)\}$ and $\{g_1(v), g_2(v), \dots, g_d(v)\}$ are linearly independent and such that the sets of coefficients $\{a_i\}$, $\{b_i\}$ so that

$$a_1 g_1(u) + \cdots + a_d g_d(u) = g_{d+1}(u) \text{ and}$$
$$b_1 g_1(v) + \cdots + b_d g_d(v) = g_{d+1}(v)$$

are distinct. It follows that the vector $\frac{1}{\sqrt{2}} g_{d+1}(u, v)$ in $\mathbb{R}^{2d}$ is not in the span of $\{\frac{1}{\sqrt{2}} g_1(u, v), \frac{1}{\sqrt{2}} g_2(u, v), \dots \frac{1}{\sqrt{2}} g_d(u, v)\}$. On the other hand, $\{\frac{1}{\sqrt{2}} g_1(u, v), \frac{1}{\sqrt{2}} g_2(u, v), \dots \frac{1}{\sqrt{2}} g_d(u, v)\}$ is linearly independent since its projection onto $\mathbb{R}^d \times \{0\}$ is $\{\frac{1}{\sqrt{2}} g_1 u, \frac{1}{\sqrt{2}} g_2 u, \dots \frac{1}{\sqrt{2}} g_d u\}$. Therefore the set

$$\{\frac{1}{\sqrt{2}} g_1(u, v), \frac{1}{\sqrt{2}} g_2(u, v), \dots, \frac{1}{\sqrt{2}} g_d(u, v), \frac{1}{\sqrt{2}} g_{d+1}(u, v)\}$$

is linearly independent, and we are done by Theorem 4 (i), provided we can find u, v, and $g_1, g_2, \dots, g_{d+1}$ as above. But according to [27], the image of every irreducible representation of a fixed point free, nonabelian group Γ contains matrices of the form

$$A = \begin{pmatrix} e^{\frac{2\pi i k}{m}} & & & & \\ & e^{\frac{2\pi i k r}{m}} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & e^{\frac{2\pi i k r^{(d/2)-1}}{m}} \end{pmatrix} \tag{50}$$
$$\text{and } B = \begin{pmatrix} 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ 0 & & & & & 1 \\ e^{\frac{2\pi i l}{n'}} & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix},$$

where k, l, m, n' and r are as in Theorem 5.5.6 in [27], $\frac{d}{2}$ plays the role of d in [27, Theorem 5.5.6], and we have used complex coordinates. So it suffices to set $u = (1, 0, \dots, 0)$, $v = (0, 1, 0, \dots, 0)$, $g_1 = Id, g_2 = A, g_3 = BA, g_4 = ABA, \dots, g_{d+1} = (BA)^{\frac{d}{2}}$. (Note that d is even.)

To quickly see that there are matrices of the form (50) in the image of every irreducible representation of a fixed point free, nonabelian group, note that such matrices are in the image of every such representation of a so called "group of type 1" (Theorem 5.5.6 and 5.5.10 in [27]) and that other nonabelian fixed point free groups contain groups of type 1 as subgroups [27, p. 204-208] . q.e.d.

Proof of (iv). View $S^{n+d}(1)$ as the join $S^n(1) * S^{d-1}(1)$, and view $\rho \oplus \sigma$ as the join of ρ and σ. Then every orbit of $\rho \oplus \sigma$ is contained in the join of an orbit of ρ with an orbit of σ. Since the orbits of ρ are all contained in proper invariant totally geodesic subspheres of S^n, the orbits of $\rho \oplus \sigma$ are contained in the joins of proper great subspheres of S^n with S^{d-1}.

q.e.d.

Proof of (v). For example, if k is so large that the order of Γ is less than $n + 1$, then every orbit is automatically contained in an invariant subspace. q.e.d.

Acknowledgements

It is my pleasure to thank Carlos Duran, Detlef Gromoll, Karsten Grove, Luis Guijarro, Vitali Kapovich, Grisha Perelman, and Michael Weiss for stimulating conversations and correspondence on this subject. I especially thank Karsten Grove and Grisha Perelman for discovering mistakes in earlier drafts of the paper, and Grisha Perelman for making me aware of Lemma 8, and thereby greatly simplifying the proof of the main lemma. I would also like to thank Paula Bergen for her expert job of copy editing. Finally, I wish to acknowledge that the beautiful work in [8] and [9] has been a true inspiration to me.

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UNIVERSITY OF CALIFORNIA, RIVERSIDE