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THE ATIYAH-JONES CONJECTURE FOR CLASSICAL GROUPS AND BOTT PERIODICITY

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0. Introduction

In this note we construct a *L-stratification* of $\mathcal{M}_k(G)$, the based G-moduli space of instantons (or equivalently anti instantons) of charge k over S^4. The structure groups we concern ourself with here are $SO(n)$ for $n > 6$ and $Sp(n)$. Then the general machinery of [6] is applied to prove

Theorem A. *For $G = SO(n)$ with $n > 6$ or $G = Sp(n)$ and for all $k > 0$ and all primes p, the Taubes inclusion map $\iota_k : \mathcal{M}_k(G) \longrightarrow \mathcal{M}_{k+1}(G)$ induces an isomorphism in homology*

$$(\iota_k)_t : H_t(\mathcal{M}_k(G); \mathbb{A}) \cong H_t(\mathcal{M}_{k+1}(G); \mathbb{A})$$

for $t \leq q = q(k) = [k/2] - 1$ and $\mathbb{A} = \mathbb{Z}$ or $\mathbb{Z}/p$.

The stratifications also lead naturally to the computation of the fundamental groups. We shall show that $\mathcal{M}_k(SO(n))$ are all simply connected for $n > 6$, and that the fundamental groups of $\mathcal{M}_k(SP(n))$ are always $\mathbb{Z}/2$. The argument of [6] can be applied to improve Theorem A as to state that ι_k induces a homotopy equivalence through dimension at least $[k/2] - 1$ for $SO(n), n > 6$ and at least $[k/2] - 2$ for $Sp(n)$. As a consequence of it, we are able to confirm the Atiyah-Jones conjecture that relates the homotopy of $\mathcal{M}_k$ to that of $\Omega^3_0 G$, a connected component of 3-fold loop space of group G (see [4]). To be precise, we shall prove

Theorem B. *For all positive integers k, the induced map (from ϑ_k)*

$$(\vartheta_k)_t : \pi_t(\mathcal{M}_k(G)) \longrightarrow \pi_t(\Omega^3_0 G)$$

on homotopy groups is an isomorphism for $t \leq q(k) = [k/2] - 1$ if $G = SO(n)$, $n > 6$; and for $t \leq q(k) = [k/2] - 2$ if $G = Sp(n)$.

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Following Kirwan [15], we also give the best possible ranges $q(k)$ for all classical groups except for a few possible cases when the sizes of the groups are small.

The stable version of the Atiyah-Jones conjecture was proved by Taubes in the general context where the structure group can be any compact simple Lie group and the base manifold can be any compact oriented Riemannian four-manifold (see [21], [11]). In the case where the base manifold is S^4, Boyer, Hurtubise, Mann and Milgram gave the proof of the Atiyah-Jones conjecture for the most important case where the structure group is $SU(2)$ (see [6]). Following their ideas, the author was able to prove the conjecture for general $SU(n)$ using the method that will be presented here (see [22]). Independently, Kirwan [15] used an equivariant cohomology technique to prove the conjecture for $SU(n)$ when $n > 2$. Although her method failed to cover the crucial $SU(2)$ case, the range $q(k)$ she got is best possible when $n > 3$. She also obtained the similar stabilization result when $n \to \infty$ induced by ϑ mapping from $H^*(\mathcal{M}_k(SU(n)))$ to $H^*(BU(k))$. This last fact has recently also be proved by Sanders [20] for $SU(n)$ and $Sp(n)$ using a quite simple method. We will use this stabilization in order to get the Bott Periodicity. As for our main purpose, a combination of the previous results for $G = SU(n)$ [6], [15], [22] with Theorem B yields the proof of the conjecture for all simple classical groups of $SO(n), SU(n)$ and $Sp(n)$. Note also that the result for $Spin(n)$ follows automatically from that of $SO(n)$. It is now conceivable that the result for any compact simple Lie group can be done similarly. Much more difficult to prove is the conjecture for general four-manifolds. Recently, Hurtubise and Milgram have proved a homological Atiyah-Jones conjecture for $G = SU(2)$ and for manifold being a ruled surface [14]. However, compared with the stability theorem of Taubes, all these results suffer from a common defect - they all depend on the particular (nice) choices of the metrics for the underlying four-manifolds. It seems that to prove the conjecture for S^4 equipped with a conformally non-flat Riemannian metric may be very difficult, and it may be crucial in understanding the conjecture in general cases.

The organization of the paper is as follows: Sections 1 through 4 mainly deal with the case where the structure group is $SO(n)$. In Section 1 we review a theorem of Donaldson which gives a holomorphic description of the moduli spaces using monads for $\mathbb{CP}^2$. We then realize his result in concrete linear algebra. In Section 2 we show that an instanton bundle is completely determined, up to "framed" isomorphism, by its behaviour restricted to a finite number of "jumping lines". This relates $\mathcal{M}_k$ to certain labelled configuration spaces, the fact that plays a central role in proving Theorem A. In Section 3 we give a L-stratification and outline the proof of Theorems A and B. In Section 4 we give the dimension estimates for the strata obtained in Section 3. We also show that $\mathcal{M}_k$ is simply connected for $G = SO(n)$, $n > 6$. In Section 5, we realize Donaldson's theorem about $\mathcal{M}_k$ in the case where $G = Sp(n)$, again in explicit linear algebra. We outline the stratification and state the results. In the final section, we make use of the stabilization processes as both $k \to \infty$ and $n \to \infty$. The observation we make is that the Atiyah-Jones conjecture yields Bott Periodicity, a fact that may lie at the heart of the Atiyah-Jones Conjecture.

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1. A theorem of Donaldson

The well-known ADHM construction gives an explicit description of moduli spaces of instantons in terms of linear algebra, or more precisely, in terms of the monad description of holomorphic bundles over the complex projective space $\mathbb{CP}^3$. There are real but not complex quadratic constraints involved in these descriptions which in many circumstances are difficult to deal with. Thus it is remarkable that Donaldson deduced a purely complex algebraic description of these moduli spaces from ADHM construction using geometric invariant theory. We now recall his result.

Let G be one of the classical group $SU(n), SO(n)$ or $Sp(n)$, and let $k \geq 0$ be minus half the first Pontryagin index of a G bundle P over $S^4 = \mathbb{R}^4 \cup \{\infty\}$. Denote by $\mathcal{M}_k(G)$ the framed moduli space whose points represent isomorphism classes of pairs:

$$(\text{anti self-dual } G\text{-connection on } P, \text{ isomorphism } P_\infty \simeq G).$$

Let $\tilde{\mathcal{M}}_k(G^{\mathbb{C}})$ denote the moduli spaces of holomorphic bundles on $\mathbb{CP}^2$ for the associated complex group, trivial on a fixed line l_∞ and with a fixed holomorphic trivialization there. Then Donaldson showed

Theorem 1.1 [10]. *There is a natural one-to-one correspondence between $\mathcal{M}_k(G)$ and $\tilde{\mathcal{M}}_k(G^{\mathbb{C}})$.*

