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DISCRETE SURFACES WITH CONSTANT NEGATIVE GAUSSIAN CURVATURE AND THE HIROTA EQUATION

ALEXANDER BOBENKO & ULRICH PINKALL

1. Introduction

Surfaces with constant curvature (especially, with constant negative curvature $K = -1$, which we will call shortly K-surfaces) were one of the favorite objects of investigation in differential geometry in the 19th century (see [2], [10]). During this classical period many properties, which nowadays might be called *integrable* were discovered and many explicit examples of surfaces were constructed.

Later on the fashion of constructing explicit examples changed to proving that certain examples do not exist. The structure of the spaces of surfaces with constant curvature was partially clarified. Typical examples of the results were theorems proving that the only surface satisfying some prescribed assumptions is a round sphere.

A modern period of interest in this theory started with the paper [24] by Wente, where the simplest tori with constant mean curvature (abbreviated to CMC) were constructed. This turned out to be an interesting alternative to the theorems mentioned above. Further progress is mostly due to the theory of integrable equations - theory of solitons, which appeared in 1960's. Though this theory was oriented basically towards problems of mathematical physics, it deals in many cases with the same equations as differential geometry. The sine-Gordon equation

$$\phi_{xt} - \sin\phi = 0, \tag{1.1}$$

which is the Gauss equation for the K-surfaces, is one of the fundamental examples in this theory [11]. A characteristic result obtained with

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the help of this theory is the classification and explicit description of all CMC tori [21],[5].

One should mention also the role of calculations in this theory. Of course, people were always interested in seeing appearances of the surfaces described by explicit formulas, or differential equations or variational principles which they have. In the pre-computer era this was really a difficult problem (see, for example, the impressive calculation tables in [23]). Often these explicit formulas involve complicated functions, and differential equations and variational principles usually allow to investigate surfaces only locally. Computers simplify these problems a lot, but solve completely the problem with explicit formulas only. As a top achievment in this area one should probably consider the software for calculations on hyperelliptic Riemann surfaces, developed by M.Heil for SFB 288, which makes visible, in particular, all the CMC tori, by visualizing the theta functional formulas of [4],[5].

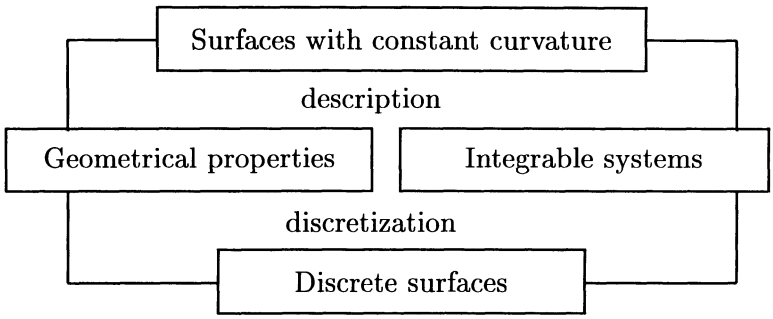
On the other hand the question of proper discretization, which is of the main importance for numerical solution of differential equations or variational problems describing the surfaces, can hardly be considered as completely solved. One can suggest various discrete problems, which locally have the same continuous limit and nevertheless have quite different global properties. For example, taking a proper discrete variational principle for the minimal surfaces one can speed up the area-minimizing process a lot [20].

Taking into account the connection between surfaces with constant curvature and integrable systems one can suggest two approaches to define proper discrete analogues to the K-surfaces:

(i) to postulate natural discrete analogues of some geometrical properties,

(ii) to construct a discrete integrable system, corresponding to a continous one. Schematically these two mechanisms are shown in a diagram below.

In the present paper we show that these two approaches yield *the same* definition of the discrete surfaces with constant negative curvature, which we call discrete K-surfaces.



Actually both methods mentioned above were already applied to the problem under consideration. Let $F(x, y)$ be an asymptotic line parametrization of a K-surface, forming a Chebyshev net (see for details §2). It means that the images of the straight lines $x = const$ or $y = const$ under the map $F : \mathbf{R}^2 \to \mathbf{R}^3$ are asymptotic lines and

$$F_{xx}, F_{yy} \perp N, \ |F_x| = |F_y| = 1, \tag{1.2}$$

where N is the Gauss map. The angle $\phi(x, y)$ between the asymptotic lines satisfies the sine-Gordon equation (1.1).

Wunderlich in [25] suggested the following natural geometrical discretization $F : \mathbf{Z}^2 \to \mathbf{R}^3$ (see §3):

1) Every point $F_{n,m}$ and its 4 neighbors $F_{n-1,m}, F_{n,m-1}, F_{n+1,m}, F_{n,m+1}$ belong to one plane $\mathcal{P}_{n,m}$.

2) All edges of the net $F_{n,m}$ are of the same length. In [25] important geometrical properties of discrete K-surfaces were established and a discrete analogue of the Bäcklund transformation was constructed, but a discrete analogue of equation (1.1) was not discussed.

On the other hand, Hirota in [14] without using any relation to geometry constructed a discrete integrable analogue to equation (1.1):

$$\begin{aligned} &\sin(\frac{1}{2}(h_{n+1,m} + h_{n,m+1} - h_{n+1,m+1} - h_{n,m})) \\ &= k \sin(\frac{1}{2}(h_{n+1,m+1} + h_{n,m} + h_{n+1,m} + h_{n,m+1})). \end{aligned} \tag{1.3}$$

In the present paper we show that the papers [25] and [14] studied essentially the same mathematical problem. Using a quaternionic description of surfaces in $\mathbf{R}^3$ we formulate geometrical properties of the discrete K-surfaces in a modern language of integrable systems and show that the angles between the edges of discrete K-surfaces satisfy the equation

$$\begin{aligned}\phi_{n+1,m+1} + \phi_{n,m} - \phi_{n+1,m} - \phi_{n,m+1} \\ = 2\arg(1 - ke^{-i\phi_{n+1,m}}) + 2\arg(1 - ke^{-i\phi_{n,m+1}}),\end{aligned} \tag{1.4}$$

or in our notation of the diagonally oriented lattice (see (6.2))

$$\phi_u + \phi_d - \phi_l - \phi_r = 2\arg(1 - ke^{-i\phi_l}) + 2\arg(1 - ke^{-i\phi_r}),$$

which is natural to be called a discrete sine-Gordon equation. §§3-6 are devoted to the description of geometrical properties of the discrete K-surfaces, in particular, the relation between equations (1.3) and (1.4) is established in §6. It turns out that the discrete sine-Gordon equation arises by averaging from the Hirota equation.

As in the smooth case the discrete K-surfaces are closely related to the Lorentz-harmonic maps. The Gauss map $N(x, y)$ of the K-surface, parametrized as above, comprises a Lorentz-harmonic Chebyshev net in S^2, which means

$$|N_x| = |N_y| = 1, \ N_{xy} \parallel N. \tag{1.5}$$

The first condition in the definition of a discrete K-surface allows us to define the Gauss map naturally in the discrete case as a unit normal to the plane $\mathcal{P}_{n,m}$. In §3 we show that this map is a Lorentz-harmonic discrete Chebyshev net in S^2. This means that all the scalar products

$$< N_{n,m}, N_{n+1,m} >=< N_{n,m+1}, N_{n,m} >= \cos\Delta \tag{1.6}$$

are independent of n, m, and

$$\begin{aligned}N_{n+1,m+1} - N_{n+1,m} - N_{n,m+1} + N_{n,m} \\ \parallel N_{n+1,m+1} + N_{n+1,m} + N_{n,m+1} + N_{n,m}.\end{aligned} \tag{1.7}$$

The properties (1.6) and (1.7) are clearly natural discrete analogues of (1.5).

§5 makes this geometrical picture familiar for specialists in the theory of integrable systems. We introduce a "spectral parameter" λ, interpret it as describing deformations preserving geometrical properties and prove the Sym formula

$$F_{n,m} = \Psi_{n,m}^{-1} \frac{\partial}{\partial \lambda} \Psi_{n,m} \tag{1.8}$$

for the discrete K-map $F : \mathbf{Z}^2 \to \mathbf{R}^3$ in terms of the corresponding extended (i.e., depending on λ) quaternionic frame $\Psi_{n,m}(\lambda) \in SU(2)$.

A simple mechanism to solve the Cauchy problem and to reconstruct step by step a discrete K-surface starting from Cauchy data

$$N_{n,m}, N_{n+1,m}, N_{n+1,m+1}, \dots , N_{n+N,m+M} = N_{n,m} \tag{1.9}$$

is presented at the end of §6. In the next section it is used to describe discrete K-surfaces with two straight asymptotic lines, which in turn leads to a discrete analogue of the Painlevé III equation.

Unfortunately, this simple geometrical method does not allow us to control the global behavior of the surface (for example, to control the periodicity of the evolution of the initial broken loop (1.9)). A typical discrete K-cylinder, obtained in this way is shown in Figure 1:

For this reason, starting in §8, we use methods from the theory of integrable equations, which are based on an analytic solution of the problem. We calculate explicitly the solutions to the Hirota equation and the correponding extended frame $\Psi_{n,m}(\lambda)$. We determine in this way the discrete K-surface $F_{n,m}$ and its Gauss map $N_{n,m}$.

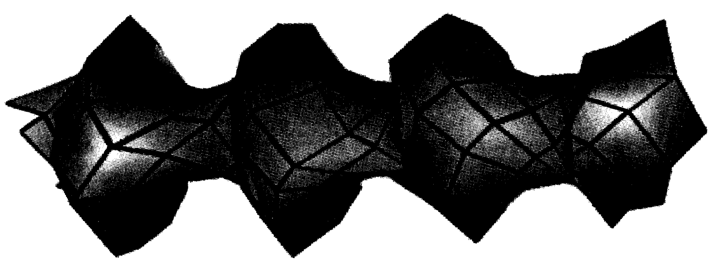


FIGURE 1. A typical discrete K-cylinder.

In §8 the Bäcklund transformation for discrete K-surfaces is presented. If some solution $h^{old}_{n,m}$ of the Hirota equation together with the corresponding function $\Psi^{old}_{n,m}(\lambda)$ is known, this procedure allows us to construct new $h^{new}_{n,m}, \Psi^{new}_{n,m}(\lambda)$, which solve the same equations and describe a new discrete K-surface. Applying this transformation to the trivial solution $h^{old}_{n,m} = 0$ we construct the discrete pseudospheres.

The rest of the paper is devoted to the finite-gap integration of the Hirota equation. Geometrically, we are interested in discrete K-surfaces, whose Gauss map $N_{n,m}$ is periodic with some period (N, M) (1.9). In §9 it is shown how multiplying the rotations of the frame along the loop (1.9) one associates a transfer matrix $T^{SG}_{n,m}(\lambda)$ and a spectral curve

$$\det(T^{SG}_{n,m}(\lambda) - \mu) = 0$$

to such a surface. The spectral curve is an invariant of the discrete K-surface and is a hyperelliptic Riemann surface of finite genus.

An extended frame $\Psi_{n,m}(\lambda)$, which is also an eigenfunction of the monodromy matrix, is called a Baker-Akhiezer function. For the usual sine-Gordon equation (1.1) this function was first introduced by A.Its (see [3]), who obtained a theta function formula for it. It is a function with essential singularities and poles. In the discrete case the Baker-Akhiezer function is a meromorphic function on the spectral curve X. The analytical properties of $\Psi_{n,m}(\lambda)$ established in §10 allow us to reconstruct it in §11 by explicit formulas in terms of theta functions and abelian differentials of X.

The Baker-Akhiezer function constructed in §11 is complex valued and is parametrized by the following parameters: the spectral curve X, singularities $P_\infty, P_0 \in X$ and vector $D \in Jac(X)$ in the Jacobi variety of X. It has the geometrical meaning of a frame only if it lies in $SU(2)$ for real λ. The corresponding specification of the parameters X, P_∞, P_0, D, which guaranties $\Psi_{n,m}(\lambda \in \mathbf{R}) \in SU(2)$ is obtained in §12. §13 presents a final formula for the discrete K-surface generated by the finite-gap solution.

Generally, this map is quasiperiodic. §14 contains the periodicity conditions for $F_{n,m}$ to be periodic with the period (N, M), and also the simples examples, generated by spectral curves of genus $g = 1$. As in the smooth case [4] the periodicity conditions are formulated in terms of the spectral curve X and do not involve the vector $D \in \mathbf{R}^g$. In contrast to the smooth case, a change of D induces a non-trivial deformation of the corresponding discrete K-surface even in the case where the spectral curve is of low genus $g = 1, 2$.

The simplest compact discrete K-surfaces are constructed in §15. The spectral curve is of genus 2 in this case and is of the same symmetry type as the curves generating the Wente tori.

We hope that the remarkable similarity of the geometrical properties, as well as of the analytical constructions and formulas, in the smooth and the discrete cases, which is visible throughout the present whole paper, convince the reader that the discretization which we study here is a proper one.

In our paper [9] in a similar way we define discrete analogues of the CMC surfaces. This case is more difficult for investigation since the corresponding Gauss equation is an elliptic version

$$\Delta u + \sinh u = 0,$$

of the sine-Gordon equation, which is more difficult to discretize. Concequently, the geometrical properties of the discrete CMC surfaces, which we postulate, are not so transparent as in the case of discrete K-surfaces and would not have been guessed without using the theory of integrable systems.

We would like to mention here also two recent papers [7], [8], which show that the applications of the discrete sine-Gordon equation (1.4) extend beyond differential geometry. This equation, written in exponential form (see §6 for notation)

$$Q_u Q_d = \frac{Q_l - k}{1 - kQ_l}\frac{Q_r - k}{1 - kQ_r}, \tag{1.10}$$

can be considered over a finite field. An integrable cellular automaton with a Lax representation was defined in this way in [7]. Another interesting application is the quantum version of (1.10)

$$Q_u Q_d = \frac{e^{i\gamma}Q_l - k}{1 - ke^{i\gamma}Q_l}\frac{e^{i\gamma}Q_r - k}{1 - ke^{i\gamma}Q_r}, \tag{1.11}$$

which was obtained in [8], based on the results of [12]. Here the fields Q's are unitary operators, which do not commute anymore. Actually the commutation rules are such that among the operators Q's on a horizontal zigzag line only the nearest neighbors do not commute:

$$Q_d Q_l = e^{2i\gamma} Q_l Q_d, \; Q_d Q_r = e^{2i\gamma} Q_r Q_d.$$

The equation (1.11) determines an integrable quantum evolution and is called the quantum discrete sine-Gordon equation. For the periodic solution of period 2 one gets the discrete quantum pendulum [8] - a quantum mechanical system generalizing the Hofstadter hamiltonian.

2. Smooth surfaces with constant negative Gaussian curvature

First we consider smooth surfaces with constant negative Gaussian curvature and present here some fragments of their theory, most of which are classical. A more detailed presentation of the theory is given in [17], [6].

Let us consider a surface $\mathcal{F}$ with negative Gaussian curvature. For each regular point of $\mathcal{F}$ there are 2 directions, called asymptotic directions, where the normal curvature vanishes. We use asymptotic line parametrizations of $\mathcal{F}$

$$F : (x, y) \in \mathbf{R}^2 \to \mathbf{R}^3.$$

For such a parametrization the vectors F_x, F_t, F_{xx}, F_{yy} are orthogonal to the normal vector N, i.e.,

$$F_x, F_y, F_{xx}, F_{yy} \perp N. \tag{2.1}$$

The fundamental forms are as follows:

$$I = < dF, dF >= A^2(dx)^2 + 2AB\cos\phi dxdy + B^2(dy)^2,$$
$$II = - < dF, dN >= 2 < F_{xy}, N > dxdy,$$

where ϕ is the angle between the asymptotic lines and

$$A = |F_x|, \qquad B = |F_y|.$$

For the constant negative Gaussian curvature case:

$$K = \det II / \det I = -1,$$

we get

$$II = 2AB\sin\phi dxdy$$

and the following Gauss-Codazzi equations:

$$\phi_{xy} - AB\sin\phi = 0, \tag{2.2}$$
$$A_y = B_x = 0. \tag{2.3}$$

If A and B do not vanish we call this parametrization a weak Chebyshev net.

The Gauss-Codazzi equations are invariant with respect to the transformations

$$A \to \lambda A, \qquad B \to \lambda^{-1} B, \qquad \lambda \in \mathbf{R}. \tag{2.4}$$

This fact implies the following well known theorem.

Theorem 1. *Every surface with constant negative Gaussian curvature posesses a one-parameter family of deformations preserving the second fundamental form, the Gaussian curvature and the angle ϕ between the asymptotic lines. This deformation is described by the transformation* (2.4).

This one-parameter family of surfaces is called an associated family.

Equations (2.2, 2.3) can be represented as the compatibility condition

$$U_y - V_x + [U, V] = 0$$

for the following system

$$\Psi_x = U\Psi, \qquad \Psi_y = V\Psi, \tag{2.5}$$

$$\begin{aligned} U &= \frac{i}{2}\begin{pmatrix} \phi_x/2 & -A\lambda e^{-i\phi/2} \\ -A\lambda e^{i\phi/2} & -\phi_x/2 \end{pmatrix}, \\ V &= \frac{i}{2}\begin{pmatrix} -\phi_y/2 & B\lambda^{-1}e^{i\phi/2} \\ B\lambda^{-1}e^{-i\phi/2} & \phi_y/2 \end{pmatrix}. \end{aligned} \tag{2.6}$$

Equations (2.5, 2.6) are the equations for the moving frame of the asymptotically parametrized surface with $K = -1, |F_x| = \lambda A, |F_y| = \lambda^{-1}B$ in the $su(2)$ representation. To show this (for more details see [6]) let us identify a 3-dimensional vector space $\mathbf{R}^3$ with the space of imaginary quaternions $su(2)$

$$\begin{aligned} &X = (X_1, X_2, X_3) \in \mathbf{R}^3 \longleftrightarrow X = -i\sum_{\alpha=1}^{3} X_\alpha \sigma_\alpha, \\ &\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{2.7}$$

From now on we represent vectors in this matrix form. The scalar product is written as follows

$$< X, Y >= -\frac{1}{2} \operatorname{tr} XY.$$

Let us remark that for $\lambda \in \mathbf{R}$ the matrices $U(\lambda), V(\lambda) \in su(2)$, therefore there is a solution of (2.5): $\Psi(x,t,\lambda) \in SU(2)$.

It can be easily checked that the following formulas describe the moving frame of a surface with $K = -1$, $|F_x| = \lambda A$, $|F_y| = \lambda^{-1}B$:

$$F_x = -i\lambda A \Psi^{-1} \begin{pmatrix} 0 & e^{-i\phi/2} \\ e^{i\phi/2} & 0 \end{pmatrix} \Psi, \tag{2.8}$$

$$F_y = -i\frac{B}{\lambda}\Psi^{-1} \begin{pmatrix} 0 & e^{i\phi/2} \\ e^{-i\phi/2} & 0 \end{pmatrix} \Psi, \tag{2.9}$$

$$N = -i\Psi^{-1}\sigma_3\Psi, \tag{2.10}$$

where $\Psi \equiv \Psi(x,t,\lambda) \in SU(2)$. Moreover, as was first observed by A. Sym [22], the dependence on λ allows us to integrate the formulas (2.8), (2.9) for the moving frame.

Theorem 2. *Let $\phi(x,y), A(x), B(y)$ be a solution of* (2.2). *Then the corresponding immersion with $K = -1, |F_x| = \lambda A, |F_y| = \lambda^{-1}B$ is given by*

$$F = 2\Psi^{-1}\frac{\partial \Psi}{\partial t}, \qquad \lambda = e^t, \tag{2.11}$$

where $\Psi(x,y,\lambda = e^t) \in SU(2)$ is a solution of (2.5), (2.6). *The Gauss map is given by*

$$N = -i\Psi^{-1}\sigma_3\Psi. \tag{2.12}$$

To prove this theorem we note that F determined by (2.11) lies in $su(2)$ and

$$F_x = 2\Psi^{-1}\frac{\partial U}{\partial t}\Psi, \quad F_y = 2\Psi^{-1}\frac{\partial V}{\partial t}\Psi,$$

following from (2.5), (2.11), coincide with (2.8), (2.9).

We consider not only immersions but more generally weakly regular surfaces, i.e., the surfaces with $A \neq 0, B \neq 0$ for all x, y. In this case the change of coordinates $x \to \tilde{x}(x), y \to \tilde{y}(y)$, conformal with respect to the second fundamental form, reparametrizes the surface so that the asymptotic lines are parametrized by arc-lengths (generally different for x and y directions)

$$A = |F_x| = \text{ const}, \quad B = |F_y| = \text{ const}. \tag{2.13}$$

We will call this parametrization an anisotropic Chebyshev net. In this parametrization the Gauss equation and the system (2.5), (2.6) become the sine-Gordon equation with the standard Lax representation [11] (the deformation parameter λ is called a spectral parameter in the theory of integrable equations). If $A = B$ then the parametrization is called a Chebyshev net. The associated family of an anisotropic Chebyshev net contains exactly one Chebyshev net.

At last we mention also a well known fact, which also can be easily checked.

Proposition 1. *The Gauss map $N : \boldsymbol{R}^2 \to S^2$ of the surface with $K = -1$ is Lorentz-harmonic, i.e.,*

$$N_{xy} = \rho N, \quad \rho : \boldsymbol{R}^2 \to \boldsymbol{R}. \tag{2.14}$$

It forms in S^2 the same kind of Chebyshev net as the immersion function does in $\boldsymbol{R}^3$:

$$|N_x| = \lambda A, \quad |N_y| = \lambda^{-1} B. \tag{2.15}$$

3. Discrete weak Chebyshev net and its Gauss map

By a discrete surface we mean a map $F : \mathbf{Z}^2 \to \mathbf{R}^3$. We use the following notation for the elements of discrete surfaces (n, m are integer labels):

$F_{n,m}$ - for the vertices,

$[F_{n+1,m}, F_{n,m}], [F_{n,m+1}, F_{n,m}]$ - for the edges,

$(F_{n+1,m+1}, F_{n+1,m}, F_{n,m}, F_{n,m+1})$ - for the elementary quadrilaterals comprised by the indicated vertices. We define a discrete weak Chebyshev net (discrete surface with constant Gaussian curvature) using natural discrete analogs of the properties (2.1), (2.3) in the smooth case.

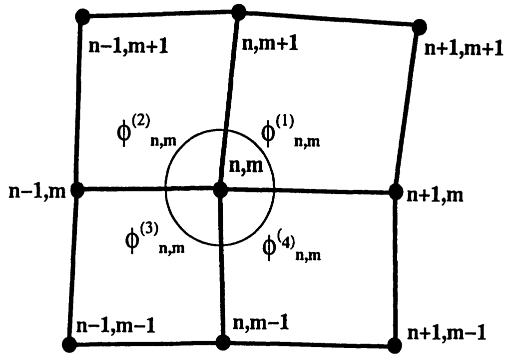


FIGURE 2. A piece of the surface with the indexes of the vertices indicated.

Definition 1. A discrete surface with constant Gaussian negative curvature (we will call these surfaces discrete K-surfaces) is a map

$$F : \mathbf{Z}^2 \to \mathbf{R}^3$$

possessing the following properties:

i) For each point $F_{n,m}$ there is a plane $\mathcal{P}_{n,m}$ such that

$$F_{n,m}, F_{n+1,m}, F_{n-1,m}, F_{n,m+1}, F_{n,m-1} \in \mathcal{P}_{n,m}.$$

ii) The lenghts of the opposite edges of an elementary quadrilateral are equal

$$|[F_{n+1,m}, F_{n,m}]| = |[F_{n+1,m+1}, F_{n,m+1}]| = A_n \neq 0,$$
$$|[F_{n,m+1}, F_{n,m}]| = |[F_{n+1,m+1}, F_{n+1,m}]| = B_m \neq 0,$$

where we have incorporated into the notation that A_n does not depend on m and B_m not on n.

Remark. The easiest example of the discrete K-surface is a map $F : \mathbf{Z}^2 \to \mathbf{R}^2$ possessing the property (ii). All these planar surfaces are made out of parallelograms and can be easily reconstructed from two broken lines $F_{n,0}$ and $F_{0,m}$; $n, m \in \mathbf{Z}$. We do not consider planar surfaces here.

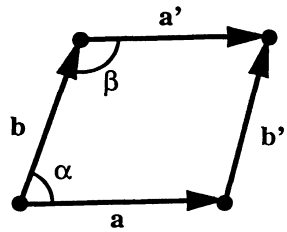


FIGURE 3. An elementary quadrilateral.

