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ON LIMITS OF TAME HYPERBOLIC 3-MANIFOLDS

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Abstract

We show that if a purely hyperbolic Kleinian group is the strong limit of a sequence of topologically tame purely hyperbolic Kleinian groups, then it is topologically tame. We then apply this result to purely hyperbolic *algebraic* limits of purely hyperbolic topologically tame Kleinian groups. We observe that such a limit is topologically tame if its quotient is not homotopy equivalent to a compression body, or if it has non-trivial domain of discontinuity.

1. Introduction

A hyperbolic 3-manifold is said to be *topologically tame* if it is homeomorphic to the interior of a compact 3-manifold. This seemingly topological property has strong geometric and analytic consequences (see theorem 1.1), and it is conjectured that all hyperbolic 3-manifolds with finitely generated fundamental groups are topologically tame [29]. One approach to this conjecture is to try to prove it for hyperbolic manifolds obtained as limits, in various senses, of manifolds already known to be tame. This approach is partially motivated by another conjecture (see Bers [4] or Thurston [40]) that every hyperbolic 3-manifold with finitely generated fundamental group is an algebraic limit of geometrically finite manifolds (which are in particular topologically tame).

In the 1970's, Bill Thurston proved that if M is a compact 3-manifold with incompressible boundary, then any type-preserving algebraic limit of a sequence of geometrically finite hyperbolic 3-manifolds homeomorphic to the interior of M is topologically tame and is itself homeomorphic to $int(M)$. In the 1980's, Francis Bonahon [9] made use of

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Thurston's techniques in his proof that *every* hyperbolic 3-manifold homotopy-equivalent to a compact 3-manifold with incompressible boundary is topologically tame.

Our main theorem is that, in the absence of parabolics, *strong limits* of topologically tame hyperbolic 3-manifolds are topologically tame. We will further show that, again in the absence of parabolics, any algebraic limit of a sequence of topologically tame hyperbolic 3-manifolds, whose fundamental group is not a free product of surface groups and free groups, is topologically tame. To obtain this generalization of our main theorem, we make use of the ideas of Ohshika [33] and Paulin and the results of Anderson-Canary [2]. We hope that our ideas might be useful to others attempting to prove the entire conjecture.

Before continuing, we recall for the reader some of the geometric and analytic consequences of topological tameness.

Theorem 1.1. *Let $N = \mathbf{H}^3/\Gamma$ be a topologically tame hyperbolic 3-manifold, and let Λ_Γ be the limit set of Γ.*

1. *([13], [39]) Either $\Lambda_\Gamma = S^2_\infty$ or Λ_Γ has measure zero. Moreover, if $\Lambda_\Gamma = S^2_\infty$, Γ acts ergodically on S^2_∞.*
2. *([13], [38], [39]) The geodesic flow of N is ergodic if and only if $\Lambda_\Gamma = S^2_\infty$.*
3. *([12]) N is geometrically finite if and only if $\lambda_0(N) \neq 0$, where $\lambda_0(N) = \inf spec(-\Delta)$, and Δ denotes the Laplacian acting on $L^2(N)$.*

We now introduce some of the notation which will be necessary to make more formal statements of our results. Let M be a compact 3-manifold, and let $\{\rho_i : \pi_1(M) \to Isom_+(\mathbf{H}^3)\}$ be a sequence of discrete faithful representations. The sequence$\{\rho_i\}$ is said to converge *algebraically* to a representation $\rho_\infty : \pi_1(M) \to Isom_+(\mathbf{H}^3)$ if it converges as a sequence of representations; that is, if for each $g \in \pi_1(M)$, $\{\rho_i(g)\}$ converges, in $Isom_+(\mathbf{H}^3)$, to $\rho_\infty(g)$. In this case, Jørgenson [25] proved that ρ_∞ is also discrete and faithful (except in the case that $\pi_1(M)$ is abelian, which has been completely analyzed by Jørgensen [24]).

Let $\Gamma_i = \rho_i(\pi_1(M))$, $\Gamma_\infty = \rho_\infty(\pi_1(M))$, $N_i = \mathbf{H}^3/\Gamma_i$ and $N_\infty = \mathbf{H}^3/\Gamma_\infty$. Since ρ_i is an isomorphism onto its image there is an inverse isomorphism $\rho_i^{-1} : \Gamma_i \to \pi_1(M)$ (and similarly for ρ_∞).

We say that $\{\rho_i\}$ converges *strongly* to ρ_∞ if it converges algebraically and $\{\Gamma_i\}$ converges geometrically to Γ_∞ (see section 3). This implies, roughly, that larger and larger portions of N_i look more and more like portions of N_∞. We will discuss this in more detail in section 3.

We will call a representation $\rho : \pi_1(M) \to Isom_+(\mathbf{H}^3)$ *purely hyperbolic* if $\rho(\pi_1(M))$ consists entirely of hyperbolic elements, i.e., contains no elliptics or parabolics. A discrete faithful representation $\rho : \pi_1(M) \to Isom_+(\mathbf{H}^3)$ is said to be topologically tame if $N = \mathbf{H}^3/\rho(\pi_1(M))$ is topologically tame.

Main Theorem. *Let M be a compact irreducible 3-manifold, and let $\{\rho_i : \pi_1(M) \to Isom_+(\mathbf{H}^3)\}$ be a sequence of purely hyperbolic, discrete, faithful, topologically tame representations converging strongly to a purely hyperbolic representation $\rho_\infty : \pi_1(M) \to Isom_+(\mathbf{H}^3)$. Then ρ_∞ is topologically tame. Moreover, for all sufficiently large i there exists a homeomorphism $\Phi_i : N_i \to N_\infty$, such that $(\Phi_i)_* = \rho_\infty \circ \rho_i^{-1}$.*

Ohshika [33] independently proved a result similar to our main theorem, by techniques that differ substantially from ours. In particular he makes key use of the work of Otal [34] on pleated surfaces in compression bodies and his proof is modeled more closely on the original proof of Thurston in the case where $\pi_1(M)$ is freely indecomposable. We note that Otal previously proved Theorem 1.2 in the cases where $\rho_1(\pi_1(M))$ is a function group isomorphic to either $\pi_1(S_1) * \mathbf{Z}$ or $\pi_1(S_1) * \pi_1(S_2)$, where S_i are closed surfaces.

Theorem 1.2. *(Ohshika [33]) Let M be a compact irreducible 3-manifold which is not a handlebody. Let $\{\rho_i : \pi_1(M) \to Isom_+(\mathbf{H}^3)\}$ be a sequence of quasiconformally conjugate, convex cocompact representations of $\pi_1(M)$ converging strongly to a purely hyperbolic representation $\rho_\infty : \pi_1(M) \to Isom_+(\mathbf{H}^3)$. Then ρ_∞ is topologically tame.*

Ohshika then combines Theorem 1.2 with the clever observation that, for a purely hyperbolic algebraic limit ρ_∞ of a sequence of quasiconformally conjugate convex cocompact Kleinian groups, if the domain of discontinuity of ρ_∞ is non-empty, then ρ_∞ is in fact a strong limit. As a consequence he observes that the limit set of ρ_∞ either has measure zero or is the entire sphere. This represents progress towards the Ahlfors measure conjecture. (Although Theorem 1.2 and part (1) of Corollary 1.3 are not explicitly stated in his paper, they seem to be implicit consequences of his techniques. Ohshika actually states part (2) of Corollary 1.3 as his main theorem.)

Corollary 1.3. *(Ohshika [33]) Let M be a compact irreducible 3-manifold and let $\{\rho_i : \pi_1(M) \to Isom_+(\mathbf{H}^3)\}$ be a sequence of quasi-*

conformally conjugate, convex cocompact representations of $\pi_1(M)$ *converging algebraically to a purely hyperbolic representation* $\rho_\infty : \pi_1(M) \to Isom_+(\mathbf{H}^3)$.

1. *If* $\rho_\infty(\pi_1(M))$ *has nontrivial domain of discontinuity, then* ρ_∞ *is topologically tame.*
2. *The limit set of* $\rho_\infty(\pi_1(M))$ *either has measure zero or is the entire 2-sphere.*

Motivated by Ohshika's work and the ideas of Paulin, we use results of Anderson-Canary [2] to generalize Ohshika's Corollary 1.3. Recall that a compression body is a compact 3-manifold M with a boundary component T such that the inclusion map $\phi : T \to M$ induces a surjection $\phi_* : \pi_1(T) \to \pi_1(M)$ (see Bonahon [7] or McCullough-Miller [30].) If M is a compression body, then $\pi_1(M)$ is the free product of surface groups and free groups (again see [7] or [30].)