Donaldson actually gave an explicit monad description of $\mathcal{M}_k$ for $G = SU(n)$ and indicated how in principle this could be generalized to $SO(n)$ and $Sp(n)$. For our purposes we shall now realize his theorem in concrete linear algebraic data. In this section we shall give the result for $G = SO(n)$. Actually, we shall only need the result for $n > 6$ since $Spin(3) \simeq SU(2), Spin(4) \simeq SU(2) \times SU(2)$ which is not simple, $Spin(5) \simeq Sp(2)$ and $Spin(6) \simeq SU(4)$. Note that $SO(n)^{\mathbb{C}} = SO(n, \mathbb{C})$. It can be shown that an $SO(n, \mathbb{C})$ monad over $\mathbb{CP}^2$ with minus half the first Pontryagin index k is of the form (see [5], [19]):

$$\mathbb{C}^{2k} \xrightarrow{A(p)} \mathbb{C}^{4k+n} \xrightarrow{A(p)^T} \mathbb{C}^{2k},$$

where $p = [x, y, z]$ are homogeneous coordinates on $\mathbb{CP}^2$, $A(p) = A_x x + A_y y + A_z z$ is a linear matrix function in x, y, and z, and $A(p)^T$ is the transpose of $A(p)$. It is subject to the following two constraints:

$$A(p)^T A(p) = 0 \text{ for all } p \in \mathbb{CP}^2, \tag{1.2}$$

$$A(p) \text{ is injective for all } p \in \mathbb{CP}^2. \tag{1.3}$$

Obviously $GL(n, \mathbb{C})$ acts on $A(p)$ from the right, and $O(4k+n, \mathbb{C})$ acts on $A(p)$ from the left. It turns out that two monads give an equivalent holomorphic bundle over $\mathbb{CP}^2$ if and only if they differ by such an action. By Donaldson's theorem, we are only concerned with those stable bundles which have an additional property that the bundles represented by these monads must be trivial over the fixed line $l_\infty = \{z = 0\}$ with a fixed trivialization. This can be reformulated as follows

(1.4) $A_x^T A_y = -A_y^T A_x$ is an isomorphism, and only the subgroup G_0 of $O(4k + n, \mathbb{C}) \times GL(n, \mathbb{C})$ whose action preserves the natural framing over $z = 0$ is permitted.

The moduli space $\mathcal{M}_k$ is thus the quotient of the set of these monads with properties 1.2 through 1.4 by the group G_0. Let us put aside the framing for the moment. Then we can first put A_x into the form

$$\begin{pmatrix} I_{2k} \\ \alpha \\ a \end{pmatrix}$$

using an appropriate action since A_x has to be injective. By 1.2 we have $A_x^T A_x = 0$, i.e., $\alpha^T \alpha + a^T a = -I_{2k}$. So in virtue of the left action by $O(4k + n, \mathbb{C}), A_x$ can be put into the form

$$A_x = \begin{pmatrix} I_{2k} \\ -iI_{2k} \\ 0 \end{pmatrix}. \tag{1.5}$$

The isotropy group of the action for 1.5 can be worked out to have the form

(1.6)

$$\begin{pmatrix} I_{2k} & 0 & 0 \\ -iI_{2k} & I_{2k} & 0 \\ 0 & 0 & I_n \end{pmatrix} \begin{pmatrix} g & h & -igu^T o_n \\ 0 & (g^{-1})^T & 0 \\ 0 & u & o_n \end{pmatrix} \begin{pmatrix} I_{2k} & 0 & 0 \\ iI_{2k} & I_{2k} & 0 \\ 0 & 0 & I_n \end{pmatrix} \times g^{-1}$$

which belongs to $O(4k+n,\mathbb{C}) \times GL(2k,\mathbb{C})$, and where $o_n \in O(n,\mathbb{C}), h = \frac{i}{2}[g-(g^{-1})^T] + ig(-\frac{1}{2}u^Tu + s)$, u is an arbitrary $n \times 2k$ matrix and s is an arbitrary $2k \times 2k$ skew-symmetric matrix. We shall denote this isotropy group by G_1. Write

$$A_y = \begin{pmatrix} \beta_1 \\ \beta_2 \\ b \end{pmatrix}.$$

Then $\beta_1^T\beta_1 + \beta_2^T\beta_2 + b^Tb = 0$ by 1.2, and $(\beta_1 - i\beta_2)^T = -(\beta_1 - i\beta_2)$ is an isomorphism by 1.4. By choosing $u = -io_nb(\beta_1 - i\beta_2)^{-1}$ we can assume $b = 0$. By choosing $s = 2(\beta_1 + i\beta_2)(\beta_1 - i\beta_2)^{-1}$ which is skew-symmetric and using the fact that $\beta_1^T\beta_1 + \alpha_2^T\beta_2 = 0$ ($b = 0$ now), we can assume that $\beta_2 = -i\beta_1 = \beta$ is a skew-symmetric isomorphism. Now the subgroup of G_1 that keeps this form invariant is

$$\left\{ \begin{pmatrix} \frac{1}{2}[g+(g^{-1})^T] & \frac{i}{2}[g-(g^{-1})^T] & 0 \\ -\frac{i}{2}[g-(g^{-1})^T] & \frac{1}{2}[g+(g^{-1})^T] & 0 \\ 0 & 0 & o_n \end{pmatrix} \times g^{-1} : g \in GL(2k,\mathbb{C}), o_n \in O(n,\mathbb{C}) \right\}.$$

Under the action of this group, $\beta \longmapsto (g^{-1})^T\beta g^{-1}$ which clearly can be given as

$$i\sigma = i\begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}.$$

Throughout this paper we assume σ is always so defined. Thus we have deduced special choices for A_x and A_y such that

$$A_x = \begin{pmatrix} I_{2k} \\ -iI_{2k} \\ 0 \end{pmatrix}, A_y = \begin{pmatrix} \sigma \\ i\sigma \\ 0 \end{pmatrix}. \tag{1.7}$$

Considering the condition 1.4, we find that it is the same as requiring o_n to be the identity I_n. The symmetry group G_1 is therefore reduced to

$$\left\{ \begin{pmatrix} \frac{1}{2}[g+(g^{-1})^T] & \frac{i}{2}[g-(g^{-1})^T] & 0 \\ -\frac{i}{2}[g-(g^{-1})^T] & \frac{1}{2}[g+(g^{-1})^T] & 0 \\ 0 & 0 & I_n \end{pmatrix} \times g^{-1} : g \in GL(2k,\mathbb{C}), g^{-1} = -\sigma g^T\sigma \right\}.$$

Now write

$$A_z = \begin{pmatrix} \gamma_1 + \gamma_2 \\ -i\gamma_1 + i\gamma_2 \\ c \end{pmatrix},$$

and go through the conditions 1.2, 1.3 and 1.4 to obtain

Proposition 1.8. *For a fixed group $SO(n)$, the moduli space $\mathcal{M}_k$ is the quotient of the subset of the set $\{(\gamma_1, \gamma_2, c) \in M_{2k\times 2k}(\mathbb{C}) \times M_{2k\times 2k}(\mathbb{C}) \times M_{n\times 2k}(\mathbb{C})\}$ that subjects to the following conditions:*

1.8.a) $\gamma_1^T = -\sigma\gamma_1\sigma \quad (i.e., (\sigma\gamma_1)^T = -\sigma\gamma_1)$,
1.8.b) $\gamma_2^T = -\gamma_2$,
1.8.c) $2(\gamma_1^T\gamma_2 + \gamma_2^T\gamma_1) + c^T c = 0$,

by the group action

$$\gamma_1 \longmapsto g\gamma_1 g^{-1}, \gamma_2 \longmapsto (g^{-1})^T\gamma_2 g^{-1}, c \longmapsto cg^{-1},$$

for $g \in G = \{g \in GL(2k, \mathbb{C}) : g^{-1} = -\sigma g^T \sigma\} = Sp(k, \mathbb{C})$.

2. Superposition

Much of this section will follow along the same line as the previous work [22], and it is slightly more complicated but nevertheless elementary. For clarity, I shall give a sufficiently detailed treatment.

Lemma 2.1. *Let γ_1, G and the action of G on γ_1 be the same as in 1.8. Then for each such γ_1 there exists a $g \in G$ such that*

$$g\gamma_1 g^{-1} = \begin{pmatrix} J & 0 \\ 0 & J^T \end{pmatrix},$$

where J is a $k \times k$ Jordan canonical form.

