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LORENTZIAN GEODESIC FLOWS

JENS CHR. LARSEN

1. Introduction

In this paper we consider time oriented Lorentzian manifolds (M, g) satisfying condition C_Q, i.e., (M, g) is

(1) future 1-connected, nonspacelike complete

(2) the sectional curvatures $K(\pi) \geq Q^2$ for every timelike two plane π

for some $Q \geq 0$. Recall that (M, g) is future 1 connected if any two smooth timelike curves from p to q are homotopic through smooth timelike curves from p to q. Also a Lorentzian manifold is a smooth, connected Hausdorff manifold with a countable base and a metric g of signature $(-, +, .., +)$. The Riemannian inclined reader may benefit from the remark that the curvature assumption (2) corresponds in some respects to negative Riemannian curvature.

The main results of this paper are the following:

(1) A density theorem for the timelike geodesic flow, cf. Theorem 8.4. Here it is proven that the closed timelike geodesics are dense in the quotient of the future timelike unit tangent bundle with a vicious group of isometries.

(2) A rigidity theorem for C_Q surfaces, cf. Theorem 10.3. More precisely we prove that an orientable C_Q surface with a vicious isometry group and Q positive must have constant curvature.

These results will follow from structure theorems for geometrically defined subsets of (M, g), notably Theorem 7.4 and Theorem 7.7. In fact the future null cone $K^+(p)$ of any point $p \in M$ is a smooth hypersurface of constant signature (0,+,...,+). The implication is that the boundary $N_\omega(N_\alpha)$ of the past (future) of a complete timelike geodesic γ is a C^1 hypersurface in M of constant signature (0,+,...,+). In other

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words N_ω is a singular semi-Riemannian manifold. Some theory is available for the geometry of singular semi-Riemannian manifolds, cf. [20], [22], [23], [24]. N_ω is the union of null colines to γ. These null colines are null axes of a hyperbolic isometry if the induced isometry on the Riemannian manifold $N_\alpha \cap N_\omega$ has a fixed point, where α and ω are endpoints of a timelike axis for the hyperbolic isometry. These results are derived in chapter 7. The main tool is the null theory from section 6. This in turn follows from section 2, deriving a triangle comparison lemma for C_Q manifolds. Theorem 5.3 proves the crucial fact that a hyperbolic isometry has a timelike axis.

On the constantly curved C_Q manifolds there are properly discontinuous groups of isometries acting on the future timelike unit tangent bundle, cf. section 9. If this group is proper, the geodesic flow induced on the quotient is mixing, hence ergodic. It has a transitive geodesic and dense periodic orbits. In dimension two a horocycle flow is induced on the quotient. It is mixing of all degrees. These results are derived from the Riemannian theory, cf. also [17], [25] and [30]. The Riemannian theory started in the 1920's, cf. [19] and [29].

Chapter nine sets the context for the neighbouring sections. A density theorem for C_Q manifolds with vicious Deck transformation group is presented in section 8. It relies on the definition of the timelike future and the timelike past of a C_Q manifold, developed in sections 3 and 4.

The same assumption for the isometry group of a C_Q surface forces the curvature to be constant.

To avoid confusion it is emphasized that throughout we shall use the convention that a mapping F from a subset A of a manifold M is C^r, $1 \leq r \leq +\infty$ if for every $q \in A$ there is a C^r map G, defined on an open neighbourhood U of q, whose restriction to $A \cap U$ coincides with the restriction of F to this set. Also the curvature tensor is

$$R_{XY}Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z.$$

The domain of definition of a mapping f is denoted by $\mathbb{D}(f)$. Maximal geodesics with initial velocity v are denoted by γ_v.

2. Timelike geodesic triangles

This section is fundamental. It provides the main tool in this paper, namely a triangle comparison lemma for timelike geodesic triangles in a C_Q manifold (M, g), i.e., a Lorentz manifold satisfying condition C_Q.

Recall that two points p and q in M are causally related, i. e., $p < q$, sometimes written $q > p$, provided there exists a nonspacelike future directed curve from p to q. Also $p << q$, sometimes written $q >> p$, provided there is a timelike future directed curve from p to q. As usual

$$I^+(p) = \{q \in M \mid p << q\}, \qquad I^-(p) = \{q \in M \mid q << p\},$$
$$J^+(p) = \{q \in M \mid p < q \text{ or } \ p = q\}, J^-(p) = \{q \in M \mid q < p \text{ or } p = q\}.$$

Lemma 2.1. *For any point p in M, the map*

$$\Lambda^+(p) \to I^+(p) \ , \ v \mapsto \exp_p(v),$$
$$\Lambda^+(p) = \{w \in T_pM \mid w \textit{ timelike and future directed } \}$$

is a C^∞ diffeomorphism.

Proof. In view of [13] we need only show that $\exp_p(\Lambda^+(p)) = I^+(p)$. Take any $q \in I^+(p)$ and a timelike future directed curve c from $c(0) = p$ to $c(a) = q$. If $q \notin \exp_p(\Lambda^+(p))$ define

$$S = \{t \in]0, a] \mid c(t) \notin \exp_p(\Lambda^+(p))\},$$
$$s_* = \inf S > 0.$$

Let γ denote a timelike geodesic from some $c(t) = \gamma(0), \ t \in [0, s_*[$ to $c(s_*) = \gamma(b), \ b \in \mathbb{D}(\gamma) \cap \mathbb{R}_+$. Define

$$t_* = \inf\{t \in \ ]0, b[\ \mid \gamma(t) \notin \exp_p(\Lambda^+(p))\}.$$

By $\phi(t), \ t \in \ ]0, t_*[$ we denote the unique vector in $\Lambda^+(p)$ such that $\exp_p(\phi(t)) = \gamma(t)$. $t \mapsto \phi(t), \ t \in]0, t_*[$ is then a timelike future directed curve in $\Lambda^+(p)$. The function

$$g(t) = (-\langle\phi(t), \phi(t)\rangle)^{\frac{1}{2}} \quad t \in \ ]0, t_*[$$

is smooth and concave according to [9]. g is then bounded above. Since $\phi(t) >> \phi(0) \in \Lambda^+(p)$ we deduce that $\phi(t)$ is contained in a compact set in T_pM, hence $\phi(t_n) \to w$ for a suitable sequence $t_n \to t_*$ and some $w \in \Lambda^+(p)$. Thus $\exp_p(w) = \gamma(t_*)$ in contradiction.

To prove the triangle comparison lemma, let p, q and r be three causally related points in a C_Q manifold $(M, g = \langle \ , \ \rangle)$. This means that $p << q,\ q << r$. Throughout the paper a TF (respectively TP) geodesic is a timelike, complete, unit speed geodesic, which is future (respectively past) directed. According to Lemma 2.1 there are unique TF geodesics γ_1, γ_2 and γ_3 from p to $q = \gamma_1(a), q$ to $r = \gamma_2(b)$ and p to $r = \gamma_3(c),\ a, b, c \in \mathbb{R}_+$. Define

$$\begin{aligned} A_p &= \langle \gamma_1'(0), \gamma_3'(0) \rangle \leq -1, \\ -A_q &= \langle \gamma_1'(a), \gamma_2'(0) \rangle \leq -1, \\ A_r &= \langle \gamma_2'(b), \gamma_3'(c) \rangle \leq -1. \end{aligned} \tag{2.1}$$

When $Q = 0,\ M_Q$ denotes Minkowski space $\mathbb{R}_1^n$, whereas

$$M_Q = \{x = (x_1, .., x_{n+1}) \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = -x_1^2 + x_2^2 + .. + x_{n+1}^2 = 1/Q^2\}$$

with metric induced by the Minkowski metric $\langle\ ,\ \rangle$ when $Q > 0$. According to Lemma 2.1, $c \geq a + b$. This means there are causally related points $p_Q, q_Q, r_Q \in M_Q$ such that

$$a = d_Q(p_Q, q_Q), \quad b = d_Q(q_Q, r_Q), \quad c = d_Q(p_Q, r_Q). \tag{2.2}$$

Here d_Q is the Lorentzian distance function in M_Q, and d will always denote the Lorentzian distance function in (M, g), cf. [5, Chapter 3].

Lemma 2.2. *Let p, q, r be causally related points in a C_Q manifold (M, g), where $Q \geq 0$. When p_Q, q_Q and r_Q are causally related points in M_Q, satisfying* (2.2), *then*

$$A_{p_Q} \leq A_p, \quad A_{q_Q} \leq A_q, \quad A_{r_Q} \leq A_r.$$

Proof. We shall use Karcher's method, see [16]. Define

$$r(x) = d(p, x), \qquad x \in I^+(p),$$

and let y_Q denote the solution to

$$y_Q'' = Q^2 y_Q, \quad y_Q(0) = 0, \quad y_Q'(0) = 1.$$

We claim that

$$H(r)(v, v) \leq -\langle v, v \rangle y_Q'(r(x))/y_Q(r(x))$$

for all $v \perp$ grad r_x, where $H(r)$ denotes the hessian of r. To this end let $\gamma : I \to I^+(p)$ denote a geodesic with $\gamma'(0) = v$ and let

$$\begin{aligned}\gamma(s) &= \exp_p(v(s)),\\ \alpha(t,s) &= \exp_p(tv(s)), \qquad t > 0, \quad s \in I.\end{aligned}$$

$N =$ Im α is a Lorentz surface with $K^N \geq Q^2$. A straightforward differentiation yields

$$H(r)(v,v) = (r \circ \gamma)''(0) = -\langle v, \alpha_{st}(1,0)\rangle / r(x).$$

Let E denote a parallel vector field in N orthogonal to $t \mapsto \alpha(t,0)$. Then $\alpha_s(t,0) = v(t)E(t)$ for some smooth function v satisfying

$$v'' = vK^N r(x)^2.$$

By standard Liouville theory we have

$$\begin{aligned}H(r)(v,v) &= -\langle v,v\rangle v'(1)/(v(1)r(x))\\ &\leq -\langle v,v\rangle y'_{Qr(x)}(1)/(y_{Qr(x)}(1)r(x))\\ &= -\langle v,v\rangle y'_Q(r(x))/y_Q(r(x)).\end{aligned}$$

Notice that $H(r)(v,v) = 0$ when $v \parallel$ grad r_x. Now define

$$S(t) = \begin{cases} \frac{1}{2}t^2, & Q = 0,\\ (y'_Q(t) - 1)/Q^2, & Q > 0,\end{cases}$$

and $r_s = S \circ r$. Then

$$\begin{aligned}H(r_s)(v,v) &= S' \circ r H(r)(v,v)\\ &\leq -\langle v,v\rangle y'_Q(r(x)) = -\langle v,v\rangle(Q^2 r_s(x) + 1), \qquad v \perp \text{ grad } r_x,\\ H(r_s)(v,v) &= S''(r(x))\, v[r]^2 = -\langle v,v\rangle(Q^2 r_s(x) + 1), \qquad v \parallel \text{grad } r_x.\end{aligned}$$

Now let $f(t) = r_s \circ \gamma_2(t), \ \ t \in [0,b]$. Then

$$f''(t) \leq Q^2 f(t) + 1.$$

Let $p_Q << q_Q << r_Q$ denote a timelike geodesic triangle in M_Q with side lengths a, b and c and sides γ_1^Q, γ_2^Q and γ_3^Q. Also let $r_Q(x) = d(p_Q, x)$ and $f_Q(t) = S \circ r_Q \circ \gamma_2^Q(t), \ \ t \in [0,b]$. Then

$$f''_Q(t) = Q^2 f_Q(t) + 1.$$

Since $f(0) = f_Q(0)$ and $f(b) = f_Q(b)$, we deduce that

$$f \geq f_Q,$$

and hence

$$r \circ \gamma_2(t) \geq r_Q \circ \gamma_2^Q(t), \quad t \in [0, b].$$

Finally

$$\begin{aligned}(r \circ \gamma_2)'(0) &= A_q \geq A_{q_Q} = (r_Q \circ \gamma_2^Q)'(0),\\ (r \circ \gamma_2)'(b) &= -A_r \leq -A_{r_Q} = (r_Q \circ \gamma_2^Q)'(b).\end{aligned}$$

Time reversal produces $A_{p_Q} \leq A_p$.

We shall present a different argument of competitive simplicity. If $A_q = 1$, then $A_p = A_{p_Q}$, so to prove $A_{p_Q} \leq A_p$ we can assume that $A_q \neq 1$. Define future directed vectors $v(s)$ in T_pM with

$$\begin{aligned}\gamma_2(s) &= \exp_p(v(s)), \quad s \in [0, b],\\ \alpha(t, s) &= \exp_p(tv(s)), \quad t > 0.\end{aligned}$$

Choose $p_Q^* \in M_Q$ and an isometry

$$I : T_pM \to T_{p_Q^*}M_Q\ .$$

Define

$$\begin{aligned}\alpha^Q(t, s) &= \exp_{p_Q^*}(tI \circ v(s)), \quad t > 0, \quad s \in [0, b],\\ Y_s(t) &= \alpha_s(t, s),\\ Y_s^Q(t) &= \alpha_s^Q(t).\end{aligned}$$

Then

$$I(\lim_{t \to 0} Y_s'(t)) = I(v'(s)) = Y_s^{Q\prime}(0), \tag{2.3}$$

where $Y_s'(t)$ denotes the induced covariant derivative of Y_s in $N = \operatorname{Im} \alpha$. Take a unit parallel vector field E^Q orthogonal to $t \mapsto \alpha^Q(t, s)$ such that

$$Y_s^{Q\perp}(t) = y_Q(t)E^Q(t).$$

Let E denote a unit parallel vector field orthogonal to $t \mapsto \alpha(t, s)$ in N. Define

$$y = \langle Y_s, E \rangle.$$

Then

$$\begin{aligned}y_Q'' - \alpha^2 Q^2 y_Q &= 0, \quad -\alpha^2 = \langle \alpha_t, \alpha_t \rangle(s),\\ y'' - \alpha^2 K^N y &= 0.\end{aligned}$$

Because of (2.3) we can assume $y'(0) = y'_Q(0) > 0$ and $y(0) = y_Q(0) = 0$. Standard Liouville theory yields

$$y_Q \leq y,$$

hence

$$||Y_s(1)|| = ||\gamma_2'(s)|| \leq ||\alpha_s^Q(1,s)||.$$

Thus

$$b = L(\gamma_2) \leq \int_0^b ||\alpha_s^Q(1,s)||ds \leq d_Q(q_Q^*, r_Q^*) = b_Q,$$

where $q_Q^* = \alpha^Q(1,0)$, $r_Q^* = \alpha^Q(1,b)$. When p_Q, q_Q, r_Q are vertices in a timelike geodesic triangle in M_Q with side lenghts a, b and c we find

$$\text{(2.4)} \qquad \begin{aligned} A_{p_Q} &= \begin{cases} \frac{\cosh(Qb)-\cosh(Qa)\cosh(Qc)}{\sinh(Qa)\sinh(Qc)}, & Q > 0 \\ (b^2 - a^2 - c^2)/2ac, & Q = 0 \end{cases} \\ &\leq A_{p_Q^*} = A_p. \end{aligned}$$

Time reversal produces $A_{r_Q} \leq A_r$. The same method can be used to prove $A_{q_Q} \leq A_q$ for small b. This angle inequality follows for arbitrary b by a subdivision of γ_2 and an induction argument.

Remark 2.3. We shall now briefly indicate how to combine Lemma 2.2 and Lemma 6.1 to show that a C_0 manifold (M, g) is globally hyperbolic; see also [13]. To verify that (M, g) is strongly causal at some $p \in M$ take some TF geodesic γ through $\gamma(0) = p$. Given an open neighbourhood U of p use Lemma 2.2 to find a positive ϵ such that the causally convex neighbourhood $I(\gamma(-\epsilon), \gamma(\epsilon))$ of p is contained in U.

If $p, q \in M$ are causally related, then $J(p,q) \triangleq J^+(p) \cap J^-(q) \subset J^+(p_*) \cap I^-(q_*)$ for any $p_* << p$ and $q << q_*$. Now we use Lemma 2.2 and a $Q = 0$ version of Lemma 6.1 to show that the counterimage of $J(p,q)$ by the restriction of $\exp_{p_*}$ to the future cone is closed and bounded. $J(p,q)$ is then compact.

