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ADDENDUM AND CORRECTION TO "A COMBINATION THEOREM FOR NEGATIVELY CURVED GROUPS"

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0. Introduction

The purpose of this note is to give an algebraic formulation of the main theorem in our paper [1, Theorem (combination theorem)] and to correct an omission in a corollary [1, Corollary (HNNs over virtually cyclics)]. We also state a slightly stronger version of the combination theorem in the last section. Refer to [1] for notation that is not introduced here. We would like to thank Olga Kharlampovich and Alexei Myasnikov for pointing out the omission.

1. Combination Theorem (Algebraic Version)

In this section we give an algebraic formulation of our main theorem [1, Theorem (combination theorem)] which gives sufficient conditions for a graph of negatively curved groups (in the sense of Gromov [3]) to be negatively curved. We start with notation and definitions. Let $\mathcal{G}$ be a finite graph of groups with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. The group associated to a vertex v is denoted by G_v, and to an edge e by G_e. These groups are assumed to be finitely generated with fixed finite generating sets. If $v = \iota(e)$ is the initial vertex of e, we are also given a monomorphism $f_e : G_e \to G_v$. The edge e with opposite orientation is denoted by $\bar{e}$.

Definitions 1.1. An *annulus of length* $2m$ consists of:

(1) an edge-path $e_{-m}e_{-m+1} \cdots e_0 \cdots e_m$ in $\mathcal{G}$, and

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(2) a sequence

$$\epsilon_{-m}, \nu_{-m}, \epsilon_{-m+1}, \nu_{-m+1}, \ldots, \epsilon_{-1}, \nu_{-1}, \epsilon_0, \nu_0, \ldots, \epsilon_{m-1}, \nu_{m-1}, \epsilon_m$$

with $\epsilon_i \in G_{e_i}$ and $\nu_i \in G_{\iota(e_{i+1})}$, and

$$\nu_i f_{\overline{e}_i}(\epsilon_i)\nu_i^{-1} = f_{e_{i+1}}(\epsilon_{i+1}), i = -m, \ldots, m-1.$$

An annulus of length $2m$ is *essential* if whenever $e_{i+1} = \overline{e}_i$, then $\nu_i \notin Im(f_{e_{i+1}})$. An annulus is *$\rho$-thin* if for each i the word-length of ν_i in $G_{\iota(e_{i+1})}$ is $\leq \rho$. The *girth* of an annulus is the word-length of ϵ_0 in G_{e_0}. An annulus is *λ-hyperbolic* if $\max\{| e_{-m} |, | e_m |\} \geq \lambda \mid e_0 \mid$. (Here $| \cdot |$ denotes the word-length in the appropriate edge-group.) $\mathcal{G}$ satisfies the *qi-embedded condition* if for every edge e the monomorphism f_e is a quasi-isometric embedding. $\mathcal{G}$ satisfies the *annuli flare condition* if there are numbers $\lambda > 1$ and $m \geq 1$ such that for all ρ there is a constant $H = H(\rho)$ such that any ρ-thin essential annulus of length $2m$ and girth at least H is λ-hyperbolic.

Example. Let $G := < a > *_{a^2=b^2} < b >$. The sequence $a^2, a, a^2 = b^2, b, b^2$ is the sequence of group elements corresponding to an essential annulus of length 2. By extending this sequence in the obvious fashion, essential annuli of arbitrary even length may be constructed. Note that, even though the amalgamating groups in this example are negatively curved, the group G is not.

Given a graph of groups for a group G, a graph of spaces with fundamental group G may be constructed by assembling the union of the presentation complexes for the vertex groups together with the product of the interval with the union of the presentation complexes for the edge groups; cf. [4]. We may apply [1, Theorem (combination theorem)] and the remark at the end of the paper to obtain:

Theorem 1.2 (Algebraic combination theorem). *Let $\mathcal{G}$ be a finite graph of negatively curved groups satisfying the qi-embedded and annuli flare conditions. Then $\pi_1(\mathcal{G})$ is negatively curved.*

2. HNN Extensions over a virtually cyclic group

In this section, we correct an omission in Corollary (HNNs over virtually cyclics) in [1]. We will need two lemmas. For basic facts about G-trees see [4].

Lemma 2.1. *Given groups G and G', let T be a G-tree and T' a G'-tree. Let $(\phi : G \to G', f : T \to T')$ be a morphism of trees. Suppose the following:*

(1) *the induced map of graphs $\overline{\phi} : T/G \to T'/G'$ is locally injective,*

(2) *for edge e of T and each vertex v of e, $\phi(\mathrm{Stab}(v)) \cap \mathrm{Stab}(fe) = \phi(\mathrm{Stab}(e))$, and*

(3) *the restriction of ϕ to a vertex stabilizer is injective.*

Then ϕ is injective.