Proof. We shall write both γ_1 and g in block form with each of their blocks being a $k \times k$ square matrix. For example, we shall write $\gamma_1 = \begin{pmatrix} B & C \\ D & B^T \end{pmatrix}$. It is then easy to see that if C and D (which are necessarily skew-symmetric) of γ_1 are zero matrices, then the lemma is true. Thus we only need to show that we can make γ_1 into a diagonal-block matrix using the action of the group G. We prove it by induction on k. The case $k = 1$ is trivial. Take $k = 2$ and without loss of generality we can assume $B = \begin{pmatrix} \lambda_1 & \delta \\ 0 & \lambda_2 \end{pmatrix}, C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$, where $\delta = 0, 1$, and $c \neq 0$ or $d \neq 0$. We may simply assume $c \neq 0$. So if $\lambda_1 = \lambda_2 = \delta = d = 0$, then $g = \begin{pmatrix} e_1 & -I_2 \\ e_2 & I_2 \end{pmatrix} \in G$ will make it work, where $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Otherwise we can choose g to be of the form $g = \begin{pmatrix} tI_2 & vf \\ wf^{-1} & uI_2 \end{pmatrix}$, where $t, u, v, w \in \mathbb{C}$ such that $tu - vw = 1$ to ensure that g belongs to G, and where f is a suitable 2×2 invertible matrix. It is a direct calculation that we can make γ_1 into a diagonal-block matrix under the action g by an appropriate choice of the parameters involved.

Now assume the claim is true for $k \geq 2$, and we shall show that the claim is also true for $k+1$. The idea is to use the result for $k = 2$ k times to inductively make the last row and column of C and D zero. Each time we choose a g that is modified from the identity matrix by changing at most sixteen elements, the size of g when $k = 2$. For example, the modification for g_1 takes place at those elements that are located at the intersections of both rows and columns being numbered $1, k+1, k+2$ and $2(k+1)$. The modification is so chosen that the resulting matrix will have the property that the elements of C and D located at $(1, k+1)$ and hence at $(k+1, 1)$ will be zero (since C and D are skew-symmetric). For g_2, the modification takes place at the intersections of rows and columns numbered $2, k+1, k+3$ and $2(k+1)$. The result is to make the elements of C and D located at $(2, k+1)$ and $(k+1, 2)$ zero. Thus after k steps we have derived a new matrix γ_1 that has the property that the last row and column of its block C and block D are zero. So by the induction hypothesis there exists a g to make the remaining parts of C and D zero.

Now choose $\gamma_1 = \begin{pmatrix} J & 0 \\ 0 & J^T \end{pmatrix}$, and write $\gamma_2 = \begin{pmatrix} D_1 & D_2 \\ -D_2^T & D_3 \end{pmatrix}$ and $c = (a \quad b)$ block-wise accordingly. If we assume $J = diag(J_1(x_1), \ldots, J_r(x_r))$ with $x_1, \ldots, x_r$ being r distinct eigenvalues and $J_1(x_1), \ldots, J_r(x_r)$ standard Jordan blocks, we shall also write D_k, a and b in obvious blocks denoted as $D_k(i,j), a(i)$ and $b(i)$ respectively for $k = 1, 2, 3$, and $i, j = 1, \ldots, r$. We also need to introduce an equivalence relation on these Jordan canonical forms. We say $J = diag(J_1(x_1), \ldots, J_r(x_r))$ and $J' = diag(J_1'(x_1'), \ldots, J_r'(x_r'))$ are equivalent, denoted as $J(x) \simeq J'(x')$, if and only if $r = r'$ and $J_i(0) \sim J_i'(0)$ for all i after some permutations of the diagonal blocks of $J'(x')$ if necessary, where $\sim$ is the symbol of similarity relation. Denote then by $\mathcal{J}$ the set of equivalence classes of Jordan canonical forms; the set $\mathcal{J}$ is finite. We write $[J]$ for an element of $\mathcal{J}$. Thus we can write $\mathcal{M}_k = \coprod_{[J]} \mathcal{M}_k([J])$ as a finite disjoint union of sets according to the Jordan canonical forms of γ_1.

For convenience, we choose a fixed representation for each $[J]$, denoted by J by abuse of notation, as follows: Write $J = diag(J_1(x_1), \ldots ,$

$J_r(x_r))$. Then $J_i(x_i)$ is chosen in such a way that $J_i(0) = J_j(0)$ if $J_i(0) \sim J_j(0)$. Moreover, the order of the diagonal blocks $J_i(\ )$ is fixed. Let $S(J)$ denote the permutation group that permutes the diagonal blocks and leaves $diag(J_1(0), \dots, J_r(0))$ fixed. So such J's are uniquely represented up to the action of $S(J)$. Evidently $S(J)$ can be thought as a subgroup of G. Denote G_J the subgroup of G that fixes $\begin{pmatrix} J & 0 \\ 0 & J^T \end{pmatrix}$, which can be seen to split as a product $G_{J_1} \times \cdots \times G_{J_r}$ (see [22]). Particularly, G_J and G_{J_i} are independent of the eigenvalues of J and J_i respectively. Furthermore, the actions of $S(J)$ and G_J commute. Thus we have $G_J \times S(J) \subset G$. Finally, we denote by Δ the diagonal variety $\mathbb{C}^r$, that is the subset of $\mathbb{C}^r$ with at least two coordinates being the same. With the notation just introduced, we can formulate $\mathcal{M}_k([J])$ in terms of J, D_1, D_2, D_3, a and b according to 1.8 as follows:

Lemma 2.2. *For $n > 6$, $\mathcal{M}_k([J])$ is the quotient of the set $\{(J, D_1, D_2, D_3, a, b) \in (\mathbb{C}^r \backslash \Delta) \times M_{k\times k}(\mathbb{C}) \times M_{k\times k}(\mathbb{C}) \times M_{k\times k}(\mathbb{C}) \times M_{n\times k}(\mathbb{C}) \times M_{n\times k}(\mathbb{C})\}$ which subject to the following conditions:*

(2.2a) D_1 *and* D_3 *are skew-symmetric,*

(2.2b)
$$\begin{aligned} &2(J^T D_1 - D_1 J) + a^T a = 0,\\ &2[J^T, D_2] + a^T b = 0,\\ &2(JD_3 - D_3 J^T) + b^T b = 0, \end{aligned}$$

(2.2c)
$$\begin{pmatrix} J + xI_k & 0 \\ 0 & J^T + xI_k \\ D_1 & D_2 + yI_k \\ -D_2^T - yI_k & D_3 \\ a & b \end{pmatrix}$$
has rank $2k$ for all $x, y \in \mathbb{C}$,

by the group action

$$\begin{pmatrix} D_1 & D_2 \\ -D_2^T & D_3 \end{pmatrix} \longmapsto g \begin{pmatrix} D_1 & D_2 \\ -D_2^T & D_3 \end{pmatrix} g^{-1}, (a \quad b) \longmapsto (a \quad b)\, g^{-1},$$

for $g \in G_J \times S(J) \subset G$.

Before we give the main theorem of this section we shall need one more notation. We define FJ_i to be the subset of $\mathcal{M}_{k_i}$ (as defined in 1.8) such that $\gamma_1 \sim J_i(0)$, where $J_i(0)$ is a $k_i \times k_i$ Jordan block with eigenvalue 0.

trivial holomorphic fibration:

$$\pi : \mathcal{M}_k([J]) \longrightarrow \mathbb{DP}^r(\mathbb{C}),$$

with fiber at any point

$$FJ = \coprod_{1}^{n([J])} FJ_1 \times FJ_2 \times \cdots \times FJ_r,$$

the disjoint union of $n([J])$ *copies of* FJ*, where* $n([J]) = \frac{r!}{|S(J)|}$.

Proof. The map π just associates to a point in $\mathcal{M}_k([J])$ the distinct eigenvalues of J, thought of as unordered. Obviously, there are $n([J])$ distinct J's up to the action of $G_J \times S(J)$ that map to the same point in $\mathbb{DP}^r(\mathbb{C})$. So we have seen that a fiber over each point is $n([J])$ disjoint sets. To understand the fibers, we need to analyze Lemma 2.2. Two facts which were essentially proved in [22, Lemma 2.11 and 2.12] are needed to show this theorem.