3. The timelike coray condition

In this section we consider a TF geodesic γ in a C_0 manifold (M, g). Recall from [4, p. 33] that a future coray from

$$x \in I^-(\gamma) = \{q \in M \mid \exists\, t \in \mathbb{R} \ : \ q << \gamma(t)\}$$

to γ is a future directed, inextendible, nonspacelike limit curve $\beta : I \to M$ through x of a sequence of TF geodesics from x_n to $\gamma(r_n)$ where $\{x_n\}_{n\in\mathrm{N}}$ and $\{r_n\}_{n\in\mathrm{N}}$ are sequences in M and $\mathbb{R}$ respectively such that $x_n \to x$, $x_n << \gamma(r_n)$ and $r_n \to +\infty$. Here I is an open interval. We can and will require that $0 \in I$ and $\beta(0) = x$. A smooth curve β_* is a past coray from

$$y \in I^+(\gamma_*) = \{q \in M \mid \exists\, t \in \mathbb{R} : q >> \gamma_*(t)\}$$

to a TP geodesic γ_* provided β_* is a future coray from $y \in I^+(\gamma_*)$ to γ_* in (M, g) with time orientation reversed. There is a future coray to γ through every $x \in I^-(\gamma)$ according to [5, Proposition 2.18]. This coray definition coincides with the definition in [4]; see also [7]. There are other definitions in [11], [12], [28] and [3]. By definition (M, g) satisfies the timelike coray condition if all future and past corays are timelike; cf. [4, Definition 3.1]. Corays are pregeodesics according to [5, Proposition 2.21, Remark 2.22, Lemma 3.5 and Theorem 3.13].

Lemma 3.1. *The timelike coray condition holds for any TF geodesic γ in a C_0 manifold (M, g).*

Proof. Assume that $x \in I^-(\gamma)$ has a future coray β to γ, which is null. Then there are sequences $\{x_n\}$ in $I^-(\gamma)$, $\{r_n\}$ in $\mathbb{R}$ and TF geodesics β_n from x_n to $\gamma(r_n)$ such that β is a future directed, inextendible, nonspacelike limit curve through x for $\{\beta_n\}$. We can suppose that x and the $x_n \in I^-(\gamma(r_0))$ for some $r_0 < r_n$. According to Lemma 2.1 there is a TF geodesic σ_n and σ from x_n and x respectively to $\gamma(r_0) = q$. Put

$$a_n = d(x_n, q), \qquad a = d(x, q), \qquad b_n = d(x_n, \gamma(r_n)), \qquad c_n = r_n - r_0.$$

Looking at the timelike geodesic triangle $x_n, q = q_n, \gamma(r_n)$ we obtain

$$1 \leq (b_n/c_n)^2 \leq 1 + (a_n/c_n)^2 + 2\,(a_n/c_n)\,A_{q_n} \to 1$$

for $n \to +\infty$. Hence $b_n/c_n \to 1$ for $n \to +\infty$. We have used that A_{q_n} is bounded and that $a_n \to a$. Adding two of the cosine laws give us that

$$-1 \geq A_{x_n} \geq A_{x_n^Q} = -a_n/b_n - c_n/b_n A_{q_n^Q} \geq -a_n/b_n - c_n/b_n A q_n, \quad Q = 0,$$

contradicting the fact that A_{x_n} is unbounded.

We can now define the Buseman function

$$b^+ \ : \ I^-(\gamma) \to \mathbb{R}\ , \quad x \mapsto \lim_{r \to +\infty} \{r - d(x, \gamma(r))\}.$$

The Buseman function is continuous, because (M, g) satisfies the timelike coray condition, cf. [5].

Let β denote a unit speed future coray from $p \in I^-(\gamma)$ to some TF geodesic γ. Proposition 3.2 below shows that it is unique. A dual statement applies to assert the uniqueness of unit speed past corays through $y \in I^+(\gamma_*)$ to some TP geodesic γ_*.

There exists by definition an s_0 such that $p << \gamma(s)$ for all $s \geq s_0$. In view of Lemma 2.1 this means that for all $s \geq s_0$ there exists a unique future directed timelike unit vector v_s such that $\gamma(s) = \exp_p(tv_s)$ for some $t > 0$.

Proposition 3.2. *$v_s \to \beta'(0)$ as $s \to +\infty$.*

Proof. We shall consider two timelike geodesic triangles $p, \gamma(s_0), \gamma(s_1)$ and $p, \gamma(s_1), \gamma(s_2)$ where $s_0 < s_1 < s_2$. Let us for notational convenience rename them p, p_0, p_1 and q, q_1, q_2 respectively. The side lengths are

$$a_1 = d(p, \gamma(s_2)), \quad a_2 = d(p, \gamma(s_1)) = b_1,$$
$$a_3 = d(\gamma(s_1), \gamma(s_2)), \quad b_2 = d(p, \gamma(s_0)), \quad b_3 = s_1 - s_0.$$

Lemma 2.2 gives us

$$a_1^2 = a_2^2 + a_3^2 + 2a_2a_3A_{q_1^Q},$$
$$a_3^2 = a_1^2 + a_2^2 + 2a_1a_2A_{q^Q},$$

which combine to $-A_{q^Q} \leq A_{q_1^Q}$. Notice that $A_{q_1} = -A_{p_1}$, hence $-A_{q^Q} \leq -A_{p_1^Q}$. Applying Lemma 2.2 once again provides

$$b_1^2 \leq b_2^2 + b_3^2 + 2b_2b_3A_{p_0}.$$

Given $\epsilon > 0$ we can make $b_3/b_1 \geq 1 - \epsilon$ for all s_1 sufficiently large. Use

$$b_2^2 = b_1^2 + b_3^2 + 2b_1b_3A_{p_1^Q}$$

to conclude that $A_{q^Q} \geq A_{p_1^Q} \geq -1 - \delta$ for all s_1 sufficiently large. The proposition now follows from the fact that we can take a sequence $\{s_n\}_{n \in \mathrm{N}}$ of real numbers $s_n \to +\infty$ such that

$$v_{s_n} \to \beta'(0).$$

Lemma 3.3. *Let $\gamma_i, i = 1, 2$, be two TF geodesics and*

$$\Omega = \{t \in \mathbb{R} \mid \gamma_1(t) << \gamma_2(t)\},$$
$$f(t) = d(\gamma_1(t), \gamma_2(t)), \qquad t \in \Omega.$$

Then f is C^∞ and concave, i.e., $f'' \leq 0$.

Proof. See [9].

4. The timelike future

We shall now define the timelike future and the timelike past from the sets Ω_{TF} and Ω_{TP} of TF and TP geodesics respectively in a C_0 manifold (M, g). We need a preliminary lemma to assert that the coray definition is translation invariant in the geodesic affine parameter.

Lemma 4.1. *If $\gamma_i \in \Omega_{TF}$, $i = 1, 2$, are future corays to $\gamma_3 \in \Omega_{TF}$ through $\gamma_1(0) << \gamma_2(0)$, then*

$$\gamma_1(t) << \gamma_2(t)$$

for all $t \in \mathbb{R}_+$, and there exists a $K > 0$ such that for these values of t

$$d(\gamma_1(t), \gamma_2(t)) < K.$$

Proof. Let c_s, d_s denote TF geodesics from $c_s(0) = \gamma_1(0), d_s(0) = \gamma_2(0)$ to

$$c_s(t_s) = d_s(u_s) = \gamma_3(s)$$

for all s exceeding some $s_0 > 0$. For $t \in A = \{t \geq 0 | c_s(t) << d_s(t)\}$ define

$$f_s(t) = d(c_s(t), d_s(t)).$$

For these values of t let β_t^s denote the TF geodesic from $\beta_t^s(0) = c_s(t)$ to $\beta_t^s(a_t^s) = d_s(t)$, and let

$$B_s(t) = \langle c_s'(t), \beta_t^{s\prime}(0)\rangle,$$
$$C_s(t) = \langle d_s'(t), \beta_t^{s\prime}(a_t^s)\rangle.$$

Define

$$h_s(u) = d(\beta_t^s(u), \gamma_3(s)), \quad u \in [0, a_t^s].$$

We have seen that $h_s'' \leq 0$, hence

$$B_s(t) = h_s'(0) \geq h_s'(a_t^s) = C_s(t),$$

and thus

$$f_s'(t) = B_s(t) - C_s(t) \geq 0.$$

It follows that $[0, u_s] \subset A$. For

$$t \in B = \{t \in [0, +\infty[\mid \gamma_1(t) << \gamma_2(t)\}$$

let η_t denote the TF geodesic from $\eta_t(0) = \gamma_1(t)$ to $\eta_t(b_t) = \gamma_2(t)$ and

$$f(t) = d(\gamma_1(t), \gamma_2(t)).$$

Then

$$f_s'(t) \to \langle \gamma_2'(t), \eta_t'(b_t) \rangle - \langle \gamma_1'(t), \eta_t'(0) \rangle = f'(t) \geq 0$$

as $s \to +\infty$ and then $B = [0, +\infty[$.

The Buseman function b^+ for γ_3 is differentiable; see [12], with

$$\langle \mathrm{grad} b^+, \mathrm{grad} b^+ \rangle = -1,$$

hence

$$b^+ \circ \eta_t(s) \geq s + b^+(\gamma_1(t)), \quad s \in [0, b_t],$$

and then

$$K = b^+(\gamma_2(0)) - b^+(\gamma_1(0)) = b^+ \circ \eta_t(b_t) - b^+ \circ \eta_t(0) \geq b_t = d(\gamma_1(t), \gamma_2(t)).$$

The lemma follows.

Proposition 4.2. *If $\gamma_1 \in \Omega_{TF}$ is a future coray to $\gamma_2 \in \Omega_{TF}$ through $\gamma_1(0)$ in a C_0 manifold (M, g), then γ_1 is a future coray to γ_2 through $\gamma_1(t)$ for every $t \in \mathbb{R}$.*

Proof. Let a $t \in \mathbb{R}$ be given. Since $\gamma_1 \in \Omega_{TF}$ is a future coray to γ_2 through $\gamma_1(0)$ we can find an $s \in \mathbb{R}$ such that $\gamma_1(t) << \gamma_2(s)$. This follows from definitions, when $t \leq 0$ and from Lemma 4.1 for $t > 0$. Define

$$c = c(a) = d(\gamma_1(t + a), \gamma_2(s + a)) > 0\ , \qquad b = d(\gamma_1(t), \gamma_2(s + a))\ ,$$

where $a \in \mathbb{R}_+$. The function c has an upper bound by Lemma 4.1. Looking at the timelike geodesic triangle $\gamma_1(t), \gamma_2(s), \gamma_2(s+a)$ we find

$$b^2 \leq a^2 + c^2 + 2acA_q \tag{4.4}$$

with $q = \gamma_2(s)$. It follows from (4.4) that $b/a \geq 1$ is close to 1, when a is sufficiently big. Let a positive ϵ be given. Looking at the timelike geodesic triangle $p = \gamma_1(t), \gamma_1(t+a), \gamma_2(s+a)$ we see that

$$A_p \geq A_{p_Q} = (c^2 - a^2 - b^2)/2ab \geq -\frac{1}{2}(1 + (b/a)^2) \geq -1 - \epsilon$$

taking a sufficiently large. This means that γ_1 is a future coray to γ_2 through $\gamma_1(t)$.

We can now adopt

Definition 4.3. $\gamma_1 \in \Omega_{TF}$ is a future coray to $\gamma_2 \in \Omega_{TF}$ provided γ_1 is a future coray to γ_2 through some and hence any $\gamma_1(t)$, $t \in \mathbb{R}$.

Two future corays have the same past. In fact we have

Lemma 4.4. *If $\gamma_1 \in \Omega_{TF}$ is a future coray to $\gamma_2 \in \Omega_{TF}$ in a C_0 manifold , then $I^-(\gamma_1) = I^-(\gamma_2)$.*

Proof. Since $\gamma_1 \in \Omega_{TF}$ is a future coray to $\gamma_2 \in \Omega_{TF}$, there exists an $s \in \mathbb{R}$ such that

$$\gamma_1(0) << \gamma_2(s) = q.$$

We denote by β the TP geodesic through $\gamma_2(s)$ and $\gamma_1(0) = \beta(a)$, $a \in \mathbb{R}_+$. Assume for contradiction there is no $u \in \mathbb{R}_+$ such that $\gamma_1(u) \in I^+(q)$. The nonempty subset

$$A = \{t \in [0,a] \mid \text{There exists no positive } s \text{ such that } \gamma_{\beta(t)}(s) \in I^+(q)\}$$

of $[0,a]$ has an infimum $z > 0$. We are using the notation $\gamma_{\beta(t)} \in \Omega_{TF}$, $t \in [0,a]$ for the future coray to γ_2 through $\beta(t) \in I^-(\gamma_2)$. $z \in A$ because the relation $<<$ is open, see [27, Proposition 14.3]. This proposition also implies that we can take $v \in [0, z[$ and $u > 0$ such that

$$\beta(v) << \gamma_{\beta(z)}(u).$$

Since $v \notin A$, there exists $r > 0$ such that

$$\gamma_{\beta(v)}(r) \in I^+(q).$$

We can now apply Lemma 4.1 to get

$$q << \gamma_{\beta(v)}(r) << \gamma_{\beta(z)}(u+r),$$

which contradicts the fact that $z \in A$, thus asserting the existence of a $\gamma_1(s) \in I^+(q), s \in \mathbb{R}_+$. Hence $I^-(\gamma_1) \subset I^-(\gamma_2)$. A second application of Lemma 4.1 yields the reverse inclusion, thereby proving the lemma.

We now define a relation $\underset{+}{\rightarrow}\sim(\underset{-}{\rightarrow}\sim)$ in $\Omega_{TF}(\Omega_{TP})$ by requiring that $\gamma_1 \underset{+}{\rightarrow}\sim\gamma_2(\gamma_1 \underset{-}{\rightarrow}\sim\gamma_2)$ provided γ_1 is a future (past) coray to γ_2. This is an equivalence relation. It is reflexive by Proposition 3.2. Symmetry and transitivity follows from Lemma 4.4, Lemma 4.1, Proposition 3.2 and an application of Lemma 2.2. We can then define the timelike future $M^+(\infty)$ and the timelike past $M^-(\infty)$ to be the quotient spaces of Ω_{TF} and Ω_{TP} under the coray equivalence relations $\underset{+}{\rightarrow}\sim$ and $\underset{-}{\rightarrow}\sim$ respectively

$$M^+(\infty) = \Omega_{TF}/\underset{+}{\rightarrow}\sim, \qquad M^-(\infty) = \Omega_{TP}/\underset{-}{\rightarrow}\sim.$$

Equivalence classes in $M^+(\infty)$ and $M^-(\infty)$ will be denoted $[\gamma]_+$ and $[\gamma]_-$ respectively. Given $\omega = [\gamma]_+ \in M^+(\infty)$, $\alpha = [\beta]_- \in M^-(\infty)$ we adopt the convention

$$I^-(\omega) = I^-(\gamma), \qquad I^+(\alpha) = I^+(\beta),$$

which is well defined by Lemma 4.4. It will be convenient to have the following.

Proposition 4.5. *Given $\omega = [\gamma]_+ \in M^+(\infty)$ and $p \in I^-(\omega)$ in a C_0 manifold, then there exists a TF geodesic β through $\beta(0) = p$ such that $[\beta]_+ = \omega$. If σ is a TF geodesic through $\sigma(0) = p$ such that $[\sigma]_+ = \omega$, then $\sigma = \beta$.*

Proof. The existence of β follows from the fact that (M, g) satisfies the timelike coray condition. Suppose σ is a TF geodesic with $\sigma(0) = \beta(0)$ and $[\sigma]_+ = \omega$. Then $\beta(0) << \sigma(s)$ for a fixed positive s. Now apply Lemma 4.1 to assert the existence of a positive K such that

$$\beta(t) << \sigma(s+t) \qquad r = r(t) = d(\beta(t), \sigma(s+t)) \leq K$$

for all $t \geq 0$. In the timelike geodesic triangle $p = \beta(0), \beta(t), \sigma(t+s)$, using Lemma 2.2 we have the following estimates

$$A_p \geq A_{pQ} \geq \frac{-t^2 + (t+s)^2}{2t(t+s)}.$$

The right-hand side converges to -1 as $t \to +\infty$, hence $A_p = -1$. Consequently $\sigma = \beta$.

