J. DIFFERENTIAL GEOMETRY
44 (1996) 214-261

A SPHERE THEOREM WITH A PINCHING CONSTANT BELOW $\frac{1}{4}$

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Dedicated to M. Berger

Abstract

In this article we establish a topological sphere theorem for compact, simply connected, odd dimensional manifolds M^n with pinched sectional curvature. The pinching constant δ_{odd} that appears in our hypotheses is $< \frac{1}{4}$, explicit, and independent of the dimension. Furthermore, in the even dimensional case we can find a pinching constant $\delta_{\text{ev}} < \frac{1}{4}$ which guarantees that the integral cohomology groups of M^n coincide up to torsion groups of odd order with the cohomology groups of a sphere or a projective space.
Both results are reduced by means of the diameter sphere theorem of Grove and Shiohama to proving Berger's horse shoe conjecture under a suitable condition on the diameter. The geometric arguments rely on refined Jacobi field estimates, which might be useful in other contexts as well.

Introduction

Berger's rigidity theorem provides a classification of all compact, simply connected manifolds M^n which carry a Riemannian metric with positive, weakly $\frac{1}{4}$-pinched sectional curvature [6]. This result has already been known in 1961. Then it has taken more than twenty years before the tools have been available to analyze weaker pinching conditions.

In fact, the pinching below-$\frac{1}{4}$ theorem for even dimensional manifolds [10] obtained by Berger in 1983 has been the first major application of

1991 *Mathematics Subject Classification.* Primary 53C20; Secondary 53C21, 53C23.

Key words and phrases. Curvature and Topology, Pinching, Sphere Theorems.

Received November 21, 1994. Both authors like to thank MSRI for its hospitality and support during the initial period of this work. Additional support has been provided by the European Union under the GADGET–II project, contract number CHRX–CT92–0050.

Gromov's compactness theorem [18, 30]. The hypothesis about the parity of the dimension has been necessary, since at the time an appropriate lower bound for the injectivity radius was not available for odd dimensional, compact, simply connected manifolds with sectional curvature pinched below $\frac{1}{4}$. In [1] we have established such an estimate, and, as a first application, we have extended Berger's result correspondingly.

Both results, Berger's pinching below-$\frac{1}{4}$ theorem for even dimensional manifolds and the sphere theorem for odd dimensional manifolds in [1], are based on Gromov's compactness theorem and on limiting arguments. Therefore the pinching constants in these results are inherently inexplicit, and they depend on the dimension.

However, the sphere theorem for odd dimensional manifolds can be improved substantially.

Theorem A. *There exists a constant* $\delta_{\text{odd}} \in (0, \frac{1}{4})$ *such that any odd dimensional, compact, simply connected Riemannian manifold* M^n *with* δ_{odd}*-pinched sectional curvature is homeomorphic to the sphere* $\mathbb{S}^n$.

We emphasize that the constant δ_{odd} is independent of the dimension and explicit. In fact, our proof works for $\delta_{\text{odd}} = \frac{1}{4}(1+\varepsilon_{\text{odd}})^{-2}$ where $\varepsilon_{\text{odd}} = 10^{-6}$. It is an interesting question whether similar techniques can be used to improve Berger's pinching below-$\frac{1}{4}$ theorem as well. We have the following partial result on the cohomology level:

Theorem B. *There exists a constant* $\delta_{\text{ev}} \in (0, \frac{1}{4})$ *such that for any even dimensional, compact, simply connected Riemannian manifold* M^n *with* δ_{ev}*-pinched sectional curvature the cohomology rings* $H^*(M^n; R)$ *with coefficients* $R \in \{\mathbb{Q}, \mathbb{Z}_2\}$ *are isomorphic to the corresponding cohomology rings of one of the compact, rank one, symmetric spaces* $\mathbb{S}^n$, $\mathbb{CP}^{n/2}$, $\mathbb{HP}^{n/4}$, $\mathbb{C}aP^2$, *or the rings* $H^*(M^n; R)$ *are truncated polynomial rings generated by an element of degree* 8.

Again the constant δ_{ev} is explicit and independent of the dimension. In fact, our proof works for $\delta_{\text{ev}} = \frac{1}{4}(1+\varepsilon_{\text{ev}})^{-2}$ where $\varepsilon_{\text{ev}} = \frac{1}{27\,000}$.

Recall that $H^*(\mathbb{C}aP^2; R) = R[\xi_R]/(\xi_R^3)$ where $\deg \xi_R = 8$. However, we cannot exclude the possibility that $H^*(M^n; R) = R[\xi_R]/(\xi_R^{m+1})$ where $\deg \xi_R = 8$ and $m > 2$. For instance, we cannot apply J. Adem's result [3, Theorem 2.2], since we do not have enough control on the cohomology ring of M^n with coefficients $\mathbb{Z}_3$.

The *proofs of Theorems A and B* do not rely on Gromov's compactness theorem at all. They are rather based on direct comparison methods and on some results from algebraic topology. In fact, the geometric arguments for Theorems A and B are essentially the same, whereas the topological arguments reflect the substantial difference between the even and the odd dimensional case.

The main ideas will be explained in more detail in the next section. There we shall also describe the organisation of the paper. We proceed discussing the context of Theorems A and B.

In low dimensions much *stronger results* are known. By the Gauß–Bonnet theorem any compact, simply connected 2–manifold with positive sectional curvature is diffeomorphic to the sphere. For compact, simply connected 3–manifolds we refer to the work of Hamilton.

Theorem 1.1. (R. S. Hamilton 1982 [21]) *Let* (M^3, g) *be a compact, connected 3–manifold with Ricci curvature* $\mathrm{ric} > 0$ *everywhere. Then* g *can be deformed in the class of metrics with* $\mathrm{ric} > 0$ *into a metric with constant sectional curvature.*

It follows that any compact, simply connected 3–manifold (M^3, g_0) with positive Ricci curvature is *diffeomorphic* to the sphere. The 4–dimensional case has been studied by Seaman.

Theorem 1.2. (W. Seaman 1989 [33]) *Let* M^4 *be a 4–dimensional, compact, oriented, connected Riemannian manifold without boundary. If the sectional curvature* K_M *of* M^4 *satisfies*

$$0.188 \approx \frac{1}{1+3\sqrt{1+2^{5/4}\cdot 5^{-1/2}}} \leq K_M \leq 1 \quad ,$$

then M^4 *is homeomorphic to* $\mathbb{S}^4$ *or* $\mathbb{CP}^2$.

We do not know the optimal value of the pinching constant in Theorems A and B above. *Examples* of compact, simply connected manifolds M^n *other than spheres and projective spaces* that have strictly positive sectional curvature are *scarce.*

As shown by Berger [7], the only other normally homogeneous examples are the spaces $M^7 = \mathrm{Sp}(2)/\mathrm{SU}(2)$ and $M^{13} = \mathrm{SU}(5)/(\mathrm{Sp}(2)\times\mathbb{S}^1)$. Their pinching constants $\delta_M := \min K_M / \max K_M$ are $\frac{1}{37}$ and $\frac{16}{29\cdot 37}$, respectively [14, 22]. Further homogeneous, odd dimensional examples are the Aloff–Wallach spaces $M^7_{k,l} = \mathrm{SU}(3)/\mathbb{S}^1_{k,l}$ where the integers k, l label the various embeddings of $\mathbb{S}^1$ into the maximal torus $\mathbb{T}^2 \subset \mathrm{SU}(3)$. It has been computed in [23] that the pinching constant of the left–invariant metric defined in [4] approaches $\frac{16}{29\cdot 37}$ as $\frac{k}{l} \to 1$. Bérard Bergery has shown that there do not exist any other odd dimensional, homogeneous spaces of positive curvature [5].

The even dimensional, homogeneous spaces with $K_M > 0$ have been classified by Wallach [36]. Besides the spheres and the projective spaces there are just the three flag manifolds $M^6 = \mathrm{SU}(3)/\mathbb{T}^2$, $M^{12} = \mathrm{Sp}(3)/(\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2))$, and $M^{24} = \mathrm{F}_4/\mathrm{Spin}(8)$. As shown in [38], the pinching constant for the Wallach spaces M^6, M^{12} and M^{24} are all equal to $\frac{1}{64}$. This value is optimal in the class of homogeneous metrics.

In addition, there is one 6–dimensional inhomogeneous orbit space and there is an infinite family of 7–dimensional inhomogeneous orbit spaces [15, 16]. These examples are constructed as quotients of SU(3) by a twosided $\mathbb{T}^2$–action or by an infinite family of twosided $\mathbb{S}^1$–actions, respectively. They resemble the Aloff–Wallach examples in many respects. Recently, extending Eschenburg's construction, Bazaikin [37] has obtained an infinite family $\mathbb{S}^1_{p_1,\dots,p_5}\backslash U(5)/(\mathrm{Sp}(2)\times\mathbb{S}^1)$ of 13-dimensional double quotients which are related to the Berger example M^{13} mentioned above. Yet, the twosided actions that lead to spaces of positive curvature are not completely classified.

In view of these facts the pinching constant in Theorem A must be strictly greater than $\frac{1}{37}$, and the pinching constant in Theorem B must be greater than $\frac{1}{64}$.

2. Main ideas

Optimal injectivity radius estimates have already been an essential ingredient in the proof of the classical sphere theorem due to Klingenberg [25]. They are also crucial for our main results. The pinching condition that is required for these estimates depends in a significant way on the parity of the dimension. Currently the best results are the following two theorems.

Theorem 2.1. (W. Klingenberg [24]) *Let M^n be a compact, simply connected, even dimensional Riemannian manifold of positive sectional curvature. Then its injectivity radius* $\mathrm{inj}\, M^n$ *is controlled in terms of its conjugate radius* $\mathrm{conj}\, M^n$

$$\mathrm{inj}\, M^n \;=\; \mathrm{conj}\, M^n \;\geq\; \pi/\sqrt{\max K_M} \quad .$$

Theorem 2.2. (c.f. Theorem 1.1 in [1]) *There exists a constant $\delta_{\mathrm{inj}} \in (\frac{1}{9}, \frac{1}{4})$ such that the injectivity radius* $\mathrm{inj}\, M^n$ *and the conjugate radius* $\mathrm{conj}\, M^n$ *of any compact, simply connected Riemannian manifold M^n with δ_{inj}–pinched sectional curvature coincide:*

$$\mathrm{inj}\, M^n \;=\; \mathrm{conj}\, M^n \;\geq\; \pi/\sqrt{\max K_M} \quad .$$

Recall that the latter theorem holds for $\delta_{\mathrm{inj}} = \frac{1}{4}(1+\varepsilon_{\mathrm{inj}})^{-2}$ where $\varepsilon_{\mathrm{inj}} = 10^{-6}$. Pinching constants $\delta_{\mathrm{inj}} \leq \frac{1}{9}$ are obstructed by the Berger metrics on odd dimensional spheres. Notice that the same lower bound for $\mathrm{inj}\, M^n$ has been established by Klingenberg in [25] under the stronger hypothesis that the sectional curvature of M^n is strictly $\frac{1}{4}$–pinched, and it is this result that has played a major role in the proof of the classical sphere theorem.

Since the injectivity radius provides a lower bound for the diameter of the manifold, the classical sphere theorem follows when combining the preceding injectivity radius estimates with the following diameter sphere theorem.

Theorem 2.3. (K. Grove, K. Shiohama [20]) *Let M^n be a complete Riemannian manifold with sectional curvature $K_M \geq \delta > 0$ and diameter* $\operatorname{diam} M^n > \frac{\pi}{2\sqrt{\delta}}$. *Then M^n is homeomorphic to the sphere $\mathbb{S}^n$.*

In our context the diameter sphere theorem and the injectivity radius estimate do not fit together that well. Nevertheless, Theorems 2.1–2.3 provide a considerable reduction; it turns out that it is sufficient to prove the following version of the horse shoe conjecture discussed by Berger in 1962 [8].

Theorem 2.4. (Horse Shoe Inequality) *There exists a constant $\delta \in (0, \frac{1}{4})$ such that for any complete Riemannian manifold M^n with*

$$\delta \leq K_M \leq 1 \quad \textit{and} \quad \pi \leq \operatorname{inj} M^n \leq \operatorname{diam} M^n \leq \tfrac{\pi}{2\sqrt{\delta}}$$

the following holds: for any $p_0 \in M^n$ and any $v \in \mathbb{S}^{n-1} \subset T_{p_0}M$ the distance of the antipodal points $\exp_{p_0}(-\pi v)$ and $\exp_{p_0}(\pi v)$ is bounded by π:

$$\operatorname{dist}_{M^n}(\exp_{p_0}(-\pi v), \exp_{p_0}(\pi v)) < \pi \quad .$$

In fact, the constant δ in this theorem is explicit and independent of the dimension. Our proof even shows that the horse shoe inequality holds for $\delta = \frac{1}{4}(1+\varepsilon)^{-2}$ where $\varepsilon = \frac{1}{27\,000}$.

It should be pointed out that a horse shoe inequality has already been established in the work of Durumeric [12, Lemma 6]. However, his version of the inequality is only valid for manifolds which are not simply connected; the basic idea of his proof is to analyze the geometry of the Dirichlet cells in the universal covering of M^n.

The major work in this paper, which will be carried out in §4 and §5, is to prove Theorem 2.4. We have to develop an *entirely new approach*, in order to cover simply connected manifolds as well. It has already been known to Berger that the preceding theorem imposes strong restrictions on the topology of M^n. In particular, he already knew how to deduce Theorems A and B. The main steps are as follows:

Corollary 2.5. (c.f. Proposition 2 in [8]) *Let $\delta \in (0, \frac{1}{4})$ be the constant from Theorem 2.4. Then any compact Riemannian manifold*

M^n with

$$\delta \le K_M \le 1 \quad \text{and} \quad \pi \le \operatorname{inj} M^n \le \operatorname{diam} M^n \le \tfrac{\pi}{2\sqrt{\delta}}$$

admits a continuous, piecewise smooth map $f\colon \mathbb{RP}^n \to M^n$ of degree 1.

Here $\deg f$ denotes the standard integral mapping degree, if M^n is odd dimensional and orientable. Otherwise $\deg f$ has to be understood as the $\mathbb{Z}_2$–mapping degree.

The next two results are of purely algebraic topological nature. The first of these theorems goes back to Samelson [32], whereas the second one can be extracted from [9, p. 135ff].

Theorem 2.6. *Let M^n be a simply connected, compact, odd dimensional manifold. Suppose that there exists a continuous map $f\colon \mathbb{RP}^n \to M^n$ with $\deg_{\mathbb{Z}} f = 1$. Then M^n is a homology sphere.*

Theorem 2.7. *Let M^n be a simply connected, compact, even dimensional manifold. Suppose that there exists a continuous map $f\colon \mathbb{RP}^n \to M^n$ with $\deg_{\mathbb{Z}_2} f = 1$. Furthermore, let $R \in \{\mathbb{Q}, \mathbb{Z}_2\}$. Then*

$$H^*(M^n; R) = R[\xi_R]/(\xi_R^{m+1})$$

where $m \ge 1$ and ξ_R is a homogeneous element of degree $\deg \xi_R = \frac{n}{m}$. Moreover, $\deg \xi_R \in \{2, 4, 8\}$ for $m \ge 2$.

The fake projective spaces discovered by Eells and Kuiper [13] show that in the even dimensional case it would not be sufficient to recover the integral cohomology rings to recognize the manifold M^n up to homeomorphism, and the assertion of Theorem 2.7 is even weaker. The theorem determines the integral cohomology ring $H^*(M^n; \mathbb{Z})$ only up to the Serre class of torsion groups of odd order. This ambiguity reflects the fact that we can only work with the modulo 2 mapping degree of f. The corresponding freedom for the homeomorphism type of M^n is illustrated in Examples 3.6.

In the presence of the preceding results Theorems A and B can be established as follows:

Proof of Theorem A. By Theorem 1.1 it is sufficient to consider manifolds M^n of dimension $n \ge 5$. We define δ_{odd} as the maximum of the constants δ_{inj} and δ from Theorems 2.2 and 2.4. For ease of exposition we scale the metric on M^n such that $\delta_{\text{odd}} \le K_M \le 1$.

Because of the diameter sphere theorem of Grove and Shiohama we only need to consider manifolds with $\operatorname{diam} M^n \le \pi/(2\sqrt{\delta_{\text{odd}}})$. By Theorem 2.2 $\operatorname{inj} M^n \ge \pi$, and thus it follows from Corollary 2.5 that there exists a continuous, piecewise smooth map $f\colon \mathbb{RP}^n \to M^n$ of degree $\deg_{\mathbb{Z}} f = 1$. With Theorem 2.6 we conclude that the manifold M^n is a homology sphere. Since by hypothesis M^n is simply connected, Smale's solution of the Poincaré conjecture in dimensions $n \geq 5$ can be applied [28, p. 109]. q.e.d.

Proof of Theorem B. The argument is very similar to the preceding proof. Since the injectivity radius can be estimated by means of Theorem 2.1 rather than Theorem 2.2, we can define δ_{ev} as the constant δ from Theorem 2.4. As above, we obtain a continuous, piecewise smooth map $f : \mathbb{RP}^n \to M^n$.

In the present case, however, we only know that $\deg_{\mathbb{Z}_2} f = 1$. This is still sufficient to apply Theorem 2.7, and Theorem B follows since the truncated polynomial rings $R[\xi_R]/(\xi_R^{m+1})$ are precisely the cohomology rings of $\mathbb{S}^n$, $\mathbb{CP}^{n/2}$, or $\mathbb{HP}^{n/4}$, if the degree of the generator is n, 2, or 4, respectively. q.e.d.

The paper is *organized* as follows: for convenience we provide complete proofs for Corollary 2.5 and Theorems 2.6 and 2.7. These arguments will be given in §3.

The actual work, however, will be to establish Theorem 2.4. The proof of this theorem turns out to be surprisingly involved. Most of the arguments are contained in §4; however, the new technical tools that are required are developed in §5.

The basic idea for the proof of Theorem 2.4 is quite simple though. The diameter of M^n is an upper bound for the distance of the points $\exp_{p_0}(-\frac{1}{2}(1+\varrho_\varepsilon)\pi v)$ and $\exp_{p_0}(\frac{1}{2}(1+\varrho_\varepsilon)\pi v)$ for any number $\varrho_\varepsilon > 0$. Our goal is to derive from this inequality an upper bound for the distance $d(\exp_{p_0}(-\pi v), \exp_{p_0} \pi v)$ that is smaller than the injectivity radius of M^n. For this purpose we shall consider a ruled surface Σ with a conical singularity at p_0. The construction and the basic properties of Σ will be explained in §4. It then turns out that we need *refined estimates* that relate the lengths of various circles of latitude in a ruled surface with a conical singularity like Σ.

These estimates, however, do not follow from standard Jacobi field estimates. They rather comprise a *new tool* in comparison geometry, which may be useful in other contexts as well and which will be developed in §5. The results are summarized in Theorem 5.4. The proofs involve bounds for the angular velocity[1] of Jacobi fields, which are derived by means of the Riccati comparison theorems. It seems that there is no simple argument which is based on the maximum principle instead.

The new estimates are used to establish Proposition 4.5, which is the crucial step in the proof of Theorem 2.4. In this context, it also becomes clear for which parameters the new inequalities provide a significant gain over the standard Jacobi field estimates.

[1]c.f. Proposition 5.12.

In the appendix the comparison functions that have been introduced in order to state Theorem 5.4 are analyzed in more detail. Formally, these results are neither required for the proof of the theorem itself nor for the proof of the horse shoe theorem. However, they provide a better understanding of the theorem and of the way in which it is used in the proof of Theorem 2.4. In particular, Proposition A.6 justifies Figure 2 on page 237, and Corollary A.7 provides some background information concerning the numerical computations in the proof of Proposition 4.5. Moreover, the results in the appendix may be useful for other applications of Theorem 5.4.

3. Arguments from topology

The purpose of this section is to prove Corollary 2.5 and Theorems 2.6 and 2.7. We begin with the construction of the map $f\colon \mathbb{RP}^n \to M^n$.

Proof of Corollary 2.5. We fix some point $p_0 \in M^n$. Then for any unit vector $v \in \mathbb{S}^{n-1} \subset T_{p_0}M^n$ it is asserted by Theorem 2.4 that

$$\operatorname{dist}_{M^n}(\exp_{p_0}(-\pi v), \exp_{p_0}(\pi v)) < \pi \leq \operatorname{inj} M^n \quad ,$$

and hence there exists a unique minimizing geodesic $c_v\colon [-\pi, \pi] \to M^n$ from the point $c_v(-\pi) := \exp_{p_0}(-\pi v)$ to $c_v(\pi) := \exp_{p_0}(\pi v)$. Evidently, the geodesics c_v depend differentiably on the unit vector $v \in \mathbb{S}^{n-1} \subset T_{p_0}M^n$. Thus we can define a continuous, piecewise differentiable map $\tilde{f}\colon \overline{B(0,2\pi)} \subset T_{p_0}M^n \to M^n$ by means of

$$\tilde{f}(w) := \begin{cases} \exp_{p_0}(w) & \text{for } 0 \leq |w| \leq \pi \ , \\ c_{\frac{w}{|w|}}(2\pi - |w|) & \text{for } \pi \leq |w| \leq 2\pi \ . \end{cases}$$

By construction $\tilde{f}(w) = c_{w/|w|}(0) = c_{-w/|w|}(0) = \tilde{f}(-w)$ for any $w \in \underline{S(0,2\pi)} \subset T_{p_0}M^n$, and therefore $\tilde{f}$ factors over a map $f\colon \mathbb{RP}^n \cong \overline{B(0,2\pi)}/_\sim \to M^n$ which is still continuous and piecewise smooth.

It remains to *verify* that this map f has degree 1 provided that the orientations are chosen appropriately. Since the mapping degree can be computed locally, it is sufficient to show that the preimage $f^{-1}(p_0)$ consists of precisely one point. By construction $d(p_0, c_v(-\pi)) = d(p_0, c_v(\pi)) = \pi$, and thus the triangle inequality shows that $d(p_0, c_v(t)) \geq \frac{1}{2}\pi > 0$ for any $t \in [-\pi, \pi]$ and any $v \in \mathbb{S}^{n-1}$. Hence $\tilde{f}(w) \neq p_0$ for any w in the annulus $\overline{B(0,2\pi)} \backslash B(0,\pi)$, and the proof is complete as $\operatorname{inj} M^n \geq \pi$. q.e.d.

The proofs of Theorems 2.6 and 2.7 make repeated use of the Poincaré duality theorem, the universal coefficient theorem, and the Steenrod squares. For this material we refer to the books by Dold [11] and Spanier [35]. In the proof of Theorem 2.7 we shall also employ J. F. Adams's results on secondary cohomology operations and the Hopf invariant one problem [2].

The first issue is to understand the structure of the cohomology ring $H^*(M^n; Z_2)$. For this purpose we analyze the map $f^*\colon H^*(M^n;\mathbb{Z}_2) \to H^*(\mathbb{RP}^n;\mathbb{Z}_2)$. As usual, ω denotes the generator of $H^1(\mathbb{RP}^n;\mathbb{Z}_2)$.