Fact 2.3a): Equations 2.2*b*) are equivalent to

$$\begin{aligned}
&2(J_i^T D_1(i,i) - D_1(i,i)J_i) + a(i)^T a(i) = 0,\\
&2[J_i^T, D_2(i,i)] + a^T(i)b(i) = 0,\\
&2(J_i D_3(i,i) - D_3(i,i)J_i^T) + b(i)^T b(i) = 0,
\end{aligned}$$

$$\text{for } i = 1, \dots, r.$$

Fact 2.3b): The rank condition 2.2*c*) is equivalent to the condition that

$$\begin{pmatrix}
J_i(0) & 0\\
0 & J_i^T(0)\\
D_1(i,i) & D_2(i,i) + yI\\
-D_2^T(i,i) - yI & D_3(i,i)\\
a(i) & b(i)
\end{pmatrix}$$

has rank k_i for all $y \in \mathbb{C}$, and $i = 1, \dots, r$.

The proof of fact 2.3*a*) requires one to solve $D_k(i,j)$, for $i \neq j$, in terms of J_i, J_j, $D_k(i,i)$, $D_k(j,j)$, $a(i), a(j), b(i)$ and $b(j)$. This can be done by a method of bootstrapping as in [22]. A simple formula will be given in the last section. The proof of fact 2.3*b*) can be outlined as follows. We note that 2.2*c*) is equivalent to the condition that

$$\begin{pmatrix}
J - x_i I_k & 0\\
0 & J^T - x_i I_k\\
D_1 & D_2 + yI_k\\
-D_2^T - yI_k & D_3\\
a & b
\end{pmatrix} \tag{2.4}$$

has rank $2k$ for all $y \in \mathbb{C}$, and for $i = 1, \ldots, r$. Denote by $D_k(i)$ the matrix obtained from D_k by replacing all blocks of D_k by zero except those j-column blocks $D_k(i,j)$ where $i \neq j$, and let K_i be the matrix obtained from $diag(J_1(x_1 - x_i), \ldots, J_r(x_r - x_i))$ by changing $J_i(0)$ to I_{k_i}. Let

$$\tilde{D}(i) = \begin{pmatrix} -D_1(i) & -D_2(i) \\ D_2(i)^T & -D_3(i) \end{pmatrix}, \quad \tilde{K}_i = \begin{pmatrix} K_i^T & 0 \\ 0 & K_i \end{pmatrix}, \quad \tilde{c} = \begin{pmatrix} \frac{1}{2}a^T \\ \frac{1}{2}b^T \end{pmatrix}.$$

Then

$$\begin{pmatrix} I_{2k} & 0 & 0 \\ \tilde{D}(i) & \tilde{K}_i & \tilde{c} \\ 0 & 0 & I_n \end{pmatrix}$$

is nonsingular. Left multiplying it by (2.4) reveals that the submatrix of the resulting matrix formed by the columns where $J_i(0)$ and $J_i^T(0)$ are located, deleting the obvious zero-block rows, is precisely the matrix in 2.3*b*) (note: 2.2*b*) is used in the calculation). This verifies the fact 2.3*b*). Since both 2.3*a*) and 2.3*b*) actually do not involve the eigenvalues of J, we see that the fibers at different points are precisely FJ. Moreover, it is now clear to see the natural local holomorphic trivial fibration from the map π.

So far we have been only doing linear algebra and leaving the more interesting geometric counterpart unmentioned. To give an indication of the geometry behind our analysis, we shall make some remarks. Remember that these monads construct certain semi-stable holomorphic bundles over $\mathbb{CP}^2$. If we choose appropriate coordinates, then we can see that x parameterizes all lines other than l_∞ through a fixed point in l_∞. The bundle is trivial restricted to all but finite number of these lines. A line to which the bundle restricted is nontrivial is called a *jumping line*. The jumping lines are precisely given by $x = -$eigenvalues of γ_1. J, the Jordan canonical form of γ_1, can be thought of as an invariant of the versal deformations of the jumping lines in their formal neighborhoods. It encodes the same information as the *graphs* associated to the jumping lines introduced in [6]. The *multiplicity* (see [6]) of a jump is the same as the corresponding size of the Jordan block. Thus the sum of the multiplicities of the jumping lines is precisly k, the size of α_i, which is the second Chern class of the bundle. Theorem 2.3 shows a striking superposition phenomenon of these instanton spaces, namely, such a bundle up to isomorphism is constructed by grafting "jumps" to a trivial bundle. In particular, we can give an explicit formula for Taubes' grafting map:

$$\iota_k : \mathcal{M}_k \longrightarrow \mathcal{M}_{k+1}, \tag{2.5}$$

using Theorem 2.3. For example, we can use dilation to map $\mathcal{M}_k$ conformally into a subset of it in such a way that all "jumps" will be inside a unit disk. We then add a jump of "multiplicity" one outside of the unit disk (see [22, p.125], $|\alpha_1|$ should be defined using eigenvalues of α_1, e.g. $\max\{|\text{eigenvalue of } \alpha_1|\}$). We shall adopt such a convention. It can be shown that the space from direct limit process using the aforementioned map is really the one obtained by Taubes up to homotopy (see [6]). Thus the following simplified version of a theorem of Taubes can be applied.

Theorem [21] **2.6.** *Let G be a compact, simple Lie group. Let $\mathcal{M}_k$ be corresponding instanton moduli space over S^4 with minus half the first Pontryagin index k. Let $\mathcal{M}_\infty$ be the direct limit of the $\mathcal{M}_k'$s and let $\vartheta : \mathcal{M}_\infty \longrightarrow \Omega_0^3 SU(n)$ be the direct limit of maps ϑ_k. Then ϑ is a homotopy equivalence.*

3. Stratification

For a given Jordan block J_i, FJ_i as defined in the previous section is not necessarily a manifold. However, it is a simple fact from algebraic geometry that FJ_i can be decomposed uniquely as a finite number of disjoint smooth manifolds

$$FJ_i = \coprod_{s=1}^{l(J_i)} FJ_i^s \tag{3.1}$$

such that $\dim_{\mathbb{C}} FJ_i^s < \dim_{\mathbb{C}} FJ_i^{s-1}$ (see [22]). Here the FJ_i^s are obviously in descending order according to s. From now on we shall always write $J_i(0) = diag(D^1, \dots, D^{i_t})$ such that the D^j are in descending order according to their sizes, where the D^j are obvious (indecomposable) Jordan blocks. Let $\hat{J}_i(0) = diag(\hat{D}^1, \dots, \hat{D}^{i_u})$. Then we say $J_i \le \hat{J}_i$ if the size of J_i is greater than or equal to that of $\hat{J}_i$, and the size of D^j is greater than or equal to that of $\hat{D}^j$ for all $j = 1, \dots, i_t$. (Note that this ordering is slightly different from the previous one in [22], but it works equally well). For each minus half the first Pontryagin index k, we can now introduce our index set

$$\mathbb{I}_k = \{J^* = (J_1^{s_1}, J_2^{s_2}, \dots, J_r^{s_r})\}, \tag{3.2}$$

where the $J_i^{s_i}$ are in descending order first according to the order of J_i we have just defined, then according to s_i as indicated in 3.1.