Let us consider an elementary quadrilateral of a discrete K-surface and denote the vectors forming its edges as in Figure 3. The definition of a discrete K-surface yields

$$|\mathbf{a}| = |\mathbf{a}'|, \ |\mathbf{b}| = |\mathbf{b}'|, \ \mathbf{a} + \mathbf{b}' = \mathbf{b} + \mathbf{a}', \tag{3.1}$$

which implies the equality of the opposite angles of the quadrilateral.

First, we consider discrete K-immersions [1], i.e., assume that in each plane $\mathcal{P}_{n,m}$ the four edges are cyclically ordered as in Figure 2, and quadrilaterals do not degenerate

$$0 < \alpha, \beta < \pi. \tag{3.2}$$

Let $N_{n,m}$ denote a unit normal vector to $\mathcal{P}_{n,m}$ (Gauss map) whose direction is chosen according to the orientation of the surface:

$$\begin{aligned} N_{n,m} &= \frac{[\mathbf{a} \times \mathbf{b}]}{|\mathbf{a}||\mathbf{b}| \sin\alpha}, \\ N_{n+1,m} &= \frac{[\mathbf{a} \times \mathbf{b}']}{|\mathbf{a}||\mathbf{b}'| \sin\beta}, \\ N_{n,m+1} &= \frac{[\mathbf{a}' \times \mathbf{b}]}{|\mathbf{a}'||\mathbf{b}| \sin\beta}, \\ N_{n+1,m+1} &= \frac{[\mathbf{a}' \times \mathbf{b}']}{|\mathbf{a}'||\mathbf{b}'| \sin\alpha}, \end{aligned} \tag{3.3}$$

[1] later on a more general class of discrete K-surfaces will be considered

where $[\times]$ denotes the vector product. The discrete analog of (2.13) is given by the following proposition.

Proposition 2. *The Gauss map of the discrete K-immersion comprises a discrete weak Chebyshev net in S^2, i.e., the angles between the normals associated with the opposite sides are equal:*

$$< N_{n,m}, N_{n+1,m} >=< N_{n,m+1}, N_{n+1,m+1} >= \cos \Delta_n^u, \tag{3.4}$$

$$< N_{n,m}, N_{n,m+1} >=< N_{n+1,m}, N_{n+1,m+1} >= \cos \Delta_m^v, \tag{3.5}$$

where $<,>$ denotes the scalar product, and we have incorporated into the notation that Δ_n^u does not depend on m, and Δ_m^v not on n.

Proof. Two equalities (3.4), (3.5) are proved in the same way. We present a proof of (3.4). Due to (3.3) it is equivalent to

$$< [\mathbf{a} \times \mathbf{b}], [\mathbf{a} \times \mathbf{b}'] >=< [\mathbf{a}' \times \mathbf{b}], [\mathbf{a}' \times \mathbf{b}'] > . \tag{3.6}$$

Introducing three vectors

$$\mathbf{A} = \mathbf{a} + \mathbf{a}', \ \mathbf{B} = \mathbf{b} + \mathbf{b}', \ \mathbf{C} = \mathbf{a}' - \mathbf{a} = \mathbf{b}' - \mathbf{b}$$

and using (3.1), we see that

$$\mathbf{a} = \frac{\mathbf{A} - \mathbf{C}}{2}, \ \mathbf{a}' = \frac{\mathbf{A} + \mathbf{C}}{2}, \ \mathbf{b} = \frac{\mathbf{B} - \mathbf{C}}{2}, \ \mathbf{b}' = \frac{\mathbf{B} + \mathbf{C}}{2} \tag{3.7}$$

and

$$< \mathbf{A}, \mathbf{C} >=< \mathbf{B}, \mathbf{C} >= 0. \tag{3.8}$$

Rewriting (3.6) in terms of $\mathbf{A}, \mathbf{B}, \mathbf{C}$, we get

$$\begin{aligned} &< [\mathbf{A} \times \mathbf{B}] + [(\mathbf{B} - \mathbf{A}) \times \mathbf{C}], [\mathbf{A} \times \mathbf{B}] + [(\mathbf{B} + \mathbf{A}) \times \mathbf{C}] > \\ &< [\mathbf{A} \times \mathbf{B}] - [(\mathbf{B} + \mathbf{A}) \times \mathbf{C}], [\mathbf{A} \times \mathbf{B}] + [(\mathbf{A} - \mathbf{B}) \times \mathbf{C}] > . \end{aligned} \tag{3.9}$$

The orthogonality (3.8) implies the proportionality

$$[\mathbf{A} \times \mathbf{B}] = q\mathbf{C}, \quad q \in \mathbf{R},$$

which, combined with the trivial identity $< \mathbf{C}, [X \times \mathbf{C}] >= 0$, proves (3.9).

Definition 1 of the discrete K-surface implies that if for some quadrilateral the directions of two normals coincide, then the whole surface lies in a plane. We choose Δ's in the interval $(0, \pi)$ so that

$$0 < \Delta^u_n, \Delta^v_m < \pi.$$

The trivial identities

$$< N_{n,m} + N_{n+1,m+1}, N_{n,m} - N_{n+1,m+1} >= 0,$$
$$< N_{n+1,m} + N_{n,m+1}, N_{n+1,m} - N_{n,m+1} >= 0,$$

combined with the equalities

$$< N_{n,m} + N_{n+1,m+1}, N_{n+1,m} - N_{n,m+1} >= 0,$$
$$< N_{n+1,m} + N_{n,m+1}, N_{n,m} - N_{n+1,m+1} >= 0,$$

following from (3.4), (3.5), show that for a weak Chebyshev net in S^2 one of the two possibilities

$$(i) \quad N_{n,m} + N_{n+1,m+1} \parallel N_{n+1,m} + N_{n,m+1}, \tag{3.10}$$
$$(ii) \quad N_{n,m} - N_{n+1,m+1} \parallel N_{n+1,m} - N_{n,m+1} \tag{3.11}$$

holds.

The following definition is a natural discrete analog of (2.14).

Definition 2. A map $N : \mathbf{Z}^2 \to S^2$ is called Lorentz-harmonic if for any n, m

$$\begin{aligned} & N_{n+1,m+1} - N_{n+1,m} - N_{n,m+1} + N_{n,m} \\ & \rho_{n,m}(N_{n+1,m+1} + N_{n+1,m} + N_{n,m+1} + N_{n,m}), \quad \rho : \mathbf{Z}^2 \to \mathbf{R}. \end{aligned} \tag{3.12}$$

Proposition 3. *The Gauss map of the immersed discrete K-surface is Lorentz-harmonic.*

Proof. Due to (3.3) we have

$$N_{n,m} + N_{n+1,m+1} = \frac{[\mathbf{a} \times \mathbf{b}] + [\mathbf{a}' \times \mathbf{b}']}{|\mathbf{a}||\mathbf{b}| \sin \alpha},$$
$$N_{n+1,m} + N_{n,m+1} = \frac{[\mathbf{a}' \times \mathbf{b}] + [\mathbf{a} \times \mathbf{b}']}{|\mathbf{a}||\mathbf{b}| \sin \beta}.$$

Rewriting the vector products in terms of the vectors (3.7) we get

$$\begin{aligned}
&[\mathbf{a}\times\mathbf{b}]+[\mathbf{a}'\times\mathbf{b}'] \\
&\qquad=\frac{1}{4}([(\mathbf{A}-\mathbf{C})\times(\mathbf{B}-\mathbf{C})]+[(\mathbf{A}+\mathbf{C})\times(\mathbf{B}+\mathbf{C})]) \\
&\qquad=\frac{1}{2}[\mathbf{A}\times\mathbf{B}], \\
&[\mathbf{a}\times\mathbf{b}]+[\mathbf{a}\times\mathbf{b}'] \\
&\qquad=\frac{1}{4}([(\mathbf{A}+\mathbf{C})\times(\mathbf{B}-\mathbf{C})]+[(\mathbf{A}-\mathbf{C})\times(\mathbf{B}+\mathbf{C})]) \\
&\qquad=\frac{1}{2}[\mathbf{A}\times\mathbf{B}].
\end{aligned}$$

It proves the proportionality of the vectors $N_{n,m}+N_{n+1,m+1}$ and $N_{n+1,m}+N_{n+1,m}$, which is equivalent to (3.12).

Proposition 4. *Any Lorentz-harmonic weak discrete Chebyshev net $N\colon \mathbb{Z}^2 \to S^2$ is the Gauss map of a discrete K-surface, which is determined by N uniquely up to a homothety and translations.*

Proof. The edge $[F_{n+1,m}, F_{n,m}]$ is orthogonal to both normal vectors $N_{n+1,m}$ and $N_{n,m}$, the vector product of which does not vanish. This implies for the edges of the elementary quadrilateral of the discrete K-surface:

$$\begin{aligned}
\mathbf{a}&=\alpha[N_{n+1,m}\times N_{n,m}], \\
\mathbf{b}&=\beta[N_{n,m}\times N_{n,m+1}], \\
\mathbf{a}'&=\alpha'[N_{n+1,m+1}\times N_{n,m+1}], \\
\mathbf{b}'&=\beta'[N_{n+1,m}\times N_{n+1,m+1}].
\end{aligned}$$

The closing condition $\mathbf{a}+\mathbf{b}'=\mathbf{b}+\mathbf{a}'$ yields

$$\begin{aligned}
&[N_{n+1,m}\times(\alpha N_{n,m}+\beta' N_{n+1,m+1})] \\
&\qquad\qquad=[(\beta N_{n,m}+\alpha' N_{n+1,m+1})\times N_{n,m+1}].
\end{aligned}\tag{3.13}$$

Since all the vectors

$$N_{n+1,m},\ \alpha N_{n,m}+\beta' N_{n+1,m+1},\ \beta N_{n,m}+\alpha' N_{n+1,m+1},\ N_{n,m+1}$$

in this expression do not vanish, they all belong to one plane. Since due to the nondegeneracy of the quadrilateral we assumed (3.2), the planes generated by $N_{n+1,m}, N_{n,m+1}$ and $N_{n,m}, N_{n+1,m+1}$ are different, and their line of intersection is parallel to $N_{n+1,m} + N_{n,m+1} \parallel N_{n,m} + N_{n+1,m+1}$. This yields the proportionality

$$\alpha N_{n,m} + \beta' N_{n+1,m+1} \parallel \beta N_{n,m} + \alpha' N_{n+1,m+1} \parallel N_{n,m} + N_{n+1,m+1},$$

or equivalently, $\alpha = \beta'$, $\beta = \alpha'$. The weak Chebyshev property implies $\alpha = \pm\alpha', \beta = \pm\beta'$. The case

$$\alpha = -\alpha', \beta = -\beta' \tag{3.14}$$

is degenerated, since

$$[(N_{n+1,m} + N_{n,m+1}) \times (N_{n,m} + N_{n+1,m+1})] = 0$$

together with

$$[(N_{n+1,m} - N_{n,m+1}) \times (N_{n,m} + N_{n+1,m+1})] = 0,$$

following from (3.13), (3.14), show that $N_{n+1,m}$ and $N_{n,m+1}$ are parallel. This makes the surface planar and contradicts our assumption. Finally, we have

$$\alpha = \beta = \alpha' = \beta',$$

which finishes the proof:

$$\begin{aligned} F_{n+1,m} - F_{n,m} &= \alpha[N_{n+1,m} \times N_{n,m}], \\ F_{n,m+1} - F_{n,m} &= -\alpha[N_{n,m+1} \times N_{n,m}]. \end{aligned} \tag{3.15}$$

Thus the proof of Proposition 4 is complete.

4. Quaternionic description of discrete Lorentz-harmonic weak Chebyshev nets in S^2 and the Hirota equation

The Gauss map can be represented in the following matrix form (compare with (2.9)):

$$N_{n,m} = -i\Phi_{n,m}^{-1}\sigma_3\Phi_{n,m}, \quad \Phi_{n,m} \in SU(2). \tag{4.1}$$

Introduce the matrices

$$\mathcal{U}_{n,m} = \Phi_{n+1,m}\Phi_{n,m}^{-1},$$
$$\mathcal{V}_{n,m} = \Phi_{n,m+1}\Phi_{n,m}^{-1}$$

lying in $SU(2)$. The conditions

$$\begin{aligned}
\cos\Delta_n^u &= < N_{n,m}, N_{n+1,m} >= -\frac{1}{2}\operatorname{tr}(N_{n,m}N_{n+1,m}) \\
&= \frac{1}{2}\operatorname{tr}(\mathcal{U}_{n,m}^{-1}\sigma_3\mathcal{U}_{n,m}\sigma_3), \\
\cos\Delta_m^v &= < N_{n,m}, N_{n,m+1} >= -\frac{1}{2}\operatorname{tr}(N_{n,m}N_{n,m+1}) \\
&= \frac{1}{2}\operatorname{tr}(\mathcal{V}_{n,m}^{-1}\sigma_3\mathcal{V}_{n,m}\sigma_3)
\end{aligned}$$

allow us to parametrize $\mathcal{U}$ and $\mathcal{V}$ as follows:

$$\mathcal{U} = \begin{pmatrix} a & ib \\ i\bar{b} & \bar{a} \end{pmatrix}, \quad a = \cos\frac{\Delta^u}{2}e^{i\alpha}, b = \sin\frac{\Delta^u}{2}e^{i\beta}, \tag{4.2}$$

$$\mathcal{V} = \begin{pmatrix} c & id \\ i\bar{d} & \bar{c} \end{pmatrix}, \quad c = \cos\frac{\Delta^v}{2}e^{i\gamma}, d = \sin\frac{\Delta^v}{2}e^{i\delta}. \tag{4.3}$$

Let us consider an elementary quadrilateral and denote it by

$$\mathcal{U} = \mathcal{U}_{n,m}, \ \mathcal{V} = \mathcal{V}_{n,m}, \ \mathcal{U}' = \mathcal{U}_{n,m+1}, \ \mathcal{V}' = \mathcal{V}_{n+1,m},$$

the matrices associated with its edges. Substituting (4.2, 4.3) into the compatibility condition

$$\mathcal{V}'\mathcal{U} = \mathcal{U}'\mathcal{V}, \tag{4.4}$$

we get

$$\cos\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}(e^{i(\alpha'+\gamma)}-e^{i(\alpha+\gamma')})=\sin\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}(e^{i(\beta'-\delta)}-e^{i(\delta'-\beta)}),$$
$$\cos\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}(e^{i(\alpha'+\delta)}-e^{i(\delta'-\alpha)})=\sin\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}(e^{i(\gamma'+\beta)}-e^{i(\beta'-\gamma)}).$$

Rewriting the above equations as

$$\begin{aligned}&\cos\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}\sin\frac{1}{2}(-\alpha+\alpha'+\gamma-\gamma')\\ &=\exp\frac{i}{2}(-\alpha-\alpha'-\beta+\beta'-\gamma-\gamma'-\delta+\delta')\\ &\quad\times\sin\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}\sin\frac{1}{2}(\beta+\beta'-\delta-\delta'),\end{aligned}\tag{4.5}$$

$$\begin{aligned}&\cos\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}\sin\frac{1}{2}(\alpha+\alpha'+\delta-\delta')\\ &=\exp\frac{i}{2}(\alpha-\alpha'+\beta+\beta'-\gamma+\gamma'-\delta-\delta')\\ &\quad\times\sin\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}\sin\frac{1}{2}(\beta-\beta'+\gamma+\gamma'),\end{aligned}\tag{4.6}$$

we see that in the case

$$\alpha+\alpha'+\delta-\delta'\not\equiv 0,\ \alpha-\alpha'-\gamma+\gamma'\not\equiv 0,\tag{4.7}$$

equations (4.5), (4.6)

$$\begin{aligned}&\beta+\beta'-\delta-\delta'\not\equiv 0, &&-\alpha-\alpha'-\beta+\beta'-\gamma-\gamma'-\delta+\delta'\equiv 0,\\ &\beta-\beta'+\gamma+\gamma'\not\equiv 0, &&\alpha-\alpha'+\beta+\beta'-\gamma+\gamma'-\delta-\delta'\equiv 0,\end{aligned}\tag{4.8}$$

where we have used the notation $\epsilon\equiv\epsilon'$ for $\epsilon\equiv\epsilon'(\bmod 2\pi)$ or, equivalently, $e^{i\epsilon}=e^{i\epsilon'}$. yield

Equations (4.7), (4.8) imply

$$\begin{aligned}&\alpha+\alpha'+\delta-\delta'\equiv\alpha-\alpha'-\gamma+\gamma'\\ &\equiv\beta+\beta'-\delta-\delta'\equiv\beta-\beta'+\gamma+\gamma'\equiv\pi.\end{aligned}$$

Substituting the above equations in (4.5), (4.6) we obtain

$$\begin{aligned}\cos\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}&=\pm\sin\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2},\\ \cos\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}&=\mp\sin\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}\end{aligned}$$

with two possible choices of signs. The last equations can be rewritten in the form

$$\cos\frac{\Delta^u \pm \Delta^v}{2} = 0, \qquad \sin\frac{\Delta^u \pm \Delta^v}{2} = 0.$$

This contradiction means that the assumption (4.7) was wrong and one of the expressions in (4.7) must vanish. So there are two cases to consider:

$$\begin{aligned}
&(i)\ \alpha + \alpha' + \delta - \delta' \equiv 0, \quad \beta - \beta' + \gamma + \gamma' \equiv 0,\\
&\cos\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}\left(e^{i(\alpha'+\gamma)} - e^{i(\alpha+\gamma')}\right) \qquad (4.9)\\
&\quad = \sin\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}\left(e^{i(\beta'-\delta)} - e^{i(\delta'-\beta)}\right),\\
&(ii)\ \alpha - \alpha' - \gamma + \gamma' \equiv 0, \quad \beta + \beta' - \delta - \delta' \equiv 0,\\
&\cos\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}\left(e^{i(\alpha'+\delta)} - e^{i(\delta'-\alpha)}\right) \qquad (4.10)\\
&\quad = \sin\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}\left(e^{i(\gamma'+\beta)} - e^{i(\beta'-\gamma)}\right).
\end{aligned}$$

We postpone the discussion of the second case (which turns out to correspond to the possibility (3.11)) until Appendix and henceforth assume that (i) is valid.

The matrices $\Phi_{n,m}$ (and $\mathcal{U}_{n,m}$ and $\mathcal{V}_{n,m}$ as a corollary) are determined by (4.1) up to the gauge transformation

$$\begin{aligned}
&\Phi_{n,m} \to \exp(i\epsilon_{n,m}\sigma_3)\Phi_{n,m},\\
&\mathcal{U}_{n,m} \to \exp(i\epsilon_{n+1,m}\sigma_3)\mathcal{U}_{n,m}\exp(-i\epsilon_{n,m}\sigma_3), \qquad (4.11)\\
&\mathcal{V}_{n,m} \to \exp(i\epsilon_{n,m+1}\sigma_3)\mathcal{V}_{n,m}\exp(-i\epsilon_{n,m}\sigma_3).
\end{aligned}$$

Let us choose the gauging in such a way that

$$\beta_{n,m} \equiv \gamma_{n,m} \equiv 0 \qquad (4.12)$$

for all n, m. Given $\epsilon_{0,0}$, the condition (4.12) specifies all $\epsilon_{n,m}$ in a unique way. The equations (4.9) in this gauge become as follows:

$$\alpha + \alpha' \equiv \delta' - \delta, \qquad (4.13)$$

$$\cos\tfrac{\Delta^u}{2}\cos\tfrac{\Delta^v}{2}\left(e^{i\alpha'} - e^{i\alpha}\right) = \sin\tfrac{\Delta^u}{2}\sin\tfrac{\Delta^v}{2}\left(e^{-i\delta} - e^{i\delta'}\right). \qquad (4.14)$$

The first equation can be easily resolved

$$\alpha_{n,m} \equiv h_{n+1,m} - h_{n,m}, \quad \delta_{n,m} \equiv h_{n,m+1} + h_{n,m}, \tag{4.15}$$

where $h_{n,m}$ now can be associated with the corresponding vertices. Finally, $h_{n,m}$ satisfy the equation

$$\begin{aligned} &\exp(ih_{n+1,m+1} + ih_{n,m}) - \exp(ih_{n+1,m} + ih_{n,m+1}) \\ &= \tan\frac{\Delta_n^u}{2}\tan\frac{\Delta_m^v}{2} \\ &\quad \times(1 - \exp(ih_{n+1,m+1} + ih_{n+1,m} + ih_{n+1,m} + ih_{n,m})), \end{aligned} \tag{4.16}$$

or equivalently,

$$\begin{aligned} &\sin(\frac{1}{2}(h_{n+1,m} + h_{n,m+1} - h_{n+1,m+1} - h_{n,m})) \\ &\quad = k_{n,m}\sin(\frac{1}{2}(h_{n+1,m+1} + h_{n,m} + h_{n+1,m} + h_{n,m+1})), \\ &k_{n,m} = \tan\frac{\Delta_n^u}{2}\tan\frac{\Delta_m^v}{2}. \end{aligned} \tag{4.17}$$

This equation is clearly a discrete analogue of the sine-Gordon equation, which can be obtained as a limit $k \to 0$. Equation (4.17) first appeared in Hirota [14] without any relation to geometry. We prefer to treat not this but another equation as a discrete sine-Gordon equation. In the smooth case the sine-Gordon equation describes the angles between the asymptotic lines (see §2), and, as we have shown, the Hirota equation has another geometrical meaning. Therefore in §6 we derive an equation describing the angles between the edges of discrete K-surfaces and call it a discrete sine-Gordon equation, which is simply related to the Hirota equation.

Remark. There is an ambiguity in the definition of $h_{n,m}$ from geometrical data:

$$\begin{aligned} h_{2n,2m} &\to h_{2n,2m} + S + T, \\ h_{2n+1,2m} &\to h_{2n+1,2m} + S - T, \\ h_{2n,2m+1} &\to h_{2n,2m+1} - S + T, \\ h_{2n+1,2m+1} &\to h_{2n+1,2m+1} - S - T \end{aligned} \tag{4.18}$$

with the same S and T for all n, m. The origin of S is the nonuniqueness in the definition (4.15) of $h_{n,m}$ for given α, δ, whereas the variation of $\epsilon_{0,0}$ in the gauging induces T in (4.18).

Proposition 5. *The Gauss map* (4.1) *determined by a solution of the Hirota equation forms a Lorentz-harmonic weak Chebyshev net in* S^2.

Proof. Up to a common rotation $\Phi_{n,m}^{-1} \ldots \Phi_{n,m}$ the normal vectors at the vertices of an elementary quadrilateral are as follows:

$$\begin{aligned} N_{n,m} &= i\sigma_3, \\ N_{n+1,m} &= i\mathcal{U}^{-1}\sigma_3\mathcal{U}, \\ N_{n,m+1} &= i\mathcal{V}^{-1}\sigma_3\mathcal{V}, \\ N_{n+1,m+1} &= i(\mathcal{V}'\mathcal{U})^{-1}\sigma_3\mathcal{V}'\mathcal{U}. \end{aligned}$$

To prove the parallelism $N_{n,m} + N_{n+1,m+1} \parallel N_{n+1,m} + N_{n+1,m}$ we show that

$$\sigma_3 + (\mathcal{V}'\mathcal{U})^{-1}\sigma_3\mathcal{V}'\mathcal{U} = r(\mathcal{U}^{-1}\sigma_3\mathcal{U} + \mathcal{V}^{-1}\sigma_3\mathcal{V})$$

with some real r, or equivalently,

$$\mathcal{U}\sigma_3\mathcal{V}^{-1} + \mathcal{V}'^{\,-1}\sigma_3\mathcal{U}' = r(\sigma_3\mathcal{U}\mathcal{V}^{-1} + \mathcal{U}\mathcal{V}^{-1}\sigma_3).$$

Calculation of both sides using (4.13) yields

$$\mathcal{U}\sigma_3\mathcal{V}^{-1} + \mathcal{V}'^{\,-1}\sigma_3\mathcal{U}' = \begin{pmatrix} p & 0 \\ 0 & -\bar{p} \end{pmatrix},$$

$$\sigma_3\mathcal{U}\mathcal{V}^{-1} + \mathcal{U}\mathcal{V}^{-1}\sigma_3 = \begin{pmatrix} q & 0 \\ 0 & -\bar{q} \end{pmatrix},$$

where

$$\begin{aligned} p &= \cos\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}(e^{i\alpha} + e^{i\alpha'}) - \sin\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}(e^{-i\delta} + e^{i\delta'}), \\ q &= 2\cos\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}e^{i\alpha} + 2\sin\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}e^{-i\delta}. \end{aligned}$$

The addition of a vanishing term to q gives

$$\begin{aligned} q &= q + \cos\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}(e^{i\alpha'} - e^{i\alpha}) + \sin\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}(e^{i\delta'} - e^{-i\delta}) \\ &= \cos\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}(e^{i\alpha} + e^{i\alpha'}) + \sin\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}(e^{-i\delta} + e^{i\delta'}), \end{aligned}$$

which proves the equality of the arguments

$$\arg p = \arg q \quad (\bmod \pi),$$

which finishes the proof of the parallelism $N_{n,m} + N_{n+1,m+1} \parallel N_{n+1,m} + N_{n+1,m}$. The property to be a weak Chebyshev net in S^2 follows from (4.2, 4.3).

