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INTEGRAL GEOMETRY OF PLANE CURVES AND KNOT INVARIANTS

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Abstract

We study the integral expression of a knot invariant obtained as the second coefficient in the perturbative expansion of Witten's Chern-Simons path integral associated with a knot. One of the integrals involved turns out to be a generalization of the classical Crofton integral on convex plane curves, and it is related with the invariants of generic plane curves recently defined by Arnold, with deep motivations in symplectic and contact geometry. Quadratic bounds on these plane curve invariants are derived using their relationship with the knot invariant.

1. Introduction

The first and second order terms in the perturbative expansion of Witten's Chern-Simons path integral associated with a knot in the 3-space were first analyzed by Guadagnini, Martellini and Mintchev [8] as well as by Bar-Natan [4] shortly after Witten's seminal work. In an announcement which appeared in 1992, Kontsevich perceived a construction of a vast family of knot invariants which, presumably, contains the same information as the family of coefficients in the perturbative expansion of the Chern-Simons path integral associated with a knot [9]. In a recent paper [7], Bott and Taubes explored this construction in a much more detailed manner. At this stage, it seems that a rigorous foundation has been laid for studying the perturbative expansion of the Chern-Simons path integral associated with a knot. But, as it seems to us, we still lack a study of each individual knot invariant in this family in a way as concrete and thorough as possible. The first term in the perturbative expansion turns out to be a classical quantity associated with a space curve with nowhere vanishing curvature, which was studied extensively under the name of the Călugăreanu-Pohl-White self-linking

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formula. Although the second term as a knot invariant is also classical, we find that the approach suggested by the perturbative theory of the Chern-Simons path integral provides deep insight into some of its previously unknown geometric and topological content. This will serve as the prototype of our further investigation.

The knot invariant we study here is, modulo a certain constant, the second coefficient of the Conway polynomial of a knot. This is a Vassiliev invariant of second order. So we denote it by v_2. Perturbative expansion of the Chern-Simons path integral leads to an expression of v_2 as the difference of two integrals, I_X and I_Y, over the knot thought of as a space curve. Or rather, the functional $I_X - I_Y$ on simple closed space curves can be proved to be a knot invariant and identified with the second coefficient of the Conway polynomial, modulo a certain constant. When the knot approaches its plane projection, the first integral I_X will be concentrated at the crossings. On the other hand, the second integral I_Y turns out to be well defined on the plane projection. In fact, it defines a functional on the space of generic plane curves. Using the fact that $I_X - I_Y$ is a knot invariant, we show that I_Y is constant on each component of the space of generic plane curves. And we can go further to understand how I_Y jumps when we pass through the discriminant (non-generic plane curves). This relates I_Y and the invariants of plane curves constructed by Arnold [1], [2], with deep motivations in symplectic and contact geometry. We believe that this relationship is what Arnold expected in [2].

We mentioned above that v_2 and the second coefficient of the Conway polynomial can be identified only modulo a certain constant. This constant is the value of I_Y on a circle. It was first calculated by Guadagnini, et al. [8]. But their computation is lengthy and not illuminating. As we couldn't understand their computation, we started to look for our own. It seems that one should think of I_Y as a 3-dimensional generalization of the Crofton formula (dating from 1868) for convex plane curves in integral geometry. Our computation of I_Y on a circle is almost parallel to the classical proof of the Crofton formula. As the classical proof of the Crofton formula yields many consequences in the integral geometry of plane curves (e.g., it implies that the measure of the set of lines intersecting a simple plane curve is equal to the length of that curve), we can't help but seek similar consequences of our generalized Crofton formula. Notice that Bott and Taubes [7] have observed that the construction of these knot invariants looks rather similar to the construction in classical integral geometry. Our study here seems to make this observation more concrete.

In our investigation of the knot invariant v_2, we noticed a quadratic bound for the values of v_2 on knots with n crossings. It is derived from a combinatorial formula for v_2. Such a quadratic bound agrees with the point of view that Vassiliev invariants should be thought of as polynomial functions on the set of knots. This observation led us to conjecture (which was proved by Bar-Natan shortly afterward) that Vassiliev invariants of order k grows like polynomial of order k with respect to the number of crossings. See Section 4. The combinatorial formula for v_2 also leads to quadratic bounds on Arnold's invariants of plane curves via their relations with I_Y.

The combinatorics of Arnold's invariants has also been studied by Polyak [11], Viro [14] and Shumakovich [16] independently. Also, the relationship between plane curve invariants and knot invariants has been studied by Polyak from the point of view of their common combinatorics. We will make reference to their work at appropriate places in the paper.

The paper is organized as follows. In Section 2, we will define the integrals I_X and I_Y and present some simple calculations and generalizations. In Section 3, we evaluate the integral I_Y on a round circle in the plane. This is done by imitating the classical proof of the Crofton formula. In Section 4, we study the combinatorics of the knot invariant v_2 by considering certain limiting behaviors of the integrals I_X and I_Y. In Section 5, we relate the integral I_Y with invariants of the so-called *unicursal* plane curves defined by Arnold [1].

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2. An integral knot invariant

For $x = (x^\mu) \in \mathbb{R}^3 \setminus \{0\}$, we denote by

$$\omega(x) = \frac{1}{8\pi}\frac{[x, dx, dx]}{|x|^3} = \frac{1}{8\pi}\frac{\epsilon_{\mu\nu\sigma}x^\mu\, dx^\nu\, dx^\sigma}{|x|^3}$$

the unit area form of the unit 2-sphere S^2, where $[\cdot\,, \cdot\,, \cdot]$ is the mixed product in $\mathbb{R}^3$.

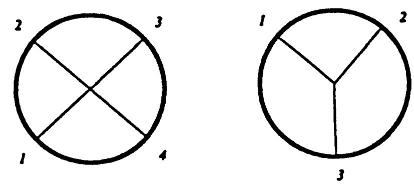

FIGURE 1. Diagrams for X-integral and Y-integral.