Now define $\mathbb{I}_k^1 = \{(J_1^{s_1}, \dots, J_r^{s_r}) \in \mathbb{I}_k :$ the size of J_r is $1\}$, and $\mathbb{I}_k^2 = \mathbb{I}_k \setminus \mathbb{I}_k^1$. Then we can give the index set $\mathbb{I}_k$ a well-ordering as follows:

First $\mathbb{I}_k^1 < \mathbb{I}_k^2$, then order both $\mathbb{I}_k^1$ and $\mathbb{I}_k^2$ by the natural lexicographical ordering defined above. We denote this well-ordering by $(\mathbb{I}_k \leq)$. So $\mathbb{I}_k$ with this ordering is denoted by $(\mathbb{I}_k, \leq)$. The particular choice of the order $\mathbb{I}_k^1 < \mathbb{I}_k^2$ is important here. It is mainly due to the effect of the Taubes grafting map 2.6, whose impact can been see through the proof of Theorem A (see [6]). Roughly speaking, because the image of the map 2.6 only lies in the strata whose indices belong to $\mathbb{I}_{k+1}^1$, and because the (complex) codimension of the each stratum whose index belongs to $\mathbb{I}_k^2$ is greater than or equal to $[k/2]$ as we will see in section 4, these strata whose indices belong to $\mathbb{I}_k^2$ have no effect to the topology of $\mathcal{M}_k$ in complex dimension less than $[k/2]$ (or real dimension 2[k/2]). Using Theorem 2.3, we can easily see that

$$\mathcal{M}_k = \coprod_{J^* \in \mathbb{I}_k} \mathcal{M}_k(J^*), \tag{3.3}$$

where $\mathcal{M}_k(J^*)$ are now smooth manifolds. This is because when restricted to the fibration 2.3 both the base and the fiber of $\mathcal{M}_k(J^*)$ are now manifolds. Thus we have obtained a desired stratification of the manifold $\mathcal{M}_k$ — the *L-stratification* (see [6]).

Theorem 3.4. *The stratification* 3.3 *of $\mathcal{M}_k$ is a L-stratification: i.e.,*

3.4*a* *The index set $\mathbb{I}_k$ is finite with a fixed well ordering $\leq$.*

3.4*b* *If $J^{(0)}$ is the smallest element in $(\mathbb{I}_k, \leq)$, then $\mathcal{M}_k(J^{(0)})$ is an open dense subset of $\mathcal{M}_k$.*

3.4*c* *For all $J^* \in \mathbb{I}_k$ the union of the submanifolds of the same or smaller order*

$$Z(J^*) = \bigcup_{K \leq J^*} \mathcal{M}_k(K)$$

is an open dense submanifold of $\mathcal{M}_k$.

3.4*d* *For all $J^* \in \mathbb{I}_k$ the normal bundle $\nu(J^*)$ of $\mathcal{M}_k(J^*)$ in $\mathcal{M}_k$ is orientable.*

The proof of the theorem is straightforward, and we shall omit it (see [22]). Two important features of this stratification are as follows: Firstly, the stratification of $\mathcal{M}_k$ leads naturally to a homology Leray spectral sequence converging to filtrations of the homology of $\mathcal{M}_k$. The expression of the E^1 term is rather simple (see [6]). Secondly, the stratifications for $\mathcal{M}_k$ and $\mathcal{M}_{k+1}$ are preserved by the map ι_k, i.e., the map

ι_k sends a stratum in $\mathcal{M}_k$ into a stratum in $\mathcal{M}_{k+1}$. It is thus a beautiful idea of [6] that in order to prove Theorem A of the introduction, all we need to do is to analyze the dimensions of the strata and the isomorphism ranges of the map ι_k induced on homology of these strata. We shall refer the reader to [6] for details. A sketch of the proof was also outlined in [22]. The dimension analysis will be given in the next section. Particularly, the Corollary 4.3a) is all we need, As to the latter, a crucial lemma whose proof is rather involved is given by BHMM as follows.

Lemma 3.5. [6, Lemma 7.8]. *For all k and all primes p, the natural inclusion ι restricted to stratum J^* induces an isomorphism in homology*

$$(\iota(J^*))_t : \ H_t(\mathcal{M}_k(J^*); \mathbb{Z}/p) \cong H_t(\mathcal{M}_{k+1}(\iota(J^*)); \mathbb{Z}/p)$$

for $t \leq q(k, J^*) = [j/2]$, where j is the number of J_i in $J = diag(J_1, \ldots, J_r)$ whose size is 1.

In the next section we will also show that the moduli spaces $\mathcal{M}_k$ are simply connected for $n > 6$. Since $\pi_1(\Omega_0^3 SO(n)) = \pi_4(SO(n)) = 0$ for $n > 5$, the proof of the Atiyah-Jones conjecture reduces to the proof of a conjecture of its homology version. Combining Theorem A and Theorem 2.6 of Taubes, we then get our main result

Theorem 3.6. *For $n > 6$ and $k > 0$, the induced map (from ϑ_k)*

$$(\vartheta_k)_t : \pi_t(\mathcal{M}_k) \longrightarrow \pi_t(\Omega_0^3 SO(n))$$

is an isomorphism for $t \leq q(k) = [k/2] - 1$.

4. Dimension counting and the fundamental group

In this section we give an upper bound on the dimension of each stratum which suffices for proving Theorem A, and show that the $\mathcal{M}_k$ are all simple connected for $n > 6$. Unless stated otherwise, dimension will be complex dimension.

From Theorem 2.3 we have $\dim \mathcal{M}_k(J^*) = r + \dim FJ_1^{s_1} + \cdots + \dim FJ_r^{s_r} \leq r + \dim FJ_1^1 + \cdots + \dim FJ_r^1$. So we need to analyze the dimensions of FJ_i. Assume that size$J_i = m_i$. To do so, we shall return to Lemma 2.2. Observe that if we consider equations 2.2b) as systems of linear equations for D_1, D_2 and D_3, then by comparing the defining equations for G_{J_i} (linearized at the identity) with those of D_i's reveals that for $a = 0$ and $b = 0$, the total number of parameters of the solution space for D_1, D_2 and D_3 is $2m_i$ less than the dimension of group G_{J_i}. Since the rank condition 2.2c) is an open condition (this is the case only if FJ_i is nonempty which is true when $n > 6$; see the proof of Proposition

4.4). This implies that $\dim FJ_i = \dim \mathcal{BA}(J_i) - 2m_i$, where $\mathcal{BA}(J_i) = \{(a,b) \in M_{n\times m_i}(\mathbb{C}) \times M_{n\times m_i}(\mathbb{C}) : a \text{ and } b \text{ solve } 2.2b)\}$. It turns out that $\mathcal{BA}$ is given by a system of homogeneous quadratic equations. If we write $J_i = diag(D^1, \dots, D^m)$ as in previous section, it is easy to verify that

$$\mathcal{BA}(J_i) \subset \mathcal{BA}(D^1) \times \cdots \times \mathcal{BA}(D^m). \tag{4.1}$$

Lemma 4.2. *Let D^i be a size k_i block as in* 4.1. *Then the dimension of $\mathcal{BA}(D^i)$ is $(2n-3)k_i$.*

Proof Recall that D^i is a $(k_i \times k_i)$-matrix whose only nonzero elements are 1's located just above the main diagonal. If we write $a = (a_1, \dots, a_{k_i})$ and $b = (b_1, \dots, b_{k_i})$, then the defining equations for $\mathcal{BA}(D^i)$ can now be written as

4.2a) $\sum_j a^T_{i-j+1} a_j = 0$, for $1 = 1, \dots, k_i$,

4.2b) $\sum_j b^T_{i-j+k_i} b_j = 0$, for $1 = 1, \dots, k_i$,

4.2c) $\sum_j b^T_{i+j-1} a_j = 0$, for $1 = 1, \dots, k_i$.