Corollary A. *Let M be a compact irreducible 3-manifold, and let* $\{\rho_i : \pi_1(M) \to Isom_+(\mathbf{H}^3)\}$ *be a sequence of purely hyperbolic, topologically tame discrete faithful representations of* $\pi_1(M)$ *converging algebraically to the purely hyperbolic representation* $\rho_\infty : \pi_1(M) \to Isom_+(\mathbf{H}^3)$.

1. *If* $\rho_\infty(\pi_1(M))$ *has nontrivial domain of discontinuity, then* ρ_∞ *is the strong limit of* $\{\rho_i\}$ *and* $\rho_\infty(\pi_1(M))$ *is topologically tame.*
2. *If M is not homotopy equivalent to a compression body, then* ρ_∞ *is the strong limit of* $\{\rho_i\}$, *and* $\rho_\infty(\pi_1(M))$ *is topologically tame.*
3. *The limit set of* $\rho_\infty(\pi_1(M))$ *either has measure zero or is the entire 2-sphere.*

One nearly immediate consequence of part 2 of corollary A is the following:

Corollary B. *Let Γ be a torsion-free convex cocompact Kleinian group and let $QC(\Gamma)$ denote its quasiconformal deformation space. If $\mathbf{H}^3/\Gamma$ is not homotopy equivalent to a compression body, then there is a dense G_δ in $\partial QC(\Gamma)$ consisting entirely of topologically tame, purely hyperbolic, geometrically infinite hyperbolic 3-manifolds.*

A more concise way of stating our main theorem is simply to say that the set of topologically tame purely hyperbolic representations is a closed subset of all purely hyperbolic representations in the strong topology. This is stated formally in Corollary C, in section 9.

A further consequence, given in Corollary D, is that the bottom λ_0

of the spectrum of the Laplacian is a continuous function on the set of topologically tame purely hyperbolic representations, in the algebraic topology. We remark (see section 9) that it is known that λ_0 is not a continuous function in the algebraic topology on the whole representation space. It is conjectured that λ_0 is continuous with respect to the strong topology.

Outline of the argument

Geometric convergence of $\{N_i\}$ to N_∞ implies (see section 3) that there are bilipschitz diffeomorphisms f_i that take large portions V_i of N_i to subsets of N_∞. Moreover, the sequence $\{f_i(V_i)\}$ is an exhaustion of N by compact submanifolds.

If E_i are ends of N_i, then each E_i has a neighborhood U_i which is homeomorphic to a product $F \times \mathbf{R}_+$ where F is a closed surface, and we may ask why this product structure cannot simply be pushed via f_i into the limit manifold. The main trouble is that the product neighborhoods U_i may be disjoint from the regions V_i where f_i is defined. An additional topological difficulty is posed by the fact that it is not sufficient to show that a manifold may be exhausted by compact cores in order to prove that it is topologically tame. Both of these difficulties are illustrated by the example below.

The following construction of a non-tame 3-manifold, analogous to a construction of Whitehead, appears essentially in [41] (see also [36]). Let H denote a handlebody of genus 2, and let $g : H \to H$ be an embedding of H into its own interior which is a knotted homotopy equivalence. That is, $g_* : \pi_1(H) \to \pi_1(H)$ is the identity, ∂H and $g(\partial H)$ are both incompressible in $H - g(H)$, and $\pi_1(\partial H)$ and $\pi_1(g(\partial H))$ generate a proper subgroup of $\pi_1(H - g(H))$. (See Figure 1).

For $n \geq 0$ let H_n be a homeomorphic copy of H equipped with an identification $\psi_n : H \to H_n$, and let $g_n = \psi_{n+1} \circ g \circ \psi_n^{-1}$. The direct limit H_∞ of the sequence of inclusions $H_0 \xrightarrow{g_0} H_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{n-1}} H_n \xrightarrow{g_n} \cdots$ will then be a topologically non-tame 3-manifold with finitely generated fundamental group.

More explicitly, let B_n denote $H_n - int(g_{n-1}(H_{n-1}))$ for $n > 0$, and let $B_0 = H_0$. Defining $\partial_0 B_n = g_{n-1}(\partial H_{n-1})$ for $n > 0$ and $\partial_1 B_n = \partial H_n$ for $n \geq 0$, we see that the map $g'_n = g_n|_{\partial_1 B_n}$ identifies $\partial_1 B_n$ to $\partial_0 B_{n+1}$

for each $n \geq 0$. The union $H'_k = \bigcup_{n=0}^{k} B_n$, with these identifications, is homeomorphic to H_k, and the infinite union $\bigcup_{n=0}^{\infty} B_n$ is H_∞.

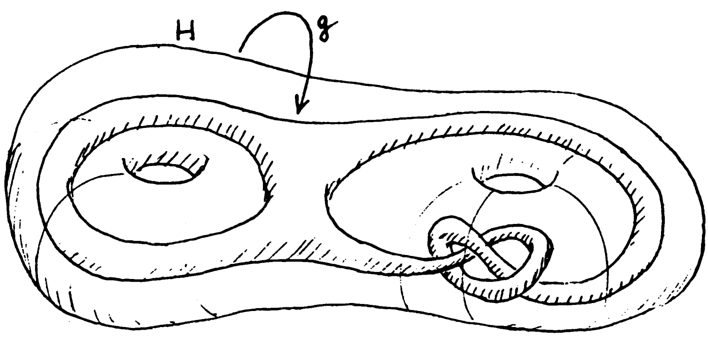

FIGURE 1

Note that the inclusion of H_0 into H_∞ is a homotopy equivalence, since the inclusion of H_0 into every H'_k is a homotopy equivalence. Thus $\pi_1(H_\infty)$ is the free group on two generators, and H_0 (in fact every H'_k) is a compact core. However, note also that the knotting of the compact core H_0 in H'_k gets progressively more complicated as $k \to \infty$.

In fact it is easy to see that $H_\infty - H_0$ has infinitely generated fundamental group, and it follows that H_∞ must not be topologically tame.

We can metrize these spaces so that H_∞ is a "geometric limit" of the manifolds H'_k. Let σ be any complete Riemannian metric on H_∞. Let σ_k be a complete metric on the interior of H'_k which is equal to σ on H'_{k-1}. Then, for each k, H'_{k-1} is a region in (H'_k, σ_k) which is isometric to the corresponding region in (H_∞, σ), and as $k \to \infty$ these regions contain arbitrarily large neighborhoods of a pre-chosen basepoint in H_0. (Compare Lemma 3.1.)

The example suggests that, in the hyperbolic context, we need a way to "pull down" the product structure neighborhoods of the ends of N_i so that they meet uniformly bounded neighborhoods of the basepoints. We use a tool developed by Bonahon [9], known as simplicial hyperbolic surfaces, which have two important features. They are determined by triangulations, on which we can perform elementary moves, and they have intrinsic metrics of negative curvature, which allow us to bound their diameter.

In each end of N_i there exists a simplicial hyperbolic surface, homotopic to a level surface in a product neighborhood of the end. We show how to homotope this surface to one that lies uniformly near the basepoint. For example, if the surface is compressible, we perform a sequence of elementary moves on the triangulation which terminate in a triangulation with a compressible curve. The resulting homotopy of simplicial hyperbolic surfaces eventually reaches a bounded neighborhood of the basepoint. We then use results of Freedman, Hass and Scott to obtain nearby embedded surfaces which are also homotopic to level surfaces. Classical results about 3-manifolds imply that the embedded surfaces bound product neighborhoods of the ends of N_i.

We can moreover obtain (after restricting to a suitable subsequence, reindexed as N_n) large product regions X_n in N_n, such that one boundary component $\partial_0 X_n$ lies uniformly near the basepoint and the other, $\partial_1 X_n$, is increasingly distant (but still in the domain of the bilipschitz map f_n). These regions give rise to regions $Y_n = f_n(X_n)$ in N_∞, whose geometry is similarly controlled. A topological argument then shows that the Y_n "interlock": $\partial_1 Y_n$ is isotopic in Y_{n+1} to a level surface. We conclude that $\bigcup Y_n$ contains a product neighborhood for an end of N_∞.