Let $\gamma : S^1 \to \mathbb{R}^3$ be a smooth imbedding, where we identify S^1 with $\mathbb{R}/\mathbb{Z}$. We denote

$$\Delta_4 = \{(t_1, t_2, t_3, t_4)\,;\, 0 < t_1 < t_2 < t_3 < t_4 < 1\}$$

and

$$\begin{aligned}\Delta_3(\gamma) = \quad & \{(t_1, t_2, t_3, z)\,;\, 0 < t_1 < t_2 < t_3 < 1, \\ & \qquad z \in \mathbb{R}^3 \setminus \{\gamma(t_1), \gamma(t_2), \gamma(t_3)\}\,\}\,.\end{aligned}$$

Define

$$I_X(\gamma) = \int_{\Delta_4} \omega(\gamma(t_3) - \gamma(t_1)) \wedge \omega(\gamma(t_4) - \gamma(t_2)) \tag{2.1}$$

and

$$I_Y(\gamma) = \int_{\Delta_3(\gamma)} \omega(z - \gamma(t_1)) \wedge \omega(z - \gamma(t_2)) \wedge \omega(z - \gamma(t_3)). \tag{2.2}$$

We will call these integrals the X-*integral* and Y-*integral* respectively. They get their names from the diagrams they correspond to. See Figure 1.

First, we want to simplify the expressions of these two integrals somewhat.

Lemma 2.1. *We have*

$$\begin{aligned} I_X(\gamma) = -\frac{1}{(4\pi)^2} \int_{\Delta_4} & \frac{[\gamma(t_3) - \gamma(t_1), \dot\gamma(t_3), \dot\gamma(t_1)]}{|\gamma(t_3) - \gamma(t_1)|^3} \\ & \cdot \frac{[\gamma(t_4) - \gamma(t_2), \dot\gamma(t_4), \dot\gamma(t_2)]}{|\gamma(t_4) - \gamma(t_2)|^3}\, dt_1\, dt_2\, dt_3\, dt_4. \end{aligned} \tag{2.3}$$

Notice that $[\gamma(t) - \gamma(t'), \dot\gamma(t), \dot\gamma(t')]$ is the oriented volume of the parallelepiped spanned by $\gamma(t) - \gamma(t')$, $\dot\gamma(t)$ and $\dot\gamma(t')$.

Lemma 2.2. *Let*

$$E(z, t) = \frac{(z - \gamma(t)) \times \dot\gamma(t)}{|z - \gamma(t)|^3}.$$

Then,

$$I_Y(\gamma) = -\frac{1}{(4\pi)^3}\int_{\Delta_3(\gamma)} [E(z,t_1),E(z,t_2),E(z,t_3)]\; d^3z\, dt_1\, dt_2\, dt_3. \tag{2.4}$$

Notice that $E(z,t)dt = dB$, where $B = B(z)$ is the magnetic field induced by the current $\gamma(t)$.

Both of these lemmas come from a straightforward computation.

Theorem 2.3. *Let*

$$v_2(\gamma) = I_X(\gamma) - I_Y(\gamma). \tag{2.5}$$

Then v_2 is invariant under an isotopy of γ.

So v_2 is a knot invariant. This was first proved rigorously by Bar-Natan [4] in his Princeton thesis. See also [7]. Furthermore, this knot invariant satisfies a crossing change formula which identifies it with the second coefficient of the Conway polynomial modulo the constant v_2(unknot). This also justifies the notation of v_2.

For readers' convenience, let us compare the formula here with that in [7]. As in [7], we denote

$$\theta_{ij} = \omega(x_j - x_i).$$

Let C_k be the configuration space of k distinct ordered points on S^1, and $C_{k,l} = C_{k,l}(\gamma)$ be the configuration space of k distinct ordered points on S^1 and l distinct ordered points in $\mathbb{R}^3$ away from the first k points on $\gamma(S^1)$. Both C_k and $C_{k,l}$ inherit orientations from that of $(S^1)^k$ and $(S^1)^k \times (\mathbb{R}^3)^l$, respectively. Then, it is given in [7] that

$$v_2(\gamma) = \frac{1}{4}\int_{C_4} \theta_{13}\theta_{24} - \frac{1}{3}\int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34}. \tag{2.6}$$

Notice that Δ_4 and $\Delta_3(\gamma)$ are fundamental domains of C_4 and $C_{3,1}$ under cyclic permutations of points on S^1, respectively. Moreover, the forms $\theta_{13}\theta_{24}$ and $\theta_{14}\theta_{24}\theta_{34}$ are fixed under cyclic permutations of points on S^1. Therefore, (2.5) and (2.6) agree.

The first step in the proof of this theorem is to show that both integrals $I_X(\gamma)$ and $I_Y(\gamma)$ are finite. This is done in [7] by compactifying the domains of integration and showing that the integral forms extend as smooth forms to the compactifications of the domains. From this consideration, the following lemma should be quite obvious.

Lemma 2.4. *For an immersion $\gamma : S^1 \to \mathbb{R}^3$ with only finitely many singularities (say, transverse double points), the integral $I_Y(\gamma)$ is finite.*

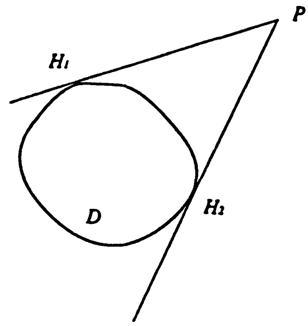

FIGURE 2. The setting of the Crofton formula.

Proof. When γ is an imbedding, $\Delta_3(\gamma)$ is the total space of a fibration over $\Delta_3 = \{(t_1, t_2, t_3)\,;\, 0 < t_1 < t_2 < t_3 < 1\}$ whose fibre is $\mathbb{R}^3$ with three distinct points deleted. In our case, we may define $\Delta_3(\gamma)$ similarly and it will have degenerate fibres over a measure zero set of Δ_3. Similar to the case where γ is an imbedding, $\Delta_3(\gamma)$ with degenerate fibres deleted can be compactified and the integral form of $I_Y(\gamma)$ extends to the compactification. This implies that the integral $I_Y(\gamma)$ is still finite when γ is an immersion with only finitely many singularities.

3. A generalized Crofton formula

Some remarkable integral formulae associated with convex sets in the plane are obtained by simple computations of the density $dx\,dy$ in different coordinate systems. The classical Crofton formula is such an example [15].