The Jacobian of these equations is $\begin{pmatrix} 2A & 0 \\ 0 & 2B' \\ B & A' \end{pmatrix}$, where

$$A = \begin{pmatrix} a_1^T & & & 0 \\ a_2^T & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{k_i}^T & \cdots & a_2^T & a_1^T \end{pmatrix}, B = \begin{pmatrix} b_{k_i}^T & & & 0 \\ b_{k_i-1}^T & \ddots & & \\ \vdots & \ddots & \ddots & \\ b_1^T & \cdots & b_{k_i-1}^T & b_{k_i}^T \end{pmatrix},$$

$$A' = \begin{pmatrix} 0 & & & a_1^T \\ & & \cdot^{\cdot^{\cdot}} & a_2^T \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ a_1^T & a_2^T & \cdots & a_{k_i}^T \end{pmatrix}, B' = \begin{pmatrix} b_{k_i}^T & b_{k_i-1}^T & \cdots & b_1^T \\ & \ddots & \ddots & \vdots \\ & & \ddots & b_{k_i-1}^T \\ 0 & & & b_{k_i}^T \end{pmatrix}.$$

For these points of $\mathcal{BA}(D^i)$ that also satisfy rank condition 2.2c), i.e., rank $\begin{pmatrix} a_1 & b_{k_i} \end{pmatrix} = 2$, we can easily see that the corresponding Jacobian is of maximal rank $3k_i$. Thus the claim of the lemma follows.

Corollary 4.3. *a): Let $J = diag(J_1, \dots, J_r)$ be as before. Then* $\dim \mathcal{M}_k(J^*) \leq (2n-5)k + r$ *(note:* $\dim \mathcal{M}_k = (2n-4)k$*). Moreover, the equality holds if* $\operatorname{rank}(J - xI) \geq \operatorname{size}(J) - 1$, *for all $x \in \mathbb{C}$. b): If $J^* \in \mathbb{I}_k^2$, then $r \leq [\frac{k}{2}]$. This implies that* $\dim \mathcal{M}_k(J^*) \leq 2(n-2)k - [\frac{k}{2}]$.

In the remaining of this section we shall show

Proposition 4.4. *The fundamental group of $\mathcal{M}_k$ is trivial for all k and all $n > 6$.*

Proof. The idea is the same as the one given in [22, Theorem 4.14] followed from [13], so we shall sketch the proof. Let $\mathcal{M}_k(J^{(0)})$ and $\mathcal{M}_k(J^{(1)})$ denote the first two strata of $\mathcal{M}_k$ as given in 3.4. Then the complement of $\mathcal{M}_k(J^{(0)}) \cup \mathcal{M}_k(J^{(1)})$ in $\mathcal{M}_k$ has complex codimension at least two, so $\pi_1(\mathcal{M}_k) = \pi_1(\mathcal{M}_k(J^{(0)}) \cup \mathcal{M}_k(J^{(1)}))$. Thus by a transversality argument we conclude that any loop $L \subset \mathcal{M}_k(J^{(0)}) \cup \mathcal{M}_k(J^{(1)})$ that represents a homotopy can be chosen to lie in $\mathcal{M}_k(J^{(0)})$ such that the homotopy of L intersects $\mathcal{M}_k(J^{(1)})$ in finite discrete points. If we restrict the projection map π in 2.3 to the union of these two strata, the image of the projection can be easily seen to be simply connected. Thus the image of L is contractible. Local triviality of the fibration $\pi : \mathcal{M}_k(J^{(0)}) \longrightarrow \mathbb{DP}^k(\mathbb{C})$ permits us to shrink the loop L in the "horizontal" direction. This process only stops at finitely many points where the loop becomes "near" those points in $\mathcal{M}_k(J^{(1)})$. But as we can show (see [22]), the loop can actually be continuously pushed through these points, and eventually the loop is homotoped to the one on the fiber of the first stratum, $FJ^{(0)}$. This shows that the fundamental group is a quotient of the fundamental group of $FJ^{(0)}$. It is clear from 2.3 that $FJ^{(0)}$ is homotopic to a k-fold product of $\mathcal{M}_1$. To finish the proof we now show that $\pi_1(\mathcal{M}_1) = 0$. From its definition it is easy to see that $\mathcal{M}_1$ is homotopic to the quotient space $\{(a,b) \in \mathbb{C}^n \times \mathbb{C}^n : a^Ta = b^Tb = a^Tb = 0, \mathrm{rank}\,(a \quad b) = 2\}/SL(2,\mathbb{C})$. Up to homotopy, we may assume that a, b belong to the spheres of radius two and four in $\mathbb{C}^n$ respectively. Then the condition that $a^Ta = 0$ and $b^Tb = 0$ imply that $a \in T_1S_1^{n-1}$, the unit tangent bundle of the unit $(n-1)$-sphere (or Stiefel manifold $V(2,n)$), and that $b \in T_2S_2^{n-1}$, the length two tangent bundle of the $(n-1)$-sphere of radius two. Since $SL(2,\mathbb{C})$ is homotopically a three-sphere, using the homotopy long exact sequence for a fibration we can see that our problem becomes to calculate the fundamental group of the set $V_n = \{(a,b) \in T_1S_1^{n-1} \times T_2S_2^{n-1} \subset \mathbb{C}^n \times \mathbb{C}^n : \mathrm{rank}\,(a \quad b) = 2, a^Tb = 0\}$ which turns out to be a manifold. Moreover, the projection of V_n to $T_1S_1^{n-1}$ by its first factor is a fibration. The fiber is homotopically $T_1S_1^{n-3}$. Applying the homotopy exact sequence for a fibration again, we find that $\pi_1(V_n) = \{0\}$ for $n > 5$.

5. $Sp(n)$ case

Now we sketch the proof for $Sp(n)$. The stratification becomes simpler than the previous case.

Proposition 5.1. *The moduli space $\mathcal{M}_k(Sp(n))$ is the quotient of the set of $(\alpha_1, \alpha_2, a, b) \in M_{k\times k}(\mathbb{C}) \times M_{k\times k}(\mathbb{C}) \times M_{n\times k}(\mathbb{C}) \times M_{n\times k}(\mathbb{C})$*

satisfying the following conditions:

5.1a) α_1, α_2 *are symmetric,*
5.1b) $[\alpha_1, \alpha_2] + a^T b - b^T a = 0$,
5.1c) $\begin{pmatrix} \alpha_1 + xI_k \\ \alpha_2 + yI_k \\ a \\ b \end{pmatrix}$

has rank k for all $x, y \in \mathbb{C}$ by the action of the group $O(k, \mathbb{C})$ given by

$$\alpha_i \mapsto g\alpha_i g^{-1}, a \mapsto ag^{-1}, b \mapsto bg^{-1}.$$

Proof We could follow the same steps as in Section 1 to prove this proposition, although Donaldson's treatment (see [10]) is more appealing. We shall sketch it here. The ADHM construction (which constructs the moduli space of instanton bundles over $\mathbb{CP}^3$) gives the following description of $\mathcal{M}_k(Sp(n))$ (see [1]): The data consists of $\{B, \Lambda\}$, where B is a quaternion valued $k \times k$ matrix, and Λ is a quaternion valued $n \times k$ matrix. The constraints on the data are a): $\Lambda^*\Lambda + B^*B$ is real, where $*$ stands for quaternion conjugation and transpose; b): B is symmetric; c): For any quaternion x, $\begin{pmatrix} B - xI_k \\ \Lambda \end{pmatrix} \chi = 0$ implies the quaternion vector $\chi = 0$. There is an $O(k)$ action given by $B \mapsto gBg^T, \Lambda \mapsto \Lambda g^T$. If we write $\Lambda = \bar{a} + bj, B = \bar{\alpha}_1 + \alpha_2 j$, then b) is obviously the same as 5.1a). The reality condition a) can be rewritten as follows:
5.2a) $[\alpha_1, \alpha_2] + a^T b - b^T a = 0$,
5.2b) $[\bar{\alpha}_1, \alpha_1] + [\bar{\alpha}_2, \alpha_2] + a^* a - a^T \bar{a} + b^* b - b^T \bar{b} = 0$.
Denote $c = \begin{pmatrix} \alpha_1 \\ a \end{pmatrix}$ and $d = \begin{pmatrix} \alpha_2 \\ b \end{pmatrix}$. Then the rank condition c) is the same as the condition that rank $\begin{pmatrix} c & -\bar{d} \\ d & \bar{c} \end{pmatrix} = 2k$. Using 5.2$a$) we have

$$\begin{pmatrix} d & \bar{c} \\ -c & \bar{d} \end{pmatrix}^T \begin{pmatrix} c & -\bar{d} \\ d & \bar{c} \end{pmatrix} = \begin{pmatrix} 0 & -d^T \bar{d} - c^T \bar{c} \\ \bar{c}^T c + \bar{d}^T d & 0 \end{pmatrix}.$$

