J.DIFFERENTIAL GEOMETRY
Vol. 43, No.1 January, 1996

TOWARDS THE HOMOLOGY OF HURWITZ SPACES

STEVEN DIAZ & DAN EDIDIN

Abstract

The purpose of this paper is to construct a complex which computes the homology of Hurwitz spaces of branched covers of $\mathbb{P}^1$. We also compute some of the low dimensional homology groups of a compactification of the Hurwitz space, and report on computer calculations performed in specific examples.

1. Introduction

In this paper we develop a cell complex that computes the homology of the Hurwitz space of branched covers of $\mathbb{P}^1$. The motivation for this construction is to compute the Picard group of $SH_{k,b}$, the Hurwitz space parametrizing degree-k covers of $\mathbb{P}^1$ simply branched over b points. In particular, the second author conjectures that $Pic(SH_{k,b}) \otimes \mathbb{Q} = 0$, and using calculations made with our complex we present evidence for the validity of this conjecture.

Our complex also easily explains some known facts about the homology of $SH_{k,b}$, leading us to believe that the complex is natural and worthy of further investigation. For example, our complex comes from a representation of a compactification of $SH_{k,b}$ as a cell complex with no cells in codimension b or higher contained in $SH_{k,b}$. It follows that $H^i(SH_{k,b}) = 0$ for $i > b$, a fact which is also a consequence of $SH_{k,b}$ being affine. Likewise, calculating the top homology of our compactification easily reduces to the classical combinatorial problem associated

Received June 20, 1994. The first aurhor partially supported by NSF grant DMS-8922795 and the second author partially supported by an NSF postdoctoral fellowship.

with counting the number of connected components of $SH_{k,b}$ (this number is well known to be 1 - [2], [9]).

One reason to be interested in the Picard groups of Hurwitz spaces is the connection with the Picard groups of moduli spaces of curves. A k-sheeted cover of $\mathbb{P}^1$ simply branched over b points, is, by the Riemann-Hurwitz formula, a smooth curve of genus $g = \frac{b}{2} - k + 1$. Thus there is a map of $SH_{k,b} \to \mathcal{M}_g$ which, when $k > 2g - 2$, has uniform fibers. The methods of [12] show that in this case $Pic(SH_{k,b}) \otimes \mathbb{Q} = 0$ if and only if $Pic(\mathcal{M}_g) \otimes \mathbb{Q} = \mathbb{Q}$.

Since Harer [7] has established that $Pic(\mathcal{M}_g) \otimes \mathbb{Q} = \mathbb{Q}$ for $g \geq 3$, the Picard group of $SH_{k,b}$ is torsion when b is large relative to k. Thus, it is natural to conjecture that it is torsion for all values of k and b, and we show this in some examples. Conversely, a proof of the conjecture mentioned above, for all values of k and b, would yield a new proof of Harer's theorem on $Pic(\mathcal{M}_g)$.

The contents of the paper are as follows. Section 1 is the introduction. In Section 2 we define Hurwitz spaces as well as compactifications of them, which we will decompose into cells. The advantage of compactifying is that it is easier to calculate homology on compact topological spaces. We also prove statements which are sufficient to relate the homology of Hurwitz compactifications to the group of codimension-one algebraic cycles on $SH_{k,b}$. In Section 3 we describe the map from Hurwitz space to the moduli space of curves, and prove that if $k > 2g - 2$ then $Pic(SH_{k,b}) \otimes \mathbb{Q} = 0$ if and only if $Pic(\mathcal{M}_g) \otimes \mathbb{Q} = \mathbb{Q}$.

Section 4 describes the complex which calculates the homology of Hurwitz spaces. The idea is to use the fact that Hurwitz spaces are étale covers of configuration spaces of points in $\mathbb{P}^1$ corresponding to configurations of the branch points. Likewise, the compactifications we consider are defined so that they are branched covers of simple compactifications of configuration spaces. Since it is relatively easy to give a cell decomposition of a compactified configuration space where the complement of the open configuration space is a subcomplex, we obtain, by lifting, a cell decomposition of $\overline{SH}_{k,b}$ (the compactification of $SH_{k,b}$) where $\overline{SH}_{k,b} - SH_{k,b}$ is a subcomplex. We then use the monodromy of the covering map to give a method for explicitly describing the boundary maps on the lifted cell complex. The calculations made here are the key to the paper and are what enable us to compute what we know about the homology of Hurwitz spaces.

Using the techniques from Section 4, we prove in Section 5 a theorem about some low dimensional homology groups of the compactifications $\overline{SH}_{k,b}$ for all values of k and b. In Section 6 we use our complex to compute the top homology group of $\overline{SH}_{k,b}$ and show that the calculation reduces to the classical combinatorial problem associated with counting the number of components of $SH_{k,b}$. The section also contains a summary of computer calculations which we made using our complex. We also discuss various conjectures and counterexamples on the homology, and Picard groups of Hurwitz spaces.

2. Hurwitz spaces and compactifications

In this section we define the different Hurwitz spaces that we will be considering in the paper. Each parametrizes degree k coverings of $\mathbb{P}^1$ simply branched over b distinct points.

- $H_{k,b}$ - the space of covers branched over b ordered points in $\mathbb{P}^1$. It is an étale cover of $(\mathbb{P}^1)^b - D$; here D is the union of all diagonals.
- $SH_{k,b}$ - the space of covers branched over b unordered points in $\mathbb{P}^1$. $SH_{k,b} = H_{k,b}/S_b$, and is an étale cover of $\mathbb{P}^b - D$; here D is the discriminant divisor and S_b is the symmetric group on b letters.
- $AH_{k,b}$ - the quotient of $H_{k,b}$ by $\mathbb{P}GL(2,\mathbb{C})$, the automorphism group of $\mathbb{P}^1$. It is an étale cover of $\mathcal{M}_{0,b}$ the open moduli space of smooth $b-$pointed curves of genus 0.

When it is clear which Hurwitz space we are using, we will often drop the sub and superscripts.

We will also consider the following compactifications of Hurwitz spaces; each is a normal but usually singular variety.

- $\overline{H}_{k,b}$ - The normalization of $(\mathbb{P}^1)^b$ in the function field of $H_{k,b}$.
- $\overline{SH}_{k,b}$ - The normalization of $\mathbb{P}^b$ in the function field of $SH_{k,b}$.

Remark. Unfortunately, it does not seem possible to interpret the boundaries of these compactifications as representing any geometrically significant functor. In his thesis, S. Mochizuki constructed a compactification of $AH_{k,b}$ using Harris and Mumford's theory of admissible covers. However, it seems out of reach to construct a cell complex for his compactification for which it is possible to make explicit computations.

Proposition 2.1. *The space $\overline{SH}_{k,b}$ is the quotient of $\overline{H}_{k,b}$ by the* *(non-free) action of S_b induced from the action on $(\mathbb{P}^1)^b$.*

Proof. If A is a ring which is integrally closed in its function field, and G is a finite group acting on A, then it is easy to see that the invariant subring A^G is integrally closed in its function field. Thus, taking a quotient by a finite group preserves normality, so $\overline{H}_{k,b}/S_b$ is a normal variety. By construction, $\overline{H}_{k,b}/S_b$ and $\overline{SH}_{k,b}$ have the same function field and both admit finite maps to $\mathbb{P}^b$. Since $\overline{SH}_{k,b}$ is defined to be the normalization of $\mathbb{P}^b$ in the function field of $SH_{k,b}$, there is a finite surjective map $\overline{H}_{k,b}/S_b \to \overline{SH}_{k,b}$. On the other hand this map has degree 1, and both varieties are normal, so it must be an isomorphism. q.e.d.

Let H and $\overline{H}$ refer to the Hurwitz spaces and their compactifications. Also let Δ denote $\overline{H} - H$. Consider the long exact sequence of a pair

$$\ldots \to H_k(\Delta) \to H_k(\overline{H}) \to H_k(\overline{H}, \Delta) \to \ldots .$$

Since the open Hurwitz spaces are smooth, Lefschetz duality implies that the term on the right can be identified $H^{2b-k}(H)$, (the real dimension of $SH_{k,b}$ and $H_{k,b}$ is $2b$) In particular we will look at the terms

$$\ldots \to H_{2b-2}(\Delta) \to H_{2b-2}(\overline{H}) \to H^2(H) \to \ldots . \tag{2.1}$$

Because Δ has dimension $2b-2$, the first term is just the number of irreducible components of Δ [6, Lemma 19.1.1].

Proposition 2.2. *If the map $i_* : H_{2b-2}(\Delta) \to H_{2b-2}(\overline{H})$ is surjective, then $H^2_{alg}(H) = 0$, where $H^2_{alg}(H)$ is the image of $A_{b-1}(H)$ in H^2.*

Proof. The proof is straightforward given the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} H_{2b-2}(\Delta) & \to & H_{2b-2}(\overline{H}) & \to & H^2(H) & & \\ \uparrow & & \uparrow & & \uparrow & & \\ A_{b-1}(\Delta) & \to & A_{b-1}(\overline{H}) & \to & A_{b-1}(H) & \to & 0 \end{array}$$

If the map $H_{2b-2}(\Delta) \to H_{2b-2}(\overline{H})$ is surjective, then the map $H_{2b-2}(\overline{H}) \to H^2(H)$ is zero. Commutativity thus implies that the map $A_{b-1}(H) \to H^2(H)$ is zero, which is what we needed to show. q.e.d.

Proposition 2.2 gives a concrete way of checking that $H^2_{alg}(H)$ is zero. All we need to do is to check that $H_{2b-2}(\overline{H})$ is generated by cycles coming from Δ. In the sequel, we will give cell decompositions for which Δ is a subcomplex, and give an algorithm for computing the homology. The next proposition shows that $Pic(H)$ may be computable as well.

Proposition 2.3. *Assume the hypothesis of Proposition 2.2 above. If in addition, $H_{2b-1}(\overline{H}) = 0$, then $Pic(H) = H^2_{alg}(H) = 0$.*

Proof. Let $H^{sm} \subset \overline{H}$ be the smooth locus. Because $\overline{H}$ is normal, $\overline{H} - H^{sm}$ has real codimension at least 4. Thus, by the long exact sequence of a pair and Lefschetz duality, $H_{2b-1}(\overline{H}) = H^1(H^{sm})$. Now let $X \to \overline{H}$ be a desingularization which is an isomorphism over H, and consider the exact sequence:

$$\ldots \to H_{2b-1}(X - H^{sm}) \to H_{2b-1}(X) \to H^1(H^{sm}) \to \ldots .$$

The groups at the ends are both zero, so $H_{2b-1}(X) = 0$. Since X is smooth, $H^1(X) = H_{2b-1}(X)$, by Poincaré duality. Since $\overline{H}$ is projective (see Section 4.4), we may assume that X is projective. It then follows from the exponential sequence and the Hodge decomposition that $Pic(X)$ injects into $H^2(X)$.