Proof. Recall that an isometry of a tree is *elliptic* if it fixes a point; otherwise it is *hyperbolic*. Conditions (1) and (2) are equivalent to f being locally injective at each vertex. Thus, the image under ϕ of a hyperbolic element of G is hyperbolic in G'. Hence, any element of the kernel of ϕ is elliptic. Since, by Condition (3), ϕ is injective on elliptics, ϕ is injective.

Recall that for $m, n \geq 1$ the Baumslag-Solitar group $BS(m,n)$ is defined via the presentation

$$BS(m,n) = \langle x, t | tx^m t^{-1} = x^n \rangle.$$

Lemma 2.2. *Let A be a virtually cyclic group, C and C' two infinite subgroups of A, and $\phi : C \to C'$ an isomorphism. Then $G = A*_\phi$ contains some $BS(m,n)$.*

Proof. If $A = \mathbb{Z}$, then $G = BS(m,n)$ for some m, n. By the structure theorem for 2-ended groups [4] every infinite virtually cyclic group admits an epimorphism with finite kernel to either $\mathbb{Z}$ or the infinite dihedral group $\mathbb{D}$.

Case 1. There is an epimorphism $\alpha : A \to \mathbb{Z}$. Since the kernel of α is the set of elements of finite order in A, ϕ induces an isomorphism $\phi' : \alpha(C) = m\mathbb{Z} \to \alpha(C') = n\mathbb{Z}$. Thus there is an epimorphism $\hat{\alpha} : G = A*_\phi \to \mathbb{Z}*_{\phi'} = BS(m,n)$. Choose a section $\sigma : \mathbb{Z} \to A$ of α. Let Z denote $\sigma(\mathbb{Z})$. Note that there is a $k \neq 0$ such that ϕ restricts to an isomorphism $kmZ \to knZ$. (For example, consider $\phi : \phi^{-1}(\phi(C \cap Z) \cap Z) \to \phi(C \cap Z) \cap Z$. The ratio of indices in Z of the groups in the previous sentence is m/n since ϕ covers ϕ'.) In particular, we have homomorphisms

$$BS(m,n) \to BS(km, kn) \cong Z*_{\phi|mkZ} \to G \xrightarrow{\hat{\alpha}} BS(m,n),$$

where the first map is given by

$$s \mapsto s^k, \ t \mapsto t.$$

Note that the composition $BS(m,n) \to BS(m,n)$ of the above maps is given by the same formula and corresponds to a k-fold covering map between presentation 2-complexes. Thus, this composition is injective and G contains $BS(m,n)$.

Case 2. There is an epimorphism $\alpha : A \to \mathbb{D}$. Let $\mathbb{Z} \subset \mathbb{D}$ be the elements fixing the ends of $\mathbb{D}$. The kernel of α is the set of elements of finite order in A that fix the ends of A. We may use the inclusion $C \subset A$ to identify the ends of C with those of A. Similarly, we may identify (via inclusion) the ends of C' with those of A, the ends of $\mathbb{Z}$ with those of $\mathbb{D}$ (via inclusion), and the ends of A with those of $\mathbb{D}$ (via α). Thus ϕ induces an isomorphism $\phi' : \alpha(C) \to \alpha(C')$. Let $\tilde{A}$ denote $\alpha^{-1}(\mathbb{Z})$, $\tilde{C}$ denote $C \cap \tilde{A}$, and $\tilde{C}'$ denote $C' \cap \tilde{A}$. The map ϕ restricts to an isomorphism from $\tilde{C}$ to $\tilde{C}'$ since ϕ sends end-preserving elements to end-preserving elements. Thus, we have a homomorphism $\tilde{A}*_{\phi|\tilde{C}} \to A*_\phi$ which is injective by Lemma 2.1. By Case 1, $\tilde{A}*_{\phi|\tilde{C}}$ contains some $BS(m,n)$ and hence G does also.

We are now ready to state and prove our correction.

Corollary 2.3 (HNN over virtually cyclic). *Let $G := A*_\phi$ where A is negatively curved and $\phi : C \to C'$ is an isomorphism between virtually cyclic subgroups of A. Then* (1), (2), *and* (3) *below are equivalent.*

(1) *The group G contains no $BS(m,n)$.*
(2) *For all $x \in A$, $CC'(x) := \{c \in C \mid xcx^{-1} \in C'\}$ is finite and one of (a) and (b) below must hold.*
 (a) *For all $x \in A \setminus C$, $C(x) := \{c \in C \mid xcx^{-1} \in C\}$ is finite.*
 (b) *For all $x \in A \setminus C'$, $C'(x) := \{c \in C' \mid xcx^{-1} \in C'\}$ is finite.*
(3) *The group G is negatively curved.*

Remarks. The second part of condition (2), whose analogue correctly appears in [1, Corollary (free products over virtually cyclics)], was inadvertently omitted from the original version of the Corollary 2.3 [1, Corollary (HNNs over virtually cyclics)]. Notice that the group $< a, b, t | t^{-1}a^2t = b^2 >$ supplied to us by O. Kharlampovich and A. Myasnikov as a counterexample to the original version satisfies the first, but not the second, part of condition (2) of Corollary 2.3 and is not negatively curved (it contains $< a^2, tbt^{-1}a > \cong \mathbb{Z} \oplus \mathbb{Z}$).