This implies that c) is the same as 5.1c). Now take the data $(\alpha_1, \alpha_2, a, b)$ as in the proposition and define a Hermitian metric by $||(\alpha_1, \alpha_2, a, b)||^2 = ||\alpha_1||^2 + ||\alpha_2||^2 + 2||a||^2 + 2||b||^2$. Let $O(k, \mathbb{C})$ act on these data as in the proposition. Then obviously the subgroup of $O(k, \mathbb{C})$ that fixes this metric is precisely $O(k)$. Moreover, the moment map of this action is $\mu = [\bar{\alpha}_1, \alpha_1] + [\bar{\alpha}_2, \alpha_2] + a^* a - a^T \bar{a} + b^* b - b^T \bar{b}$ which is the left-hand side of 5.2b). Donaldson then showed that the conditions 5.1b) and 5.1c)

imply that the orbits of the action of $O(k,\mathbb{C})$ are stable. By geometric invariant theory this implies that the above ADHM description is equivalent to the proposition.

Starting from this proposition, the L-stratification becomes much simpler. We can put α_1 into diagonal block form using the action of $O(k,\mathbb{C})$ according to the eigenvalues of α_1 and their multiplicities. This amounts to choosing an orthonormal basis (with respect to the standard symmetric form) for each minimal invariant subspace of α_1. Denote a diagonal block of α_1 by $N(i)+\mu_i I$ such that $N(i)$ is a nilpotent matrix and μ_i is a scalar. Denote by $\alpha(ij)$ a block of α_2. Then the $\alpha(ij)$ satisfy the following equation:

$$[(\mu_i-\mu_j)I+N(i)]\alpha(ij)=\alpha(ij)N(j)+L,$$

where $L=b(i)^Ta(j)-a(i)^Tb(j)$. The important issue is that the formulation of L does not involve any $\alpha(ij)$. Note that from the above formula we inductively have

$$[(\mu_i-\mu_j)I+N(i)]^n\alpha(ij)=\alpha(ij)N(j)^n+L'.$$

Again L' is formulated without involving any $\alpha(ij)$. Since $N(j)$ is nilpotent $N(j)^n=0$ if n is large enough. $[(\mu_i-\mu_j)I+N(i)]$ is nonsingular if $i\neq j$ since $\mu_i\neq\mu_j$ and $N(i)$ is nilpotent. Thus $\alpha(ij)$ can be uniquely determined by diagonal blocks. With obvious steps we get a theorem similar to Theorem 2.8 in [22] (or Theorem 3.5 in [13]). By stratifying the 'fibers' into smooth manifolds abstractly we can derive the desired stratification. Theorem A is again a formal consequence of the stratification. However, as we can check that the moduli spaces $\mathcal{M}_k(Sp(n))$ are not simply connected. In fact, $\pi_1(\mathcal{M}_k(Sp(n)))=\mathbb{Z}/2$, which is the same as $\pi_1(\Omega^3_0Sp(n))$, since $\mathcal{M}_1(Sp(n))$ is easily seen to be homotopy equivalent to $\mathbb{RP}^{4n-1}$. Thus the argument of [6], [7] can be applied to give the proof of the following theorem:

Theorem 5.3. *For all positive integers n and k, the map (induced from ϑ_k)*

$$(\vartheta_k)_t:\pi_t(\mathcal{M}_k(Sp(n)))\longrightarrow\pi_t(\Omega^3_0Sp(n))$$

is an isomorphism for $t\leq q(k)=[k/2]-2$.

Remark. We have now superficially three different stratifications for group $SU(2)=Sp(1)$: [6], [22] and one that follows from Proposition 5.1. It is natural to expect that all of them are equivalent from the geometry underlying the stratifications.

6. The Atiyah-Jones conjecture and Bott periodicity

The purpose of this section is to point out a connection between the Atiyah-Jones conjecture, Bott Periodicity and some "partial symmetries" present in these moduli spaces. To begin with, note from the monad description of the moduli spaces that there are natural inclusion maps

$$j(K(n),k):\ \mathcal{M}_k(K(n)) \to \mathcal{M}_k(K(n+1))$$

for $K(n) = SU(n), Sp(n)$ and $SO(n)$ respectively. These maps can be described by adding appropriate rows of zeros to the appropriate matrices. Geometrically, it amounts to the Whitney sum of adding a trivial bundle. Kirwan [15] first showed that these maps $j(K(n),k)$ stabilize homotopically to $BU(k)$, the classifying space for $U(k)$ when $K(n) = SU(n)$. We shall call this stabilization *Kirwan's stabilization*. Her result has recently been extended to symplectic groups by Sanders [20]. Using the concrete description of $\mathcal{M}_k(K(n))$ which we have obtained, Sander's short proof fits nicely here. So I shall include it here.

Theorem 6.1. [15], [20] *Let $\mathcal{M}_k(SU), \mathcal{M}_k(Sp)$ and $\mathcal{M}_k(SO)$ denote the direct limits of the moduli spaces $\mathcal{M}_k(SU(n)), \mathcal{M}_k(Sp(n))$ and $\mathcal{M}_k(SO(n))$ (as $n \to \infty$) respectively. Then we have the following homotopy equivalences:*

$$\mathcal{M}_k(SU) \sim BU(k), \mathcal{M}_k(Sp) \sim BO(k), \mathcal{M}_k(SO) \sim BSp(k).$$

Here $BU(k), BO(k)$ and $BSp(k)$ are the classifying spaces for $U(k), O(k)$ and $Sp(k)$ respectively.

Proof. From the monad description of the moduli spaces in [10] and the preceding sections, we have seen that all those moduli spaces are obtained by taking quotient of various affine algebraic varieties (say $V_k(K(n))$) by the corresponding complex algebraic groups (say $G_K(k)$). All the actions of the groups are free. $G_K(k)$ remains the same when n (the size of the structure group) changes. Note that the image of $V_k(K(n))$ under the inclusion $j(K(n+lk-1),k) \circ \cdots \circ j(K(n),k)$ is contractible (can be coned off) to a point that lies outside of the image, where $l = 1$ for $Sp(n)$, $l = 2$ for $SU(n)$ and $l = 4$ for $SO(n)$. This shows that the direct limit of $V_k(K(n))$ as n goes to infinity is a contractible CW complex. By the definition of the classifying spaces, this proves the theorem.