Let D be a bounded convex set in the plane. Through each point P exterior to D there pass two supporting lines of D. Let s and s' respectively be the lengths of the line segments from P to the corresponding supporting points H_1 and H_2, and let α be the angle H_1PH_2 between the supporting lines. See Figure 2. Then

$$\int_{P \notin D} \frac{\sin \alpha}{s \cdot s'}\, dx\, dy = 2\pi^2. \tag{3.1}$$

Let $A = A(P)$ be the area of the parallelogram spanned by PH_2 and PH_1. Then (3.1) can be written as

$$\int_{P \notin D} \frac{A}{s^2 \cdot s'^2}\, dx\, dy = 2\pi^2. \tag{3.2}$$

The Crofton formula (3.1) or (3.2) yields many consequences in the integral geometry of plane curves. For example, it implies that the

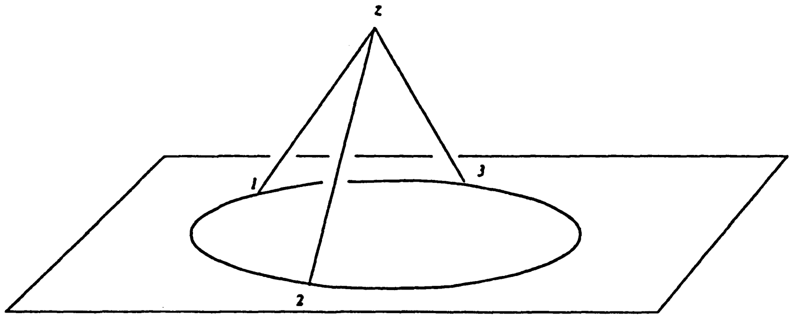

FIGURE 3. The setting of the generalized Crofton formula.

measure of the set of lines intersecting a simple plane curve is equal to the length of that curve.

It seems amazing that the Crofton integral (left-hand side of (3.1) or (3.2)) is independent of the shape of C. Assuming this for the moment, let us try to evaluate the Crofton integral when the boundary of D is the circle $\{(\cos 2\pi t, \sin 2\pi t)\,;\, 0 \leq t \leq 1\}$ in the plane. This is done by a certain change of coordinates.

Let $\phi : \mathbb{R}^2 \setminus D = \{(x,y)\,;\, x^2+y^2 > 1\} \longrightarrow S^1 \times S^1$ be the map defined by sending each point $P \notin D$ to the pair of angles (θ_1, θ_2) with $\theta_1 < \theta_2$ of the supporting points. It is easy to check:

- ϕ is one-one;
- if P approaches the infinity in the direction of angle θ_0, then $\phi(P)$ approaches $(\theta_0 + \pi/2, \theta_0 + 3\pi/2)$;
- if P approaches the point $(\cos\theta_0, \sin\theta_0)$, then $\phi(P)$ approaches (θ_0, θ_0).

Therefore, the image of ϕ covers exactly one half of the torus $S^1 \times S^1$. By a direct computation, the Crofton integral is the same as the signed area covered by the image of ϕ. It follows that the Crofton integral is equal to $1/2 \cdot (2\pi)^2 = 2\pi^2$.

In general, the same argument will go through since the angle of a supporting line of the convex set D is well defined once a center of D is chosen. See [15].

Let γ be a simple closed plane curve. The X-integral I_X of γ is 0 (see (2.3)). The fact that $I_X - I_Y$ is a knot invariant implies that the Y-integral $I_Y(\gamma)$ of γ is invariant under deformation of γ in the space of simple closed plane curves. The computation of the Y-integral for the circle was first done by [8]. We will provide a computation here which is similar to the proof of the Crofton formula described above.

Denote $\mathbb{R}^3_+ = \{(z_1, z_2, z_3) \in \mathbb{R}^3\,;\, z_3 > 0\}$ and $\mathbb{R}^2 = \{z_3 = 0\} \subset \mathbb{R}^3$.

Theorem 3.1. (Generalized Crofton formula) *Let $\gamma = \gamma(t) : S^1 \longrightarrow \mathbb{R}^2$ be a simple closed plane curve in $\mathbb{R}^2$. Then*

$$\int_{\Delta_3} \int_{z \in \mathbb{R}^3_+} \frac{V}{\prod_{i=1}^3 |z - \gamma(t_i)|^3} \, d^3z \, dt_1 \, dt_2 \, dt_3 = \frac{4\pi^3}{3}, \tag{3.3}$$

where V is the oriented volume of the parallelepiped spanned by $(z - \gamma(t_i)) \times \dot{\gamma}(t_i)$, $i = 1, 2, 3$.

We will call the integral in (3.3) *the generalized Crofton integral.* See Figure 3.

Proof. Compare (3.3) with (2.4). The integral in (3.3) is equal to a constant multiple of $I_Y(\gamma)$. Since $I_Y(\gamma) = -v_2(\gamma)$ in this case, $I_Y(\gamma)$ is invariant when γ is deformed by an isotopy of the plane. Therefore, it suffices to prove the theorem for the circle $\{(\cos 2\pi t, \sin 2\pi t, 0)\,;\, 0 \leq t \leq 1\}$ in $\mathbb{R}^2$.

Let $\phi : \mathbb{R}^3_+ \times (S^1)^3 \longrightarrow S^2_+ \times S^2_+ \times S^2_+$ be the map defined by sending (z, t_1, t_2, t_3) to

$$\left(\frac{z - \gamma(t_1)}{|z - \gamma(t_1)|}, \frac{z - \gamma(t_2)}{|z - \gamma(t_2)|}, \frac{z - \gamma(t_3)}{|z - \gamma(t_3)|} \right).$$

Then the generalized Crofton integral on the circle is equal to the signed volume of the part of $(S^2_+)^3$ covered by the image of the map ϕ multiplied by $(4\pi)^3/3!$.

Claim. There is a subset $A \subset \mathrm{Im}(\phi)$ of full measure in $(S^2_+)^3$ such that $\phi|\phi^{-1}(A)$ is one-to-one.

It follows from the claim that the generalized Crofton integral on the circle is $\pm(4\pi)^3/3! \cdot (1/2)^3 = \pm 4\pi^3/3$. By checking the orientations, we know the integral equals $\pi^3/6$.