Now consider the diagram of exact sequences:

$$\begin{array}{ccccc}
H_{2b-2}(X - H) & \xrightarrow{i_*} & H_{2b-2}(X) & \xrightarrow{j^*} & H^2(H) \\
\uparrow \simeq cl & & \uparrow cl & & \uparrow cl \\
A_{b-1}(X - H) & \xrightarrow{i_*} & A_{b-1}(X) & \xrightarrow{j^*} & A_{b-1}(H) \to 0
\end{array}$$

Since the lower j^* is surjective, any element in $A_{b-1}(H)$ is of the form j^*x for $x \in A_{b-1}(X)$. Suppose that $cl(j^*x) = 0$. Then $j^*cl(x) = 0$, so $cl(x) = i_*(y)$ for $y \in H_{2b-2}(X - H)$. Since the leftmost vertical arrow is an isomoprhism, $cl(x) = cl(i_*z)$ for some $z \in A_{b-1}(X - H)$. On the other hand, the middle vertical map is injective, so $x = i_*z$, and thus $j^*x = 0$. This proves that the map $cl : A_{b-1}(H) \to H^2(H)$ is injective. On the other hand, we proved in Proposition 2.2 that the map was zero. Therefore, $Pic(H) = A_{b-1}(H) = 0$. q.e.d.

Remark. Since the proofs of Propositions 2.2 and 2.3 do not depend on the coefficients, they also hold with rational coefficients; i.e., facts about the rational homology of Hurwitz spaces imply statements about $H^2_{alg} \otimes \mathbb{Q}$ and $Pic \otimes \mathbb{Q}$.

3. The map from Hurwitz space to moduli space

Each of the Hurwitz spaces $H_{k,b}, SH_{k,b}, AH_{k,b}$ maps to $\mathcal{M}_g$, the moduli space of curves of genus $g = \frac{b}{2} - k + 1$.

The purpose of this section is to relate *Pic* or H^2_{alg} of any of these Hurwitz spaces to the corresponding groups for $\mathcal{M}_g$. Since the arguments are virtually identical for either group, we will, for the rest of the section, use the generic notation $\mathcal{H}^2(X)$ to consistently refer to either of *Pic* or H^2_{alg}. Rational coefficients are assumed throughout.

In particular, we prove the following theorem which is due to Mochizuki [12].

Theorem 3.1. *Assume that $k > 2g - 2$. Then the following hold:*

(1) $\mathcal{H}^2(H) = 0$ implies $\mathcal{H}^2(\mathcal{M}_g) = \mathbb{Q}$; here H refers to any of the Hurwitz spaces mentioned above.

(2) $\mathcal{H}^2(\mathcal{M}_g) = \mathbb{Q}$ implies $\mathcal{H}^2(SH_{k,b}) = 0$.

Remark. The proof of this theorem is contained in [12] using the language of stacks. We go over the proof by the language of schemes. The argument presented is similar to that given in [3] for the maps from Severi varieties of plane curves to $\mathcal{M}_g$.

Proof of Theorem 3.1. Set $\mathcal{G}_{k,b} = AH_{k,b}/S_b$; this quotient parametrizes g^1_k's with exactly b simple branch points. The following proposition is the first step in the proof.

Proposition 3.1.

(1) $\mathcal{H}^2(H) = 0$ implies $\mathcal{H}^2(\mathcal{G}_{k,b}) = 0$

(2) $\mathcal{H}^2(\mathcal{G}_{k,b}) = 0$ implies that $\mathcal{H}^2(SH_{k,b}) = 0$.

Proof of Proposition 3.1. Consider the following Cartesian diagram:

$$\begin{array}{ccc} H_{k,b} & \to & SH_{k,b} \\ \downarrow & & \downarrow \\ AH_{k,b} & \to & \mathcal{G}_{k,b} \end{array}$$

The horizontal maps are finite étale, so the pullbacks are injective on all cohomology groups. The vertical maps are $\mathbb{P}GL(2,\mathbb{C})$ bundles. Since the first characteristic class of a principal $\mathbb{P}GL(2,\mathbb{C})$ bundle is zero, the vertical pullbacks are isomorphisms on (complex) codimension-one cycles ([15]). Combining the facts above proves the proposition.

To complete the proof of Theorem 3.1 we now prove

Proposition 3.2. *Assume $k > 2g - 2$. Then $\mathcal{H}^2(\mathcal{G}_{k,b}) = 0$ if and only if $\mathcal{H}^2(\mathcal{M}_g) = \mathbb{Q}$.*

Proof of Proposition 3.2. If $\mathcal{C} \to \mathcal{M}$ is a family of curves, let $J^k(\mathcal{M})$ denote the Picard scheme parametrizing degree k line bundles on the fibers of $\mathcal{C}$ over $\mathcal{M}$. If that family has a section, then there is a Poincaré bundle $\mathcal{L}$ on $J^k(\mathcal{M}) \times_{\mathcal{M}} \mathcal{C}$ ([13, p. 22]). Let $\mathcal{M}' \xrightarrow{f} \mathcal{M}_g$ be a finite cover over which the pullback, $\mathcal{C}'$, of the universal curve has a section. Let $\mathcal{L}$ be the Poincaré bundle over $J^k(\mathcal{M}') \times_{\mathcal{M}'} \mathcal{C}'$, and set $\mathcal{G}^1_k(\mathcal{C}') = Gr(2, \pi_*\mathcal{L})$ where $\pi : J^k(\mathcal{M}') \times_{\mathcal{M}'} \mathcal{C}' \to J^k(\mathcal{M}')$ is the projection. Set $\mathcal{G}'_{k,b} = \mathcal{G}_{k,b} \times_{\mathcal{M}_g} \mathcal{M}'$. The arguments of [3, Lemma 3] show that $\mathcal{G}'_{k,b} \subset \mathcal{G}^1_k(\mathcal{C}')$ is an open subscheme. Furthermore, the arguments of [12] show that $\mathcal{G}^1_k(\mathcal{C}') - \mathcal{G}'_{k,b}$ consists of three irreducible components of codimension 1, which are independent of $\mathcal{H}^2$. The remainder of the proof is essentially identical to [3, Section 5]. q.e.d.

4. The cell complex

In this section we show how to express certain compactified Hurwitz spaces as cell complexes. Both the cells and the boundary maps are explicitly described. The cell complexes have the property that the points added to compactify a Hurwitz space form a subcomplex. This allows the computation of certain homology groups as well as in certain cases the map i_* in Proposition 2.2 of Section 2. These calculations are described in Sections 5 and 6.

The basic idea is as follows. Observe that in the map of a compactified Hurwitz space to the base product of $\mathbb{P}^1$'s or to a single $\mathbb{P}^b$, ramification can only change when more branch points in the curves being parametrized come together. Construct a cellular decomposition of the base such that whenever more branch points come together there is a new cell. It is then possible to use the cell decomposition of the base to induce a cellular decomposition of the covering Hurwitz space.

We now proceed to make these ideas precise and explicitly carry them out. We first do it for ordered branch points, and then by analyzing the action of the symmetric group we are able to give a cell decomposition in the case of unordered branch points.

4.1. Cell decomposition of the base. The cell decomposition given here is similar to the cell decompositions for configuration spaces; for references to the literature on this topic see the book by Vassiliev [14].

We first define a cellular decomposition of $(\mathbb{P}^1)^b$.

View $\mathbb{P}^1$ as $\mathbb{C} \cup \infty$ and let $z_j = a_j + \sqrt{-1}c_j$ be a holomorphic coordinate on the finite part of the j'th $\mathbb{P}^1$.

$(2b)$-cells: There are $b!$ of these, one for each ordering of $\{1, \dots, b\}$. Let $\sigma \in S_b$. Then the open cell corresponding to σ is defined as:

$$\{(z_1, \dots, z_b) | a_{\sigma(1)} > a_{\sigma(2)} > \dots > a_{\sigma(b)}\}$$

(note that all of these cells are contained in the finite part of $(\mathbb{P}^1)^b$).

All cells of lower dimension are obtained from the $(2b)$-cells by a finite sequence of the following three types of operations.

(1) Set one complex coordinate equal to ∞.

(2) Set two real coordinates equal to each other; this is only allowed if the two real coordinates are adjacent in the ordering given by the prior cell. One must then order the imaginary coordinates corresponding to the equal real coordinates.

(3) Set two imaginary coordinates equal; this is only allowed if the corresponding real coordinates are equal as in (2) and the two imaginary coordinates are adjacent in the ordering of the prior cell.

Thus a cell will be described as follows: some of the coordinates will be ∞, among the other coordinates given any two a_i, a_j we have $a_i = a_j$, $a_i < a_j$ or $a_i > a_j$; within each group of equal a's given any two c_i, c_j, $c_i = c_j$, $c_i < c_j$ or $c_i > c_j$. A typical cell:

$$\begin{aligned}\{(\infty, z_2, z_3, z_4, \infty, z_6, z_7, z_8) | a_3 = a_4 = a_6 < a_2 = a_7 < a_8,\\ c_4 = c_6 < c_3, c_2 < c_7\},\end{aligned}$$

(this cell has codimension 8). We now give a list of all the types of cells of low codimension and low dimension, as we will compute with them later on.

$(2b-1)$ cells all have the form:

$$\begin{aligned}\{(z_1, \dots, z_b) | a_{\sigma(1)} > a_{\sigma(2)} \dots > \dots a_{\sigma(l)}\\ = a_{\sigma(k)} > \dots > a_{\sigma(b)}, c_{\sigma(l)} < c_{\sigma(k)}\}.\end{aligned}$$

$(2b-2)$ cells are of three types with some subtypes.

Infinity: $\{(z_1, \dots, \infty, \dots z_b) | a_{\sigma(1)} > a_{\sigma(2)} > \dots > \hat{a}_{\sigma(l)} > \dots a_{\sigma(b)}\}$, where as usual $\hat{a}$ means omit.

Complex diagonal:

$$\begin{aligned}\{(z_1, \dots, z_b) | a_{\sigma(1)} > a_{\sigma(2)} > \dots > a_{\sigma(l)} = a_{\sigma(k)} > \dots > a_{\sigma(b)},\\ c_{\sigma(l)} = c_{\sigma(k)}\}.\end{aligned}$$

Two real diagonals: these may be adjacent or non-adjacent.

$$\begin{aligned}\{(z_1,\dots,z_b)|a_{\sigma(1)} > a_{\sigma(2)} > \dots > a_{\sigma(l)} \\ = a_{\sigma(k)} = a_{\sigma(m)} > \dots > a_{\sigma(b)}, \\ c_{\sigma(l)} < c_{\sigma(k)} < c_{\sigma(m)}\}\end{aligned}$$

or

$$\begin{aligned}\{(z_1,\dots,z_b)|a_{\sigma(1)} > a_{\sigma(2)} \dots > \dots a_{\sigma(l)} \\ = a_{\sigma(k)} > \dots a_{\sigma(m)} = a_{\sigma(n)} > \dots > a_{\sigma(b)}, \\ c_{\sigma(k)} < c_{\sigma(l)}, c_{\sigma(m)} < c_{\sigma(n)}\}.\end{aligned}$$

$(2b-3)$-cells are of three types, with many subtypes.