5. Associated family and formula for immersion

The Hirota equation is invariant with respect to the transformation

$$(5.1)\qquad \begin{aligned} \tan\frac{\Delta_n^u}{2} &\to \tan\frac{\Delta_n^u(\lambda)}{2} = \lambda \tan\frac{\Delta_n^u}{2}, \\ \tan\frac{\Delta_m^v}{2} &\to \tan\frac{\Delta_m^v(\lambda)}{2} = \frac{1}{\lambda}\tan\frac{\Delta_m^v}{2}, \end{aligned}$$

which is an analog of the Lorentz transformation (2.4) in the smooth case. This transformation changes the matrices $\mathcal{U}_{n,m}, \mathcal{V}_{n,m}$ as

$$\mathcal{U}_{n,m} \to \overset{0}{U}_{n,m}(\lambda) = \begin{pmatrix} \cos\frac{\Delta_n^u(\lambda)}{2} e^{ih_{n+1,m}-ih_{n,m}} & i\sin\frac{\Delta_n^u(\lambda)}{2} \\ i\sin\frac{\Delta_n^u(\lambda)}{2} & \cos\frac{\Delta_n^u(\lambda)}{2} e^{-ih_{n+1,m}+ih_{n,m}} \end{pmatrix},$$

$$\mathcal{V}_{n,m} \to \overset{0}{V}_{n,m}(\lambda) = \begin{pmatrix} \cos\frac{\Delta_m^v(\lambda)}{2} & i\sin\frac{\Delta_m^v(\lambda)}{2} e^{ih_{n,m+1}+ih_{n,m}} \\ i\sin\frac{\Delta_m^v(\lambda)}{2} e^{-ih_{n,m+1}-ih_{n,m}} & \cos\frac{\Delta_m^v(\lambda)}{2} \end{pmatrix}.$$

Due to Propositions 4, 5, the matrices $\overset{0}{U}_{n,m}(\lambda), \overset{0}{V}_{n,m}(\lambda)$ describe a one-parameter deformation family of discrete K-surfaces and their Gauss maps. To describe this family more precisely it is convenient to present the matrices $\overset{0}{U}_{n,m}(\lambda), \overset{0}{V}_{n,m}(\lambda)$ in the form

$$(5.2)\qquad \overset{0}{U}_{n,m} = \frac{1}{\sqrt{\cot^2\frac{\Delta_n^u}{2} + \lambda^2}} U_{n,m}, \quad \overset{0}{V}_{n,m} = \frac{1}{\sqrt{\lambda^{-2}\tan^2\frac{\Delta_m^v}{2} + 1}} V_{n,m},$$

$$U_{n,m}(\lambda) = \begin{pmatrix} \cot \frac{\Delta_n^u}{2} e^{ih_{n+1,m}-ih_{n,m}} & i\lambda \\ i\lambda & \cot \frac{\Delta_n^u}{2} e^{-ih_{n+1,m}+ih_{n,m}} \end{pmatrix}, \tag{5.3}$$

$$V_{n,m}(\lambda) = \begin{pmatrix} 1 & i\frac{1}{\lambda} \tan \frac{\Delta_m^v}{2} e^{ih_{n,m+1}+ih_{n,m}} \\ i\frac{1}{\lambda} \tan \frac{\Delta_m^v}{2} e^{-ih_{n,m+1}-ih_{n,m}} & 1 \end{pmatrix}. \tag{5.4}$$

The compatibility condition

$$V_{n+1,m}U_{n,m} = U_{n,m+1}V_{n,m}$$

is nothing else but the Lax representation for the Hirota equation.

Theorem 3. *Let $h_{n,m}$ be a solution of* (4.16). *Then the immersion of the corresponding discrete K-surface with the angles $\Delta_n^u(\lambda), \Delta_m^v(\lambda)$ given by*

$$\tan \frac{\Delta_n^u(\lambda)}{2} = \lambda \tan \frac{\Delta_n^u}{2},$$

$$\tan \frac{\Delta_m^v(\lambda)}{2} = \frac{1}{\lambda} \tan \frac{\Delta_m^v}{2}$$

is described up to a homothety by the formula

$$F_{n,m} = 2 \overset{0}{\Psi}{}^{-1}_{n,m} \frac{\partial \overset{0}{\Psi}_{n,m}}{\partial t}, \qquad \lambda = e^t, \tag{5.5}$$

where $\overset{0}{\Psi}_{n,m}$ $(\lambda \in \boldsymbol{R}) \in SU(2)$ is a solution of the system

$$\overset{0}{\Psi}_{n+1,m} = \overset{0}{U}_{n,m} \overset{0}{\Psi}_{n,m}, \quad \overset{0}{\Psi}_{n,m+1} = \overset{0}{V}_{n,m} \overset{0}{\Psi}_{n,m}. \tag{5.6}$$

Moreover, the Gauss map is given by

$$N_{n,m} = -i \overset{0}{\Psi}{}^{-1}_{n,m} \sigma_3 \overset{0}{\Psi}_{n,m}. \tag{5.7}$$

Proof. We see that (5.7) coincides with (4.1). Since the Gauss map determines the discrete K-surface up to homothety (Propositions 4, 5), to prove the theorem we have to show only that (5.5) describes a discrete K-surface and (5.7) is its Gauss map. For the vectors of edges we have

$$
\begin{aligned}
F_{n+1,m} - F_{n,m} &= 2\, \overset{0}{\Psi}{}^{-1}_{n,m} \Big(\overset{0}{U}{}^{-1}_{n,m} \frac{\partial \overset{0}{U}_{n,m}}{\partial t}\Big)\, \overset{0}{\Psi}_{n,m},\\
F_{n,m+1} - F_{n,m} &= 2\, \overset{0}{\Psi}{}^{-1}_{n,m} \Big(\overset{0}{V}{}^{-1}_{n,m} \frac{\partial \overset{0}{V}_{n,m}}{\partial t}\Big)\, \overset{0}{\Psi}_{n,m},\\
F_{n-1,m} - F_{n,m} &= -2\, \overset{0}{\Psi}{}^{-1}_{n,m} \Big(\frac{\partial \overset{0}{U}_{n-1,m}}{\partial t}\, \overset{0}{U}{}^{-1}_{n-1,m}\Big)\, \overset{0}{\Psi}_{n,m},\\
F_{n,m-1} - F_{n,m} &= -2\, \overset{0}{\Psi}{}^{-1}_{n,m} \Big(\frac{\partial \overset{0}{V}_{n,m-1}}{\partial t}\, \overset{0}{V}{}^{-1}_{n,m-1}\Big)\, \overset{0}{\Psi}_{n,m} .
\end{aligned}
$$

All these vectors as well as $N_{n,m}$ (forming a frame associated with the vertex $F_{n,m}$) have common factors $\overset{0}{\Psi}{}^{-1}_{n,m}$ on the left and $\overset{0}{\Psi}_{n,m}$ on the right, which describe a rotation of this frame as a whole. Considering the local geometry of the frame we can neglect this rotation. Direct calculation yields [2]

$$
\begin{aligned}
2\, \overset{0}{U}{}^{-1}_{n,m} \frac{\partial \overset{0}{U}_{n,m}}{\partial t} &= i \sin \Delta^u_n(\lambda) \begin{pmatrix} 0 & e^{ih_{n,m}-ih_{n+1,m}} \\ e^{-ih_{n,m}+ih_{n+1,m}} & 0 \end{pmatrix},\\
2\, \overset{0}{V}{}^{-1}_{n,m} \frac{\partial \overset{0}{V}_{n,m}}{\partial t} &= -i \sin \Delta^v_m(\lambda) \begin{pmatrix} 0 & e^{ih_{n,m}+ih_{n,m+1}} \\ e^{-ih_{n,m}-ih_{n,m+1}} & 0 \end{pmatrix},\\
-2 \frac{\partial \overset{0}{U}_{n-1,m}}{\partial t}\, \overset{0}{U}{}^{-1}_{n-1,m} &= -i \sin \Delta^u_n(\lambda) \begin{pmatrix} 0 & e^{ih_{n,m}-ih_{n-1,m}} \\ e^{-ih_{n,m}+ih_{n-1,m}} & 0 \end{pmatrix},\\
-2 \frac{\partial \overset{0}{V}_{n,m-1}}{\partial t}\, \overset{0}{V}{}^{-1}_{n,m-1} &= i \sin \Delta^v_m(\lambda) \begin{pmatrix} 0 & e^{ih_{n,m}+ih_{n,m-1}} \\ e^{-ih_{n,m}-ih_{n,m-1}} & 0 \end{pmatrix},
\end{aligned} \tag{5.8}
$$

[2] Doing this calculation it is more convenient to work with the matrices $U_{n,m}(\lambda)$, $V_{n,m}(\lambda)$, which differ from $\overset{0}{U}_{n,m}, \overset{0}{V}_{n,m}$ by a scalar factor (see the Remark at the end of this section).

where we have used the identities

$$\sin \Delta_n^u(\lambda) = \frac{2\lambda \cot \frac{\Delta_n^u}{2}}{\lambda^2 + \cot^2 \frac{\Delta_n^u}{2}}, \qquad \sin \Delta_m^v(\lambda) = \frac{2\lambda^{-1} \tan \frac{\Delta_m^v}{2}}{1 + \lambda^{-2} \tan^2 \frac{\Delta_m^v}{2}},$$

following from (5.1). These vectors are orthogonal to $-i\sigma_3$, which proves the orthogonality of the corresponding edges to $N_{n,m}$. The property (ii) of the definition of the discrete K-surfaces is also evidently satisfied since the lengths of the edges $|[F_{n+1,m}, F_{n,m}]|, |[F_{n,m+1}, F_{n,m}]|$ are independent of m and n respectively.

Formulas (3.4), (3.5), (5.8) imply the following corollary, which follows also from Propositions 2,4.

Corollary 1. *Under the action of the one-parameter deformation family (associated family) the edges and the normals of the discrete K-surface transform as follows:*

$$\begin{aligned} &< N_{n+1,m}, N_{n,m} > = \cos \Delta_n^u(\lambda), && |[F_{n+1,m}, F_{n,m}]| = \sin \Delta_n^u(\lambda), \\ &< N_{n,m+1}, N_{n,m} > = \cos \Delta_m^v(\lambda), && |[F_{n,m+1}, F_{n,m}]| = \sin \Delta_m^v(\lambda), \end{aligned}$$

where the angles $\Delta(\lambda)$ are determined by (5.1).

Corollary 2. *The vectors of the normals and edges of the discrete K-surface, described in Theorem 3 are related as follows:*

$$\begin{aligned} F_{n+1,m} - F_{n,m} &= [N_{n+1,m} \times N_{n,m}], \\ F_{n,m+1} - F_{n,m} &= -[N_{n,m+1} \times N_{n,m}]. \end{aligned} \tag{5.9}$$

Proof. To prove this specification of Proposition 4 let us write down the vector product as a matrix commutator, using the isomorphism (2.4),

$$[A \times B] = \frac{1}{2}[A, B].$$

In the moving frame used in the proof of Theorem 3 the normal vectors are equal to

$$N_{n,m} = -i\sigma_3, \qquad N_{n+1,m} = -i \overset{0}{U}{}^{-1}_{n,m} \sigma_3 \overset{0}{U}_{n,m} .$$

Calculating the vector product and using (5.8), we get

$$\begin{aligned}
[N_{n+1,m} \times N_{n,m}] &= -\frac{1}{2}[\overset{0}{U}{}^{-1}_{n,m}\sigma_3 \overset{0}{U}_{n,m}, \sigma_3] \\
&= i \sin \Delta^u_n(\lambda) \begin{pmatrix} 0 & e^{ih_{n,m}-ih_{n+1,m}} \\ e^{-ih_{n,m}+ih_{n+1,m}} & 0 \end{pmatrix} \\
&= 2 \overset{0}{U}{}^{-1}_{n,m} \partial \overset{0}{U}_{n,m} /\partial t = F_{n+1,m} - F_{n,m}.
\end{aligned}$$

The second equality in (5.9) follows from the first one and Proposition 4, but it also can be easily checked by a direct calculation.

Remark. To find $F_{n,m}, N_{n,m}$ it is enough to determine $\overset{0}{\Psi}_{n,m}(\lambda)$ up to a scalar factor. Indeed, if $\Psi_{n,m}$ differs from $\overset{0}{\Psi}_{n,m}$ by a scalar factor,

$$\overset{0}{\Psi}_{n,m}(\lambda) = f_{n,m}(\lambda)\Psi_{n,m}(\lambda),$$

the immersion and the Gauss map can be written in terms of $\Psi_{n,m}$ as follows:

$$\begin{aligned}
F_{n,m} &= 2\Psi_{n,m}{}^{-1}\frac{\partial \Psi_{n,m}}{\partial t} + 2I\frac{\partial}{\partial t}\log f_{n,m} \\
&= 2\left[\Psi_{n,m}{}^{-1}\frac{\partial \Psi_{n,m}}{\partial t}\right]^{\mathrm{tr}=0}, \\
N_{n,m} &= -i\Psi^{-1}_{n,m}\sigma_3\Psi_{n,m},
\end{aligned}$$

where

$$[X]^{\mathrm{tr}=0} = X - \frac{1}{2}I \ \ \mathrm{tr}X.$$

6. The discrete sine-Gordon equation

Formulas (5.8) allow us to determine the angles between all edges (see Figure 2 for the notation of the angles)

$$\begin{aligned}
\phi^{(1)}_{n,m} &\equiv -h_{n,m+1} - h_{n+1,m} + \pi, \\
\phi^{(2)}_{n,m} &\equiv h_{n-1,m} + h_{n,m+1}, \\
\phi^{(3)}_{n,m} &\equiv -h_{n,m-1} - h_{n-1,m} + \pi, \qquad (6.1)\\
\phi^{(4)}_{n,m} &\equiv h_{n+1,m} + h_{n,m-1}.
\end{aligned}$$

Let us consider again a small piece of the discrete K-surface and derive a difference equation for the angles between edges, which can be regarded as a difference analog of the sine-Gordon equation (2.2). Now if we orient the lattice diagonally (Figure 4), the following theorem holds.

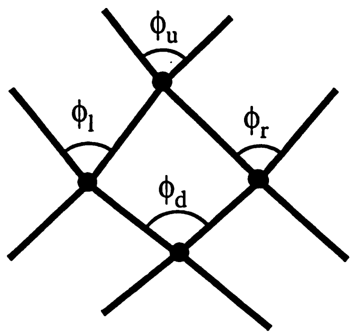


FIGURE 4. The angles between edges of discrete K-surface.

Theorem 4. *The neighboring angles between the edges of a discrete K-surface satisfy the equation*

$$\phi_u + \phi_d - \phi_l - \phi_r \equiv 2\arg(1 - k_l e^{-i\phi_l}) + 2\arg(1 - k_r e^{-i\phi_r}), \tag{6.2}$$

where

$$k_l = \tan\frac{\Delta_l^u}{2}\tan\frac{\Delta_l^v}{2},\ k_r = \tan\frac{\Delta_r^u}{2}\tan\frac{\Delta_r^v}{2}$$

are the products associated with the quadrilaterals, corresponding to ϕ_l and ϕ_r respectively.

Proof. There are two angles $\phi^{(1)}_{n,m}$ and $\phi^{(2)}_{n+1,m}$ associated with the quadrilateral $(F_{n+1,m+1}, F_{n+1,m}, F_{n,m}, F_{n,m+1})$. The other two are the same because of the symmetry of the quadrilateral

$$\phi^{(1)}_{n,m} = \phi^{(3)}_{n+1,m+1},\ \phi^{(2)}_{n+1,m} = \phi^{(4)}_{n,m+1}. \tag{6.3}$$

Due to (6.1) the Hirota equation (4.16) relates these two angles

$$\exp(i\phi^{(1)}_{n,m} + i\phi^{(2)}_{n+1,m}) + 1 = k_{n,m}(\exp(i\phi^{(1)}_{n,m}) + \exp(i\phi^{(2)}_{n+1,m})),$$
$$k_{n,m} = \tan\frac{\Delta^u_n}{2}\tan\frac{\Delta^v_m}{2},$$

or equivalently,

$$
\begin{aligned}
\phi^{(1)}_{n,m} + \phi^{(2)}_{n+1,m} &\equiv \pi + 2\arg(1 - k_{n,m}\exp(i\phi^{(2)}_{n+1,m})) \\
&\equiv \pi + 2\arg(1 - k_{n,m}\exp(i\phi^{(1)}_{n,m})).
\end{aligned} \tag{6.4}
$$

The sum of angles around a vertex is equal to 2π, so

$$\phi^{(1)}_{n,m} + \phi^{(2)}_{n,m} + \phi^{(3)}_{n,m} + \phi^{(4)}_{n,m} \equiv 0,$$

or using (6.3) we get

$$\phi^{(1)}_{n,m} + \phi^{(1)}_{n-1,m-1} + \phi^{(2)}_{n,m} + \phi^{(2)}_{n+1,m-1} \equiv 0.$$

The Hirota equation (6.4) allows us to rewrite this as an equation for $\phi^{(1)}$'s

$$
\begin{aligned}
\phi^{(1)}_{n,m} + \quad & \phi^{(1)}_{n-1,m-1} - \phi^{(1)}_{n-1,m} - \phi^{(1)}_{n,m-1} \\
&= 2\arg(1 - k_{n-1,m}\exp(-i\phi^{(1)}_{n-1,m})) \\
&\quad +2\arg(1 - k_{n,m-1}\exp(-i\phi^{(1)}_{n,m-1})).
\end{aligned}
$$

Turning the lattice by 45^0 we get this equation in the form (6.2). Note that by symmetry all the angles $\phi^{(2)}$'s, $\phi^{(3)}$'s, $\phi^{(4)}$'s satisfy the same equation (6.2).

In the exponential form $Q = e^{i\phi}$ equation (6.2) reads as follows:

$$Q_u Q_d = \frac{Q_l - k_l}{1 - k_l Q_l}\frac{Q_r - k_r}{1 - k_r Q_r}. \tag{6.5}$$

Let us consider now a small piece of the Lorentz-harmonic Chebyshev net in S^2 generated by the Gauss map $N_{n,m}$ (see Figure 5). The difference equation for the angles between the arcs of the big circles in S^2, generated by the corresponding normals can be easily derived from equation (6.2).

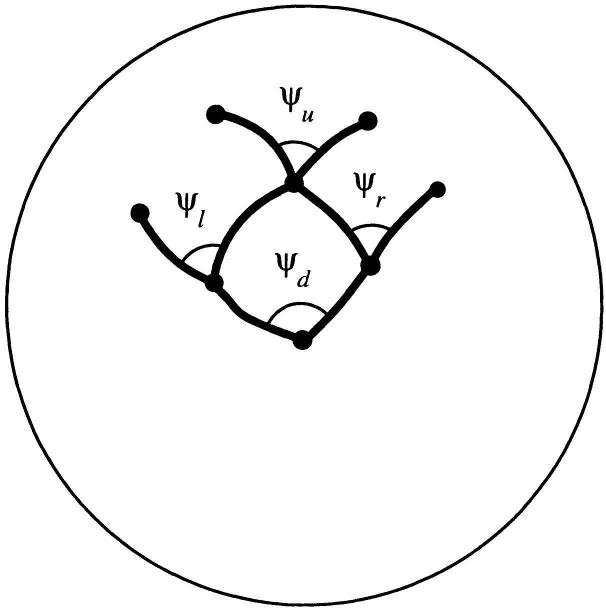


FIGURE 5. Lorentz-harmonic Chebyshev net in S^2.

Theorem 5. *The neighboring angles between the arcs (see Figure 5) of the Lorentz-harmonic Chebyshev net in S^2 satisfy the equation*

$$\psi_u + \psi_d - \psi_l - \psi_r \equiv 2\arg(1 + k_l e^{-i\psi_l}) + 2\arg(1 + k_r e^{-i\psi_r}). \tag{6.6}$$

Proof. Let us denote by $\mathcal{A}_{n,m}$ the plane containing the vectors $N_{n,m}$ and $N_{n+1,m}$, by $\mathcal{B}_{n,m}$ the plane containing the vectors $N_{n,m}$ and $N_{n,m+1}$, and by $\psi^{(1)}_{n,m}$ the angle between these two planes (see Figure 6).

The vectors $\mathbf{a} = F_{n+1,m} - F_{n,m}$ and $\mathbf{b} = F_{n,m+1} - F_{n,m}$ are orthogonal to $\mathcal{A}_{n,m}$ and $\mathcal{B}_{n,m}$ respectively. Moreover, formulas (5.9) (see also Figure 6) imply the following relation of $\psi^{(1)}_{n,m}$ with the angle $\phi^{(1)}_{n,m}$ between the edges $[F_{n+1,m}, F_{n,m}]$ and $[F_{n,m+1}, F_{n,m}]$ (see Figure 2):

$$\psi^{(1)}_{n,m} \equiv \phi^{(1)}_{n,m} + \pi.$$

Then (6.6) follows from (6.2).

Now we are already in a position to construct discrete K-surfaces, solving the Cauchy problem for equation (6.6). Let us consider some initial staircase loop

$$N_{n,m}, N_{n,m+1}, N_{n+1,m+1}, N_{n+1,m+2}, \dots, N_{n+N,m+N} = N_{n,m}$$

in S^2. Equation (6.6) describes a discrete evolution of this loop. Indeed, the initial loop provides us with the angles

$$\psi^{(2)}_{n+1,m}, \psi^{(2)}_{n+1,m+1}, \psi^{(2)}_{n+2,m+1}, \psi^{(2)}_{n+2,m+2}, \dots, \psi^{(2)}_{n+N,m+N} = \psi^{(2)}_{n,m}$$

(the angle $\psi^{(2)}$ is related with the angle $\phi^{(2)} \equiv \psi^{(2)} + \pi$ and satisfies the same equation (6.6) for ψ). Using the evolution equation (6.2) we can then determine $\psi^{(2)}_{k,l}$ for all k, l and, as a corollary (since the lengths of all the arcs are known), the coordinates $N_{k,l}$ of all points of the discrete Gauss map.

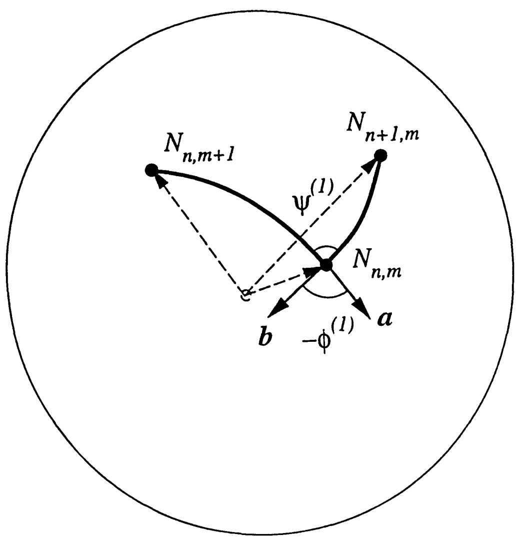


FIGURE 6. Relation between the angles ϕ and ψ.

Even simpler, one can reconstruct all $N_{n,m}$, $n, m \in \mathbf{Z}$ using the property (3.12) of N to be Lorentz-harmonic. Equation (3.12) uniquely determines $N_{n,m+1}$ by $N_{n,m}, N_{n+1,m}, N_{n+1,m+1}$:

$$N_{n,m+1} = -N_{n+1,m} + \frac{< N_{n+1,m}, N_{n,m} + N_{n+1,m+1} >}{1 + < N_{n,m}, N_{n+1,m+1} >}(N_{n,m} + N_{n+1,m+1}).$$

Finally, the formulas (5.9) describe the corresponding discrete K-surface.

Remark. One cannot apply the same recepie to the initial staircase loop

$$F_{n,m}, F_{n,m+1}, F_{n+1,m+1}, F_{n+1,m+2}, \ldots, F_{n+N,m+N} = F_{n,m}.$$

Such a loop can not be chosen arbitrary and should satisfy the restrictions (5.9). Indeed, if the neighboring edges are not parallel, they uniquely determine the Gauss map

$$N_{n,m}, N_{n,m+1}, N_{n+1,m+1}, N_{n+1,m+2}, \ldots, N_{n+N,m+N} = N_{n,m},$$

which should be related to the edges as described by (5.9).