Given $p \in M$ we shall say that $p << \omega \in M^+(\infty)$ provided there exists $\gamma \in \Omega_{TF}$ such that $\gamma(0) = p$, $[\gamma]_+ = \omega$. Similarly $p >> \alpha \in M^-(\infty)$ if there exists $\gamma \in \Omega_{TP}$ such that $\gamma(0) = p$, $[\gamma]_- = \alpha$. We can then define subsets

$$I^+_\infty(p) = \{\omega \in M^+(\infty) \mid p << \omega\}, \qquad I^-_\infty(p) = \{\alpha \in M^-(\infty) \mid p >> \alpha\}$$

of $M^+(\infty)$ and $M^-(\infty)$ respectively. Also $\alpha << \omega$, $\alpha \in M^-(\infty)$, $\omega \in M^+(\infty)$ provided there exists $p \in M$ such that $p >> \alpha$, $p << \omega$. In this case α and ω are causally related.

A sequence $\{\omega_n\}_{n\in\mathbb{N}}$ in $M^+(\infty)$ converges to $\omega \in I^+_\infty(p)$ with respect to $p \in M$ if there exists an $n_0 \in \mathbb{N}$ such that $\omega_n \in I^+_\infty(p)$ for all $n \geq n_0$ and

$$c_n'(0) \to \gamma'(0)$$

as $n \to +\infty$. Here c_n is the unique TF geodesic from p to ω_n and γ is the unique TF geodesic from p to $\omega = [\gamma]_+$. As usual time reversal produces a definition of convergence for a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ in $M^-(\infty)$.

We will adopt the notation

$$\omega_n \to_p \omega, \qquad \alpha_n \to_p \alpha,$$

when $\{\omega_n\}_{n\in\mathbb{N}}$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ converge with respect to p to ω and α respectively.

Also a sequence $\{p_n\}_{n\in\mathbb{N}}$ in M converges to $\omega \in I^+_\infty(p)$ with respect to p provided there exists an $n_0 \in \mathbb{N}$ such that $p_n \in I^+(p)$ and

$$c_n'(0) \to \gamma'(0), \qquad d(p, p_n) \to +\infty,$$

as $n \to +\infty$. Here c_n and γ are the TF geodesics from $c_n(0) = p = \gamma(0)$ to p_n and $[\gamma]_+ = \omega$ respectively. We write

$$p_n \to_p \omega, \qquad p_n \to_p \alpha,$$

whenever p_n converges with respect to p to ω and $\alpha \in M^-(\infty)$ respectively.

In the rest of this section (M, g) is a C_Q manifold with $Q > 0$.

Proposition 4.6. *If $\omega_n \to_p \omega$, then $\omega_n \to_q \omega$ for every $q \in M$ such that $\omega \in I^+_\infty(q)$.*

Proof. Take TF geodesics $\gamma_i,\ i=1,2,$ with $\gamma_1(0)=p,\ \gamma_2(0)=q$ and $[\gamma_1]_+=\omega=[\gamma_2]_+$. Given an $\epsilon>0$, take $r>0$ such that

$$\cosh^2(Qr)/\sinh^2(Qr)<1+\epsilon.$$

Now γ_1 is a future coray to γ_2 through p. We can therefore find a $t>r$ such that $p<<\gamma_2(t)$. But γ_2 is also a coray to γ_1 through $\gamma_2(t)$. There is then an $s\in\mathbb{R}$ such that $\gamma_2(t)<<\gamma_1(s)$. Recall from [27, p.403] that $I^+(\gamma_2(t))$ is an open neighbourhood of $\gamma_1(s)$. Since $\omega_n\to_p\omega$, there exist an $n_0\in\mathbb{N}$ and TF geodesics c_n for $n\geq n_0$ having $c_n(0)=p,\ [c_n]_+=\omega_n$. Their initial tangent vectors converge to $\gamma_1'(0)$. We can therefore assume n_0 is chosen to render $c_n(s)\in I^+(\gamma_2(t))$ for all $n\geq n_0$. It implies that $\omega_n\in I^+_\infty(q)$ for these values of n. Looking at the timelike geodesic triangles $q,\gamma_2(t),c_n(s)$ with side lengths a_n,b_n and e_n we can estimate

$$\begin{aligned}A_q\geq A_{qQ}&=\frac{\cosh(Qb_n)-\cosh(Qe_n)\cosh(Qa_n)}{\sinh(Qe_n)\sinh(Qa_n)}\\&\geq-\frac{\cosh(Qe_n)\cosh(Qt)}{\sinh(Qe_n)\sinh(Qt)}>-1-\epsilon,\end{aligned}$$

showing that $\omega_n\to_q\omega$.

The proof of the next proposition is quite similar to that of the previous one and is omitted.

Proposition 4.7. *If $p_n\to_p\omega$, then $p_n\to_q\omega$ for every $q\in M$ such that $\omega\in I^+_\infty(q)$.*

Time reversal of a TF geodesic produces a TP geodesic γ_-, that is, $\gamma_-(t)=\gamma(-t)$ for all $t\in\mathbb{R}$. Translations on the real line are denoted $\tau_a(t)=t+a,\ a,t\in\mathbb{R}$.

Proposition 4.8. *Let $\alpha\in M^-(\infty)$ and $\omega\in M^+(\infty)$ be causally related in a C_Q manifold (M,g) where $Q>0$. There exists a TF geodesic γ with $[\gamma_-]_-=\alpha$ and $[\gamma]_+=\omega$. If σ is a TF geodesic with $[\sigma_-]_-=\alpha$ and $[\sigma]_+=\omega$, then $\sigma=\gamma\circ\tau_a$ for some $a\in\mathbb{R}$.*

Proof. According to the definitions there exists a $p\in M$ such that $\alpha<<p<<\omega$. In other words there exist a TF geodesic γ_1 and a TP geodesic γ_2 such that

$$[\gamma_1]_+=\omega,\qquad[\gamma_2]_-=\alpha,\qquad\gamma_i(0)=p.$$

For all s larger than some positive s_0 the TF geodesic γ^s from $\gamma^s(0)=\gamma_2(s)$ to $\gamma^s(a_s)=\gamma_1(s)$, $a_s\in\mathbb{R}_+$ gives rise to the definition

$$t_s=\sup\{\ t\geq0\mid\gamma^s(t)\in J(\gamma_2(s),p)\ \}.$$

Due to [27, 14.1 and 14.5] there exists an NP geodesic β_s from $\beta_s(0) = p$ to $\beta_s(1) = \gamma_s(t_s)$. Lemma 6.1 implies that

$$-\langle \beta_s'(0), \gamma_2'(0)\rangle \leq \frac{\cosh(Qs_0)}{Q\sinh(Qs_0)},$$

which means we can take a sequence of real numbers $s_n \to +\infty$, indexed by $n \in \mathbb{N}$, such that the sequence $\{\beta_{s_n}'(0)\}$ is convergent with limit v. But then $\gamma^{s_n}(t_{s_n}) \to q \triangleq \exp_p(v)$ as $n \to +\infty$. Define $\beta_n = d_{s_n} \circ \tau_{s_n}$. The sequences $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\beta_{n-}\}_{n\in\mathbb{N}}$ have unit speed, future directed limit curves ξ_1 and ξ_2. They are by definition corays to γ_1 and γ_2 respectively. For an appropriate sequence $\{n_k\}$ in $\mathbb{N}$ we have convergence of

$$\beta_{n_k}'(0)\ ,\quad {\beta_{n_k}}_-'(0)$$

to $\xi_1'(0)$ and $-\xi_2'(0)$, which means that $\xi_1 = \gamma$ is a future coray to γ_1, and γ_- is a past coray to γ_2. This proves the existence.

In the uniqueness proof we may assume $\sigma(0) << \gamma(0)$. Due to Lemma 4.1 $\sigma(t) << \gamma(t)$ for all $t \in \mathbb{R}$ and for these values of t we may then define

$$f(t) = d(\sigma(t), \gamma(t)).$$

Since f is concave, $f'(t) \geq 0$ for all $t \in \mathbb{R}$. Let α_t denote the TF geodesic from $\sigma(t)$ to $\gamma(t)$ and define

$$N(s) = \frac{\partial\alpha}{\partial t}(s,0) + \langle\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial t}\rangle\frac{\partial\alpha}{\partial s}(s,0).$$

Using [5, p.374] we can estimate

$$f''(0) \leq -Q^2\int_0^{f(0)} \langle N, N\rangle(s)ds,$$

where $\langle N, N\rangle(0) = -1 + \langle\sigma'(0), \frac{\partial\alpha}{\partial s}(0)\rangle^2 \geq 0$. If this scalar product is nonzero, then $f''(0) < 0$, leading to the existence of a $t_1 \in \mathbb{R}$ with $f'(t_1) > 0$, in contradiction. We conclude that $\sigma'(0) = \frac{\partial\alpha}{\partial s}(0)$ and hence

$$\sigma(t + f(0)) = \gamma(t)$$

for all $t \in \mathbb{R}$. Hence the Proposition follows.

5. Hyperbolic isometries

Recall that the time orientation of our C_0 manifold (M, g) is a timelike continuous vector field X on M.

Definition 5.1. An isometry μ of (M, g) is time orientation preserving provided

$$\langle T\mu(X), X\rangle(\mu(p)) < 0$$

for all p in M.

Associated with a time orientation preserving isometry μ on (M, g) is a natural map on $M^+(\infty)$ and $M^-(\infty)$ defined by

$$\begin{aligned} \mu_+ &: M^+(\infty) \to M^+(\infty), \quad [\gamma]_+ \to [\mu \circ \gamma]_+, \\ \mu_- &: M^-(\infty) \to M^-(\infty), \quad [\gamma]_- \to [\mu \circ \gamma]_-. \end{aligned}$$

μ_+ and μ_- have inverses ${\mu^{-1}}_+$ and ${\mu^{-1}}_-$.

Definition 5.2. A time orientation preserving isometry μ on (M, g) is hyperbolic, provided there exists a p in M such that

$$\mu(p) << p \quad \text{or} \quad p << \mu(p). \tag{5.1}$$

A timelike axis γ of an isometry μ is a timelike TF geodesic or TP geodesic such that

$$\mu \circ \gamma(t) = \gamma(t + d_\mu)$$

for all $t \in \mathbb{R}$ and some $d_\mu \in \mathbb{R}_+$.

A null axis β of an isometry μ is an NF geodesic or NP geodesic such that

$$\mu \circ \beta(t) = \beta(\lambda t + d_\mu)$$

for all $t \in \mathbb{R}$ and some $\lambda, d_\mu \in \mathbb{R}$.

Theorem 5.3. *A hyperbolic isometry μ on a C_Q manifold, $Q > 0$, has a timelike axis.*

Proof. Since μ is hyperbolic, there exists a $p \in M$ such that (5.1) holds. By considering μ^{-1} instead if necessary we can suppose that $p << \mu(p)$, hence

$$p << \mu(p) << \dots. << \mu^n(p) << \dots$$

By $c_n, n \geq 1$, we denote the TF geodesic through $c_n(0) = p$ and $\mu^n(p)$, and d_n denotes the TF geodesic through $d_n(0) = \mu(p)$ and $\mu^n(p), n \geq 2$.

We claim that $\{c'_n(0)\}_{n\geq 1}$ and $\{d'_n(0)\}_{n\geq 2}$ are convergent sequences. To this end notice that

$$r = d(\mu^n(p), \mu^{n+1}(p)) = d(p, \mu(p)), \qquad n \geq 1,$$

and define $s_n \triangleq d(p, \mu^n(p)), n \geq 1$. The timelike geodesic triangle $p\mu^n(p)\mu^{n+1}(p)$ gives us

$$\begin{aligned} \cosh\ \theta_i &\triangleq -A_p \leq -A_{pQ} \\ &= \frac{-\cosh(Qr) + \cosh(Qs_n)\cosh(Qs_{n+1})}{\sinh(Qs_n)\sinh(Qs_{n+1})} \\ &\leq (1 + 2\exp(-2Qs_n))^2(1 + 2\exp(-2Qs_{n+1}))^2 \\ &\leq 1 + \alpha\exp(-2Qnr) \triangleq 1 + x_n \end{aligned}$$

for all n greater than or equal to some $n_0 \in \mathbb{N}$, because $s_n \geq nr$. Here θ_i is a nonnegative real number and $x_n, \alpha \in \mathbb{R}_+$. But then

$$\begin{aligned} \textstyle\sum_{n\geq n_0}\theta_n &\leq \textstyle\sum_{n\geq n_o}\log(1 + x_n + ((1+x_n)^2 - 1)^{\frac{1}{2}}) \\ &\leq \textstyle\sum_{n\geq n_0}\log(1 + \beta\exp(-nQr)) \\ &\leq \textstyle\sum_{n\geq n_0}\beta\exp(-nQr) \\ &= \beta\exp(-n_0Qr)/(1 - \exp(-Qr)) \end{aligned} \tag{5.2}$$

for some sufficiently large positive β. Let Λ_p^+ denote the future time cone in T_pM, and $T_p^{-1}M^+$ the set of unit length future directed vectors in T_pM. According to [27, pp. 144 and 156],

$$d^+ \ :\ T_p^{-1}M^+ \times T_p^{-1}M^+ \to \mathbb{R}, \quad (x,y) \mapsto \cosh\,{}^{-1}_{|\mathbb{R}\setminus\mathbb{R}_-}(-\langle x, y\rangle)$$

is well defined and a metric on $T_p^{-1}M^+$. Due to (5.2) the sequence $\{c'_n(0)\}_{n\in\mathrm{N}}$ is contained in the compact set

$$\{w \in T_p^{-1}M^+ \mid d^+(w, c'_1(0)) \leq R\},$$

when $R \in \mathbb{R}_+$ is large enough. d^+ induces a complete metric space structure on this set. According to (5.2), $\{c'_n(0)\}_{n\in\mathrm{N}}$ is a Cauchy sequence in this metric space and hence convergent as claimed. It follows that also $\{d'_n(0)\}_{n\geq 2}$ is convergent.

By c and d we denote the TF geodesics with initial velocities

$$\lim_{n\to+\infty} c'_n(0) \quad \text{and} \lim_{n\to+\infty} d'_n(0)$$

respectively. To show that c and d are future corays argue as in Lemma 4.1 to verify that $c(t) << d(t)$ for all $t > 0$ and that there exists a positive real number K such that

$$d(c(t), d(t)) \leq K$$

for all $t \geq 0$. Hence $[c]_+ = [d]_+$. Notice that $d(p, \mu^n(p)) \geq nr$ by the reverse triangle inequality. By the definition of convergence in Chapter four we have

$$\mu^n(p) \to_p [c]_+.$$

Proposition 4.7 yields

$$\mu^{n+1}(p) \to_{\mu(p)} [c]_+.$$

Since we also have

$$\mu(\mu^n(p)) \to_{\mu(p)} [\mu \circ c]_+,$$

we deduce that $[c]_+$ is a fixed point for μ_+. Time orientation reversal produces a fixed point $[e]_-$, $e \in \Omega_{TP}$, for μ_-. Proposition 4.8 asserts the existence of a TF geodesic γ with $[\gamma]_+ = [c]_+$ and $[\gamma_-]_- = [e]_-$. This γ is an axis for μ due to the uniqueness part in Proposition 4.8, i.e., $\mu \circ \gamma = \gamma \circ \tau_{d_\mu}$. To see that $d_\mu > 0$ let s_* denote the smallest real number such that $\gamma(s_*) \in J^+(p)$, hence $\gamma(s_*) \notin I^+(p)$. Since $p << \mu(p)$, we conclude that

$$\gamma(s_*) \notin J^+(\mu(p)).$$

But $\mu \circ \gamma(s_*) = \gamma(s_* + d_\mu) \in J^+(\mu(p))$. Using [27, Corollary 14.1] we deduce that $d_\mu > 0$, and the Theorem follows.

Example 5.4. The linear map with matrix representation

$$\begin{pmatrix} \cosh\phi & \sinh\phi & 0 \\ \sinh\phi & \cosh\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \phi \in \mathbb{R}_+,$$

in the standard basis in $\mathbb{R}^3_1$ is an isometry of $\mathbb{R}^3_1$. The restriction of this isometry to $M_Q, Q > 0$ is a hyperbolic isometry μ. For appropriately chosen causally related points p and q a suitable conformal change of the metric on $I(p,q)$ and its translates by μ result in a nonconstantly curved C_Q manifold with a hyperbolic isometry and $Q > 0$.