Infinity and a real diagonal.

Complex diagonal and a real diagonal; these may be adjacent or nonadjacent.

Three real diagonals: these may be all three adjacent, two of the three adjacent, or none adjacent.

We now describe the cells of low dimension.

3-cells

Set j of the coordinates equal to infinity, where $0 \leq j \leq b-2$. Among coordinates not set equal to infinity, set all real coordinates equal to each other. Divide the imaginary coordinates into two disjoint nonempty groups. Within each group set all the imaginary coordinates equal to each other. Finally, declare one group of imaginary coordinates to be greater than the other group. For instance:

$$\begin{aligned}\{(z_1,\dots,z_{b-j},\infty,\dots\infty)|a_1 = \dots = a_{b-j}, \\ c_1 = \dots = c_l > c_{l+1} = \dots = c_{b-j}\}.\end{aligned}$$

2-cells

These are quite similar to 3-cells. Set j of the coordinates equal to infinity, where $0 \leq j \leq b-1$. Among coordinates not set equal to infinity set all real coordinates equal, and set all imaginary coordinates equal.

1-cells

None

0-cells

$\{(\infty,\dots,\infty)\}$.

It is rather clear, though perhaps tedious to write down in complete detail that $(\mathbb{P}^1)^b$ is a disjoint union of these cells and that each cell is homeomorphic to an open ball of the appropriate dimension. We leave this to the reader.

We now define our cell decomposition for $\mathbb{P}^b = (\mathbb{P}^1)^b/S_b$. It is easy to see that for any cell C in the decomposition of $(\mathbb{P}^1)^b$ and any $\sigma \in S_b$, either σ acts as the identity on C or maps C homeomorphically onto another cell C'. From this it follows that we have a cell decomposition of $\mathbb{P}^b$ where cells are equivalence classes of cells of $(\mathbb{P}^1)^b$ under the action of S_b (two cells being equivalent if and only if some element of S_b establishes a homeomorphism between them). As we did for $(\mathbb{P}^1)^b$ we shall describe explicitly the cells with which we shall later compute.

$(2b)$ cells

There is only one of these which we may represent as: $[1 > 2 > \ldots > b]$, where $1 > 2$ is short hand for $a_1 > a_2$.

$(2b-1)$ cells

There are $b-1$ of these which may represent as: $[1 = 2 > 3 > \ldots > b], [1 > 2 = 3 > \ldots > b], \ldots, [1 > 2 > 3 \ldots > b-1 = b]$, where $1 = 2$ is short hand for $a_1 = a_2$. Having said $a_i = a_j$ we do not need to say whether $c_i < c_j$ or $c_j < c_i$ because these two cells are equivalent under the S_b action.

$(2b-2)$ cells. We now have several different types.

Infinity: only 1 cell $[\infty \times 2 > 3 > \ldots > b]$.

Complex diagonal: $b-1$ cells $[1 \equiv 2 > 3 > \ldots > b], \ldots [1 > 2 > 3 \ldots b-1 \equiv b]$, where $1 \equiv 2$ is short hand for $a_1 = a_2$ and $c_1 = c_2$.

Two real diagonals: $\frac{1}{2}(b-1)(b-2)$ cells, choose 2 $>$'s in $[1 > 2 > \ldots > b]$ and turn them into $=$'s. Again there is no need to order the c's.

$(2b-3)$ cells.

Infinity and a real diagonal: b-2 cells $[\infty \times 2 = 3 > \ldots > b], \ldots, [\infty \times 2 > 3 \ldots > b-1 = b]$. Again no need to order the c's.

Complex diagonal with additional real diagonal. This has one slightly tricky point. $\{a_1 > \ldots > a_i = a_j > \ldots > a_k = a_l > \ldots > a_b, c_i = c_j\}$ and $\{a_1 > \ldots > a_i = a_j > \ldots > a_k = a_l > \ldots > a_b, c_k = c_l\}$ are not equivalent. Likewise, $\{a_1 > \ldots > a_i = a_j = a_k > \ldots > a_b, c_i < c_j = c_k\}$ and $\{a_1 > \ldots > a_i = a_j = a_k > \ldots > a_b, c_i = c_j < c_k\}$ are not equivalent. There are $(b-1)(b-2)$ cells. Starting from $[1 > 2 \ldots > b]$ one has $(b-1)$ choices for turning a $>$ into $\equiv$, and then $(b-2)$ choices for turning one of the remaining $>$'s to a $=$. In this notation, $1 \equiv 2 = 3$ means $a_1 = a_2 = a_3$ and $c_1 = c_2 < c_3$.

Three real diagonals: $\frac{1}{6}(b-1)(b-2)(b-3)$ cells; choose 3 $>$'s and turn them into $=$'s. Again there is no need to order the c's.

$3 - cells$. There are $\frac{1}{2}b(b-1)$ cells: $[1 = 2 \equiv 3 \equiv \ldots \equiv b]$, $[1 \equiv 2 = 3 \equiv 4 \ldots \equiv b], \ldots$ $[1 \equiv 2 \equiv 3 \ldots \equiv b-1 = b]$, $[\infty \times 2 = 3 \equiv \ldots \equiv b], \ldots,$ $[\infty \times \ldots \times \infty \times b-1 = b]$.

$2 - cells$. There b such cells:

$[1 \equiv 2 \equiv \ldots \equiv b]$, $[\infty \times 2 \equiv 3 \ldots \equiv b] \ldots, ,$ $[\infty \times \ldots \infty \times \mathbb{C}]$.

$1 - cells$. None

$0 - cells$. $[\infty \times \ldots \times \infty]$

4.2. Cellular decomposition of the Hurwitz space

Lemma 4.1. *Consider the map $\overline{\pi}_{k,b} : \overline{H}_{k,b} \to (\mathbb{P}^1)^b$ (or $s\overline{\pi}_{k,b} : \overline{SH}_{k,b} \to \mathbb{P}^b$). Let C be a cell in our decomposition of $(\mathbb{P}^1)^b$ (or $\mathbb{P}^b$). Then the inverse image of C under $\overline{\pi}_{k,b}$ (or $s\overline{\pi}_{k,b}$) has finitely many connected components, each of which connected component is homeomorphic to C via the restriction of $\overline{\pi}_{k,b}$ (or $s\overline{\pi}_{k,b}$).*

Proof. Since $\overline{\pi}_{k,b}$ and $s\overline{\pi}_{k,b}$ are finite algebraic, and the cells are homeomorphic to open balls, all we need to show is that over each cell the number of preimages of points not counting multiplicity is constant. The arguments for $\overline{\pi}_{k,b}$ and $s\overline{\pi}_{k,b}$ are essentially the same. Denote by $D \subset (\mathbb{P}^1)^b$ the locus of points where two or more coordinates are equal. Let C be a cell in $(\mathbb{P}^1)^b$, and p and q two points in C. From the way the cellular decomposition was constructed, at all points of C the same coordinates are equal to each other. Thus, we may assume that centered at p we have holomorphic coordinates $w_1, \ldots, w_b$ with D given by $w_1 = \ldots = w_j$, and near q we have holomorphic coordinates $v_1, \ldots, v_b$ with D given by $v_1 = \ldots = v_j$. The map given by $f(w_i) = v_i$ yields an isomorphism of a small neighborhood W of p with a small neighborhood V of q, taking p to q and mapping $W \cap D$ isomorphically onto $V \cap D$. (In $\mathbb{P}^b$ the local picture is simply the quotient of this picture by the symmetric group). The covering $\pi_{k,b} : H_{k,b} \to (\mathbb{P}^1)^b - D$ is determined by the local combinatorial data, which is the same for $W - D$ and $V - D$. This gives a commutative diagram:

$$\begin{array}{ccc} \pi_{k,b}^{-1}(W-D) & \xrightarrow{\cong} & \pi_{k,b}^{-1}(V-D) \\ \Big\downarrow \pi_{k,b} & & \pi_{k,b} \Big\downarrow \\ W-D \subset W & \xrightarrow{\cong} & V-D \subset V \end{array}$$

$\overline{H}_{k,b}$ is defined as the normalization of $(\mathbb{P}^1)^b$ in the function field of $H_{k,b}$. Normalizations can be constructed locally and patched together (see [4] expose XII for the analytic case). Therefore, the preceeding commuative diagram gives rise to the following commutative diagram:

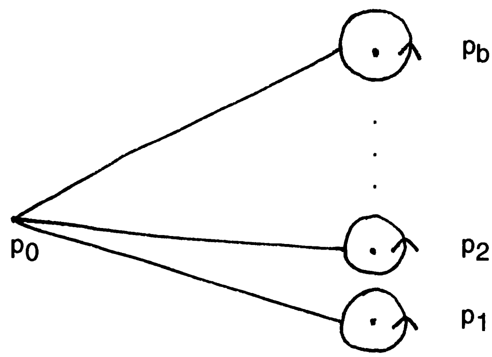
$$\begin{array}{ccc} \overline{\pi}_{k,b}^{-1}(W) & \xrightarrow{\cong} & \overline{\pi}_{k,b}^{-1}(V) \\ \Big\downarrow & & \Big\downarrow \\ W & \xrightarrow{\cong} & V \end{array}$$

So there are certainly the same number of points over p as over q. q.e.d.

With this lemma proven we may define a cellular decomposition of $\overline{H}_{k,b}$ or $\overline{SH}_{k,b}$ the open cells of which are exactly the connected components of the inverse images via $\overline{\pi}_{k,b}$ or $s\overline{\pi}_{k,b}$ of the open cells in our cellular decomposition of $(\mathbb{P}^1)^b$ or $\mathbb{P}^b$.

We now show how to keep track of the connected components of the inverse image of a given open cell. This will allow us to give explicit names to all the cells of our cellular decomposition of a Hurwitz space. This is needed to be able to perform explicit homology computations.

The first step here is to describe the fiber of $\pi_{k,b}$ (or $s\pi_{k,b}$) over a point away from all diagonals. It has been known for a long time how to do this; see for instance [1] or [5] for details and proofs. Let $p = (p_1, \dots, p_b) \in (\mathbb{P}^1)^b - D$ (Remember D consists of all diagonals). Choose a base point $p_0 \in \mathbb{P}^1$ distinct from $\{p_1, \dots, p_b\}$. For each $j = 1, \dots, b$ choose an oriented path starting at p_0 and travelling once around p_j then returning to p_0 and not enclosing any of the other p_i. Orient the loops so that their product is the identity. This is sometimes called a lolly pop diagram.



Once such a choice of lolly pop diagram has been made there is a bijection between:

- $\pi_{k,b}^{-1}(p)$
- ordered b–tuples $(\sigma_1, \ldots, \sigma_b)$ of simple transpositions in S_k such that $\sigma_1\sigma_2\ldots\sigma_b = id$, and $\{\sigma_1, \ldots, \sigma_b\}$ generates a transitive subgroup of S_k; modulo the equivalence relation $(\sigma_1, \ldots, \sigma_b) \simeq (\tau_1, \ldots \tau_b)$ iff there exists $\sigma \in S_k$ such that $\sigma_i = \sigma\tau_i\sigma^{-1}$ for all i.