Beside various cylinders one can construct by these elementary methods also discrete analogue of the Amsler surface. These surfaces are constructed in the next Section.

Unfortunately, it seems to be impossible to construct compact discrete K-surfaces in this way. This simple geometrical method does not allow us to control a global behavior of the surface (for example, to control the periodicity). For this reason, starting §8, we will use methods from the theory of integrable equations, which are based on an analytic solution of the problem. We calculate explicitly the solutions to the Hirota equation, the corresponding function $\overset{0}{\Psi}_{n,m}(\lambda)$ and determine in this way the immersion $F_{n,m}$ and its Gauss map $N_{n,m}$.

From now on we consider mostly special cases of discrete weak Chebyshev net, which we call as follows:

$\Delta_n^u = \Delta^u, \Delta_m^v = \Delta^v$ - a discrete anisotropic Chebyshev net,

$\Delta^u = \Delta^v$ - a discrete Chebyshev net.

In these cases $k = k_l = k_r$ in (6.2) is the same for all the surface:

$$k = \tan\frac{\Delta^u}{2}\tan\frac{\Delta^v}{2}.$$

7. The discrete Amsler surface

Let us return for a moment to smooth Chebyshev nets. For such a net we normalize $A = B = 1$ in (2.2), and the angle between the asymptotic lines is described by the sine-Gordon equation

$$\phi_{xy} - \sin\phi = 0.$$

If the solution $\phi(x,y)$ depends on $r=\sqrt{xy}$ only, i.e.,

$$\phi(x,y)=\phi(r), \tag{7.1}$$

then the equation reduces to the third Painlevé equation (PIII)

$$\phi_{rr}+\frac{\phi_r}{r}=4\sin\phi. \tag{7.2}$$

The corresponding surfaces were first studied by Bianchi [2]. It turns out that they possess nice geometrical properties. Amsler computed and sketched [1] a surface with $K=-1$ having two straight asymptotic lines L^x, L^y and showed that this surface belongs to the class (7.1). More precisely, there is a one-parametric family of the Amsler surfaces, parametrized by the angle ϕ between L^x and L^y, which can be chosen arbitrarily. This one-parametric family corresponds to the one-parametric family (see [16]) of the smooth solutions of the PIII equation (7.2).

Coming back to the discrete case, let us consider two great circles C^u, C^v on the sphere S^2 with the angle ψ between them, and suppose that the images of the normals $N_{n,m}$ for $n=0$ and $m=0$ belong to these circles

$$N_{0,k}\in C^u, \qquad N_{k,0}\in C^v, \qquad \forall n\in\mathbf{Z}. \tag{7.3}$$

Relations (5.9) show that the edges $[F_{k,0},F_{k+1,0}]$ and $[F_{0,k},F_{0,k+1}]$ for all $k\in\mathbf{Z}$ lie on two straight lines L^u and L^v in $\mathbf{R}^3$, and $\phi=\psi+\pi$ is the angle between them.

Starting with the normals (7.3) and using the property (5.9) of the Gauss map to be Lorentz-harmonic one can easily reconstruct step by step $N_{n,m}$ for all $n,m\in\mathbf{Z}$

$$N_{n+1,m+1}=-N_{n,m}+\frac{<N_{n,m},N_{n+1,m}+N_{n,m+1}>}{1+<N_{n+1,m},N_{n,m+1}>}(N_{n+1,m}+N_{n,m+1}).$$

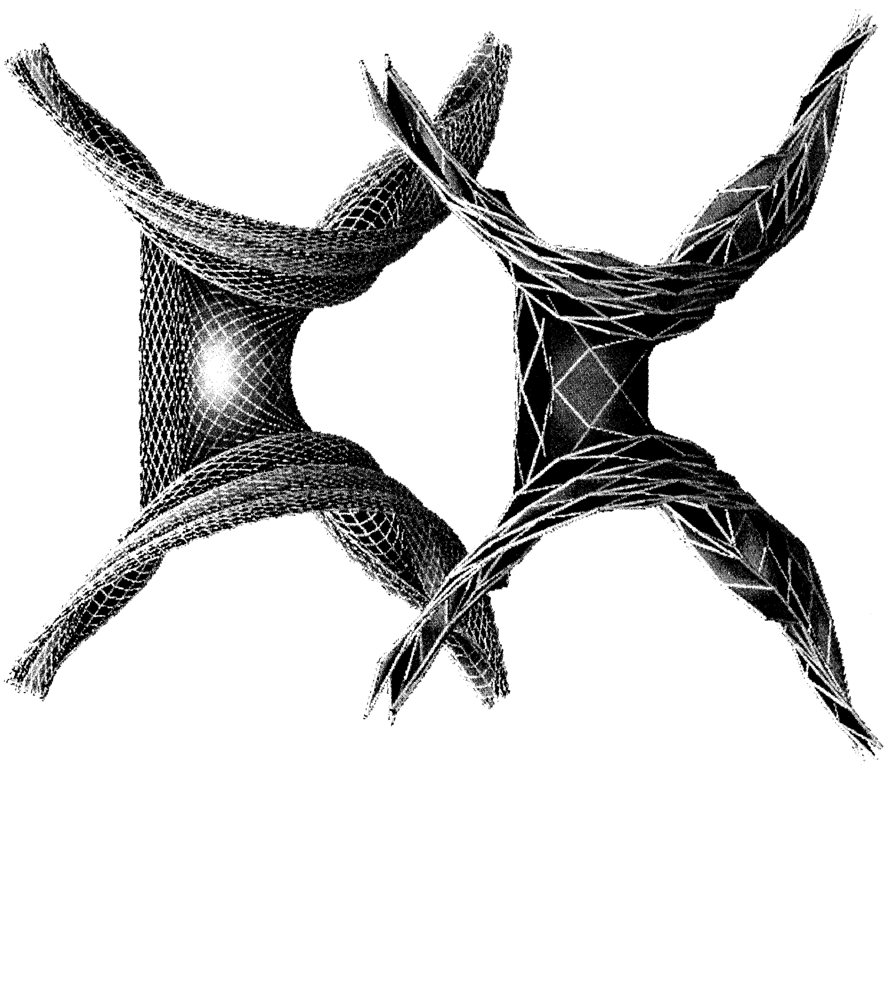


FIGURE 7. Smooth and discrete Amsler surfaces, $\phi = \pi/2$.

In the case of a discrete Chebyshev net $\Delta_n^u = \Delta_m^v = \Delta$ for fixed Δ one gets a one-parametric family of surfaces with two straight "asymptotic lines" $F_{k,0} \in L^u, F_{0,k} \in L^v, k \in \mathbf{Z}$. The parameter here is the angle ϕ between L^u and L^v. The smooth and discrete Amsler surfaces with $\phi = \pi/2$ are presented in Figure 7[3]. Having in mind the relation to the PIII equation in the smooth case, it is natural to expect, that the discrete K-surfaces just described generate discrete Painlevé equations, may be exactly those, which were recently found in [19]. Using the results of [18] Tim Hoffmann was able to show that the angles between the edges of the discrete Amsler surfaces in addition to the discrete sine-Gordon equation satisfy the restriction

$$\begin{aligned} &n(e^{i\phi^{(1)}_{n,m}} - e^{i\phi^{(3)}_{n,m}} + e^{-i\phi^{(2)}_{n,m}} - e^{-i\phi^{(4)}_{n,m}}) \\ &+m(e^{i\phi^{(3)}_{n,m}} - e^{i\phi^{(1)}_{n,m}} + \epsilon^{-i\phi^{(2)}_{n,m}} - e^{-i\phi^{(4)}_{n,m}}) = 0. \end{aligned} \tag{7.4}$$

Combined with (6.2) this restriction gives rise to a difference equation describing the angles $\phi_k = \phi_{k,k}$ of the discrete Amsler surfaces at the diagonal $n = m = k$, which is a discrete analogue of the PIII equation.

8. Dressing up procedure (Bäcklund transformations)

Here we describe some version of the Bäcklund transformation for discrete K-surfaces. For this purpose we use the analytical formalism of the dressing up procedure, suggested in the theory of integrable equations (see [15]). If some solution $h^{old}_{n,m}$ of the Hirota equation together with the corresponding solution $\Psi^{old}_{n,m}$ of the system

$$\Psi_{n+1,m} = U_{n,m}\Psi_{n,m}, \; \Psi_{n,m+1} = V_{n,m}\Psi_{n,m} \tag{8.1}$$

is known, this procedure allows us to construct new

$$h^{new}_{n,m}, \; \Psi^{new}_{n,m},$$

which solve the same equations.

[3]The authors are grateful to T. Hoffmann, who produced this figure.

The matrices

$$(8.2)\qquad \begin{aligned} U_{n,m} &= i\lambda\sigma_1 + u\begin{pmatrix} e^{i\alpha_{n,m}} & 0 \\ 0 & e^{-i\alpha_{n,m}} \end{pmatrix}, \\ V_{n,m} &= I + \frac{i}{\lambda v}\begin{pmatrix} 0 & e^{i\delta_{n,m}}, \\ e^{-i\delta_{n,m}} & 0 \end{pmatrix} \\ u &= \cot(\frac{\Delta^u}{2}), \quad v \;=\; \cot(\frac{\Delta^v}{2}) \end{aligned}$$

(we come back to the notation (4.15)) satisfy the following symmetries:

$$(8.3)\qquad X(-\lambda) = \sigma_3 X(\lambda)\sigma_3,$$

$$(8.4)\qquad X(\bar{\lambda}) = \sigma_2\overline{X(\lambda)}\sigma_2.$$

Let $\Psi^{old}_{n,m}(\lambda)$ be a solution of (8.1), satisfying both these reductions. The function $\Psi^{new}_{n,m}(\lambda)$ is represented in the form

$$\Psi^{new}(\lambda) = Q(\lambda)\Psi^{old}(\lambda),$$

where $Q(\lambda)$ is a matrix polynomial in λ, satisfying the symmetries (8.3), (8.4):

$$Q(\lambda) = Q_N\lambda^N + ... + Q_0$$

with fixed leading coefficient

$$Q_{N=even} = I \quad \text{and} \quad Q_{N=odd} = i\sigma_1.$$

The matrix $Q(\lambda)$ is determined by the conditions that

$$\Psi^{new}_{n+1,m}(\Psi^{new}_{n,m})^{-1}, \quad \Psi^{new}_{n,m+1}(\Psi^{new}_{n,m})^{-1}$$

are nonsingular at the zeros of $\det Q(\lambda)$.

Theorem 6. *Let $\Lambda^b \cup \Lambda^k \cup S^b \cup S^k$ be a set of paremeters:*

$$\begin{aligned} \Lambda^b &= \{\lambda^b_1, \lambda^b_2, ..., \lambda^b_K, \ \lambda^b \in \boldsymbol{C}\}, \\ S^b &= \{s^b_1, s^b_2, ...s^b_K, \ s^b \in \boldsymbol{C}\}, \\ \Lambda^k &= \{\lambda^k_1, \lambda^k_2, ..., \lambda^k_L, \ \lambda^k \in i\boldsymbol{R}\}, \\ S^k &= \{s^k_1, s^k_2, ...s^k_L, \ |s^k| = 1\}, \\ N &= 2K + L, \end{aligned}$$

in general position, and $\Psi^{old}(\lambda)$ *be a solution of* (8.1) *with some* $h_{n,m}$ *as coefficients, satisfying the reductions* (8.3), (8.4)*, where by general position we mean that* $\det \Psi^{old}(\lambda)$ *does not vanish at the points in* Λ*,* λ_i'*s do not coinside,* $\Lambda \cap \bar{\Lambda} = \Lambda \cap (-\Lambda) = \emptyset$*. Then the function*

$$\Psi^{new} = Q\Psi^{old},$$

where the coefficients of Q *are determined by the linear system*

$$\Psi^{new}(\lambda_i^b)\begin{pmatrix} 1 \\ s_i^b \end{pmatrix} = 0, \ i = 1, ..., K, \tag{8.5}$$

$$\Psi^{new}(\lambda_j^k)\begin{pmatrix} 1 \\ s_j^k \end{pmatrix} = 0, \ j = 1, ..., L, \tag{8.6}$$

satisfies the system (8.1) *with some* $h_{n,m}^{new}$.

Proof. The symmetries (8.3), (8.4) for Q imply

$$Q_{2n} = \begin{pmatrix} q_{2n} & 0 \\ 0 & \bar{q}_{2n} \end{pmatrix}, \ Q_{2n+1} = \begin{pmatrix} 0 & q_{2n+1} \\ -\bar{q}_{2n+1} & 0 \end{pmatrix}.$$

The system (8.5), (8.6) is a system of N linear non-homogeneous equations for N variables (we calculate the complex dimensions). To calculate the number of equations we note that each vector equation (8.5) represents two scalar equations, whereas (8.6) results in one scalar equation. As a matter of fact, due to the reductions (8.3), (8.4) ($\lambda^k = -\bar{\lambda}^k$)

$$\Psi^{new}(\lambda^k) = \sigma_1 \overline{\Psi^{new}(\lambda^k)} \sigma_1,$$

vanishing of the first component of the vector

$$\Psi^{new}(\lambda_j^k)\begin{pmatrix} 1 \\ s_j^k \end{pmatrix}$$

implies that the second component also vanishes.

The conditions (8.5), (8.6) mean that in the neighborhood of Λ the function Ψ^{new} can be represented as follows:

$$\Psi_{n,m}^{new}(\lambda) \stackrel{\lambda \sim \lambda_i}{=} \hat{\Psi}_{n,m}(\lambda) \begin{pmatrix} \lambda - \lambda_i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -s_i & 1 \end{pmatrix},$$

where $\hat{\Psi}_{n,m}$ is holomorphic and invertible. This representation shows non-singularity of

$$U^{new}_{n,m}(\lambda) = \Psi^{new}_{n+1,m}(\lambda)(\Psi^{new}_{n,m}(\lambda))^{-1} \stackrel{\lambda \sim \lambda_i}{=} \hat{\Psi}_{n+1,m}(\lambda)\hat{\Psi}^{-1}_{n,m}(\lambda)$$

at points in Λ. The equations (8.5), (8.6) together with the symmetries (8.3), (8.4) give all $2N$ zeroes of $\det Q(\lambda)$, which are

$$\Lambda^b \cup (-\Lambda^b) \cup \overline{\Lambda^b} \cup (-\overline{\Lambda^b}) \cup \Lambda^k \cup (-\Lambda^k).$$

The arguments above and the reductions (8.3,8.4) prove non-singularity of $U^{new}_{n,m}$ at all these points, which implies the following general form for $U^{new}_{n,m}$:

$$U^{new}_{n,m} = \lambda A_{n,m} + B_{n,m}.$$

Substituting the asymptotics at $\lambda \to \infty$ in

$$U^{new}_{n,m} = Q_{n+1,m} U^{old}_{n,m} Q^{-1}_{n,m}$$

and taking into account the reductions (8.3,8.4), we get (8.2) with some $\alpha^{new}_{n,m}$. The same arguments yields the form (8.2) for

$$V^{new}_{n,m} = Q_{n,m+1} V^{old}_{n,m} Q^{-1}_{n,m}.$$

We apply the dressing up procedure to the vacuum solution of the Hirota equation $h^{old}_{n,m} = 0$. The corresponding Ψ-function is equal to

$$\Psi^{old}_{n,m} = U^n V^m =$$
$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} (u+i\lambda)^n(1+\frac{i}{v\lambda})^m & 0 \\ 0 & (u-i\lambda)^n(1-\frac{i}{v\lambda})^m \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

To construct discrete pseudospheres we take only one point $\lambda^k \in \Lambda$, $N = 1, L = 1, K = 0$,

$$Q = \begin{pmatrix} q_{n,m} & i\lambda \\ i\lambda & \bar{q}_{n,m} \end{pmatrix}.$$

Equation (8.6) yields

$$\begin{aligned} q_{n,m} &= re^{i\phi_{n,m}}, \\ \phi_{n,m} &= 2\arg(1 + iSg_{n,m}(r)), \quad S \in \mathbf{R}, \\ g_{n,m} &= \left(\frac{u+r}{u-r}\right)^n \left(\frac{vr-1}{vr+1}\right)^m. \end{aligned}$$

The calculation of $(\Psi^{new})^{-1}\Psi^{new}_{\lambda}(\lambda = 1)$, using the concluding Remark of the §3, gives the following formula for the immersion:

$$F_{n,m} = i\sigma_3 \left(\frac{r}{r^2+1} \cos\phi_{n,m} + \frac{nu}{u^2+1} - \frac{mv}{v^2+1} \right)$$
$$+ \frac{r}{r^2+1} \sin\phi_{n,m} \begin{pmatrix} 0 & -e^{-i\theta_{n,m}} \\ e^{i\theta_{n,m}} & 0 \end{pmatrix},$$

where

$$4e^{i\theta_{n,m}} = \left(\frac{u+i}{u-i} \right)^n \left(\frac{v+i}{v-i} \right)^m .$$

Finally, in the most symmetric case $u = v, r = 1$, we get

$$\begin{aligned} F_1 &= \frac{1}{2} \sin\phi_{n,m} \sin a(n+m), \\ F_2 &= -\frac{1}{2} \sin\phi_{n,m} \cos a(n+m), \\ F_3 &= \frac{1}{2} \cos\phi_{n,m} + \frac{u}{u^2+1}(n-m), \end{aligned} \tag{8.7}$$

where

$$\begin{aligned} \phi_{n,m} &= 2\arg\left(1 + iS\left(\tfrac{u+1}{u-1}\right)^{n-m}\right), \\ a &= 2\arg(1+iu^{-1}). \end{aligned} \tag{8.8}$$

If a is a rational factor of π, then $F_{n,m}$ given by (8.7) is a periodic function of $n+m$, and the corresponding surface closes up in this direction. We call these surfaces discrete pseudospheres, because they are discrete analogues of the pseudosphere, which is the simplest smooth surface with the Gaussian curvature $K = -1$. The asymptotic line parametrization $F(x, y)$ of the pseudosphere is given by the formulas

$$\begin{aligned} F_1 &= \frac{1}{2} \sin\phi \sin(x+y), \\ F_2 &= -\frac{1}{2} \sin\phi \cos(x+y), \\ F_3 &= \frac{1}{2} \cos\phi + (x-y), \\ \phi &= 2\arg(1+ise^{x-y}), \end{aligned}$$

and the corresponding surface is presented in Figure 8.

To compare, Figure 9 presents a discrete pseudosphere with relatively small edges.

For a given $a = \pi/N$ ($\mathbf{Z}_N$ rotational symmetry) the discrete pseudosphere depends on an additional parameter S, and one gets a one-parametric family of deformations. In the smooth case the corresponding deformation of the pseudosphere is trivial and is just its reparametrization. Figure 10 demonstrates the dependence of the discrete pseudosphere on S.

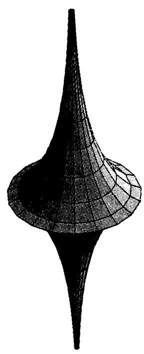


FIGURE 8. The smooth pseudosphere.

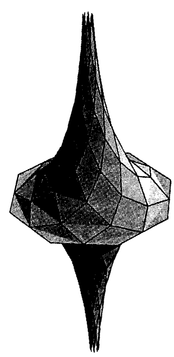


FIGURE 9. The discrete pseudosphere for $u = cot\frac{\pi}{16} \leftrightarrow a = \frac{\pi}{8}$ and $S = 1$.

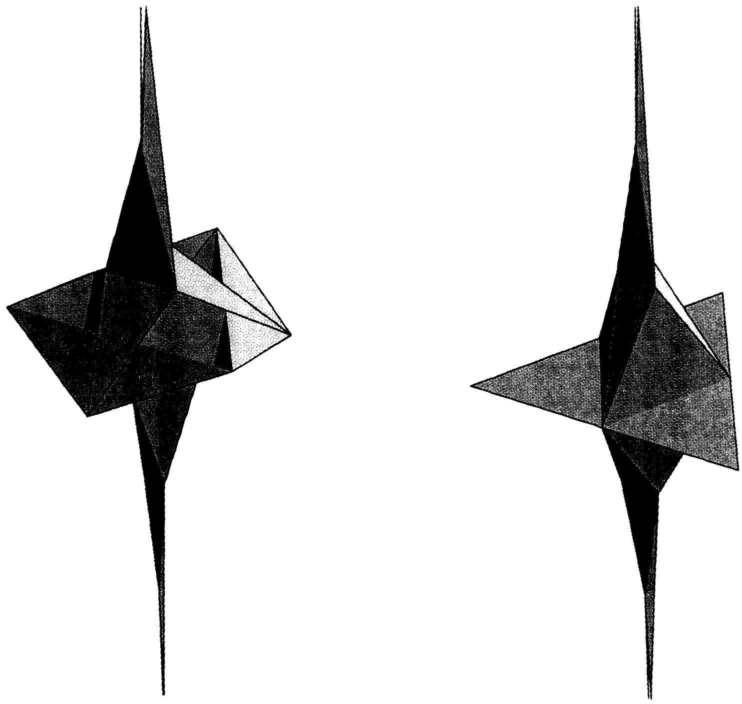


FIGURE 10. The discrete pseudospheres for $u = cot\frac{\pi}{6} \leftrightarrow a = \frac{\pi}{3}$ and $S = 1$ and for $S = 0.535898$

9. The spectral curve

Now we start to describe a wide class of discrete K-surfaces containing, in particular, all the surfaces with periodic Gauss map $N_{n,m}$. For this purpose the finite-gap integration technique from the theory of integrable equations is used.

Let us consider a discrete K-surface, the Gauss map of which is periodic with a period (N, M)

$$N_{n+N,m+M} = N_{n,m}.$$

Arguments of §6 show that all the N-loops of the surface with the period (N, M) are closed. To fix ideas, let us consider the loop

$$N_{n,m}, N_{n+1,m} \cdots , N_{n+N,m}, N_{n+N,m+1}, \dots N_{n+N,m+M}$$

or, equivalentely (see (5.9)), the brocken line

$$F_{n,m}, F_{n+1,m}, \dots , F_{n+N,m}, F_{n+N,m+1}, \dots , F_{n+N,m+M}$$

of the corresponding discrete K-surface. In general this broken line is not closed, but the angles between its edges are periodic with the same period (N, M). Schematically this broken line is shown in Figure 11. To simplify the notation we set $n = m = 0$.

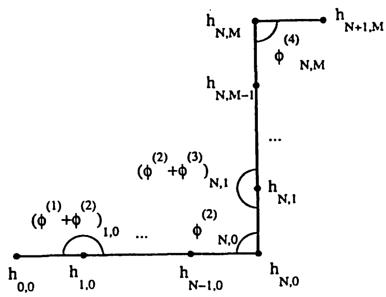


FIGURE 11. The period of the broken line.

The relations (6.1) between the angles and the Hirota variables $h_{n,m}$ show that $h_{0,0}$ and $h_{1,0}$ can be chosen arbitrary [4], all other $h_{n,m}$ are then uniquely determined by the geometrical data:

$$\begin{aligned}
(\phi^{(1)} + \phi^{(2)})_{k,0} &\equiv h_{k-1,0} - h_{k+1,0} + \pi, \\
\phi^{(2)}_{N,0} &\equiv h_{N,1} + h_{N-1,0}, \\
(\phi^{(2)} + \phi^{(3)})_{N,k} &\equiv h_{N,k+1} - h_{N,k-1} + \pi, \\
\phi^{(4)}_{N,M} &\equiv h_{N+1,M} + h_{N,M-1}.
\end{aligned}$$

Multiplying the matrices (5.3, 5.4) successively along the corresponding edges of the broken loop we get the transfer matrix of the Hirota model

$$T^H_{n,m} = V_{n+N,m+M-1} \dots V_{n+N,m} U_{n+N-1,m} \dots U_{n,m}.$$

Obviously this matrix satisfies the relations

$$\begin{aligned}
T^H_{n+1,m} &= U_{n+N,m+M} T^H_{n,m} U^{-1}_{n,m}, \\
T^H_{n,m+1} &= V_{n+N,m+M} T^H_{n,m} V^{-1}_{n,m}.
\end{aligned} \tag{9.1}$$

[4]This corresponds to the umbiguity (4.18)

Although the geometrical data of the K-surface are periodic, the Hirota variables, and consequently the matrices $U_{n,m}$, $V_{n,m}$, in general are not. To bring the equations (9.1) to the Lax-form (see below) we introduce the transfer matrix of the discrete sine-Gordon model.

Definition 3. The matrix

$$T^{SG}_{n,m} = e^{i\frac{\sigma_3}{2}\Theta_{n,m}} T^H_{n,m},$$

where

$$\Theta_{n,m} = h_{n+N+1,m+M} - h_{n+N,m+M} - h_{n+1,m} + h_{n,m}$$

is called the transfer matrix of the discrete sine-Gordon model.