Proposition 3.1. *Let $f\colon \mathbb{RP}^n \to M^n$ be a continuous map into some compact, connected manifold M^n, and let $\ell := \inf\{k > 0 \mid H^k(f;\mathbb{Z}_2) \neq 0\}$. Suppose that $\deg_{\mathbb{Z}_2} f = 1$. Then*

(i) *$f^*\colon H^*(M^n;\mathbb{Z}_2) \to H^*(\mathbb{RP}^n;\mathbb{Z}_2)$ is injective,*
(ii) *$\operatorname{im} f^* = \mathbb{Z}_2[\omega^\ell]/(\omega^{n+1}) \subset H^*(\mathbb{RP}^n;\mathbb{Z}_2)$,*
(iii) *ℓ divides the dimension n of the manifold, and*
(iv) *$\ell = n$ or ℓ is a power of 2.*

Proof. i) Let $x_k \in H^k(M^n;\mathbb{Z}_2)$. The Poincaré duality theorem implies that there exists an element $y_{n-k} \in H^{n-k}(M^n;\mathbb{Z}_2)$ such that $x_k \cup y_{n-k} = \zeta_M$ where ζ_M stands for the generator of $H^n(M^n;\mathbb{Z}_2)$. Note that
$H^n(\mathbb{RP}^n;\mathbb{Z}_2)$ is generated by ω^n. Thus

$$f^k(x_k) \cup f^{n-k}(y_{n-k}) \;=\; f^n(\zeta_M) \;=\; \deg_{\mathbb{Z}_2}(f)\cdot\omega^n \;\neq\; 0 \quad ,$$

and hence $f^k(x_k) \neq 0$.

ii) and iii) Since $f^n(\zeta_M) = \deg_{\mathbb{Z}_2}(f)\cdot\omega^n = \omega^n$, it follows that Poincaré duality holds for the subalgebra $\operatorname{im} f^* \subset H^*(\mathbb{RP}^n;\mathbb{Z}_2) = \mathbb{Z}_2[\omega]/(\omega^{n+1})$. More concretely,

$$\omega^k \in \operatorname{im} f^* \quad\Leftrightarrow\quad \omega^{n-k} \in \operatorname{im} f^* \quad .$$

Hence $\operatorname{im} f^k = 0$ for $n-\ell < k < n$. Finally, we observe that for $k \leq n-\ell$ the multiplication by ω^ℓ defines an injective map $\operatorname{im} f^k \to \operatorname{im} f^{k+\ell}$.

iv) We assume that $\ell \neq n$. Thus (iii) implies that $n \geq 2\ell$. This case can be analyzed with the help of the Steenrod squares Sq^i. Since $\omega^\ell \in \operatorname{im} f^*$, it follows directly from the axioms for Sq^* that

$$\omega^\ell \cup (1{+}\omega)^\ell \;=\; \left(\mathrm{Sq}^*\,\omega\right)^\ell \;=\; \mathrm{Sq}^*(\omega^\ell) \in \operatorname{im} f^* \quad .$$

Using (ii) we conclude that $\binom{\ell}{\mu} \equiv 0 \mod 2$ for $0 < \mu < \ell$, hence the claim. q.e.d.

Corollary 3.2. *Let M^n be a compact, simply connected manifold. Suppose that there is a continuous map $f\colon \mathbb{RP}^n \to M^n$ of degree $\deg_{\mathbb{Z}_2} f = 1$. Then*

$$H^*(M^n;\mathbb{Z}_2) = \mathbb{Z}_2[\xi]/(\xi^{m+1})$$

where $m \geq 1$ *and where* ξ *is homogeneous element of degree* $\deg \xi = \frac{n}{m} > 1$. *Furthermore,* $\deg \xi = n$ *if* n *is odd.*

Proof. Using Proposition 3.1, we find $\xi \in H^\ell(M^n;\mathbb{Z}_2)$ such that $f^\ell(\xi) = \omega^\ell$. The case $\ell = 1$ is ruled out, since $H^1(M^n;\mathbb{Z}_2) = \mathrm{Hom}(\pi_1(M^n),\mathbb{Z}_2) = 0$. q.e.d.

Substantial information about $H^*(M^n;\mathbb{Z})$ can be recovered from the very special structure of $H^*(M^n;\mathbb{Z}_2)$, using just the universal coefficient theorem and the Poincaré duality theorem.

Proposition 3.3. *Let* M^n *be a compact, connected manifold. Suppose that its cohomology* $H^*(M^n;\mathbb{Z}_2)$ *is a truncated polynomial ring generated by a homogeneous element* ξ *of degree* $\deg \xi > 1$. *Then*

(i) *the natural homomorphisms* $H^k(M^n;\mathbb{Z}) \to H^k(M^n;\mathbb{Z}_2)$ *are surjective,*

(ii) *the groups* $H^k(M^n;\mathbb{Z})$ *do not have any 2–torsion, and*

(iii) $\mathrm{rk}_{\mathbb{Z}}\, H^k(M^n;\mathbb{Z}) = \dim_{\mathbb{Z}_2} H^k(M^n;\mathbb{Z}_2)$.

Remark 3.4. Consider two primitive elements $x_i \in H^{m_i}(M^n;\mathbb{Z})$, $i = 1,2$, of infinite order such that $m_1 + m_2 \leq n$. Then their cup product $x_1 \cup x_2 \in H^{m_1+m_2}(M^n;\mathbb{Z})$ is again an element of infinite order. It is an odd multiple of a primitive element.

Proof of Proposition 3.3. Since $\deg \xi > 1$, $H^1(M^n;\mathbb{Z}_2) = 0$ and thus M^n is orientable. Hence $H^0(M^n;\mathbb{Z}) = \mathbb{Z}$ and $H^n(M^n;\mathbb{Z}) \cong \mathbb{Z}$. It is therefore sufficient to prove statements (i)–(iii) for $0 < k < n$.

The *first step* is to establish (i)–(iii) for any k such that $H^k(M^n;\mathbb{Z}_2) = 0$. Note that in this case statement (i) is trivial. By the universal coefficient theorem there exists an injective homomorphism $H^k(M^n;\mathbb{Z}) \otimes \mathbb{Z}_2 \to H^k(M^n;\mathbb{Z}_2)$. Hence $H^k(M^n;\mathbb{Z}) \otimes \mathbb{Z}_2 = 0$, and we obtain (ii) and (iii) as well.

It *remains* to handle the case that $H^k(M^n;\mathbb{Z}_2) \neq 0$ for some k. By hypothesis $H^*(M^n;\mathbb{Z}_2) = \mathbb{Z}_2[\xi]/(\xi^{m+1})$ for some m, and $\deg \xi > 1$. Hence k and n are multiples of $\deg \xi$. This in turn implies that $H^{n-k+1}(M^n;\mathbb{Z}_2) = 0$. By Poincaré duality the torsion subgroup of $H^k(M^n;\mathbb{Z})$ is isomorphic to the torsion subgroup of $H^{n-k+1}(M^n;\mathbb{Z})$, and by the preceding step we already know that the 2–torsion of $H^{n-k+1}(M^n;\mathbb{Z})$ vanishes. Thus the proof of (ii) is complete. In particular, $\mathrm{Tor}(H^{k+1}(M^n;\mathbb{Z}),\mathbb{Z}_2) = 0$. Hence the map $H^k(M^n;\mathbb{Z}) \otimes \mathbb{Z}_2 \to H^k(M^n;\mathbb{Z}_2)$ that appears in the universal coefficient theorem is an isomorphism, and statements (i) and (iii) follow using (ii). q.e.d.

Proof of Theorem 2.6. The manifold M^n is simply connected. We shall see that M^n is in addition a *homology sphere.* Hence the theorem follows from Smale's solution of the Poincaré conjecture in dimensions $n \geq 5$ [28, p. 109], [34].

We have to show that $H^k(M^n;\mathbb{Z}) = 0$ for $0 < k < n$. Let us *assume* the converse. Suppose that there exists some $x_k \in H^k(M^n;\mathbb{Z})$, $0 < k < n$, and some prime field $\mathbb{F}$ such that $x_k \otimes 1 \in H^k(M^n;\mathbb{Z}) \otimes \mathbb{F}$ is different from zero.

Then on one hand $f_{\mathbf{Z}}^k(x_k) = 0$. In fact, Corollary 3.2 and Proposition 3.3 show that x_k is a torsion element and that $\operatorname{ord} x_k$ is odd. But $H^k(\mathbb{RP}^n;\mathbb{Z}_2)$ does not contain any torsion element of odd order.

On the other hand the following argument shows that $f_{\mathbf{Z}}^k(x_k) \neq 0$. To begin with we observe that

$$f_{\mathbf{Z}}^n(\zeta_M) \;=\; \deg_{\mathbf{Z}}(f) \cdot \zeta_{\mathbf{RP}^n} \;=\; \zeta_{\mathbf{RP}^n} \quad ,$$

where $\zeta_M \in H^n(M^n;\mathbb{Z})$ and $\zeta_{\mathbf{RP}^n} \in H^n(\mathbb{RP}^n;\mathbb{Z})$ denote the generators that represent the given orientations. By the universal coefficient theorem the corresponding generators of $H^n(M^n;\mathbb{F})$ and $H^n(\mathbb{RP}^n;\mathbb{F})$ are the elements

$$\begin{aligned} \zeta_M \otimes 1 &\in H^n(M^n;\mathbb{Z}) \otimes \mathbb{F} \subset H^n(M^n;\mathbb{F}) \quad , \\ \zeta_{\mathbf{RP}^n} \otimes 1 &\in H^n(\mathbb{RP}^n;\mathbb{Z}) \otimes \mathbb{F} \subset H^n(\mathbb{RP}^n;\mathbb{F}) \quad . \end{aligned}$$

For the same reason $H^k(M^n;\mathbb{Z}) \otimes \mathbb{F}$ embeds into $H^k(M^n;\mathbb{F})$. Thus by Poincaré duality there exists some $y_{n-k}^{\mathbf{F}} \in H^{n-k}(M^n;\mathbb{F})$ such that $(x_k \otimes 1) \cup y_{n-k}^{\mathbf{F}} = \zeta_M \otimes 1$. Applying $f_{\mathbf{F}}^*$, we conclude that

$$\begin{aligned} (f_{\mathbf{Z}}^k(x_k) \otimes 1) \cup f_{\mathbf{F}}^{n-k}(y_{n-k}^{\mathbf{F}}) \;&=\; f_{\mathbf{F}}^n((x_k \otimes 1) \cup y_{n-k}^{\mathbf{F}}) \\ &=\; f_{\mathbf{F}}^n(\zeta_M \otimes 1) \;=\; f_{\mathbf{Z}}^n(\zeta_M) \otimes 1 \;=\; \zeta_{\mathbf{RP}^n} \otimes 1 \;\neq\; 0 \quad , \end{aligned}$$

and hence $f_{\mathbf{Z}}^k(x_k) \neq 0$.

Proof of Theorem 2.7. We begin with the case that $R = \mathbb{Z}_2$. By Corollary 3.2 $H^*(M^n;\mathbb{Z}_2)$ is a truncated polynomial ring generated by some homogeneous element $\xi_{\mathbf{Z}_2}$ of degree $\deg \xi_{\mathbf{Z}_2} > 1$. It remains to improve the restrictions for the degree of this generator if $m \geq 2$. As explained in [29, page 134], the claim follows from J. F. Adams's work on secondary cohomology operations and the Hopf invariant one problem [2].

For coefficients in $R = \mathbb{Q}$ the claimed structure of the cohomology ring follows from the result for $\mathbb{Z}_2$–coefficients by means of Proposition 3.3. q.e.d.

Remarks 3.5. a) All compact, simply connected manifolds M^n have the same integral cohomology groups in degrees 0, 1, $n-1$, and n:

$$H^0(M^n;\mathbb{Z}) = \mathbb{Z} \;, \quad H^1(M^n;\mathbb{Z}) = H^{n-1}(M^n;\mathbb{Z}) = 0 \;, \\ \text{and} \quad H^n(M^n;\mathbb{Z}) \cong \mathbb{Z} \;.$$

Furthermore, $H^2(M^n;\mathbb{Z})$ is always torsionfree.

b) The integral cohomology ring of a compact, simply connected 4–manifold is determined by the second Betti number and by the intersection form. Theorem B asserts that this Betti number is ≤ 1, and thus the classification result due to Freedman [17] implies that M^4 is homeomorphic to $\mathbb{S}^4$ or $\mathbb{CP}^2$.

c) Seaman's proof is based on Freedman's classification, too. However, Bochner techniques are used to establish the upper bound for the second Betti number.

Yet, the conclusion of Theorem B is the *best possible result* in dimensions $n \geq 6$.

Example 3.6. (communicated to us by M. Kreck) i) Let $2 < k \leq \frac{n}{2}$, and let $\ell > 1$. Then there exists a compact, simply connected manifold $S^n_{k,\ell}$ with

$$H^q(S^n_{k,\ell};\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q \in \{0\,,n\}\ , \\ \mathbb{Z}_\ell & \text{if } q \in \{k\,,n-k+1\}\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

ii) Consider a finite number of manifolds $S^n_{k_j,\ell_j}$ as in (i) where $2 < k_j \leq \frac{n}{2}$, $\ell_j > 1$, and $\ell_j \equiv 1\,(2)$. Then the connected sum M^n of a compact, simply connected manifold V^n like $\mathbb{S}^n$, $\mathbb{CP}^{n/2}$, $\mathbb{HP}^{n/4}$, or $\mathbb{C}aP^2$ with the $S^n_{k_j,\ell_j}$ is simply connected, and the projection $M^n \to V^n$ induces isomorphisms

$$H^*(V^n;\mathbb{Q}) \overset{\cong}{\to} H^*(M^n;\mathbb{Q}) \quad \text{and} \quad H^*(V^n;\mathbb{Z}_2) \overset{\cong}{\to} H^*(M^n;\mathbb{Z}_2) \quad .$$

But $H^*(M^n;\mathbb{Z}) \ncong H^*(V^n;\mathbb{Z})$.

Construction of the Manifolds $S^n_{k,\ell}$. Consider the cell complex $X_{k,\ell}$ obtained by attaching a k–dimensional cell e^k to the sphere $\mathbb{S}^{k-1}$ by means of a map of degree ℓ. We may view $X_{k,\ell}$ as the $(k-2)$–fold suspension of $X_{2,\ell}$. By hypothesis $n \geq k+3$, and thus there exists a PL–embedding $\iota^n_{k,\ell}\colon X_{k,\ell} \to \mathbb{R}^n$. Let $Y^n_{k,\ell} \subset \mathbb{R}^n$ be a small tube around $\iota^n_{k,\ell}(X_{k,\ell})$ with smooth boundary $\partial Y^n_{k,\ell}$. The compact manifold $S^n_{k,\ell}$ is then obtained as the double of $Y^n_{k,\ell}$ along its boundary.

Computation of $\pi_1(S^n_{k,\ell})$ *and* $H^*(S^n_{k,\ell};\mathbb{Z})$ By construction the image of $\iota^n_{k,\ell}$ is a deformation retract of $Y^n_{k,\ell}$, and therefore

$$\pi_1(Y^n_{k,\ell}) \;=\; (\iota^n_{k,\ell})_\sharp \pi_1(X_{k,\ell}) \;=\; 0 \quad .$$

Here the second equality sign makes use of the assumption that $k > 2$. Since $\partial Y^n_{k,\ell}$ is connected, the Seifert–van Kampen theorem is applicable, and hence the double of $Y^n_{k,\ell}$ is simply connected.

Observe that $\tilde{H}_{k-1}(X_{k,\ell};\mathbb{Z}) = \mathbb{Z}_\ell$. The other reduced homology groups of $X_{k,\ell}$ vanish. Since $\iota^n_{k,\ell}(X_{k,\ell})$ is a deformation retract of $Y^n_{k,\ell}$ and of $Y^n_{k,\ell}\backslash\partial Y^n_{k,\ell}$, we can compute $H^*(Y^n_{k,\ell};\mathbb{Z})$ and $H^*(Y^n_{k,\ell},\partial Y^n_{k,\ell};\mathbb{Z})$ from the universal coefficient theorem and the Alexander duality theorem, respectively. The nontrivial groups are

$$H^q(Y^n_{k,\ell}) = \begin{cases}\mathbb{Z} & \text{if } q=0\ ,\\ \mathbb{Z}_\ell & \text{if } q=k\ ,\end{cases} \qquad H^q(Y^n_{k,\ell},\partial Y^n_{k,\ell}) = \begin{cases}\mathbb{Z} & \text{if } q=n\ ,\\ \mathbb{Z}_\ell & \text{if } q=n-k+1\ .\end{cases}$$

By excision $H^*(S^n_{k,\ell},Y^n_{k,\ell}) \overset{\cong}{\to} H^*(Y^n_{k,\ell},\partial Y^n_{k,\ell})$. Since $k \le \frac{n}{2}$, there are short exact sequences

$$0 \longrightarrow H^q(Y^n_{k,\ell},\partial Y^n_{k,\ell};\mathbb{Z}) \longrightarrow H^q(S^n_{k,\ell};\mathbb{Z}) \longrightarrow H^q(Y^n_{k,\ell};\mathbb{Z}) \longrightarrow 0 \quad ,$$

hence the claim.

4. The geometric setup for Theorem 2.4

The purpose of this section is to prove Theorem 2.4 up to some estimates for ruled surfaces that will be provided in the two subsequent subsections. The argument will be given as a sequence of lemmas and propositions.

Throughout the entire section we suppose that M^n is a complete Riemannian manifold with $0 < \delta \le K_M \le 1$ and $\pi \le \operatorname{inj} M^n \le \operatorname{diam} M^n \le \frac{\pi}{2\sqrt{\delta}}$. Clearly, these assumptions imply that $\delta \le \frac{1}{4}$. We find it more convenient to write $\delta = \frac{1}{4(1+\varepsilon)^2}$ and think of $\varepsilon \in [0,\infty)$ as the independent variable.

It will be necessary to impose stronger and stronger bounds for ε as we proceed.

Configuration (Horse shoe). Let $p_0 \in M^n$ and $v \in \mathbb{S}^{n-1} \subset T_{p_0}M$, and consider the points $p_1 := \exp_{p_0}(-\pi v)$ and $p_2 := \exp_{p_0}(\pi v)$.

In order to prove Theorem 2.4, we have to show that $d_{M^n}(p_1,p_2) < \pi$.

Construction. Suppose that $0 < \varepsilon \le \frac{1}{5}$. We define a number $\varrho_\varepsilon \in (0,\frac{1}{2})$ by means of the equation $\sin(\frac{1}{2}\varrho_\varepsilon\pi) = \sin(\frac{1}{4}\varepsilon^{1/3}\pi)^{-1}\cdot\sin(\frac{1}{2}\varepsilon\pi)$, and introduce the points

$$q_1^\varepsilon := \exp_{p_0}(-\tfrac{1}{2}(1+\varrho_\varepsilon)\pi v) \quad ,$$
$$q_2^\varepsilon := \exp_{p_0}(\ \tfrac{1}{2}(1+\varrho_\varepsilon)\pi v) \quad .$$

Furthermore, we consider a minimizing geodesic $c^\varepsilon\colon [0,1] \to M^n$ joining $c^\varepsilon(0) := q_1^\varepsilon$ to $c^\varepsilon(1) := q_2^\varepsilon$.

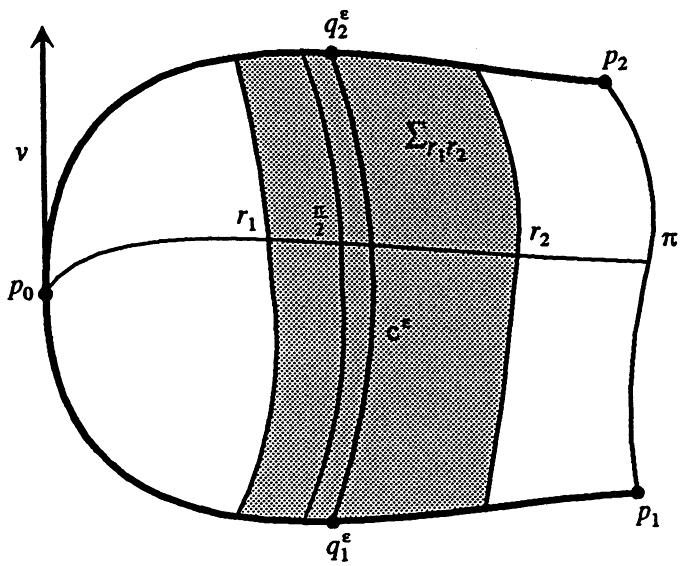

FIGURE 1. Configuration of the Horse Shoe

The geometric setup described so far is depicted in Figure 1. In this Figure we have also indicated some circles of latitude that will be constructed later using the estimates from Lemmas 4.1–4.3 in combination with Assumption 4.4.

By hypothesis $L(c^\varepsilon) = d_{M^n}(q_1^\varepsilon, q_2^\varepsilon) \leq \operatorname{diam} M^n \leq \pi(1+\varepsilon)$. Further information about the geodesic c^ε is provided in the following three lemmas.

Lemma 4.1. *Let $0 < \varepsilon \leq \frac{1}{5}$, and consider the horse shoe $p_1p_0p_2$ and the geodesic c^ε constructed above. Then $d_{M^n}(p_0, c^\varepsilon(t)) \geq \frac{1}{2}(1 - \frac{1}{2}\varepsilon^{1/3})\pi$ for any $t \in [0,1]$.*

In particular, the geodesic $c^\varepsilon\colon [0,1] \to M^n$ does not intersect the metric ball $B(p_0, \frac{7}{16}\pi)$ if $0 < \varepsilon \leq \frac{1}{64}$, and it does not intersect the ball $B(p_0, \frac{59}{120}\pi)$ provided that $0 < \varepsilon \leq \frac{1}{27\,000}$.

Lemma 4.2. *Let $0 < \varepsilon \leq \frac{1}{5}$, and consider the horse shoe $p_1p_0p_2$ and the geodesic c^ε constructed above. Suppose that $d(p_1, p_2) \geq \pi$. Then the angles*

$$\alpha_1^\varepsilon := \sphericalangle\left(\tfrac{\partial}{\partial t}c^\varepsilon(t)|_{t=0}\,, \tfrac{\partial}{\partial r}\exp_{p_0}(-rv)|_{r=\frac{1}{2}(1+\varrho_\varepsilon)\pi}\right)$$
$$\alpha_2^\varepsilon := \sphericalangle\left(-\tfrac{\partial}{\partial t}c^\varepsilon(t)|_{t=1}\,, \tfrac{\partial}{\partial r}\exp_{p_0}(\ rv)|_{r=\frac{1}{2}(1+\varrho_\varepsilon)\pi}\right)$$

can be bounded from below by means of the inequalities:

$$\cos\alpha_i^\varepsilon \;\leq\; \sqrt{2}\cdot\sin\!\left(\tfrac{\pi}{2}\tfrac{\varepsilon}{1+\varepsilon}\right)\sin\!\left(\tfrac{\pi}{4}\tfrac{1-\varrho_\varepsilon}{1+\varepsilon}\right)^{-1} \qquad \textit{for } i = 1, 2\,.$$

Lemma 4.3. *Let $0 < \varepsilon \leq \frac{1}{64}$, and consider the horse shoe $p_1p_0p_2$ and the geodesic c^ε constructed above. Suppose that $d(p_1, p_2) \geq \pi$. Then $d(p_0, c^\varepsilon(t)) < \frac{3}{4}\pi$ for all $t \in [0,1]$.*

Proof of Lemma 4.1. Since $K_M \leq 1$ and $\operatorname{inj} M^n \geq \pi$, it is standard to define a weak contraction $\Phi\colon M^n \to \mathbb{S}^n$ as follows: we fix a point $\bar{p}_0 \in \mathbb{S}^n$ and a linear isometry $I\colon T_{p_0}M^n \to T_{\bar{p}_0}\mathbb{S}^n$, and set

$$\Phi(p) := \begin{cases} \exp_{\bar{p}_0} \circ I \circ \exp_{p_0}^{-1}(p) & \text{if } d(p, p_0) < \pi \ , \\ -\bar{p}_0 & \text{otherwise ;} \end{cases}$$

here $-\bar{p}_0$ denotes the antipodal point of $\bar{p}_0 \in \mathbb{S}^n$.