6. Null colines

The concept null coline is crucial to the structure theorems in chapter 7. Their definition relies on Lemmas 6.1 and 6.2 to be derived. To this end let γ_1 and γ_3 be TF geodesics, and γ_2 an NF geodesic in a C_Q manifold, $Q > 0$. So γ_2 is a future directed complete null geodesic. The three geodesics γ_1, γ_2 and γ_3 form a nonspacelike geodesic triangle with vertices

$$\begin{aligned} \gamma_1(0) &= p, & \gamma_1(a) &= q, \\ \gamma_2(0) &= q, & \gamma_2(1) &= r, \\ \gamma_3(0) &= p, & \gamma_3(c) &= r. \end{aligned}$$

Let us introduce the following notation:

$$e = -\langle \gamma_1'(a), \gamma_2'(0) \rangle, \quad d = -\langle \gamma_2'(1), \gamma_3'(c) \rangle,$$

Then we have the following inequalities.

Lemma 6.1.

$$\begin{aligned} &\cosh(Qc) \leq \cosh(Qa) + Qe\sinh(Qa), \\ (6.1, 2, 3) \qquad &1 \leq \cosh(Qa)\cosh(Qc) + \sinh(Qa)\sinh(Qc)A_p, \\ &\cosh(Qa) \leq \cosh(Qc) - \sinh(Qc)Qd. \end{aligned}$$

Proof. According to [15, Corollary 2.5]. there are open neighbourhoods U of $\gamma_2'(0)$ and V of $\gamma_3(c)$ such that

$$F = \exp_{q|U} U \to V$$

is a diffeomorphism. Define nonnegative reals c_s and d_s by

$$\begin{aligned} c_s^2 &= -\langle \exp_p^{-1}(\gamma_3(s)), \exp_p^{-1}(\gamma_3(s)) \rangle, \\ d_s^2 &= -\langle F^{-1}(\gamma_3(s)), F^{-1}(\gamma_3(s)) \rangle \end{aligned}$$

for $s \geq c$. Looking at the timelike geodesic triangle $p, q = q_s, \gamma_3(s)$ we find that

$$\sinh(Qd_s)A_{q_s} \to Qe$$

for $s \to c$. Then Lemma 2.2 yields

$$\cosh(Qc_s) \leq \cosh(Qd_s)\cosh(Qa) + \sinh(Qd_s)\sinh(Qa)A_{q_s}.$$

Convergence to $s = c$ leads to (6.1). (6.2) and (6.3) are similar.

Suppose we are given an NF geodesic and a point $p \in M$ such that $p << \beta(s_*), s_* \in \mathbb{R}$. We can then define v_s by

$$\beta(s) = \exp_p(v_s\ d(p, \beta(s))) \tag{6.4}$$

for $s \geq s_*$.

Lemma 6.2. *There exists a $v \in T_p^{-1}M^+$ such that*

$$d(p, \beta(s)) \to +\infty, \qquad v_s \to v,$$

as $s \to +\infty$.

Proof. Letting $a = d(p, \beta(s_*))$, $B_s = \langle v_{s_*}, v_s \rangle$ and $c_s = d(p, \beta(s))$ for $s > s_*$ we find, in consequence of Lemma 6.1,

$$\begin{aligned} B_s &\geq -\cosh(Qa)\cosh(Qc_s)/[\sinh(Qa)\sinh(Qc_s)] \\ &\geq -\cosh^2(Qa)/\sinh^2(Qa) = K. \end{aligned}$$

This follows from the fact that $s \mapsto c_s$ is smooth for $s > s_*$ with

$$\frac{d}{ds}c_s = -\langle \beta'(s), T_{v_s} \exp_p(v_s) \rangle.$$

We can then take a sequence $\{s_k\}_{k \in \mathbb{N}}$, $s_k \geq s_*$ such that v_{s_k} converges to some $v \in T_pM$ as $k \to +\infty$. If the Lorentzian distance from p to $\beta(s)$ were bounded by some d we would have

$$\begin{aligned} &\beta(s_k) \in \exp_p(C), \\ &C = \{sw | w \in T_p^{-1}M^+,\ K \leq \langle w, v_{s_*} \rangle \leq -1, 0 \leq s \leq d\} \end{aligned}$$

for all k larger than some $k_0 \in \mathbb{N}$. This however contradicts [27, 14.13] because β is future inextendible and (M, g) is strongly causal by Remark 2.3.

To show the second statement in this lemma use Lemma 6.1 in the estimate

$$\langle v_{t_1}, v_{t_2} \rangle \geq -\cosh(Qc_{t_1})\cosh(Qc_{t_2})/(\sinh(Qc_{t_1})\sinh(Qc_{t_2})).$$

Given $K \leq -1$, the right-hand side is greater than or equal to K for all t_1, t_2 larger than some t_*. This means that $\{v_s\}_{s \geq s_*}$ is a Cauchy net in the metric space

$$T_p^{-1}M^+ = \{w \in T_pM \mid \langle w, w \rangle = -1,\ \langle v_{s_*}, w \rangle < 0\}$$

with metric

$$d(v,w) = \cosh^{-1}_{\mathbb{R}-\mathbb{R}_-}(\langle v,w\rangle), \quad v,w \in T_p^{-1}M^+ :$$

cf. [27, p.156]. From this the second statement in the lemma follows.

Definition 6.3. The NF geodesic β is a future null coline to the TF geodesic γ through $\gamma(0) = p$, provided

$$\lim_{s\to+\infty} v_s = \gamma'(0).$$

Fortunately, we have

Proposition 6.4. *If β is a future null coline to γ, then β is a future null coline to any TF geodesic σ, which is a future coray to γ.*

Proof. According to Lemma 6.2 and the definitions,

$$\beta(n) \to_p \omega \triangleq [\gamma]_+$$

as $n \to +\infty$, where $p = \gamma(0)$. Proposition 4.7 tells us that

$$\beta(n) \to_q \omega = [\sigma]_+$$

as $n \to +\infty$ where now $q = \sigma(0)$. Thus the proposition follows.

7. Structure theorems

The future null cone in a C_Q manifold M of a point $p \in M$ is by definition

$$K^+(p) = \{q \in M \mid p < q,\ q \notin I^+(p)\}. \tag{7.1}$$

Also define

$$\mathbb{D}^+ = \{(p,q) \in M \times M \mid p < q,\ q \notin I^+(p)\}. \tag{7.2}$$

We shall show that (7.1) and (7.2) are C^∞ submanifolds of M and $M \times M$ respectively. $K^+(p)$ is degenerate of constant signature (0,+,...,+). This will imply that the square of the Lorentzian distance function is smooth on

$$C^+ = \{(p,q) \in M \times M \mid p \leq q\}. \tag{7.3}$$

We start with

Lemma 7.1. *Suppose $q = \exp_p(v)$ for some future directed null vector $v \in T_pM$ in a C_Q manifold (M, g), $Q \geq 0$. Then $q \notin I^+(p)$.*

Proof. Assume for contradiction that $p << q$. According to Lemma 2.1 there exists a TF geodesic γ from p to $q = \gamma(a), a \in \mathbb{R}_+$. Take open neighbourhoods U around v in $\mathbb{D}(\exp_p)$ and V around q such that the restriction of $\exp_p$ to U is a diffeomorphism onto V. An open interval I around a is mapped by γ into V. Define

$$\begin{aligned} \sigma \ &: \ I \to T_pM, \quad t \mapsto \exp_{p|U}^{-1} \circ \gamma(t), \\ f(t) &= \langle \sigma(t), \sigma(t) \rangle, \qquad t \in I. \end{aligned}$$

Notice that $a\gamma'(0) \notin U$, since this would imply that the timelike vector $a\gamma'(0)$ is equal to the null vector v. Since the scalar product

$$\langle \gamma'(a), T_{\sigma(a)} \exp_p(\sigma(a)) \rangle = \frac{1}{2} f'(a)$$

of two future directed nonspacelike vectors is negative, there exists a positive $t \in I$ such that $\sigma(t) \neq t\gamma'(0)$ is a timelike future directed vector. But this means that

$$\exp_p(\sigma(t)) = \gamma(t) = \exp_p(t\gamma'(0)),$$

contradicting Lemma 2.1.

We continue with a lemma, involving

$$\Lambda^{0+} = \{w \in TM \mid \langle w, w \rangle \leq 0, \ \langle w, X \rangle \leq 0\}. \tag{7.4}$$

Dually Λ^{0-} consists of the set of w in TM such that $-w \in \Lambda^{0+}$.

Lemma 7.2. *Let v be a future directed null vector in a C_Q manifold, $Q \geq 0$. Then there exists an open neighbourhood W around v in $\mathbb{D}(E)$ such that*

$$E(w) \notin \mathbb{C}_*^+ = \{(p, q) \in M \times M \mid p < q\} \tag{7.5}$$

for any $w \in W \backslash \Lambda^{0+}$.

Proof. Take a timelike future directed vector

$$w = T_v \exp_p(z),$$

where $p = \pi(v)$. Let Y denote a smooth vector field on an open neighbourhood $\mathbb{D}(Y)$ of z in TM with $Y(v) = z$. We can assume that $T\pi(Y) \equiv 0$. Since

$$\Lambda^0 = \{x \in TM \mid x \text{ is a future directed null vector }\}$$

is a hypersurface in TM, we can take a local flow

$$\Phi \ :]-\epsilon, \epsilon\, [\ \times U \to \mathbb{D}(Y)$$

for Y around v and adapted to Λ^0. Since $Y(v) = z$, we can assume that

$$\exp \circ \Phi_w \ :]-\epsilon, \epsilon\, [\ \to M$$

is a smooth timelike future directed curve for all $w \in U$, by taking a smaller U and $\epsilon > 0$ if necessary. We can now define F to be the restriction of Φ to $]-\epsilon, \epsilon\, [\ \times \Lambda^0_U$, where $\Lambda^0_U = \Lambda^0 \cap U$. By adjusting the domain of definition we can assume that F is a diffeomorphism, because $T_{(0,v)}F$ is an isomorphism. In fact $Y(v) \notin T_v\Lambda^0$ due to the fact that $Y(v) = z$; cf. [13, Proposition 2.2]. If the domain of definition of F is sufficiently small, the restriction of

$$E \ : \mathbb{D}(E) \to M \times M, \quad v \mapsto (\pi(v), \exp(v))$$

to $W = \operatorname{Im} F$ will be a diffeomorphism onto its open image; cf. [13, Proposition 2.1].

Suppose $w \in W \backslash \Lambda^{0+}$, that is $w = F(t,u)$, $(t,u) \in \mathbb{D}(F)$. By construction F_u is a smooth timelike future directed curve in T_qM, $q = \pi(w)$. If t was nonnegative, using the causality relations in T_qM we would have

$$0_q < u \quad \begin{cases} = F(t,u), & t = 0, \\ << F(t,u), & t > 0. \end{cases}$$

Consequently w is in the causal future $J^+(0_q)$ of the zero vector 0_q in T_qM. This contradicts the fact that $w \in W \backslash \Lambda^{0+}$. Thus $t < 0$. If $(x,y) = E(w)$ was in $\mathbb{C}^+_*$, we would have

$$x < y = \exp(F(t,u)) << \exp(F(0,u)) = \exp(u),$$

contradicting Lemma 7.1. Consequently $E(w) \notin \mathbb{C}^+_*$ and the lemma follows.

From Lemmas 7.1 and 7.2 we deduce

Proposition 7.3. *The square of the Lorentzian distance function* $d^2 : M \times M \to \mathbb{R}$ *is smooth on* C^+.

Proof. Let us first consider $(p, q) = (p, \exp_p(v)) \in \mathbb{D}^+ \subset C^+$, where $v \in \Lambda^0$. Take an open neighbourhood W around v in TM such that the restriction of E to W is a diffeomorphism onto its open image and such that (7.5) holds. A careful choice of W ensures that $E(W)$ has empty intersection with the diagonal in $M \times M$. Define a smooth function F on $E(W)$ by

$$F(x, y) = -\langle w, w\rangle \quad , \quad w = E^{-1}_{|W}(x, y). \tag{7.6}$$

If $(x, y) \in C^+ \cap E(W)$, then by [5, Theorem 10.16] we have

$$d(x, y)^2 = F(x, y), \tag{7.7}$$

whenever $x << y$. We can obtain (7.7) by (7.5) when $x < y$, $y \notin I^+(x)$. The remaining cases $p << q$ and $p = q$ follow from the openness of $<<$, the strong causality of (M, g) and [5, Theorem 10.16].

Given a point p in a C_Q manifold (M, g), $Q \geq 0$, we can now prove

Theorem 7.4. $\mathbb{D}^+$ *and* $K^+(p)$ *are* C^∞ *hypersurfaces of* $M \times M$ *and* M *respectively. The metric induced on* $K^+(p)$ *has constant signature* $(0,+,\dots,+)$.

Proof. If $(p, q) \in \mathbb{D}^+$, then according to [27, p. 404], there exists a future directed null vector $v \in T_pM$ such that $q = \exp_p(v)$. By Lemma 7.2 there exists an open neighbourhood W around v such that the restriction of E to W is a diffeomorphism onto its open image and

$$E(w) \notin \mathbb{C}^+_*, \tag{7.8}$$

whenever $w \in W \backslash \Lambda^{0+}$. We claim that

$$E(W \cap \Lambda^0) = E(W) \cap \mathbb{D}^+. \tag{7.9}$$

The left-hand side is a subset of the right-hand side according to Lemma 7.1. The reverse inclusion follows from (7.8). Combining (7.9) with the fact that Λ^0 is a hypersurface in TM we conclude that $\mathbb{D}^+$ is a hypersurface in $M \times M$.

To show that the smooth submanifold $M(p) = \{p\} \times M$ of $M \times M$ is transversal to $\mathbb{D}^+$ take $(p, q) = E(v)$, $v \in \Lambda^0$ and observe that

$$T_{(p,q)}\mathbb{D}^+ + T_{(p,q)}M(p) = T_vE(T_v\Lambda^0 + T_vi(T_pM)),$$

where $i : T_pM \to TM$ denotes the inclusion. If we define $\alpha(t) = v + tw$ for some timelike $w \in T_{\pi(v)}M$, then $T_v i(\alpha'(0)) \notin T_v\Lambda^0$. Thus $\mathbb{D}^+$ is transversal to $M(p)$. It follows that

$$\mathbb{D}^+ \cap M(p) = \{p\} \times K^+(p)$$

is a smooth submanifold of $M \times M$. A codimension count shows that $K^+(p)$ is a C^∞ hypersurface of M.

The squared Lorentzian distance function f is smooth on C^+ by Proposition 7.3 and

$$T_qK^+(p) = \text{grad } f_p(q)^\perp.$$

Since grad $f_p(q)$ is null, the last statement of the lemma follows.

Lemma 7.5. *C^+ is closed.*

Proof. Let $(p,q) \in M \times M$ denote the limit point of some sequence $\{(p_n, q_n)\}_{n\in\mathbb{N}}$ from C^+, converging in $M \times M$. To show that (p,q) belongs to C^+ take $r \in M$ such that $p, q \in I^-(r)$. According to Proposition 7.3 there exist an $n_0 \in \mathbb{N}$ and $K > 0$ such that $p_n << r$ and $c_n = d(p_n, r) \geq K$ for all $n \geq n_0$. Let β_n and γ_n denote nonspacelike or constant geodesics from p_n to $q_n = \beta_n(1)$ and $r = \gamma_n(c_n)$ respectively. It follows from Lemmas 2.1 and 6.1 that

$$-\langle \beta_n'(0), \gamma_n'(0)\rangle \tag{7.1}$$

is bounded above by some $C > 0$ for all $n \geq n_0$. We can assume that the sequence $\{p_n\}_{n\geq n_0}$ belongs to the domain of some orthonormal frame $E_1, ..., E_n$ with E_1 timelike, future directed, and write

$$v_n = \beta_n'(0) = \sum_i \lambda_i E_i \quad w_n = \gamma_n'(0) = \sum_i \mu_i E_i.$$

We have an upper bound D on μ_1 since w_n is a convergent sequence. We can now use (7.10) and the Schwartz inequality to get

$$\lambda_1\mu_1 \leq C + \lambda_1(\mu_1^2 - 1)^{\frac{1}{2}},$$

and hence

$$\lambda_1^2 - 2C\lambda_1(D^2 - 1)^{\frac{1}{2}} - C^2 \leq 0.$$

This shows that there is an upper bound to the absloute value of the λ_i. A subsequence $\{v_{n_k}\}$ of $\{v_n\}$ will then converge to some nonspacelike or zero vector v showing that $(p,q) = (p, \exp_p(v)) \in C^+$.