Proof of the claim. Let v be a vector in the upper hemisphere $S^2_+ \subset \mathbb{R}^3$, and $\phi_1 : \mathbb{R}^3_+ \times S^1 \longrightarrow S^2_+$ be the map defined by sending (z, t) to

$$\frac{z - \gamma(t)}{|z - \gamma(t)|}.$$

Then ${\phi_1}^{-1}(v)$ is the half-infinite cylinder

$$C_v = \{\gamma(t) + sv\,;\, 0 \leq t \leq 1,\ s \geq 0\}.$$

Let (v_1, v_2, v_3) be a point in $(S^2_+)^3$. Then $\phi^{-1}(v_1, v_2, v_3)$ is in one-to-one correspondence with the set of intersections of the three half-infinite cylinders C_{v_1}, C_{v_2}, and C_{v_3}. If $v_1 \neq v_2$, then C_{v_1}, C_{v_2} intersect in an arc lying on both C_{v_1} and C_{v_2} whose ends are a pair of antipode points of the circle γ. If v_1, v_2 and v_3 are pairwise distinct, then C_{v_1} and C_{v_2} intersect in an arc A_{12} on C_{v_1}, and similarly, C_{v_1} and C_{v_3}

intersect in an arc A_{13} on C_{v_1}. Let $p : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection. If $p(v_2) - p(v_1)$ and $p(v_3) - p(v_1)$ are not collinear ($p(v_1)$, $p(v_2)$ and $p(v_3)$ are in general position), then the ends of A_{12} and A_{13} form two distinct pairs of antipode points on the circle. This implies that A_{12} and A_{13} intersect at exactly one point in R^3_+. So if $p(v_1)$, $p(v_2)$ and $p(v_3)$ are in general position, then $\phi^{-1}(v_1, v_2, v_3)$ is a single point in $\mathbb{R}^3_+ \times (S^1)^3$. The complement in $(S^2_+)^3$ is of measure 0. This proves the claim and thus the theorem.

It seems very likely that the argument in the proof of the claim above also works if γ is a convex curve in $\mathbb{R}^2$. If this is so, we will have a direct proof of the generalized Crofton formula for convex plane curves. Here is a very interesting intuitive interpretation of the claim in the proof of Theorem 3.1. Imagine that a fixed circle in the plane starts to move in the plane with three non-collinear constant velocities respectively, so that we will see three round circles of the same radius at any instant. Then there will be exactly one instant when these three circles have exactly one point of intersection. Intuitively, this should also be true if we start with a convex curve. We are not sure whether such a phenomenon has been discussed in the literature.

Corollary 3.2. $v_2(\text{unknot}) = -1/24$.

Proof. It is easy to see that on a circle, I_Y is equal to the generalized Crofton integral on the circle times $2 \cdot 1/(2\pi)^3 = 1/4\pi^3$. So I_Y on a round circle is $1/24$.

4. The combinatorics of the integral knot invariant v_2

As we mentioned before, the knot invariant v_2 can be identified, modulo the constant $v_2(\text{unknot}) = -1/24$, with the second coefficient of the Conway polynomial via a crossing change formula. From this identification, one can draw most of the conclusions about v_2 in this section. In fact, there exist already several other approaches in studying the combinatorics of Vassiliev invariants. See [10] and [12]. Therefore, the main interest of this section is to see how one can understand the combinatorics of v_2 by studying certain limiting behaviors of the integrals I_X and I_Y. Such a consideration actually led us to the discovery of the relationship between the integral I_Y and Arnold's invariants of plane curves discussed in the next section.

Let $\gamma : S^1 \longrightarrow \mathbb{R}^3$ be an imbedding. Since it is very difficult to compute v_2 by evaluating both integrals $I_X(\gamma)$ and $I_Y(\gamma)$ directly, we study a limiting situation when the curve is pushed into a plane via a regular projection. It turns out that the limits of both integrals can be computed, and this gives us a new way of studying the knot invariant v_2. The same analysis applied to the Gauss linking formula gives us the well-known combinatorial formula for the linking number.

When an imbedded curve γ in $\mathbb{R}^3$ acquires a double point, the X-integral of γ blows up. On the other hand, the Y-integral is still meaningful by Lemma 2.4.

Proposition 4.1. *Let $\gamma : S^1 \longrightarrow \mathbb{R}^3$ be an immersion with only transverse double points. Then the following hold:*

(1) $I_Y(\gamma)$ does not depend on the parameterization or the orientation of γ.

(2) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be an affine similarity. Then $I_Y(T \circ \gamma) = I_Y(\gamma)$.

The proof is immediate.

Let K be the knot type of the imbedding $\gamma : S^1 \longrightarrow \mathbb{R}^3$. Thought of as a knot diagram, K can be drawn inside the plane except around each crossing. Assume that when we make an over-pass at a crossing of K, we go along a semi-circle of radius ϵ perpendicular to the plane. The other parts of K lie completely in the plane. Denote such a diagram of K by K_ϵ. Also, we associate to each crossing a sign ± 1 by the usual right-hand-rule.

When ϵ approaches 0, K_ϵ limits to a closed plane curve with only transverse double points. Denote this limiting plane curve by K_0. We associate to each double point the sign of the corresponding crossing, and we call K_0 with these signs the *signed limiting plane curve.* In general, let C be an immersed circle in the plane with only transverse double points. If each double point is associated with a sign, then C is called *a signed immersed circle.* For each closed plane curve with only transverse double points, we have a chord diagram defined as follows.

Definition 4.2. Let C be an immersed circle with only transverse double points in the plane. The *chord diagram* (or *Gauss diagram*) of C is the combinatorial pattern of a finite collection of straight chords with both ends on the round circle, connecting the points sent by the immersion to the same double point of the immersed circle. If each chord is associated with a sign, the chord diagram is called a *signed chord diagram.*

If K_0 is the limiting plane curve of a knot diagram K_ϵ, then we associate to each chord of the chord diagram of K_0 the sign of the corresponding double point. For each pair of chords crossing each other in the signed chord diagram of K_0, we assign to this pair of chords a sign equal to the product of the signs of the two chords.

Proposition 4.3. *Let K be a knot diagram with n crossings. Then*

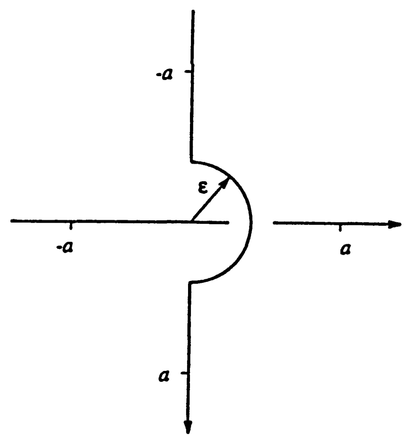

FIGURE 4. The local picture of a knot projection at a crossing.

the limit

$$\lim_{\epsilon \to 0} I_X(K_\epsilon)$$

exists. Let $I_X(K_0)$ denote this limit. Then

$$I_X(K_0) = \frac{n}{16} + \frac{(c_+ - c_-)}{4} \tag{4.1}$$

where c_+ (respectively, c_-) is the number of pairs of chords in the signed chord diagram of K_0 with a positive (respectively, negative) sign.