This equivalence relations is known as simultaneous conjugation. Denote the equivalence class of $(\sigma_1, \ldots, \sigma_b)$ by $[\sigma_1, \ldots, \sigma_b]$.

Fix a base point $p_0 \in \mathbb{P}^1$. For each open cell C of $(\mathbb{P}^1)^b$ that is not contained in D fix a point P none of whose coordinates is equal to p_0, and fix a lolly pop diagram for P with base point p_0. From Lemma 4.1 we see that each connected component of $\pi_{k,b}^{-1}(C)$ contains exactly one element of $\pi_{k,b}^{-1}(P)$. Thus to designate an open cell of $\overline{H}_{k,b}$ not lying over D one gives the cell of $(\mathbb{P}^1)^b$ over which it lies, and an equivalence class $[\sigma_1, \ldots, \sigma_b]$.

Cells that lie over of D are more difficult to keep track of since the map $\pi_{k,b}$ can branch over them. We first tell how to do this for cells in D of dimension $2b-2$. In the list of $(2b-2)$ cells in Section 4.1 these are the complex diagonal cells. We will express cells over

$$C = \{a_{\sigma(1)} > \ldots > a_{\sigma(l)} = a_{\sigma(k)} > \ldots a_{\sigma(b)}, c_{\sigma(l)} = c_{\sigma(k)}\}$$

as equivalence classes of cells over

$$C' = \{a_{\sigma(1)} > \ldots > a_{\sigma(l)} = a_{\sigma(k)} > \ldots a_{\sigma(b)}, c_{\sigma(l)} > c_{\sigma(k)}\}.$$

Some of the sheets of $\overline{H}_{k,b}$ over C' might come together over C.

Lemma 4.2. *For C , C' just defined, each cell lying over C is uniquely determined by listing all cells lying over C' having it in their closure.*

Proof. It is sufficient to show that each cell lying over C' has exactly one cell lying over C in its closure. First note that $C \cup C'$ is simply connected. Assume to the contrary that we have a cell E' lying over C' with two distinct cells E and F lying over C in its closure. We may then find points $p' \in C'$, $p \in C$ and two paths γ_1 and γ_2 from p' to p such that γ_1 lifts to a path starting in E' and ending in E while γ_2 lifts to a path starting in E' and ending in F. Because $C' \cup C$ is simply connected we may continuously deform γ_1 into γ_2 leaving the endpoints fixed. This would induce a deformation of the lifting of γ_1 into the lifting of γ_2 leaving the endpoints fixed. This is impossible because the liftings have distinct endpoints. q.e.d.

Thus a cell over C can be unambiguously identified by giving the cells C and C' and a list of those cells over C' having it in their closure. This list is the previously mentioned equivalence class. For purposes of explicit computations we also need to be able to say which cells lying over C' have the same cells lying over C in their closure. We first prove a lemma stating essentially that which sheets come together is determined by the local monodromy of the cover.

Lemma 4.3. *Let C be any open cell in D (not necessarily of dimension $2b-2$). Pick a point $p \in C$, a small open neighborhood (say any open ball) B of $(\mathbb{P}^1)^b$ containing p, and a point $q \in B - D$. The fundamental group of $B - D$ with base point q acts via monodromy on $\pi_{k,b}^{-1}(q)$. Define an equivalence relation on $\pi_{k,b}^{-1}(q)$ by saying that two points are equivalent if and only if they can be taken to each other by monodromy actions. For B sufficiently small the following are true.*

1. Two points of $\pi_{k,b}^{-1}(q)$ lie in the same monodromy equivalence class iff they lie in the same connected component of $\pi_{k,b}^{-1}(B - D)$.

2. The closure of each connected component of $\pi_{k,b}^{-1}(B-D)$ in $\overline{\pi}_{k,b}^{-1}(B)$ has exactly one point over p.

3. The closures of the connected components of $\pi_{k,b}^{-1}(B - D)$ in $\overline{\pi}_{k,b}^{-1}(B)$ are all disjoint from each other.

Proof. 1. and 2. are standard monodromy facts. For 3. we need to use the fact that $\overline{H}_{k,b}$ was defined as a normalization. The disjoint union of the closures of the connected components of $\pi_{k,b}^{-1}(B - D)$ in

$\overline{\pi}_{k,b}^{-1}(B)$ must be dominated by the normalization which is $\overline{\pi}_{k,b}^{-1}(B)$. Thus they are equal. q.e.d.

Another way of stating the lemma is that the points of $\pi_{k,b}^{-1}(q)$ come together over p if and only if they are equivalent under the monodromy action.

Now back to our specific C, C' of dimensions $(2b-2)$ and $(2b-1)$. Going from C' to C involves $p_{\sigma(l)}$ and $p_{\sigma(k)}$ coming together. Thus the monodromy involves $p_{\sigma(l)}$ and $p_{\sigma(k)}$ moving around each other (At this point the reader might find it useful to review the foundational material in [1] Sections 1 and 2). For simplicity of illustration we assume that $\sigma(i) = i$ in C, C'.

We start with the chosen lolly pop diagram.

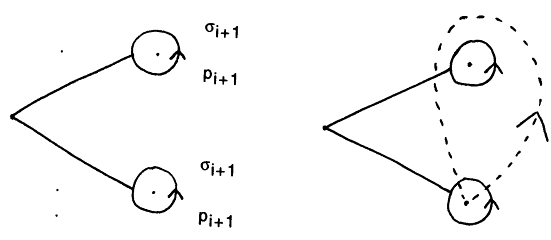
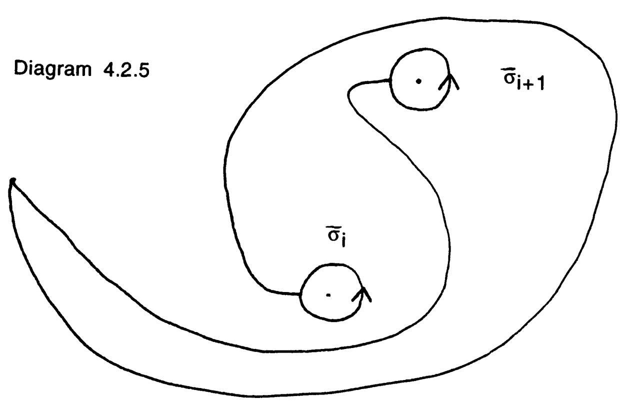


Diagram 4.2.4

Then p_i takes a trip around p_{i+1} ending back where it started. The paths σ_i and σ_{i+1} must deform as p_i moves so that the points never cross the paths. The end result is:

Diagram 4.2.5



notice that $\overline{\sigma}_i = \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}$ and $\overline{\sigma}_{i+1} = \sigma_i\sigma_{i+1}\sigma_i^{-1}$.

Start with a sheet over C' given with respect to the standard lolly pop diagram 4.2.4 by $[\tau_1, \dots, \tau_b]$. The monodromy element induced by p_i going around p_{i+1} as shown takes that sheet to a sheet which with respect to the new lolly pop diagram 4.2.5 is given by $[\tau_1, \dots, \tau_b]$. Since sheets over C' are named according to the standard lolly pop diagram, we must determine how this sheet is named with respect to the standard diagram. Let $[\overline{\tau_1}, \dots, \overline{\tau_b}]$ be its name with respect to the standard lolly pop diagram. From [1, Lemma 1.4] we see that we want $\sigma_j \mapsto \overline{\tau}_j$ $j = 1, \dots, b$ and $\overline{\sigma}_j \mapsto \tau_j$ $j = 1, \dots, b$ to induce the same homomorphism from $\pi_1(\mathbb{P}^1 - \{p_1, \dots, p_b\}, p_0) \to S_k$. This is achieved by setting

$$\begin{aligned} \overline{\tau}_j &= \tau_j,\ j \neq i, i+1, \\ \overline{\tau}_i &= \tau_{i+1}^{-1}\tau_i\tau_{i+1}, \\ \overline{\tau}_{i+1} &= \tau_{i+1}^{-1}\tau_i^{-1}\tau_{i+1}\tau_i\tau_{i+1}. \end{aligned} \tag{4.1}$$

Thus a cell over $\{a_1 > \dots > a_i = a_{i+1} > \dots > a_b, c_i = c_{i+1}\}$ can be designated as

$$\{a_1 > \dots > a_i = a_{i+1} > \dots > a_b, c_i = c_{i+1},\ [[\tau_1, \dots, \tau_b]]\}$$

where the double square brackets mean equivalence class not only with respect to simultaneous conjugation but also with respect to the equivalence relation generated by the monodromy substitution 4.1. Similar considerations apply to all cells of dimension $2b-2$ lying over D.

Cells of dimension $2b-3$ lying over D are quite easy. Referring to our list in Section 4.1 we see that in $(\mathbb{P}^1)^b$ they are all of the type "complex diagonal with real diagonal". To each such cell C of dimension $2b-3$ associate the cell C' of dimension $2b-2$ which simply in the description of C leaves out the extra real diagonal. For example:

$$\begin{aligned} C = \{a_1 > \dots > a_i = a_{i+1} > \dots > a_j = a_{j+1} > \dots > a_b,& \\ c_i = c_{i+1}, c_j < c_{j+1}\},& \end{aligned}$$

$$C' = \{a_1 > \dots > a_i = a_{i+1} > \dots > a_b, c_i = c_{i+1}\}.$$

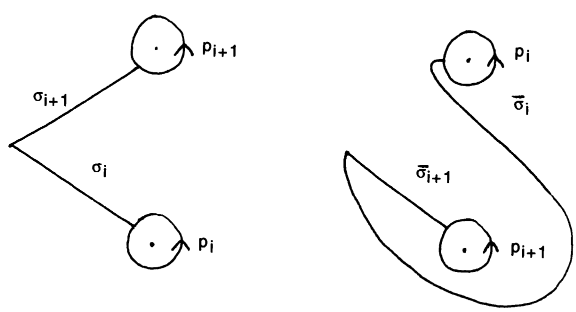
The number of sheets over C and C' are exactly the same. Each sheet over C is in the closure of a unique sheet over C'. Thus to designate a cell over C we name C, the associated cell C', and a cell over C'.

Cells of dimension $2b-4$ lying over D are again more difficult. Given a cell C of dimension $2b-4$ in D one looks for a cell C' of dimension $2b-3$, not necessarily in D such that for C, C' we can prove a lemma analogous to Lemma 4.2. One then designates cells over C as equivalence classes of cells over C' modulo a monodromy equivalence relation.

We stop here but the ambitious reader could keep going.