Lemma 7.6. *$F : \Lambda^{0+} \to C^+ v \mapsto (\pi(v), \exp(v))$ is a homeomorphism.*

Proof. To prove injectivity suppose that $F(v_1) = F(v_2)$, so that $p = \pi(v_1) = \pi(v_2)$. If v_1 and v_2 are timelike, then $v_1 = v_2$ by Lemma 2.1. $v_1 = 0$ and v_2 nonspacelike contradicts the strong causality of (M, g). v_1 timelike and v_2 null is impossible by Lemma 7.1. It remains to consider the case, where v_1 and v_2 are both null vectors. To this end define $\beta_i(s) = \exp_p(sv_i)$, $i = 1, 2$ and observe that

$$\beta_i'(1) \in T_{\beta_1(1)}K^+(p).$$

It follows from Theorem 7.4 that $\beta_1'(1) = \beta_2'(1)\lambda$ for some $\lambda \neq 0$. The strong causality of (M, g) implies that $\lambda = 1$, so that $v_1 = v_2$.

Since F is onto by [27, p. 402] we conclude that F is a bijection with inverse G. Due to Lemma 2.1, G is smooth on some open neighbourhood of the image by F of any timelike future directed vector.

1) We now insist that G is smooth in the image by F of some zero vector $v \in \Lambda^{0+}$ by taking an open neighbourhood V around 0 in TM. We shall require that $0_{\pi(v)} \in V$ whenever $v \in V$ and also that the restriction of E to V is a diffeomorphism onto its open image. Take a causally convex open neighbourhood U of $\pi(v)$ such that $U \times U \subset E(V)$ and define $W = E_{|V}^{-1}(U \times U)$.

Suppose $\exp(w) \in J^+(\pi(w))$ for some $w \in W$. By definition this means that either $w = 0$ or there exists some smooth nonspacelike curve $\alpha : [0, a] \to M$ from $\alpha(0) = \pi(w) = q$ to $\exp_q(w) = \alpha(a) \neq q$. Since U is causally convex we can define

$$\beta(t) = E_{|W}^{-1}(q, \alpha(t)), \quad t \in [0, a],$$

which is a smooth curve in the future causal cone of T_qM by [27, Lemma 5.33], hence $\beta(a) = w \in \Lambda^{0+}$. We have shown

$$E(w) \notin C^+, \tag{7.11}$$

when $w \in W \backslash \Lambda^{0+}$.

2) Around any null vector v there is an open neighbourhood W in $\mathbb{D}(E)$ such that (7.11) holds and such that the restriction of E to W is a diffeomorphism onto its open image. This follows from Lemma 7.2.

In both cases the restriction of G to $E(W) \cap C^+$ coincides with the restriction of $E_{|W}^{-1}$ to $E(W) \cap C^+$. G is hence smooth in the image by F of any null or zero vector. The lemma follows.

Theorem 7.7. *Let $\omega = [\gamma]_+$ and $\alpha = [\gamma_-]_-$ belong to the timelike future $M^+(\infty)$ and the timelike past $M^-(\infty)$ of a C_Q manifold (M, g) with $Q > 0$. Then the following hold:*

1) $\partial I^-(\omega)$ $(\partial I^+(\alpha))$ is a C^1 hypersurface in M of constant signature $(0,+,\dots,+)$. Through every point in $\partial I^-(\omega)$ there is a future null coline β to γ.

2) The intersection $\partial I^-(\omega) \cap \partial I^+(\alpha)$ is a C^1 Riemannian manifold of dimension dim $M - 2$.

Proof. To prove 1) for any $\omega = [\gamma]_+ \in M^+(\infty)$, suppose $x \in \partial I^-(\omega)$.

We claim that there exists a future null coline β to γ through $\beta(0) = x$. To this end let ζ denote some TF geodesic through $\zeta(0) = x$. $\zeta(t)$ is in $I^-(\gamma)$ for $t < 0$ because then $I^+(\zeta(t))$ is an open neighbourhood of x. For a suitable increasing sequence $\{u_n\}$ converging to 0, the sequence $v(u_n)$ with

$$v(u) \triangleq -V \circ \zeta(u)/\langle V, \zeta' \rangle, \quad u < 0$$

will converge to some $v \in T_xM$. Here V denotes the vector field, that assigns to each $x \in I^-(\gamma)$ the tangent vector to the future coray from x to γ. We need to know that for every $t > 0$ such that $x \notin J^+(\gamma(t))$, the set

$$K_t = I^-(\omega) \cap (K^+(\gamma(t)) \cup \{\gamma(t)\})$$

is contained in a compact set, C_t say. But to any $q \in K_t$ there exists an NF geodesic β from $\gamma(t)$ to q. Since $q \in I^-(\omega)$ there is a TF geodesic σ from q to some $\gamma(t + c) = \sigma(a)$ with $\cosh(Qc)/\sinh(Qc) \le 2$. The existence of a compact set C_t containing K_t now follows from Lemma 6.1 showing that

$$-\langle \beta'(0), \gamma'(t) \rangle \le (\cosh(Qc) - \cosh(Qa))/Q\sinh(Qc) \le 2/Q.$$

For every $u < 0$ there exists an $s_u > 0$ such that

$$\exp(s_u v(u)) \in I^+(\gamma(t + 1)),$$

and hence also a $t_u \in [0, s_u[$ such that

$$q_u = \exp(t_u v(u)) \in K_{t+1}.$$

If D is a compact neighbourhood of x, then for all $u < 0$ sufficiently close to 0, we have

$$t_u \; v(u) = F^{-1}(\zeta(u), q_u) \in F^{-1}((D \times C_{t+1}) \cap C^+).$$

This set is compact by Lemma 7.5 and Lemma 7.6. This means that the convergence of $\{t_{u_n} v(u_n)\}_{n\in\mathrm{N}}$ to some w can be assumed by taking a subsequence of $\{u_n\}$ if necessary. The nonspacelike geodesic $\beta(s) = \exp_x(sw)$ is then in $I^+(\gamma(t))$ for all $s \geq 1$. Looking at the timelike geodesic triangle $\gamma(0)\gamma(t)\beta(s)$ with side lengths t, c and b we find

$$\begin{aligned}\langle c_s'(0), \gamma'(0)\rangle &= (\cosh(Qc) - \cosh(Qb)\cosh(Qt))/[\sinh(Qb)\sinh(Qt)] \\ &\geq -\cosh^2(Qt)/\sinh^2(Qt).\end{aligned}$$

Here c_s is the TF geodesic from $c_s(0) = \gamma(0)$ to $c_s(b) = \beta(s)$. Since $x \notin I^-(\omega)$ we infer that β is a future null coline to γ through x, thereby proving the claim. Notice that $\sigma(s) \in \partial I^-(\omega)$ for all $s \in \mathbb{R}$, due to the convergence of $\{t_{u_n} v(u_n)\}$ to w.

Define $p = \beta(s), q = \beta(-s)$ and $f_s : J^-(\beta(s)) \to \mathbb{R}$ by

$$f_s(y) = d(y, \beta(s))$$

for $s > 0$. Take some $z \in I^+(p)$. Since $z >> x$, there exists some past directed timelike vector v in T_zM such that $\gamma_v(t_*) = x$ for some $t_* > 0$. We shall now prove a sequence of four claims, leading to a proof of the first statement.

First claim. We claim the existence of an open neighbourhood U of v in the set $T_z^{-1}M^-$ of past directed timelike unit vectors in T_zM and a C^∞ function t_s on U such that $t_s(v) = t_*$ and for all w in U we have

$$\gamma_w(t_s(w)) \in J^-(\beta(s)) \quad , \quad f_s(\gamma_w(t_s(w))) = 0.$$

To see this take an open neighbourhood V of $-s\beta'(s)$ in $TM\backslash\Lambda^{0+}$ such that the restriction of $\exp_p$ to V is a diffeomorphism onto its open image and such that $\exp_p(w) \notin J^-(p)$ for any $w \in V\backslash\Lambda^{0-}$; cf. Lemma 7.2. Define

$$g_s(y) = -\langle w, w\rangle \quad , \quad w = \exp_{p|V}^{-1}(y)$$

for $y \in \exp_p(V)$. For some open neighbourhood Ω of (v, t_u) in $T_z^{-1}M^- \times \mathbb{R}$, $\exp_z(wt) \in \exp_p(V)$ for all $(w, t) \in \Omega$. The claim now follows from an application of the inverse function theorem to the function

$$G_s \ : \ \Omega \to \mathbb{R} \ , \quad (t, w) \mapsto g_s(\exp_z(wt)).$$

Second claim. For all $w \in U$ and all $u > s$ there exists a unique $t_u(w) \in [0, t_s(w)]$ such that

$$\gamma_w(t_u(w)) \in J^-(\beta(u)), \quad f_u(\gamma_w(t_u(w))) = 0. \tag{7.12}$$

Notice that $\gamma_w(t_s(w)) \in J^-(\beta(s)) \subset J^-(\beta(u))$; cf. [27, 14.1 and 14.6]. This means that

$$t_u(w) = \inf\{t \in [0, t_s(w)] \mid \gamma_w(t) \in J^-(\beta(u))\} > 0$$

satisfies (7.12). If some $t \in]0, t_u(w)[$ also satisfies the claim, then

$$\gamma_w(t_u(w)) << \gamma_w(t) < \beta(u).$$

Consequently $\gamma_w(t_u(w)) \in I^-(\beta(u))$. Since this is untrue, the uniqueness of $t_u(w)$ follows.

Third claim. The function $u \mapsto t_u(w) = t_w(u) > 0, u > s$ is decreasing, hence convergent.

This follows from the above definition of $t_u(w)$ and the fact that $J^-(\beta(u_1)) \subset J^-(\beta(u_2))$, whenever $s < u_1 < u_2$.

We can now define a function t on U by declaring

$$t(w) = \inf_{u \geq s} t_u(w), \qquad w \in U.$$

Clearly $r = \gamma_w(t(w)) \in$ closure $I^-(\omega)$; cf. [27, 14.6 (2)]. Assume for contradiction that $r \in I^-(\omega)$; i.e., $r << \gamma(a)$ for some $a \in \mathbb{R}$. Since β is a null coline to γ, $\beta(b) \in I^+(\gamma(a))$ for some $b > s$. Define

$$A = \{t \in [0, t(w)] \mid \exp_z(tw) \in J(r, \beta(b))\},$$

which is closed by global hyperbolicity of (M, g). Hence $t_* \in A$, where $t_* = \inf A \in [0, t(w)[$. Since $<<$ is open,

$$\exp(t_* w) \in J^-(\beta(b)), \quad f_b(\exp(t_* w)) = 0,$$

contradicting the definition of $t(w)$. Thus $r = \gamma_w(t(w)) \in \partial I^-(\omega)$.

Fourth claim. t is C^1.

To see that t is differentiable in $w \in U$ define $y = \exp_z(wt(w)) \in \partial I^-(\omega)$. We know that there exists a future null coline β_y to γ through $\beta_y(0) = y$. Take some $s > 0$ and some $z \in I^+(\beta_y(s))$. According to the first claim and its past dual around any $w \in U$ there exist an open neighbourhood U_w of w and C^∞ functions t^+ and t^- such that $t^+(w) = t^-(w) = t(w)$ and

$$\gamma_u(t^+(u)) \in K^-(\beta_y(s)), \quad \gamma_u(t^-(u)) \in K^+(\beta_y(-s))$$

for all $u \in U_w$. Since $\gamma_u(t^+(u)) \in J^-(\beta_y(s)) \subset$ closure $I^-(\omega)$, $t^+(u) \geq t(u)$.

Lemma 14.1 in [27] tells us that there is no $t \geq 0$ such that $\beta_y(t) \in I^-(\omega)$. This will not happen for a negative t either according to Lemma 7.1. Using [27, 14.1] once more we deduce that $\gamma_u(t^-(u)) \notin I^-(\omega)$ and hence $t^-(u) \leq t(u)$. From the fact that $t^-(w) = t(w) = t^+(w)$ it follows that t is differentiable.

To see that t is C^1 in $w \in U$ let s denote a C^∞ function on an open neighbourhood Ω of $(w, \beta_y(1))$ in $T_z^{-1}M^- \times M$ such that $s(w,y) = t(w)$ and

$$\gamma_v(s(v,x)) \in K^-(x)$$

for every $(v,x) \in \Omega$. Give some open neighbourhood W of y in $\partial I^-(\omega)$ a Riemannian metric h. The tangent vectors to differentiable curves through q in W span a subspace

$$A_q = T_qK^-(\beta_y(1))$$

in T_qW. It has signature (0,+,..,+) according to Theorem 7.4. We can then define a vector field X on W. The value of X at q is the unique future directed null vector of unit Riemannian length in A_q. Notice that $\beta_{X(q)}$ is a future null coline to γ for all $q \in W$.

If X is not continuous at some $q \in W$, there exists a sequence $\{q_n\}$ in W such that $\{q_n\}$ and $\{X(q_n)\}$ converge to q and $Y_q \neq X(q)$ respectively. For some $s < 0$, $\beta_{Y(q)}(s) \in I^-(\omega)$, because $Y(q)$ and $X(q)$ are linearly independant null vectors and $\beta_{X(q)}(1) \in \partial I^-(\omega)$. Continuity of the geodesic flow implies that

$$\beta_{X(q_n)}(s) \in I^-(\omega)$$

for some sufficiently large n. This contradicts Lemma 7.1. Thus

$$v \mapsto dt_v = ds_{(v, \beta_{X(\exp_z(t(v)v))}(1))}$$

is continuous.

The map $w \mapsto \exp_z(wt(w))$ from U to $\partial I^-(\omega)$ gives rise to a chart in a C^1 submanifold structure on $\partial I^-(\omega)$. The tangent space to $\partial I^-(\omega)$ at $x \in \partial I^-(\omega)$ coincides with the tangent space to $K^-(\beta_x(s))$, where β_x is a future null coline to γ through x and $s > 0$. Theorem 7.3. now tells us that $\partial I^-(\omega)$ has constant signature (0,+,...,+).

2) To verify that $\partial I^-(\omega) \cap \partial I^+(\alpha)$ is nonempty let σ denote some TF geodesic with $\sigma(0) = \gamma(0)$ and

$$A = \langle \sigma'(0), \gamma'(0) \rangle \neq -1.$$

Choose $t > 0$ subject to the requirement

$$A^2 - \cosh^2(Qt)/\sinh^2(Qt) > 0.$$

The reverse triangle inequality shows that we can find some s_* such that $\sigma(s_*) \in J^-(\gamma(t)) \backslash I^-(\gamma(t))$. Using Lemma 6.1 we derive

$$(\cosh^2(Qs) - 1)\sinh^2(Qt)A^2 \leq (\cosh(Qt)\cosh(Qs) - 1)^2. \tag{7.13}$$

We conclude that $\cosh(Qs) \leq K$ for some positive K regardless of the value of t. It follows that $y = \sigma(s^*) \in \partial I^-(\omega)$ for some $s^* > 0$. We have seen that there exists a future null coline β_y to γ through $\beta_y(0) = y$ and that the counter image by the restriction of $\exp_y$ to $\Lambda^{0-}(y) = \Lambda^{0-} \cap T_yM$ of $K^-(y) \cap I^+(\alpha)$ is contained in some compact set in T_yM. We can then find a $t_* < 0$ such that

$$x = \beta_y(t_*) \in \partial I^+(\alpha) \cap \partial I^-(\omega).$$

Combining Proposition 4.8. and Lemma 7.1 we see that $\beta_y'(t_*)$ cannot belong to the tangent space to $\partial I^+(\alpha)$ at x, because the signature of this vector space is (0,+,..,+). We conclude that $\partial I^-(\omega)$ and $\partial I^+(\alpha)$ have nonempty transversal intersection in a C^1 submanifold N of M. There is also a past null coline β_x through x to γ_-. N is Riemannian because

$$\beta_y'(t_*) - \beta_x'(0)$$

is a timelike vector orthogonal to T_xN. Hence the Theorem follows.

Example 7.8. There are C_Q Robertson Walker spacetimes of nonconstant sectional curvature $Q > 0$. In fact let g_Q denote the metric on $M_Q \subset \mathbb{R}^{n+1}$, and f_Q the restriction to M_Q of a smooth function f on $\mathbb{R}^{n+1}$, depending only on the first coordinate. With a suitable choice of f the timelike sectional curvatures of $(M_Q, f_Q g_Q)$ will be bounded below from zero by some $Q_* \in]0, Q[$. Nonspacelike completeness of $(M_Q, f_Q g_Q)$ follows from [26, Lemma 14.13]. It is future 1-connected, because this is invariant under conformal changes of the metric. Dynamic properties of the geodesic flow on Lorentzian manifolds have been considered in [1], [10] and [32].