Clearly, the points $\bar{q}_1^\varepsilon := \Phi(q_1^\varepsilon)$, $\bar{p}_0 = \Phi(p_0)$, and $\bar{q}_2^\varepsilon := \Phi(q_2^\varepsilon)$ lie on a great circle and $d(\bar{p}_0, \bar{q}_1^\varepsilon) = d(\bar{p}_0, \bar{q}_2^\varepsilon) = \frac{1}{2}(1 + \varrho_\varepsilon)\pi$. Moreover, the geodesic c^ε maps to a curve $\bar{c}^\varepsilon\colon [0,1] \to \mathbb{S}^n$ such that $L(\bar{c}^\varepsilon) \leq L(c^\varepsilon) \leq \pi(1+\varepsilon)$ and $d(\bar{p}_0, \bar{c}^\varepsilon(t)) = d(p_0, c^\varepsilon(t))$ for all $t \in [0,1]$. Hence

(1)
$$d(p_0\,, c^\varepsilon(t)) \ \geq \ \inf\{d(\bar{p}_0, \bar{q}) \mid \bar{q} \in \mathbb{S}^n,\ d(\bar{q}, \bar{q}_1^\varepsilon) + d(\bar{q}, \bar{q}_2^\varepsilon) \leq \pi(1+\varepsilon)\}$$

for all $t \in [0,1]$. The infimum on the right hand side is actually a minimum, which is achieved at some point $\bar{q}^\varepsilon \in \mathbb{S}^n$ such that $d(\bar{q}^\varepsilon, \bar{q}_1^\varepsilon) = d(\bar{q}^\varepsilon, \bar{q}_2^\varepsilon) = \frac{1}{2}(1 + \varepsilon)\pi$. The great circle through $\bar{p}_0$ and $\bar{q}^\varepsilon$ intersects the great circle through $\bar{q}_1^\varepsilon$, $\bar{p}_0$, and $\bar{q}_2^\varepsilon$ orthogonally, and by the Law of Cosines we obtain

$$\cos\left(\tfrac{1}{2}(1+\varepsilon)\pi\right) \ = \ \cos\left(\tfrac{1}{2}(1+\varrho_\varepsilon)\pi\right) \cdot \cos d(\bar{p}_0, \bar{q}^\varepsilon) \quad ,$$

or equivalently:

$$\sin\left(\tfrac{1}{2}\pi - d(\bar{p}_0, \bar{q}^\varepsilon)\right) \ = \ \frac{\sin(\frac{1}{2}\varepsilon\pi)}{\sin(\frac{1}{2}\varrho_\varepsilon\pi)} \ = \ \sin\left(\tfrac{1}{4}\varepsilon^{1/3}\pi\right) \quad .$$

Thus $d(\bar{p}_0, \bar{q}^\varepsilon) = \frac{1}{2}(1 - \frac{1}{2}\varepsilon^{1/3})\pi$, and the lemma follows with inequality (1).

Proof of Lemma 4.2. By symmetry it is sufficient to establish the lower bound for the angle α_1^ε. Recall that $K_M \geq \delta = \frac{1}{4(1+\varepsilon)^2}$. We consider a comparison triangle $\bar{p}_0\bar{p}_1\bar{p}_2$ in the sphere $\mathbb{S}_\delta^2$ of constant curvature δ such that

$$\begin{aligned} d(\bar{p}_0\,, \bar{p}_1) \ &= \ d(p_0\,, p_1) \ = \ \pi \ , \\ d(\bar{p}_0\,, \bar{p}_2) \ &= \ d(p_0\,, p_2) \ = \ \pi \ , \\ d(\bar{p}_1\,, \bar{p}_2) \ &= \ \pi \ \leq \ d(p_1\,, p_2) \ . \end{aligned}$$

On the edge $\bar{p}_0\bar{p}_2$ we consider the point $\bar{q}_2^\varepsilon := \exp_{\bar{p}_0}\big(\frac{1}{2}(1+\varrho_\varepsilon)\cdot\exp_{\bar{p}_0}^{-1}(\bar{p}_2)\big)$. The Toponogov triangle comparison theorem yields

$$d(p_1\,,q_2^\varepsilon) \;\geq\; d(\bar{p}_1\,,\bar{q}_2^\varepsilon) \quad .$$

In order to compute the right hand side, we introduce the mid point $\bar{m}$ on $\bar{p}_0\bar{p}_2$ and apply the Law of Cosines to the triangles $\bar{p}_1\bar{m}\bar{p}_2$ and $\bar{p}_1\bar{m}\bar{q}_2^\varepsilon$, which have angles $\frac{\pi}{2}$ at the vertex $\bar{m}$. We obtain

$$\begin{aligned}
\cos\big(\tfrac{1}{2(1+\varepsilon)}\,d(p_1\,,q_2^\varepsilon)\big) \;&\leq\; \cos\big(\tfrac{1}{2(1+\varepsilon)}\,d(\bar{p}_1\,,\bar{q}_2^\varepsilon)\big)\\
&=\; \cos\big(\tfrac{1}{2(1+\varepsilon)}\,d(\bar{p}_1\,,\bar{m})\big)\cdot\cos\big(\tfrac{1}{2(1+\varepsilon)}\,d(\bar{m}\,,\bar{q}_2^\varepsilon)\big)\\
&=\; \cos\big(\tfrac{\pi}{4(1+\varepsilon)}\big)^{-1}\cos\big(\tfrac{\pi}{2(1+\varepsilon)}\big)\cdot\cos\big(\tfrac{\pi\varrho_\varepsilon}{4(1+\varepsilon)}\big)\\
&\leq\; \sqrt{2}\cdot\sin\big(\tfrac{\pi}{2}\tfrac{\varepsilon}{1+\varepsilon}\big) \quad .
\end{aligned} \tag{2}$$

The next step is to apply Toponogov's theorem to the triangle $p_1q_1^\varepsilon q_2^\varepsilon$. Using the Law of Cosines we conclude that

$$\begin{aligned}
&\cos(\alpha_1^\varepsilon)\\
&\leq\; \frac{\cos\big(\frac{1}{2(1+\varepsilon)}\,d(p_1,q_2^\varepsilon)\big)-\cos\big(\frac{1}{2(1+\varepsilon)}\,d(p_1,q_1^\varepsilon)\big)\cdot\cos\big(\frac{1}{2(1+\varepsilon)}\,d(q_1^\varepsilon,q_2^\varepsilon)\big)}{\sin\big(\frac{1}{2(1+\varepsilon)}\,d(p_1,q_1^\varepsilon)\big)\cdot\sin\big(\frac{1}{2(1+\varepsilon)}\,d(q_1^\varepsilon,q_2^\varepsilon)\big)}\\
&\leq\; \frac{\sqrt{2}\cdot\sin\big(\frac{\pi}{2}\frac{\varepsilon}{1+\varepsilon}\big)-\cos\big(\frac{\pi}{4}\frac{1-\varrho_\varepsilon}{1+\varepsilon}\big)\cdot\cos\big(\frac{1}{2(1+\varepsilon)}\,L(c^\varepsilon)\big)}{\sin\big(\frac{\pi}{4}\frac{1-\varrho_\varepsilon}{1+\varepsilon}\big)\cdot\sin\big(\frac{1}{2(1+\varepsilon)}\,L(c^\varepsilon)\big)} \quad .
\end{aligned}$$

The bounds for ε imply that $0<\sqrt{2}\,\sin(\frac{\pi}{2}\frac{\varepsilon}{1+\varepsilon})<\cos(\frac{\pi}{4}\frac{1-\varrho_\varepsilon}{1+\varepsilon})$, and thus the right hand side is a monotonically increasing function of $L(c^\varepsilon)$. The lemma follows, since this length is bounded by $\pi(1+\varepsilon)$.

Proof of Lemma 4.3. We consider the hinges $p_0q_1^\varepsilon c^\varepsilon(t)$ if $t\in[0,\frac{1}{2}]$ and $p_0q_2^\varepsilon c^\varepsilon(t)$ if $t\in[\frac{1}{2},1]$. In either case Lemma 4.2 provides lower bounds for the exterior angles α_i^ε of the hinge, and applying Toponogov's theorem with the sphere $\mathbb{S}_\delta^2$ of constant curvature δ as the model space, we obtain the inequality

$$\begin{aligned}
\cos\big(\tfrac{1}{2(1+\varepsilon)}\,d(p_0,c^\varepsilon(t))\big) \;&\geq\; \cos\big(\tfrac{\pi(1-|2t-1|)}{4}\big)\cdot\cos\big(\tfrac{\pi}{4}\tfrac{1+\varrho_\varepsilon}{1+\varepsilon}\big)\\
&\qquad -\cos(\alpha_i^\varepsilon)\cdot\sin\big(\tfrac{\pi(1-|2t-1|)}{4}\big)\cdot\sin\big(\tfrac{\pi}{4}\tfrac{1+\varrho_\varepsilon}{1+\varepsilon}\big)\\
&\geq\; \tfrac{1}{\sqrt{2}}\cos\big(\tfrac{\pi}{4}\tfrac{1+\varrho_\varepsilon}{1+\varepsilon}\big)\\
&\qquad -\sin\big(\tfrac{\pi}{2}\tfrac{\varepsilon}{1+\varepsilon}\big)\sin\big(\tfrac{\pi}{4}\tfrac{1-\varrho_\varepsilon}{1+\varepsilon}\big)^{-1}\cdot\sin\big(\tfrac{\pi}{4}\tfrac{1+\varrho_\varepsilon}{1+\varepsilon}\big)\\
&\geq\; \tfrac{1}{\sqrt{2}}\cos\big(\tfrac{\pi(1+\varrho_\varepsilon)}{4}\big)-\tfrac{1+\varrho_\varepsilon}{1-\varrho_\varepsilon}\sin\big(\tfrac{1}{2}\varepsilon\pi\big) \quad .
\end{aligned}$$

Since by definition ϱ_ε depends monotonically on ε, it is easy to compute that $\varrho_\varepsilon \le \frac{1}{12}$, and thus we conclude that

$$\begin{aligned}\cos\left(\tfrac{1}{2(1+\varepsilon)}\, d(p_0, c^\varepsilon(t)\right) &\ge \tfrac{1}{2}\cos\left(\tfrac{1}{4}\varrho_\varepsilon\pi\right) - \tfrac{1}{2}\sin\left(\tfrac{1}{4}\varrho_\varepsilon\pi\right) - \tfrac{13}{11}\sin\left(\tfrac{1}{2}\varepsilon\pi\right) \\ &\ge 0.498 - 0.033 - 0.030 = 0.435 \\ &\ge 0.400 \ge \cos\left(\tfrac{32}{65}\cdot\tfrac{3}{4}\pi\right) \quad .\end{aligned}$$

Lemmas 4.1 and 4.3 suggest that we *prove Theorem 2.4 indirectly.* For this purpose we impose the following inequality:

Assumption 4.4. Suppose that $d(p_1, p_2) \ge \pi$ for the horse shoe constructed above.

Construction (continued). Suppose in addition that $0 < \varepsilon \le \frac{1}{64}$. Then it follows from Lemmas 4.1 and 4.3 that the geodesic $c^\varepsilon\colon [0,1] \to M^n$ can be lifted under $\exp_{p_0}$ to a curve

$$\tilde{c}^\varepsilon\colon [0,1] \to \quad \overline{B(0, \tfrac{3}{4}\pi)}\backslash B(0, \tfrac{7}{16}\pi) \subset T_{p_0}M^n \quad .$$

The corresponding unit vector field $\tilde{v}^\varepsilon(t) := |\tilde{c}^\varepsilon(t)|^{-1}\, \tilde{c}^\varepsilon(t)$ defines a differentiable map

$$\begin{aligned}\gamma^\varepsilon\colon [0,\pi] \times [0,1] &\to M^n \\ (r\,,\,t) &\mapsto \exp_{p_0}(r\cdot\tilde{v}^\varepsilon(t)) \quad .\end{aligned} \tag{3}$$

Notice that the geodesic c^ε lifts under γ^ε to the graph of the function $\tilde{r}^\varepsilon\colon t \mapsto |\tilde{c}^\varepsilon(t)|$. More precisely, $c^\varepsilon(t) = \gamma^\varepsilon(\tilde{r}^\varepsilon(t), t)$ for all $t \in [0,1]$. In fact, $\gamma^\varepsilon|_{[0,\pi)\times[0,1]}$ describes an *immersed ruled surface with a conical singularity at* p_0.

By construction $\tilde{v}^\varepsilon(0) = -v$ and $\tilde{v}^\varepsilon(1) = v$, and thus the *total angle* at p_0 is bounded from below by π:

$$\varphi_0^\varepsilon := \int_0^1 \left|\tfrac{\partial}{\partial t}\tilde{v}^\varepsilon(t)\right| dt \ge \pi \quad . \tag{4}$$

For any $r \in [0,\pi]$ we denote by $\ell^\varepsilon(r)$ the length of the corresponding *circle of latitude*:

$$\ell^\varepsilon(r) := \int_0^1 \left|\tfrac{\partial}{\partial t}\gamma^\varepsilon(r,t)\right| dt \quad . \tag{5}$$

Since $0 < \delta \le K_M \le 1$, the standard Jacobi field estimates reveal that these length are related to the total angle φ_0^ε as follows:

$$\varphi_0^\varepsilon \cdot \sin(r) \le \ell^\varepsilon(r) \le \varphi_0^\varepsilon \cdot \mathrm{sn}_\delta(r) \quad , \tag{6}$$

where $\mathrm{sn}_\delta(r) := \frac{1}{\sqrt{\delta}}\cdot\sin(\sqrt{\delta}\, r)$ as usual.

Proposition 4.5. *Let $0 < \varepsilon \le \frac{1}{27\,000}$. Suppose that Assumption 4.4 holds, and consider the ruled surface γ^ε constructed above. Then there is a uniform lower bound for the length $\ell^\varepsilon(\frac{59}{120}\pi) = (1 + a_\varepsilon)\,\pi \cdot \sin(\frac{59}{120}\pi)$ given in terms of the inequality $a_\varepsilon \ge \frac{13}{8000}$.*

Proof. By construction the total angle φ_0^ε of the ruled surface $\gamma^\varepsilon\colon [0,\pi] \times [0,1] \to M^n$ and the length $\ell^\varepsilon(\pi)$ of the small circle segment at radius π are bounded from below as follows:

$$\varphi_0^\varepsilon \ge \pi \qquad \text{and} \qquad \ell^\varepsilon(\pi) \ge d(p_1, p_2) \ge \pi \quad .$$

Now the idea is to apply Theorem 5.4 with $\lambda := \delta \le \frac{1}{4}$, $\Lambda := 1$, $r_1 := \frac{59}{120}\pi$, and $r_2 := \pi$. Using the monotonicity properties from Lemma 5.3(iv), we conclude that

$$\ell^\varepsilon(r_1) \;\ge\; \Psi_{r_1 r_2}(\pi, \pi\,; \delta, 1) \;=\; \pi \cdot \psi_{r_1 r_2}(1\,; \delta, 1) \quad , \tag{7}$$

where $\psi_{r_1 r_2}$ and $\Psi_{r_1 r_2}$ are the comparison functions introduced via equations (20)–(23) in the next section. For clarity, we have extended the argument lists of these functions by the curvature bounds δ and 1. It remains to evaluate the function $h(\delta) := \sin(r_1)^{-1}\psi_{r_1 r_2}(1\,;\delta,1)$. We have to show that $h(\delta) \ge 1 + \frac{13}{8000} = 1.001\,625$ for $\frac{1}{4(1+\varepsilon)^2} \le \delta \le \frac{1}{4}$. Our plan is to compute $h(\frac{1}{4})$ explicitly and thereafter to proceed using the continuity and monotonicity of h.

When computing $h(\frac{1}{4})$, the parameters in equations (20)–(22) are $\lambda = \frac{1}{4}$ and $\Lambda = 1$, and the function $\bar{w}$ simplifies as follows:

$$\bar{w}(r_0, r) \;=\; \cos^2(\tfrac{1}{2}r_0) - 2\sin^2(\tfrac{1}{2}r_0) \cdot \big(1 - \tfrac{\sin(\frac{1}{2}r_0)}{\sin(\frac{1}{2}r)}\big) \quad . \tag{8}$$

Observe that $\sin(\frac{1}{2}r)^{-1}\sin(\frac{1}{2}r_0)$ lies in $[\sin(\frac{1}{2}r_0)\,, 1]$ for $0 \le r_0 \le r \le \pi$. It has been shown in Lemma 5.1(i) that $\bar{w} \ge 0$ on $[0,\pi] \times (0,\pi]$. Furthermore, the expression for the function $\bar{y}$ defined in (21) turns into

$$\bar{y}(r_0, r) \;=\; \begin{cases} \sin(r) & \text{if } r \le r_0 \;, \\ \sin(\frac{1}{2}(r + r_0)) \\ \quad + \cos(r_0)\sin(\frac{1}{2}(r - r_0)) & \text{if } r_0 \le r \text{ and } n = 2 \;, \\ 2\sin(\frac{1}{2}r) \cdot \bar{w}(r_0, r)^{1/2} & \text{if } r_0 \le r \text{ and } n > 2 \;. \end{cases} \tag{9}$$

By Lemma 5.2(iv) the equation $\bar{y}(\hat{r}_0, \pi) = 1$ has a unique solution $\hat{r}_0$ in $(0,\pi)$. With the preceding expressions for $\bar{w}$ and $\bar{y}$ we may rewrite this equation as follows:

$$\begin{cases} 1 = 2\cos^3(\frac{1}{2}\hat{r}_0) & \text{if } n = 2 \;, \\ \frac{1}{4} = \cos^2(\frac{1}{2}\hat{r}_0) - 2(1 - \sin(\frac{1}{2}\hat{r}_0)) \cdot \sin^2(\frac{1}{2}\hat{r}_0) & \text{if } n > 2 \;. \end{cases} \tag{10}$$

The solution $\hat{r}_0$ can be characterized by means of the equation

$$\begin{cases} \cos(\tfrac{1}{2}\hat{r}_0) = \frac{1}{\sqrt[3]{2}} & \text{if } n = 2 \,, \\ \sin(\tfrac{1}{2}\hat{r}_0) = \tfrac{1}{2} + \sin(\tfrac{1}{18}\pi) & \text{if } n > 2 \,, \end{cases} \tag{11}$$

and thus $\hat{r}_0 \leq 0.416\,304\,\pi$ if $n = 2$, and $\hat{r}_0 \leq 0.470\,548\,\pi$ if $n > 2$. In either case we have $\hat{r}_0 < r_1 = \frac{59}{120}\pi$. Using formula (22) and Lemma 5.2(iv), we conclude that $\psi_{r_1 r_2}(1\,;\delta,1) = \bar{y}(\hat{r}_0, r_1\,;\delta,1) > \sin(r_1)$, provided that δ is sufficiently close to $\frac{1}{4}$. With the help of equations (8) and (9) it is easy to compute that $h(\frac{1}{4}) \geq 1.020\,616$ if $n = 2$, and $h(\frac{1}{4}) \geq 1.001\,663$ if $n > 2$.

By Proposition 5.5 the map $\delta \mapsto h(\delta)$ is nondecreasing, and a numerical computation based on the original definition of $\psi_{r_1 r_2}$ via equations (20)–(22) reveals that for $\varepsilon_0 := \frac{1}{27\,000}$ we have:

$$h\big(\tfrac{1}{4(1+\varepsilon_0)^2}\big) \geq \begin{cases} 1.020\,612 > 1.001\,625 & \text{if } n = 2 \,, \\ 1.001\,661 > 1.001\,625 & \text{if } n > 2 \,. \end{cases}$$

The interesting feature of the preceding proposition is the factor $1 + a_\varepsilon$ in the expression on the right hand side. By our next result this little gain turns out to be a significant improvement over the standard estimate (6). For this purpose we consider for any radius $r_1 \in (0, \frac{3}{4}\pi)$ and any angle $\tilde{\varphi}_0 > 0$ the spherical ribbon

$$\Sigma(r_1, \tilde{\varphi}_0) := ([r_1\,, \tfrac{3}{4}\pi] \times [0, \tilde{\varphi}_0]\,, \bar{g}) \quad , \tag{12}$$

where $\bar{g} := dr^2 + \sin(r)^2 d\varphi^2$ is the standard metric of constant curvature 1.

Proposition 4.6.(A Weak Contraction) *Let* $0 < \varepsilon \leq \frac{1}{64}$ *and* $0 < r_1 < \frac{3}{4}\pi$. *Set* $\tilde{\varphi}^\varepsilon := \sin(r_1)^{-1} \cdot \ell^\varepsilon(r_1)$, *and consider the ruled surface* γ^ε *and the spherical ribbon* $\Sigma(r_1, \tilde{\varphi}^\varepsilon)$. *Then there exists a diffeomorphism* $\Phi^\varepsilon\colon [0,1] \to [0, \tilde{\varphi}^\varepsilon]$ *such that the map*

$$\mathrm{id} \times \Phi^\varepsilon\colon ([r_1\,, \tfrac{3}{4}\pi] \times [0,1]\,, (\gamma^\varepsilon)^* g) \to ([r_1\,, \tfrac{3}{4}\pi] \times [0, \tilde{\varphi}^\varepsilon]\,, \bar{g})$$

is nonexpanding.

Proof. The pull–back metric $(\gamma^\varepsilon)^* g$ is given by $dr^2 + |Y(r,\vartheta)|^2\, d\vartheta^2$ where $Y(r,\vartheta) := \frac{\partial}{\partial\vartheta}\gamma^\varepsilon(r,\vartheta)$. Hence it is sufficient to show that

$$\sin(r)^{-1} \cdot |Y(r,\vartheta)| \;\geq\; |\tfrac{\partial}{\partial\vartheta}\Phi^\varepsilon(\vartheta)| \tag{13}$$

for all $(r,\vartheta) \in [r_1\,, \frac{3}{4}\pi] \times [0,1]$. It follows from the definition of $\ell^\varepsilon(r_1)$ that the expression $\Phi^\varepsilon(\vartheta) := \int_0^\vartheta \sin(r_1)^{-1}|Y(r_1,t)|\,dt$ yields a bijective, differentiable map $\Phi^\varepsilon\colon [0,1] \to [0, \tilde{\varphi}^\varepsilon]$ such that inequality (13) holds on $\{r_1\} \times [0,1]$.