Proof. Our first observation is that when $\dot\gamma(t)$ and $\dot\gamma(t')$ are co-planar, then the Gaussian form $\omega(\gamma(t) - \gamma(t'))$ is 0.

We may assume that the i-th crossing of K_ϵ looks like the crossing depicted in Figure 4. Let C^i_ϵ be the semi-circle of radius ϵ at that crossing, and A^i_a be a line segment $[-a, a]$ with a fixed small $a > 0$ on K_ϵ running under C^i_ϵ. Furthermore, denote by $C^i_{\epsilon,a}$ the union of C^i_ϵ and line segments $[-a, -a+\epsilon]$ and $[a-\epsilon, a]$.

By the above observation, if both $\gamma(t)$, $\gamma(t')$ lie outside $\cup_{i=1}^n C^i_\epsilon$, or they both lie inside some $C^i_{\epsilon,a}$, then $\omega(\gamma(t) - \gamma(t')) = 0$. If $\gamma(t) \in C^i_\epsilon$, and $\gamma(t')$ lies outside $C^i_{\epsilon,a} \cup A^i_a$, then $|\gamma(t) - \gamma(t')|$ is bounded from below by a constant. It follows that when ϵ approaches 0, the integral I_X over all those pairs goes to 0. Note that we need to fix a base point in order to evaluate the X-integral I_X for a curve. So we choose a base point on K_ϵ which is not in any $C^i_{\epsilon,a}$ or A^i_a. Then all nonzero limits come from the following two cases:

(1) $\gamma(t_1), \gamma(t_2) \in C^i_\epsilon$, and $\gamma(t_3), \gamma(t_4) \in A^i_a$;

(2) $\gamma(t_1) \in C^i_\epsilon, \gamma(t_3) \in A^i_a$ or vice versa, and $\gamma(t_2) \in C^j_\epsilon, \gamma(t_4) \in A^j_a$ or vice versa, with $i \neq j$.

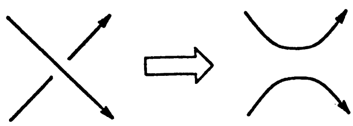

FIGURE 5. An oriented surgery.

By a direct computation, the limit for case (1) is always 1/16. It is independent of the sign of the crossing.

Since $t_1 < t_2 < t_3 < t_4$, case (2) is possible if and only if the chords corresponding to i and j cross each other. In this case, the limit is $\epsilon_i \epsilon_j / 4$, where ϵ_i, ϵ_j are the signs of the i-th and j-th crossings. This completes the proof.

Let C be an immersed circle in the plane with only transverse double points. Then we can resolve C to knots by changing each double point to a crossing. There are 2^n resolutions if C has n double points.

Corollary 4.4. *Let C be an immersed circle in the plane with only transverse double points.*

(1) *If C is resolved to two knots K^1 and K^2 of the same knot type, then $I_X(K^1_\epsilon)$ and $I_X(K^2_\epsilon)$ have the same limit.*

(2) *$I_Y(C)$ is invariant when C is deformed in the plane without changing its chord diagram.*

(3) *If C is resolved to an unknot K^u, then*

$$I_Y(C) = \frac{1}{24} + \frac{n}{16} + \frac{(c_+ - c_-)}{4},$$

where $c_\pm$ are computed using the signed chord diagram of K^u.

Corollary 4.4 (2) implies thousands of integral formulae like the generalized Crofton formula.

Proof. (1) This is because the limit of $I_Y(K^i_\epsilon)$ when ϵ approaches 0 is $I_Y(C)$, for $i = 1, 2$, and $v_2(K^1) = v_2(K^2)$.

(2) Resolve C to a knot K. Since both $I_X(K_0)$ and $v_2(K)$ are invariant when C is deformed in the plane without changing its chord diagram, so is $I_Y(C) = I_X(K_0) - v_2(K)$.

(3) This is a direct consequence of Corollary 3.2 and Lemma 4.3.

Let K be a knot diagram. At each crossing of K, the modification of the knot diagram depicted in Figure 5 changes K into a link of two oriented components. Such a modification is called an *oriented surgery* at a crossing.

Corollary 4.5. *Let K be a knot diagram with n crossings. Let K^u be an unknot obtained from K by changing some crossings.*

(1) *Let l_i be the linking number of the two-component link obtained from the oriented surgery at the i-th crossing of K, and l_i^u be the corresponding linking number from K^u. Then*

$$v_2(K) = \frac{1}{2}\sum_{i=1}^{n}(l_i - l_i^u) - \frac{1}{24}; \tag{4.2}$$

(2) $|v_2(K)| \le n(n-1)/4 + 1/24$;

(3) $v_2(K) + 1/24$ *is an integer;*

(4) $v_2(K)$ *is independent of the orientation of K.*

Proof. Let K_0 and K_0^u be the limiting plane curves of K and K^u respectively. By Lemma 4.3 and Corollary 4.4 (3),

$$v_2(K) = \frac{c_+(K_0) - c_-(K_0)}{4} - \frac{c_+(K_0^u) - c_-(K_0^u)}{4} - \frac{1}{24}. \tag{4.3}$$

The difference between the signed chord diagrams of K_0 and K_0^u is that some signs of chords are changed.

(2) As the chord diagrams of K_0 and K_0^u both have n-chords, there are at most $n(n-1)/2$ intersections among chords. Thus, (4.3) implies $|v_2(K)| \le n(n-1)/4 + 1/24$.

(3) Let d_i be the number of intersections of all chords with the i-th chord counted with signs. Then $d_i = 2l_i$, so d_i is always an even integer.

Claim. Let K_0^1 be a signed plane curve obtained from K_0 by changing one sign of the double points. Then $I_X(K_0) - I_X(K_0^1)$ is an integer.

Proof of the claim. By (4.1), we need only count the changes of the intersections between the chord corresponding to the double point where the sign is changed and other chords, divided by 4. When the sign of the i-th chord is changed, d_i is changed to $-d_i$. As d_i is even, the total change $2d_i$ is divisible by 4.