8. Density of timelike periodic geodesics

In this section we shall show that the timelike periodic geodesics are dense in the future timelike unit tangent bundle

$$T^{-1}M^{+} = \{v \in TM | \langle v, v\rangle = -1, \; v \text{ future directed}$$

of a C_Q manifold (M, g) with Q positive and with vicious Deck transformation group. We shall proceed to define this concept.

Let π denote the tangent bundle projection, and $D : TTM \to TM$ the connection map; see [18]. The tangent bundle to M at $v \in TM$ decomposes into

$$T_vTM = \text{ HOR } T_vTM \oplus \text{ VER } T_vTM,$$

where HOR T_vTM is the kernel of D_v, and VER T_vTM is the kernel of $T_v\pi$. Any $w \in T_vTM$ decomposes uniquely as

$$w = w^h + w^v, \qquad w^h \in \text{ HOR } T_vTM, \;\; w^v \in \text{ VER } T_vTM.$$

The tangent bundle to M carries a canonical metric G, defined by

$$G(w_1, w_2) = g(w_1^h, w_2^h) + g(w_1^v, w_2^v) \qquad w_1, w_2 \in T_vTM$$

and this induces a Lorentzian metric on the future timelike unit tangent bundle $T^{-1}M^{+}$.

Definition 8.1. A group Γ of isometries on (M, g) acting properly discontinuous on $T^{-1}M^{+}$ is vicious provided $T^{-1}M^{+}/\Gamma = N$ is time orientable and totally vicious, that is,

$$I^{+}(p) \cap I^{-}(p) = N$$

for all $p \in N$.

Remark 8.2. Suppose Γ is a group of isometries on (M, g) acting properly discontinuous on $T^{-1}M^{+}$. According to [27, p. 191] there is a unique differentiable structure and metric on $T^{-1}M^{+}/\Gamma$ making the natural map $\pi_\Gamma \ : T^{-1}M^+ \to T^{-1}M^+/\Gamma = N$ a semi Riemannian covering map. N will always be given this differentiable structure. The reader may consult remark 9.6 for examples of vicious groups of isometries.

Suppose $\omega \in M^+(\infty)$ and $\alpha \in M^-(\infty)$ are causally related, i.e., there exists a $p \in M$ such that $p << \omega$ and $p >> \alpha$ respectively. According to Proposition 4.5 there exist a unique TF geodesic γ_1 and a unique TP geodesic γ_2 from p to $[\gamma]_+ = \omega$ and $[\gamma_-]_- = \alpha$ respectively. It will be convenient to define

$$B_\epsilon(\omega,p) = \{[\gamma_w]_+ \in M^+(\infty) \mid w \in T_p^{-1}M^+ \ \ \langle w, \gamma_1'(0)\rangle \geq -1-\epsilon\},$$
$$B_\epsilon(\alpha,p) = \{[\gamma_w]_- \in M^-(\infty) \mid w \in T_p^{-1}M^- \ \ \langle w, \gamma_2'(0)\rangle \geq -1-\epsilon\}.$$

Proposition 8.3. *If Γ is a totally vicious group of isometries on (M,g) and $p >> \alpha$, $p << \omega$ for some $p \in M$, then for every $\epsilon > 0$ there exists a TF axis γ_ξ of a hyperbolic isometry $\xi \in \Gamma$ such that*

$$[\gamma_\xi]_+ \in B_\epsilon(\omega,p) \quad [\gamma_{\xi_-}]_- \in B_\epsilon(\alpha,p).$$

Proof. To prove this we first apply Proposition 4.5 to give us a TF geodesic γ_1 and a TP geodesic γ_2 with $\gamma_1(0) = \gamma_2(0) = p$, $[\gamma_1]_+ = \omega$ and $[\gamma_2]_- = \alpha$. We will first show that there exist isometries μ_+ and μ_- in Γ such that

$$\mu_+(p) \in I^+(\gamma_1(t)), \quad \mu_-(p) \in I^-(\gamma_2(t)). \tag{8.1}$$

We have chosen $t > 0$ to satisfy

$$\cosh^2(Qt)/\sinh^2(Qt) < 1+\epsilon.$$

It will suffice to find a $\mu_+ \in \Gamma$ satisfying (8.1). To introduce notation let X, Y and Z denote the time orientations of $T^{-1}M^+, T^{-1}M^+/\Gamma = N$ and M respectively. These time orientations may be compatible or incompatible at some $v \in T^{-1}M^+, \pi(v) = \gamma_1(t)$. That is,

$$\langle T\pi(X(v)), Z(\pi(v))\rangle\langle Y(\pi_\Gamma(v)), T_v\pi_\Gamma(X)\rangle$$

may be either (i) positive or (ii) negative. Since N is totally vicious, there exists a smooth timelike curve $\beta : [0,1] \to T^{-1}M^+/\Gamma$ from

$\pi_\Gamma(v) = \beta(0)$ to some $\pi_\Gamma(w) = \beta(1)$, $w \in T_p^{-1}M^+$ with $\pi(w) = p$. We can assume it is future directed in case (i), and past directed in case (ii). The projection $\pi \circ \eta$ to M of the lift $\eta : I \to T^{-1}M^+$ of β through $\eta(0) = v$ is then a future directed smooth timelike curve in $T^{-1}M^+$ by definition of the metric on $T^{-1}M^+$. But this means that there exists a $\mu_+ \in \Gamma$ such that $T\mu_+(w) = \eta(1)$ hence $\gamma_1(t) << \mu_+(p)$ as claimed.

Having found $\mu_- \in \Gamma$ satisfying (8.1) by logically equivalent reasoning we define $\xi = \mu_+ \circ \mu_-^{-1}$ and combine

$$p << \mu_+(p), \qquad p << \mu_-^{-1}(p)$$

to assert that $p << \xi(p)$. Let the TF geodesic γ_ξ denote a timelike axis for ξ with $\xi \circ \gamma_\xi = \gamma_\xi \circ \tau_{d_\xi}$; its existence is guaranteed by Theorem 5.3. Recall that we can assume that $[\gamma_\xi]_+ \in I_\infty^+(p)$ and $[\gamma_{\xi_-}]_- \in I_\infty^-(p)$. Combining

$$\langle T\xi(\gamma_\xi'(0)), X\rangle = \langle \gamma_\xi'(\tau_{d_\xi}(0)), X\rangle < 0$$

with the fact that $T^{-1}M^+$ is path connected we conclude that $T\xi$ preserves time orientation. Let σ denote some TF geodesic through $\sigma(0) = p$. Then

$$p << \mu_+(p) << \mu_+([\sigma]_+).$$

Looking at the timelike geodesic triangle $p\ \gamma_1(t)\ \mu_+ \circ \sigma(s)$ with sidelengths $t, v = d(p, \mu_+ \circ \sigma(s))$ and $u = d(\gamma_1(t), \mu_+ \circ \sigma(s))$ we find

$$\begin{aligned} A_p \geq A_{pQ} &= (\cosh(Qu) - \cosh(Qv)\cosh(Qt))/[\sinh(Qv)\sinh(Qt)] \\ &\geq -\cosh^2(Qt)/\sinh^2(Qt) \geq -1 - \epsilon. \end{aligned}$$

We deduce that $\mu_+(I_\infty^+(p)) \subset B_\epsilon(\omega, p)$. Similarly $\mu_-(I_\infty^-(p)) \subset B_\epsilon(\alpha, p)$. Hence also

$$[\gamma_\xi]_+ \in \xi_+(I_\infty^+(p)) \subset B_\epsilon(\omega, p), \qquad [\gamma_{\xi_-}]_- \in \xi_-^{-1}(I_\infty^-(p)) \subset B_\epsilon(\alpha, p),$$

and the proposition follows.

We can now prove the density of timelike periodic geodesics in the future timelike unit tangent bundle of a C_Q manifold.

Theorem 8.4. *Let (M, g) denote a C_Q manifold , $Q > 0$, with a vicious group of isometries acting on the future timelike unit tangent bundle. Given an open set U in $T^{-1}M^+$, there exists a $v \in U$ such that the geodesic with initial velocity $\pi_\Gamma(v)$ is periodic.*

Proof. We shall prove more, namely: On any C_Q manifold with $Q > 0$, the tangent vectors to TF geodesics joining any pair $[\gamma_{v-}]_- = \alpha \in M^-(\infty)$ and $[\gamma_v]_+ = \omega \in M^+(\infty)$ depend continuously on the endpoints. By this we mean that to any neighbourhood U around v in $T^{-1}M^+$ there exists an $\epsilon > 0$ such that any TF geodesic γ joining $[\gamma]_+ = \omega_* \in B_\epsilon(\alpha, p)$ and $[\gamma_-]_- = \alpha_* \in B_\epsilon(\omega, p)$ has a tangent vector in U.

Choose some $t_2 > 0$ and open neighbourhoods W, U_1 and U_2 around $t_2 v$ in Λ^+ and $p_1 = \gamma_v(0) = \gamma_v(t_1),\ p_2 = \gamma_v(t_2)$ in M such that

$$E_{|W} \ :\ W \to U_1 \times U_2 \qquad w \mapsto (\pi(w), \exp(w))$$

is a C^∞ diffeomorphism. We can assume that $x << y$ for all $x \in U_1$ and $y \in U_2$ cf. [27] p.404 and also that $u/||u|| \in U$ for all u in W. Let $E_1 = \gamma_v', ..., E_n$ denote a parallel orthonormal basis along γ_v. There exists $b_j > 0$ such that any $z_{p_j} \in T_{p_j}M,\ j = 1, 2$ satisfying

$$|\ \langle z_{p_j}, E_i(0)\rangle\ | <\ 2b_j$$

for all $i = 1, .., n$ is mapped into U_j by $\exp_{p_j}$.

We claim that there exists an $A_i > 1$ such that for any TF geodesic β with

$$\begin{aligned} &[\beta]_+ = [c_i]_+ \quad [\beta_-]_- = [d_i]_-, \quad c_i \in \Omega_{TF},\ d_i \in \Omega_{TP}, \\ &\langle c_i'(0), d_i'(0)\rangle < A_i, \quad c_i(0) = d_i(0) = p_i, \end{aligned} \tag{8.2}$$

there exists a past directed null or zero vector $z_i \in T_{p_i}M$ with $\exp_{p_i}(z_i) \in \beta(\mathbb{R})$ and

$$|\ \langle z_i, d_i'(0)\rangle\ | < b_i.$$

To prove this claim choose $A_i > 1$ such that

$$f(x) \triangleq (1 - x + (x^2 - 1)^{\frac{1}{2}})/Q < b/2$$

whenever $x \in [1, A_i[$. This A_i will work. To see this we denote by d_s the TF geodesic from $d_i(s) = d_s(0)$ to $c_i(s) = d_s(u_s),\ s > 0$, and define

$$t_s = \sup\{t \geq 0 \mid d_s(t) \in J(d(s), p)\}.$$

Then $d_s(t_s) \in J^-(p)\backslash I^-(p)$ by global hyperbolicity of (M, g). According to [27, 14.5] there exists a past directed null or zero vector $z_i(s)$ satisfying the requirement

$$\exp_{p_i}(z_i(s)) = d_s(t_s).$$

If $z_i = 0$, the claim follows. Otherwise define $\eta(t) = \exp_p(tz_i)$. Lemma 6.1 gives us the following inequalities involving

$$h = u_s - t_s, \quad l = t_s, \quad d = \langle d_s'(l), \eta'(1)\rangle = e, \quad d_* = -\langle \eta'(0), d'(0)\rangle,$$

namely

$$\begin{aligned} \cosh(Qs) &\le \cosh(Qh) - \sinh(Qh)Qd, \\ \cosh(Qs) &\le \cosh(Ql) + \sinh(Ql)Qe, \\ \cosh(Ql) &\le \cosh(Qs) - \sinh(Qs)Qd_* \end{aligned} \tag{8.3, 4, 5}$$

Combine (8.3) and (8.4) to get

$$\cosh(Qs)(\sinh(Ql) + \sinh(Qh)) \le \sinh(Q(l+h)). \tag{8.6}$$

We will also need to combine

$$\cosh(Q(h+l)) \le \cosh^2(Qs)(1 + A_p) - A_p$$

with (8.6) to yield

$$\begin{aligned} (\cosh(Q(h+l)) + A_p)\ &(\sinh(Ql) + \sinh(Qh))^2 \\ &\le (1 + A_p)\sinh^2(Q(l+h)). \end{aligned}$$

Squaring the brackets and rearranging the terms we obtain

$$\begin{aligned} \cosh(Q(h+l))(\sinh^2(Ql) + \sinh^2(Qh)) &+ 2\sinh^2(Ql)\sinh^2(Qh) \\ \le\ \cosh^2(Ql)\sinh^2(Qh) &+ \sinh^2(Ql)\cosh^2(Qh) \\ +2A_p\sinh(Ql)\sinh(Qh)&(\cosh(Q(h+l)) - 1), \end{aligned}$$

and then finally

$$\sinh^2(Qh) - 2A_p\sinh(Ql)\sinh(Qh) + \sinh^2(Ql) \le 0.$$

We deduce immediately that

$$\sinh(Qh)/\sinh(Ql) \le A_p + (A_p^2 - 1)^{\frac{1}{2}} \triangleq \alpha.$$

The reverse triangle inequality tells us that $h \ge s$. For s greater than or equal to some s_0, we may then compute from (8.5)

$$\begin{aligned} d_* &\le \frac{\sinh(Ql)\cosh\ (Qh) - \cosh(Qs)\sinh(Ql)}{Q\sinh(Qh)\sinh(Qs)} \\ &\le \cosh(Qs)(1 - 1/\alpha)/(Q\sinh(Qs)) \\ &= \cosh(Qs)f(A_p)/\sinh(Qs) < b/2 \end{aligned} \tag{8.7}$$

when s_0 is sufficiently large. From (8.7) it follows that the $\{z_i(s)\}_{s \geq s_0}$ lie in a compact subset of $T_{p_i}M$. We can then take a sequence of real numbers $s_n \geq s_0$ converging to $+\infty$ and such that $z_i(s_n) \to z_i$ as $n \to +\infty$. Now $\exp_{p_i}(z_i) \in \beta(\mathbb{R})$ because $d_{s_n} \circ \tau_{t_{s_n}} = \beta^n$ and β^n_- have nonspacelike limit curves ξ and ζ through $d_{s_n}(t_{s_n})$, which are TF geodesics with $\xi'(0) = -\zeta'(0)$ and $\xi(0) = \zeta(0) = \exp_{p_i}(z_i)$. This is due to the fact that

$$\exp_{p_i}(z_i(s_n)) = d_{s_n}(t_{s_n}) \to \exp_{p_i}(z_i)$$

for $n \to +\infty$. Thus ξ is a future coray to c, and ξ_- a past coray to d. By the uniqueness in Proposition 4.8 we conclude that $\beta = \xi \circ \tau_a$ for some $a \in \mathbb{R}$ and hence $\exp_{p_i}(z_i) = \xi(0) = \beta(-a)$. This establishes the claim.

Due to the claim there are $A_i = \cosh a_i$, $a_i > 0$, such that the conclusion following (8.2) is true. We can also assume that $b_i A_i + (A_i^2 - 1)^{\frac{1}{2}} b_i < 2b_i$, $i = 1, 2$. Now take $s_1 < t_1$, $s_2 > t_2$ subject to the requirement that any $c_j \in \Omega_{TF}$ and $d_j \in \Omega_{TP}$ with

$$[c_j]_+ \in I^+(\gamma_v(s_2)), \quad [d_j]_- \in I^-(\gamma_v(s_1)), \quad c_j(0) = d_j(0) = p_j$$

satisfy the inequalities

$$-\langle c_j'(0), \gamma_v'(t_j)\rangle \leq \cosh(a_j/2), \qquad \langle d_j'(0), \gamma_v'(t_j)\rangle \leq \cosh(a_j/2). \tag{8.8}$$

There exists a TF geodesic β with

$$\begin{aligned} [c_j]_+ &= [\beta]_+ \in I^+_\infty(\gamma_v(s_2)), \\ [d_j]_- &= [\beta_-]_- \in I^-_\infty(\gamma_v(s_1)), \end{aligned} \tag{8.9}$$

where $c_j(0) = p_j$. Now (8.8) implies that $\langle c_j', d_j'\rangle \leq A_j$. According to the claim (8.2) there are past directed null or zero vectors $z_j \in T_{p_j}M$ and $s_j \in \mathbb{R}$ such that

$$\exp_{p_j}(z_j) = \beta(s_j), \quad |\, \langle z_j, d_j'(0)\rangle \,| < b_j, \quad j = 1, 2.$$

It follows that

$$|\, \langle z_j, \gamma_v'(t_j)\rangle \,| \leq A_j b_j + (A_j^2 - 1)^{\frac{1}{2}} b_j < 2b_j.$$

Hence $\beta(s_j) \in U_j$. Since $\beta(s_1) << \beta(s_2)$ we conclude that

$$(s_2 - s_1)\beta'(s_1) = E_{|W}{}^{-1}(\beta(s_1), \beta(s_2)) \in W,$$

so that $\beta'(s_1) \in U$. Due to Proposition 8.3, we can assume that β is an axis of an isometry $\mu \in \Gamma$. This finishes the proof.