Now for $\overline{SH}_{k,b}$, cells in $\overline{SH}_{k,b}$ are equivalence classes of cells in $\overline{H}_{k,b}$ under the action of S_b. We describe this action in Section 4.3 in conjunction with our calculations of boundary maps. One should also be aware that if monodromy calculations are done directly in $\overline{SH}_{k,b}$, they are different in $\overline{H}_{k,b}$. The quotient map $(\mathbb{P}^1)^b \to \mathbb{P}^b$ ramifies to order two at the diagonals where exactly two of the coordinates are equal. The simple loops which we constructed in $(\mathbb{P}^1)^b$ pushforward to double loops in $\mathbb{P}^b$. In [1, Section 2] simple loops in $\mathbb{P}^b$ were constructed. Points only need to go half way around each other in $(\mathbb{P}^1)^b$ to go completely around each other in $\mathbb{P}^b$.

For example:



Here $\overline{\sigma_i} = \sigma_i\sigma_{i+1}\sigma_i^{-1}$, $\overline{\sigma}_{i+1} = \sigma_i$ and the monodromy substitution is: $\overline{\tau_i} = \tau_{i+1}$, $\overline{\tau}_{i+1} = \tau_{i+1}^{-1}\tau_i\tau_{i+1}$.

4.3. The boundary maps. We proceed in a manner similar to the way which we proceeded for the description of the cellular decomposition. We first describe the boundary maps for the base $(\mathbb{P}^1)^b$ then use this to describe them for $\overline{H}_{k,b}$. After that we factor out by the symmetric group and get boundary maps for $\mathbb{P}^b$ and $\overline{SH}_{k,b}$.

First, let us describe the boundary maps for the base $(\mathbb{P}^1)^b$. Set theoretically this is quite simple. Recall the three operations which we gave when first defining the cell complex. There are three codimension-one things we can do to a cell to obtain another cell.

(1) Set two adjacent real coordinates equal to each other (remember to order the corresponding imaginary coordinates).

(2) Set some complex coordinates equal to infinity under the following rules. We start with a maximal set of complex coordinates all of whose real coordinates are equal. Within this we find a maximal proper subset S whose imaginary coordinates are also equal. Set all complex coordinates in S equal to infinity. Be careful about the ordering of the remaining coordinates.

(3) Set two adjacent imaginary coordinates equal, provided that the corresponding real coordinates are already equal.

Set theoretically the boundary of a cell consists of all those cells of one smaller dimension that can be obtained via the operations above. The signs can be easily obtained by writing down local coordinates and using standard conventions found in most basic algebraic topology books.

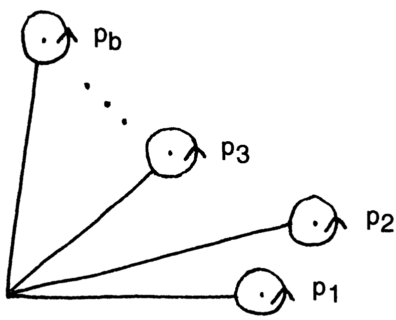
The boundary maps for $\overline{H}_{k,b}$ are pretty much the same except that we must keep track of the sheets of the cover. Let $\tilde{C}$ be a cell in $\overline{H}_{k,b}$ lying over the cell C in $(\mathbb{P}^1)^b$. For each cell C' in the set theoretic boundary of C we get a cell $\tilde{C}'$ lying over C' in the boundary of $\tilde{C}$ (set theoretically). The signs in the boundary maps are the same as down in $(\mathbb{P}^1)^b$. However, we must keep track of the sheets! To do so, we will be more explicit and use the explicit names for $\tilde{C}$ and $\tilde{C}'$ given in terms of conjugacy classes of $b-$tuples of transpositions in S_k.

The precise description of how to do this breaks down into several cases depending on how the chosen lolly pop diagrams for C and C' relate, and whether C and/or C' are in D.

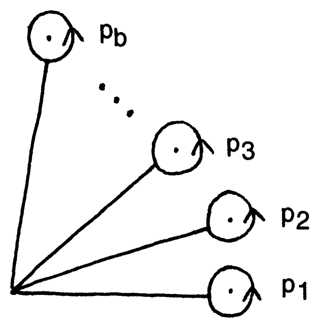
We first do the easiest case where neither C nor C' is contained in D. C has a given lollypop diagram as does C'. In the diagrams the points $p_1, \dots, p_b$ are situated as they must be to represent a point of C or of C'. Move the points in the diagram for C in a continuous manner until they become the points in the diagram for C'. As you move the points make sure that they always represent a point of C until the end of your movement where they represent a point of C'. Deform the lolly pop diagram along with the points. The end lolly pop diagram is a possible lolly pop diagram for C'. Compare it with the chosen lolly pop diagram for C' and make a translation as was done in the preceding monodromy computations. An example should (hopefully!) make the process clear.

Consider a cell lying over the cell $\{a_2 > a_1 > a_3 > \ldots > a_b\}$. Such a cell would be designated as $\{a_2 > a_1 > a_3 \ldots > a_b, [\tau_1, \ldots , \tau_b]\}$. Among the cells in the boundary of $\{a_2 > a_1 > a_3 > \ldots > a_b\}$ are $\{a_1 = a_2 > a_3 > \ldots > a_b, c_1 < c_2\}$ and $\{a_1 = a_2 > a_3 > \ldots > a_b, c_1 > c_2\}$.

Suppose that the standard lolly pop diagrams for these cells have been chosen as follows.

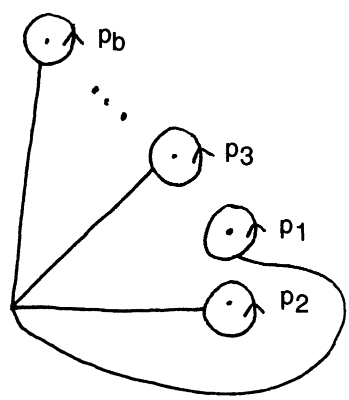


$\{a_2>a_1>a_3>\ldots>a_b\}$



$\{a_1=a_2>a_3>\ldots>a_b,c_1<c_2\}$

Diagram 4.3.1



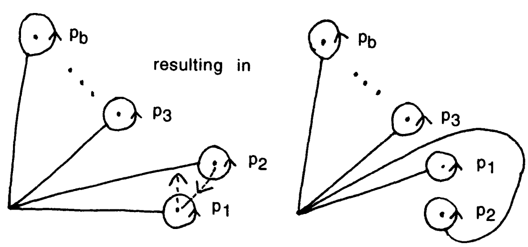
$\{a_1=a_2>a_3>\ldots>a_b,c_1>c_2\}$

To move the points in the lolly pop diagram for $\{a_2 > a_1 > a_3 > \ldots > a_b\}$ into those for $\{a_1 = a_2 > a_3 > \ldots > a_b, c_1 < c_2\}$ one simply moves p_2 to the left until it gets to the proper place. With this movement one may deform the lollypop diagram for $\{a_2 > a_1 > a_3 > \ldots > a_b\}$ into that for $\{a_1 = a_2 > a_3 > \ldots > a_b, c_1 < c_2\}$.

Since the deformed lolly pop diagram agrees with the chosen lolly pop diagram, no translation of permutations is needed. Thus one of the terms in the boundary of $\{a_2 > a_1 > a_3 \ldots > a_b, [\tau_1, \ldots, \tau_b]\}$ will be $\{a_1 = a_2 > a_3 > \ldots > a_b, c_1 < c_2, [\tau_1, \ldots, \tau_b]\}$.

For $\{a_1 = a_2 > a_3 > \ldots > a_b, c_1 > c_2\}$ the situation is not so simple. Since the movement must stay inside $a_2 > a_1$ until its end, p_1 can not travel over p_2 as would be needed to obtain a deformed lolly pop diagram equal to the chosen lolly pop diagram for $\{a_1 = a_2 > a_3 > \ldots > a_b, c_1 > c_2\}$.

Consider the following movement.



If $\sigma_1, \ldots, \sigma_b$ are the elements of the fundamental group represented by lolly pops in the standard diagram for $\{a_1 = a_2 > a_3 > \ldots > a_b, c_1 > c_2\}$, and $\alpha_1, \ldots, \alpha_b$ are those in the deformed lolly pop diagram then $\alpha_1 = \sigma_2^{-1}\sigma_1\sigma_2$, $\alpha_2 = \sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_2$, and $\alpha_i = \sigma_i$ otherwise. Solving for the σ_j's in terms of the α_j's we have $\sigma_1 = \alpha_1\alpha_2\alpha_1\alpha_2^{-1}\alpha_1^{-1}$ and $\sigma_2 = \alpha_1\alpha_2\alpha_1^{-1}$. With the same argument used to write down the monodromy action, we conclude that one of the terms in the boundary of $\{a_2 > a_1 > a_3 \ldots > a_b, [\tau_1, \ldots, \tau_b]\}$ is $\{a_1 = a_2 > a_3 > \ldots > a_b, c_1 > c_2, [\tau_1\tau_2\tau_1\tau_2^{-1}\tau_1^{-1}, \tau_1\tau_2\tau_1^{-1}, \tau_3, \ldots, \tau_b]\}$.

In choosing the standard lolly pop diagrams it is natural to try to make, as many as possible, standard ones be identical to deformed ones. No matter how clever one is, one can not make this always happen. This is a reflection of the fact that the cover $\overline{H}_{k,b} \to (\mathbb{P}^1)^b$ has non-trivial monodromy.

When one or both of C and C' are contained in D things can be more complicated. It always comes down to finding a way to compare the lolly pop diagrams used in designating cells over C and C'. We do a typical example.

Both of the cells $C_1 = \{a_1 = a_2 > a_3 > \ldots > a_b, c_1 < c_2\}$ and $C_2 = \{a_1 = a_2 > a_3 > \ldots > a_b, c_1 > c_2\}$ have $C_3 = \{a_1 = a_2 > a_3 > \ldots > a_b, c_1 = c_2\}$ in their boundaries. Sheets over C_3 are represented as equivalence classes of sheets over C_1. This makes the boundary map from sheets over C_1 to sheets over C_3 quite simple. A sheet over C_1 has in its boundary the sheet over C_3 which is its equivalence class. The boundary map from sheets over C_2 to sheets over C_3 will involve lolly pop diagram computations. In computing the boundary map from sheets over $C_0 = \{a_2 > a_1 > a_3 > \ldots > a_b\}$ to the sheets over C_1 and C_2 we saw how to compare sheets over C_0 with sheets over both C_1 and C_2. From this (and the fact that $C_0 \cup C_1 \cup C_2$ is simply connected) we see that the boundary map from sheets over C_2 to sheets over C_3 is: the sheet $[\tau_1, \ldots, \tau_b]$ over C_2 goes to the sheet $[[\tau_1\tau_2\tau_1\tau_2^{-1}\tau_1^{-1}, \tau_1\tau_2\tau_2^{-1}, \tau_3, \ldots, \tau_b]]$, where as before [[ ]] means equivalence with respect to both simultaneous conjugation and the monodromy substitution 4.1 with $i = 1$.