Now we finish the proof of (3) as follows: choose a sequence of double points so that when we change their signs one after another we get a sequence of signed plane curves $K_0^1, \cdots, K_0^s$ with $K_0 = K_0^1$ and K_0^s is the limit of an unknot. Then

$$v_2(K) + \frac{1}{24} = I_X(K_0) - I_X(K_0^s) = \sum_{i=1}^{s-1}\left(I_X(K_0^i) - I_X(K_0^{i+1})\right).$$

Thus the proof of (3) is completed.

Finally, it is clear that (1) follows from the proof of (3). It is also clear that (4) can be derived in many ways and one way is via the formula (4.2) since the linking number will not change if one changes the orientations of both components of a link.

The bound in Corollary 4.5 (2) is not the sharpest. One may get a sharper bounds for v_2 using the combinatorial formula for v_2 in [12]. But it is the order of such a bound which interests us the most. Recall that a knot invariant is of finite type or a Vassiliev invariant if it vanishes on "higher order differences of knots". One may think of such a knot invariant as a "polynomial function" on the set of knot types. See, for example, [5]. Since v_2 is known to be a Vassiliev invariant of order 2, the bound for its values on knots with n crossings agrees with such a point of view. Notice that using the combinatorial formula for a Vassiliev invariant of order 3 in [10] or the one in [12], we can get a similar bound for values of Vassiliev invariants of order 3 on knots with n crossings. This leads us to conjecture the following theorem in the original version of this paper, which was proved by Bar-Natan shortly afterward [6].

Theorem 4.6. (Bar-Natan) *For every Vassiliev invariant of order k, say v_k, there is a constant C such that if K is a knot with n crossings, then*

$$|v_k(K)| < Cn^k.$$

5. Invariants of unicursal curves

Consider the space $\mathcal{M}$ of all immersions of S^1 into the plane. By a classical theorem of Whitney, the components of this space can be indexed by $\mathbb{Z}$ using the winding number. We will denote by $\mathcal{M}_w$ the component of $\mathcal{M}$ whose members all have winding number or index w. If we want to look at $\mathcal{M}$ more carefully, we will see generic immersions and non-generic ones. Generic immersions are those with only transverse double points. The set Σ of non-generic immersions or the discriminant of $\mathcal{M}$ can be thought of as a stratified space whose top stratum has three components of particular interest to us. One component consists of immersions with exactly one transverse triple point, and the other two consist, respectively, of immersions with exactly one direct or inverse self-tangent point where the two tangent branches of the curve have distinct curvature. A self-tangent point is called *direct* if the two tangent vectors of the curve at the tangent point are in the same direction, and it is called *inverse* otherwise. It turns out that these three components of the top stratum of Σ corresponding respectively to immersions with exactly one transverse triple point, one direct self-tangent point or one inverse self-tangent point are all well "co-oriented". This means that we can talk about the positive or negative side of these three component of the top stratum of Σ in $\mathcal{M}$. It is quite easy to see that a path in $\mathcal{M}$

can be perturbed so that it only crosses Σ transversally through these three components at finitely many places. For a detailed study of the topology of $\Sigma \subset \mathcal{M}$, see [1].

We will call a generic immersion of S^1 into the plane a *unicursal curve.* Two unicursal curves are equivalent if they belong to the same component of $\mathcal{M} \setminus \Sigma$. It is not hard to see that a path in $\mathcal{M} \setminus \Sigma$ is the same as a deformation of a unicursal curve without changing its chord diagram. An *invariant of unicursal curves* assigns values to every unicursal curve and equivalent unicursal curves should be assigned the same value.

Lemma 5.1. *$C \to I_Y(C)$ is an invariant of unicursal curves.*

This is simply a restatement of Corollary 4.4 (2).

In [1], Arnold constructed three basic invariants of unicursal curves, St and $J^{\pm}$. Up to an additive constant, they are completely determined by the way they jump when a deformation of unicursal curves crosses through a triple point, or a direct or inverse self-tangent point. To describe these invariants, we need to first make some definitions.

Definition 5.2. (1) A transversal crossing of a self-tangent point is *positive* if the number of double points grows (by 2).

(2) A transversal crossing of a triple point is *positive* if the newly born vanishing triangle is positive.

Here, for a given a unicursal curve C, a *vanishing* triangle of C is a triangle formed by three branches of C and no other branches of C are allowed to run into such a triangle. At a transversal crossing of a triple point, one sees the death of one vanishing triangle and the birth of another one. The sign of a vanishing triangle is defined as follows. The orientation of the immersed circle defines a cyclic ordering of the sides of the vanishing triangle. Hence the sides of the triangle acquire orientations induced by the ordering. But each side also has its own direction which might coincide, or not, with the orientation defined by the ordering. For each vanishing triangle, let q be the number of sides equally oriented by the ordering and their directions. Then the *sign* of a vanishing triangle is $(-1)^q$. It is easy to check that at a transversal crossing of a triple point, the dying vanishing triangle and the newly born vanishing triangle always have opposite signs.

Theorem 5.3. (Arnold) (1) *There exists a unique (up to an additive constant) invariant of unicursal curves of fixed index whose value remains unchanged at a transversal crossing of a self-tangent point, but increases by 1 at a positive transversal crossing of a triple point. This invariant is denoted by St with an appropriate normalization.*

(2) *There exists a unique (up to an additive constant) invariant of*

unicursal curves of fixed index whose value remains unchanged at a transversal crossing of a triple point or an inverse (respectively, direct) self-tangent point, but increases by 2 (respectively, -2) at a positive transversal crossing of a direct (respectively, inverse) self-tangent point. This invariant is denoted by J^+ (respectively, J^-) with an appropriate normalization. The invariants J^+ and J^- are related by $J^+ - J^- = n$ on unicursal curves with n double points.

(3) *These invariants are independent of the orientation of unicursal curves.*

Here a normalization means to choose a unicursal curve C_w for each index $w \in \mathbb{Z}$ and the value of the invariant in question on C_w. One may think of these three invariants St and $J^\pm$ of unicursal curves as dual to those three components of the top stratum of Σ corresponding to one triple point, one direct self-tangent point and one inverse self-tangent point respectively. Any invariant of unicursal curves which jumps by a constant at a transversal crossing of a triple point and a self-tangent point can be expressed uniquely as a linear combination of St and $J^\pm$, modulo a constant depending on the index.