Remarks. 1. Many of these considerations were unnecessary in Thurston's proof of tameness for limits of hyperbolic 3-manifolds whose compact cores have incompressible boundary. In this setting, there is no possibility of knotting of the compact core. In particular, it suffices to obtain a sequence of surfaces homotopic to the boundary of the compact core, exiting every compact set. For example, if the approximating manifolds N_i are geometrically finite, let λ be a limit of the pleating loci of the boundaries of the convex cores of N_i corresponding to a given end E of N_∞. If E is geometrically infinite then, using a sequence γ_n of simple closed curves coverging to λ, Thurston constructs from their geodesic representatives in N_∞ a sequence of pleated surfaces leaving every compact set in N_∞. (In general when the approximating ends are geometrically infinite, λ can be taken to be the limit of the sequence of ending laminations.)

2. Extending our main theorem to the case where parabolics occur seems to be more difficult than one might expect. The main problem is the possibility of "new" parabolics: elements of $\pi_1(M)$ which are taken to parabolics by ρ_∞ but not by ρ_i. In this case the uniform diameter bounds on simplicial hyperbolic surfaces provided by Corollary 5.3 are not valid. Attempts to circumvent this encounter surprising topological difficulties. For example, it may not be true that a curve in a cusp is

homotopic, in the complement of the compact core, into the boundary of the compact core.

We conclude the introduction with a brief section-by-section summary. Section 2 contains some background topological results which we believe are well-known. In particular we describe the compact core of a 3-manifold, and discuss the results of Freedman-Hass-Scott.

In section 3 we discuss geometric limits, and show that, if C_∞ is a compact core of N_∞, then the preimages $C_i = f_i^{-1}(C_\infty)$ are compact cores for N_i, for large enough i (where f_i are the bilipschitz maps discussed above). It is important for topological reasons that the homotopies described above be performed in the complements of these compact cores.

In section 4 we introduce simplicial hyperbolic surfaces, and elementary moves on their triangulations. The goal of the section is Proposition 4.5, which states that any given end-homotopic simplicial hyperbolic surface can be deformed through a continuous family of such surfaces to one that intersects a uniformly bounded neighborhood of C_i.

In section 5 we prove a bounded diameter lemma that implies in particular that the final surface produced by Proposition 4.5 lies entirely in a bounded neighborhood of C_i. This lemma will be also used later, to control the diameter of surfaces further out in the ends.

Section 6 is quite technical. Here we pay the price for the ease of using simplicial hyperbolic surfaces instead of Thurston's related notion of pleated surfaces. The surfaces which naturally arise in a geometrically finite manifold as the boundaries of the convex core are pleated surfaces, so in this section we show how to approximate these by simplicial hyperbolic surfaces. We write the proof in somewhat greater generality than we require, in the hope that it may be of use elsewhere.

In section 7 we briefly discuss geometrically finite and infinite ends, and apply the results of section 6 to obtain simplicial hyperbolic surfaces in neighborhoods of the ends of the approximating manifolds.

Section 8 contains the proof of the main theorem. The proofs of Corollaries A through D are found in section 9.

2. Topological preliminaries

In this section we collect together some of the topological background which will be used in the proof. We imagine that all the results in this section are well-known.

Let us briefly recall some of the language of 3-manifolds. A 3-manifold M is said to be *irreducible* if every embedded 2-sphere in M bounds a region homeomorphic to a ball. Let $f : S \to M$ be a map of a closed orientable surface (of positive genus) into an orientable 3-manifold M. f is said to be *π_1-injective* if $f_* : \pi_1(S) \to \pi_1(M)$ is injective. If f is a π_1-injective embedding, it is said to be *incompressible.* We will abuse notation by referring to any map $f : S \to M$ such that f_* is not injective as *compressible.*

The *ends* of a topological space X can be defined in terms of their *neighborhoods* (see [9]). That is, an end of X is a maximal family $\{U_i\}_{i\in I}$ of open sets, called its neighborhoods, with the following properties: Each U_i has compact boundary but non-compact closure, and for each i, j there exists k so that $U_k \subset U_i \cap U_j$. An end of the interior of a compact 3-manifold is just the family of neighborhoods of a boundary component.

Peter Scott [35] proved that every irreducible 3-manifold N with finitely generated fundamental group contains a compact submanifold C, called the *compact core*, such that the inclusion map of C into N is a homotopy equivalence. (Notice that this implies that C itself is irreducible.) Moreover, if N is an open 3-manifold, each component of $N - C$ is a neighborhood of exactly one end of N (see Proposition 1.3 in [9]). Let N be a topologically tame 3-manifold and C a compact core for N. We will say that a continuous map $f : S \to N$ is *end-homotopic* if $f(S)$ is contained within a component of $N - C$ whose associated end has a neighborhood U homeomorphic to $S \times (0, \infty)$ and $f(S)$ is homotopic (within $N - C$) to an embedding with image $S \times \{1\}$. (We note that the existence of a fixed choice of compact core will be implicit in our use of the term end-homotopic.)

We will make use on several occasions of the following fundamental result of Waldhausen.

Proposition 2.1. *(Corollary 5.5 in [42]) Let M be an irreducible 3-manifold and let $f : S \to M$ and $g : S \to M$ be two disjoint, homotopic, incompressible surfaces. Then there is a region bounded by $f(S)$ and*

$g(S)$ which is homeomorphic to $S \times I$.

Waldhausen's result will first be used in our proof that any end-homotopic surface is incompressible in $N - C$ and homeomorphic to the appropriate boundary component of C.

Proposition 2.2. *Let N be a topologically tame, orientable, irreducible 3-manifold with compact core C. Let $g : S \to N$ be an end-homotopic map. Let V be the component of $N - C$ which contains $g(S)$ and let F be the boundary of V. Then:*

1. S is homeomorphic to F, and

2. g is π_1-injective as a map from S to $N - C$.

Proof of 2.2. Since N is topologically tame, we may identify it with the interior of a compact 3-manifold M. Let $\widehat{V}$ be the component of $M - C$ containing V. Notice that S is homeomorphic to the boundary component S' of $\widehat{V}$ which lies within ∂M, and that $g(S)$ is homotopic within $M - C$ to S'.

The proof of both parts of our proposition make use of the following lemma:

Lemma 2.3. *Let Σ be a boundary component of a compact irreducible 3-manifold M. If Σ is homotopic into a compact submanifold K of $int(M)$, then Σ is incompressible in $M - K$.*

Proof of 2.3. Suppose that Σ is compressible in $M - K$. Then Dehn's Lemma (see [20], for example) would imply that there exist a simple, homotopically non-trivial curve β on Σ and a properly embedded disk D bounding β which lies entirely in $M - K$. The Seifert-van Kampen theorem then guarantees, if M_1 is the component of $M - D$ containing K, that $\pi_1(M)$ can be written as a non-trivial free product $G_1 * G_2$ where $G_1 = \pi_1(M_1)$, and the inclusion $(i_\Sigma)_*(\pi_1(\Sigma))$ is not conjugate to a subgroup of G_1 or G_2. This would contradict the fact that Σ is homotopic into K. q.e.d.

Returning to the proof of 2.2, Theorem 2 of McCullough-Miller-Swarup [31] asserts that if C_1 and C_2 are any two compact cores of an irreducible, orientable 3-manifold M, then there exists a homeomorphism $h : C_1 \to C_2$ such that $h_* = (i_2)_*^{-1} \circ (i_1)_*$, where $i_j : C_j \to M$ is the inclusion map. Notice that M is certainly a compact core for itself. Therefore, there exists a homeomorphism $h : M \to C$ which is homotopic (in M) to the identity map. Thus, there exists a component F' of the boundary of C such that $h(S') = F'$. If $F' = F$, then statement

1 holds.

Suppose that $F' \neq F$. Theorem 1.1 of McCullough-Miller [30] (see also [9]) asserts that there exists a (possibly disconnected) incompressible surface T separating F from F'. Therefore, T is an incompressible surface in M separating F' from S'. Let M_1 be the component of $M - T$ which contains S'. Lemma 2.3 guarantees that S' is incompressible in M_1 since it is homotopic to a surface disjoint from M_1. Since T is incompressible, $\pi_1(M_1)$ injects in $\pi_1(M)$. Therefore, S' is also incompressible in M. Hence, F' is incompressible in M as well. Proposition 2.1 guarantees that the region C' between S' and F' is homeomorphic to $S' \times I$.