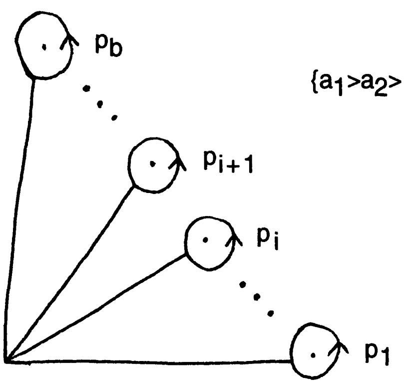
That is as much as we will say about how to compute boundary maps in $\overline{H}_{k,b}$. The boundary maps in $\overline{SH}_{k,b}$ are based on these, but also involve factoring out by the action of S_b. We now explain how to compute the boundary maps for $\overline{SH}_{k,b}$.

First we state a general rule for calculating boundaries. For a cell C in $\overline{H}_{k,b}$ denote by $[C]$ the cell in $\overline{SH}_{k,b}$ corresponding to the S_b orbit of C. Say $dC = \sum_{i=1}^{n} a_i C_i$, and suppose that among the C_i there are m equivalence classes $[C_1], \ldots, [C_m]$. Then $d[C] = \sum_{i=1}^{m} b_i [C_i]$ where the b_i are determined as follows. For each C_j, $j > m$ choose an element $\sigma_j \in S_b$ that fixes C and takes C_j to some C_l with $l \leq m$, (It is easy to see that one can always do this.) b_i is a_i plus the sum over all $j > m$ such that $[C_j] = [C_i]$ of $a_j \text{sign}\, \sigma_j$. Here, by sign σ_j we mean its sign not as a permutation, but rather as a homeomorphism from C_j to C_i; it is $+1$ if it preserves orientation and -1 if it reverses orientation. The correctness of this is clear if one thinks about what the quotient map $\overline{H}_{k,b} \to \overline{SH}_{k,b}$ looks like near $[C_i]$. Notice that sign σ_j is independent of the choice of σ_j; for suppose σ_j and σ_j' had opposite signs, then $\sigma_j^{-1}\sigma_j'$ would be orientation reversing from C_j to itself, contradicting the fact that the action always either takes a cell to another cell, or a cell identically to itself.

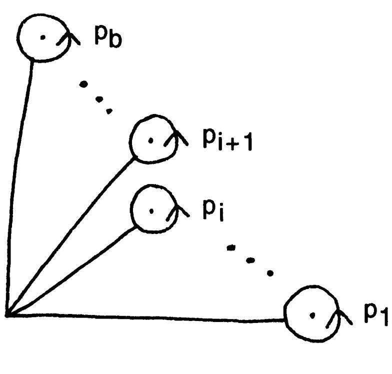
It is very easy to determine when two cells of $(\mathbb{P}^1)^b$ are equivalent under the action of S_b. For $\overline{H}_{k,b}$ it is not so easy, one must do lolly pop diagram computations similar to those done before. Here we show how to do what we need to compute the homology groups which were calculated with the computer.

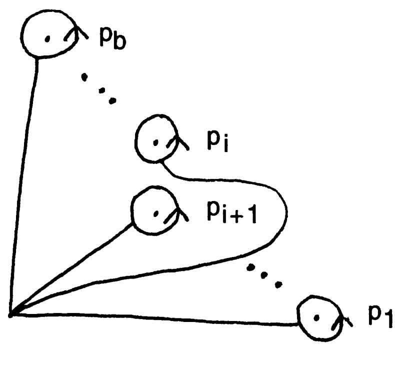
The boundary of the unique $2b$−cell $[a_1 > \ldots > a_b]$ will be a sum of terms of the form $[a_1 > \ldots > a_i = a_{i+1} > \ldots > a_b, c_i < c_{i+1}]$ and $[a_1 > \ldots > a_i = a_{i+1} > \ldots > a_b, c_i > c_{i+1}]$. Those two cells represent the same cell in $\mathbb{P}^b$. The question is when do the cells over them in $\overline{H}_{k,b}$ represent the same cell in $\overline{SH}_{k,b}$. That is, when will $[a_1 > \ldots > a_i = a_{i+1} > \ldots > a_b, c_i < c_{i+1}, [\tau_1, \ldots, \tau_b]]$ and $[a_1 > \ldots > a_i = a_{i+1} > \ldots > a_b, c_i > c_{i+1}, [\tau_1', \ldots, \tau_b']]$ represent the same cell in $\overline{SH}_{k,b}$?

Suppose we have chosen standard lolly pop diagrams as follows.



$\{a_1>a_2>\ldots>a_b\}$





$\{a_1>a_2>\ldots>a_i=a_{i+1}>\ldots>a_b, c_i<c_{i+1}\}$ $\{a_1>a_2>\ldots>a_i=a_{i+1}>\ldots>a_b, c_i>c_{i+1}\}$

Think of computing $d[a_1 > a_2 > \dots a_b, [\tau_1, \dots, t_b]]$ in $\overline{H}_{k,b}$. For the two terms involving $a_i = a_{i+1}$ we have chosen standard lolly pop diagrams which are equal to the deformed lolly pop diagrams, so we get terms $[a_1 > a_2 > \dots > a_i = a_{i+1} > \dots > a_b, c_i < c_{i+1}, [\tau_1, \dots, \tau_b]]$ and $[a_1 > a_2 > \dots > a_i = a_{i+1} > \dots > a_b, c_i > c_{i+1}, [\tau_1, \dots, \tau_b]]$. Notice that in the two lollypop diagrams for $c_i < c_{i+1}$ and $c_i > c_{i+1}$ the sets $\{p_1, \dots, p_b\}$ are equal. The fibers of $\overline{H}_{k,b}$ over each of these points are therefore equal - both identified with covers of $\mathbb{P}^1 - \{p_1, \dots, p_b\}$. The fiber of $\overline{SH}_{k,b}$ over the single point represented as the equivalence class containing those two points is again just covers of $\mathbb{P}^1 - \{p_1, \dots, p_b\}$. When identifying two fibers in $\overline{H}_{k,b}$ to get the single fiber in $\overline{SH}_{k,b}$ we must identify a cover in one fiber with the same cover in the other fiber. When representing these covers as equivalence classes $[\tau_1, \dots, \tau_b]$ we must take into account that we have different lolly pop diagrams for the two points in $(\mathbb{P}^1)^b$ which become one point in $\mathbb{P}^b$. This involves writing the cycles in one diagram in terms of the cycles in the other as we have done many times before. The end result is:

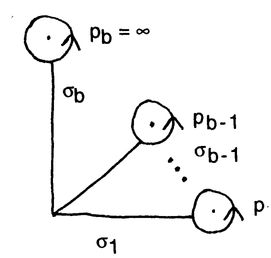
$$\begin{aligned} &d[a_1 > \dots > a_b; [\tau_1, \dots, \tau_b]] \\ &\quad = \sum_{i=1}^{b-1} \pm [a_1 > \dots > a_i = a_{i+1} > \dots > a_b, c_i < c_{i+1}, [\tau_1, \dots, \tau_b]] \\ &\qquad \pm [a_1 > \dots > a_i = a_{i+1} \dots > a_b, c_i < c_{i+1}, \\ &\qquad\quad [\tau_1, \dots, \tau_{i-1}, \tau_{i+1}, \tau_{i+1}^{-1} \tau_i \tau_{i+1}, \tau_{i+2}, \dots, \tau_b]]. \end{aligned}$$

Next we think about the boundaries of $2b - 1$ cells. From the list at the beginning of Section 4.1 we see that all $b - 1$ cells are more or less of the same type. Let us think about taking the boundary of $[a_1 > \dots > a_i = a_{i+1} \dots > a_b, c_i < c_{i+1}, [\tau_1, \dots, \tau_b]]$. In its boundary we will have all the types of $2b - 2$ cells on the list from Section 4.1. Computing the boundary for cells that add a new real diagonal not adjacent to the one, we already have, proceeds almost exactly as the $2b$ to $2b - 1$ boundary proceeded. Adding a real diagonal adjacent to $a_i = a_{i+1}$ also proceeds very similarly except that the lolly pop diagrams are a bit more complicated because three points are permuted. The only really new things are when one gets a complex diagonal or an infinity.

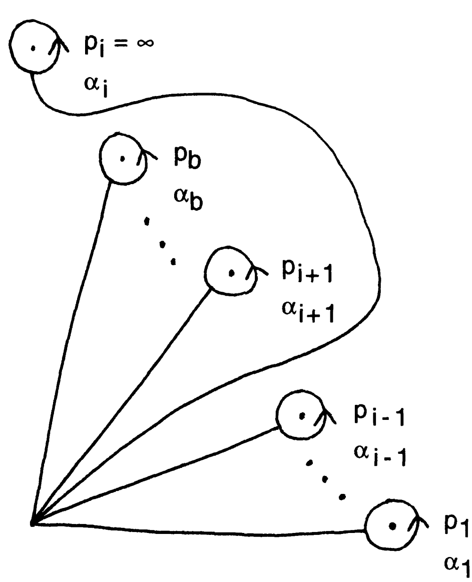
The complex diagonal term is quite easy. Sheets over $[a_1 > \dots > a_i = a_{i+1} \dots > a_b, c_i = c_{i+1}]$ are identified as equivalence classes of sheets over $[a_1 > \ldots > a_i = a_{i+1} \ldots > a_b, c_i < c_{i+1}]$. The boundary map sends each sheet to its equivalence class.

Boundary terms that involve infinity are much more complicated. The boundary of $\{a_1 > \ldots > a_i = a_{i+1} \ldots > a_b, c_i = c_{i+1}\}$ in $(\mathbb{P}^1)^b$ contains both $\{a_1 > \ldots > a_{i-1} \times \infty > a_{i+1} > \ldots a_b\}$ and $\{a_1 > \ldots > a_i \times \infty > a_{i+2} > \ldots a_b\}$. As i ranges from 1 to $b-1$ we get infinity in all positions. However, under the action of S_b all these are equivalent. We must see how the sheets over them in $\overline{H}_{k,b}$ are equivalent under the action of S_b. Given what we already know how to do, this is not very difficult. Suppose we choose as our representative cell in $\overline{SH}_{k,b}$ the cell of $\overline{H}_{k,b}$ where ∞ is in the b'th position.

Its standard lolly pop diagram could be chosen as:



Now suppose ∞ is in the i'th position. Then its standard lolly pop diagram could be chosen as:



Thus we have $\alpha_j = \sigma_j$ for $j < i$; $\alpha_i = \sigma_i\sigma_{i+1}\dots\sigma_{b-1}\sigma_b\sigma_{b-1}^{-1}\dots\sigma_i^{-1}$ and $\alpha_j = \sigma_{j-1}$ for $j > i$. We can therefore make the appropriate transformations on the equivalence classes $[\tau_1, \dots, \tau_b]$ to identify sheets over the two cells - as we have done many times before.

The boundary maps of higher codimension become more complicated, but we can always do them by paying careful attention to lolly pop diagrams.

4.4. This cell complex does compute homology. Notice that we have never referred to our cell complex as a CW-complex. Indeed it is not obvious that it is a CW-complex. Thus, there could be a question as to whether or not it can actually be used to compute the homology of $\overline{H}_{k,b}$ or $\overline{SH}_{k,b}$, which for the rest of the section will be referred to as H. We now show that it does indeed compute the homology.