By Lemmas 4.1 and 4.3 the geodesic c^ε lies in the annulus $B(p_0, \frac{3}{4}\pi)\backslash B(p_0, \frac{7}{16}\pi)$, and thus its lift $\tilde{c}^\varepsilon$ cannot be radial. In particular, the vectors $\tilde{c}^\varepsilon(\vartheta)$ and $\frac{\partial}{\partial\vartheta}\tilde{c}^\varepsilon(\vartheta)$ are linearly independent for all $\vartheta \in [0,1]$, and the map Φ^ε defined above is a diffeomorphism.

Finally, we apply the infinitesimal version of the Rauch comparison theorem to conclude that the left hand side of (13) is a nondecreasing function of r.

Corollary 4.7. *Let $0 < \varepsilon \le \frac{1}{27\,000}$, and suppose that Assumption 4.4 holds. Then the length of the geodesic c^ε is bounded from below as follows:*

$$L(c^\varepsilon) \;\ge\; \left(1 + \tfrac{13}{16\,000}\sqrt{2} - \varepsilon^{2/3}\right)\pi \;>\; \pi(1+\varepsilon) \quad .$$

Proof of Theorem 2.4. The preceding corollary states that

$$\pi(1+\varepsilon) \;<\; L(c^\varepsilon) \;\le\; d(q_1^\varepsilon, q_2^\varepsilon) \;\le\; \operatorname{diam} M^n$$

for $0 \le \varepsilon \le \frac{1}{27\,000}$, which is the required contradiction because of the given upper bound for the diameter of M^n.

Proof of Corollary 4.7. We set $r_1 := \frac{59}{120}\pi$ and $\tilde{\varphi}^\varepsilon := \sin(r_1)^{-1} \cdot \ell^\varepsilon(r_1)$. By Proposition 4.5 $\tilde{\varphi}^\varepsilon = (1 + a_\varepsilon)\pi$ with $a_\varepsilon \ge \frac{13}{8000}$. We consider the spherical ribbon $\Sigma(r_1, \tilde{\varphi}^\varepsilon)$ introduced in (12). Combining the current bound for ε with Lemmas 4.1 and 4.3, we see that the geodesic c^ε is contained in the annulus $B(p_0, \frac{3}{4}\pi)\backslash B(p_0, r_1)$. Notice that the points $\tilde{q}_1 := (\frac{1}{2}(1 + \varrho_\varepsilon)\pi, 0)$ and $\tilde{q}_2 := (\frac{1}{2}(1 + \varrho_\varepsilon)\pi, \tilde{\varphi}^\varepsilon)$ are the images under $\mathrm{id} \times \Phi^\varepsilon$ of the end points $(\gamma^\varepsilon)^{-1}(c^\varepsilon(0))$ and $(\gamma^\varepsilon)^{-1}(c^\varepsilon(1))$. Hence Proposition 4.6 asserts that $L(c^\varepsilon) \ge \operatorname{dist}_{\Sigma(r_1, \tilde{\varphi}^\varepsilon)}(\tilde{q}_1, \tilde{q}_2)$.

Replacing a curve $c\colon t \mapsto (r(t), \varphi(t)) \in \Sigma(r_1, \tilde{\varphi}^\varepsilon)$ connecting the points $\tilde{q}_1$ and $\tilde{q}_2$ by the curve $t \mapsto (\max\{r(t), \pi - r(t)\}, \varphi(t))$, we conclude that

$$(14) \qquad L(c^\varepsilon) \;\ge\; \operatorname{dist}_{\Sigma(r_1, \tilde{\varphi}^\varepsilon)}(\tilde{q}_1, \tilde{q}_2) \;=\; \operatorname{dist}_{\Sigma(\frac{1}{2}\pi, \tilde{\varphi}^\varepsilon)}(\tilde{q}_1, \tilde{q}_2) \quad .$$

Thus the basic step in the proof of the corollary is to *determine a minimizing curve* $c\colon [0,1] \to \Sigma(\frac{1}{2}\pi, \tilde{\varphi}^\varepsilon)$ from $\tilde{q}_1$ to $\tilde{q}_2$. Since $\tilde{\varphi}^\varepsilon > \pi$, the ribbon $\Sigma(\frac{1}{2}\pi, \tilde{\varphi}^\varepsilon)$ does not contain a segment of a great circle connecting $\tilde{q}_1$ and $\tilde{q}_2$. Hence $c|_{(0,1)}$ meets the boundary of the ribbon.

The only nonconvex piece of this boundary is the arc that lies on the small circle $\{r = \frac{3}{4}\pi\}$. Moreover, the radial projection of $\Sigma(\frac{1}{2}\pi, \tilde{\varphi}^\varepsilon)$ onto this arc is a weak contraction. We conclude that any minimizing curve c from $\tilde{q}_1$ to $\tilde{q}_2$ consists of three arcs $\tilde{q}_1\tilde{x}_1$, $\tilde{x}_1\tilde{x}_2$, and $\tilde{x}_2\tilde{q}_2$. The arc $\tilde{x}_1\tilde{x}_2$ lies on the small circle $\{r = \frac{3}{4}\pi\}$, and $\tilde{q}_1\tilde{x}_1$ and $\tilde{x}_2\tilde{q}_2$ are segments of great circles which touch the small circle tangentially. In particular, c is uniquely determined, and $L(\tilde{q}_1\tilde{x}_1) = L(\tilde{x}_2\tilde{q}_2)$.

It remains to estimate the length $L(c) = 2\,L(\tilde{q}_1\tilde{x}_1)+L(\tilde{x}_1\tilde{x}_2)$ from below. Notice that the great circle through $\tilde{x}_1$ and $\tilde{q}_1$ intersects the equator $\{r = \frac{1}{2}\pi\}$ in some point $(\frac{1}{2}\pi, -\tilde{\xi}_0)$ at an angle of $\frac{\pi}{4}$. We consider the triangle with vertices $(\frac{1}{2}\pi, -\tilde{\xi}_0)$, $(\frac{1}{2}\pi, 0)$, and $\tilde{q}_1 = (\frac{1}{2}(1+\varrho_\varepsilon)\pi, 0)$. Observe that the segment from $(\frac{1}{2}\pi, 0)$ to $\tilde{q}_1$ has length $\frac{1}{2}\varrho_\varepsilon\pi$. Let $\tilde{\xi}_1$ be the length of the segment from $\tilde{q}_1$ to $(\frac{1}{2}\pi, -\tilde{\xi}_0)$. An elementary calculation based on the Laws of Sines and Cosines shows that

$$\sin(\tilde{\xi}_0) = \tan(\tfrac{1}{2}\varrho_\varepsilon\pi) \qquad \text{and} \qquad \sin(\tilde{\xi}_1) = \sqrt{2}\cdot\sin(\tfrac{1}{2}\varrho_\varepsilon\pi) \quad , \tag{15}$$

where ϱ_ε is the number that appears in the definition of the points q_1^ε and q_2^ε. Clearly, $L(\tilde{q}_1\tilde{x}_1) = \frac{1}{2}\pi - \tilde{\xi}_1$. Furthermore, $\tilde{x}_1 = (\frac{3}{4}\pi, \frac{1}{2}\pi - \tilde{\xi}_0)$, $\tilde{x}_2 = (\frac{3}{4}\pi, \tilde{\varphi}^\varepsilon - \frac{1}{2}\pi + \tilde{\xi}_0)$, and thus $L(\tilde{x}_1\tilde{x}_2) = \frac{1}{\sqrt{2}}(\tilde{\varphi}^\varepsilon - \pi + 2\tilde{\xi}_0) = \frac{1}{\sqrt{2}}(a_\varepsilon\pi + 2\tilde{\xi}_0)$. Hence

$$\begin{aligned} L(c^\varepsilon) &\geq \operatorname{dist}_{\Sigma(\frac{1}{2}\pi, \tilde{\varphi}^\varepsilon)}(\tilde{q}_1\,, \tilde{q}_2) \\ &= \pi + \tfrac{1}{\sqrt{2}}\,a_\varepsilon\pi - \left(2\tilde{\xi}_1 - \sqrt{2}\,\tilde{\xi}_0\right) \\ &\geq \left(1 + \tfrac{13}{16\,000}\sqrt{2} - \varepsilon^{2/3}\right)\pi \quad . \end{aligned}$$

Here we have used Proposition 4.5 and the equations in (15), in order to control the terms $\tilde{\varphi}^\varepsilon - \pi$ and $2\tilde{\xi}_1 - \sqrt{2}\,\tilde{\xi}_0$, respectively.

5. Estimates for ruled surfaces with a conical singularity

The estimates to be presented in this section provide a new tool that might be useful in other contexts as well. So we find it appropriate to work with a generic Riemannian manifold M^n and just assume that its sectional curvature K_M is bounded from above and below, i. e., $\lambda \leq K_M \leq \Lambda$. We shall consider *ruled surfaces* in M^n which are given as differentiable maps

$$\gamma\colon [0, r_2] \times [0,1] \to M^n \tag{16}$$

such that each curve $\gamma_\vartheta = \gamma(.,\vartheta)$ is a normal geodesic emanating from some fixed point $p_0 \in M^n$, i. e.,

$$\tfrac{\nabla}{\partial r}\tfrac{\partial\gamma}{\partial r} \equiv 0\ , \quad \left|\tfrac{\partial\gamma}{\partial r}\right| \equiv 1\ , \quad \text{and} \quad \gamma(0,\vartheta) = p_0 \quad .$$

In general, r_2 can be any positive number. However, if $\Lambda > 0$, we suppose that $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$. For any $r \in (0, r_2]$ we consider the length

$$(17) \qquad \ell(r) := \int_0^1 \left| \tfrac{\partial \gamma}{\partial \vartheta}(r, \vartheta) \right| d\varphi$$

of the corresponding small circle arc. By the infinitesimal version of the Rauch comparison theorem the functions $r \mapsto \mathrm{sn}_\lambda(r)^{-1} \left| \frac{\partial \gamma}{\partial \vartheta}(r, \vartheta) \right|$ are nonincreasing, and thus $\ell(r_1) \geq \frac{\mathrm{sn}_\lambda(r_1)}{\mathrm{sn}_\lambda(r_2)} \ell(r_2)$ for $0 < r_1 \leq r_2$. The preceding *lower bound for* $\ell(r_1)$ can be improved by means of the total angle

$$(18) \qquad \varphi_0 := \int_0^1 \left| \tfrac{\nabla}{\partial \vartheta} \tfrac{\partial \gamma}{\partial r}(0, \vartheta) \right| d\vartheta$$

at the conical singularity p_0 as follows:

$$(19) \qquad \ell(r_1) \geq \max\{\varphi_0 \cdot \mathrm{sn}_\Lambda(r_1)\,,\, \tfrac{\mathrm{sn}_\lambda(r_1)}{\mathrm{sn}_\lambda(r_2)} \cdot \ell(r_2)\} \quad .$$

We are mainly interested in the case that $\varphi_0 \cdot \mathrm{sn}_\Lambda(r_2) < \ell(r_2) < \varphi_0 \cdot \mathrm{sn}_\lambda(r_2)$. This condition means in particular that $\lambda < \Lambda$, and the right hand side of (19), when considered as a function of r_1, has a corner at some point $r_c \in (0, r_2)$. The *principal goal* in this section is to improve inequality (19) for all r_1 in some interval $(r_0, r_2) \subset (0, r_2)$ containing r_c.

To begin with, we introduce geometrically better adapted comparison functions. For this purpose we restrict to the case $\lambda < \Lambda$ and consider the real analytic maps $\bar{w}\colon [0, r_2] \times (0, r_2] \to \mathbb{R}$ given by

$$(20)\, \bar{w}(r_0, r) := \frac{\mathrm{sn}_\Lambda^2(r_0)}{\mathrm{sn}_\lambda^2(r_0)} - 2 \cdot \det \begin{pmatrix} \mathrm{cn}_\lambda(r_0) & \mathrm{cn}_\Lambda(r_0) \\ \mathrm{sn}_\lambda(r_0) & \mathrm{sn}_\Lambda(r_0) \end{pmatrix} \cdot \int_{r_0}^{r} \frac{\mathrm{sn}_\Lambda(\varrho)}{\mathrm{sn}_\lambda^3(\varrho)}\, d\varrho \quad .$$

Notice that the determinant factor has a zero of third order at $r_0 = 0$, and thus $\bar{w}(0, r) = 1$ for all $r \in (0, r_2]$. In fact, $\bar{w} \to 1$ as $\lambda \to \Lambda$.

Lemma 5.1. *Let* $\lambda < \Lambda$ *and* $r_2 > 0$. *Suppose that* $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$ *if* $\Lambda > 0$. *Then the function* $\bar{w}$ *introduced in* (20) *has the following properties:*

(i) *the expression* $\mathrm{sn}_\lambda(r)\bar{w}(r_0, r)$ *extends as a real analytic function to the closure* $[0, r_2] \times [0, r_2]$ *of the domain of* $\bar{w}$, *and*

$$\lim_{r \to 0} \mathrm{sn}_\lambda(r)\bar{w}(r_0, r) = 2 \cdot \det \begin{pmatrix} \mathrm{cn}_\lambda(r_0) & \mathrm{cn}_\Lambda(r_0) \\ \mathrm{sn}_\lambda(r_0) & \mathrm{sn}_\Lambda(r_0) \end{pmatrix} \geq 0 \quad ,$$

(ii) $\bar{w}$ *is nonnegative, and* $\bar{w}(r_0, r) = 0$ *if and only if* $\Lambda > 0$ *and* $r_0 = r = \frac{\pi}{\sqrt{\Lambda}}$,

(iii) *if* $\Lambda > 0$ *and* $r_2 = \frac{\pi}{\sqrt{\Lambda}}$, *then the first couple of terms in the Taylor expansion*

$$\bar{w}(r_0, r) = \sum a_{\mu\nu} \cdot \left(r_0 - \tfrac{\pi}{\sqrt{\Lambda}}\right)^{\mu} \left(r - \tfrac{\pi}{\sqrt{\Lambda}}\right)^{\nu}$$

are given by $a_{00} = a_{10} = a_{20} = a_{30} = 0$ *and* $a_{\mu 1} = 0$ *for all* $\mu \geq 0$, *whereas* a_{40} *and* a_{02} *are positive. More precisely,* $a_{40} = \frac{1}{4}(\Lambda - \lambda) a_{02}$ *and* $a_{02} = \mathrm{sn}_\lambda^{-2}(\frac{\pi}{\sqrt{\Lambda}})$.

The proof of this lemma will be given in §5.1. With the help of $\bar{w}$ we can now introduce a continuous map $\bar{y}\colon [0, r_2] \times [0, r_2] \to [0, \infty)$ by means of

$$(21)\quad \bar{y}(r_0, r) := \begin{cases} \mathrm{sn}_\Lambda(r) & \text{if } r \leq r_0 , \\ \mathrm{sn}_\Lambda(r_0)\,\mathrm{cn}_\lambda(r - r_0) \\ \qquad + \mathrm{cn}_\Lambda(r_0)\,\mathrm{sn}_\lambda(r - r_0) & \text{if } r_0 \leq r \text{ and } n = 2 , \\ \mathrm{sn}_\lambda(r) \cdot \bar{w}(r_0, r)^{1/2} & \text{if } r_0 \leq r \text{ and } n > 2 . \end{cases}$$

Set $\mathcal{Z}_{\bar{y}} := \{(0,0), (r_2, r_2)\}$ if $\Lambda > 0$ and $r_2 = \frac{\pi}{\sqrt{\Lambda}}$, and $\mathcal{Z}_{\bar{y}} := \{(0,0)\}$ otherwise. Notice that the restriction of $\bar{y}$ to $[0, r_2] \times [0, r_2] \setminus \mathcal{Z}_{\bar{y}}$ is piecewise analytic.

Lemma 5.2. *Let* $\lambda < \Lambda$ *and* $r_2 > 0$. *Suppose that* $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$ *if* $\Lambda > 0$. *Then the function* $\bar{y}$ *defined in* (21) *has the following properties:*

(i) *the function* $\bar{y}\colon [0, r_2] \times [0, r_2] \to [0, \infty)$ *is of class* $C^{1,1}$,

(ii) $\frac{\partial}{\partial r_0} \bar{y}(r_0, r) < 0 \quad$ *if* $0 < r_0 < r \leq r_2$,
$\frac{\partial}{\partial r_0} \bar{y}(r_0, r) = 0 \quad$ *if* $r_0 = 0$ *or* $r_0 = r$,

(iii) $\frac{\partial^2}{\partial r_0^2} \bar{y}(0, r) = -(\Lambda - \lambda)\,\mathrm{sn}_\lambda(r) \quad$ *for* $0 < r \leq r_2$,

(iv) *for any* $r \in (0, r_2]$ *the restriction of* $\bar{y}(\,.\,, r)$ *to the interval* $[0, r]$ *is a surjective, strictly decreasing, real analytic map* $\bar{\bar{y}}_r\colon [0, r] \xrightarrow{\approx} [\mathrm{sn}_\Lambda(r), \mathrm{sn}_\lambda(r)]$.

The proof of this result is also postponed to §5.1. The lemma implies in particular that for $\lambda < \Lambda$ and any $r_1 \in (0, r_2]$ there exists a unique, surjective, nondecreasing $C^{1,1}$–map $\psi_{r_1 r_2}\colon [\mathrm{sn}_\Lambda(r_2), \mathrm{sn}_\lambda(r_2)] \to [\mathrm{sn}_\Lambda(r_1), \mathrm{sn}_\lambda(r_1)]$ such that

$$(22)\qquad \psi_{r_1 r_2} \circ \bar{y}(\,.\,, r_2) = \bar{y}(\,.\,, r_1) \quad .$$

The complete setup for the definition of $\psi_{r_1 r_2}$ is shown in Figure 2.

Observe that $\psi_{r_1 r_2}$ maps the subinterval $[\mathrm{sn}_\Lambda(r_2), \bar{y}(r_1, r_2)]$ to the single point $\mathrm{sn}_\Lambda(r_1)$, whereas $\psi'_{r_1 r_2} > 0$ on the subinterval $(\bar{y}(r_1, r_2), \mathrm{sn}_\lambda(r_2)]$. The restriction of $\psi_{r_1 r_2}$ to the punctured interval $[\mathrm{sn}_\Lambda(r_2), \mathrm{sn}_\lambda(r_2)] \setminus \{\bar{y}(r_1, r_2)\}$ is real analytic, and $\psi'_{r_1 r_2}(\mathrm{sn}_\lambda(r_2)) =$

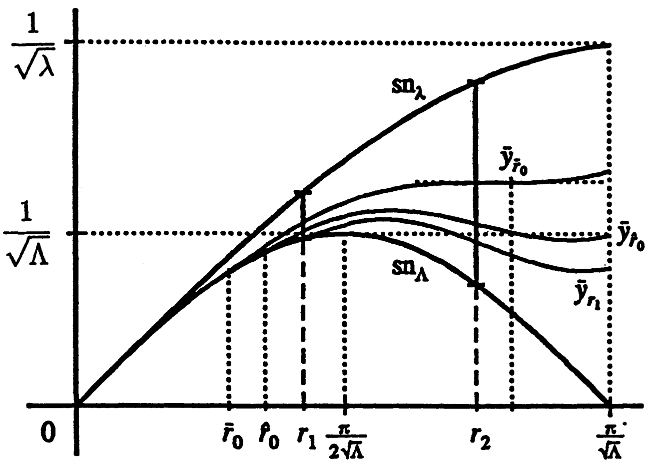

FIGURE 2. On the Map $\psi_{r_1 r_2}$: The picture shows the qualitative properties of the functions sn_λ, sn_Λ, and $\bar{y}_{r_0}$ for fixed parameters $0.186\Lambda < \lambda < \frac{1}{4}\Lambda$ and for various values of r_0. The graphs of the functions $\bar{y}_{r_0}$ are schematic; they are justified by Proposition A.6. Note, however, that in reality $\bar{r}_0 \in (0.452\, r_\Lambda, \frac{1}{2} r_\Lambda)$ and $\hat{r}_0 \in (0.470\, r_\Lambda, \frac{1}{2} r_\Lambda)$ where $r_\Lambda := \frac{\pi}{\sqrt{\Lambda}}$. For further details we refer to Corollary A.7.

$\frac{\mathrm{sn}_\lambda(r_1)}{\mathrm{sn}_\lambda(r_2)}$. In the limit $r_1 \to r_2$ the functions $\psi_{r_1r_2}$ converge to the identity map.

Next we define the comparison functions $\Psi_{r_1r_2}\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ that will replace the right hand side of (19). We suppose that $\lambda \leq \Lambda$ and $0 < r_1 \leq r_2$, where $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$ if $\Lambda > 0$, and set:

(23)

$$\Psi_{r_1r_2}(\alpha,\eta) := \begin{cases} \alpha \cdot \mathrm{sn}_\Lambda(r_1) & \text{if } \eta \leq \alpha \cdot \mathrm{sn}_\Lambda(r_2)\ , \\ \alpha \cdot \psi_{r_1r_2}\left(\frac{\eta}{\alpha}\right) & \text{if } \alpha \cdot \mathrm{sn}_\Lambda(r_2) < \eta \leq \alpha \cdot \mathrm{sn}_\lambda(r_2)\ , \\ \frac{\mathrm{sn}_\lambda(r_1)}{\mathrm{sn}_\lambda(r_2)} \cdot \eta & \text{if } \alpha \cdot \mathrm{sn}_\lambda(r_2) \leq \eta\ . \end{cases}$$

The strict inequality in the condition on the middle line implies that in this case $\alpha > 0$ and $\lambda < \Lambda$, and thus the term $\alpha \cdot \psi_{r_1r_2}(\alpha^{-1}\eta)$ is well–defined. In the case $\lambda = \Lambda$ the definition of $\Psi_{r_1r_2}$ actually reduces to the first and third line. Since $\psi_{r_1r_2}(\mathrm{sn}_\Lambda(r_2)) = \mathrm{sn}_\Lambda(r_1)$, it is clear that $\Psi_{r_1r_2}$ is continuous.

Lemma 5.3. *Let $\lambda < \Lambda$ and $0 < r_1 \leq r_2$, where $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$ if $\Lambda > 0$. Then the function $\Psi_{r_1r_2}\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ introduced in* (23) *has the following properties:*

(i) *$\Psi_{r_1r_2}$ is linear on each line through the origin,*
(ii) *$\Psi_{r_1r_2}$ is continuously differentiable except at $(\alpha,\eta) = (0,0)$,*
(iii) *$\Psi_{r_1r_2}$ is weakly convex,*
(iv) *$\Psi_{r_1r_2}$ is nondecreasing with respect to both variables,*
(v) *$\Psi_{r_1r_2}(\alpha,\eta) \geq \max\{\alpha \cdot \mathrm{sn}_\Lambda(r_1)\, ,\, \frac{\mathrm{sn}_\lambda(r_1)}{\mathrm{sn}_\lambda(r_2)} \cdot \eta\}$ for all (α,η).*

This lemma will also be established in §5.1, where the basic analytical properties of the functions $\bar{w}$, $\bar{y}$, $\psi_{r_1r_2}$, and $\Psi_{r_1r_2}$ are investigated in detail. Notice that — despite of the significant difference in the explicit expression for $\bar{y}$ in (21) — the essential qualitative properties of the functions $\bar{y}$, $\psi_{r_1r_2}$, and $\Psi_{r_1r_2}$ are independent of the dimension n of the Riemannian manifold.