Now since F' is incompressible, $\pi_1(C')$ injects into $\pi_1(M)$, and since C' contains a compact core C for M, $\pi_1(C')$ surjects onto $\pi_1(M)$. Therefore, C' is a compact core for M. Since C is homeomorphic to C' (by the above result of McCullough-Miller-Swarup), every boundary component of C is homeomorphic to S'. In particular, F is homeomorphic to S'. This completes the proof of part 1.

We may also apply Lemma 2.3 to prove part 2. Since C is a compact core for M, S' is homotopic into C. Lemma 2.3 then implies that S' is incompressible in $M - C$. Recalling that $g(S)$ is homotopic to S' (within $M - C$), we have completed the proof of part 2. q.e.d.

We also make use of the following fact, due to E.H. Brown [11].

Theorem 2.4. *(Theorem 7.2 in [11]) Let S be a closed surface and let $f : S \to S \times [0,1]$ be an embedding such that $f(S)$ separates the boundary components of $S \times [0,1]$. Then $f(S)$ is isotopic to $S \times \{\frac{1}{2}\}$.*

Often the end-homotopic surfaces produced by our techniques will fail to be embedded. We now observe that near any end-homotopic surface there lies an end-homotopic embedded surface. Bonahon (Lemma 1.22 in [8]) was the first to observe that the following type of result was implicit in the work of Freedman-Hass-Scott [18]:

Theorem 2.5. *(Freedman-Hass-Scott [18]) Let $g : S \to M$ be a π_1-injective map of a closed (orientable) surface S into a compact, orientable, irreducible 3-manifold M such that g is homotopic to an embedding. If W is any open neighborhood of $g(S)$, then g is homotopic to an embedding $g' : S \to M$ with image in W.*

We can apply Theorem 2.5 to the situation where $g : S \to N$ is an end-homotopic surface.

Proposition 2.6. *Let N be a topologically tame hyperbolic 3-manifold* *and let C be a compact core for N. If $g : S \to N$ is an end-homotopic map, then g is homotopic, within $N - C$, to an embedding $g' : S \to N$, such that $g'(S) \subset \mathcal{N}(1, g(S))$, where $\mathcal{N}(1, g(S))$ denotes a neighborhood of $g(S)$ of radius 1.*

Proof of 2.6. Since N is topologically tame, we may identify it with the interior of a compact, orientable, irreducible 3-manifold M. Let V be the closure of the component of $M - C$ containing $g(S)$. Then V is a compact 3-manifold and, by Proposition 2.2, $g : S \to V$ is an incompressible surface in V, which is homotopic to a component of ∂V. To complete the proof, we apply Theorem 2.5 to the open neighborhood $W = \mathcal{N}(1, g(S)) \cap V$ of $g(S)$ in V.

Remark. As a statement of Theorem 2.5 is not contained in the paper [18], we will sketch the proof. The main theorem of [18] guarantees that, under our hypotheses, if M is given a metric with convex boundary, then there exists a minimal area surface homotopic to g, which is either embedded or double covers the base surface of a twisted I-bundle X embedded in M such that $g(S)$ is homotopic to the boundary of X. We therefore choose a metric on M which is "very large" in the complement of W. Properly done, this guarantees that the minimal area surface homotopic to g must lie in W. We note that if the minimal area surface double covers the base surface of a twisted I-bundle in M, then there exists an embedded surface homotopic to g in an arbitrarily small neighborhood of the base, so this poses no additional difficulties.

It is technically simpler to make this proof precise using the parallel theory of PL-minimal surfaces developed by Jaco and Rubinstein [23]. Jaco and Rubinstein showed that all the results of [18] held for PL-minimal surfaces. Let W' be an open neighborhood of $g(S)$ such that the closure of W' is contained in W. One may assume that M has been triangulated so that the closure of W' is a subcomplex K of M. In order to insure that the PL-minimal surface homotopic to g lies entirely in W it suffices to subdivide the simplices of $M - K$ a large number of times.

3. Geometric convergence and a uniform choice of compact core

In this section we recall the basic definitions and characterizations of geometric convergence. We will apply these, in Proposition 3.3, to obtain a "uniform" choice of compact cores for the approximating manifolds in a strongly convergent sequence of hyperbolic 3-manifolds. We will then prove Lemma 3.4, which gives a uniform bound on the diameter of a compact subset in an approximating manifold, in terms of the distance of its boundary from the basepoint.

A sequence of closed subsets X_i of a locally compact metric space $\mathcal{X}$ *converges geometrically* to a subset X_∞ if every point in X_∞ is a limit of a convergent sequence $\{x_i \in X_i\}$, and every accumulation point of any sequence $\{x_i \in X_i\}$ is in X_∞. We recall (see e.g. Proposition 3.1.2 in [15]) that the set of closed subsets of $\mathcal{X}$ is compact in the topology of geometric convergence.

As a special case we obtain the *geometric topology* on the set of Kleinian groups, viewed as closed subsets of $Isom(\mathbf{H}^3)$. The following lemma indicates the geometric significance, on the level of the quotient manifolds, of the geometric convergence of a sequence of Kleinian groups. (For a proof, see Theorem 3.2.9 of [15], and Theorem E.1.13 and Remark E.1.19 of [3].)

Lemma 3.1. *A sequence of torsion-free Kleinian groups $\{\Gamma_i\}$ converges geometrically to a torsion-free Kleinian group Γ_∞ if and only if there exists a sequence $\{(R_i, K_i)\}$ and a sequence of maps $\widetilde{f}_i : B_{R_i}(0) \to \mathbf{H}^3$, where $B_R(0)$ is a ball of radius R centered on the origin in $\mathbf{H}^3$, such that the following hold:*

1. *$R_i \to \infty$ and $K_i \to 1$.*
2. *The map $\widetilde{f}_i$ is a K_i-bilipschitz diffeomorphism onto its image, $\widetilde{f}_i(0) = 0$, and for any compact set A, $\widetilde{f}_i|_A$ converges to the identity.*
3. *If $N_i = \mathbf{H}^3/\Gamma_i$ and $N_\infty = \mathbf{H}^3/\Gamma_\infty$, then $\widetilde{f}_i$ descends to a map $f_i : V_i \to N_\infty$, where $V_i = B_{R_i}(0)/\Gamma_i$ is a submanifold of N_i. Moreover f_i is also a K_i-bilipschitz diffeomorphism onto its image.*

For the remainder of the paper we shall fix a compact irreducible 3-manifold M, and let $\{\rho_i : \pi_1(M) \to Isom_+(\mathbf{H}^3)\}$ be a sequence of topologically tame, purely hyperbolic discrete faithful representations converging strongly to a purely hyperbolic representation $\rho_\infty : \pi_1(M) \to Isom_+(\mathbf{H}^3)$. We will maintain the following notation: $\Gamma_i = \rho_i(\pi_1(M))$, $\Gamma_\infty = \rho_\infty(\pi_1(M))$, $N_i = \mathbf{H}^3/\Gamma_i$, and $N_\infty = \mathbf{H}^3/\Gamma_\infty$. Let f_i, V_i, K_i, and R_i be as given in Lemma 3.1. Let $b_i \in N_i$ be the images of the origin $0 \in \mathbf{H}^3$, for $i = 1, \ldots, \infty$. Note that $f_i(b_i) = b_\infty$.

Throughout the paper, we shall denote by $\mathcal{N}(r, X)$ the open neighborhood of radius r of X, where X is either a subset or single point in a metric space.

In particular, the image $f_i(V_i)$ contains $\mathcal{N}(R_i/K_i, b_\infty)$ in N_∞, by virtue of the following simple property of bilipschitz diffeomorphisms, which we state without proof. We will be using this fact throughout the paper.

Lemma 3.2. *Let M and M' be two Riemannian manifolds of the same dimension. Suppose that $U \subset M$ and $f : U \to M'$ is a K-bilipschitz diffeomorphism (onto its image). If $R > 0$, $x \in M$ and $\mathcal{N}(R, x) \subset U$, then $\mathcal{N}(R/K, f(x)) \subset f(U)$.*

We are now prepared to produce the desired sequence $\{C_i\}$ of compact cores of the N_i.