To simplify the terminology, we define several operations on unicursal curves analogous to the Reidemeister moves in knot theory. See Figure 6. A *type* I *move* on a unicursal curve kills one small kink on it. A *type* II$^+$ (II$^-$, respectively) *move* is a positive transversal crossing of a direct (inverse, respectively) self-tangent point. Finally, A *type* III *move* is a positive transversal crossing of a triple point.

Definition 5.4. Let C be a unicursal curve. Then we define

$$\alpha(C) = I_Y(C) + \frac{n}{16} - \frac{1}{24},$$

where n is the number of double points of C.

As both the Y-integral and the number of double points are invariants of unicursal curves, so is α. If C is resolved to an unknot K^u, then

$$\alpha(C) = \frac{n}{8} + \frac{c_+ - c_-}{4}, \tag{5.1}$$

where $c_\pm$ are computed using the signed chord diagram of K_0^u. From this formula, we see that each double point of C contributes $1/8$ to α, and each pair of intersecting chords contributes $\pm 1/4$ to α.

Theorem 5.5. *The invariant α of unicursal curves has the following properties:*

(1) *α equals 0 for every simple closed plane curve;*

(2) *α is decreased by $1/8$ if a type* I *move is performed;*

(3) *α is unchanged if a type* II$^+$ *move is performed;*

(4) *α is decreased by 1/4 if a type* III *move is performed.*

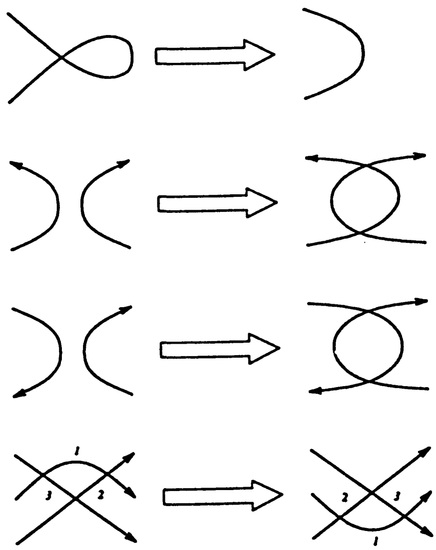

FIGURE 6. Elementary moves among unicursal curves.

As a consequence of (2), (3), and (4), α is increased by $1/4$ if a type II^- move is performed. Furthermore, we have

(5) $|\alpha(C)| \leq n^2/8$, *where n is the number of double points on C;*

(6) $\alpha(C)$ *is independent of the orientation of C.*

Proof. They are all consequences of (5.1).

(1) This is obvious.

(2) Note that the chord corresponding to the double point in a type I move is an isolated chord (it does not intersect any other chord). So this double point contributes only $1/8$ to α.

(3) We may assume that after a type II^+ move, the two new double points are resolved with opposite signs. They contribute $1/4$ to α. But the two chords of the new double points intersect and every other chord either intersects them both or misses them both. So their contribution to α is $-1/4$. This implies (3)

(4) Assume that a type III move changes C to C' and the vanishing triangles on C and C' are resolved as depicted in Figure 7. This is done by choosing a base point and resolving double points according to the rule that the branch one walks through first is always above the branch one walks through second. The resulting knot is an unknot.

On the level of chord diagrams, there are two cases to study. See again Figure 7. In both cases, the chord diagrams for C and C' have the same number of chords. Let $\{a, b, c\}$ and $\{a', b', c'\}$ be the signed chords at the vertices of the vanishing triangles of C and C' respectively.

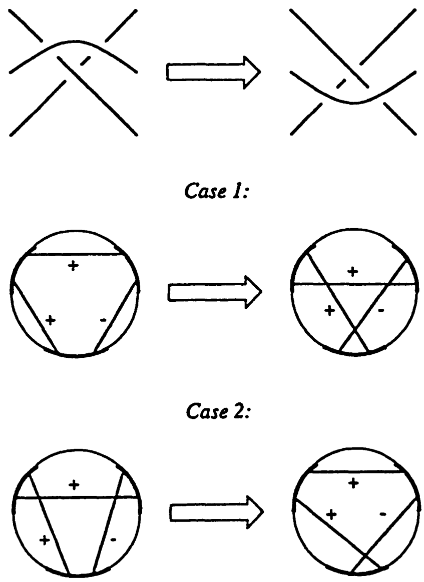

FIGURE 7. The proof of Theorem 5.5 (4).

The edges of the vanishing triangle of C (respectively, C') correspond to three disjoint arcs on S^1 and each chord in $\{a, b, c\}$ (respectively, $\{a', b', c'\}$) connects end points of two distinct arcs. No other chords of C (respectively, C') will touch these arcs. Furthermore, when a type III move changes C to C', the chord diagram of C is changed to the chord diagram of C' by switching every two end points of $\{a, b, c\}$ paired as the end points of those arcs on S^1. The signs of chords will not be changed. So the contribution of $\{a, b, c\}$ to $\alpha(C)$ will be $1/4$ more than the contribution of $\{a', b', c'\}$ to $\alpha(C')$. Furthermore, if another chord intersects one chord in $\{a, b, c\}$, it will also intersect one in $\{a', b', c'\}$ with the same sign. Therefore, $\alpha(C') = \alpha(C) - 1/4$.

(5) This is a consequence of (5.1) and the bound $|c_+ - c_-| \leq n(n-1)/2$.

(6) This also follows easily from (5.1).

Now by Theorem 5.4, it is very easy to compute α, and consequently $I_Y(C)$ for any unicursal curve C. For example, $\alpha(\infty) = 1/8$. It follows that $I_Y(\infty) = 5/48$. Here the symbol ∞ is used to denote a unicursal curve of the same shape.

Corollary 5.6. *We have*

$$\alpha = -\frac{2St + J^-}{8}. \tag{5.2}$$

Proof. The invariant on the right-hand side of (5.2) changes in the same way as α does under type $\mathrm{II}^{\pm}$ and III moves. Checking the initial values of St and $J^{\pm}$ given in [1] verifies (5.2).

A result of F. Aicardi (see [1]) states that $2St + J^{+} = 0$ holds if the chord diagram of C has no intersecting chords. The following is a generalization of this result.