Theorem 5.4. *Let M^n be a complete Riemannian manifold with $\lambda \leq K_M \leq \Lambda$, and let $0 < r_1 \leq r_2$, where $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$ if $\Lambda > 0$. Consider a ruled surface $\gamma\colon [0,r_2] \times [0,1] \to M^n$ generated by normal geodesics $\gamma_\vartheta = \gamma(.,\vartheta)$ emanating from a fixed point $p_0 \in M^n$. Then the lengths $\ell(r_1)$, $\ell(r_2)$ and the total angle φ_0 introduced in* (17) *and* (18) *obey the following inequality:*

$$\ell(r_1) \;\geq\; \Psi_{r_1r_2}(\varphi_0\,, \ell(r_2)) \quad .$$

In the proof of Proposition 4.5 it has already become apparent in which sense this theorem is a *substantial improvement* over inequality (19). Of course, Theorem 5.4 is the integrated version of a corresponding estimate for Jacobi fields which we shall state in Theorem 5.6. Although these two theorems provide subtle curvature controlled estimates, they are by nature not comparison theorems in the strong sense. In particular, in dimensions $n > 2$ there are except for trivial cases no appropriate model spaces on which the estimates turn out to be sharp. This will be explained further in the addendum to Theorem 5.4.

Nevertheless, the comparison functions $\Psi_{r_1 r_2}$ show the geometrically expected monotonicity with respect to the curvature bounds λ and Λ.

Proposition 5.5. *Let $0 < r_1 \leq r_2$. Then the functions $\Psi_{r_1 r_2}$ introduced via equations (20)–(23) are nondecreasing with respect to the parameter λ and nonincreasing with respect to Λ, as long as $\lambda \leq \Lambda \leq r_2^{-2}\pi^2$.*

This proposition will be established in the appendix.

Theorem 5.6. *Let M^n be a complete Riemannian manifold with $\lambda \leq K_M \leq \Lambda$, and let $0 < r_1 \leq r_2$, where $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$ if $\Lambda > 0$. Consider a geodesic $\gamma\colon [0, r_2] \to M^n$ parametrized by arc length and a normal Jacobi field $Y\colon [0, r_2] \to TM$ along γ with initial value $Y(0) = 0$. Then the following inequality holds:*

$$|Y(r_1)| \geq \Psi_{r_1 r_2}(|\tfrac{\nabla}{\partial r} Y(0)|, |Y(r_2)|) \quad .$$

We consider this result as a mixed Jacobi field estimate, since the initial value $Y(0) = 0$ is augmented by two additional pieces of information, knowledge of the absolute value of the initial derivative $\frac{\nabla}{\partial r} Y(0)$ and knowledge of the absolute value $|Y(r_2)|$ at a second boundary point r_2, in order to obtain a refined lower bound for the absolute value $|Y(r_1)|$ at some intermediate point r_1.

Addendum to Theorem 5.6. *Consider the case $n > 2$, and suppose that $r_1 < r_2$. Then $\Psi_{r_1 r_2}(|\frac{\nabla}{\partial r} Y(0)|, |Y(r_2)|) = |Y(r_1)|$ if and only if the restriction of the Jacobi field Y to the interval $[0, r_2]$ is of the form $\mathrm{sn}_\lambda \cdot W$ or $\mathrm{sn}_\Lambda \cdot W$ where W is a parallel vector field along γ.*

The proof of Theorem 5.6 is given in §§5.1–5.3. The cases $n = 2$ and $n > 2$ are substantially different, since for surfaces the Jacobi field equation reduces to a scalar differential equation of second order rather than to a system of second order differential equations. The analytical aspects of the argument are contained in Proposition 5.10, whereas the geometric ingredients are covered by Proposition 5.12. The Addendum follows immediately from the equality discussion in Remark 5.14.

Proof of Theorem 5.4. Consider the geodesics γ_ϑ which generate the ruled surface γ and apply Theorem 5.6 to the normal Jacobi fields $Y_\vartheta := \frac{\partial\gamma}{\partial\vartheta}(.\,,\vartheta)$ along these geodesics to obtain the inequality

$$|Y_\vartheta(r_1,\vartheta)| \geq \Psi_{r_1r_2}\big(|\tfrac{\nabla}{\partial r}Y_\vartheta(0,\vartheta)|\,,\,|Y_\vartheta(r_2,\vartheta)|\big) \quad .$$

By Lemma 5.3(iii) the function $\Psi_{r_1r_2}\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ is weakly convex. Hence we can apply Jensen's inequality to conclude that

$$\begin{aligned}
\ell(r_1) &\geq \int_0^1 \Psi_{r_1r_2}\big(|\tfrac{\nabla}{\partial r}Y_\vartheta(0,\vartheta)|\,,\,|Y_\vartheta(r_2,\vartheta)|\big)\,d\vartheta \\
&\geq \Psi_{r_1r_2}\Big(\int_0^1 |\tfrac{\nabla}{\partial r}Y_\vartheta(0,\vartheta)|\,d\vartheta\,,\,\int_0^1 |Y_\vartheta(r_2,\vartheta)|\,d\vartheta\Big) \\
&= \Psi_{r_1r_2}(\varphi_0\,,\ell(r_2)) \quad .
\end{aligned}$$

5.1. Basic analytical properties of the comparison functions

The purpose of this subsection is to establish the basic analytical properties of all four functions $\bar{w}$, $\bar{y}$, $\psi_{r_1r_2}$, and $\Psi_{r_1r_2}$. In particular, we want to prove Lemmas 5.1–5.3. In the next subsection we shall characterize the maps $\bar{y}_{r_0} := \bar{y}(r_0,.)$ by means of differential equations of second order, and in §5.3 we shall use these differential equations to link the geometrical properties of the ruled surface $\gamma\colon [0,r_2]\times[0,1] \to M^n$ to the comparison functions $\Psi_{r_1r_2}$.

Proof of Lemma 5.1. i) First we observe that the terms $\mathrm{sn}_\lambda^{-2}(r_0)\,\mathrm{sn}_\Lambda^2(r_0)$ and $D(r_0) := \det\begin{pmatrix}\mathrm{cn}_\lambda(r_0) & \mathrm{cn}_\Lambda(r_0)\\ \mathrm{sn}_\lambda(r_0) & \mathrm{sn}_\Lambda(r_0)\end{pmatrix}$ on the right hand side of equation (20) are bounded, real analytic functions on the interval $[0,r_2]$. Moreover, $D(0) = 0$. Since $D'(r_0) = (\Lambda-\lambda)\,\mathrm{sn}_\lambda(r_0)\,\mathrm{sn}_\Lambda(r_0)$ for any $r_0 \in [0,r_2]$, we see that D is a positive, strictly increasing function on the interval $[0,r_2]$ with a zero of third order at $r_0 = 0$. Note that $D'''(0) = 2(\Lambda-\lambda)$.

In order to understand the integral $I(r_0,r) := \int_{r_0}^r \mathrm{sn}_\lambda^{-3}(\varrho)\,\mathrm{sn}_\Lambda(\varrho)\,d\varrho$, we make use of the fact that $\mathrm{sn}_\lambda^{-3}(\varrho)\,(\mathrm{sn}_\lambda(\varrho)-\mathrm{sn}_\Lambda(\varrho))$ is a positive, even, real analytic function on $[-r_2,r_2]$. In particular, the expression

$$\hat{I}(r_0,r) := \int_{r_0}^r \mathrm{sn}_\lambda^{-3}(\varrho)\cdot(\mathrm{sn}_\lambda(\varrho)-\mathrm{sn}_\Lambda(\varrho))\,d\varrho$$

defines a real analytic function on the closed square $[0,r_2]\times[0,r_2]$ which vanishes on the diagonal $\{r_0 = r\}$. Furthermore, $I(r_0,r) = \mathrm{ct}_\lambda(r_0) - \mathrm{ct}_\lambda(r) - \hat{I}(r_0,r)$.

The properties of the functions D and I established above imply that the expression $\mathrm{sn}_\lambda(r)\bar{w}(r_0,r)$ extends analytically to the closed square, and an easy computation shows that

$$\lim_{r\to 0} \mathrm{sn}_\lambda(r)\bar{w}(r_0,r) = \lim_{r\to 0} -2D(r_0)\,\mathrm{sn}_\lambda(r)\cdot(\mathrm{ct}_\lambda(r_0)-\mathrm{ct}_\lambda(r)) = 2D(r_0) \quad .$$

ii) We continue to denote the determinant and the integral in the expression for $\bar{w}$ by D and I, and compute the partial derivatives of $\bar{w}$:

(24) $$\tfrac{\partial}{\partial r_0}\bar{w}(r_0,r) = -2(\Lambda-\lambda)\,\mathrm{sn}_\lambda(r_0)\,\mathrm{sn}_\Lambda(r_0)\cdot I(r_0,r) \quad ,$$

(25) $$\tfrac{\partial}{\partial r}\bar{w}(r_0,r) = -2D(r_0)\cdot \mathrm{sn}_\lambda^{-3}(r)\,\mathrm{sn}_\Lambda(r) \quad .$$

Clearly, $\frac{\partial}{\partial r}\bar{w} \le 0$ on the entire domain $[0,r_2]\times(0,r_2]$. Recall that the function $\hat{I}(r_0,r) = \mathrm{ct}_\lambda(r_0) - \mathrm{ct}_\lambda(r) - I(r_0,r)$ is real analytic on $[0,r_2]\times[0,r_2]$. Hence

(26) $$\tfrac{\partial}{\partial r_0}\bar{w}(0,r) = 0 \qquad \text{if } 0 < r \le r_2 \ ,$$

(27) $$\tfrac{\partial}{\partial r_0}\bar{w}(r_0,r) < 0 \qquad \text{if } 0 < r_0 < r \le r_2 \ .$$

Since $\bar{w}(0,r) = 1$ and $\bar{w}(r,r) = \mathrm{sn}_\lambda^{-2}(r)\,\mathrm{sn}_\Lambda^2(r)$ for any $0 < r \le r_2$, we conclude that $0 \le \bar{w}(r,r) < \bar{w}(r_0,r) \le \bar{w}(r_0,r_0) < 1$ provided that $0 < r_0 < r \le r_2$.

iii) Equation (25) asserts that $\frac{\partial}{\partial r}\bar{w}(r_0,r)|_{r=\pi/\sqrt{\Lambda}} = 0$ for $0 \le r_0 \le r_2$, and thus all the coefficients $a_{\mu 1}$ vanish. The other five coefficients are obtained by differentiating equations (24) and (25) a few more times.

Proof of Lemma 5.2. i) Recall that the restriction of $\bar{y}$ to $[0,r_2]\times[0,r_2]\backslash\mathcal{Z}_{\bar{y}}$ is piecewise analytic. Its partial derivatives are given by

(28)

$$\tfrac{\partial}{\partial r_0}\bar{y}(r_0,r) = \begin{cases} 0 & \text{if } r \le r_0 \ , \\ -(\Lambda-\lambda)\,\mathrm{sn}_\Lambda(r_0)\,\mathrm{sn}_\lambda(r-r_0) & \text{if } r_0 \le r \text{ and } n = 2 \ , \\ \mathrm{sn}_\lambda(r)\bar{w}(r_0,r)^{-1/2}\,\tfrac{1}{2}\tfrac{\partial}{\partial r_0}\bar{w}(r_0,r) & \text{if } r_0 \le r \text{ and } n > 2 \ , \end{cases}$$

and

(29)

$$\tfrac{\partial}{\partial r}\bar{y}(r_0,r) = \begin{cases} \mathrm{cn}_\Lambda(r) & \text{if } r \le r_0 \ , \\ \mathrm{cn}_\Lambda(r_0)\,\mathrm{cn}_\lambda(r-r_0) & \text{if } r_0 \le r \\ -\lambda\,\mathrm{sn}_\Lambda(r_0)\,\mathrm{sn}_\lambda(r-r_0) & \quad\text{and } n = 2 \ , \\ \mathrm{cn}_\lambda(r)\bar{w}(r_0,r)^{1/2} & \text{if } r_0 \le r \\ +\,\mathrm{sn}_\lambda(r)\bar{w}(r_0,r)^{-1/2}\,\tfrac{1}{2}\tfrac{\partial}{\partial r}\bar{w}(r_0,r) & \quad\text{and } n > 2 \ . \end{cases}$$

If $n = 2$, it is evident that the preceding expressions are consistent along the diagonal $\{r_0 = r\}$, and thus $\bar{y}$ is a $C^{1,1}$–function on the closed square $[0, r_2] \times [0, r_2]$.

If $n > 2$, it follows from equations (24) and (25) that the preceding expressions are still consistent along the diagonal. In this case, however, the consistency in (28) and (29) only implies that the restriction of $\bar{y}$ to $[0, r_2] \times [0, r_2] \backslash \mathcal{Z}_{\bar{y}}$ is locally of class $C^{1,1}$. It remains to analyze the behavior of $\bar{y}$ in a neighborhood of $\mathcal{Z}_{\bar{y}}$.

In order to investigate the regularity of $\bar{y}$ in a neighborhood of the point $(0,0)$, we need to show that the second derivatives of $\bar{y}$ are uniformly bounded on a sufficiently small triangle $\Delta_0(\zeta) := \{0 < r_0 < r < \zeta\}$. For this purpose we employ Lemma 5.1(i) to write $\bar{y}(r_0, r)^2 = \mathrm{sn}_\lambda(r)^2 \bar{w}(r_0, r)$ as $2rD(r_0) + r^2 h(r_0, r)$, where h is a suitable, real analytic function on the closed square $[0, r_2] \times [0, r_2]$. Since $\bar{w}(0, r) = 1$ for $0 < r \le r_2$, we conclude that $h(0,0) = 1$. If $\zeta > 0$ is sufficiently small, it follows that $h|_{\Delta_0(\zeta)} \ge \frac{1}{2}$. Therefore the quotients $\frac{r}{\bar{y}(r_0,r)}$, $\frac{D(r_0)}{\bar{y}(r_0,r)}$, $\frac{D(r_0)}{\bar{y}(r_0,r)^2}$, and $\frac{D'(r_0)}{\bar{y}(r_0,r)}$ are uniformly bounded on the triangle $\Delta_0(\zeta)$, and an elementary computation yields:

$$
\begin{aligned}
\frac{\partial^2}{\partial r_0^2}\,\bar{y} &= \frac{r}{\bar{y}}\left(D'' + \frac{r}{2}\frac{\partial^2 h}{\partial r_0^2}\right) - \frac{r^2}{\bar{y}}\left(\frac{D'}{\bar{y}} + \frac{r}{2\bar{y}}\frac{\partial h}{\partial r_0}\right)^2 \\
\frac{\partial^2}{\partial r_0 \partial r}\,\bar{y} &= \frac{D'}{\bar{y}} + \frac{r}{\bar{y}}\left(\frac{\partial h}{\partial r_0} + \frac{r}{2}\frac{\partial^2 h}{\partial r_0 \partial r}\right) - \frac{r}{\bar{y}}\left(\frac{D'}{\bar{y}} + \frac{r}{2\bar{y}}\frac{\partial h}{\partial r_0}\right)\left(\frac{D}{\bar{y}} + \frac{r}{\bar{y}}\,h + \frac{r^2}{2\bar{y}}\frac{\partial h}{\partial r}\right) \\
\frac{\partial^2}{\partial r^2}\,\bar{y} &= \frac{r^3}{\bar{y}^3}\left(h\,\frac{\partial h}{\partial r} + h\,\frac{r}{2}\frac{\partial^2 h}{\partial r^2} - \frac{r}{4}\frac{\partial h}{\partial r}\frac{\partial h}{\partial r} + D\,\frac{\partial^2 h}{\partial r^2}\right) + 3\,\frac{r^2}{\bar{y}^2}\frac{D}{\bar{y}}\frac{\partial h}{\partial r} - \frac{D^2}{\bar{y}^3} \quad .
\end{aligned}
$$

As explained before all terms on the right hand sides are uniformly bounded on the triangle $\Delta_0(\zeta)$, and thus we are finished, unless $\Lambda > 0$ and $r_2 = \frac{\pi}{\sqrt{\Lambda}}$.

If $\Lambda > 0$ and $r_2 = \frac{\pi}{\sqrt{\Lambda}}$, we use a similar argument to analyze the behavior of $\bar{y}$ in a neighborhood of the point (r_2, r_2). In this case we refer to Lemma 5.1(iii) to conclude that

$$
\bar{w}\left(\tfrac{\pi}{\sqrt{\Lambda}} - t_0\,,\, \tfrac{\pi}{\sqrt{\Lambda}} - t\right) \;=\; t_0^4\, h_0(t_0) + t^2\, h(t_0, t) \quad ,
$$

where h_0 and h are real analytic functions on the closed interval $[0, r_2]$ and the closed rectangle $[0, r_2] \times [0, \frac{1}{2} r_2]$, respectively. Moreover, $h_0(0) = a_{40} > 0$ and $h(0,0) = a_{02} > 0$. A similar argument as in the preceding case shows that the second derivatives of the square root of $t_0^4\, h_0(t_0) + t^2 h(t_0, t)$ are uniformly bounded on a sufficiently small triangle $\Delta_1(\zeta) := \{0 < t < t_0 < \zeta\}$.

ii) For $(r_0, r) \notin \mathcal{Z}_{\bar{y}}$ the assertion follows directly from (24) and (26)–(28). For $(r_0, r) \in \mathcal{Z}_{\bar{y}}$ we can then refer to the established continuity of $\frac{\partial}{\partial r_0}\bar{y}$.

iii) If $n = 2$, we merely need to differentiate equation (28) once more. In the case $n > 2$ we differentiate (28) and, simplifying the resulting expression with the help of (26), we see that $\frac{\partial^2}{\partial r_0^2}\bar{y}(0,r) = \mathrm{sn}_\lambda(r)\,\frac{1}{2}\,\frac{\partial^2}{\partial r_0^2}\bar{w}(0,r)$ for $0 < r \leq r_2$. In order to evaluate the right–hand side, we differentiate equation (24) once more. Thus we obtain

$$\begin{aligned}\frac{\partial^2}{\partial r_0^2}\bar{w}(r_0,r) &= 2(\Lambda-\lambda)\,\mathrm{sn}_\lambda^{-2}(r_0)\,\mathrm{sn}_\Lambda^2(r_0)\\ &\quad -2(\Lambda-\lambda)(\mathrm{cn}_\lambda(r_0)\,\mathrm{sn}_\Lambda(r_0)+\mathrm{sn}_\lambda(r_0)\,\mathrm{cn}_\Lambda(r_0))\cdot I(r_0,r) \quad .\end{aligned}$$

Since $I(r_0,r) = \mathrm{ct}_\lambda(r_0) - \mathrm{ct}_\lambda(r) - \hat{I}(r_0,r)$ where $\hat{I}$ is a real analytic function on the closed square, we eventually conclude that $\frac{\partial^2}{\partial r_0^2}\bar{w}(0,r) = -2(\Lambda-\lambda)$ for $0 < r \leq r_2$.

iv) The regularity of the partial maps $\bar{\bar{y}}_r \colon [0,r] \to [\mathrm{sn}_\Lambda(r), \mathrm{sn}_\lambda(r)]$ follows directly from formula (21), unless $n > 2$, $\Lambda > 0$, and $r = r_2 = \frac{\pi}{\sqrt{\Lambda}}$. In the latter case we have to refer in addition to Lemma 5.1(iii) in order to establish the analyticity of $\bar{\bar{y}}_r$ on the entire domain $[0,r]$ and not just on $[0,r)$. By assertion (ii) the partial maps $\bar{\bar{y}}_r$ are strictly decreasing. Thus for any $r \in (0,r_2]$ the range of $\bar{\bar{y}}_r$ is the interval $[\bar{\bar{y}}_r(r)\,,\bar{\bar{y}}_r(0)] = [\mathrm{sn}_\Lambda(r)\,,\mathrm{sn}_\lambda(r)]$.

Lemma 5.7. *Let $\lambda < \Lambda$ and $0 < r_1 < r_2$ where $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$ if $\Lambda > 0$. Then the $C^{1,1}$-function $\psi_{r_1 r_2} \colon [\mathrm{sn}_\Lambda(r_2)\,,\mathrm{sn}_\lambda(r_2)] \to [\mathrm{sn}_\Lambda(r_1)\,,\mathrm{sn}_\lambda(r_1)]$ introduced in* (22) *has the following properties:*

(i) $\psi'_{r_1 r_2}(\mathrm{sn}_\Lambda(r_2)) = 0$ *and* $\psi'_{r_1 r_2}(\mathrm{sn}_\lambda(r_2)) = \frac{\mathrm{sn}_\lambda(r_1)}{\mathrm{sn}_\lambda(r_2)}$,

(ii) $\psi_{r_1 r_2}$ *is convex and strictly convex on* $[\bar{y}(r_1,r_2)\,,\mathrm{sn}_\lambda(r_2)]$,

(iii) $\psi_{r_1 r_2}(\eta) \geq \max\{\mathrm{sn}_\Lambda(r_1)\,, \frac{\mathrm{sn}_\lambda(r_1)}{\mathrm{sn}_\lambda(r_2)}\cdot\eta\}$ *for* $\eta \in [\mathrm{sn}_\Lambda(r_2)\,,\mathrm{sn}_\lambda(r_2)]$.

Proof. i) Both values are computed from Lemma 5.2(ii) and 5.2(iii) by means of the chain rule.

ii) By Lemma 5.2(iv) the function $\psi_{r_1 r_2}$ is of class $C^{1,1}$ and its restriction to the interval $[\mathrm{sn}_\Lambda(r_2)\,,\bar{y}(r_1,r_2)]$ is constant. Thus it is sufficient to verify that $\psi''_{r_1 r_2}(\eta) > 0$ for all $\eta \in (\bar{y}(r_1,r_2)\,,\mathrm{sn}_\lambda(r_2)) = \{\bar{y}(r_0,r_2) \mid 0 < r_0 < r_1\}$. Since $\frac{\partial}{\partial r_0}\bar{y}(.\,,r_2) < 0$ on $(0,r_1)$, it is equivalent to show that

$$(30)\quad \frac{\partial}{\partial r_0}(\psi'_{r_1 r_2}\circ\bar{y}(r_0,r_2)) \;\equiv\; (\psi''_{r_1 r_2}\circ\bar{y}(r_0,r_2))\cdot\frac{\partial}{\partial r_0}\bar{y}(r_0,r_2) \;<\; 0$$

for any $r_0 \in (0,r_1)$.