Proposition 6. *The eigenvalues of the tranfer matrix $T^{SG}_{n,m}(\lambda)$ do not depend on the normalization* (4.18) *of the Hirota variables and are uniquely determined by the geometry of the initial contour $N_{n,m}, \dots, N_{n+N,m+M}$.*

The transfer matrix satisfies the equations

$$T^{SG}_{n+1,m} = U_{n,m} T^{SG}_{n,m} U^{-1}_{n,m}, \qquad T^{SG}_{n,m+1} = V_{n,m} T^{SG}_{n,m} V^{-1}_{n,m}, \tag{9.2}$$

implying the independence of the eigenvalues of $T^{SG}_{n,m}(\lambda)$ on n and m.

Proof. The matrices $U_{n,m}$, $V_{n,m}$ can be reduced to the form

$$\begin{aligned}
U_{n,m} &= e^{\frac{i}{2}\sigma_3(h_{n+1,m} - h_{n,m})} U e^{\frac{i}{2}\sigma_3(h_{n+1,m} - h_{n,m})}, \\
V_{n,m} &= e^{\frac{i}{2}\sigma_3(h_{n,m+1} + h_{n,m})} V e^{-\frac{i}{2}\sigma_3(h_{n,m+1} + h_{n,m})}, \\
U &= \begin{pmatrix} \cot\frac{\Delta^u}{2} & i\lambda \\ i\lambda & \cot\frac{\Delta^u}{2} \end{pmatrix}, \quad V = \begin{pmatrix} 1 & \frac{i}{\lambda}\tan\frac{\Delta^v}{2} \\ \frac{i}{\lambda}\tan\frac{\Delta^v}{2} & 1 \end{pmatrix}.
\end{aligned}$$

For the transfer matrix $T^{SG}_{0,0}$ this implies (again to simplify the notation we set $n = m = 0$)

$$\begin{aligned}
T^{SG}_{0,0} &= V_{N,M-1} \cdots V_{N,0} U_{N-1,0} \cdots U_{0,0} \\
&= e^{-\frac{i}{2}\sigma_3\Theta_{0,0}} e^{-\frac{i}{2}\sigma_3(h_{N,M} + h_{N,M-1})} V e^{-\frac{i}{2}\sigma_3(-h_{N,M} + h_{N,M-2})} \\
&\qquad \cdots V e^{-\frac{i}{2}\sigma_3(-h_{N,1} + h_{N-1,0})} U e^{-\frac{i}{2}\sigma_3(h_{N,0} - h_{N-2,0})} \\
&\qquad \cdots U e^{-\frac{i}{2}\sigma_3(h_{1,0} - h_{0,0})} \\
&= e^{-\frac{i}{2}\sigma_3(h_{0,0} - h_{1,0})} T_{0,0} e^{-\frac{i}{2}\sigma_3(h_{1,0} - h_{0,0})},
\end{aligned}$$

where

$$T_{0,0} = (-1)^{N+M} e^{-\frac{i}{2}\sigma_3 \phi^{(4)}_{N,M}} V e^{-\frac{i}{2}\sigma_3 (\phi^{(2)} + \phi^{(3)})_{N,M-1}} \\ \cdots V e^{-\frac{i}{2}\sigma_3 \phi^{(2)}_{N,0}} U e^{-\frac{i}{2}\sigma_3 (\phi^{(1)} + \phi^{(2)})_{N-1,0}} U \cdots U.$$

The eigenvalues of $T^{SG}_{0,0}$ and $T_{0,0}$ coincide, which proves the first statement of Proposition 6. The equalities (9.2) can be checked by direct calculation.

The characteristic polynominal

$$\det(T^{SG}_{n,m}(\lambda) - \mu) = 0$$

defines an algebraic curve $\tilde{X}$, which, due to Proposition 6, is an invariant of the discrete K-surface. The substitution

$$\mathcal{M} = \mu - \frac{\text{tr } T^{SG}(\lambda)}{2}$$

reduces this curve to the hyperelliptic form

$$\mathcal{M}^2 = \frac{(\text{ tr } T^{SG}(\lambda))^2}{4} - \det T^{SG}(\lambda).$$

Because of the symmetries

$$U_{n,m}(-\lambda) = \sigma_3 U_{n,m} \sigma_3, \quad V_{n,m}(-\lambda) = \sigma_3 V_{n,m}(\lambda) \sigma_3, \tag{9.3}$$

the transfer matrix satisfies the reduction

$$T^{SG}(-\lambda) = \sigma_3 T^{SG}(\lambda) \sigma_3. \tag{9.4}$$

Both $\text{tr } T^{SG}(\lambda)$ and $\det T^{SG}(\lambda)$ are functions of $\Lambda = \lambda^2$:

$$\text{tr } T^{SG}(\lambda) = t(\Lambda), \quad \det T^{SG}(\lambda) = d(\Lambda).$$

The curve $\tilde{X}$ by virtue of (9.4) posesses the involution

$$\tau_1 : (\lambda, \mu) \to (-\lambda, \mu).$$

The quotient $X = \tilde{X}/\tau_1$ is an algebraic curve

$$\mathcal{M}^2 = \frac{(t(\Lambda))^2}{4} - d(\Lambda), \tag{9.5}$$

which is called *the spectral curve.* This curve is central for our further calculations.

Proposition 7. *The spectral curve X is of the form*

$$\mathcal{M}^2 = \sum_{k=-M}^{N} g_k \Lambda^k,$$

with the coefficients g_N and g_{-M} given by

$$\begin{aligned} g_N &= -1, && N \text{ odd}, \\ g_N &= -\sin^2 \tfrac{1}{2}(h_{n+N+1,m+M} - h_{n+N,m+M} - h_{n+1,m} + h_{n,m}), && N \text{ even}, \\ g_{-M} &= -g^2, && M \text{ odd}, \\ g_{-M} &= -g^2 \sin^2 \tfrac{1}{2}(h_{n+N+1,m+M} + h_{n+N,m+M} - h_{n+1,m} - h_{n,m}), && M \text{ even}, \end{aligned}$$

where

$$g = (\cot \frac{\Delta^u}{2})^N (i \tan \frac{\Delta^v}{2})^M.$$

Proof. The formulas for g_N follow from the asymptotics of T^{SG} at $\Lambda \to \infty$

$$T^{SG}_{n,m} = (i\lambda)^N e^{\frac{i}{2}\sigma_3 \Theta_{n,m}} \sigma_1^N + O(\lambda^{N-1}). \tag{9.6}$$

In the same way the asymptotics at $\Lambda \to 0$ yields the formula for g_{-M}.

Remark. Due to Proposition 7 the coefficients g_N, g_{-M} can be interpreted geometrically. Let us present this interpretation for a staircase loop with even $N = M = 2n$, $N_{n+2k,m+2k} = N_{n,m}$. In this case both g_{2k}, g_{-2k} are nontrivial. Both $h_{4k+1} - h_1$ and $h_{4k+2} - h_2$ are integrals of the surface and can be expressed as alternating sums of angels (see the numeration in Figure 12).

$$\begin{aligned} h_{4k+1} - h_1 &= -\phi_2^{(2)} + \phi_4^{(2)} - \ldots - \phi_{4k-2}^{(2)} + \phi_{4k}^{(2)}, \\ h_{4k+2} - h_2 &= \phi_1^{(4)} - \phi_3^{(4)} + \ldots + \phi_{4k-3}^{(4)} - \phi_{4k-1}^{(4)}. \end{aligned}$$

Remark. The parities of N and M influence the branching of the covering $X \to \Lambda$ at $\Lambda = \infty$ and $\Lambda = 0$. This leads to a slight difference in the consideration of the four cases of even or odd N and M, which should be treated separately, although in similar ways. From now on we restrict ourself to the case of both N and M odd i.e., to the case where

$$N = 2k + 1, \quad M = 2l + 1.$$

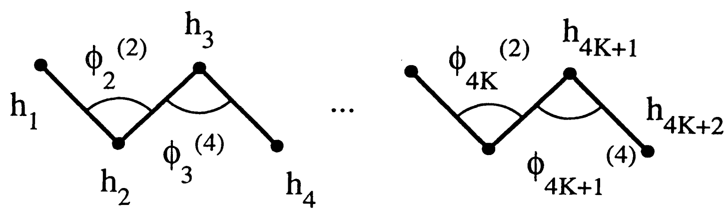


FIGURE 12. Additional invariants of the staircase loops with even number of stairs.

The spectral curve X in this case is of the genus

$$g = k + l + 1 \tag{9.7}$$

and has the branch points at $\Lambda = \infty$ and $\Lambda = 0$.

10. The Baker-Akhiezer function. Analytic properties

Let us denote by u and v the square root factors in $\mathring{U}_{n,m}, \mathring{V}_{n,m}$ (5.2)

$$u^2 = \Lambda + \cot^2 \frac{\Delta^u}{2}, \qquad v^2 = 1 + \Lambda^{-1} \tan^2 \frac{\Delta^v}{2} \tag{10.1}$$

and by $\hat{\Psi}_{n,m}(\lambda, u, v)$ the matrix solution of

$$\hat{\Psi}_{n+1,m} = \mathring{U}_{n,m}\, \hat{\Psi}_{n,m}, \qquad \hat{\Psi}_{n,m+1} = \mathring{V}_{n,m}\, \hat{\Psi}_{n,m}.$$

with the normalization $\hat{\Psi}_{0,0}(\lambda, u, v) = I$:

$$\hat{\Psi}_{n,m} = \mathring{V}_{n,m-1} \cdots \mathring{V}_{n,0} \mathring{U}_{n-1,0} \cdots \mathring{U}_{0,0}\, .$$

Let also $H(\mathcal{M}, \lambda)$ be an eigenvector of $T^{SG}_{0,0}(\lambda)$ with the first component H_1 normalized to $H_1 = 1$. These two functions $\hat{\Psi}_{n,m}(\lambda, u, v)$ and $H(\mathcal{M}, \lambda)$ are defined on a 8-sheeted covering $\hat{X}$ of the spectral curve

X. Namely the covering $\hat{X} \to X$ is defined by the equations (10.1) and $\lambda^2 = \Lambda$. The double-valued functions λ, u, v on X become single-valued on $\hat{X}$

$$\hat{X}(\mathcal{M}, \lambda, u, v) \xrightarrow{8:1} X(\mathcal{M}, \Lambda).$$

The Riemann surface $\hat{X}$ possesses the involutions

$$\begin{aligned} \tau_1 &: \lambda \to -\lambda, \\ \tau_2 &: u \to -u, \\ \tau_3 &: v \to -v, \\ \tau_4 &: \mathcal{M} \to -\mathcal{M}, \end{aligned}$$

and the quotient of $\hat{X}$ with respect to the group $\{\tau_1, \tau_2, \tau_3\}$ generated by the involutions τ_1, τ_2, τ_3 is $X = \hat{X}/\{\tau_1, \tau_2, \tau_3\}$.

Whereas in the case of both N and M odd the covering $\tilde{X} \to X$ defined by the function $\lambda = \sqrt{\Lambda}$ considered in the previous section is unramified, the covering $\hat{X} \to X$ is ramified at the zeroes of u and v. In Figure 13 the hyperelliptic Riemann surface X is presented. The branch points $\Lambda = \infty$ and $\Lambda = 0$ are connected by a cut $[0, \infty]$. The contour $\mathcal{L}$ surrounds all other branch points, the contour l_0 connects two points P_0^+ and P_0^- on X with the same Λ coordinate $\Lambda(P_0^\pm) = -\tan^2 \frac{\Delta^v}{2}$, and the contour l_∞ connects the points $P_\infty^\pm$ with $\Lambda(P_\infty^\pm) = -\cot^2 \frac{\Delta^u}{2}$ (we prefer to think about the point $\Lambda = \infty$ as a usual point of X and to draw P_∞ to the right of it).

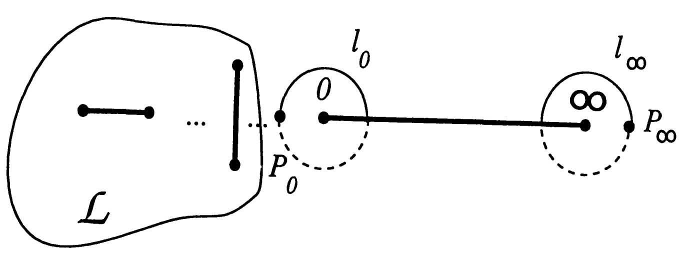


FIGURE 13. The spectral curve X.

We fix the branches of λ, u, v on $X \setminus \{\mathcal{L}, \ell_\infty, \ell_0\}$ by saying that λ changes its sign on $\mathcal{L}$ and is positive on the upper edge of the cut

$[0,\infty]$ on the first sheet of the covering $X \to \Lambda$, u and v change sign when one crosses ℓ_∞ or ℓ_0 respectively and are normalized to behave as

$$\frac{u}{\lambda} \to 1, \qquad v \to 1 \tag{10.2}$$

in some small neighborhood of infinity $\mathcal{U}_\infty$. If the contour ℓ_∞ chosen as in Figure 13, then both u and v are positive at $\Lambda = 1$, i.e.,

$$u(\Lambda = 1) > 0, \qquad v(\Lambda = 1) > 0.$$

Let P denote a point of $\hat{X}$ with coordinates $(\lambda, \mathcal{M}, u, v)$.

Definition 4. The vector-function

$$\psi_{n,m}(P) = \hat{\Psi}_{n,m}(\lambda, u, v) H(\mathcal{M}, \lambda)$$

is called a Baker-Akhiezer function (BA).

A BA function is a rational function of $\mathcal{M}, \lambda, u, v$, satisfying the equations

$$\psi_{n+1,m} = \overset{\circ}{U}_{n,m}\, \psi_{n,m}, \qquad \psi_{n,m+1} = \overset{\circ}{V}_{n,m}\, \psi_{n,m}, \qquad T^{SG}_{n,m} \psi_{n,m} = \mu \psi_{n,m}.$$

Perhaps only the latter needs some comments. This equation is a consequence of the identity

$$T^{SG}_{n,m} \hat{\Psi}_{n,m} = \hat{\Psi}_{n,m} T^{SG}_{0,0},$$

which follows from (9.2) and the definition of $\hat{\Psi}_{n,m}$.

The BA function transformes in a very simple way under the involutions τ_1, τ_2, τ_3:

$$\begin{aligned} \psi_{n,m}(\tau_1 P) &= \sigma_3 \psi_{n,m}(P), \\ \psi_{n,m}(\tau_2 P) &= (-1)^n \psi_{n,m}(P), \\ \psi_{n,m}(\tau_3 P) &= (-1)^m \psi_{n,m}(P). \end{aligned} \tag{10.3}$$

Two last identities are trivial and to prove the first identity let us write down the formula for the eigenvector $H(\mathcal{M}, \lambda)$:

$$\begin{aligned} H(\mathcal{M}, \lambda) &= \begin{pmatrix} 1 \\ h(\mathcal{M}, \lambda) \end{pmatrix}, \\ h(\mathcal{M}, \lambda) &= \frac{\mu - A(\lambda)}{B(\lambda)} = \frac{2\mathcal{M} + D(\lambda) - A(\lambda)}{2B(\lambda)}, \end{aligned} \tag{10.4}$$

where we have used the following notation for the elements of $T^{SG}_{0,0}$:

$$T^{SG}_{0,0}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}.$$

The symmetry

$$\sigma_3 \hat{\Psi}(-\lambda, u, v)\sigma_3 = \hat{\Psi}(\lambda, u, v),$$

or equivalently

$$\begin{array}{ll} A(-\lambda) = A(\lambda), & B(-\lambda) = -B(\lambda), \\ C(-\lambda) = -C(\lambda), & D(-\lambda) = D(\lambda), \end{array} \tag{10.5}$$

derived from (5.2),(9.1), imply

$$\sigma_3 \psi(\tau_1 P) = \sigma_3 \hat{\Psi}(-\lambda, u, v) \begin{pmatrix} 1 \\ \dfrac{2\mathcal{M} + D(-\lambda) - A(-\lambda)}{2B(-\lambda)} \end{pmatrix} = \psi(\tau_1 P).$$

Finally, we may also consider the BA function as an analytic function on $X \setminus \{\mathcal{L}, \ell_\infty, \ell_0\}$ satisfying the symmetry relations

$$\begin{aligned} \sigma_3 \psi^+_{n,m} &= \psi^-_{n,m}|_{P\in\mathcal{L}}, \\ \psi^+_{n,m} &= (-1)^n \psi^-_{n,m}|_{P\in\ell_\infty}, \\ \psi^+_{n,m} &= (-1)^m \psi^-_{n,m}|_{P\in\ell_0}. \end{aligned} \tag{10.6}$$

In other words, $\psi_{n,m}$ acquires *a* factor

$$(-1)^{n<\gamma,\ell_\infty>+m<\gamma,\ell_0>} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{<\gamma,\mathcal{L}>} \end{pmatrix}$$

along a loop γ on X. Here $<,>$ is the intersection number.

The polar divisor $\mathcal{D}$ of $\psi_{n,m}$ on $X \setminus \{P_0^\pm, P_\infty^\pm\}$ is defined by H (since $\hat{\Psi}_{n,m}$ is regular there) and consequently does not depend on n, m. If both N and M are odd, then the highest order terms λ^N and λ^{-M} of the matrix $T^{SG}_{0,0}$ at $\lambda \to \infty$ and $\lambda \to 0$ are off-diagonal, i.e.,

$$A(\lambda) = \sum_{k=-M+1}^{N-1} a_k \lambda^k, \qquad B(\lambda) = \sum_{k=-M}^{N} b_k \lambda^k,$$
$$C(\lambda) = \sum_{k=-M}^{N} c_k \lambda^k, \qquad D(\lambda) = \sum_{k=-M+1}^{N-1} d_k \lambda^k,$$

where b_N, b_{-N}, c_M, c_{-M} do not vanish. Combined with the equation (9.5) of the curve, this implies the regularity of H at $\lambda = \infty$ and $\lambda = 0$. The poles of H are situated at the $(N+M)/2$ zeroes of the polynomial

$$\tilde{B}(\Lambda) = b_n \Lambda^{\frac{N+M}{2}} + \ldots + b_{-M} = \lambda^M B(\lambda)$$

(by taking into account the symmetry (10.5)). Over each zero Λ_z of $\tilde{B}(\Lambda)$ there lie two points Λ_z^+, Λ_z^- of the curve X. Moreover, for one of these two points the numerator $\mathcal{M} - \frac{A-D}{2}$ in (10.4) vanishes too. This fact follows from the curve equation

$$\left(\frac{A-D}{2} - \mathcal{M}\right)\left(\frac{A-D}{2} + \mathcal{M}\right) = -BC,$$

which shows that one of the factors on the left-hand side must vanish at the zeroes $\Lambda_z^{\pm}$ of B. On the other hand, these two factors differ by the hyperelleptic involution $\mathcal{M} \to -\mathcal{M}$, and therefore their zeroes are Λ_z^+ and Λ_z^- respectively. Finally we get a non-special [5] polar divisor $\mathcal{D}$ of degree $(N+M)/2 = k + \ell + 1 = g$ coinciding with the genus of X (see(9.7)).

To investigate the behavior of the BA function at the points $P_\infty^{\pm}, P_0^{\pm}$ let us introduce a function $\tilde{\psi}_{n,m}$ which differs from $\psi_{n,m}$ by a scalar factor, so that

$$\tilde{\psi}_{n,m} = u^n v^m \psi_{n,m} = R_{n,m} H, \qquad R_{n,m} = V_{n,m-1} \ldots V_{n,0} U_{n-1,0} \ldots U_{0,0}.$$

Written in terms of the matrix elements of

$$R_{n,m} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

it takes the form

$$\tilde{\psi}_{n,m} = \begin{pmatrix} r_{11} + r_{12} h \\ r_{21} + r_{22} h \end{pmatrix},$$

which is an eigenvector of $T_{n,m}^{SG}$. Quite the same arguments as those used for the function h show that the second component of the normalized

[5]The divisor of degree g on the hyperelliptic surface of genus g is called special if it contains a pair of points P^+, P^- interchanged by the hyperelliptic involution $\Lambda(P^+) = \Lambda(P^-)$, and non-special otherwise. There is no nontrivial function with non-special pole divisor of degree g.

(the first component is 1) function $\tilde{\psi}_{n,m}$

$$\frac{r_{21}+r_{22}h}{r_{11}+r_{12}h}$$

has a pole divisor of degree g. Since the pole divisors $\mathcal{D}+n\infty+m0$ of the functions $r_{11}+r_{12}h$ and $r_{21}+r_{22}h$ coincide ($\Lambda=\infty$ is a pole of degree $n, \Lambda=0$ is a pole of degree m) and are of degree $g+n+m$, the zero divisors of these two functions have a common part $\mathcal{A}_{n,m}$ of degree $n+m$. Moreover, it is clear that

$$\mathcal{A}_{n+1,m}=\mathcal{A}_{n,m}+P^u_{n,m}, \qquad \mathcal{A}_{n,m+1}=\mathcal{A}_{n,m}+P^v_{n,m},$$

where $P^u_{n,m}, P^v_{n,m}$ are some points.

To determine $\mathcal{A}_{n,m}$ let us introduce a matrix

$$\tilde{\Psi}_{n,m}=(\tilde{\psi}_{n,m}, \tau_4^*\tilde{\psi}_{n,m}).$$

Taking determinants of the equality $\tilde{\Psi}_{n+1,m}=U_{n,m}\tilde{\Psi}_{n,m}$,

$$\frac{\lambda \det \tilde{\Psi}_{n+1,m}}{\mathcal{M}}=\det U_{n,m}\frac{\lambda \det \tilde{\Psi}_{n,m}}{\mathcal{M}}, \tag{10.7}$$

we note that the left-hand side as well as both factors on the right-hand side are invariant with respect to the involutions τ_1 and τ_4. Therefore, they are rational functions of Λ. On the Λ-plane the function $\det U_{n,m}$ has a pole at $\Lambda=\infty$ and a zero at $\Lambda=P_\infty$. The function $\lambda \det \tilde{\Psi}_{n,m}/\mathcal{M}$ has a divisor of poles $\pi(\mathcal{D})+m0+n\infty$ and a divisor of zeros $\pi(\mathcal{A}_{n,m})+\frac{M+1}{2}0+\frac{N-1}{2}\infty$, where $\pi : X \to \Lambda$ is the projection on the Λ-plane. Finally, comparing the poles and zeros of the right- and the left-hand sides of (10.7), we get

$$\pi(P^u_{n,m})=P_\infty.$$

The equality

$$\pi(P^v_{n,m})=P_0$$

is proved in the same way. Now we come back to the BA-function

$$\psi_{n,m}=\frac{1}{u^n v^m}\tilde{\psi}_{n,m},$$

and suppose $P^+_\infty=P^u_{0,0}$, $P^+_0=P^v_{0,0}$. The function $\psi_{1,1}$ has zeros at P^+_∞, P^+_0 and poles at P^-_∞, P^-_0. It is easy to see that the presence of P^-_∞

among $P^u_{n,m}$ (as well as the presence of P_0^- among $P^v_{n,m}$) corresponds to the degeneracy of the discrete K-surface. Indeed $P^u_{n,m} = P^+_\infty, P^u_{n+1,m} = P^-_\infty$ yield $\psi_{n+2,m} = \psi_{n,m}$ or, equivalently, $\overset{\circ}{U}_{n+1,m}\overset{\circ}{U}_{n,m} = I$. The lines $F_{n+2,m}$ and $F_{n,m}$, $m \in \mathbf{Z}$ of the discrete K-surface, coincide in this case. Therefore from now on we set

$$\mathcal{A}_{n,m} = nP^+_\infty + mP^+_0,$$

and the function $\psi_{n,m}$ has poles at $nP^-_\infty + mP^-_0$ and zeros at $nP^+_\infty + mP^+_0$.

Remark. In the case of N, M odd, which we consider, the ambiquity (4.18) in the geometrical definition of the Hirota variables can be used to normalize them to be periodic, so that

$$h_{n+N,m+M} = h_{n,m}, \qquad \forall n, m.$$

From now we use this normalization.