9. Constant curvature

In this section we will show that there are discrete groups of isometries acting on the future timelike unit tangent bundle of the complete C_Q manifolds of constant sectional curvature $Q > 0$. To do this consider

$$\begin{aligned} \mathbb{X} = \{(x,v) \in \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \mid & \langle x,x\rangle = 1/Q^2,\ \langle x,v\rangle = 0, \\ & \langle v,v\rangle = -1/Q^2,\ v_1 > 0\}, \\ \mathbb{Y} = \{(y,w) \in \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \mid & \langle y,y\rangle = -1/Q^2,\ \langle y,w\rangle = 0, \\ & \langle w,w\rangle = 1/Q^2,\ y_1 > 0\}. \end{aligned}$$

Riemannian hyperbolic space is denoted by

$$M_H = \{x \in \mathbb{R}^{n+1}_1 \mid \langle x,x\rangle = -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = -1/Q^2\}.$$

We have the natural maps $G_{\mathbb{X}}$ and $G_{\mathbb{Y}}$ from the future timelike unit tangent bundle

$$T^{-1}M_Q^+ = \{v \in TM_Q \mid \langle v,v\rangle = -1,\ \langle v,X\rangle < 0\}$$

of M_Q and unit tangent bundle T^1M_H of M_H to $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, defined by the sequences

$$\begin{aligned} T^{-1}M_Q^+ &\to T\mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, \\ T^1M_H &\to T\mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}. \end{aligned}$$

In each row the first map is the inclusion, the second map the natural identification. The map that takes $(x,v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ to $(x, v \cdot 1/Q)$ is denoted by h_Q. Notice that $h_Q \circ G_{\mathbb{X}}$ and $h_Q \circ G_{\mathbb{Y}}$ map onto $\mathbb{X}$ and $\mathbb{Y}$. This means that

$$\begin{aligned} F_{\mathbb{X}} &: T^{-1}M_Q^+ \to \mathbb{X}\ ,\quad v \mapsto h_Q \circ G_{\mathbb{X}}(v), \\ F_{\mathbb{Y}} &: T^1M_H \to \mathbb{Y}\ ,\quad v \mapsto h_Q \circ G_{\mathbb{Y}}(v) \end{aligned}$$

are diffeomorphisms to $\mathbb{X}$ and $\mathbb{Y}$ with their submanifold structures from the ambient $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Also

$$G : \mathbb{X} \to \mathbb{Y} \quad (x,v) \mapsto (v,x)$$

is a diffeomorphism, showing that $T^{-1}M_Q^+$ and T^1M_H are diffeomorphic via the composition

$$\Psi = F_{\mathbf{Y}}^{-1} \circ G \circ F_{\mathbf{X}}$$

of diffeomorphisms. We have geodesic flows

$$\begin{aligned} \Phi_Q &: \mathbb{R} \times T^{-1}M_Q^+ \to T^{-1}M_Q^+, \\ \Phi_H &: \mathbb{R} \times T^1M_H \to T^1M_H \end{aligned}$$

on M_Q and M_H respectively. The diffeomorphism Ψ conjugates these two flows. In fact, we have

Proposition 9.1. *$\Psi \circ \Phi_Q(t,v) = \Phi_H(t,\Psi(v))$ for all $v \in T^{-1}M_Q^+$ and all $t \in \mathbb{R}$.*

Proof. Given $t \in \mathbb{R}$ and $F_{\mathbf{X}}(v) = (x,y) \in \mathbb{X}$ define

$$\begin{aligned} \gamma(t) &= x\,\cosh(Qt) + y\,\sinh(Qt), \\ \beta(t) &= x\,\sinh(Qt) + y\,\cosh(Qt). \end{aligned}$$

They are geodesics in M_Q and M_H with initial velocities $\gamma'(0) = v$ and $\Psi(v) = \beta'(0)$. The proposition follows from a direct computation, showing that $F_{\mathbf{Y}} \circ \Psi(\gamma'(t)) = F_{\mathbf{Y}}(\beta'(t))$.

The tangent maps of a properly discontinuous group Γ of isometries on M_H induce a properly discontinuous group Γ_H of diffeomorphisms of T^1M_H. The properly discontinuous groups of diffeomorphisms

$$\begin{aligned} \Gamma_{\mathbf{Y}} &= \{F_{\mathbf{Y}} \circ \zeta \circ F_{\mathbf{Y}}^{-1} \mid \zeta \in \Gamma_H\}, \\ \Gamma_{\mathbf{X}} &= \{G^{-1} \circ \zeta \circ G \mid \zeta \in \Gamma_{\mathbf{Y}}\}, \\ \Gamma_Q &= \{F_{\mathbf{X}}^{-1} \circ \zeta \circ F_{\mathbf{X}} \mid \zeta \in \Gamma_{\mathbf{X}}\} \end{aligned}$$

give rise to the following commutative diagram:

$$\tag{9.1} \begin{array}{ccccccc} T^{-1}M_Q^+ & \to & \mathbb{X} & \to & \mathbb{Y} & \leftarrow & T^1M_H \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T^{-1}M_Q^+/\Gamma_Q & \to & \mathbb{X}/\Gamma_{\mathbf{X}} & \to & \mathbb{Y}/\Gamma_{\mathbf{Y}} & \leftarrow & T^1M_H/\Gamma_H \end{array}$$

where the vertical maps are the natural maps, and the maps in the bottom row are induced by the maps $F_{\mathbf{X}}, G$ and $F_{\mathbf{Y}}$ in the top row. The restriction maps

$$R_Q \;:\; I(\mathbb{R}_1^{n+1}) \to I(M_Q), \qquad R_H \;:\; O_1^{++}(n+1) \cup O_1^{+-}(n+1) \to I(M_H)$$

from the isometry group $I(\mathbb{R}^{n+1}_1)$ of $\mathbb{R}^{n+1}_1$ to the isometry groups $I(M_Q)$ and $I(M_H)$ of M_Q and M_H are isomorphisms, according to [27, 9.8]. Hence

$$\Gamma_Q = \{T\xi \ : \ T^{-1}M_Q^+ \to T^{-1}M_Q^+ \mid \mu \in \Gamma \subset I(M_H), \quad \xi = R_Q(R_H^{-1}(\mu))\}. \tag{9.2}$$

Thus Γ_Q is a properly discontinuous group of tangent maps of isometries of M_Q. It follows that the commutative diagram in (9-1) provides a link between the geometries of Riemannian and Lorentzian hyperbolic manifolds, making available Riemannian theory applicable to Lorentzian hyperbolic manifolds.

Remark 9.2. It is also clear that the composition ψ of diffeomorphisms from left to right in the bottom row of diagram (9.1) conjugate the geodesic flows ψ_Q and ψ_H of $T^{-1}M_Q^+/\Gamma_Q$ and T^1M_H/Γ_H respectively.

Propositions 9.3 and 9.4 below will enable us to deduce results about the dynamic properties of the geodesic (horocycle) flow on $T^{-1}M_Q^+/\Gamma_Q$. These results set the context for Theorem 8.4; see Remark 9.6.

We shall now show that Ψ and hence ψ preserve Liouville measures τ_Q and τ_H on $T^{-1}M_Q^+/\Gamma_Q$ and T^1M_H/Γ_H when M_H/Γ is orientable.

Proposition 9.3. *$\psi_*\tau_H = \lambda\tau_Q$, for some $\lambda \in \mathbb{R}\backslash\{0\}$.*

Proof. It is clear that for some $v \in T^{-1}M_Q^+$ we have

$$\Psi_*\zeta_H(v) = \lambda\zeta_Q(v)$$

for some $\lambda \in \mathbb{R}\backslash\{0\}$. We have used ζ_Q and ζ_H to denote Liouville measures on $T^{-1}M_Q^+$ and T^1M_H. Given $w \in T^{-1}M_Q^+ = N_Q$ we can take an orientation preserving isometry μ on M_H such that

$$T\mu(\Psi(v)) = \Psi(w) = \Psi \circ T\xi(v),$$

where $\xi = R_Q(R_H^{-1}(\mu))$. Suppressing evaluation in v, compute

$$\lambda T\xi_*\zeta_Q = \lambda\zeta_Q = \Psi_*\zeta_H = \Psi_*T\mu_*\zeta_H = T\xi_*\Psi_*\zeta_H,$$

hence $\Psi_*\zeta_H = \lambda\zeta_Q$. This property descends to the quotients.

We can define a horocycle flow on M_Q when the dimension of M_Q is two. We proceed to define it. First of all we need M_Q and M_H to have compatible orientations. The restrictions of the position vector field on $\mathbb{R}^3_1\backslash\{0\}$ to M_Q and M_H provide normal vector fields U_Q and U_H on M_Q and M_H. The orientation $\omega = -dx_1 \wedge dx_2 \wedge dx_3$ in $\mathbb{R}^3_1$ gives us orientations

$$\begin{aligned} \omega_Q(x) &= \omega(U_Q(x), \cdot, \cdot), \quad x \in M_Q, \\ \omega_H(y) &= \omega(U_H(y), \cdot, \cdot), \quad y \in M_H \end{aligned}$$

of $T_x M_Q = x^\perp$ and $T_y M_H = y^\perp$. Given $v \in T^{-1}M_Q^+$, let b_v^+ denote the Buseman function for γ_v, defined on $I^-(\omega), \omega = [\gamma_v]_+$. The horosphere

$$B_v = \{q \in I^-(\omega) \mid b_v^+(q) = b_v^+(\pi(v))\}$$

is a smooth, spacelike hypersurface of M_Q, since $\langle \text{grad } b_v^+, \text{grad } b_v^+\rangle \equiv -1$. There is a unit speed geodesic $\beta_v : \mathbb{R} \to B_v$ through $\pi(v)$ such that $\beta_v'(0)$ and v are positively oriented. The horocycle flow

$$h_Q \ : \ \mathbb{R} \times T^{-1}M_Q^+ \to T^{-1}M_Q^+$$

is then

$$h_Q(t, v) = \text{grad } b_v^+(\beta_v(t)), \qquad t \in \mathbb{R}.$$

Similarly the horocycle flow on M_H is denoted

$$h_H \ : \ \mathbb{R} \times T^1 M_H \to T^1 M_H.$$

We need to know

Proposition 9.4. $\Psi \circ h_Q(t, v) = h_H(t, \Psi(v)), \quad (t, v) \in \mathbb{R} \times T^{-1}M_Q^+$.

Proof. Let us find the horocyclic orbits of $v_0 \in T^{-1}M_Q^+$ and $w_0 \in T^1 M_H$, where $F_{\mathbb{X}}(v_0) = 1/Q(e_3, e_1)$ and $F_{\mathbb{Y}}(w_0) = 1/Q(e_1, e_3)$. Here $\{e_i\}$ denotes the canonical basis in $\mathbb{R}^3$. We find that

$$\begin{aligned} F_{\mathbb{X}}(h_{v_0}^Q(t)) &= ((-\tfrac{Q}{2}t^2, t, 1/Q - \tfrac{Q}{2}t^2), (1/Q + \tfrac{Q}{2}t^2, -t, \tfrac{Q}{2}t^2)) \\ &= G \circ F_{\mathbb{Y}}(h_{w_0}^H(t)), \end{aligned}$$

showing

$$\Psi \circ h_Q(t, v_0) = h_H(t, \Psi(v_0)). \tag{9.3}$$

Since $I(M_H)$ acts transitively on the orthonormal bases of M_H (cf. [27, 4.30]), there exists an orientation preserving isometry $\mu \in I(M_H)$

taking some $w = \Psi(v)$, $v \in T^{-1}M_Q^+$ to $T\mu(w) = w_0$. Define $\xi = R_Q(R_H^{-1}(\mu))$ and observe that

$$\begin{aligned} &T\mu \circ \Psi = \Psi \circ T\xi, \\ &T\mu \circ h_w^H(t) = h_{T\mu(w)}^H(t), \quad t \in \mathbb{R}, \\ &T\xi \circ h_v^Q(t) = h_{T\xi(v)}^Q(t), \quad t \in \mathbb{R}. \end{aligned}$$

Combining this with (9.3) we conclude

$$\Psi \circ T\xi(h_v^Q(t)) = h_{\Psi(v_0)}^H(t) = T\mu \circ \Psi(h_v^Q(t)) = T\mu \circ h_w^H(t).$$

Thus the proposition follows.

Definition 9.5. The group Γ_Q in (9.2) is proper when Γ acts properly discontinuously on M_H such that M_H/Γ is a connected, orientable Riemann surface of finite volume.

The horocycle flow descends to the quotient of the future unit timelike tangent bundle $T^{-1}M_Q^+$ with a proper group Γ_Q.

Remark 9.6. In view of Propositions 9.3. and 9.4. a number of available results are now applicable to the quotient $X = T^{-1}M_Q^+/\Gamma_Q$ of the future timelike unit tangent bundle with a proper group Γ_Q. We mention a few as follows.

1) The geodesic flow is mixing and ergodic; cf. [18].
2) The horocycle flow is mixing of all degrees; cf. [26].
3) The timelike periodic geodesics are dense in X; cf. [18].
4) There exists a transitive timelike geodesic in X; cf. [18].

Notice that Γ_Q is a vicious group of isometries.

Remark 9.7. A referee pointed out that there is another way of seeing the existence of properly discontinuous groups of isometries acting on $T^{-1}M_Q^+$. The isometry group of M_Q is $O_1(n+1)$; see [27, p. 239]. The group $O_1(n+1)$ also acts transitively on M_Q. According to [27, p. 307]

$$M_Q = O_1(n+1)/O_1(n).$$

Take

$$v_0 = e_{1_{e_{n+1}}} \in T^{-1}M_Q^+,$$

where $e_1, .., e_{n+1}$ is the canonical basis in $\mathbb{R}^{n+1}$. The isotropy group at v_0 is

$$O(n-1) = \{L \in O_1(n+1) \mid L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, c \in O(n-1)\}$$

under the transitive action of

$$O_1^+(n+1) = O_1^{++}(n+1) \cup O_1^{+-}(n+1)$$

on $T^{-1}M_Q^+$. From this group we obtain the coset manifolds

$$\begin{aligned} \rho &: O_1^+(n+1)/O(n-1) \to T^{-1}M_Q^+ \quad L\,O(n-1) \mapsto T\mu(v_0), \\ \rho' &: O_1^+(n+1)/O(n-1) \to T^1 M_H \quad L\,O(n-1) \mapsto T\xi(\Psi(v_0)), \end{aligned}$$

where

$$\begin{aligned} L \circ i_Q &= i_Q \circ \mu, \\ L \circ i_H &= i_H \circ \xi. \end{aligned}$$

Then we find

$$\begin{aligned} &\rho(L\,O(n-1)) = T\mu(v_0), \\ &\rho'(L\,O(n-1)) = T\xi(\Psi(v_0)) = \Psi(T\mu(v_0)) = \Psi(\rho(L\,O(n-1))). \end{aligned}$$

Thus

$$\Psi = \rho' \circ \rho^{-1} : T^{-1}M_Q^+ \to O_1^+(n+1)/O(n-1) \to T^1 M_H.$$

Taking a properly discontinuous group Γ of isometries of M_H we obtain a properly discontinuous group of isometries Γ_Q of $T^{-1}M_Q^+$ via ρ.

10. C_Q surfaces

The existence of null axes for a hyperbolic isometry μ on a C_Q manifold (M, g) with $Q > 0$ is related to the existence of fixed points for a Riemannian isometry in the following way.