In order to compute the term $\psi'_{r_1 r_2}\circ\bar{y}(r_0,r_2)$, we differentiate equation (22) with respect to r_0. By the chain rule $\frac{\partial}{\partial r_0}\bar{y}(r_0,r_1) = (\psi'_{r_1 r_2}\circ\bar{y}(r_0,r_2))\cdot\frac{\partial}{\partial r_0}\bar{y}(r_0,r_2)$, and combining this identity with formulas (24) and (28), we conclude that

$$(31)\qquad \psi'_{r_1r_2}\circ\bar{y}(r_0,r_2) \;=\; \begin{cases} \frac{\operatorname{sn}_\lambda(r_1-r_0)}{\operatorname{sn}_\lambda(r_2-r_0)} & \text{if } n=2\ ,\\ \frac{\operatorname{sn}_\lambda(r_1)}{\operatorname{sn}_\lambda(r_2)}\sqrt{\frac{\bar{w}(r_0,r_2)}{\bar{w}(r_0,r_1)}}\cdot\frac{I(r_0,r_1)}{I(r_0,r_2)} & \text{if } n>2\ ,\end{cases}$$

where $I(r_0,r) = \int_{r_0}^{r}\operatorname{sn}_\lambda^{-3}(\varrho)\operatorname{sn}_\Lambda(\varrho)\,d\varrho$ as before. Since $\psi'_{r_1r_2}\circ\bar{y}(r_0,r_2) > 0$, it is now sufficient to verify that

$$h_{r_1r_2}(r_0) \;:=\; \tfrac{\partial}{\partial r_0}(\ln\circ\psi'_{r_1r_2}\circ\bar{y}(r_0,r_2)) \;<\; 0$$

for any $r_0\in(0,r_1)$.

If $n=2$, we observe that $h_{r_1r_2}(r_0) = \operatorname{ct}_\lambda(r_2-r_0)-\operatorname{ct}_\lambda(r_1-r_0)$, which is indeed strictly negative. The case $n>2$, however, requires a slightly more elaborate computation. Equations (24) and (31) reveal that

$$\begin{aligned} h_{r_1r_2}(r_0) \;=\; &-(\Lambda-\lambda)\cdot\operatorname{sn}_\lambda(r_0)\operatorname{sn}_\Lambda(r_0)\cdot\left(\tfrac{I(r_0,r_2)}{\bar{w}(r_0,r_2)}-\tfrac{I(r_0,r_1)}{\bar{w}(r_0,r_1)}\right)\\ &-\operatorname{sn}_\lambda^{-3}(r_0)\operatorname{sn}_\Lambda(r_0)\cdot\left(\tfrac{1}{I(r_0,r_1)}-\tfrac{1}{I(r_0,r_2)}\right)\quad. \end{aligned}$$

We want to show that the right hand side is strictly negative for $0<r_0<r_1<r_2$. For this purpose we may consider the two terms in parentheses separately. In fact, it follows directly from the definitions that

$$\begin{aligned} \tfrac{I(r_0,r_2)}{\bar{w}(r_0,r_2)}-\tfrac{I(r_0,r_1)}{\bar{w}(r_0,r_1)} \;&=\; \operatorname{sn}_\lambda^{-2}(r_0)\operatorname{sn}_\Lambda^{2}(r_0)\cdot\tfrac{I(r_1,r_2)}{\bar{w}(r_0,r_1)\,\bar{w}(r_0,r_2)} \;>\; 0\quad,\\ \tfrac{1}{I(r_0,r_1)}-\tfrac{1}{I(r_0,r_2)} \;&=\; \tfrac{I(r_1,r_2)}{I(r_0,r_1)\,I(r_0,r_2)} \;>\; 0\quad. \end{aligned}$$

iii) Since $\psi_{r_1r_2}(\operatorname{sn}_\Lambda(r_2)) = \operatorname{sn}_\Lambda(r_1)$ and $\psi_{r_1r_2}(\operatorname{sn}_\lambda(r_2)) = \operatorname{sn}_\lambda(r_1)$, the assertion follows directly from (i) and (ii).

Proof of Lemma 5.3. i) By equation (23) it is evident that $\Psi_{r_1r_2}$ is positively homogeneous of degree 1.

ii) Combine the fact that $\psi_{r_1r_2}$ is of class $C^{1,1}$ with Lemma 5.7(i) to conclude that $\Psi_{r_1r_2}|_{\{1\}\times[0,\infty)}$ is a $C^{1,1}$–function. Because of Property (i) we conclude that $\Psi_{r_1r_2}\in C^1((0,\infty)\times[0,\infty))$. By definition the restriction of $\Psi_{r_1r_2}$ to the cone $\{\alpha\operatorname{sn}_\lambda(r_2)\le\eta\}$ is linear, and thus $\Psi_{r_1r_2}$ is smooth in a neighborhood $(0,\infty)\times\{0\}$ as well.

iii) The function $\Psi_{r_1r_2}$ is linear on the cones $\{\eta\le\alpha\bar{y}(r_1,r_2)\}$ and $\{\alpha\operatorname{sn}_\lambda(r_2)\le\eta\}$. Thus it is sufficient to verify that $\Psi_{r_1r_2}$ is weakly convex on the open cone $C:=\{(\alpha,\eta)\mid\alpha\bar{y}(r_1,r_2)<\eta<\alpha\operatorname{sn}_\lambda(r_2)\}$. Clearly, $\Psi_{r_1r_2}|_C$ is real analytic, and it is straightforward to compute that

$$\operatorname{hess}\Psi_{r_1r_2}|_{(\alpha,\eta)} \;=\; \tfrac{1}{\alpha^3}\,\psi''_{r_1r_2}\left(\tfrac{\eta}{\alpha}\right)\cdot\begin{pmatrix}\eta^2 & -\alpha\eta\\ -\alpha\eta & \alpha^2\end{pmatrix}$$

for all $(\alpha,\eta) \in \mathcal{C}$. It has been shown in Lemma 5.7(ii) that $\psi''_{r_1r_2}(\alpha^{-1}\eta) > 0$ on $\mathcal{C}$. Observe that $\mathbb{R}\cdot(\alpha,\eta) \subset \ker(\operatorname{hess}\Psi_{r_1r_2}|_{(\alpha,\eta)})$, and thus $\operatorname{hess}\Psi_{r_1r_2}|_{(\alpha,\eta)} \geq 0$.

iv) It is evident from the definition of $\Psi_{r_1r_2}$ in (23) that $\frac{\partial}{\partial\eta}\Psi_{r_1r_2}(\alpha,0) = 0$ for any $\alpha > 0$, and similarly $\frac{\partial}{\partial\alpha}\Psi_{r_1r_2}(0,\eta) = 0$ for any $\eta > 0$. Hence the monotonicity of the function $\Psi_{r_1r_2}$ with respect to both variables is a consequence of the convexity established in (iii).[2]

v) This estimate is an immediate consequence of Lemma 5.7(iii).

5.2. Differential equations for $\bar{y}$ and basic comparison results

In this subsection we explain the analytical properties of the function $\bar{y}$ that are necessary for establishing the *link to the geometry* of the ruled surface $\gamma\colon [0,r_2]\times[0,1] \to (M^n,g)$. The issue of the whole discussion is to prepare the proof of Theorem 5.6 at the end of §5.3. Notice that in the case $\lambda = \Lambda$ the theorem merely summarizes the standard Jacobi field estimates. It improves on inequality (19) only for $\lambda < \Lambda$. Hence we shall exclude the constant curvature case for the subsequent discussion.

Note that the functions $\bar{y}_{r_0} = \bar{y}(r_0,.)$ are of class $C^{1,1}$. In fact they are smooth unless $r = r_0$. Evidently for any $r_0 \in (0,r_2)$, the restriction of $\bar{y}_{r_0}$ to the interval $(0,r_0)$ is a solution of the differential equation $\frac{\partial^2}{\partial r^2}z + \Lambda\cdot z = 0$ with initial data $\bar{y}(0) = 0$ and $\frac{\partial}{\partial r}\bar{y}(0) = 1$.

Our first goal is to characterize the restriction of $\bar{y}_{r_0}$ to $(r_0,r_2) \subset (0,r_2)$ by means of a linear differential equation. Recall that $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$ if $\Lambda > 0$, and consider the differential operator $L_n\colon C^2((r_0,r_2)) \to C^0((r_0,r_2))$ given by

$$(32)\qquad L_n(z) := \begin{cases} \frac{\partial^2}{\partial r^2}z + \lambda z & \text{if } n = 2\,, \\ \frac{\partial^2}{\partial r^2}z + (\operatorname{ct}_\lambda - \operatorname{ct}_\Lambda)\frac{\partial}{\partial r}z & \\ \qquad + (\lambda - \operatorname{ct}_\lambda\cdot(\operatorname{ct}_\lambda - \operatorname{ct}_\Lambda))\,z & \text{if } n > 2\,. \end{cases}$$

Notice that $\operatorname{ct}_\lambda - \operatorname{ct}_\Lambda$ is a real analytic function on $[0,r_2)$ with a simple zero at $r = 0$.

Lemma 5.8. *Let $\lambda < \Lambda$ and $r_2 > 0$. Suppose in addition that $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$ if $\Lambda > 0$. Then the kernel of the differential operators L_n introduced in* (32) *consists of the functions*

$$z_{a_1a_2}(r) := \begin{cases} a_1\operatorname{sn}_\lambda(r) + a_2\operatorname{cn}_\lambda(r) & \textit{if } n = 2\,, \\ \operatorname{sn}_\lambda(r)\cdot\left(a_1 + a_2\int_r^{r_2}\operatorname{sn}_\lambda^{-3}(\varrho)\operatorname{sn}_\Lambda(\varrho)\,d\varrho\right) & \textit{if } n > 2\,, \end{cases}$$

[2]Of course, the monotonicity of $\Psi_{r_1r_2}$ with respect to the variable η is also a direct consequence of the monotonicity of the function $\psi_{r_1r_2}$.

where a_1 and a_2 are the constants of integration.

In particular, for any $r_0 \in [0, r_2)$ the function $\bar{y}_{r_0} = \bar{y}(r_0, .)$ introduced in (21) solves the differential equations

$$(33) \qquad \begin{cases} \frac{\partial^2}{\partial r^2} \bar{y}_{r_0} + \Lambda \bar{y}_{r_0} = 0 & \text{on } (0, r_0) \ , \\ L_2(\bar{y}_{r_0}) = 0 & \text{on } (r_0, r_2) \ \text{if } n = 2 \ , \\ L_n(\frac{1}{\mathrm{sn}_\lambda} \bar{y}^2_{r_0}) = 0 & \text{on } (r_0, r_2) \ \text{if } n > 2 \ . \end{cases}$$

Proof of the Lemma. It is straightforward to verify that $L_n(z_{a_1 a_2}) = 0$. In the case $n > 2$ it saves some work to observe that the general solution $z_{a_1 a_2}$ is obtained from the special solution $z_{1,0} = \mathrm{sn}_\lambda$ by means of the standard Wronskian trick.

Lemma 5.9.(Maximum Principle) *Let $\lambda < \Lambda$ and $r_2 > 0$. Suppose in addition that $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$ if $\Lambda > 0$, and consider the differential operators L_n introduced in* (32)*. Furthermore, let $r_0 \in (0, r_2)$ and let $z \in C^0([r_0, r_2]) \cap C^2((r_0, r_2))$ such that $z(r_0) \geq 0$, $z(r_2) \geq 0$ and $L_n(z) \leq 0$ on (r_0, r_2). Then $z \geq 0$ on the entire interval $[r_0, r_2]$.*

Proof. Notice that the function sn_λ is a solution of the linear differential equation $L_n(z) = 0$ on (r_0, r_2), and $\mathrm{sn}_\lambda > 0$ on the closed interval $[r_0, r_2]$. Hence the lemma follows as explained at the beginning of Section 2 in [31, Chap. 1].

Our main application is the following comparison result in terms of the functions $\psi_{r_1 r_2}$ introduced in (22), which in turn is crucial for our proof of Theorem 5.6 in §5.3.

Proposition 5.10. *Let $\lambda < \Lambda$ and $r_2 > 0$, and suppose that $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$ if $\Lambda > 0$. Furthermore, let $y \in C^0([0, r_2]) \cap C^2((0, r_2))$ such that $\mathrm{sn}_\Lambda \leq y \leq \mathrm{sn}_\lambda$. Consider the differential operators L_n introduced in* (32)*, and suppose that $L_2(y) \leq 0$ if $n = 2$ and $L_n(\frac{1}{\mathrm{sn}_\lambda} y^2) \leq 0$ if $n > 2$. Then $\psi_{r_1 r_2} \circ y(r_2) \leq y(r_1)$ for any $r_1 \in (0, r_2]$.*

Proof. Since $y(r_2) \in [\mathrm{sn}_\Lambda(r_2), \mathrm{sn}_\lambda(r_2)]$, it follows from Lemma 5.2(iv) that there exists a unique $r_0 \in [0, r_2]$ such that $\bar{y}_{r_0}(r_2) = y(r_2)$. By definition $\psi_{r_1 r_2} \circ \bar{y}_{r_0}(r_2)) = \bar{y}_{r_0}(r_1)$, and thus we have to show that $\bar{y}_{r_0}(r) \leq y(r)$ for $r \in (0, r_2)$. Observe that $\bar{y}_{r_0}(r) = \mathrm{sn}_\Lambda(r)$ for $0 \leq r \leq r_0$, and thus it remains to verify that $\bar{y}_{r_0}(r) \leq y(r)$ for $r_0 < r < r_2$.

If $n = 2$ we finish the proof by applying the maximum principle from Lemma 5.9 to the function $z := y - \bar{y}_{r_0}$.

If $n > 2$ we observe that $y \geq 0$ and $\bar{y}_{r_0} \geq 0$ on $[r_0, r_2]$, and therefore it is sufficient to show that

$$z := \frac{1}{\mathrm{sn}_\lambda} y^2 - \frac{1}{\mathrm{sn}_\lambda} \bar{y}^2_{r_0} \geq 0$$

on (r_0, r_2). Clearly, $z(r_0) \geq 0$ and $z(r_2) = 0$, and thus we can again apply the maximum principle from Lemma 5.9 to finish the proof.

The following observation proves to be useful for some arguments in the appendix:

Remark 5.11. Lemma 5.9 and Proposition 5.10 remain valid for the functions z and y that are merely continuous, provided that we work with upper barriers in order to make sense of the differential inequalities. In particular, if we are dealing with functions $z \in C^0([r_0, r_2]) \cap C^1((r_0, r_2))$ and $y \in C^0([0, r_2]) \cap C^1((0, r_2))$ that are piecewise of class C^2, it is sufficient to verify the differential inequalities at those points r in the open interval where the second derivatives are continuous.

To conclude this subsection we observe that the linear differential equations of second order for the functions $\bar{y}_{r_0}$ given in (33) can be transformed into differential equations of Riccati type for the functions $\bar{u}_{r_0} := \bar{y}_{r_0}^{-1} \frac{\partial}{\partial r} \bar{y}_{r_0}$:

(34)
$$\frac{\partial}{\partial r}\bar{u}_{r_0} = \begin{cases} -\Lambda - \bar{u}_{r_0}^2 & \text{on } (0, r_0)\ , \\ -\lambda - \bar{u}_{r_0}^2 & \text{on } (r_0, r_2) \text{ if } n = 2\ , \\ -\lambda - \mathrm{ct}_\lambda \, \mathrm{ct}_\Lambda + (\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda)\bar{u}_{r_0} - 2\bar{u}_{r_0}^2 & \text{on } (r_0, r_2) \text{ if } n > 2\ . \end{cases}$$

Notice that the functions $\bar{u}_{r_0}$ are the partial maps of a piecewise analytic, locally Lipschitz continuous function $\bar{u}\colon [0, r_2) \times (0, r_2] \to \mathbb{R}$. The differential inequalities corresponding to (34) will appear naturally in the discussion of the geometry of the ruled surface $\gamma\colon [0, r_2] \times [0, 1] \to M^n$ in Proposition 5.12. Further applications of the differential equations (34) can be found in the appendix.

5.3. Mixed Jacobi field estimates

The purpose of this subsection is to prove Theorem 5.6. We assume that (M^n, g) is a Riemannian manifold with $\lambda \leq K_M \leq \Lambda$. As in the theorem it will be sufficient to consider just a single normal geodesic $\gamma\colon [0, r_2] \to M^n$ rather than the entire family $(\gamma_\vartheta)_{0 \leq \vartheta \leq 1}$ that constitutes the ruled surface. Recall that we are assuming $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$ if $\Lambda > 0$.

We consider a normal Jacobi field Y along γ which has initial data $Y(0) = 0$ and $|\frac{\nabla}{\partial r} Y(0)| = 1$. Up to a constant factor these are precisely the data that arise from the ruled surface. The Rauch comparison theorem asserts that $\mathrm{sn}_\Lambda \leq |Y| \leq \mathrm{sn}_\lambda$. Therefore $y := |Y|$ is a strictly positive and hence smooth function on $(0, r_2)$, which extends as a C^1–function to the closed interval $[0, r_2]$. Its initial data are $y(0) = 0$ and $\frac{\partial}{\partial r}\, y(0) = |\frac{\nabla}{\partial r} Y(0)| = 1$.

The unit vector field $E := \frac{1}{y} Y$ along $\gamma|_{(0, r_2)}$ extends continuously to the closed interval $[0, r_2]$, and $E(0) = \frac{\nabla}{\partial r} Y(0)$. The essential ingredient in the proof of Theorem 5.6 is a bound for the angular velocity $|\frac{\nabla}{\partial r} E|$ of the unit vector field $E \perp \frac{\partial \gamma}{\partial r}$. Notice that in the 2–dimensional case E is a parallel vector field along γ. However, for higher dimensional manifolds $|\frac{\nabla}{\partial r} E|$ may be nontrivial. This difference in the qualitative behavior of E is the cause for having two distinct comparison functions $\Psi_{r_1 r_2}$ in the cases $n = 2$ and $n > 2$.

Proposition 5.12. *Let M^n be a complete Riemannian manifold with $\lambda \leq K_M \leq \Lambda$, and let $\gamma\colon [0, r_2] \to M^n$ be a geodesic parametrized by arc length. Consider a normal Jacobi field $Y\colon [0, r_2] \to TM$ along γ such that $Y(0) = 0$ and $|\frac{\nabla}{\partial r} Y| = 1$. Suppose that $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$ if $\Lambda > 0$. Then on $(0, r_2)$ the functions $y := |Y|$, $u := \frac{1}{y} \frac{\partial}{\partial r} y$, and the unit vector field $E = \frac{1}{y} Y$ obey the following inequalities:*

$$
\begin{aligned}
|\tfrac{\nabla}{\partial r} E|^2 &\leq \tfrac{1}{4} (\mathrm{ct}_\lambda - \mathrm{ct}_\Lambda)^2 - (u - \tfrac{1}{2}(\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda))^2 \\
&= -\mathrm{ct}_\lambda \, \mathrm{ct}_\Lambda + (\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda)\, u - u^2 \quad ,
\end{aligned} \tag{35}
$$

$$
-\Lambda - u^2 \leq \tfrac{\partial}{\partial r} u \leq \begin{cases} -\lambda - u^2 & \text{if } n = 2 \, , \\ -\lambda - \mathrm{ct}_\lambda \, \mathrm{ct}_\Lambda \\ \qquad + (\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda)\, u - 2u^2 & \text{if } n > 2 \, . \end{cases} \tag{36}
$$

Remarks 5.13. i) Since the left hand side of (35) is nonnegative, we recover the inequality $\mathrm{ct}_\Lambda \leq u \leq \mathrm{ct}_\lambda$, which is just another way of stating the infinitesimal version of the Rauch comparison theorem. In the 2–dimensional case the function u can be interpreted as the curvature of concentric spheres around $\gamma(0)$, and formula (36) is just the standard inequality that appears in the comparison theorem for the second fundamental forms.

ii) Formula (36) consists of the differential inequalities corresponding to the Riccati equations in (34). Thus the functions $\bar{u}_{r_0} = \frac{1}{\bar{y}_{r_0}} \frac{\partial}{\partial r} \bar{y}_{r_0}$ introduced before are comparison functions for $u = \frac{1}{y} \frac{\partial}{\partial r} y$.

iii) The right hand side in (35) vanishes for $\bar{u}_0 = \mathrm{ct}_\lambda$, and therefore this particular function solves *both* differential equations,

$$
\tfrac{\partial}{\partial r} u = -\lambda - u^2 \qquad \text{and} \qquad \tfrac{\partial}{\partial r} u = -\lambda - \mathrm{ct}_\lambda \, \mathrm{ct}_\Lambda + (\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda)\, u - 2u^2 \quad .
$$

Since $\bar{u}_0 = \frac{1}{\mathrm{sn}_\lambda} \frac{\partial}{\partial r} \mathrm{sn}_\lambda$, we see that $z_{1,0} = \mathrm{sn}_\lambda \in \ker L_n$, where L_n denotes the linear differential operator introduced in (32).

Addendum to Proposition 5.12. *Let $n > 2$ and $r_1 \in (0, r_2)$. Then the following statements are equivalent:*

(i) *The norm of $\frac{\nabla}{\partial r} E|_{r_1}$ coincides with the upper bound given by* (35).

(ii) *There exist parallel vector fields* E_λ *and* E_Λ *along* γ *such that on* $[0, r_1]$ *the following holds:*
  - *the Jacobi field* Y *is of the form* $\mathrm{sn}_\lambda \cdot E_\lambda + \mathrm{sn}_\Lambda \cdot E_\Lambda$*, and*
  - $R(E_\lambda, \frac{\partial\gamma}{\partial r})\frac{\partial\gamma}{\partial r} = \lambda\, E_\lambda$ *and* $R(E_\Lambda, \frac{\partial\gamma}{\partial r})\frac{\partial\gamma}{\partial r} = \Lambda\, E_\Lambda$.

(iii) *In formula* (35) *equality holds on the entire interval* $(0, r_1]$.

The Addendum has strong implications for the equality discussion in (36). Inspecting the final step in the proof of the latter inequality, we obtain

Remark 5.14. In addition to the assumptions of Proposition 5.12 we suppose that $n > 2$ and $\lambda < \Lambda$, and let $r_1 \in (0, r_2)$. Then the upper bound for $\frac{\partial}{\partial r}u$ given in (36) is sharp at r_1, if and only if the Jacobi field $Y|_{[0,r_1]}$ is of the form $\mathrm{sn}_\lambda \cdot E$, where E is a parallel unit vector field along γ.