The value of the BA-function at $\Lambda = \infty$ can be easily calculated. The asymptotics (9.6) of $T^{SG}_{0,0}$ gives

$$\begin{aligned} A &= o(\lambda^N), \\ B &= (i\lambda)^N + o(\lambda^N), \\ D &= o(\lambda^N), \\ \mathcal{M} &= (i\lambda)^N + o(\lambda^N), \quad \lambda \to \infty \end{aligned}$$

and the folowing value of the eigenvector:

$$H = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \Lambda = \infty, \quad \Lambda \in \mathcal{U}_\infty. \tag{10.8}$$

For the function $\hat{\Psi}_{n,m}$ with $\Lambda \in \mathcal{U}_\infty$, taking into account the fixation (10.2) of branches, one gets

$$\hat{\Psi}_{n,m} = \lim_{\Lambda \to \infty} \frac{1}{u^n v^m} (i\lambda)^n \sigma_1^n = i^n \sigma_1^n,$$

which, combined with (10.8), proves the normalization

$$\psi_{n,m} = i^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \Lambda = \infty, \quad \Lambda \in \mathcal{U}_\infty.$$

We formulate all the established analytical properties of $\psi_{n,m}$ in a theorem.

Theorem 7. *The BA-function $\psi_{n,m}$ is meromorphic on $X \setminus \{\mathcal{L}, l_\infty, l_0\}$, and the following holds:*

1) *The function* $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} u^n v^m \psi_{n,m}$ *is single-valued on X. In other words, $\psi_{n,m}$ acquires the jumps (10.6) on the contours $\mathcal{L}, \ell_\infty, \ell_0$.*
2) *The leading terms of $\psi_{n,m}$ at the points $P_\infty^\pm, P_0^\pm$ are of the form*

$$\begin{aligned} \psi_{n,m}(P) &= (\Lambda - P_\infty)^{\pm n/2} O(1), \qquad P \to P_\infty^\pm, \\ \psi_{n,m}(P) &= (\Lambda - P_0)^{\pm m/2} O(1), \qquad P \to P_0^\pm. \end{aligned} \tag{10.9}$$

3) *$\psi_{n,m}$ is normalized at $\Lambda = \infty$ as*

$$\psi_{n,m} = i^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{10.10}$$

4) *Yhe pole divisor $\mathcal{D}$ of $\psi_{n,m}$ on $X \setminus \{P_\infty^\pm, P_0^\pm\}$ is nonspecial, of degree g and independent of n, m.*

11. The Baker-Akhiezer function. Explicit formulas

The function $\psi_{n,m}(P)$ of Theorem 7 can be explicitly expressed in terms of Riemann theta functions and Abelian integrals. Moreover, in this context it is natural to extend the class of periodic solutions of the Hirota equation and to consider the class of finite-gap solutions. In this Section, motivated by Theorem 7, we show that for an arbitrary hyperelleptic Riemann surface X of genus g with branch points $\Lambda = 0, \infty$ and arbitrary non-special divisor $\mathcal{D}$ of degree g, the function $\psi_{n,m}(P)$ with the properties 1-4 of Theorem 7 is unique, and we obtain an explicit formula for this function[6]. In the next sections we specify X and $\mathcal{D}$ corresponding to the discrete K-surfaces and discuss their periodicity.

Theorem 8. *For any hyperelleptic Riemann surface X of genus g with branch points $\Lambda = 0, \infty$ and non-special divisor $\mathcal{D}$ of degree g in*

[6]The formulas of this section are similar to the formulas for the Baker-Akhiezer function of the usual sine-Gordon equation (2.1), obtained first by A.Its (see [3]).

the general position, the function $\psi_{n,m}(P)$ with the analytical properties formulated in Theorem 7 is unique.

Proof. Let us suppose that there exist two functions with the properties 1-4 listed above. Let $f'_{n,m}$ and $f''_{n,m}$ be their first components. Since $\mathcal{D}$ is non-special and in the general position, the zero divisors $\mathcal{D}'_{n,m}$ and $\mathcal{D}''_{n,m}$ of $f'_{n,m}$ and $f''_{n,m}$ on $X \setminus \{P_\infty^\pm, P_0^\pm\}$ are also non-special. But then the quotient $f''_{n,m}/f'_{n,m}$ is a meromorphic function on X with a non-special pole divisor $\mathcal{D}'_{n,m}$ of degree g. By the Riemann-Roch theorem such a function is constant. The proof for the second component of $\psi_{n,m}$ is the same.

Let $\mathcal{L}, \ell_\infty, \ell_0$ be the contours as shown in Figure 13 and let a canonical basis of cycles $a_n, b_n, n = 1, \ldots, g$, be chosen so that a- and b-cycles do not intersect the contours ℓ_∞, ℓ_0, and the cycle $\mathcal{L}$ is equal to the sum of all a-cycles, i.e.,

$$\mathcal{L} = a_1 + \ldots + a_g.$$

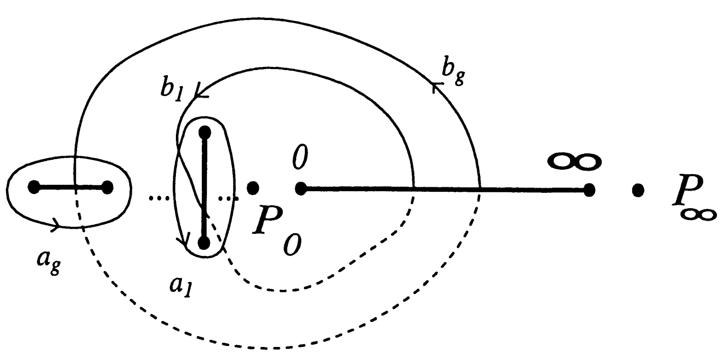


FIGURE 14. The canonical basis of cycles of the spectral curve.

The normalized holomorphic Abelian differentials

$$\int_{a_n} dw_m = 2\pi i \delta_{n,m} \tag{11.1}$$

define the period matrix $B_{n,m} = \int_{b_n} dw_m$, in terms of which the Riemann theta function is defined:

$$\theta(z) = \sum_{m \in \mathbf{Z}^g} \exp(\frac{1}{2} < Bm, m > + < z, m >), \quad z \in \mathbf{C}^g,$$

which is a quasiperiodic function

$$\theta(z+2\pi iN+BM) = \exp\{-\tfrac{1}{2}<BM,M>-<z,M>\}\theta(z), \tag{11.2}$$

on the Jacobian

$$\mathrm{Jac}(X)=\mathbf{C}^g/\{z\to z+2\pi iN+BM;\ N,M\in\mathbf{Z}^g\}.$$

Let us introduce two normalized differentials of the third kind

$$\int_{a_n} d\Omega_{\infty,0}=0 \tag{11.3}$$

with singularities of the following form:

$$\begin{aligned} d\Omega_\infty &= \pm\frac{1}{2}\frac{d\Lambda}{\Lambda-P_\infty}+O(1), \qquad P\to P_\infty^\pm, \\ d\Omega_0 &= \pm\frac{1}{2}\frac{d\Lambda}{\Lambda-P_0}+O(1), \qquad P\to P_0^\pm, \end{aligned} \tag{11.4}$$

The reciprocity law (see, for example, [13]) allows us to express the periods of Abelian differentials of the third kind in terms of the normalized holomorphic differentials:

$$U_n\equiv\int_{b_n} d\Omega_\infty=\int_{P_\infty^-}^{P_\infty^+} dw_n, \quad V_n\equiv\int_{b_n} d\Omega_0=\int_{P_0^-}^{P_0^+} dw_n. \tag{11.5}$$

Here the integration paths $[P_\infty^+,P_\infty^-]$, $[P_0^+,P_0^-]$ coincide with ℓ_∞,ℓ_0. The exponentials

$$f_\infty(P)\equiv\exp(\int_{\Lambda=\infty}^P d\Omega_\infty), \qquad f_0(P)\equiv\exp(\int_{\Lambda=\infty}^P d\Omega_0),$$

of these integrals have the following:

1)They have singularities of the form

$$\begin{aligned} f_\infty(P) &= (\sqrt{\Lambda-P_\infty})^{\pm1}O(1), \qquad \Lambda\to P_\infty^\pm, \\ f_0(P) &= (\sqrt{\Lambda-P_0})^{\pm1}O(1), \qquad \Lambda\to P_0^\pm. \end{aligned} \tag{11.6}$$

2) They have jumps on ℓ_∞ and ℓ_0:

$$f_\infty^+(P)=-f_\infty^-(P),\ P\in\ell_\infty \quad f_0^+(P)=-f_0^-(P),\ P\in\ell_0. \tag{11.7}$$

3) They are preserved when running through the loop a_n. Running through b_n they get the factors e^{U_n} and e^{V_n} so that

$$\overset{\bigcirc}{b_n} f_\infty = e^{U_n} f_\infty, \qquad \overset{\bigcirc}{b_n} f_0 = e^{V_n} f_0.$$

4) They are normalized at $\Lambda = \infty$ as

$$f_\infty(\Lambda = \infty) = f_0(\Lambda = \infty) = 1.$$

Let us note also that by the Riemann theorem (see, for example, [13]) for a vector $D \in \mathbf{C}^g$ in general position the zero divisor of

$$\theta\Big(\int^P dw + D\Big) \tag{11.8}$$

is of degree g and non-special. Moreover, any non-special divisor $\mathcal{D}$ of degree g can be represented as a zero divisor of the theta function (11.8), where $\mathcal{D}$ and D are related by the Jacobi inverse problem

$$D = \sum_{i=1}^{g} \int^{P_i} dw + K, \qquad \mathcal{D} = \sum_{i=1}^{g} P_i.$$

Here K is the Riemann constant.

Theorem 9. *The BA function $\psi_{n,m}(P)$ given by the formulas*

$$\begin{aligned}\psi^1_{n,m}(P) &= i^n \frac{\theta(\int_\infty^P dw + \Omega_{n,m})\theta(D)}{\theta(\int_\infty^P dw + D)\theta(\Omega_{n,m})} e^{\int_\infty^P (nd\Omega_\infty + md\Omega_0)},\\ \psi^2_{n,m}(P) &= i^n \frac{\theta(\int_\infty^P dw + \Omega_{n,m} + \Delta)\theta(D)}{\theta(\int_\infty^P dw + D)\theta(\Omega_{n,m} + \Delta)} e^{\int_\infty^P (nd\Omega_\infty + md\Omega_0)},\end{aligned} \tag{11.9}$$

has the analytical properties 1-4, formulated in Theorem 7. Here

$$\begin{aligned}\Delta &= \pi i(1, 1 \dots, 1),\\ \Omega_{n,m} &= Un + Vm + D,\\ U &= (U_1, \dots, U_g),\\ V &= (V_1, \dots, V_g),\\ \int_\infty^P dw &= \Big(\int_\infty^P dw_1 \dots, \int_\infty^P dw_g\Big),\end{aligned}$$

and the integration paths in (11.9) *coincide.*

Proof. The normalization (10.10) is evident. The asymptotics (10.9) follows from (11.6). The divisor $\mathcal{D}$ is produced by the denominators of (11.9). The jumps (10.6) on ℓ_∞, ℓ_0 are due to (11.7). Using the normalization (11.1,11.3) and the periodicity (11.2) of the theta function one can easily show that $\psi^1_{n,m}$ is preserved when one runs through any closed contour

$$\gamma = \sum_{i=1}^{g} n_i a_i + m_i b_i$$

on X. On the other hand the function $\psi^2_{n,m}$ aquires the factor $(-1)^{\sum_{i=1}^g m_i}$. For the basis shown in Figure 14 this factor is equal to $(-1)^{<\gamma,\mathcal{L}>}$, which shows that $\lambda\psi^2_{n,m}$ does not change sign. Hence the theorem is proved.

Theorem 10. *The function* (11.9) *solves the system*

$$\psi_{n+1,m} = \frac{1}{u} U_{n,m}\psi_{n,m}, \qquad \psi_{n,m+1} = \frac{1}{v} V_{n,m}\psi_{n,m} \tag{11.10}$$

with the matrices $U_{n,m}, V_{n,m}$ *of the form*

$$U_{n,m} = \begin{pmatrix} a_{n,m} & i\lambda \\ i\lambda & b_{n,m} \end{pmatrix}, \qquad V_{n,m} = \begin{pmatrix} 1 & \frac{i}{\lambda}c_{n,m} \\ \frac{i}{\lambda}d_{n,m} & 1 \end{pmatrix}. \tag{11.11}$$

The coeficients of these matrices are equal to

$$\begin{aligned} a_{n,m} &= \cot\frac{\Delta^u}{2}\varepsilon_\infty \frac{\theta(\Omega_{n+1,m}+\Delta)\theta(\Omega_{n,m})}{\theta(\Omega_{n+1,m})\theta(\Omega_{n,m}+\Delta)}, \\ b_{n,m} &= \cot\frac{\Delta^u}{2}\varepsilon_\infty \frac{\theta(\Omega_{n+1,m})\theta(\Omega_{n,m}+\Delta)}{\theta(\Omega_{n+1,m}+\Delta)\theta(\Omega_{n,m})}, \\ c_{n,m} &= \tan\frac{\Delta^v}{2}\varepsilon_0 \frac{\theta(\Omega_{n,m+1}+\Delta)\theta(\Omega_{n,m}+\Delta)}{\theta(\Omega_{n,m+1})\theta(\Omega_{n,m})}, \\ d_{n,m} &= \tan\frac{\Delta^v}{2}\varepsilon_0 \frac{\theta(\Omega_{n,m+1})\theta(\Omega_{n,m})}{\theta(\Omega_{n,m+1}+\Delta)\theta(\Omega_{n,m}+\Delta)}, \end{aligned} \tag{11.12}$$

where the signs $\varepsilon_\infty, \varepsilon_0 = \pm 1$ *are determined by the labeling of the points* $P^\pm_\infty$ *and* $P^\pm_0$ *(see the Remark below) and are defined by*

$$\varepsilon_\infty = -i\exp(\int_\alpha d\Omega_\infty), \qquad \varepsilon_0 = -i\exp(\int_\alpha d\Omega_0), \tag{11.13}$$

with the integration path α along a cut $[0,\infty]$ (α does not intersect the contours ℓ_∞, ℓ_0 shown in Figure 13).

Proof. Let us consider the function

$$\tilde{\psi}_{n+1,m} = \frac{1}{u}\begin{pmatrix} a_{n,m} & i\lambda \\ i\lambda & b_{n,m} \end{pmatrix} \psi_{n,m}.$$

For any $a_{n,m}, b_{n,m}$ the function $\tilde{\psi}_{n+1,m}$ possesses the properties 1,3,4 of Theorem 7 and has the asymptotics (10.9) at the points $P_0^\pm, P_\infty^-$ (one should replace n by $n+1$). Specifying $a_{n,m}$ and $b_{n,m}$ one can obtain also a higher order zero of $\tilde{\psi}_{n+1,m}$ at $P \to P_\infty^+$. Indeed, let

$$\psi_{n,m}^{1,2} = (\Lambda - P_\infty)^{n/2} f_{n,m}^{1,2}, \qquad P \to P_\infty^+.$$

Then, choosing

$$a_{n,m} = -i\frac{\lambda f_{n,m}^2}{f_{n,m}^1}(P_\infty^+), \qquad b_{n,m} = -i\frac{\lambda f_{n,m}^1}{f_{n,m}^2}(P_\infty^+),$$

one gets

$$\tilde{\psi}_{n+1,m} = (\Lambda - P_\infty)^{(n+1)/2} O(1), \qquad P \to P_\infty^+.$$

The uniqueness (due to Theorem 8) of the function with these analytic properties implies

$$\tilde{\psi}_{n+1,m} = \psi_{n+1,m}.$$

The second equation (11.10) is proved in the same way.

To get formulas (11.12) for the coefficients of the matrices $U_{n,m}, V_{n,m}$ it is more convenient to substitute the value of $\psi_{n,m}$ at $\Lambda = 0$ into (11.10). In order to calculate $\psi_{n,m}^{1,2}(\Lambda = 0)$ let us integrate $\int_\alpha dw$, $\int_\alpha d\Omega_{\infty,0}$ along some path α, which goes from $\Lambda = \infty$ to $\Lambda = 0$ along the cut $[\infty, 0]$ and does not cross the a-cycles: $\int_\alpha dw = \Delta$,

$$\psi_{n,m}^1(\Lambda = 0) = i^n \frac{\theta(\Omega_{n,m} + \Delta)\theta(D)}{\theta(D+\Delta)\theta(\Omega_{n,m})}(ig_\infty)^n (ig_0)^m,$$

(11.14)

$$\psi_{n,m}^2(\Lambda = 0) = i^n \frac{\theta(\Omega_{n,m})\theta(D)}{\theta(D+\Delta)\theta(\Omega_{n,m}+\Delta)}(ig_\infty)^n (ig_0)^m,$$

where we used the notation

$$(11.15)\qquad ig_\infty = \exp(\int_\alpha d\Omega_\infty), \qquad ig_0 = \exp(\int_\alpha d\Omega_0).$$

Because of the normalization (11.3) for a cycle $\mathcal{L} = a_1 + \ldots + a_g$ going around the cut $[\infty, 0]$ we have $\int_{\mathcal{L}} d\Omega_{\infty,0} = 0$. The cycle $\alpha - \tau_4\alpha$ is equivalent to $\mathcal{L}$, but crosses each of the contours ℓ_∞, ℓ_0 once. Therefore we have

$$\exp(\int_\alpha d\Omega_{\infty,0} - \int_{\tau_4\alpha} d\Omega_{\infty,0}) = -1.$$

On the other hand,

$$\tau_4^* d\Omega_{\infty,0} = -d\Omega_{\infty,0},$$

which implies

$$\exp(2\int_\alpha d\Omega_{\infty,0}) = -1,$$

and, finally, $g_\infty, g_0 = \pm 1$.

Substituting (11.14) into (11.10) yields

$$(11.16)\qquad \begin{aligned} a_{n,m} &= u\frac{\psi^1_{n+1,m}}{\psi^1_{n,m}}(\Lambda = 0), & b_{n,m} &= u\frac{\psi^2_{n+1,m}}{\psi^2_{n,m}}(\Lambda = 0), \\ c_{n,m} &= -i\lambda v\frac{\psi^1_{n,m+1}}{\psi^2_{n,m}}(\Lambda = 0), & d_{n,m} &= -i\lambda v\frac{\psi^2_{n,m+1}}{\psi^1_{n,m}}(\Lambda = 0). \end{aligned}$$

Taking into account

$$u(\Lambda = 0) = \cot\frac{\Delta^u}{2}\ \mathrm{sign}\, u(\Lambda = 0),$$
$$\lambda v(\Lambda = 0) = \tan\frac{\Delta^v}{2}\ \mathrm{sign}(\lambda v(\Lambda = 0)),$$

we get the formulas (11.13) with

$$(11.17)\qquad \varepsilon_\infty = g_\infty\ \mathrm{sign}\, u(\Lambda = 0), \quad \varepsilon_0 = g_0\ \mathrm{sign}(\lambda v(\Lambda = 0)).$$

For the contours ℓ_∞, ℓ_0 shown in Figure 13, $\mathrm{sign}\, u(\Lambda = 0) = \mathrm{sign}(\lambda v(\Lambda = 0)) = 1$ and consequently

$$\varepsilon_\infty = g_\infty, \qquad \varepsilon_0 = g_0.$$

Remark. The signs $\varepsilon_\infty, \varepsilon_0$ defined by (11.17) are independent of the choice of ℓ_∞ and ℓ_0, but depend on the labeling of the points $P^\pm_\infty$

and $P_0^{\pm}$ (we still have not fixed which of the two points $P_\infty^{\pm}$ is P_∞^{+} and which of $P_0^{\pm}$ is P_0^{+}). The exchange $P_\infty^{+} \leftrightarrow P_\infty^{-}$ or $P_0^{+} \leftrightarrow P_0^{-}$ implies $\varepsilon_\infty \leftrightarrow -\varepsilon_\infty$ or $\varepsilon_0 \leftrightarrow -\varepsilon_0$ respectively.

Corollary 3. *The function $h_{n,m}$ defined by*

$$e^{ih_{n,m}} = (\varepsilon_o)^m (\varepsilon_\infty)^n \frac{\theta(\Omega_{n,m} + \Delta)}{\theta(\Omega_{n,m})} \tag{11.18}$$

satisfies the Hirota equation (4.16).

Proof. The system (11.10) implies

$$(U_{n,m+1} V_{n,m} - V_{n+1,m} U_{n,m}) \psi_{n,m} = 0.$$

The matrix in this formula is independent of λ and has a λ-dependent vector $\psi_{n,m}$ in its kernel, therefore it must vanish identically:

$$U_{n,m+1} V_{n,m} - V_{n+1,m} U_{n,m} = 0. \tag{11.19}$$

The coefficients (11.11) of these matrices are of the form

$$\begin{aligned} a_{n,m} &= \cot \frac{\Delta^u}{2} X_{n,m}, & b_{n,m} &= \cot \frac{\Delta^u}{2} \frac{1}{X_{n,m}}, \\ c_{n,m} &= \tan \frac{\Delta^v}{2} Y_{n,m}, & d_{n,m} &= \tan \frac{\Delta^v}{2} \frac{1}{Y_{n,m}}. \end{aligned} \tag{11.20}$$

Substitution of these expressions into (11.19) yields

$$X_{n,m+1} Y_{n,m} = Y_{n+1,m} \frac{1}{X_{n,m}}, \tag{11.21}$$

$$X_{n,m+1} - X_{n,m} = \tan \frac{\Delta^u}{2} \tan \frac{\Delta^v}{2} \left(\frac{1}{Y_{n,m}} - Y_{n+1,m}\right). \tag{11.22}$$

Equation (11.21) can be easily solved:

$$X_{n,m} = e^{ih_{n+1,m} - ih_{n,m}}, \qquad Y_{n,m} = e^{ih_{n,m+1} + ih_{n,m}}, \tag{11.23}$$

with the usual ambiguity (5.3) in definition of $h_{n,m}$. Equation (11.22) becomes in this case the Hirota equation for $h_{n,m}$. Formula (11.18) presents a solution to system (11.23) with $X_{n,m}, Y_{n,m}$ defined by (11.20, 11.16, 11.14).

Remark. The exchange $P_\infty^+ \leftrightarrow P_\infty^-$ implies $\varepsilon_\infty \leftrightarrow -\varepsilon_\infty$ and $U \leftrightarrow -U$ in $\Omega_{n,m}$. In the same way $P_0^+ \leftrightarrow P_0^-$ implies $\varepsilon_0 \leftrightarrow -\varepsilon_0$, $V \leftrightarrow -V$. These transformations are equivalent to

$$\begin{aligned} i_\infty : \ \varepsilon_\infty \leftrightarrow -\varepsilon_\infty, \qquad & n \leftrightarrow -n; \\ i_0 : \ \varepsilon_0 \leftrightarrow -\varepsilon_0, \qquad & m \leftrightarrow -m, \end{aligned} \tag{11.24}$$

which preserve the Hirota equation.

12. Reality and formula for the angles

All the functions in Section 11 are complex-valued. Here we shall obtain restrictions on the parameters that ensure the reality of $h_{n,m}$ dictated by our geometrical problem. First of all, let us derive some properties of the spectral curve X in the periodic case. The matrices $U(\lambda), V(\lambda)$ and, as a corollary, also $T^H(\lambda)$ and $T^{SG}(\lambda)$ in Section 9 satisfy the reduction

$$\overline{T^{SG}(\lambda)} = \sigma_2 T^{SG}(\bar{\lambda})\sigma_2,$$

which implies the existence of an antiholomorphic involution

$$\tau_5 : \ (\mathcal{M}, \Lambda) \longrightarrow (\bar{\mathcal{M}}, \bar{\Lambda})$$

of the spectral curve (9.5). The set of the branch points $\Lambda_j, j = 1, \dots, 2g$ of this curve is symmetric with respect to the conjugation $\Lambda \to \bar{\Lambda}$. Moreover, one can easily see the absence of positive branch points $\Lambda_j > 0$. Indeed, for positive Λ (or, equivalently, real λ) the matrices U, V, T^{SG} are quaternions

$$T^{SG}(\lambda) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad \lambda \in \mathbf{R}.$$

This point $\lambda \in \mathbf{R}$ cannot be a branch point of X, since $\mathcal{M}$ in (9.5) does not vanish:

$$\mathcal{M}^2 = \frac{(a-\bar{a})^2}{4} - b\bar{b} < 0, \qquad \Lambda > 0.$$

Now let us consider the general finite-gap case. Let X be a hyperelliptic Riemann surface with branch points subdivided into two families:

a) Λ_j, $1 \le j \le 2k$, are negative and ordered in the following way

$$\Lambda_1 < \Lambda_2 < \dots < \Lambda_{2k} < 0,$$

b) Λ_{2k+i}, $i = 1, \dots, 2(g-k)$, are complex and

$$\Lambda_{2k+2j} = \bar{\Lambda}_{2k+2j-1}.$$

The integer k may be chosen arbitrary between $k = 0$ and $k = g$. In the case $k = 0$ ($k = g$) the branch points of the type a) (type b)) are absent.