Proposition 10.1. *Let $\alpha = [\gamma_-]_- \in M^-(\infty)$ and $\omega = [\gamma]_+ \in M^+(\infty)$, where γ denotes an axis for μ. If $x \in \partial I^-(\omega) \cap \partial I^+(\alpha) = N$, and v_x denotes a null vector in $T_x\partial I^-(\omega)$, then the following hold:*

1) $N, \partial I^-(\omega)$ and $\partial I^+(\alpha)$ are μ invariant.

2) If x is a fixed point for μ, then γ_{v_x} is a null axis for μ.

Proof. 1) $I^-(\omega)$ is μ invariant and so is $\partial I^-(\omega)$, hence also $\partial I^+(\alpha)$. The intersection N is then μ invariant. 2) Simply observe that $T\mu(v_x) = \lambda v_x$ for some $\lambda > 0$.

This fixed point problem can be solved completely on C_Q surfaces with $Q > 0$ and a volume form ω. If γ denotes a timelike axis for the orientation preserving hyperbolic isometry μ and $\omega = [\gamma]_+$, $\alpha = [\gamma]_-$ we have indeed

Proposition 10.2. *Every future null coline β_x through $x \in \partial I^-(\gamma)$ to γ is a null axis for μ. Furthermore there exists a $t_x \in \mathbb{R}$ such that $\beta_x(t_x) \in \partial I^-(\omega) \cap \partial I^+(\alpha)$ is a fixed point for μ.*

Proof. Due to the definition of a null coline there exists a positive s such that $\beta_x(s) = y \in I^+(p)$, $p = \gamma(0)$. Let β_y denote a future null coline to γ through y. We need only prove μ invariance of β_y.

To this end let E denote a timelike parallel vector field along γ with $\exp_p(sE(0)) = y$ for some $s > 0$. Define a geodesic variation

$$\alpha \; : \{(s,t) \in \mathbb{R}^2 \mid t \geq 0\} \to M, \quad (s,t) \mapsto \exp(sE(t)).$$

We claim that for every $t \geq 0$ there exists an $s(t) \geq 0$ such that

$$\alpha(s(t),t) \in \beta_y(\mathbb{R}).$$

This is true for positive t values in a neighbourhood of 0 by the implicit function theorem. If the claim is untrue we can define

$$t_* = \inf \{ \; t \geq 0 \mid \alpha(s,t) \notin \beta_y(\mathbb{R}) \text{ for all } s \geq 0 \; \} > 0.$$

Notice that $s(t) \leq K$ for some $K > 0$ and all $t \in [0, t_*[$; cf. (7.13). We can assume convergence of $\{s(t_n)\}$ to s_* for a suitable increasing sequence $\{t_n\}$ of positive real numbers, converging to t_*. There exists real numbers z_n such that

$$\alpha(s(t_n), t_n) = \beta_y(z_n).$$

Taking subsequences if necessary we can assume the convergence of $\{z_n\}$ to z_* too, because $\{z_n\}$ is a bounded sequence. This follows from global hyperbolicity and [27, 14.13]. Since

$$\beta_x'(z_*) \; , \; \frac{\partial \alpha}{\partial s}(s_*, t_*)$$

are linearly independant, we can apply the inverse function theorem to assert the existence of $s(t)$ for t values in a neighbourhood of t_*. This contradiction verifies the claim. The uniqueness of $s(t) \geq 0$ follows from Lemma 7.1 and the strong causality of (M, g).

Suppose μ translates γ with $a > 0$, i.e., $\mu \circ \gamma = \gamma \circ \tau_a$. Then $T\mu(E(0)) = E(a)$, because μ is orientation preserving, hence

$$\mu \circ \alpha(s, 0) = \alpha(s, a). \tag{10.1}$$

Now $\beta_y(\mathbb{R}) \subset \partial I^-(\omega)$. In view of Proposition 10.1 this implies that $s(0) = s(a)$, so that

$$\mu(\beta_y(u_1)) = \mu \circ \alpha(s(0), 0) = \alpha(s(a), a) = \beta_y(u_2)$$

for some $u_1, u_2 \in \mathbb{R}$. Since $\partial I^-(\omega)$ is one dimensional, we conclude that β_y is a null axis for μ.

According to the proof of Theorem 7.7 2), there exists a $t < 0$ such that $u = \beta_x(t) \in \partial I^-(\omega) \cap \partial I^+(\alpha)$. A past null coline β_u to γ through u is also μ invariant. Assume for contradiction that some $s < t$ makes $\beta_x(s) = \beta_u(v)$, $v \in \mathbb{R}$. For $v < 0$ this contradicts Lemma 7.6. For $v = 0$ it contradicts strong causality of (M, g). For $v > 0$ reach a contradiction by applying Lemma 7.2 to find a $w \in]0, v[$ such that $\beta_u(w) \notin J^+(\beta_u(v))$. The uniqueness of t just proven combined with μ invariance of β_u and β_x implies that $\beta_x(t)$ is a fixed point for μ.

Definition 10.3. If the NF (NP) geodesic β is a future coray to the TF (TP) geodesic γ, then β has future (past) endpoint

$$\omega(\beta) = [\gamma]_+, \qquad (\alpha(\beta) = [\gamma]_-).$$

We can now introduce relations $\underset{+}{\rightarrow} \sim_n$ and $\underset{-}{\rightarrow} \sim_n$ in the sets Ω_{NF} and Ω_{NP} of NF geodesics and NP geodesics respectively. For $\beta_1, \beta_2 \in \Omega_{NF}$ (Ω_{NP}) we define

$$\beta_1 \underset{+}{\rightarrow} \sim_n \beta_2 \quad (\beta_1 \underset{-}{\rightarrow} \sim_n \beta_2)$$

if

$$\omega(\beta_1) = \omega(\beta_2) \quad (\alpha(\beta_1) = \alpha(\beta_2)).$$

Since $\underset{+}{\rightarrow} \sim_n$ and $\underset{-}{\rightarrow} \sim_n$ are equivalence relations, we can finally introduce the null future and the null past as

$$M_N^+(\infty) = \Omega_{NF} / \underset{+}{\rightarrow} \sim_n,$$
$$M_N^-(\infty) = \Omega_{NP} / \underset{-}{\rightarrow} \sim_n.$$

Finally, we have

Theorem 10.3. *An orientable C_Q surface with $Q > 0$ and vicious isometry group Γ has constant curvature.*

Proof. We shall first verify that a future null coline β_x to some TF geodesic γ through $\beta_x(0) = x \in M$ maps into $\partial I^-(\gamma)$. To see this take $s > 0$ such that

$$\beta_x(s) \in I^+(\gamma(0)).$$

Let γ_s denote the TF geodesic from $\gamma_s(0) = \gamma(0)$ to $\gamma_s(a_s) = \beta_x(s), a_s > 0$. If x was not in $\partial I^-(\gamma)$, then it would neither be in $I^-(\gamma)$ according to Lemma 7.1. For some $t_* \in]0, a_s[$, $\gamma_s(t_*) \in \partial I^-(\gamma)$. Let σ_u denote the TF geodesic from $\gamma_s(t_*)$ to $\beta_x(u)$, $u \geq s$ and v be the limit of $\sigma_u'(0)$ as $u \to +\infty$; see Lemma 6.2. γ_v is not a future coray to γ because $\gamma_s(t_*) \in \partial I^-(\gamma)$. For any $t < t_*$, $\gamma_s(t) \in I^-(\gamma) \cap I^-(\gamma_v)$. By τ_v we denote the TF geodesic from $\gamma_s(t)$ to $\beta_x(v), v \geq s$. $\tau_v'(0)$ converges as $v \to +\infty$ again by Lemma 6.2. This is incompatible with the fact that $[\gamma_v]_+ \neq [\gamma]_+$, hence $x \in \partial I^-(\gamma)$.

Now take some $p \in M$ and a future directed respectively past directed null vector $w_+, w_- \in T_pM$. We need them to be linearly independant. There exists a TF geodesic σ with

$$[\sigma]_+ = \omega(\beta_{w_+}), \quad [\sigma_-]_- = \alpha(\beta_{w_-}).$$

We aim to assert that

$$\partial I^-(\sigma) = \beta_x(\mathbb{R}) \cup \beta_y(\mathbb{R}), \qquad \beta_x(\mathbb{R}) \cap \beta_y(\mathbb{R}) = \emptyset, \tag{10.2}$$

where β_x and β_y are future null colines to σ through $x, y \in \partial I^-(\sigma)$. Let $X_1 = \sigma'(0)$, X_2 denote an orthonormal basis in $T_{\sigma(0)}M$, and define

$$v = \cosh 1\ X_1 + \sinh 1\ X_2, \qquad w = \cosh 1\ X_1 - \sinh 1\ X_2.$$

We have already seen that there exists $s, t > 0$ such that $\gamma_v(s) = x$, $\gamma_w(t) = y \in \partial I^-(\sigma)$. Theorem 7.7 asserts the existence of future null colines β_x and β_y to γ through x and y. For positive s we let σ_s^1 and σ_s^2 denote the TF geodesics from $\sigma(0)$ to $\beta_x(s) >> \sigma(0)$ and $\beta_y(s) >> \sigma(0)$ respectively.

The two bases $\sigma'(0), \sigma_s^{1\prime}(0)$ and $\sigma'(0), \sigma_s^{2\prime}(0)$ have opposite orientations which do not depend on s. We have already seen that β_x and β_y

map into $\partial I^-(\sigma)$, which is a C^1 degenerate hypersurface. It follows that $\beta_x(\mathbb{R})$ and $\beta_y(\mathbb{R})$ are disjoint.

We can assume σ is an axis of a hyperbolic isometry $\mu \in \Gamma$; cf. Proposition 8.3. μ has fixed points $p_+ = \beta_x(t_x)$ and $p_- = \beta_y(t_y)$, $t_x, t_y \in \mathbb{R}$ according to Proposition 10.2. We know from Proposition 10.2 that the future null colines β_{p_+} and β_{p_-} to σ through $\beta_{p_+}(0) = p_+$ and $\beta_{p_-}(0) = p_-$ respectively are null axes for $\mu \in \Gamma$. That is,

$$\mu \circ \beta_{p_+}(s) = \beta_{p_+}(\lambda_+ s + r_+),$$
$$\mu \circ \beta_{p_-}(s) = \beta_{p_-}(\lambda_- s + r_-)$$

for real constants $\lambda_+, \lambda_-, r_+, r_-$. Here $r_+ = r_- = 0$ since p_+ and p_- are fixed points for μ. β_{p_+} being a future null coray to σ, there exists $s > 0$ such that $\beta_{p_+}(s) >> r = \sigma(0)$ and hence also $s_* \geq 0$ such that

$$\beta_{p_+}(s_*) \in J^+(r) \backslash I^+(r).$$

If $\lambda_+ \leq 1$, then we would have

$$r << \mu(r) < \mu \circ \beta_{p_+}(s_*) \leq \beta_{p_+}(s_*),$$

a contradiction, hence $\lambda_+ > 1$. Similarly $\lambda_- > 1$.

Consider now an arbitrary $q \in M$. Take an arbitrarily small open neighbourhood U around $q \in M$ on which we have defined two linearly independant smooth, future and past directed null vector fields X_+ and X_- respectively. Notice that the integral curves of X_- and X_+ are null pregeodesics. There exists a TF geodesic τ with

$$[\tau]_+ = \omega(\beta_+), \quad [\tau_-]_- = \alpha(\beta_-),$$
$$\beta_+ = \beta_{X_+(q)}, \quad \beta_- = \beta_{X_-(q)}.$$

We can assume that U is chosen to render

$$H_+(r) \triangleq \omega(\beta_{X_+(r)}) \in I^+_\infty(\tau(0)),$$
$$H_-(r) \triangleq \omega(\beta_{X_-(r)}) \in I^-_\infty(\tau(0))$$

for all $r \in U$.

For some orthonormal basis $Y_1 = \tau'(0), Y_2$ we can define invertible maps

$$I_+ : \mathbb{R} \to I^+_\infty(\tau(0)), \quad s \mapsto [\gamma_{\cosh s\ Y_1 + \sinh s\ Y_2}],$$
$$I_- : \mathbb{R} \to I^-_\infty(\tau(0)), \quad s \mapsto [\gamma_{-\cosh s\ Y_1 + \sinh s\ Y_2}].$$

Lemma 2.2 tells us that $f_+(t) = I_+^{-1} \circ H_+ \circ \beta_-(t)$ and $f_-(t) = I_-^{-1} \circ H_- \circ \beta_+(t), t \in I$, are both continuous functions. Here I is an open interval around 0 such that $\beta_+(I), \beta_-(I) \subset U$.

Assume for contradiction that $H_+ \circ \beta_-(t_1) = H_+ \circ \beta_-(t_2) = [\xi]_+$ for some TF geodesic ξ and $t_1 < t_2$ such that $\beta_-(t_1), \beta_-(t_2) \in U$. Then $\beta_-(t_2) \in \partial I^-(\xi)$ hence $\beta_-(t_1) \in I^-(\xi)$. This however contradicts Lemma 7.1. Consequently $H_+ \circ \beta_-$ and also $H_- \circ \beta_+$ are both injective on I.

It follows from the Implicit Function Theorem that there exists a smooth mapping

$$G : I_- \times I_+ \to U$$

such that $\beta_-(t_-)$, $\beta_+(t_+) \in U$, and $G(t_-, t_+)$ is a point of intersection of $\beta_{X_+(\beta_-(t_-))}$ and $\beta_{X_-(\beta_+(t_+))}$ for every $(t_-, t_+) \in I_- \times I_+$. Here I_- and I_+ are open intervals around 0. There are $\epsilon_-, \epsilon_+ > 0$ such that

$$\begin{gathered}]-\epsilon_+, \epsilon_+[\;\subset\; \mathrm{Im}\, f_+, \quad]-\epsilon_-, \epsilon_-[\;\subset\; \mathrm{Im}\, f_-, \\ f_+^{-1}(s) \in I_-, \qquad f_-^{-1}(s) \in I_+, \end{gathered}$$

whenever $s \in\,]-\epsilon_-, \epsilon_-[$, and $s \in\,]-\epsilon_+, \epsilon_+[$. According to the proof of Proposition 8.3 there exists a $\xi \in \Gamma$ such that

$$\omega_* = \xi([\sigma]_+) = I_+(s_+),$$

where $s_+ \in\,]-\epsilon_+, \epsilon_+[$. Define $s_- \in\,]-\epsilon_-, \epsilon_-[$ by

$$\alpha_* = [\tau_-]_- = I_-(s_-),$$

and also $t_- = f_+^{-1}(s_+)$, $t_+ = f_-^{-1}(s_-)$. Then using our first assertion

$$G(t_-, t_+) \in \partial I^-(\omega_*) = \xi \circ \beta_{p_+}(\mathbb{R}) \cup \xi \circ \beta_{p_-}(\mathbb{R})$$

We conclude that $G(t_-, t_+)$ is equal to $\xi \circ \beta_{p_+}(v_+)$ or $\xi \circ \beta_{p_-}(v_-)$ for some $v_+, v_- \in \mathbb{R}$. Notice that

$$\begin{aligned} K(\beta_{p_+}(v_+)) = K(\xi \circ \beta_{p_+}(v_+)) = K(\mu^{-n}(\beta_{p_+}(v_+))) \to K(p_+), \\ K(\beta_{p_-}(v_-)) = K(\xi \circ \beta_{p_-}(v_-)) = K(\mu^{-n}(\beta_{p_-}(v_-))) \to K(p_-), \end{aligned}$$

as $n \to +\infty$, hence

$$\begin{aligned} K(\beta_{p_+}(v_+)) = K(\xi \circ \beta_{p_+}(v_+)) = K(p_+), \\ K(\beta_{p_-}(v_-)) = K(\xi \circ \beta_{p_-}(v_-)) = K(p_-). \end{aligned}$$

It follows that we can take a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ in Γ and $\{v_n^+\}$ or $\{v_n^-\}$ in $\mathbb{R}$ such that $\{\xi_n \circ \beta_{p_+}(v_n^+)\}_{n\in\mathbb{N}}$ or $\{\xi_n \circ \beta_{p_-}(v_n^-)\}_{n\in\mathbb{N}}$ is a sequence in M converging to q. The sectional curvatures at the arbitrary point q is then either $K(p_-)$ or $K(p_+)$, and the Theorem follows.

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TECHNICAL UNIVERSITY OF DENMARK