Proposition 3.3. *If C_∞ is a compact core for N_∞, then there exists I such that if $i \geq I$, then $C_\infty \subset f_i(V_i)$ and $C_i = f_i^{-1}(C_\infty)$ is a compact core for N_i.*

Proof of 3.3. Note that we may assume that the basepoint b_∞ lies in C_∞. We shall make the natural identifications $\Gamma_i = \pi_1(N_i)$ for $i = 1, \ldots, \infty$, and also $\pi_1(C_\infty) = \pi_1(N_\infty)$, where the use of basepoints $b_i \in N_i$ is implicit.

Now if we choose I so that for $i \geq I$ we have $K_i < 2$ and $R_i > 2\,\mathrm{diam}(C_\infty)$, it is clear that $C_\infty \subset f_i(V_i)$, and we may set $C_i = f_i^{-1}(C_\infty)$.

Fix an element $g \in \pi_1(M)$ and let $\gamma_\infty = \rho_\infty(g)$. Represent γ_∞ by a loop α_∞ in C_∞ based at b_∞, and let $\alpha_i = f_i^{-1}(\alpha_\infty)$. Let $\gamma_i \in \Gamma_i$ be the element represented by α_i, and let $\widetilde{\alpha}_i$ be the lift of α_i joining 0 to $\gamma_i(0)$. For sufficiently large i, $\widetilde{\alpha}_i$ is contained in $B_{R_i}(0)$, and then $\widetilde{f}_i(\widetilde{\alpha}_i) = \widetilde{\alpha}_\infty$. Since $\{\widetilde{f}_i\}$ converges to the identity on compact sets, we conclude that $\{\gamma_i(0)\}$ converges to $\gamma_\infty(0)$.

On the other hand, since ρ_i converges algebraically to ρ_∞, we may apply the Margulis lemma to conclude that there is a neighborhood U of $\gamma_\infty(0)$ such that, for any i, there is at most one element β_i in Γ_i such that $\beta_i(0) \in U$. (See e.g. Lemma 3.7 in Jørgensen-Marden [26].) Since $\rho_i(g)(0)$ converges to $\rho_\infty(g)(0)$, it follows that $\gamma_i = \rho_i(g)$ for large enough i.

Let $\phi_i = f_i^{-1}|_{C_\infty} : C_\infty \to N_i$, and let $(\phi_i)_* : \pi_1(C_\infty) \to \pi_1(N_i)$ be the induced map on fundamental groups. We have shown that, for each $\gamma_\infty \in \Gamma_\infty$, $(\phi_i)_*(\gamma_\infty) = \rho_i \circ \rho_\infty^{-1}(\gamma_\infty)$ for large enough i. Since $\pi_1(C_\infty)$ is finitely generated, this implies that, for large enough i, $(\phi_i)_* = \rho_i \circ \rho_\infty^{-1}$, which is an isomorphism.

Now let $(\phi_i)'_*$ denote the map induced by ϕ_i from $\pi_1(C_\infty)$ to $\pi_1(C_i)$, and note that this is an isomorphism. Let $j_i : C_i \to N_i$ be the inclusion map. Then $(\phi_i)_* = (j_i)_* \circ (\phi_i)'_*$, and it follows that $(j_i)_*$ is an isomorphism, so that C_i is a compact core of N_i. q.e.d.

We can thus assume, by truncating the sequence if necessary, that for all i, $C_i = f_i^{-1}(C_\infty)$ exists and is a compact core for N_i. We may also assume, again by truncating if necessary, that $K_i \leq 2$ for all i. Both of these assumptions are made only to simplify later choices of constants.

The existence of a strong limit allows us to obtain various uniform estimates on our sequence of manifolds. The following lemma describes a uniform sense in which the diameter of a compact region is bounded in terms of the location of its boundary. This will be applied in the proof of the main theorem to bound the diameter of product regions in N_i, enabling us to conclude that they are contained in the regions V_i and hence map to product regions in N_∞.

Lemma 3.4. *For each $R > 0$ there exist $L(R)$ and $n_0(R)$ such that, if $i > n_0$ and $A \subset N_i$ is a compact subset for which $\partial A \subset \mathcal{N}(R, b_i)$, then $A \subset \mathcal{N}(L, b_i)$.*

Proof of 3.4. Without loss of generality, we may assume that R is sufficiently large that $C_i \subset \mathcal{N}(R, b_i)$ for all i, and that $C_\infty \subset \mathcal{N}(R/2, b_\infty)$. Furthermore we may suppose that R is simultaneously a regular value for the functions $d(b_i, \cdot)$ for all i; thus $\partial\mathcal{N}(R, b_i)$ is always a smooth surface.

Let $L_\infty = L_\infty(R)$ denote the least number such that every compact component of $N_\infty - \mathcal{N}(R, b_\infty)$ is contained in $\mathcal{N}(L_\infty(R), b_\infty)$ (such a number exists because there are only finitely many such components).

Let n_0 be sufficiently large that, for all $i > n_0$, $\mathcal{N}(2L_\infty(2R), b_i)$ is contained in the region V_i where the bilipschitz map f_i is defined.

We shall need the following observation: a component U of $N - C$, where C is a compact core of a 3-manifold N, contains no closed surface F which is non-separating in N. If there were such a surface, there would be a curve γ intersecting it exactly once, which can therefore not be deformed off of F. This contradicts the assumption that C is a compact core.

Let X be a component of $N_i - \mathcal{N}(R, b_i)$ for $i > n_0$. We claim that ∂X is connected. If it were not, none of the components would separate the manifold since $\mathcal{N}(R, b_i)$ is connected. However, by our initial assumption on R, X must be contained in a component U of $N_i - C_i$, which has no non-separating closed surfaces.

Similarly, let $Q_i = f_i(\mathcal{N}(R, b_i))$ and let Y be a component of $N_\infty - Q_i$. Since (by Lemma 3.2) $C_\infty \subset \mathcal{N}(R/2, b_\infty) \subset Q_i$, the same argument shows that ∂Y is also connected. Thus, $\partial Y = f_i(\partial X)$ for some component X of $N_i - \mathcal{N}(R, b_i)$. We may number these components $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ so that $\partial Y_j = f_i(\partial X_j)$.

Let Y_j be a compact component of $N_\infty - Q_i$. Since f_i is 2-bilipschitz, $\partial Y_j \subset \mathcal{N}(2R, b_\infty)$ which implies that $Y_j \subset \mathcal{N}(L_\infty(2R), b_\infty)$. By Lemma 3.2, this neighborhood is contained in the image of f_i, so that $f_i^{-1}(Y_j)$ must be an entire component of $N_i - \mathcal{N}(R, b_i)$, namely X_j. In particular X_j is compact as well, and contained in $\mathcal{N}(2L_\infty(2R), b_i)$.

Each non-compact Y_j contains a neighborhood of an end of N_∞, and each end of N_∞ has a neighborhood contained in some non-compact Y_j. Since C_∞ separates the ends, and lies in Q_i, each Y_j can be a neighborhood of at most one end. It follows that the non-compact components among the Y_j are in one-to-one correspondence with the ends of N_∞. The same is true, by the same argument, for the X_i and the ends of N_i. Thus the number of non-compact components is equal in both sets. It follows that X_j is compact if and only if Y_j is compact. Hence we have proved that all compact X_j are contained in $\mathcal{N}(L, b_i)$ where $L = 2L_\infty(2R)$.

Now if A is any compact subset of N_i for $i > n_0$ such that $\partial A \subset \mathcal{N}(R, b_i)$ then, for each X_j, $\partial(A \cap X_j) \subset \partial X_j$, so that either $int(X_j) \subset A$, or $int(X_j) \cap A = \emptyset$. In the first case X_j must be compact, so we may conclude that $A \subset \mathcal{N}(L, b_i)$.

4. Simplicial hyperbolic surfaces

We first recall a generalized definition of a triangulation of a surface (following Harer [22] and Hatcher [21]). Let F be a closed surface and let $\mathcal{V}$ denote a finite collection of points in F. (We will often restrict to the case where $\mathcal{V}$ is a single point.) A *curve system* $\{\alpha_1, \dots, \alpha_m\}$ is a collection of arcs with disjoint interiors and endpoints in $\mathcal{V}$, no two of which are ambient isotopic (rel $\mathcal{V}$), and none of which is homotopic to a point (rel $\mathcal{V}$). A *triangulation* $\mathcal{T}$ of $(F, \mathcal{V})$ is simply a maximal curve system for $(F, \mathcal{V})$. We will say that two triangulations are equivalent if they are ambiently isotopic (rel $\mathcal{V}$). Note that the faces of $\mathcal{T}$ are triangles, possibly with some of their vertices or edges identified.