First note that H is the normalization of a projective variety in a function field. Thus by [10, Section v.4, Theorem 4] it is also a projective variety. Next refer to [8] on the triangulation of semi-algebraic sets. From the fact that H is projective algebraic, and the way the cells were defined in terms of equalities and inequalities, it follows that the closure of any cell in the base is a closed semi-algebraic set. By [8, Theorem, p. 170] one may triangulate H in such a way that the total inverse image in H of the closure of each cell is a simplicial subcomplex. Finally [16, Section 2.9] shows how to make a cell complex that computes homology out of a simplicial complex. Our cell complex is made in that manner. Each open cell is the union of all open simplices that are contained in it.

5. Computation of some low dimensional homology groups

Theorem 5.1.
(a) $H^i(SH_{k,b}) = H^i(H_{k,b}) = 0$ *for* $i > b$,
(b) $H_1(\overline{H}_{k,b}) = H_1(\overline{SH}_{k,b}) = 0$,
(c) $H_2(\overline{SH}_{k,b}, \mathbb{Q}) = \mathbb{Q}$.

Proof. (a) follows from the fact that these spaces are affine, see [11, p. 39], but can also been seen from our cell complex.

When $i < b$ every cell of dimension i is supported in $\overline{H}_{k,b} - H_{k,b}$ (resp $\overline{SH}_{k,b} - SH_{k,b}$). Thus the relative homology groups $H_i(\overline{H}_{k,b}, \overline{H}_{k,b} - H_{k,b})$ and $H_i(\overline{SH}_{k,b}, \overline{SH}_{k,b} - SH_{k,b})$ vanish. Applying Lefschetz duality (applicable because of the triangulability mentioned in Section 4.4) it follows that

$$H^{2b-i}(\overline{H}_{k,b}) = H^{2b-i}(\overline{SH}_{k,b}) = 0$$

for $i < b$.

(b) There are no 1 cells in the complex.

(c) This requires more computation. First note that every 2-cell is closed because there are no 1-cells. To simplify notation we will use the following notation for 2-cells in $\mathbb{P}^b$.

$$\begin{aligned}
\Delta_0 &= [1 \equiv 2 \equiv 3 \equiv \ldots \equiv b], \\
\Delta_1 &= [\infty \times 2 \equiv 3 \equiv \ldots \equiv b], \\
&\ldots \\
&\ldots \\
\Delta_i &= [\infty \times \infty \cdots \times \infty \times i+1 \equiv i+2 \equiv \cdots b], \\
&\ldots \\
&\ldots \\
\Delta_{b-1} &= [\infty \times \ldots \infty \times \mathbb{C}].
\end{aligned}$$

Using this notation, we will compute the boundaries of selected 3 cells (mod 2)

$$d[1 = 2 \equiv 3 \equiv \cdots b] = \Delta_0 + \Delta_1 + \Delta_{b-1},$$

$$d[\infty \times 2 = 3 \equiv \cdots b] = \Delta_1 + \Delta_2 + \Delta_{b-1},$$

continuing until we take

$$d[\infty \times \infty \cdots \infty \times b-1 = b] = \Delta_{b-2} + \Delta_{b-1} + \Delta_{b-1}.$$

Similar relations hold in the homology of of $\overline{SH}_{k,b}$, however, matters are complicated by the fact that there are many sheets. We can however, write

$$d[1 = 2 \equiv 3 \equiv \cdots b]^{(i)} = \Delta_0^{(\sigma_0(i))} + \Delta_1^{(\sigma_1(i))} + \Delta_{b-1}^{(\sigma_{b-1}(i))},$$

where the superscripts in parentheses indicate a particular sheet of $\overline{SH}_{k,b}$ over the cell, and σ_0, σ_1 and σ_{b-1} are some integer valued functions depending on an ordering of the sheets over the cells concerned. We do know however, that every sheet over Δ_0 must occur in the boundary of some cell over $[1 = 2 \equiv 3 \ldots \equiv b]$.

Let us write down an arbitrary 2−cycle

$$\alpha = \sum_{i=0}^{b-1} \sum_j a_{i,j} \Delta_i^{(j)}.$$

From what we just said, by adding suitable terms of the form $d[1 = 2 \equiv 3 \ldots \equiv b]^{(l)}$ we see that α is homologous to a cycle of the form

$$\alpha = \sum_{i=1}^{b-1} \sum_j a_{i,j} \Delta_i^{(j)}.$$

In a similar fashion we can use terms of the form $d[\infty \times 2 = 3 \equiv \ldots \equiv b]^{(l)}$ to eliminate $i = 1$ terms from the sum. Continuing in this manner, we eventually obtain that α is homologous to a cycle of the form

$$\sum_l a_l \Delta_{b-1}^{(l)}.$$

Thus,

$$rk\ H_2(\overline{SH}_{k,b}, \mathbb{Z}/2\mathbb{Z}) \leq \text{number of sheets over } \Delta_{b-1}.$$

We now state:

Lemma 5.1. *The number of sheets over Δ_{b-1} is one.*

Given the lemma, the theorem follows almost immediately. Of course $H_2(\mathbb{P}^b, \mathbb{Q}) = \mathbb{Q}$. Since the map $s\overline{\pi}_{k,b} : \overline{SH}_{k,b} \to \mathbb{P}^b$ is a (singular) branched cover, the transfer map is injective on rational homology. Hence, $rk_{\mathbb{Q}}\ H_2(\overline{SH}_{k,b}, \mathbb{Q}) \geq 1$. This bound, together with the one above, yields $H_2(\overline{SH}_{k,b}, \mathbb{Q}) = \mathbb{Q}$ as desired. q.e.d.

Proof of Lemma 5.1. To compute the number of sheets over the cell Δ_{b-1} in $\mathbb{P}^b$, we must compute the number of sheets in $\overline{H}_{k,b}$ over the cell $\{(\infty, \ldots, \infty, z_b) : z_b \in \mathbb{C}\}$ in $(\mathbb{P}^1)^b$. (Note that this cell is in the equivalence class that represents Δ_{b-1}.) Here, we must compute, by local monodromy, the number of points in $H_{k,b}$ lying over a point in $(\mathbb{P}^1)^b$ where $z_1 = z_2 \ldots z_{b-1}$. This number is the number of equivalence classes of the monodromy group generated by loops in $(\mathbb{P}^1)^b - \Delta$ which take any 2 points, but the last one around each other.

However, due essentially to the fact that $\mathbb{P}^1$ minus a single point is still simply connected, any loop involving points going around the last point can be deformed to a loop that involves only the first $b-1$ points going around each other. Hence, the number of sheets is the number of monodromy equivalence classes of the full monodromy group. Thus, to prove the lemma, it suffices to show that $H_{k,b}$ is connected.

By classical arguments, (cf. [5]) the quotient $SH_{k,b} = H_{k,b}/S_b$ is connected. Furthermore, since $(\mathbb{P}^1)^b - \Delta$ is connected, it suffices to find a path connecting any two points in a fiber of the map $\pi_{k,b} : H_{k,b} \to (\mathbb{P}^1)^b - \Delta$.

Let $a = (z_1, \dots, z_b)$ be a point in $(\mathbb{P}^1)^b - \Delta$ and let $a_1, a_2, \dots a_N$ be the orbit of a under the action of S_b where $a_1 = a$ and $N = (b)!$. The fiber of $\pi_{k,b}$ over each point a_i represents all possible degree k covers of $\mathbb{P}^1$ simply branched over the set $\{z_1, \dots z_b\}$. Thus we can fix a single lolly pop diagram for $\mathbb{P}^1 - \{z_1, \dots, z_b\}$ and use it to represent the fibers over each a_i as equivalence classes of ordered b–tuples of simple transpositions as in Section 4. Denote by $a_i^{(1)}$ the point in the fiber over a_i represented as

$$[(12), (12), \dots, (12), (13), (13), (14), (14), \dots, (1k), (1k)], \tag{5.1}$$

where the transposition (12) is repeated $b - 2(k-2)$ times.

Because $SH_{k,b}$ is connected, any 2 points in $\pi_{k,b}^{-1}(a)$ can be connected to be points of the form $a_i^{(1)}$ and $a_j^{(1)}$ for some i and j respectively. Thus, we can reduce the proof of connectedness to showing that two points of the form $a_i^{(1)}$ and $a_j^{(1)}$ can be connected.

Fortunately, it is quite easy to construct a path connecting these two points. At the end of Section 4.2 we constructed paths in $(\mathbb{P}^1)^b$ that induced simple loops in $\mathbb{P}^b$. Denote by Γ_l these paths in $(\mathbb{P}^1)^b$. Thus Γ_l induces a path in $H_{k,b}$, that takes the point $[\sigma_1, \dots, \sigma_l, \sigma_{l+1}, \dots, \sigma_b]$ in the fiber over $(z_1, \dots, z_l, z_{l+1}, \dots, z_b)$ to the point

$$[\sigma_1, \dots, \sigma_{l-1}, \sigma_{l+1}, \sigma_{l+1}^{-1}\sigma_l\sigma_{l+1}, \sigma_{l+2}, \dots, \sigma_b]$$

in the fiber over $(z_1, \dots, z_{l-1}, z_{l+1}, z_l, z_{l+2}, \dots, z_b)$. Compute directly that Γ_l^3 induces a path in $H_{k,b}$, that takes a point represented by (5.1) over $(z_1, \dots, z_l, z_{l+1}, \dots, z_b)$ to a point again represented by (5.1) over $(z_1, \dots, z_{l-1}, z_{l+1}, z_l, z_{l+2}, \dots, z_b)$. Since S_b is generated by the transpositions $(12), (23), \dots, (b-1, b)$, using a product of the Γ_l's will take any $a_i^{(1)}$ to $a_j^{(1)}$. q.e.d.

6. High dimensional homology

6.1. Top dimensional homology. In this section we compute $H_{2b}(\overline{SH}_{k,b}, \mathbb{Z}/2\mathbb{Z})$ by means of our complex. We use $\mathbb{Z}/2\mathbb{Z}$ coefficients because the computations with them are easier and no information is lost. Since $SH_{k,b}$ is non-singular and $\overline{SH}_{k,b}$ is normal, the rank of $H_{2b}(\overline{SH}_{k,b})$ with coefficients in any non-trivial group is the number of connected components of $\overline{SH}_{k,b}$ which is the same as the number of connected components of $SH_{k,b}$.

It has been known since the 1800's that $SH_{k,b}$ is connected ([9], see also [5]), thus $H_{2b}(\overline{SH}_{k,b}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. In this section we show that using our cell complex to compute this homology group reduces to a combinatorial problem, and that this combinatorial problem is exactly the same one used in the classical proof that $SH_{k,b}$ is connected.