Proof of Proposition 5.12. In order to establish the bound for $|\frac{\nabla}{\partial r}E|$ we consider a local distance function d_{p_0} to the point $p_0 := \gamma(0)$ along the geodesic γ. Thus $A|_r := \mathrm{Hess}\, d_{p_0}|_{\gamma(r)}$ represents the second fundamental form of the sphere of radius r around p_0 along γ. Because of the initial condition $Y(0) = 0$ the following differential equations hold:

$$\tfrac{\nabla}{\partial r}Y = A \cdot Y \qquad \text{and} \qquad \tfrac{\nabla}{\partial r}A + A^2 + R(\,.\,, \tfrac{\partial\gamma}{\partial r})\tfrac{\partial\gamma}{\partial r} = 0 \quad .$$

The standard comparison argument for the Riccati equation shows

$$\mathrm{ct}_\Lambda \cdot P \le A \le \mathrm{ct}_\lambda \cdot P \quad , \tag{37}$$

where $P = \mathrm{Id} - \langle ., \frac{\partial\gamma}{\partial r}\rangle \frac{\partial\gamma}{\partial r}$ denotes the orthogonal projector onto the normal bundle of γ. Differentiating the identity $E = \frac{1}{y}Y$, we obtain

$$\tfrac{\nabla}{\partial r}E \;=\; \tfrac{1}{y}\,(\tfrac{\nabla}{\partial r}Y - uY) \;=\; A \cdot E - uE \quad ,$$

and hence

$$\tfrac{\nabla}{\partial r}E + \big(u - \tfrac{1}{2}(\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda)\big) \cdot E \;=\; \big(A - \tfrac{1}{2}(\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda)P\big) \cdot E \quad . \tag{38}$$

Because of (37) the right hand side is bounded by $\frac{1}{2}(\mathrm{ct}_\lambda - \mathrm{ct}_\Lambda)$. Moreover, the fields E and $\frac{\nabla}{\partial r}E$ are always perpendicular, and hence inequality (35) follows by taking the norm on both sides of (38).

For the proof of inequality (36) we observe that

$$\begin{aligned}\tfrac{1}{2}\,\tfrac{\partial^2}{\partial r^2}(y^2) \;&=\; \big|\tfrac{\nabla}{\partial r}Y\big|^2 + \langle Y\,, \tfrac{\nabla^2}{\partial r^2}Y\rangle \\ &=\; \big|(\tfrac{\partial}{\partial r}y)E + y\,\tfrac{\nabla}{\partial r}E\big|^2 - \langle R(Y, \tfrac{\partial\gamma}{\partial r})\tfrac{\partial\gamma}{\partial r}\,, Y\rangle \quad .\end{aligned}$$

Recall that E and $\frac{\nabla}{\partial r}E$ are perpendicular. Thus we conclude that

$$\tfrac{\partial}{\partial r}\,u + 2u^2 \;=\; \tfrac{1}{2y^2}\,\tfrac{\partial^2}{\partial r^2}(y^2) \;=\; u^2 + \bigl|\tfrac{\nabla}{\partial r}E\bigr|^2 - K_M\bigl(\operatorname{span}\{E,\tfrac{\partial\gamma}{\partial r}\}\bigr) \quad .$$

Hence

$$0 \;=\; \tfrac{\partial}{\partial r}u + u^2 + K_M\bigl(\operatorname{span}\{E,\tfrac{\partial\gamma}{\partial r}\}\bigr) - \bigl|\tfrac{\nabla}{\partial r}E\bigr|^2 \quad , \tag{39}$$

and the inequalities in (36) follow observing the hypothesis $\lambda \le K_M \le \Lambda$, the bound on $|\frac{\nabla}{\partial r}E|$ in the case $n > 2$, and the fact that $\frac{\nabla}{\partial r}E$ vanishes for $n = 2$. q.e.d.

For a *geometric interpretation* of formula (39) we think of γ as one of the geodesics γ_ϑ, $0 \le \vartheta \le 1$, of an immersed ruled surface Σ with a conical singularity. Its second fundamental form is

$$2y\bigl(\tfrac{\nabla}{\partial r}E\bigr)\;dr\,d\vartheta + \bigl(\tfrac{\nabla}{\partial r}\tfrac{\partial\gamma}{\partial\vartheta}\bigr)^{\perp}\,d\vartheta^2 \quad ,$$

and by the Gauss equations the intrinsic curvature K_Σ of the surface turns out to be

$$K_\Sigma \;=\; K_M\bigl(\operatorname{span}\{E,\tfrac{\partial\gamma}{\partial r}\}\bigr) - \bigl|\tfrac{\nabla}{\partial r}E\bigr|^2 \quad .$$

Comparing terms, (39) can be rewritten as $\frac{\partial}{\partial r}\,u + u^2 + K_\Sigma$, which is the usual Riccati equation for the curvature u of concentric spheres around $\gamma(0)$ in Σ. Of course, this interpretation provides an independent approach to equation (39).

Proof of the Addendum to Proposition 5.12.

(i) $\Rightarrow$ (ii): Analyzing the step from (38) to inequality (35), we deduce from (i) that

$$\bigl|(A - \tfrac{1}{2}(\mathrm{ct}_\lambda{+}\mathrm{ct}_\Lambda)P)\cdot E\bigr|_{|r=r_1} \;=\; \tfrac{1}{2}\bigl(\mathrm{ct}_\lambda(r_1){-}\mathrm{ct}_\Lambda(r_1)\bigr) \quad .$$

Thus $E(r_1)$ can be decomposed as $E(r_1) = E_\lambda(r_1) + E_\Lambda(r_1)$, where E_λ and E_Λ are parallel vector fields along γ such that

$$A|_{r_1}\cdot E_\lambda(r_1) = \mathrm{ct}_\lambda(r_1)\cdot E_\lambda(r_1) \quad ,$$
$$A|_{r_1}\cdot E_\Lambda(r_1) = \mathrm{ct}_\Lambda(r_1)\cdot E_\Lambda(r_1) \quad .$$

The equality discussion for the Riccati comparison argument leading to (37) reveals that on the entire interval $(0, r_1]$ we have

$$A\cdot E_\lambda = \mathrm{ct}_\lambda\cdot E_\lambda \qquad \text{and} \qquad R(E_\lambda,\tfrac{\partial\gamma}{\partial r})\tfrac{\partial\gamma}{\partial r} \;=\; \lambda\,E_\lambda \ ,$$
$$A\cdot E_\Lambda = \mathrm{ct}_\Lambda\cdot E_\Lambda \qquad \text{and} \qquad R(E_\Lambda,\tfrac{\partial\gamma}{\partial r})\tfrac{\partial\gamma}{\partial r} \;=\; \Lambda\,E_\Lambda \ .$$

(ii) $\Rightarrow$ (iii): Clearly, any Jacobi field Y of the form $\mathrm{sn}_\lambda \cdot E_\lambda + \mathrm{sn}_\Lambda \cdot E_\Lambda$ satisfies

$$\left|\tfrac{\nabla}{\partial r} Y - \tfrac{1}{2}(\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda) \cdot Y\right| = \tfrac{1}{2}(\mathrm{ct}_\lambda - \mathrm{ct}_\Lambda) \cdot |Y| \quad .$$

We conclude that

$$\left|(A - \tfrac{1}{2}(\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda)P) \cdot E\right| = \tfrac{1}{2}(\mathrm{ct}_\lambda - \mathrm{ct}_\Lambda)$$

for $E = \frac{1}{y} Y$, and thus (iii) follows with the help of formula (38).
(iii) $\Rightarrow$ (i): Evident.

Proof of Theorem 5.6. The Rauch comparison theorems assert that

$$|Y(r_1)| \geq \max\{|\tfrac{\nabla}{\partial r} Y(0)| \cdot \mathrm{sn}_\Lambda(r_1) \, , \, \tfrac{\mathrm{sn}_\lambda(r_1)}{\mathrm{sn}_\lambda(r_2)} \cdot |Y(r_2)|\} \quad .$$

Comparing the right hand side to the definition of $\Psi_{r_1 r_2}$ in (23), we see that it is sufficient to consider the case where $\lambda < \Lambda$ and

$$0 < |\tfrac{\nabla}{\partial r} Y(0)| \cdot \mathrm{sn}_\Lambda(r_2) \leq |Y(r_2)| \leq |\tfrac{\nabla}{\partial r} Y(0)| \cdot \mathrm{sn}_\lambda(r_2) \quad .$$

In the remaining case we employ the homogeneity of $\Psi_{r_1 r_2}$ as stated in Lemma 5.3(i) to scale the Jacobi field Y. Hence it is sufficient to consider normal Jacobi fields Y with $Y(0) = 0$ and $|\frac{\nabla}{\partial r} Y(0)| = 1$ and prove that

$$y(r_1) \geq \psi_{r_1 r_2} \circ y(r_2) \tag{40}$$

where $y := |Y|$. By Proposition 5.12 we see that on the interval $(0, r_2)$ the function $u := \frac{1}{y} \frac{\partial}{\partial r} y$ obeys the following differential inequality of Riccati type:

$$\frac{\partial}{\partial r} u \leq \begin{cases} -\lambda - u^2 & \text{if } n = 2 \, , \\ -\lambda - \mathrm{ct}_\lambda \, \mathrm{ct}_\Lambda + (\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda) \, u - 2u^2 & \text{if } n > 2 \, . \end{cases}$$

Since $y > 0$, this differential inequality transforms into the differential inequalities $L_2(y) \leq 0$ if $n = 2$ and $L_n(\frac{1}{\mathrm{sn}_\lambda} y^2) \leq 0$ if $n > 2$, where L_n denotes the linear differential operators[3] introduced in (32). Now we can apply Proposition 5.10 to conclude that inequality (40) holds.

[3]In the case $n > 2$ the coefficient 2 in front of the u^2–term is responsible for obtaining a linear differential equation in y^2 rather than y. The factor $\frac{1}{\mathrm{sn}_\lambda}$ has been introduced, in order to avoid additional singularities in the differential operator L_n corresponding to higher order zeroes of special solutions.

Appendix A. Further analytical properties of the comparison functions

Strictly speaking, Theorem 5.4 is not a comparison theorem, since in dimensions $n > 2$ there are no appropriate model spaces except for trivial cases. The theorem rather provides curvature controlled estimates in terms of some functions $\Psi_{r_1 r_2}$ defined via equations (20)–(23). In this appendix we want to show that the analytical properties of the comparison functions nevertheless tie in nicely with the standard comparison results for Jacobi fields.

Our first issue is to discuss how the functions $\Psi_{r_1 r_2}$ depend on the curvature bounds λ and Λ. Whenever necessary, we shall indicate this dependence by adding the parameters λ and Λ to the argument lists of the functions under consideration. We shall also use the notation $\bar{w}^{\lambda\Lambda}_{r_0}(r) := \bar{w}(r_0, r\,; \lambda, \Lambda)$, $\bar{y}^{\lambda\Lambda}_{r_0}(r) := \bar{y}(r_0, r\,; \lambda, \Lambda)$, and $\psi^{\lambda\,\Lambda}_{r_1 r_2}(\eta) := \psi_{r_1 r_2}(\eta\,; \lambda, \Lambda)$. Similarly, we shall write L^{λ}_2 and $L^{\lambda\Lambda}_n$ if $n > 2$, in order to indicate the parameters used in the definition of the differential operators in (32).

Remark A.1. Let $n = 2$ and $0 < r_0 < r_2$. Then it is not hard to verify that

$$L^{\hat{\lambda}}_2(\bar{y}^{\lambda\Lambda}_{r_0}) = \begin{cases} -(\Lambda - \hat{\lambda}) \cdot \bar{y}^{\lambda\Lambda}_{r_0} \le 0 & \text{on } (0, r_0)\,, \\ -(\lambda - \hat{\lambda}) \cdot \bar{y}^{\lambda\Lambda}_{r_0} \le 0 & \text{on } (r_0, r_2)\,, \end{cases}$$

provided that $\hat{\lambda} \le \lambda < \Lambda \le r_2^{-2}\pi^2$.

Lemma A.2. *Let* $0 < r_0 < r_2$ *and* $\hat{\lambda} \le \lambda < \Lambda \le \hat{\Lambda} \le r_2^{-2}\pi^2$. *Then for* $n > 2$ *the functions* $\bar{y}^{\lambda\Lambda}_{r_0}$ *introduced in* (21) *satisfy the differential inequalities*

(i) $L^{\hat{\lambda}\Lambda}_n\big(\frac{1}{\mathrm{sn}_{\hat{\lambda}}}(\bar{y}^{\lambda\Lambda}_{r_0})^2\big) \le 0$ *on* $(0, r_0) \cup (r_0, r_2)$,

(ii) $L^{\lambda\hat{\Lambda}}_n\big(\frac{1}{\mathrm{sn}_{\lambda}}(\bar{y}^{\lambda\Lambda}_{r_0})^2\big) \le 0$ *on* $(0, r_0) \cup (r_0, r_2)$.

Combining the remark and the preceding lemma with Proposition 5.10, we obtain the monotonicity properties of the functions $\psi^{\lambda\,\Lambda}_{r_1 r_2}$ introduced in (22).

Corollary A.3. *Let* $0 < r_1 < r_2$ *and let* $\hat{\lambda} \le \lambda < \Lambda \le \hat{\Lambda} \le r_2^{-2}\pi^2$. *Then for any* $\eta \in [\mathrm{sn}_{\Lambda}(r_2)\,, \mathrm{sn}_{\lambda}(r_2)]$ *the following inequalities hold:*

(i) $\psi^{\hat{\lambda}\,\Lambda}_{r_1 r_2}(\eta) \le \psi^{\lambda\,\Lambda}_{r_1 r_2}(\eta)$,

(ii) $\psi^{\lambda\,\hat{\Lambda}}_{r_1 r_2}(\eta) \le \psi^{\lambda\,\Lambda}_{r_1 r_2}(\eta)$.

Proof of Lemma A.2. i) A straightforward computation yields:

$$\begin{aligned} \mathrm{sn}_{\hat{\lambda}} \cdot L^{\hat{\lambda}\Lambda}_n\big(\tfrac{1}{\mathrm{sn}_{\hat{\lambda}}}(\bar{y}^{\lambda\Lambda}_{r_0})^2\big) &= \tfrac{\partial^2}{\partial r^2}\big((\bar{y}^{\lambda\Lambda}_{r_0})^2\big) - (\mathrm{ct}_{\hat{\lambda}} + \mathrm{ct}_{\Lambda}) \cdot \tfrac{\partial}{\partial r}\big((\bar{y}^{\lambda\Lambda}_{r_0})^2\big) \\ &\quad + 2(\hat{\lambda} + \mathrm{ct}_{\hat{\lambda}}\,\mathrm{ct}_{\Lambda}) \cdot (\bar{y}^{\lambda\Lambda}_{r_0})^2 \quad . \end{aligned} \tag{41}$$

On the interval $(0, r_0)$ we have $\bar{y}_{r_0}^{\lambda\Lambda} = \mathrm{sn}_\Lambda$, and hence

$$\mathrm{sn}_{\hat\lambda} \cdot L_n^{\hat\lambda\Lambda}\big(\tfrac{1}{\mathrm{sn}_{\hat\lambda}}(\bar{y}_{r_0}^{\lambda\Lambda})^2\big) \;=\; -2\,(\Lambda - \hat\lambda) \cdot \mathrm{sn}_\Lambda^2 \;\le\; 0 \quad .$$

On the interval (r_0, r_2) we employ the differential equation from (33) and the definition of $\bar{y}_{r_0}^{\lambda\Lambda}$ in terms of $\bar{w}_{r_0}^{\lambda\Lambda}$ from (21) to conclude that

$$\begin{aligned}
\mathrm{sn}_{\hat\lambda} \cdot L_n^{\hat\lambda\Lambda}\big(\tfrac{1}{\mathrm{sn}_{\hat\lambda}}(\bar{y}_{r_0}^{\lambda\Lambda})^2\big) \;&=\; \mathrm{sn}_{\hat\lambda} \cdot L_n^{\hat\lambda\Lambda}\big(\tfrac{1}{\mathrm{sn}_{\hat\lambda}}(\bar{y}_{r_0}^{\lambda\Lambda})^2\big) - \mathrm{sn}_\lambda \cdot L_n^{\lambda\Lambda}\big(\tfrac{1}{\mathrm{sn}_\lambda}(\bar{y}_{r_0}^{\lambda\Lambda})^2\big) \\
&=\; -\,(\mathrm{ct}_{\hat\lambda} - \mathrm{ct}_\lambda) \cdot \tfrac{\partial}{\partial r}\big((\bar{y}_{r_0}^{\lambda\Lambda})^2\big) \\
&\qquad + 2\,\mathrm{ct}_\Lambda\,(\mathrm{ct}_{\hat\lambda} - \mathrm{ct}_\lambda) \cdot (\bar{y}_{r_0}^{\lambda\Lambda})^2 - 2(\lambda - \hat\lambda) \cdot (\bar{y}_{r_0}^{\lambda\Lambda})^2 \\
&=\; \mathrm{sn}_\lambda^2 \cdot \big(-(\mathrm{ct}_{\hat\lambda} - \mathrm{ct}_\lambda)\,\tfrac{\partial}{\partial r}\bar{w}_{r_0}^{\lambda\Lambda} \\
&\qquad -2\,(\mathrm{ct}_{\hat\lambda} - \mathrm{ct}_\lambda)(\mathrm{ct}_\lambda - \mathrm{ct}_\Lambda) \cdot \bar{w}_{r_0}^{\lambda\Lambda} - 2(\lambda - \hat\lambda) \cdot \bar{w}_{r_0}^{\lambda\Lambda}\big) \\
&\le 2\,\mathrm{sn}_\lambda^2 \cdot (\mathrm{ct}_{\hat\lambda} - \mathrm{ct}_\lambda)\big(D^{\lambda\Lambda}(r_0) \cdot \mathrm{sn}_\lambda^{-3}\,\mathrm{sn}_\Lambda - (\mathrm{ct}_\lambda - \mathrm{ct}_\Lambda) \cdot \bar{w}_{r_0}^{\lambda\Lambda}\big) \quad .
\end{aligned}$$

In order to see that the expression on the right hand side is nonpositive, we recall that the map $r_0 \mapsto D^{\lambda\Lambda}(r_0)$ is strictly increasing. Furthermore, according to inequality (27) the map $r_0 \mapsto \bar{w}_{r_0}^{\lambda\Lambda}(r)$ is strictly decreasing for $0 < r_0 < r$. Since $\bar{w}_{r_0}^{\lambda\Lambda}(r)|_{r_0 = r} = \mathrm{sn}_\lambda(r)^{-2}\,\mathrm{sn}_\Lambda(r)^2$, we indeed obtain the inequality

$$\begin{aligned}
D^{\lambda\Lambda}(r_0) \cdot \tfrac{\mathrm{sn}_\Lambda(r)}{\mathrm{sn}_\lambda^3(r)} &- (\mathrm{ct}_\lambda(r) - \mathrm{ct}_\Lambda(r)) \cdot \bar{w}_{r_0}^{\lambda\Lambda}(r) \\
&\le\; D^{\lambda\Lambda}(r) \cdot \tfrac{\mathrm{sn}_\Lambda(r)}{\mathrm{sn}_\lambda^3(r)} - (\mathrm{ct}_\lambda(r) - \mathrm{ct}_\Lambda(r)) \cdot \tfrac{\mathrm{sn}_\Lambda^2(r)}{\mathrm{sn}_\lambda^2(r)} \;=\; 0
\end{aligned}$$

for $0 < r_0 \le r \le r_2$.

ii) In this case the computation reveals that on the interval $(0, r_0)$ we have

$$\begin{aligned}
\mathrm{sn}_\lambda \cdot L_n^{\lambda\hat\Lambda}\big(\tfrac{1}{\mathrm{sn}_\lambda}(\bar{y}_{r_0}^{\lambda\Lambda})^2\big) \;&=\; \mathrm{sn}_\lambda \cdot L_n^{\lambda\hat\Lambda}\big(\tfrac{1}{\mathrm{sn}_\lambda}\,\mathrm{sn}_\Lambda^2\big) \\
&=\; -\,2\big(\Lambda - \lambda + (\mathrm{ct}_\lambda - \mathrm{ct}_\Lambda)(\mathrm{ct}_\Lambda - \mathrm{ct}_{\hat\Lambda})\big) \cdot \mathrm{sn}_\Lambda^2 \;\le\; 0 \quad .
\end{aligned}$$

On the interval (r_0, r_2), however, we observe that $\frac{1}{\mathrm{sn}_\lambda}\,(\bar{y}_{r_0}^{\lambda\Lambda})^2 = \mathrm{sn}_\lambda \cdot \bar{w}_{r_0}^{\lambda\Lambda}$, and thus we obtain the identity:

$$\begin{aligned}
\tfrac{1}{\mathrm{sn}_\lambda}\,L_n^{\lambda\hat\Lambda}\big(\tfrac{1}{\mathrm{sn}_\lambda}\,(\bar{y}_{r_0}^{\lambda\Lambda})^2\big) \;&=\; \tfrac{1}{\mathrm{sn}_\lambda}\,L_n^{\lambda\hat\Lambda}(\mathrm{sn}_\lambda \cdot \bar{w}_{r_0}^{\lambda\Lambda}) \\
&=\; \tfrac{\partial^2}{\partial r^2}\bar{w}_{r_0}^{\lambda\Lambda} + (3\,\mathrm{ct}_\lambda - \mathrm{ct}_{\hat\Lambda}) \cdot \tfrac{\partial}{\partial r}\bar{w}_{r_0}^{\lambda\Lambda} \quad .
\end{aligned}$$

Again we employ the differential equation from (32). This time we conclude that

$$\begin{aligned}
\tfrac{1}{\mathrm{sn}_\lambda}\,L_n^{\lambda\hat\Lambda}\big(\tfrac{1}{\mathrm{sn}_\lambda}\,(\bar{y}_{r_0}^{\lambda\Lambda})^2\big) \;&=\; \tfrac{1}{\mathrm{sn}_\lambda}\,L_n^{\lambda\hat\Lambda}\big(\tfrac{1}{\mathrm{sn}_\lambda}\,(\bar{y}_{r_0}^{\lambda\Lambda})^2\big) - \tfrac{1}{\mathrm{sn}_\lambda}\,L_n^{\lambda\Lambda}\big(\tfrac{1}{\mathrm{sn}_\lambda}\,(\bar{y}_{r_0}^{\lambda\Lambda})^2\big) \\
&=\; (\mathrm{ct}_\Lambda - \mathrm{ct}_{\hat\Lambda}) \cdot \tfrac{\partial}{\partial r}\bar{w}_{r_0}^{\lambda\Lambda} \;\le\; 0 \quad ,
\end{aligned}$$

where we have used inequality (25) to determine the sign.

Proof of Corollary A.3. By continuity it is sufficient to handle the case where η lies in the open interval $(\mathrm{sn}_\Lambda(r_2), \mathrm{sn}_\lambda(r_2))$. In this context Lemma 5.2(iv) asserts that the equation $\bar{y}(\hat{r}_0, r; \lambda, \Lambda) = \eta$ has a unique solution $\hat{r}_0 \in (0, r_2)$. It follows directly from the definition in (22) that

$$\psi^{\lambda\Lambda}_{r_1 r_2}(\eta) = \bar{y}(\hat{r}_0, r_1; \lambda, \Lambda) \quad .$$

Clearly, $\mathrm{sn}_{\hat{\Lambda}} \leq \mathrm{sn}_\Lambda \leq \bar{y}^{\lambda\Lambda}_{\hat{r}_0} \leq \mathrm{sn}_\lambda \leq \mathrm{sn}_{\hat{\lambda}}$, and by Remark A.1 and Lemma A.2 we can apply the comparison result from Proposition 5.10 as extended[4] in Remark 5.11 to deduce (i) and (ii).