Next, notice that the moduli spaces of all connections with structure groups $SU(n)$, $Sp(n)$ and $SO(n)$ are naturally homotopic to $\Omega_0^3 SU(n), \Omega_0^3 Sp(n)$ and $\Omega_0^3 SO(n)$ respectively [4]. Now we can perform the following direct limits

$$\lim_{n\to\infty}\lim_{k\to\infty}\mathcal{M}_k(SU(n)) \sim \lim_{n\to\infty}\Omega_0^3 SU(n) = \Omega_0^3(SU),$$
$$\lim_{n\to\infty}\lim_{k\to\infty}\mathcal{M}_k(Sp(n)) \sim \lim_{n\to\infty}\Omega_0^3 Sp(n) = \Omega_0^3(Sp),$$
$$\lim_{n\to\infty}\lim_{k\to\infty}\mathcal{M}_k(SO(n)) \sim \lim_{n\to\infty}\Omega_0^3 SO(n) = \Omega_0^3(SO).$$

The homotopy equivalence in each of the above identifications uses the Atiyah-Jones conjecture which we proved, or more precisely, the stable Atiyah-Jones conjecture solved by Taubes. By changing the orders of the limits, we also have:

$$\lim_{k\to\infty}\lim_{n\to\infty}\mathcal{M}_k(SU(n)) \sim \lim_{k\to\infty} BU(k) = BU,$$
$$\lim_{k\to\infty}\lim_{n\to\infty}\mathcal{M}_k(Sp(n)) \sim \lim_{k\to\infty} BO(k) = BO,$$
$$\lim_{k\to\infty}\lim_{n\to\infty}\mathcal{M}_k(SO(n)) \sim \lim_{k\to\infty} BSp(k) = BSp.$$

Here the homotopy equivalence in each of the above three identifications uses Kirwan's stabilization. As two limits homotopy commute which is due to the fact that $j(K(n),k)$ and the Taubes inclusions ι_k homotopy commute (actually, by choosing the Taubes inclusions ι_k appropriately the diagrams can be made commutative), we obtain Bott Periodicity for Sp and SO and partial periodicity for SU (we get period 4 instead of 2). Assuming that we have done the calculations for $\pi_1(U(3)), \pi_2(U(3)), \pi_3(U(3))$ and $\pi_4(U(3))$, we recover Bott Periodicity for U. One can clearly see that the Atiyah-Jones conjecture is in certain sense a generalization of the Bott Periodicity. They are rooted in Morse theory.

Now let us take a closer look at these two stabilization processes and try to understand the similarity between $\mathcal{M}_k(K(n))$ and $\mathcal{M}_n(K(k))$ that the above discussion suggests. Firstly, if we consider the dimensions of the pair as quadratic polynomial functions of k and n, then the leading order terms are the same. In the case where $K(n) = SU(n)$, both $\mathcal{M}_k(SU(n))$ and $\mathcal{M}_n(SU(k))$ are of the same dimension. Secondly, as is also clear from Theorem B and Theorem 6.1, the lower homotopy groups of the pair are the same. If we assume both k and n are not too small, it is not hard to figure out the exact isomorphism range for each pair (see below for more details). Thirdly, notice that Kirwan actually obtained the best possible range $q(k) = 2k + 1$ for the Atiyah-Jones conjecture when $K(n) = SU(n)$. The $q(k)$ was obtained as the homotopy isomorphism range of the inclusion $BU(k) \to BU(k+1)$. The point which we want to make here is that the stable range $q(k)$ as $k \to \infty$ is in certain sense obtained by studying the limit spaces of $\mathcal{M}_k(SU(n))$

as $n \to \infty$. It is not hard to see that Kirwan's argument extends to the cases where $K(n) = SO(n), Sp(n)$ using the explicit description of the moduli spaces we obtained. Since $BO(k) \to BO(k+1) \to BO$ induces the homotopy isomorphism $\pi_i(BO(k)) \to \pi_i(BO(k+1))$ for $i \leq k$ and since $BSp(k) \to BSp(k+1) \to BSp$ induces the homotopy isomorphism $\pi_i(BSp(k)) \to \pi_i(BSp(k+1))$ for $i \leq 4k+3$, we can state the following result (for a proof see [15]).

Theorem 6.2. *The best possible $q(k)$ for the Atiyah-Jones conjecture are $4k+3$ if $K(n) = SO(n)$, $2k+1$ if $K(n) = SU(n)$ and k if $K(n) = Sp(n)$ with at most a few exceptions in each case when n is small.*

On the other hand, Kirwan showed that $\pi_i(\mathcal{M}_k(SU(n))) = \pi_i(BU)$ for $i \leq n-2$. We can ask the same question about the best possible isomorphism ranges, say $r(n)$, for Kirwan's stabilization $\mathcal{M}_k(K(n)) \to \mathcal{M}_k(K(\infty))$. The natural expectation would be that the partial symmetry still holds. Namely, we expect, with possibly a few exceptions, that

$$r(n) = \begin{cases} 2n-3, & \text{for } \; K(n) = SU(n), \\ n-4, & \text{for } \; K(n) = SO(n), \\ 4n-1, & \text{for } \; K(n) = Sp(n) \end{cases}$$

for the inclusion maps $\Omega_0^3(K(n)) \to \Omega_0^3(K(n+1))$ induce the homotopy equivalences in the same ranges $r(n)$ above. Now let us take a look at the simplest case where $k=1$. Then we have

$$\mathcal{M}_1(Sp(n)) \sim \mathbb{RP}^{4n-1} \to BO(1) \sim \mathbb{RP}^{\infty},$$

$$\mathcal{M}_1(SU(n)) \sim T_1\mathbb{CP}^{n-1} \to BU(1) \sim \mathbb{CP}^{\infty},$$

$$\mathcal{M}_1(SO(n)) \sim (T_1 S^{n-3} \times_{Twist} T_1 S^{n-1})/S^3 \to BSp(1) \sim \mathbb{HP}^{\infty}.$$

Here $T_1 X$ stands for the unit tangent vector bundle of X. So the above expectation for $r(n)$ holds when $K(n) = SU(n)$ and $Sp(n)$ and may be off by one in the case where $K(n) = SO(n)$. Thus it may well be true that the best possible $r(n)$ are given by

$$r(n) = \begin{cases} 2n-3, & \text{for } \; K(n) = SU(n), \\ n-5, & \text{for } \; K(n) = SO(n), \\ 4n-1, & \text{for } \; K(n) = Sp(n). \end{cases} \tag{6.3}$$

Finally we give an indication on how this partial symmetry might be seen. It is in a certain sense related to the symmetry of the Grassmanians $Gr(n, n+k)$ and $Gr(k, n+k)$. For simplicity, we shall only treat the case where $K(n) = Sp(n)$. The other two cases can be done entirely analogously. Let $E = [(\alpha_1, \alpha_2, a, b)] \in \mathcal{M}_k(Sp(n)$ and let $F = [(\beta_1, \beta_2, c, d)] \in \mathcal{M}_n(Sp(k))$. We say E and F are *related* if there are representatives $(\alpha_1, \alpha_2, a, b)$ of E and (β_1, β_2, c, d) of F such that

$$H = \begin{pmatrix} \alpha_1 & \alpha_2 & a^T & b^T \\ -\alpha_2^* & \alpha_1^* & -b^* & a^* \\ d^T & c^T & \beta_1 & \beta_2 \\ -c^* & d^* & -\beta_2^* & \beta_1^* \end{pmatrix}$$

is an automorphism of $\mathbb{C}^{2k+2n}$ and such that

$$HH^* = \begin{pmatrix} \# & 0 & 0 & 0 \\ 0 & \# & 0 & 0 \\ 0 & 0 & \# & 0 \\ 0 & 0 & 0 & \# \end{pmatrix}.$$

As usual, $*$ here stands for the complex conjugate transpose and # means that there is no further restriction on these diagonal blocks. Define $\mathcal{L}(k,n) = \{(E,F) \in \mathcal{M}_k(Sp(n)) \times \mathcal{M}_n(k); E$ and F are related $\}$. A simple counting shows that for any $E \in \mathcal{M}_k(Sp(n)$ there exist(s) $F \in \mathcal{M}_n(Sp(k))$ such that E and F are related, and vice versa. So we obtain the following diagram

$$\begin{tikzcd} & \mathcal{L}(k,n) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{M}_k(Sp(n)) & & \mathcal{M}_p(Sp(k)) \end{tikzcd}$$

where π_1 and π_2 are natural surjections to the first and second factors respectively. It seems worth investigating this partial symmetry between the pairs. One may hope that it can be helpful in understanding the topology of these moduli spaces and related questions. One can also try to do all of this for arbitrary four-manifolds. It is reasonable to hope that comparison of these two stabilizations will be useful.

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