Corollary 5.7. *The identity $2St + J^{+} = 0$ holds for a unicursal curve C with n double points if and only if $\alpha(C) = n/8$, and if and only if a certain signed chord diagram coming from an unknot resolution of C has $c_{+} = c_{-}$.*

Proof. For unicursal curves with n double points, we have $J^{+} - J^{-} = n$. So (5.2) gives us

$$\alpha = -\frac{2St + J^{+}}{8} + \frac{n}{8}. \tag{5.3}$$

on unicursal curves with n double points. This together with (5.1) proves the corollary.

Arnold's invariants St, J^{+}, J^{-} are related to each other via some other index-type invariants to be defined as follows.

Let C be a unicursal curve such that at each transverse double point the two tangent vectors of C are orthogonal. At every double point, we may divide C into two branches C_1 and C_2. In fact, the preimage of that double point on S^1 cuts S^1 into two arcs and the images of these two arcs under the immersion are C_1 and C_2, respectively. These two branches C_1 and C_2 are ordered such that if the outgoing tangent vectors of C_1 and C_2 at the corresponding double point are v_1 and v_2, respectively, then the frame $\{v_1, v_2\}$ has the same orientation as that of the plane.

Definition 5.8. The *half-index* i_1 (respectively, i_2) of a double point is the angle of the rotation of the radius-vector connecting the double point to a point moving along C_1 (respectively, C_2) from the double point to itself divided by $\pi/2$. The *index* of a double point is the difference $i = i_1 - i_2$.

The invariants $I^{\pm}$ are defined to be

$$I^{\pm} = \frac{\sum i \pm 2n}{4}, \tag{5.4}$$

where n is the number of double points, and the sum is over all double points.

Note that $I^{+} - I^{-} = n$, and we have

$$J^{\pm} = I^{\pm} - 3St \tag{5.5}$$

as shown in [1]. So, among these three invariant St, J^+ and J^-, there is only one which is essentially not of index-type. Corollary 5.10 below shows that α is such an invariant.

The following theorem is essentially in [1].

Theorem 5.9. *I^- is determined by the following properties:*

(1) *I^- of a simple closed plane curve is 0;*

(2) *I^- is decreased by $(i-2)/4$ if a type* I *move is performed, where i is the index of this double point;*

(3) *I^- is unchanged if a type* II^+ *move is performed, and I^- is decreased by 2 if a type* II^- *move is performed;*

(4) *I^- is increased by 3 if a type* III *move is performed.*

Corollary 5.10. *We have the following identities:*

(1) $St = I^- + 8\alpha$;

(2) $J^- = -2I^- - 24\alpha$;

(3) $J^+ = n - 2I^- - 24\alpha$.

Proof. The proof follows from (5.2), (5.5) and $J^+ - J^- = n$.

Let C be a unicursal curve, and x be a point not on C. The *winding number of C relative to x* is the degree of the position map

$$S^1 \longrightarrow S^1 : \ t \mapsto \frac{C(t)-x}{|C(t)-x|}.$$

The relative winding number remains unchanged if x moves in a connected component of $\mathbb{R}^2 \setminus C$. We will use the relative winding number to estimate the index of double points.

Let us notice that the oriented surgery at a crossing on a knot diagram can be generalized to the oriented surgery at a double point on a unicursal curve. The oriented surgery at a double point x on a unicursal curve C will result in two new unicursal curves C_1 and C_2 intersecting each other transversally. The component of $\mathbb{R}^2 \setminus C_1 \cup C_2$ where x lies is well defined.

Lemma 5.11. (1) *Let w_1 and w_2 be the winding numbers of C_1 and C_2 relative to x. Then the index of this double point x on C is* $4w_1 - 4w_2 - 2$.

(2) *The index i of any double point on a unicursal curve with n double points satisfies the inequality $|i| \leq 4n+6$.*

Proof. (1) If

$$i_1 = \frac{\theta_1}{\pi/2} \qquad \text{and} \qquad i_2 = \frac{\theta_2}{\pi/2},$$

then

$$w_1 = \frac{\theta_1 - \pi/2}{2\pi} \qquad \text{and} \qquad w_2 = \frac{\theta_2 + \pi/2}{2\pi}.$$

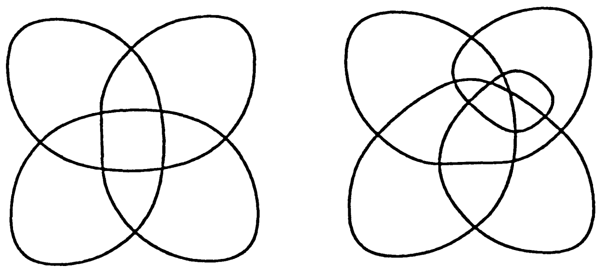

FIGURE 8. Examples of unicursal curves with $\alpha \leq 0$.

So we have $i = i_1 - i_2 = 4w_1 - 4w_2 - 2$.

(2) Let C be a unicursal curve. Then C divides the plane into many regions. Inside each region, pick a point. Then we assign to each region the winding number of C relative to this point.

Claim. If C has n double points, then the maximum of the absolute value of the winding numbers for all regions is not greater than $n+1$.

Proof of the claim. It is easy to check that if two regions are adjacent with a common edge, then their winding numbers differ by 1. If C has n double points, then C has $2n$ edges, and the plane is divided into $n+2$ regions (including the unbounded one). Note that the winding number for the unbounded region is always 0, and the claim follows.

Now the inequality in (2) follows from (1) and the claim.

Corollary 5.12. *We have the following bounds on unicursal curves with n double points:*

(1) $|I^-| \leq n^2 + 2n$;
(2) $|St| \leq 2n^2 + 2n$;
(3) $|J^-| \leq 5n^2 + 4n$.

These follow easily from Theorem 5.5 (5), Corollary 5.10 and Lemma 5.11.

Notice that sharper bounds and extremal curves (as conjectured by Arnold [1]) for the invariants St and J^- were obtained by Shumakovich [16] and Viro [14] respectively, using different combinatorial formulae for these invariants. We list their inequalities below:

(1) (Shumakovich's inequality) $|St| \leq n(n+1)/2$;
(2) (Viro's inequality) $J^{-1} \leq n^2 + 2n$.

These inequalities were also proved by Polyak using his formula for Arnold's invariants [11]. Here again, we are more interested in the order of these bounds.

The extremal values of α are still unknown. The examples in Figure

8 show that there are curves with $\alpha \leq 0$. Actually, there is no bound from below for α if we do not fix the number of double points. Notice that both curves have only positive triangles, and there are curves with only negative triangles, too.

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