On such a surface we can always construct a canonical basis of cycles (see Figure 14 for an example of such a basis), which transforms under τ_5 as follows:

$$\begin{aligned} &\tau_5 a_n = -a_n, \qquad n = 1, \dots, g, \\ &\tau_5 b_n = b_n, \qquad n = 1, \dots, k, \\ &\tau_5 a_n = b_n - a_n, \qquad n = k+1, \dots, g. \end{aligned} \tag{12.1}$$

Note that the involution does not change the sheets of X. This implies the following symmetries of the normalized differentials and of the period matrix:

$$\tau_5^* dw_n = \overline{dw_n}, \qquad n = 1, \dots, g$$

$$\begin{aligned} B_{n,m} &= \int_{\tau_5 b_n} \tau_5^* dw_m = \overline{\int_{b_n} dw_m} = \overline{B_{n,m}}, \qquad n \le k, \\ B_{n,m} &= \int_{\tau_5 b_n} \tau_5^* dw_m = \overline{\int_{b_n - a_n} dw_m} = \overline{B_{n,m}} + 2\pi i \delta_{n,m}, \qquad n > k. \end{aligned}$$

The theta function defined by such a B-matrix

$$B = B_R + J, \quad B_R \in Mat(g, \mathbf{R}), \quad J = \pi i \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & \ddots \\ 0 & & & & & 1 \end{pmatrix} \tag{12.2}$$

is conjugated as follows:

$$\overline{\theta(z)} = \theta(\bar{z} + \Delta_0), \quad \Delta_0 = \ \mathrm{diag} J = \pi i (0, \dots, 0, 1, \dots, 1) \tag{12.3}$$

(zeroes at the first k places). In the chosen basis the vectors U and V are real ($\tau_5\ell_\infty = \ell_\infty, \tau_5\ell_0 = \ell_0$). Using the representation (11.5), we get

$$U_n = \int_{\tau_5 P_\infty^-}^{\tau_5 P_\infty^+} \tau_5^* dw_n = \overline{\int_{P_\infty^-}^{P_\infty^+} dw_n} = \bar{U}_n, \qquad V_n = \bar{V}_n. \tag{12.4}$$

Now we are in a position to consider the reality conditions for $h_{n,m}$. The real-valuedness of $h_{n,m}$ given by formula (11.18) is equivalent to

$$\left|\frac{\theta(Un+Vm+D+\Delta)}{\theta(Un+Vm+D)}\right| = 1.$$

Using (12.3,12.4), this relation can be rewritten in the form

$$\frac{\theta(Un+Vm+D+\Delta)\theta(Un+Vm+\bar{D}+\Delta+\Delta_0)}{\theta(Un+Vm+D)\theta(Un+Vm+\bar{D}+\Delta_0)} = 1$$

and leads to the following restriction on the structure of the vector D:

$$\begin{aligned} &D = D_R + \tfrac{\Delta_1}{2} + \pi i N, \\ &\Delta_1 = \pi i(1,\dots,1,0,\dots,0), \quad N \in \mathbf{Z}^g, \end{aligned} \tag{12.5}$$

(zeros at the last $g-k$ places in Δ_1), where $D_R \in \mathbf{R}^g$.

Theorem 11. *Let X be a hyperelliptic Riemann surface with branch points $\Lambda = 0, \infty$, an antiholomorphic involution $\tau : \Lambda \to \bar{\Lambda}$ and a canonical basis of cycles, which is transformed by τ as indicated in* (12.1). *Then formula* (11.18) *describes real finite-gap solutions to the Hirota equation if the imaginary part of the vector D is of the form* (12.5). *The whole variety of real finite-gap solutions to the Hirota equation corresponding to the fixed spectral curve X, after factorizing by the symmetries* (11.24, 2.18) *of the equation, is subdivided into 2^k connected components fixed by the different possible choices of the vector D and labeled by the $\mathbf{Z}_2^k$-valued vector K in*

$$D = D_R + \frac{\Delta_1}{2} + \pi i(K, 0, \dots, 0), \tag{12.6}$$

where $K = (K_1, \dots, K_k) \in \mathbf{Z}_2^k$ comprises the first k components of the last vector in (12.6).

All periodic real solutions to the Hirota equation, corresponding to non-singular spectral curves, belong to the set of real finite-gap solutions described above.

The angles of the discrete K-surface, corresponding to the finite-gap solution (11.18), are as follows:

$$
\begin{aligned}
&e^{i\varphi^{(1)}_{n,m}} \\
&\quad = -\varepsilon_\infty\varepsilon_0 \frac{\theta(U(n+1)+Vm+D)\theta(Un+V(m+1)+D)}{\theta(U(n+1)+Vm+D+\Delta)\theta(Un+V(m+1)+D+\Delta)},\\
&e^{i\varphi^{(2)}_{n,m}} \\
&\quad = \varepsilon_\infty\varepsilon_0 \frac{\theta(Un+V(m+1)+D+\Delta)\theta(U(n-1)+Vm+D+\Delta)}{\theta(Un+V(m+1)+D)\theta(U(n-1)+Vm+D)},\\
&e^{i\varphi^{(3)}_{n,m}} \\
&\quad = -\varepsilon_\infty\varepsilon_0 \frac{\theta(U(n-1)+Vm+D)\theta(Un+V(m-1)+D)}{\theta(U(n-1)+Vm+D+\Delta)\theta(Un+V(m-1)+D+\Delta)},\\
&e^{i\varphi^{(4)}_{n,m}} \\
&\quad = \varepsilon_\infty\varepsilon_0 \frac{\theta(U(n+1)+Vm+D+\Delta)\theta(Un+V(m-1)+D+\Delta)}{\theta(U(n+1)+Vm+D)\theta(Un+V(m-1)+D)}.
\end{aligned}
\tag{12.7}
$$

Proof. To reduce (12.5) to the form (12.6) let us note that the solution (11.18) is invariant [7] with respect to translations of D by lattice vectors of the Jacobian:

$$D \to D + 2\pi i N + BM, \qquad N, M \in \mathbf{Z}^g.$$

Since the imaginary part of the period matrix is of the form (12.2), one can choose $M \in \mathbf{Z}^g$ to reduce (12.5) to (12.6).

In §10 considering the discrete K-surfaces with both period numbers (N, M) odd, we showed that in this case the Hirota field can be chosen periodic, and investigated the properties of the spectral curve and the BA function in this case. One can easily see that this consideration works for any N, M if the Hirota field is periodic.

Formulas (12.7) follow from (6.1).

[7] If $\sum M_i$ is odd then one has to replace e^{ih} by $-e^{ih}$, but this transformation preserves the class (4.18).

13. Formula for immersion in terms of theta functions

Taking into account formula (5.5), written in terms of the matrix valued solution $\overset{\circ}{\Psi}$ of the linear system (11.10), it is not suprising that with the help of the BA function one can not only describe the angles (12.7) of the discrete K-surface, but also derive an explicit formula for the corresponding immersion $F : \mathbf{Z}^2 \to \mathbf{R}^3$.

Indeed, the functions $\psi(P)$ and $\psi(\tau_4 P)$ correspond to the same [8] λ, u, v and therefore solve the same linear system (11.10). Combined together

$$\Psi(P) = i(\psi(P), \psi(\tau_4 P)),$$

these two functions comprise a matrix valued function

$$\Psi(P) = \begin{pmatrix} i\dfrac{\theta(\Omega + \int_\infty^P dw)}{\theta(\Omega)} & i\dfrac{\theta(\Omega - \int_\infty^P dw)}{\theta(\Omega)} \\ i\dfrac{\theta(\Omega + \Delta + \int_\infty^P dw)}{\theta(\Omega+\Delta)} & -i\dfrac{\theta(\Omega + \Delta - \int_\infty^P dw)}{\theta(\Omega+\Delta)} \end{pmatrix}$$

(13.1)

$$\times \begin{pmatrix} \alpha^n \beta^m \dfrac{\theta(D)}{\theta(D + \int_\infty^P dw)} & 0 \\ 0 & \alpha^{-n}\beta^{-m}\dfrac{\theta(D)}{\theta(D - \int_\infty^P dw)} \end{pmatrix},$$

where

$$\alpha = i e^{\int_\infty^P d\Omega_\infty}, \qquad \beta = e^{\int_\infty^P d\Omega_0} \tag{13.2}$$

and the integration path $[\infty, P]$ does note intersect the contours $\mathcal{L}, \ell_\infty$, ℓ_0. This formula needs clarification: the hyperelliptic involution π of X corresponds to the combination $\tau_1\tau_4$ of $\hat{X}$. We have chosen the branch of λ on X so that its sign is changed by π. Therefore, due to the symmetry (10.3), τ_4 is equivalent to the combination of π and $\psi \to \sigma_3\psi$. To get (13.1) one should also use the symmetries

$$\int_\infty^{\pi P} dw = -\int_\infty^P dw, \quad e^{\int_\infty^{\pi P} d\Omega_0} = e^{-\int_\infty^P d\Omega_0}, \quad e^{\int_\infty^{\pi P} d\Omega_\infty} = -e^{-\int_\infty^P d\Omega_\infty}.$$

[8] Recall that the involution τ_4 of $\hat{X}$ only changes the sign of $\mathcal{M}$ and preserves λ, u, v.

The last equality follows from the fact that one of two paths $[\infty, P]$, $[\infty, \pi P]$ crosses ℓ_∞.

Starting with the solution (13.1) one can easily get a desirable solution $\overset{\circ}{\Psi}_{n,m}(\lambda) \in SU(2)$ to be substituted into formula (5.5). The addition formula for the theta function

$$\frac{\theta(\int_\infty^P dw + \Omega)\theta(\int_\infty^P dw - \Omega + \Delta) + \theta(\int_\infty^P dw - \Omega)\theta(\int_\infty^P dw + \Omega + \Delta)}{\theta(\int_\infty^P dw)\theta(\int_\infty^P dw + \Delta)} = 2\frac{\theta(\Omega)\theta(\Omega+\Delta)}{\theta(0)\theta(\Delta)},$$

which can be found, for example, in [4] (or proved directly by analysing analytical properties of the left-hand side) allows us to calculate the determinant of the first matrix factor in (13.2). The second matrix in (13.1) can be simplified by multiplication with a factor independent of n and m.

Now let us consider the points $P = (\Lambda, \mu)$ of X with real positive Λ-coordinates. These points correspond to the associated family of the discrete K-surfaces, described in §5. Since the contour $\mathcal{L}$ is separated from the cut $[\infty, 0]$, all the points near $[\infty, 0]$ can be parametrized by $\lambda \in \mathbf{R}$, and in the next two theorems we prefer to use this notation for the point P.

Theorem 12. *The function $\overset{\circ}{\Psi}_{n,m}(\lambda) \in SU(2)$ given by the formula*

$$\overset{\circ}{\Psi}_{n,m}(\lambda) = \frac{1}{\sqrt{d}} \begin{pmatrix} \overset{\circ}{A} & \overset{\circ}{B} \\ \overset{\circ}{C} & \overset{\circ}{D} \end{pmatrix} \begin{pmatrix} \alpha^n\beta^m & 0 \\ 0 & \alpha^{-n}\beta^{-m} \end{pmatrix},$$

$$\overset{\circ}{A} = i\frac{\theta(\Omega + \int_\infty^\lambda dw)}{\theta(\Omega)}, \qquad \overset{\circ}{B} = i\frac{\theta(\Omega - \int_\infty^\lambda dw)}{\theta(\Omega)},$$

(13.3)

$$\overset{\circ}{C} = i\frac{\theta(\Omega + \Delta + \int_\infty^\lambda dw)}{\theta(\Omega+\Delta)}, \qquad \overset{\circ}{D} = -i\frac{\theta(\Omega + \Delta - \int_\infty^\lambda dw)}{\theta(\Omega+\Delta)},$$

$$d = \overset{\circ}{A}\overset{\circ}{D} - \overset{\circ}{B}\overset{\circ}{C} = \frac{2\theta(\int_\infty^\lambda dw)\theta(\int_\infty^\lambda dw + \Delta)}{\theta(0)\theta(\Delta)},$$

is a solution of the system (5.6) *with the coefficients* (11.18). *Here the point λ lies on the uper edge of the cut $[\infty, 0], \lambda > 0$, the integration* *path goes along the upper edge of* $[\infty, 0]$, α *and* β *are defined by* (13.2). *All* λ, u, v *in* (13.3, 5.6) *are positive.*

Proof. The only statement to be proved is that $\overset{\circ}{\Psi}_{n,m}(\lambda)$ defined by (13.3) belongs to $SU(2)$. All differentials are invariant with respect to τ_5, i.e.,

$$\tau_5 dw = \overline{dw}, \qquad \tau_5 d\Omega_{\infty,0} = \overline{d\Omega_{\infty,0}}.$$

For $\Lambda > 0$ we have

$$\overline{\int_\infty^\lambda dw} = \int_\infty^\lambda \tau_5^* dw = \int_\infty^{\tau_5\lambda} dw = -\int_\infty^\lambda dw,$$
$$\overline{\int_\infty^\lambda d\Omega_\infty} = -\int_\infty^\lambda d\Omega_\infty, \qquad \int_\infty^\lambda d\Omega_0 = -\int_\infty^\lambda d\Omega_0,$$

which imply

$$|\alpha| = |\beta| = 1, \qquad \overset{\circ}{A} = \overline{\overset{\circ}{D}}, \qquad \overset{\circ}{B} = -\overline{\overset{\circ}{C}}.$$

To prove the theta functional identities we use (12.3), (12.5) to obtain

$$\overline{\theta(\int_\infty^\lambda dw + \Omega_{n,m})} = \theta(\int_\infty^\lambda \overline{dw} + Un + Vm + \bar{D} + \Delta_0)$$
$$= \theta(-\int_\infty^\lambda dw + \Omega_{n,m} + \Delta).$$

Theorem 13. *The coordinates of the discrete* K*-surface (the discrete anisotropic Chebyshev net) and its Gauss map, corresponding to the finite-gap solution* (11.18) *of the Hirota equation (where* $\Delta_n^u = \Delta^u, \Delta_m^v = \Delta^v$*) are given by the formulas*

$$\begin{aligned} F_1 + iF_2 &= 4\Lambda i r_{n,m} \frac{AC_\Lambda - CA_\Lambda}{AD - BC}, \\ F_3 &= \frac{2\Lambda i}{AD - BC}(DA_\Lambda - AD_\Lambda + CB_\Lambda - BC_\Lambda) \\ &\quad + 4\Lambda i R_{n,m}, \end{aligned} \tag{13.4}$$

$$\begin{aligned} N_1 + iN_2 &= -r_{n,m}\frac{2AC}{AD - BC}, \\ N_3 &= \frac{AD + BC}{AD - BC}, \end{aligned} \tag{13.5}$$

where

$$\begin{aligned}
&A = \theta(\textstyle\int_\infty^\lambda dw + \Omega_{n,m}), \qquad B = \theta(\textstyle\int_\infty^\lambda dw - \Omega_{n,m}),\\
&C = \theta(\textstyle\int_\infty^\lambda dw + \Omega_{n,m} + \Delta), D = -\theta(\textstyle\int_\infty^\lambda dw - \Omega_{n,m} + \Delta),
\end{aligned}$$

$$\begin{aligned}
R_{n,m} &= n\frac{d\Omega_\infty}{d\Lambda} + m\frac{d\Omega_0}{d\Lambda},\\
r_{n,m} &= \alpha^{2n}\beta^{2m}, \qquad \alpha = ie^{\int_\infty^P d\Omega_\infty}, \qquad \beta = e^{\int_\infty^P d\Omega_0},\\
\Omega_{n,m} &= Un + Vm + D.
\end{aligned}$$

The lower indices Λ in (13.4) *denote the partial derivatives with respect to Λ*

$$\frac{\partial}{\partial\Lambda}\theta(\int_\infty^\lambda dw + R) = \sum_{i=1}^g \frac{\partial}{\partial z_i}\theta(\int_\infty^\lambda dw + R)\frac{dw_i}{d\Lambda},$$

and $\frac{dw_i}{d\Lambda}, \frac{d\Omega_\infty}{d\Lambda}, \frac{d\Omega_0}{d\Lambda}$ denote the values of the corresponding differentials at the point λ. The determinant $AD - BC$ can be calculated also as

$$AD - BC = -2\frac{\theta(\Omega)\theta(\Omega+\Delta)}{\theta(0)\theta(\Delta)}\theta(\int_\infty^P dw)\theta(\int_\infty^P dw + \Delta).$$

All the discrete K-surfaces with the periodic Gauss map and with both odd coordinates N, M of the period, are described by these formulas.

Proof. To derive formulas (13.4, 13.5) let us recall the Remark at the end of §5 and use a non-normalized function

$$\Psi_{n,m} = \begin{pmatrix} \mathring{A} & \mathring{B} \\ \mathring{C} & \mathring{D} \end{pmatrix} \begin{pmatrix} \alpha^n\beta^m & 0 \\ 0 & \alpha^{-n}\beta^{-m} \end{pmatrix}.$$

Then Theorem 3 implies the following formulas for the immersion in the quaternionic representation

$$F = 4\Lambda[\Psi^{-1}\Psi_\Lambda]^{\,\mathrm{tr}=0}, \qquad N = -i\Psi^{-1}\sigma_3\Psi, \qquad \Lambda = e^{2t}.$$

Substituting (13.3) into these formulas, rewriting them coordinatewise and canceling some factors, we finally get (13.4), (13.5).

It is very easy to get rid of the conditions $\Delta_n^u = \Delta^u, \Delta_m^v = \Delta^v$ and to generalize Theorem 13 to discrete weak Chebyshev nets. This generalization is used in §15 to describe compact examples.

Let $d\Omega_\infty^n, d\Omega_0^m$, $n, m \in \mathbf{Z}$ be normalized Abelian differentials of the third kind with the singularities

$$d\Omega_\infty = \pm\frac{1}{2}\frac{d\Lambda}{\Lambda - P_\infty^n} + O(1), \qquad P \to P_\infty^{n\,\pm},$$
$$d\Omega_0 = \pm\frac{1}{2}\frac{d\Lambda}{\Lambda - P_0^m} + O(1), \qquad P \to P_0^{m\,\pm},$$

at the points

$$P_\infty^n = -\cot\frac{\Delta_n^u}{2}, P_0^m = -\tan\frac{\Delta_m^v}{2},$$

and

$$U^n = \int_b d\Omega_\infty^n, \qquad V^m = \int_b d\Omega_0^m \tag{13.6}$$

be their period vectors ($U^n, V^m \in \mathbf{R}^g$ in the basis shown in Figure 14).

Theorem 14. *The coordinates of the discrete weak Chebyshev net (see §§3–6) and its Gauss map, corresponding to the finite-gap solution* (11.18) *of the Hirota equation are given by the formulas* (13.4), (13.5), *where A, B, C, D are as in Theorem 13 and*

$$R_{n,m} = \sum_{k=1}^{n}\frac{d\Omega_\infty^k}{d\Lambda} + \sum_{l=1}^{m}\frac{d\Omega_0^l}{d\Lambda},$$
$$r_{n,m} = \prod_{k=1}^{n}\alpha_k^2\prod_{l=1}^{m}\beta_l^2, \qquad \alpha_k = ie^{\int_\infty^P d\Omega_\infty^k}, \qquad \beta^l = e^{\int_\infty^P d\Omega_0^l}, \tag{13.7}$$
$$\Omega_{n,m} = \sum_{k=1}^{n}U^k + \sum_{l=1}^{m}V^l + D.$$

All discrete weak Chebyshev nets with the periodic Gauss map with the period (N, M), where both N, M are odd, are described by these formulas.

14. Periodicity and simplest examples

In general the immersion (13.4) is not periodic.

Proposition 8. *The discrete anisotropic Chebyshev net $F_{n,m}(\lambda_0)$ described in Theorem 13 with $\lambda = \lambda_0$ possesses the following periodicity properties with the period $n, m \to n + N, m + M$:*

1) *The angles $\phi_{n,m}^{(i)}$ between the edges of this surface are periodic if and only if*

$$UN + VM = B_R L, \tag{14.1}$$

where B_R is the real part of the period matrix, $L \in \boldsymbol{Z}^g$ is integer and its $g-k$ last coordinates $L_{k+1}, \dots, L_g$ are even; or equivalently, in a more invariant way, there exists a differential $d\Omega_{N,M}$ of the third kind with the singularities

$$d\Omega_{N,M} = \pm\frac{N}{2}\frac{d\Lambda}{\Lambda - P_\infty} + O(1), \qquad P \to P_\infty^\pm,$$
$$d\Omega_{N,M} = \pm\frac{M}{2}\frac{d\Lambda}{\Lambda - P_0} + O(1), \qquad P \to P_0^\pm,$$

all the periods of which are proportional to $2\pi i$

$$\frac{1}{2\pi i}\int_\alpha d\Omega_{N,M} \in \boldsymbol{Z}, \qquad \forall \alpha \in H_1(X, \boldsymbol{Z}). \tag{14.2}$$

2) *The Gauss map $N_{n,m}(\lambda_0)$ is periodic if and only if, in addition, the equality*

$$(-1)^N \exp\{2\int_\infty^{\lambda_0} d\Omega_{N,M}\} = 1 \tag{14.3}$$

or, equivalently,

$$(-1)^M \exp\{2\int_0^{\lambda_0} d\Omega_{N,M}\} = 1$$

holds.

3) *The immersion $F_{n,m}(\lambda_0)$ is periodic if and only if, in addition to the conditions stated above under 1), 2), the differential $d\Omega_{N,M}$ vanishes at the point λ_0*

$$d\Omega_{N,M}(\lambda_0) = 0. \tag{14.4}$$

Proof. Under the transformation $n, m \to n+N, m+M$ the arguments of all the theta functions in (12.7) are shifted by $UN + VM$. For the angles $\phi^{(i)}_{n,m}$ to be periodic it is necessary that this shift belongs to the lattice of the Jacobian

$$UN + VM = 2\pi i K + BL, \quad K, L \in \mathbf{Z}^g.$$

Taking into account the reality properties of U, V, B we get (14.1). If (14.1) holds, then the differential

$$Nd\Omega_\infty + Md\Omega_0 - < L, dw >$$

possesses all the properties of $d\Omega_{N,M}$ listed in 1) and vice versa: the differential $d\Omega_{N,M} - Nd\Omega_\infty - Md\Omega_0$ is holomorphic and all its a-periods are multiples of $2\pi i$, and therefore it equals $< L, dw >, L \in \mathbf{Z}^g$. The comparison of the imaginary parts of the periods $b_l, l > k$ yields the parity of $L_l, l > k$.

The coordinate N_3 of the Gauss map is always periodic if (14.1) holds. For the coordinates $N_1 + iN_2$, using the periodicity properties (11.2), we get

$$\frac{AC}{AD - BC}(\Omega + BL) = \frac{AC}{AD - BC}(\Omega) \exp(-2 < \int_\infty^{\lambda_0} dw, L >).$$

Multiplying by $\alpha^{2N}\beta^{2M}$ leads to

$$(i \exp(\int_\infty^{\lambda_0} d\Omega_\infty))^{2N}(\exp(\int_\infty^{\lambda_0} d\Omega_0))^{2M} \exp(-2 < \int_\infty^{\lambda_0} dw, L >),$$

which gives the factor (14.3). Under the conditions 1), 2) $F_1 + iF_2$ is periodic. Using the differentiated equality (11.2)

$$\begin{aligned}\frac{\partial}{\partial\Lambda}\theta(\int_\infty^\lambda dw + R + BL) &= \exp(-\frac{1}{2} < BL, L > - < \int dw + R, L >)\\ &\times(\frac{\partial}{\partial\Lambda}\theta(\int_\infty^\lambda dw + R) - < \frac{dw}{d\Lambda}, L > \theta(\int_\infty^\lambda dw + R)),\end{aligned}$$

one can get

$$\frac{DA_\Lambda - AD_\Lambda + CB_\Lambda - BC_\Lambda}{AD - BC}\Big|_\Omega^{\Omega + BL} = -2 < \frac{dw}{d\Lambda}, L > .$$

Then the periodicity of F_3 implies (14.4).