A continuous map $f : S \to N$ from a closed surface S into a hyperbolic 3-manifold N is said to be a *simplicial pre-hyperbolic surface* if there exists a triangulation $\mathcal{T}$ of S such that the image of each face of $\mathcal{T}$ is an immersed, totally geodesic, non-degenerate triangle. The map f induces a piecewise Riemannian metric on S, and f is said to be a *simplicial hyperbolic surface* if the angle about each vertex of $\mathcal{T}$ is at least 2π.

For simplicity, we will often work with what we call *useful* simplicial hyperbolic surfaces. A simplicial hyperbolic surface $p : S \to N$ (with associated triangulation $\mathcal{T}$) is *useful* if $\mathcal{T}$ has only one vertex v and if one of the edges e (called the *distinguished* edge) of $\mathcal{T}$ is mapped to a closed geodesic.

Let $f : S \to N$ be a continuous map. We will say that a homotopically non-trivial curve γ in S is *compressible* if $f(\gamma)$ is homotopically trivial in N. We will also refer to its image in $f(S)$ as compressible.

Lemma 4.1. *If $f : S \to N_i$ is an end-homotopic simplicial hyperbolic surface such that $f(S) \subset N_i - \mathcal{N}(1, C_i)$, then every compressible curve on $f(S)$ has length at least 1.*

Proof of 4.1. Recall first, from Proposition 2.2, that any end-homotopic map $f : S \to N_i$ is incompressible as a map into $N_i - C_i$. Let γ be a compressible curve on S. By subdividing γ into its intersections with the faces of the triangulation and straightening, we may replace it by a homotopic curve γ' such that $f(\gamma')$ is a polygonal curve with length at most that of $f(\gamma)$. Since γ' is compressible it bounds an immersed disk D which we may assume is triangulated by totally geodesic triangles whose vertices lie on $f(\gamma')$. D thus inherits a hyperbolic metric. Since $f(S)$ is incompressible in $N_i - C_i$, we see that D must intersect C_i in some point x. So D is a hyperbolic disk such that every point on the boundary has distance at least one from x. This implies that $f(\gamma') = \partial D$ has length at least $2\pi \sinh\, 1 > 2\pi > 1$.

q.e.d.

Let $\mathcal{T}$ be a triangulation of $(F, \mathcal{V})$ and consider a quadrilateral in $\mathcal{T}$ bounded by e_1, e_2, e_3, and e_4 and with diagonal e_5. One may obtain a new triangulation $\mathcal{T}'$ of F by replacing e_5 with the other diagonal of the quadrilateral, which one may denote e_6. $\mathcal{T}'$ is said to be obtained from $\mathcal{T}$ by performing an *elementary move.* Harer [22] (see also Hatcher [21]) proved that any two triangulations of $(F, \mathcal{V})$ are related by a finite sequence of such elementary moves.

The following sequence of lemmas, from [14], allows us to construct continuous families of simplicial hyperbolic surfaces. Lemmas 4.2 and 4.3 allow us to construct continuous families of simplicial hyperbolic surfaces joining any two useful simplicial hyperbolic surfaces whose associated triangulations differ by an elementary move. Lemma 4.4 assures us that if the associated triangulation of a useful simplicial hyperbolic surface can be changed to one which contains a "compressible edge", then one can construct a continuous family of simplicial hyperbolic surfaces in which the length of that edge converges to 0.

Lemma 4.2. *Let $h : S \to N$ be a useful simplicial hyperbolic surface with associated triangulation $\mathcal{T}$, vertex v and distinguished edge $\hat{e}$. Let $\bar{e}$ be another edge of $\mathcal{T}$ such that $h(\bar{e} \cup v)$ has a closed geodesic representative $h(\bar{e} \cup v)^*$. Then there exists a continuous family $J : S \times [0,1] \to N$ of simplicial hyperbolic surfaces joining h to a useful simplicial hyperbolic surface $\bar{h}$ with associated triangulation $\mathcal{T}$ and distinguished edge $\bar{e}$.*

Lemma 4.3. *Let $h : S \to N$ be a useful simplicial hyperbolic surface with associated triangulation $\mathcal{T}$, vertex v and distinguished edge $\hat{e}$. Let e_1, e_2, e_3 and e_4 bound a quadrilateral in $\mathcal{T}$ with diagonal $e_5 \neq \hat{e}$. Let e_6 be the other diagonal of this quadrilateral and $\mathcal{T}'$ the triangulation obtained by replacing e_5 with e_6. Assume that $h(e_6 \cup v)$ is homotopically non-trivial. Then we may construct a continuous family of simplicial hyperbolic surfaces joining h to a useful simplicial hyperbolic surface h' with associated triangulation $\mathcal{T}'$ and distinguished edge $\hat{e}$.*

Lemma 4.4. *Let $h : S \to N$ be a useful hyperbolic surface with associated triangulation $\mathcal{T}$, vertex v and distinguished edge $\hat{e}$. Let e_1, e_2, e_3 and e_4 form a quadrilateral with diagonal $e_5 \neq \hat{e}$ such that the other diagonal e_6 determines a compressible curve $e_6 \cup v$. Then there exists a continuous family $H : S \times [0,1) \to N$ of simplicial hyperbolic surfaces such that $H(\cdot, 0) = h$, and the length of $H(e_6 \times \{t\})$ converges* *monotonically to 0 as* $t \to 1$.

Given a useful end-homotopic simplicial hyperbolic surface $g : S \to N_i$ far away from C_i, we may use these lemmas to construct a continuous family of simplicial hyperbolic surfaces which eventually come near to C_i.

Proposition 4.5. *Let* $h : S \to N_i$ *be a compressible useful end-homotopic simplicial hyperbolic surface such that* $h(S) \subset N_i - \mathcal{N}(1, C_i)$. *There exists a continuous family* $H : S \times [0,1] \to N_i$, *such that* $h = H(\cdot, 0)$, $H(S \times \{t\}) \subset N_i - \mathcal{N}(1, C_i)$ *for all* $t \in [0,1]$, *and* $H(S \times \{1\}) \cap cl(\mathcal{N}(1, C_i)) \neq \emptyset$.

Proof of 4.5. Let v be the vertex of the triangulation $\mathcal{T}$ of S associated to h. By Dehn's lemma v is contained in a simple compressible curve on S, which we may extend to a triangulation $\hat{\mathcal{T}}$ of $(S, \{v\})$. Let $\mathcal{T} = \mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n = \hat{\mathcal{T}}$ be a sequence of triangulations such that $\mathcal{T}_{j+1}$ is obtained from $\mathcal{T}_j$ by an elementary move for $j < n$. Such a sequence is guaranteed to exist by Harer's theorem. Let $\mathcal{T}_j$ be the first triangulation in this sequence which has a compressible edge. Then one may use lemmas 4.2 and 4.3 to obtain a continuous family of simplicial hyperbolic surfaces $F : S \times [0,1] \to N_i$ such that $F(\cdot, 0) = h$ and $F(\cdot, 1)$ is a useful simplicial hyperbolic surface with associated triangulation $\mathcal{T}_{j-1}$. One may then use Lemma 4.4 to construct a continuous family of simplicial hyperbolic surfaces $F' : S \times [1, 2) \to N_i$ such that $F'(\cdot, 1) = F(\cdot, 1)$ and the length of the image of some compressible edge of $\mathcal{T}_j$ converges to 0 as t approaches to 2. Concatenate F and F' to form $G : S \times [0, 2) \to N_i$. Notice that Lemma 4.1 guarantees that there must exist some s such that $G(S \times \{s\}) \cap \mathcal{N}(1, C_i) \neq \emptyset$. Let

$$\delta = \inf\{s \in [0,2) | G(S \times \{s\}) \cap \mathcal{N}(1, C_i) \neq \emptyset\}.$$

Then we may define $H : S \times [0,1] \to N_i$, by $H(x, t) = G(x, t\delta)$.

5. A uniform bounded diameter lemma

We recall that the *injectivity radius* $inj_N(x)$ of a point $x \in N$ is defined to be half the length of the shortest homotopically non-trivial closed curve passing through x. We define

$$N_{thick(\epsilon)} = \{x \in N | inj_N(x) \geq \epsilon\}$$

and

$$N_{thin(\epsilon)} = \{x \in N | inj_N(x) \leq \epsilon\}.$$

This is known as the *thick-thin decomposition* of N. There exists a constant $\mathcal{M}$, called the Margulis constant (see chapter D in Benedetti-Petronio [3]) such that if $\epsilon < \mathcal{M}$ and N is a hyperbolic 3-manifold without cusps, then every component of $N_{thin(\epsilon)}$ is a tubular neighborhood of a closed geodesic. There is also a constant $s > 0$, depending only on ϵ, such that the distance between any two components of $N_{thin(\epsilon)}$ is at least s.