Proof. **(1)$\Longrightarrow$(2):** We show that the failure of condition (2) forces G to contain a $BS(m,n)$. Since G has an alternate description as $(A *_{C=C'} A)*_{A=A}$, we see that $A *_{C=C'} A$ is a subgroup of G. Thus, if the second condition of (2) fails (that is, if both (2a) and (2b) fail), G has a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ by Corollary(free products over virtually cyclics). So, assume that $CC'(x)$ is infinite for some $x \in A$. Let $\phi' : C \to x^{-1}C'x$ be given by $\phi'(c) = x^{-1}\phi(c)x$. Since $A*_{\phi'}$ and $A*_\phi$ are isomorphic, we may assume that $C \cap C'$ is infinite. Since C is virtually cyclic, there is an element c of infinite order in $C \cap C'$. Any element of $C \cup C'$ has a nontrivial power in common with c and so (since A is hyperbolic) $C \cup C'$ generates a virtually cyclic subgroup A' of A. By Lemma 2.1, $G' := A'*_\phi$ injects into G. The group G' contains some

$BS(m,n)$ by Lemma 2.2.

(2)⟹(3): We verify the conditions of Theorem 1.2(Algebraic combination theorem). Every virtually cyclic subgroup of a negatively curved group is qi-embedded [3, Corollary 8.1.D]. For concreteness, assume (2a) holds. To verify the annuli flare condition, let $\lambda = 2$ and $m = 1$. Let ρ be given. The graph corresponding to G has a single vertex v and a single edge e where Image$(f_e) = C$ and Image$(f_{\overline{e}}) = C'$. Consider a ρ-thin length 2 annulus consisting of the edge path $e_{-1}e_0e_1$ and sequence $\epsilon_{-1}\nu_{-1}\epsilon_0\nu_0\epsilon_1$. We argue that there are only finitely many possibilities for ϵ_0. This implies the claim, since we can choose H to be larger than the length of any of these elements.

We may assume (by reversing the path if necessary) that $e_0 = e$. By the definition of an annulus,

$$\nu_{-1}f_{\overline{e}_{-1}}(\epsilon_{-1})\nu_{-1}^{-1} = f_{e_0}(\epsilon_0).$$

Thus, since $f_{e_0}(\epsilon_0) \in C$, $f_{e_0}(e_0) \in CC'(\nu_{-1}^{-1})$ if $e_{-1} = e$ or $f_{e_0}(e_0) \in C(\nu_{-1}^{-1})$ if $e_{-1} = \overline{e}$. So, $f_0(\epsilon_0)$ is in the finite set

$$\bigcup_{x\in S\setminus C} C(x) \cup \bigcup_{x\in S} CC'(x),$$

where S is the set of elements of A whose A-length is $\leq \rho$.

(3)⟹(1): See [3, Corollary 8.2.C.], [2].

3. Combination theorem (geometric version)

In this section we state a stronger version of Theorem (combination theorem) [1]. Let $B = B_X$ be the function defined in [1, p.90].

Definition(3.1). The graph of spaces X satisfies the *weak annuli flare condition* if there are numbers $\lambda > 1$, $m \geq 1$, and H such that any $B_X(4)$-thin essential annulus of length $2m$ and girth at least H is λ-hyperbolic.

In [1] we actually prove the following strengthening of the main theorem there. The remark below then implies Theorem 1.2.

Theorem 3.2 Geometric combination theorem. *Let X be a finite graph of negatively curved spaces satisfying the qi-embedded and weak annuli flare conditions. Then X is negatively curved.*

Remark. Consider an annulus $\Delta : [-m,m] \times S^1 \to X$ such that for the induced hallway $\tilde{\Delta} : [-m,m] \times [0,1] \to \tilde{X}$ the following holds:

$$d_{X_{v(i)}}(\tilde{\Delta}(i,0), \tilde{\Delta}(i+1,0)) \leq \rho.$$

If we homotope Δ so that $\Delta([i, i+1] \times \{0\})$ is geodesic in the appropriate vertex space, then $\tilde{\Delta}([i, i+1] \times [0,1])$ is a τ-quasi geodesic quadrilateral in the universal cover of this vertex space and so is $B(4)$-thin. It follows that Δ is $\rho + 2B(4)$-thin.

4. Errata to [1]

Hallways flare condition, p.90. Replace the word "annulus" by the word "hallway".

Figure 3, p.92. Replace "W_i" by "W_1" in the caption. Also, in the figure, draw two "o's" at the endpoints of the segment constituting the intersection of W_1 and the shaded region. See Figure 2 for notation.

Corollary(HNN's over virtually cyclics), p.100. The statement should include the extra hypothesis indicated above.

References

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