Let us consider now the discrete K-surfaces generated by the spectral curves of genus $g = 1$. To make the surface especially symmetric we set $\Delta_n^u = \Delta_m^v = \Delta$ (a discrete Chebyshev net) and suppose that the spectral curve X possesses one more involution

$$\tau_6 \ : \Lambda \to 1/\Lambda. \qquad (14.5)$$

We distinguish τ_6 from $\pi\tau_6$ by the condition that τ_6 has 4 fixed points $\Lambda = \pm 1$ on X. This involution acts on the chosen basis of cycles as follows:

$$\tau_6 a = -a, \qquad \tau_6 b = -b.$$

In the symmetric case of the discrete Chebyshev net

$$P_\infty = 1/P_0,$$

the involution τ_6 interchanges the differentials $d\Omega_\infty$ and $d\Omega_0$:

$$\tau_6^* d\Omega_\infty = \varepsilon d\Omega_0, \qquad \varepsilon = \pm 1,$$

where the sign ε depends on the labeling of the singularities $P_\infty^\pm, P_0^\pm$. For the b-periods we get

$$U = -\varepsilon V,$$

which shows that the theta functions in formulas (13.4, 13.5) depend only on the combination

$$U(n - \varepsilon m). \tag{14.6}$$

Let us take a "basic" surface of the associated family with $t = 0$, or equivalently, $\lambda = 1$. Then $R_{n,m}$ in formula (13.4) depends only also on $n - \varepsilon m$:

$$R_{n,m} = \frac{d\Omega_\infty}{d\Lambda}(\lambda = 1)(n - \varepsilon m). \tag{14.7}$$

Here we have used the symmetry

$$\frac{d\Omega_\infty}{d\Lambda}(\lambda = 1) = \frac{\tau_6^* d\Omega_\infty}{\tau_6^* d\Lambda}(\tau_6 \lambda = 1) = -\varepsilon \frac{d\Omega_0}{d\Lambda}(\lambda = 1).$$

Since $\lambda = 1$ is a fixed point of τ_6, we have

$$\beta = \exp(\int_\infty^{\lambda=1} d\Omega_0) = \exp(\varepsilon \int_0^{\lambda=1} d\Omega_\infty) = (-\alpha/\varepsilon_\infty)^\varepsilon,$$

which shows that $r_{n,m}$ is a function of the combination $n + \varepsilon m$:

$$r_{n,m} = \alpha^{2n}\beta^{2m} = \alpha^{2(n+\varepsilon m)}.$$

We set $\varepsilon = 1$ (the opposite sign $\varepsilon = -1$ is equivalent to the replacement $m \to -m$).

Since the transformation

$$n, m \to n + K_1, m + K_1 \tag{14.8}$$

does not effect the arguments (14.6) of the theta functions and the linear term (14.7) in F_3, the surface under consideration is periodic with the period (14.8) if $\alpha^{4K_1} = 1$, i.e.,

$$\exp(4K_1 \int_0^{\lambda=1} d\Omega_\infty) = 1, \qquad K_1 \in \mathbf{Z}. \tag{14.9}$$

The Gauss map of the surface possesses the second period

$$n, m \to n + K_2, m - K_2, \qquad K_2 \in \mathbf{Z} \tag{14.10}$$

if

$$K_2 U / B_R \in \mathbf{Z} \tag{14.11}$$

in the case of a "vertical" cut ($g = 1, k = 0, \Lambda_2 = \bar{\Lambda}_1, |\Lambda_{1,2}| = 1$, the period $B = B_R + \pi i, B_R \in \mathbf{R}$). In the case of a "horizontal" cut ($g = 1, k = 1, \Lambda_1 < 0, \Lambda_2 = 1/\Lambda_1$, the period $B = B_R$ is real) the periodicity (14.10) is equivalent to

$$2K_2 U / B_R \in \mathbf{Z}. \tag{14.12}$$

Since the differential

$$d\Omega_{K_2, -K_2} = K_2 (d\Omega_\infty - d\Omega_0) + L d\omega$$

responsible for the translational periodicity (14.4) never vanishes at $\lambda_0 = 1$, the period (14.10) is a translation in $\mathbf{R}^3$. The surface is not compact and looks like a cylinder. The equalities (14.11, 14.12) are equivalent to (14.1).

The periodicity conditions (14.9) and (14.10) or (14.12) for the Gauss map are two conditions on two real parameters: the singularity $P_\infty < 0$ and the branch point Λ_1, which is either real negative ("horizontal" cut) or lies on the unit circle ("vertical" cut). Under these conditions one gets a discrete set of P_∞, Λ_1. As usual, there are no periodicity conditions on the vector $D \in \mathbf{R}$ on the Jacobian, which plays the same role as the parameter S in Section 8. Finally, we have one-parametric families of discrete K-surfaces, parametrized by $D \in \mathbf{R}$.

Figures 15 and 16 present examples of the discrete K-surfaces with "vertical" and "horizontal" cuts. When this cut shrinks to a point, the translational period of the surface becomes infinite and we get the discrete pseudospheres of Section 8.

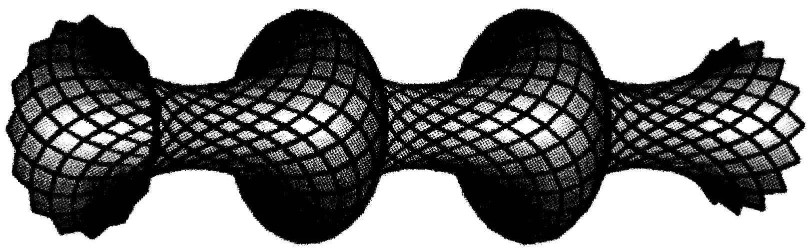


FIGURE 15. The discrete K-surface with a spectral curve of genus $g = 1$ and a "vertical" cut $\Lambda_2 = \bar{\Lambda}_1$

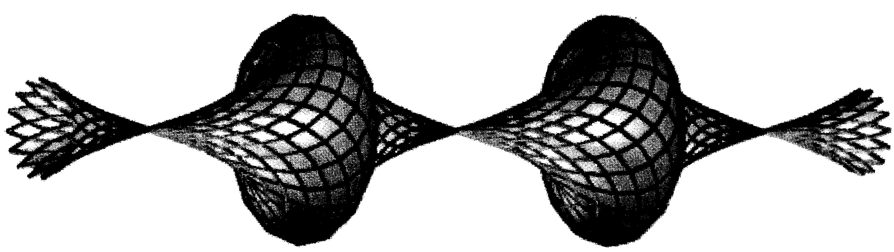


FIGURE 16. The discrete K-surface with a spectral curve of genus $g = 1$ and a "horizontal" cut $\Lambda_1 < \Lambda_2 < 0$

15. Compact examples

For the general case of a weak Chebyshev net Proposition 8 reads as follows:

Proposition 9. *The discrete weak Chebyshev net $F_{n,m}(\lambda_0)$ described in Theorem 14 with $\lambda = \lambda_0$ is periodic with the period $n, m \to n+N, m+M$ if and only if there exists a differential $d\Omega_{N,M}$ of the third kind with the singularities*

$$d\Omega_{N,M} = \pm\frac{1}{2}\frac{d\Lambda}{\Lambda - P_\infty^n} + O(1), \qquad P \to P_\infty^{n\ \pm}, \qquad n = 1, \dots, N,$$
$$d\Omega_{N,M} = \pm\frac{1}{2}\frac{d\Lambda}{\Lambda - P_0^m} + O(1), \qquad P \to P_0^{m\ \pm}, \qquad m = 1, \dots, M,$$

possessing the properties (14.2), (14.3), (14.4). *This differential equals*

$$d\Omega_{N,M} = \sum_{n=1}^{N} d\Omega_\infty^n + \sum_{m=1}^{M} d\Omega_0^m - < L, dw >, \tag{15.1}$$

where $L \in \mathbf{Z}^g$ is integer, and its $g - k$ last components are even. This implies in particular the period equality (*the periodicity condition for the angles*)

$$\sum_{n=1}^{N} U^n + \sum_{m=1}^{M} V^m = B_R L. \tag{15.2}$$

If there exist two differentials $d\Omega_{N_1,M_1}$ and $d\Omega_{N_2,M_2}$ with independent pairs $(N_1, M_1), (N_2, M_2)$, possessing all the properties (14.2, 14.3, 14.4), then the map $F_{n,m}$ is doubly periodic, i.e.,

$$F_{n+N_1,m+M_1} = F_{n+N_2,m+M_2} = F_{n,m},$$

and the corresponding discrete K-surface is compact.

To describe the simplest compact discrete K-surfaces let us consider a spectral curve X of genus 2 with an additional involution

$$\tau_6 : \Lambda \to 1/\Lambda, \tag{15.3}$$

which has 2 fixed points lying over $\Lambda = 1$. We assume that the branch-points

$$\Lambda_1, \ \Lambda_2, \ \Lambda_3 = 1/\Lambda_2, \ \Lambda_4 = 1/\Lambda_1 \tag{15.4}$$

of X are complex conjugate $\Lambda_2 = \bar{\Lambda}_1$ (see Figure 17).

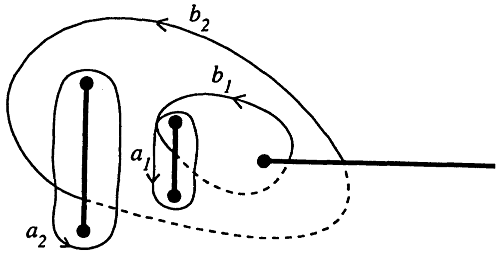


FIGURE 17. The symmetrical spectral curve of genus $g = 2$ with the basis of cycles.

Let us assume also that the edges of the discrete K-surface under consideration are of K different lengths, or, more precisely, that there are K different singularities $P^k_\infty, k = 1, \dots K$ and $P^k_0, k = 1, \dots K$, which are pairwise symmetric:

$$P^k_0 = 1/P^k_\infty. \tag{15.5}$$

The involution τ_6 acts on the canonical basis indicated in Figure 17 as follows:

$$\tau_6 a_1 = -a_2, \tau_6 a_2 = -a_1, \tau_6 b_1 = -b_2, \tau_6 b_2 = -b_1, \tag{15.6}$$

which implies

$$\tau_6^* d\omega_1 = -d\omega_2$$

for the normalized holomorphic differentials. For the differentials $d\Omega^k_{\infty,0}$ due to the symmetry (15.5), one gets $\tau_6^* d\Omega^k_\infty = \epsilon d\Omega^k_0$, where the sign $\epsilon = \pm 1$ depends on the "$\pm$" labeling of the singularities $P^{k,\pm}_{\infty,0}$. Let us fix it by setting

$$\tau_6^* d\Omega^k_\infty = d\Omega^k_0. \tag{15.7}$$

To get a doubly-periodic discrete K-surface one needs two differentials $d\Omega_{N_1,M_1}, d\Omega_{N_2,M_2}$ satisfying the periodicity properties. In the case of the symmetry (15.3) the spectral curve X covers two curves X_+, X_-

both of genus 1, defined as factors of X by the involutions τ_6 and $\tau_6\tau_4$ respectively

$$X \to X_+ = X/\tau_6, \quad X \to X_- = X/\tau_6\tau_4.$$

Here τ_4 is the hyperelliptic involution of §10. Therefore it is natural to look for the differentials

$$\begin{aligned} d\Omega_+ &\equiv d\Omega_{K_1,K_1}, \\ d\Omega_- &\equiv d\Omega_{K_2,-K_2}, \end{aligned}$$

which are symmetric and antisymmetric, respectively, with respect to τ_6 :

$$\tau_6^* d\Omega_\pm = \pm d\Omega_\pm. \tag{15.8}$$

In other words, they can be projected to X_+ and X_- respectively. In this case K_1 and K_2 are multiples of some $K \in \mathbf{N}$,

$$K_1 = NK, \quad K_2 = MK,$$

and the differentials $d\Omega_\pm$ are described by the following formulas

$$\begin{aligned} d\Omega_+ &= N\sum_{k=1}^{K}(d\Omega_\infty^k + d\Omega_0^k) - L_1(d\omega_1 - d\omega_2), \\ d\Omega_- &= M\sum_{k=1}^{K}(d\Omega_\infty^k - d\Omega_0^k) - L_2(d\omega_1 + d\omega_2). \end{aligned}$$

The period matrix of the curve with the involution (15.3) is also symmetric :

$$B = B_R + \pi i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_R = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad B_R \in Mat(2, \mathbf{R}). \tag{15.9}$$

Let us consider the "basic" discrete K-surface of the associated family $\lambda = 1$. Then the number of the periodicity conditions to be satisfied is reduced by factor of 2 due to the symmetries of the periods and of the values of the differentials and the integrals at $\lambda = 1$.

First, let us consider the intrinsic periodicity (14.1). Formulas (15.6, 15.7) imply that the $b-$periods (13.6) of $d\Omega^k_\infty$ and $d\Omega^k_0$ are related as follows:

$$U^k = (U_1^k, U_2^k),\ V^k = (V_1^k, V_2^k) = (-U_2^k, -U_1^k).$$

For the b_1-periods of $d\Omega_\pm$ this gives

$$\int_{b_1} d\Omega_+ = N\sum_{k=1}^{K}(U_1^k - U_2^k) - L_1(a - b + \pi i),$$

$$\int_{b_1} d\Omega_- = M\sum_{k=1}^{K}(U_1^k + U_2^k) - L_2(a + b + \pi i).$$

Due to the symmetry

$$\int_{b_2} d\Omega_\pm = \mp\int_{b_1} d\Omega_\pm$$

it is enough to satisfy the periodicity condition (14.1) for the period b_1 only. By Proposition 8 both L_1, L_2 must be even :

$$L_1 = 2l_1,\ L_2 = 2l_2,\ l_1, l_2 \in \mathbf{Z}.$$

This allows us to rewrite the intrinsic periodicity conditions as follows:

$$l_1 \equiv \frac{N}{2(a-b)}\sum_{k=1}^{K}(U_1^k - U_2^k) \in \mathbf{Z}, \tag{15.10}$$

$$l_2 \equiv \frac{M}{2(a+b)}\sum_{k=1}^{K}(U_1^k + U_2^k) \in \mathbf{Z}, \tag{15.11}$$

where $a, b \in \mathbf{R}$ are the elements of the period matrix (15.9).

The rotational periodicity (14.3) can be investigated in a similar way. For the differential $d\Omega_-$ it is automatically satisfied :

$$(-1)^{MK}\exp(2\int_0^1 d\Omega_-) = 1. \tag{15.12}$$

To prove (15.12), let us remark that

$$\int_0^1 d\Omega_0^k = \int_\infty^1 d\Omega_\infty^k = \int_0^1 d\Omega_\infty^k + \int_0^\infty d\Omega_\infty^k,$$

which yields

$$\exp\left(\int_0^1 d\Omega_\infty^k - \int_0^1 d\Omega_0^k\right) = \frac{1}{i\epsilon_\infty}, \quad \epsilon_\infty = \pm 1.$$

Combined with the identity

$$\exp\left(\int_0^1 d\omega_1 + d\omega_2\right) = \pm 1,$$

obtained in the same way, this finally proves (15.12). For the differential $d\Omega_+$ the same arguments

$$\exp\left(\int_0^1 d\Omega_\infty^k + d\Omega_0^k\right) = i\epsilon_\infty \exp\left(2\int_0^1 d\Omega_\infty^k\right),$$
$$\exp\left(\int_0^1 d\omega_1 - d\omega_2\right) = -\exp\left(2\int_0^1 d\omega_1\right)$$

imply the rotational periodicity condition

$$\exp\left(4N\sum_{k=1}^{K}\int_0^1 d\Omega_\infty^k - 8l_1\int_0^1 d\omega_1\right) = 1. \tag{15.13}$$

Because of the symmetry

$$\tau_6^* \frac{d\Omega_+}{d\Lambda}(\Lambda = 1) = -\frac{d\Omega_+}{d\Lambda}(\Lambda = 1)$$

the translational periodicity condition (14.4) is always satisfied by $d\Omega_+$ at $\lambda_0 = 1$. By the same reasons,

$$\frac{d\Omega_0^k}{d\Lambda}(\Lambda = 1) = -\frac{d\Omega_0^k}{d\Lambda}(\Lambda = 1), \ \frac{d\omega_1}{d\Lambda}(\Lambda = 1) = \frac{d\omega_2}{d\Lambda}(\Lambda = 1).$$

For the differential $d\Omega_-$ this condition (14.4) is formulated as follows:

$$M\sum_{k=1}^{K}\frac{d\Omega_\infty^k}{d\Lambda}(\Lambda = 1) - 2l_2\frac{d\omega_1}{d\Lambda}(\Lambda = 1) = 0. \tag{15.14}$$

The following proposition is proved.

Proposition 10. *Let X be a spectral curve of genus 2 with the branchpoints as above* (15.4), *for which the singularities $P_{\infty,0}^k$ are symmetric* (15.5) *and the periodicity conditions* (15.10), (15.11), (15.13), (15.14) *are satisfied. Then the formulas* (13.4), (13.5), (13.7) *describe a compact discrete K-surface, which is doubly periodic with the periods* (KN, KN), $(KM, -KM)$.

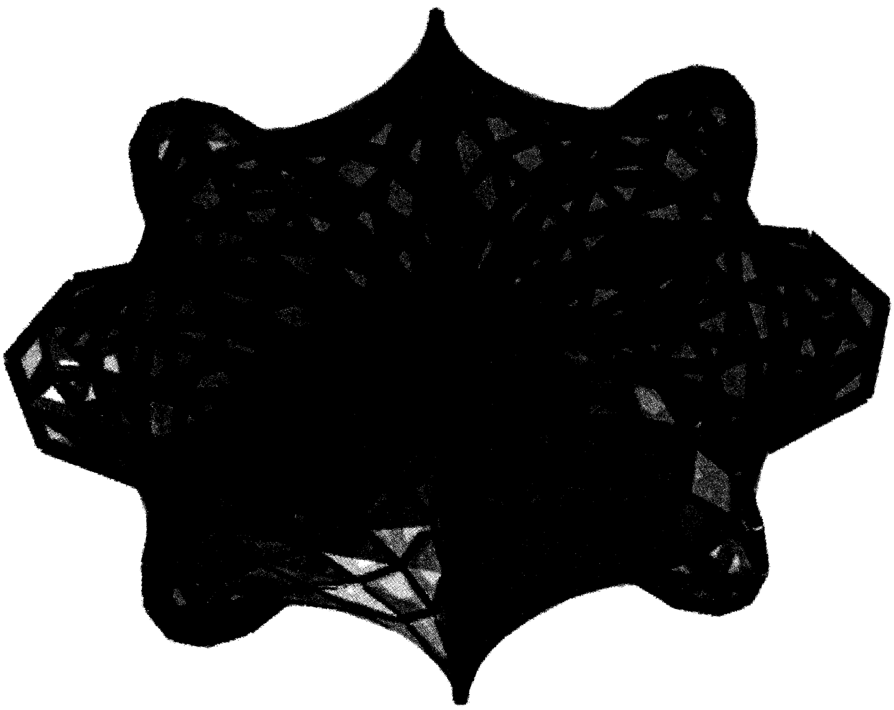


FIGURE 18. A compact discrete K-surface, which corresponds to a symmetrical spectral curve of genus $g = 2$

There are four periodicity conditions (15.10), (15.11), (15.13), (15.14) to be satisfied, and there are $K+2$ free parameters: K different singularities P_∞^k, $k=1,\dots,K$ (or, equivalently, K different lengths of edges $\Delta_1,\dots,\Delta_K$) and the real and imaginary parts of Λ_1. This indicates that there are no doubly periodic discrete K-surfaces with all edges of the same length, but one can hope to construct such surfaces comprised by edges of two different lengths.

An example of doubly periodic discrete K-surface is presented in Figure 18. The surface in this figure has edges of two different lengths. This example was constructed by Matthias Heil, who solved numerically the periodicity conditions (15.10, 15.11, 15.13, 15.14) using the software for doing calculations on hyperelliptic Riemann surfaces developed by him at SFB 288.

The smooth surfaces generated by the symmetric spectral curve considered in this section are undeformable. In contrast to this, in the discrete case the spectral curve X and the points P_∞^k, satisfying the periodicity conditions, generate families of discrete K-surfaces. In the cases of the surface, presented in the coloured picture at the end of the paper it is a 2-parametric family, parametrized by a vector $D \in \mathbf{R}^2$. By variation of D one moves the closed $F_{n,m}$ net in the meridian or parallel directions. The form of the smooth counterpart remains recognizable.

16. Appendix

We consider the postponed case (4.10) and show first that

$$N_{n+1,m+1} - N_{n,m} \parallel N_{n+1,m} - N_{n+1,m}. \tag{16.1}$$

By a suitable gauge transformation (4.11) we can obtain

$$\beta \equiv \beta' \equiv \delta \equiv \delta' \equiv 0. \tag{16.2}$$

In this gauge (4.10) looks as follows:

$$\alpha - \alpha' \equiv \gamma - \gamma', \tag{16.3}$$

$$\cos\tfrac{\Delta^u}{2}\sin\tfrac{\Delta^v}{2}(e^{i\alpha'} - e^{-i\alpha}) = \cos\tfrac{\Delta^v}{2}\sin\tfrac{\Delta^u}{2}(e^{i\gamma'} - e^{-i\gamma}). \tag{16.4}$$

The arguments of the proof of Proposition 5 show that (16.1) is equivalent to

$$\mathcal{V}'^{\,-1}\sigma_3\mathcal{U}' - \mathcal{U}\sigma_3\mathcal{V}^{-1} = r(\sigma_3\mathcal{U}\mathcal{V}^{-1} - \mathcal{U}\mathcal{V}^{-1}\sigma_3),$$

where $\mathcal{U}$ and $\mathcal{V}$ are the matrices in the gauge (16.2). By using (16.3), calculation of both sides yields

$$\mathcal{V}'^{\,-1}\sigma_3\mathcal{U}' - \mathcal{U}\sigma_3\mathcal{V}^{-1} = \begin{pmatrix} 0 & ip \\ -i\bar{p} & 0 \end{pmatrix},$$

$$\sigma_3\mathcal{U}\mathcal{V}^{-1} - \mathcal{U}\mathcal{V}^{-1}\sigma_3 = \begin{pmatrix} 0 & iq \\ -i\bar{q} & 0 \end{pmatrix},$$

where

$$p = \cos\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}(e^{i\alpha} + e^{i\alpha'}) + \sin\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}(e^{i\gamma} + e^{-i\gamma'}),$$

$$q = 2\sin\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}e^{i\gamma} - 2\cos\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}e^{i\alpha}.$$

The addition of a vanishing term to q

$$\begin{aligned} q &= q + \sin\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}(e^{-i\gamma'} - e^{i\gamma}) - \cos\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}(e^{-i\alpha'} - e^{i\alpha}) \\ &= \sin\frac{\Delta^u}{2}\cos\frac{\Delta^v}{2}(e^{i\gamma} + e^{-i\gamma'}) - \cos\frac{\Delta^u}{2}\sin\frac{\Delta^v}{2}(e^{i\alpha} + e^{-i\alpha'}) \end{aligned}$$

gives the equality of the arguments

$$\arg p = \arg q \quad (\bmod \pi),$$

and finally (16.1).

Now we prove that in the case of discrete Chebyshev net,

$$\Delta^u = \Delta^v,$$

the discrete surfaces described by (4.10) degenerate to discrete curves. Equation (16.4) reads as follows:

$$\exp(i\alpha') - \exp(-i\alpha) = \exp(i\gamma') - \exp(-i\gamma)$$

and has 3 possible solutions

$$\begin{aligned} &\text{(a)}\ \ \alpha \equiv \gamma,\ \alpha' \equiv \gamma', \\ &\text{(b)}\ \ \alpha \equiv -\gamma' + \pi,\ \alpha' \equiv -\gamma + \pi, \\ &\text{(c)}\ \ \alpha' \equiv -\alpha,\ \gamma' \equiv -\gamma,\ 2\alpha \equiv 2\gamma. \end{aligned}$$

The case (a) implies $\mathcal{U} = \mathcal{V}$, $\mathcal{U}' = \mathcal{V}'$, which proves the degeneracy of the surface

$$N_{n+1,m} = N_{n,m+1}.$$

The case (b) implies $\mathcal{U}' = -\mathcal{V}^{-1}$, $\mathcal{V}' = -\mathcal{U}^{-1}$, which gives

$$N_{n,m} = N_{n+1,m+1}.$$

The last case (c) splits into two possibilities. If $\alpha \equiv \gamma$, then $\alpha' \equiv \gamma'$ and we get a special case of (a). If $\alpha \equiv \gamma + \pi$, then $\alpha' \equiv -\alpha$, $\gamma' \equiv -\alpha + \pi$, which yields

$$\mathcal{V} = -\sigma_3 \mathcal{V} \sigma_3, \; \mathcal{U}' = \sigma_3 \mathcal{U}^{-1} \sigma_3, \; \mathcal{V}' = -\mathcal{U}^{-1},$$

and finally, $N_{n,m} = N_{n+1,m+1}$. This finishes the proof of the degeneracy.

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TECHNISCHE UNIVERSITÄT BERLIN, GERMANY