The following is a version of Bonahon's bounded diameter lemma (for a proof see [9] and [13].)

Proposition 5.1. *(Bounded Diameter Lemma) Given $\chi_0 > 0$ and $\epsilon > 0$, there exists $A(\chi_0, \epsilon)$ such that if $f : S \to N$ is a simplicial hyperbolic surface, $|\chi(S)| \leq \chi_0$, every compressible curve on $f(S)$ has length at least 1, and $x, y \in S$, then x and y may be joined by a path R such that $f(R) \cap N_{thick(\epsilon)}$ has length at most $A(\chi_0, \epsilon)$.*

This lemma is the key use of hyperbolicity of the simplicial hyperbolic surfaces. Hyperbolicity and the Gauss-Bonnet theorem give a bound on area of the surface, and this together with a lower bound on injectivity radius in the parts of the surface that map to $N_{thick(\epsilon)}$ implies the bound on diameter.

Since, for a fixed i, each component of $(N_i)_{thin(\epsilon)}$ is compact, and the separation between components is at least s, Proposition 5.1 implies a bound on the diameter of any simplicial hyperbolic surface without short compressible curves passing through a *fixed* point $x \in N_i$. By applying such reasoning to the convergent sequence $N_i \to N_\infty$ we can obtain a uniform bound:

Proposition 5.2. *(Uniform Bounded Diameter Lemma) Given $L > 0$ and $\chi_0 > 0$ there exists $D(L, \chi_0)$ such that if $g : S \to N_i$ is a simplicial hyperbolic surface, $g(S) \cap \mathcal{N}(L, b_i) \neq \emptyset$, $|\chi(S)| \leq \chi_0$ and every compressible curve on $g(S)$ has length at least 1, then $g(S)$ has diameter less than $D(L, \chi_0)$. Moreover, we may assume that $D(L, \chi_0)$ is a monotone increasing function of L (for all $\chi_0 > 0$.)*

Proof of 5.2. For $\epsilon < \mathcal{M}$ and for x, y in a hyperbolic manifold N define $d_{thick(\epsilon)}(x, y)$ to be the infimum over all paths γ from x to y of $\ell(\gamma \cap N_{thick(\epsilon)})$. Now since, for any $i = 1, \ldots, \infty$, each component of $(N_i)_{thin(\epsilon)}$ is compact and the components are uniformly separated, the function

$$R_i(K,\epsilon) = \sup\{d(b_i,x) : d_{thick(\epsilon)}(b_i,x) \le K\}$$

is finite for any i, K, ϵ. Fixing ϵ, we shall prove that these functions are eventually uniformly bounded in the sense that, for each K, there exists $I = I(K)$ such that if $i > I$, then $R_i(K,\epsilon/2) \le 2R_\infty(2K,\epsilon)$.

Given K and $R = R_\infty(2K,\epsilon)$, let I be such that for $i > I$ the region V_i contains $\mathcal{N}(2R+3\epsilon, b_i)$. Now if $R_i(K,\epsilon/2) > 2R$, then there would exist a path γ contained in $\mathcal{N}(2R+\epsilon, b_i)$ and connecting b_i to a point y such that $d(b_i,y) > 2R$ but $\ell(\gamma \cap (N_i)_{thick(\epsilon/2)}) \le K$.

However, it is easy to see that

$$f_i((N_i)_{thin(\epsilon/2)} \cap \mathcal{N}(2R+\epsilon)) \subset (N_\infty)_{thin(\epsilon)}.$$

In fact, for any $x \in (N_i)_{thin(\epsilon/2)} \cap \mathcal{N}(2R+\epsilon)$ there is a non-trivial loop β passing through x with length no more than ϵ, so that $\ell(f_i(\beta)) \le 2\epsilon$. If $f_i(\beta)$ were trivial it would span a disk which, by negative curvature, would have diameter at most ϵ and would thus (applying Lemma 3.2) be contained in $f_i(V_i)$, contradicting the non-triviality of β. It follows that $f_i(x) \in (N_\infty)_{thin(\epsilon)}$.

We conclude that

$$f_i(\gamma) \cap (N_\infty)_{thick(\epsilon)} \subset f_i(\gamma \cap (N_i)_{thick(\epsilon/2)}),$$

so that $d_{thick(\epsilon)}(b_\infty, f_i(y)) \le 2K$. But then $d(b_\infty, f_i(y)) \le R$ which contradicts our assumptions that $d(b_i,y) > 2R$ and that f_i is 2-bilipschitz.

The bound $R_i(K,\epsilon/2) \le 2R_\infty(2K,\epsilon)$ follows for $i > I(K)$ and, taking a maximum over the finitely many $i \le I(K)$, we obtain a uniform bound $R_i(K,\epsilon/2) \le R'(K)$ for all i. Now let $A_0 = A(\chi_0, \epsilon/2)$, and suppose that $g : S \to N_i$ is a simplicial hyperbolic surface such that $g(S) \cap \mathcal{N}(L, b_i) \ne \emptyset$, $|\chi(S)| \le \chi_0$ and every compressible curve on $g(S)$ has length at least 1. By Proposition 5.1 we have $d_{thick(\epsilon/2)}(b_i, y) \le L + A_0$ for any $y \in g(S)$, so applying the above argument we find that $g(S) \subset \mathcal{N}(R'(L+A_0), b_i)$, which gives the desired diameter bound, $D(L,\chi_0) = 2R'(L+A_0)$. Moreover the bound is clearly monotonic in L, by the construction. q. e.d.

Combining Proposition 5.2 with Proposition 2.2 and Lemma 4.1, we obtain:

Corollary 5.3. *Given $L > 0$, there exists $\mathcal{D}(L)$ such that if $g : S \to N_i$ is an end-homotopic simplicial hyperbolic surface, such that*

$g(S) \cap \mathcal{N}(L, b_i) \neq \emptyset$ and $g(S) \cap \mathcal{N}(1, C_i) = \emptyset$, then $g(S)$ has diameter less than $\mathcal{D}(L)$. Moreover, we may assume that $\mathcal{D}(L)$ is a monotone increasing function of L.

Proof of 5.3. We first notice that Proposition 2.2 assures us that S is homeomorphic to a boundary component of C_∞. Thus there is an upper bound χ_0 on $|\chi(S)|$ depending only on the topological type of M. Lemma 4.1 implies that, since $g(S) \cap \mathcal{N}(1, C_i) = \emptyset$, every compressible curve on $g(S)$ has length at least 1. Therefore, we may take $\mathcal{D}(L) = D(L, \chi_0)$ where D is the function obtained in Proposition 5.2.

6. Simplicial approximations to pleated surfaces

Although simplicial hyperbolic surfaces serve us for the most part as a simplified substitute for Thurston's pleated surfaces, there is one point in the next section where the use of pleated surfaces could not be avoided. To bridge the gap we provide the following technical lemmas.

A *pleated surface* in a hyperbolic 3-manifold N (see [15, 39]) is a map $g : F \to N$ which is a pathwise-isometry with respect to a hyperbolic metric σ on a surface F, totally geodesic on the complement of a σ-geodesic lamination λ on F, and maps the leaves of λ geodesically. (A *geodesic lamination* on a hyperbolic surface F is a closed set foliated by geodesics.) The smallest λ that works in the definition is known as the *pleating locus* of g.

The following lemma states that a pleated surface g can be perturbed by an arbitrarily small amount to a simplicial hyperbolic surface h, whose associated triangulation is an approximation to the pleating locus λ.

Lemma 6.1. *Let N be a hyperbolic 3-manifold and F a closed surface of genus at least 2. Given a pleated surface $g : F \to N$ and any $\epsilon > 0$ there exists a homotopy $H : F \times [0,1] \to N$ such that $g = H(\cdot, 0)$, the map $h = H(\cdot, 1)$ is a simplicial hyperbolic surface, and the trajectories $H(p \times [0,1])$ have lengths at most ϵ.*

Furthermore, the triangulation $\mathcal{T}$ of F corresponding to h contains a closed curve that is mapped to a closed geodesic, and there is a bound $\mathcal{G}$ on the number of vertices in $\mathcal{T}$, depending only on the genus of F.