Let l be the degree of the map $s\pi_{k,b} : SH_{k,b} \to \mathbb{P}^b$. From Section 4 we know that $\overline{SH}_{k,b}$ will have l cells of dimension $(2b)$ and $l(b-1)$ cells of dimension $(2b-1)$. To write down the boundary map d : $(2b)$chains $\to$ $(2b-1)$chains in compact form we first define operators Γ_j, $1 \leq j \leq b-1$ on equivalence classes of ordered b-tuples of simple transpositions which represent fibers of $s\pi_{k,b}$ as in Section 4:

$$\Gamma_j[\sigma_1, \dots, \sigma_b] = [\sigma_1, \dots, \sigma_j, \sigma_{j+1}, \sigma_{j+1}^{-1}\sigma_j\sigma_{j+1}, \sigma_{j+2}, \dots, \sigma_b]. \tag{6.1}$$

From Section 4 we deduce that

$$\begin{aligned} &d\sum_{i=1}^{l} a_i[1 > \dots > b, [\sigma_1^{(i)}, \dots, \sigma_b^{(i)}]] \\ &= \sum_{j=1}^{b-1}\sum_{i=1}^{i=l} a_i[1 > \dots > j = j+1 > \dots > b, [\sigma_1^{(i)}, \dots, \sigma_b^{(i)}]] \\ &\qquad + a_i[1 > \dots > j = j+1 > \dots > b, \Gamma_j[\sigma_1^{(i)}, \dots, \sigma_b^{(i)}]]. \end{aligned}$$

$H_{2b}(\overline{SH}_{k,b}, \mathbb{Z}/2\mathbb{Z})$ is of course the kernel of d.

Lemma 6.1. *Each Γ_j is a bijection.*

Proof. $\Gamma_j^{-1}[\sigma_1, \dots, \sigma_b] = [\sigma_1, \dots, \sigma_{j-1}, \sigma_j\sigma_{j+1}\sigma_j^{-1}, \sigma_j, \sigma_{j+2} \dots, \sigma_b]$. q.e.d.

Remark. Remembering that the σ's are transpositions one can see the following. If $\sigma_j = \sigma_{j+1}$ then Γ_j acts as the identity. If σ_j and σ_{j+1} are disjoint, then Γ_j^2 acts as the identity. If σ_j and σ_{j+1} are neither disjoint nor equal, then Γ_j^3 acts as the identity. So in any case, Γ_i^6 acts as the identity and we may take $\Gamma_j^{-1} = \Gamma_j^5$.

From the lemma it follows that each $(2b-1)$cell has exactly two $(2b)$cells in whose boundary it lies; $[1 > \dots j = j+1 > \dots b, [\sigma_1, \dots, \sigma_b]]$ lies in the boundary of exactly, $[1 > \dots > b, [\sigma_1, \dots, \sigma_b]]$ and $[1 > \dots > b, \Gamma_j^{-1}[\sigma_1, \dots, \sigma_b]]$.

Suppose that $\alpha = \sum_{i=1}^{l} a_i[1 > \dots > b, [\sigma_1^{(i)}, \dots, \sigma_b^{(i)}]]$ and $d\alpha = 0$, and further that $\Gamma_j[\sigma_1^{(t)}, \dots, \sigma_b^{(t)}] = [\sigma_1^{(s)}, \dots, \sigma_b^{(s)}]$ where of course $1 \leq s \leq l$.

Lemma 6.2. $a_t = a_s$.

Proof. The coefficient of $[1 > \dots j = j+1 > \dots b, [\sigma_1^{(s)}, \dots, \sigma_b^{(s)}]]$ in $d\alpha$ is $a_t + a_s$. q.e.d.

Suppose now that we put an equivalence relation on equivalence classes $[\sigma_1, \dots, \sigma_b]$ as follows. Two such are equivalent if there is some Γ_j taking one to the other. Then take the equivalence relation thus generated. By the remark above, one could generate the same equivalence relation with Γ_j^{-1}'s.

By Lemma 6.2 the rank of the kernel of d is at most the number of equivalence classes under that equivalence relation. Indeed since, Lemma 6.2 gives one possibly trivial linear relation among the coefficients a_i for each $(2b-1)$cell, the rank of the kernel of d is exactly the number of equivalence classes. Showing that this number is one is a purely combinatorial problem. It is exactly the combinatorial problem used to prove the connectivity of $SH_{k,b}$ (see for example [5]).

6.2. Examples and conjectures. Using computer programs written by the second author, we were able to determine homology information about the Hurwitz spaces $SH_{3,4}$, $SH_{3,6}$, $SH_{3,8}$ and $SH_{4,6}$ as well as their compactifications. We summarize the results of these calculations below. All computations were done with rational coefficients.

$SH_{3,4}$ is a 4 sheeted cover of $\mathbb{P}^4 - D$ where D is the discriminant. We found $H^1(SH_{3,4}) = \mathbb{Q}$ and $H^2(SH_{3,4}) = 0$. After compactifying we found $H_7(\overline{SH}_{3,4}) = 0$ and $H_6(\overline{SH}_{3,4}) = \mathbb{Q}$. Since $H^2(SH_{3,4}) = 0$ it follows from exact sequence 2.1 that the map i_* in Proposition 2.2 is surjective. Thus, $H^2_{alg}(SH_{3,4}) = 0$. Since $H_7(\overline{SH}_{3,4}) = 0$, $Pic(SH_{3,4}) = 0$ as well by Proposition 2.3.

$SH_{3,6}$ is a 40 sheeted cover of $\mathbb{P}^6 - D$, and $H^1(SH_{3,6}) = \mathbb{Q}$ while

$H^2(SH_{3,6}) = 0$. We also found $H_{11}(\overline{SH}_{3,6}) = 0$ and $H_{10}(\overline{SH}_{3,6}) = \mathbb{Q}^2$. Thus, as above $H^2_{alg}(SH_{3,6}) = Pic(SH_{3,6}) = 0$.

$SH_{3,8}$ is a 364 sheeted cover of $\mathbb{P}^8 - D$, and $H^1(SH_{3,8}) = \mathbb{Q}$ and $H^2(SH_{3,8}) = 0$. We also found $H_{15}(\overline{SH}_{3,8}) = 0$ and $H_{14}(\overline{SH}_{3,8}) = \mathbb{Q}^2$. As in the previous examples, we can then conclude $H^2_{alg}(SH_{3,8}) = Pic(SH_{3,8}) = 0$.

$SH_{4,6}$ is a 40 sheeted cover of $\mathbb{P}^6 - D$ and $H^1(SH_{4,6}) = \mathbb{Q}^2$ while $H^2(SH_{4,6}) = \mathbb{Q}$. This example shows that $H^2(SH_{k,b})$ is not in general 0, as one might have hoped. We also found $H_{11}(\overline{SH}_{4,6}) = 0$ and $H_{10}(\overline{SH}_{4,6}) = \mathbb{Q}^2$. Because $H^2(SH_{4,6}) \neq 0$, we can not conclude from the exact sequence 2.1 that the map i_* from Proposition 2.2 is surjective. Nevertheless we are able to show by direct calculation that $H_2(\overline{SH}_{4,6})/im(H_2(\overline{SH}_{4,6} - SH_{4,6})) = 0$, so the map i_* is surjective and $H^2_{alg}(SH_{4,6}) = 0$. Since $H_{11}(\overline{SH}_{4,6}) = 0$, $Pic(SH_{4,6}) = 0$ as well.

Since $SH_{k,b}$ is far from being simply connected, we expect that $H^1(SH_{k,b})$ can be arbitrarily large for $k, b >> 0$. Based on the first examples involving $SH_{3,4}, SH_{3,6}, SH_{3,8}$ we wondered if $H^2(SH_{k,b})$ would vanish. However, as noted above $H^2(SH_{4,6}) \neq 0$, and we expect that $H^2(SH_{k,b})$ can become arbitrarily large.

In all of the examples we calculated that $H_{2b-1}(\overline{SH}_{k,b}) = 0$ leading the first author to ask, and the second author to conjecture[1] the following:

Conjecture 1. *$H_{2b-1}(\overline{SH}_{k,b}) = 0$ for all k and b.*

Likewise, in all examples where it was possible to for us to compute $H_{2b-2}(\overline{SH}_{k,b})$ was generated by the image of $H_{2b-2}(\overline{SH}_{k,b} - SH_{k,b})$. We believe this happens for all k and b, which is consistent with the following conjecture[1]

Conjecture 2. *$H^2_{alg}(SH_{k,b}) = 0$ for all k and b.*

As proved in Section 3, Harer's theorem implies the conjecture for k large relative to b.

Combining Conjecture 1 and Conjecture 2 yields by Proposition 2.3

Conjecture 3. *$Pic(SH_{k,b}) \otimes \mathbb{Q} = 0$.*

Again, Harer's theorem proves this conjecture when k is large relative to b.

---

[1]Again, the first author does not make conjectures.

References

[1] E. Arbarello, *On subvarieties of the moduli space of curves of genus g defined in terms of Weierstrass points*, Lincei-Mem. Sci. Fis., ecc. - 1978 - S. VIII, Vol. XV, Sez. I, 1.

[2] C. Clebsch, *Zür theorie der Riemann'schen fläche*, Math. Ann. **6** (1872) 216-230.

[3] D. Edidin, *Picard groups of Severi varieties*, Comm. Algebra **22**(6) (1994) 2073-2081.

[4] A. Grothendieck, et. al., *Revetements étale et Groupe Fondemental*, Lecture Notes in Math. Vol. 224, Springer, Berlin, 1971.

[5] W. Fulton, *Hurwitz schemes and the irreducibility of moduli of algebraic curves*, Ann. Math **90** (1969) 542-575.

[6] _____, *Intersection Theory*, Ergebnisse, (3) Vol. 2, Springer, Berlin, 1984.

[7] J. Harer, *The second homology of the mapping class group of an orientable surface*, Invent. Math. **72** (1983) 221-239.

[8] H. Hironaka, *Triangulations of algebraic sets*, Algebraic Geometry Arcata, Amer. Math. Soc. Proc. Sympos. Pure Math. Vol. 29, 1975, 165-184.

[9] A. Hurwitz, *Über Riemann'sche Flächen mit gegeberen Verzweigungspunkten*, Math. Ann. **39** (1891) 1-61.

[10] S. Lang, *Introduction to Algebraic Geometry*, Interscience, New York, 1958.

[11] J. Milnor, *Morse Theory*, Princeton University Press, 1963.

[12] S. Mochizuki, Thesis, Princeton, 1992.

[13] D. Mumford, *Geometric Invariant Theory*, Ergebnisse, N. F. Vol. 34, Springer, Berlin, 1965.

[14] V.A. Vassilev, *Complements of Discriminants of Smooth Maps: Topology and Applications*, Transl. Math. Monographs, Vol. 98, Amer. Math. Soc., 1992.

[15] A. Vistoli, *Chow groups of quotient varieties,* J. Algebra **107** (1987) 410-424.

[16] A. Wallace, *Algebraic Topology: homology and cohomology*, W.A.

Benjamin, New York, 1970.

Syracuse University
University of Chicago