Proof of Proposition 5.5. Because of the homogeneity property from Lemma 5.3(i) it is sufficient to consider the restriction of the map $\Psi_{r_1 r_2}$ to the ray $\{1\} \times [0, \infty)$. According to Lemma 5.3(iv) this restriction is a nondecreasing function of η. Hence we only need to establish the monotonicity with respect to the curvature bounds in the three cases $0 \leq \eta < \mathrm{sn}_\Lambda(r_2)$, $\mathrm{sn}_\Lambda(r_2) < \eta < \mathrm{sn}_\lambda(r_2)$, and $\mathrm{sn}_\lambda(r_2) < \eta$, respectively.

Notice that $\mathrm{sn}_\Lambda(r_1)$ is independent of λ and strictly decreasing with respect to Λ. Similarly, the quotient $\frac{\mathrm{sn}_\lambda(r_1)}{\mathrm{sn}_\lambda(r_2)}$ is independent of Λ and strictly increasing with respect to λ. Hence in the cases $0 \leq \eta < \mathrm{sn}_\Lambda(r_2)$ and $\mathrm{sn}_\lambda(r_2) < \eta$ the assertion follows directly from the definition of $\Psi_{r_1 r_2}$ in (23).

In the remaining case the monotonicity with respect to λ as well as Λ follows from Corollary A.3.

Our next issue is to understand the relationship between Theorem 5.4 and the infinitesimal version of the Rauch comparison theorem. For this purpose we shall discuss how the functions $\Psi_{r_1 r_2}$ depend on the parameter r_1, i.e., we shall analyze the qualitative properties of the maps $r_1 \mapsto \bar{y}_{r_0}(r_1)$.

Lemma A.4. *Let $\lambda < \Lambda$ and $r_2 > 0$. Suppose in addition that $r_2 \leq \frac{\pi}{\sqrt{\Lambda}}$ if $\Lambda > 0$. Then the logarithmic derivative $\bar{u} := \frac{1}{\bar{y}} \frac{\partial}{\partial r} \bar{y}$ has the following properties:*

- (i) $\frac{\partial}{\partial r_0} \bar{u}(r_0, r) < 0$ *for* $0 < r_0 < r < r_2$,
- (ii) *for any $r \in (0, r_2)$ the restriction of the function $\bar{u}(., r)$ to the interval $[0, r]$ is a surjective, strictly decreasing, real analytic map* $\bar{\bar{u}}_r \colon [0, r] \xrightarrow{\cong} [\mathrm{ct}_\Lambda(r), \mathrm{ct}_\lambda(r)]$,

---

[4]Since $\bar{y}^{\lambda\Lambda}_{\hat{r}_0} \in C^2((0, \hat{r}_0) \cup (\hat{r}_0, r_2))$, it is actually possible to avoid this extension. However, additional arguments are necessary in order to analyze how the point $\hat{r}_0$ depends on λ and Λ.

(iii) *the functions* $\bar{u}_{r_0}$ *depend smoothly on the parameter* $\lambda \in (-\infty, \Lambda)$. *More precisely,* $\frac{\partial}{\partial\lambda}\bar{u}_{r_0}(r_0) = 0$ *and* $\frac{\partial}{\partial\lambda}\bar{u}_{r_0} < 0$ *for* $0 < r_0 < r < r_2$.

Remarks A.5. i) The second assertion of the lemma implies that

$$\mathrm{ct}_\Lambda(r) \le \tfrac{\partial}{\partial r} \ln \circ \bar{y}_{r_0}(r) \le \mathrm{ct}_\lambda(r) \qquad \text{for } 0 < r < r_2 .$$

These bounds for $\frac{\partial}{\partial r}(\ln \circ \bar{y}_{r_0})$ coincide with the upper and lower bounds for the logarithmic derivative of the norm of the Jacobi field Y as stated in the infinitesimal version of the Rauch comparison theorem.

ii) In case $n > 2$ and $r_2 = \frac{\pi}{\sqrt{\Lambda}}$, $\Lambda > 0$, the second assertion in the lemma does not extend to hold for $r = r_2$. From equations (21) and (25) we rather deduce the identity

$$\lim_{r\to r_2} \bar{u}(r_0, r) = \mathrm{ct}_\lambda(r_2) - \bar{w}(r_0, r_2)^{-1/2}\, \tfrac{1}{2}\tfrac{\partial}{\partial r}\bar{w}(r_0, r)|_{r=r_2} = \mathrm{ct}_\lambda(r_2) \quad ,$$

provided that $0 \le r_0 < r_2$. In fact, $r = r_2$ is a regular singular point of the Riccati equation for $\bar{u}_{r_0} = \bar{u}(r_0, .)$, and the assignment $r_0 \mapsto \lim_{r\to r_2} \bar{u}'_{r_0}(r)$ defines a bijection $[0, r_2) \to [\mathrm{ct}'_\lambda(r_2), \infty)$. Thus the properties of $\bar{u}$ at $r = r_2$ differ significantly from the properties of $\bar{y}$ established in Lemma 5.2(iv).

iii) Of course, assertion (iii) of the lemma implies a corresponding inequality for the sign of $\frac{\partial}{\partial\lambda}\bar{y}^{\lambda\Lambda}_{r_0}$. In fact, when combining this inequality with Lemma 5.2(ii), we obtain a second proof for Corollary A.3(i).

Proof of the Lemma. i) Differentiating the Riccati equation in (34) with respect to the variable r_0, we obtain the following linear differential equation of first order for $\frac{\partial}{\partial r_0}\bar{u}(r_0, .)$:

$$\frac{\partial}{\partial r}\Big(\frac{\partial}{\partial r_0}\bar{u}_{r_0}\Big) = \begin{cases} -2\bar{u}_{r_0}\, \frac{\partial}{\partial r_0}\bar{u}_{r_0} & \text{if } n = 2 , \\ -(4\bar{u}_{r_0} - \mathrm{ct}_\lambda - \mathrm{ct}_\Lambda)\, \frac{\partial}{\partial r_0}\bar{u}_{r_0} & \text{if } n > 2 , \end{cases}$$

provided that $r \in (r_0, r_2)$. Since its coefficients are locally bounded, we conclude that $\frac{\partial}{\partial r_0}\bar{u}(r_0, .)$ does not change its sign on (r_0, r_2). If $n = 2$, we differentiate equation (21) and conclude that

$$\lim_{r\searrow r_0} \frac{\partial}{\partial r_0}\bar{u}(r_0, r) = \Big(\frac{1}{\bar{y}}\frac{\partial^2 \bar{y}}{\partial r_0\, \partial r} - \frac{1}{\bar{y}^2}\frac{\partial \bar{y}}{\partial r_0}\frac{\partial \bar{y}}{\partial r}\Big)_{|r=r_0} = -(\Lambda - \lambda) < 0 \quad .$$

It requires only slightly more effort to establish the corresponding inequality in the case $n > 2$. Equation (21) yields that $\bar{u}(r_0, r) = \mathrm{ct}_\lambda(r) + \bar{w}(r_0, r)^{-1}\,\frac{1}{2}\frac{\partial}{\partial r}\bar{w}(r_0, r)$ on the triangle $\{r_0 < r\}$, and with the help of formula (24) we see that

$$\lim_{r\searrow r_0} \frac{\partial}{\partial r_0}\bar{u}(r_0, r) = \frac{1}{2}\Big(\frac{1}{\bar{w}}\frac{\partial^2 \bar{w}}{\partial r_0\, \partial r} - \frac{1}{\bar{w}^2}\frac{\partial \bar{w}}{\partial r_0}\frac{\partial \bar{w}}{\partial r}\Big)_{|r=r_0} = -(\Lambda - \lambda) < 0 \quad .$$

ii) By assertion (i) it is sufficient to observe that $\bar{u}(r,r) = \mathrm{ct}_\Lambda(r)$ and $\bar{u}(0,r) = \mathrm{ct}_\lambda(r)$, in order to determine the ranges of the maps $\bar{\bar{u}}_r$.

iii) Since $\bar{u}_{r_0}(r_0) = \mathrm{ct}_\Lambda(r_0)$, it is clear that $\frac{\partial}{\partial\lambda}\bar{u}_{r_0}(r_0) = 0$. Hence it follows from the differential equation (34) that the function $\bar{u}_{r_0}$ depends smoothly on the parameter λ, and it is straightforward to compute that

$$\frac{\partial}{\partial r}\left(\frac{\partial}{\partial\lambda}\bar{u}_{r_0}\right) = \begin{cases} -1 - 2\bar{u}_{r_0}\,\frac{\partial}{\partial\lambda}\bar{u}_{r_0} & \text{if } n = 2\,, \\ -1 + (\bar{u}_{r_0} - \mathrm{ct}_\Lambda)\,\frac{\partial}{\partial\lambda}\,\mathrm{ct}_\lambda & \\ \quad - (4\bar{u}_{r_0} - \mathrm{ct}_\lambda - \mathrm{ct}_\Lambda)\,\frac{\partial}{\partial\lambda}\bar{u}_{r_0} & \text{if } n > 2\,. \end{cases}$$

Since $\frac{\partial}{\partial\lambda}\,\mathrm{ct}_\lambda < 0$ and $\bar{u}_{r_0} - \mathrm{ct}_\Lambda \geq 0$ by assertion (ii), we see that in either case the right hand side of the preceding differential equation for $\frac{\partial}{\partial\lambda}\bar{u}_{r_0}$ consists of a term, which is ≤ -1, and a term, which is linear in $\frac{\partial}{\partial\lambda}\bar{u}_{r_0}$. Therefore Gronwall's lemma applies. q.e.d.

With these preparations we can now understand the critical points of the comparison functions $\bar{y}_{r_0}$, which in turn helps in understanding Figure 2 on page 24. If $\Lambda \leq 0$, it follows directly from Remark A.5(i) that the functions $\bar{y}_{r_0}$ do not have any critical points in $[0,\infty)$. If $\Lambda > 0$, it is sufficient to consider the case that $r_2 = \frac{\pi}{\sqrt{\Lambda}}$. Clearly, the functions $\bar{y}_{r_0}\colon [0,r_2] \to [0,\infty)$ still do not have any critical points in the interval $[0,\frac{1}{2}r_2)$.

Proposition A.6. *Let $\Lambda > 0$, $\lambda < \Lambda$, and $r_2 = \frac{\pi}{\sqrt{\Lambda}}$. Suppose that $n > 2$. Then the functions $\bar{y}_{r_0}$ defined in (21) have at most two critical points. More precisely, there are the following four cases:*

(i) *If $\frac{1}{4}\Lambda < \lambda < \Lambda$, then each function $\bar{y}_{r_0}$ has precisely one critical point in the interval $(0,r_2)$, which is a nondegenerate local maximum.*

(ii) *If $\lambda = \frac{1}{4}\Lambda$, then each function $\bar{y}_{r_0}$ has a critical point at $r = r_2$. In fact, there exists a number $\bar{r}_0 \in (0,\frac{1}{2}r_2)$ such that*

  (a) *for $r_0 \in [0,\bar{r}_0]$ the function $\bar{y}_{r_0}$ is strictly increasing with $\bar{y}'_{r_0} > 0$ on $[0,r_2)$, and its critical point at $r = r_2$ is nondegenerate unless $r_0 = \bar{r}_0$,*

  (b) *for $r_0 \in (\bar{r}_0, r_2]$ the function $\bar{y}_{r_0}$ has precisely two critical points, a nondegenerate local maximum in $[\frac{1}{2}r_2, r_2)$ and a nondegenerate local minimum at $r = r_2$.*

(iii) *If $0 < \lambda < \frac{1}{4}\Lambda$, then there exist $\bar{r}_0 \in (0,\frac{1}{2}r_2]$ and $r^c \in [\frac{1}{2}r_2, r_2)$ such that*

  (a) *for $r_0 \in [0,\bar{r}_0)$ the function $\bar{y}_{r_0}$ is strictly increasing with $\bar{y}'_{r_0} > 0$,*

  (b) *for $r_0 = \bar{r}_0$ the function $\bar{y}_{r_0}$ remains to be strictly increasing, but there is a degenerate critical point at $r = r^c$,*

(c) *for* $r_0 \in (\bar{r}_0, r_2)$ *the function* $\bar{y}_{r_0}$ *has precisely two critical points, a nondegenerate local maximum in* $(\frac{1}{2}r_2, r^c)$ *and a nondegenerate local minimum in* (r^c, r_2).

(iv) *If* $\lambda \leq 0$, *then the following holds:*

(a) *for* $r_0 \in [0, \frac{1}{2}r_2)$ *the function* $\bar{y}_{r_0}$ *is strictly increasing with* $\bar{y}'_{r_0} > 0$,

(b) *for* $r_0 = \frac{1}{2}r_2$ *the function* $\bar{y}_{r_0}$ *remains to be strictly increasing, but there is a critical point at* $r = \frac{1}{2}r_2$,

(c) *for* $r_0 \in (\frac{1}{2}r_2, r_2)$ *the function* $\bar{y}_{r_0}$ *has precisely two critical points, a nondegenerate local maximum at* $r = \frac{1}{2}r_2$ *and a nondegenerate local minimum in* $(r_0, r_2]$.

It seems to be an intriguing fact that for $\Lambda > 0$ and $n > 2$ the qualitative appearance of the comparison functions $\bar{y}_{r_0}$ changes significantly at $\lambda = \frac{1}{4}\Lambda$. These changes, however, do not occur if $n = 2$. Furthermore, we like to point out that for $n > 2$ the nondegenerate local minimum of $\bar{y}_{r_0}$ that appears as the parameter r_0 gets larger than $\bar{r}_0$ persists as long as $r_0 < r_2$. If $n = 2$ and $\lambda < 0$, the functions $\bar{y}_{r_0}$ also exhibit a nondegenerate local minimum as r_0 gets larger than $\frac{1}{2}r_2$, but this minimum leaves the interval $[0, r_2]$ long before r_0 gets close to r_2.

Corollary A.7. *Let* $n > 2$ *and* $\Lambda > 0$, *and suppose that* $r_2 = \frac{\pi}{\sqrt{\Lambda}}$. *Then for* $0 < \lambda \leq \frac{1}{4}\Lambda$ *the following holds:*

(i) *the number* $\bar{r}_0 = \sup\{r_0 \in [0, r_2] \mid \bar{y}'_{r_0} \geq 0\}$ *introduced in Proposition A.6 is a strictly decreasing function of the parameter* λ, *which is in fact a bijection* $(0, \frac{1}{4}\Lambda] \overset{\approx}{\to} [\bar{r}_0^{\Lambda/4}, \frac{1}{2}r_2)$ *with* $\bar{r}_0^{\Lambda/4} \approx 0.452\,731\, r_2$,

(ii) *the equation* $\bar{y}^{\lambda\Lambda}(\hat{r}_0^{\lambda}, r_2) = \frac{1}{\sqrt{\Lambda}}$ *has a unique solution* $\hat{r}_0^{\lambda} \in (\bar{r}_0, r_2)$. *This solution is a strictly decreasing function of the parameter* λ, *which maps* $(0, \frac{1}{4}\Lambda]$ *into the interval* $[\hat{r}^{\lambda/4}, r_2)$ *where* $\hat{r}^{\lambda/4} \approx 0.470\,547\, r_2 < \frac{1}{2}r_2)$. *The value* $\frac{1}{2}r_2$ *is attained at* $\lambda \approx 0.185\,048\,\Lambda$.

Proof of Proposition A.6 i) The bounds from Lemma A.4(ii) imply that that for any $r_0 \in [0, r_2]$ all the zeroes of $\bar{u}_{r_0} = \frac{1}{\bar{y}_{r_0}} \frac{\partial}{\partial r}\bar{y}_{r_0}$ lie in the interval $[\frac{1}{2}r_2, \frac{1}{2}r_3] \subset [\frac{1}{2}r_2, r_2]$ where $r_3 := \frac{\pi}{\sqrt{\lambda}}$. Since $\frac{\partial}{\partial r}\bar{u}_{r_0}|_{r=r_2} \leq \mathrm{ct}_\lambda(r_2) < 0$, it is sufficient to show that $\frac{\partial}{\partial r}\bar{u}_{r_0} < 0$ at all the zeroes of the function $\bar{u}_{r_0}$ itself.

In view of the Riccati equation in (34) we only need to verify that the expression $\mathrm{cn}_\lambda(r)\,\mathrm{cn}_\Lambda(r) + \lambda\,\mathrm{sn}_\lambda(r)\,\mathrm{sn}_\Lambda(r)$ is strictly positive for $r \in [\frac{1}{2}r_2, \frac{1}{2}r_3] \subset [\frac{1}{2}r_2, r_2]$. By differentiation it is easy to see that on this particular interval the preceding expression is a strictly decreasing function of r. Moreover, its value at $r = \frac{1}{2}r_3$ is strictly positive.

ii) In this case all relevant expressions can be computed explicitly:

$$\bar{w}(r_0, r) = \operatorname{cn}^2_\Lambda(\tfrac{1}{2}r_0) - 2\Lambda \operatorname{sn}^2_\Lambda(\tfrac{1}{2}r_0) \cdot \left(1 - \frac{\operatorname{sn}_\Lambda(\frac{1}{2}r_0)}{\operatorname{sn}_\Lambda(\frac{1}{2}r)}\right) \quad , \tag{42}$$

and thus

$$\begin{aligned} \tfrac{\partial}{\partial r}(\bar{y}_{r_0}(r)^2) = 4\operatorname{cn}_\Lambda(\tfrac{1}{2}r) \cdot \big[(\operatorname{cn}^2_\Lambda(\tfrac{1}{2}r_0) - 2\Lambda \operatorname{sn}^2_\Lambda(\tfrac{1}{2}r_0)) \operatorname{sn}_\Lambda(\tfrac{1}{2}r) \\ + \Lambda \operatorname{sn}^3_\Lambda(\tfrac{1}{2}r_0)\big] \end{aligned}$$

for $r_0 < r \le r_2$. The factor $\operatorname{cn}_\Lambda(\frac{1}{2}r)$ vanishes at $r = r_2$, which establishes the critical point of $\bar{y}_{r_0}$ at this particular boundary point.[5] Any other critical point of $\bar{y}_{r_0}$ in (r_0, r_2) corresponds to a zero of the factor in square brackets, and thus it is straightforward to compute that the assertion holds, provided that $\bar{r}_0 \in (0, \frac{1}{2}r_2)$ is the solution of the equation $\sin(\frac{1}{2}\sqrt{\Lambda}\,\bar{r}_0) = 1 - 2\sin\frac{\pi}{18} \approx 0.652\,704$. Here the right hand side comes as the root of the cubic $1 - 3x^2 + x^3$ which lies in $(0, 1)$.

iii) and iv) As in (i) our starting point is the observation that a critical point of $\bar{y}_{r_0}$ corresponds to a zero of $\bar{u}_{r_0}$, whereas a degenerate critical point of $\bar{y}_{r_0}$ corresponds to a common zero of $\bar{u}_{r_0}$ and its derivative. Hence by (34) a degenerate critical point of $\bar{y}_{r_0}$ can only occur at a zero of the map $r \mapsto \operatorname{cn}_\lambda(r)\operatorname{cn}_\Lambda(r) + \lambda \operatorname{sn}_\lambda(r)\operatorname{sn}_\Lambda(r)$. This map is strictly decreasing on $[0, r_2]$, and thus it is easy to verify that it has a unique zero $r^c \in (0, r_2)$. Clearly, any critical point of $\bar{y}_{r_0}$ that lies between $\max\{r_0, r^c\}$ and r_2 is a nondegenerate local minimum, and any critical point between 0 and $\max\{r_0, r^c\}$ is a nondegenerate local maximum. Moreover, Remark A.5(ii) implies that $\bar{y}'_{r_0}(r_2) = \operatorname{ct}_\lambda(r_2) \cdot \bar{y}_{r_0}(r_2) > 0$ for $0 \le r_0 < r_2$.

If $\lambda > 0$, we actually find that $r^c \in (\frac{1}{2}r_2, r_2)$, and thus $\operatorname{ct}_\Lambda(r^c) < 0 < \operatorname{ct}_\lambda(r^c)$. Hence by Lemma A.4(ii) there exists a unique number $\bar{r}_0 \in (0, r^c)$ such that $\bar{u}_{\bar{r}_0}(r^c) = 0$. The preceding discussion shows that for these particular numbers $\bar{r}_0$ and r^c we indeed get statements (a)–(c) in assertion (iii). In particular, $\bar{y}_{\bar{r}_0}$ is strictly increasing on $[0, r^c)$, and therefore $\bar{r}_0 \in (0, \frac{1}{2}r_2) \subset (0, r^c)$ as claimed.

If on the other hand $\lambda \le 0$, we find that $r^c \in (0, \frac{1}{2}r_2]$, and thus assertion (iv) follows immediately from the preceding discussion.

Proof of Corollary A.7 i) The monotonicity of $\bar{r}_0$ with respect to λ is a direct consequence of Lemma A.4(iii). The proof of assertion (ii) in Proposition A.6 reveals that $\sin(\frac{1}{2}\sqrt{\Lambda}\,\bar{r}_0^{\Lambda/4}) = 1 - 2\sin\frac{\pi}{18} \approx 0.652\,704$. Similarly, the proof of assertion (iii) implies that $r^c \to \frac{1}{2}r_2$ as $\lambda \to 0$, and by Lemma A.4(iii) we conclude that $\bar{r}_0 \to \frac{1}{2}r_2$.

[5] This critical point could also have been established referring to Remark A.5(ii), since under the current hypotheses $\operatorname{ct}_\lambda(r_2) = 0$.

ii) Note that $\operatorname{sn}_\lambda(r_2) < \frac{1}{\sqrt{\Lambda}} < \operatorname{sn}_\lambda(r_2)$. Therefore existence and uniqueness of a solution $\hat{r}_0^\lambda \in [0, r_2)$ follow directly from Lemma 5.2(iv). Since $\frac{1}{\sqrt{\Lambda}} = \operatorname{sn}_\Lambda(\frac{1}{2}r_2)$, Proposition A.6(iii) yields that $\max \operatorname{sn}_\Lambda < \max \bar{y}_{\hat{r}_0^\lambda}$, and hence $\bar{r}_0 < \hat{r}_0^\lambda$. The monotonicity of $\hat{r}_0^\lambda$ with respect to λ is an immediate consequence of Corollary A.3(i). Equation (42) showss that the number $\sin(\frac{1}{2}\sqrt{\Lambda}\,\hat{r}_0^{\Lambda/4})$ is a root of the cubic $8x^3 - 12x^2 + 3$. Thus $\sin(\frac{1}{2}\sqrt{\Lambda}\,\hat{r}_0^{\Lambda/4}) = \frac{1}{2} + \sin\frac{\pi}{18} \approx 0.673\,648 < \frac{1}{2}\sqrt{2} = \sin(\frac{1}{4}\pi)$, and hence $\hat{r}_0^{\Lambda/4} < \frac{1}{2}r_2$ as claimed. Finally, a numerical computation reveals that $\frac{1}{2}r_2 = \hat{r}_0^\lambda$ for $\lambda \approx 0.185\,048\,\Lambda$

Acknowledgements

We like to thank D. Gromoll and M. Kreck for helpful discussions. In particular, we are indebted to M. Kreck for suggesting Example 3.6.

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