Proof of 6.1. In the following arguments, we work with the hyperbolic metric σ on F described in the definition of a pleated surface. Let

λ be the pleating locus of g. It is known (see [15, Theorem 4.2.8]) that λ can be written uniquely as $\lambda_0 \cup \ell_1 \cup \cdots \cup \ell_m$ with $m \geq 0$, where λ_0 is a non-empty union $\lambda_1 \cup \cdots \cup \lambda_k$ of *minimal* laminations and ℓ_i are geodesic leaves that accumulate on λ_0. (A minimal lamination is one that has no proper (closed) sublamination.) A hyperbolic area argument (see §4 in [16]) bounds the number $m+k$ of these components in terms of the genus of F.

Let l be a leaf of λ. Let $P_l : l \to T_1(N)$ be the map which takes a point $x \in l$ to the point in the unit tangent bundle $T_1(N)$ associated to the tangent vector to $g(l)$ at $g(x)$. Notice that P_l is well-defined up to a choice of orientation of l.

Pick $\epsilon_1 > 0$. If λ_i is not a closed geodesic, then as in Lemma 4.2.15 in [15] we may approximate λ_i by a geodesic subarc γ_i' of a leaf l_i of λ_i of length at least 1 together with a "jump" γ_i'' of length less than ϵ_1, such that $\lambda_i \subset \mathcal{N}(\epsilon_1, \gamma_i')$. Moreover, we may assume that if v_i^1 and v_i^2 are the endpoints of γ_i', then $d(P_{l_i}(v_i^1), P_{l_i}(v_i^2)) < \epsilon_1$. Let $\gamma_i''' = \gamma_i' \cup \gamma_i''$, and let γ_i be the geodesic representative of γ_i''' in F. If λ_i is itself a closed geodesic we choose $\gamma_i = \gamma_i''' = \lambda_i$.

Basic hyperbolic trigonometry (see for example Theorem 4.2.10 in [15], or Lemma 5.5 in [9]) then implies that γ_i is ϵ_2-near to γ_i''', and that the geodesic representative $g(\gamma_i)^*$ of $g(\gamma_i)$ in N is ϵ_2-near to $g(\gamma_i)$, where $\epsilon_2 = O(\epsilon_1)$. (We say that two paths α, β in N are *ϵ-near* if there are lifts $\tilde{\alpha}, \tilde{\beta}$ to the universal cover $\tilde{N}$ such that $\tilde{\alpha} \subset \mathcal{N}(\epsilon, \tilde{\beta})$ and $\tilde{\beta} \subset \mathcal{N}(\epsilon, \tilde{\alpha})$.)

We can then define a homotopy $H : \gamma_i \times [0,1] \to N$ such that $H(\gamma_i, 0) = g(\gamma_i)$ and $H(\gamma_i, 1) = g(\gamma_i)^*$. Moreover, $H(x \times [0,1])$ can be made to have length $O(\epsilon_1)$ for each $x \in \gamma_i$.

For each of the additional arcs ℓ_j of λ, the intersection $\ell_j \cap (F - \mathcal{N}(\epsilon_1, \lambda_0))$ consists of just one arc α_j, if ϵ_1 has been chosen sufficiently small. We may also assume that α_j has length at least 1. We may append arcs of length at most $2\epsilon_1$ between the endpoints of α_j and the curves γ_p and γ_q nearest those endpoints, and straighten the resulting arc to get a geodesic segment ℓ_j' which is ϵ_3-near to α_j, where $\epsilon_3 = O(\epsilon_1)$.

Let X denote the 1-complex obtained by adjoining the ℓ_j' to $\bigcup_i \gamma_i$ along their endpoints, and note that, because geodesics minimize intersection number, X is embedded in F. (If a component of X consists of an isolated γ_i which meets no ℓ'_j, we may add one vertex on γ_i at an arbitrary point.) Let $g(l'_j)^*$ denote the geodesic arc in N homotopic to $g(l'_j)$ with endpoints fixed. Notice that $g(l'_j)^*$ is ϵ_4-near to $g(\alpha_j)$ where $\epsilon_4 = O(\epsilon_1)$. Thus we may extend H to a homotopy $H : X \times [0,1] \to N$ such that $H(\cdot, 1)$ maps the edges of X to geodesics, and so that for any $x \in X$, the trajectory $H(x \times [0,1])$ is a path of length $O(\epsilon_1)$.

Say that two curves α and β in N or F are *ϵ-nearly parallel* if their lifts to the unit tangent bundle (with suitable orientation) are ϵ-near to each other. Notice that ϵ_3 may be chosen so that any edge of X is ϵ_3-nearly parallel to a segment of λ, and that $\lambda \subset \mathcal{N}(\epsilon_3, X)$.

Let R denote the subsurface $F - X$. Since each component of R is convex, we may triangulate it by adding geodesic arcs whose endpoints lie on vertices of X. If β is such an arc, let J_0, J_1 be the components of $\beta \cap \mathcal{N}(\epsilon_3, X)$ containing the endpoints of β, and let $\beta' = \beta - (J_0 \cup J_1)$. Then any component of $\beta' \cap \mathcal{N}(\epsilon_3, X)$ is $O(\epsilon_3)$-nearly parallel to a segment of X. It follows that, at any point $p \in \beta' \cap \lambda$, the angle between β' and λ is $\theta(p) = O(\epsilon_3)$. Thus there is a constant ℓ_0, independent of ϵ_1 for sufficiently small ϵ_1, such that p is the midpoint of a segment of β of length $2\ell_0$ which is ϵ_5-near to a segment of the leaf of λ containing p, for $\epsilon_5 = O(\epsilon_1)$. Further, if $J \subset \beta'$ is any segment lying ϵ_5-near to a segment μ of λ, then J can be lengthened on both sides by ℓ_0 to a segment J' which is ϵ_6-near to $\mu' \supset \mu$, for $\epsilon_6 = O(\epsilon_5)$.

It follows that β' can be divided into segments $I_1, \ldots, I_k$, each of length at least ℓ_0, such that each I_j is either in $F - \lambda$, or lies ϵ_6-near to a segment μ of λ. Thus we may form a chain of geodesics I'_j (ϵ_6-near to I_j) such that $g(I'_j)$ are geodesics too, and successive endpoints are separated by short jumps, as before – if $v_j = I_j \cap I_{j+1}$ and v'_j, v''_j are the corresponding endpoints of I'_j, I'_{j+1}, then $d(P_{I'_j}(v'_j), P_{I'_{j+1}}(v''_j)) < \epsilon_7 = O(\epsilon_1)$.

Hence, $g(\beta')$ is ϵ_8-near to $g(\beta')^*$, for $\epsilon_8 = O(\epsilon_1)$. To extend this estimate to $g(\beta)$, consider one of the terminal segments, say J_0. Let a, b be the endpoints of J_0 where $b = J_0 \cap \beta'$, and we may assume (possibly lengthening β' and shortening J_0) that $b \in \lambda$, and that a punctured neighborhood of b in β' is disjoint from λ.

Let δ be the length of J_0, and let μ and μ' be segments of λ of length at least δ passing through a and b, respectively, which are are ϵ_9-nearly parallel, where $\epsilon_9 = O(\epsilon_3)$. It is easy to see that the acute angle of intersection θ of μ' and β at b is $O(\epsilon_3/\delta)$.

Since g is a pleated map, an arc from μ to μ' of length $O(\epsilon_3)$ maps to an arc in N of the same length, and thus the images $g(\mu)$ and $g(\mu')$ are still $O(\epsilon_3)$-nearly parallel. If $\nu \subset N$ is the geodesic homotopic to $g(J_0)$ with endpoints fixed, then the projection of ν to $g(\mu)$ has length at least $\delta - \epsilon_{10}$ where $\epsilon_{10} = O(\epsilon_3)$, and therefore that ν makes an acute angle with $g(\mu')$ of $\theta' = O(\epsilon_3/\delta)$.

Thus, the geodesics $g(\beta')^*$ and ν meet at an angle $\pi - O(\theta)$. Now, it is easy to see that, if ABC is a hyperbolic triangle such that $AB \geq \delta - O(\epsilon_3)$ and $\angle B \geq \pi - O(\epsilon_3/\delta)$, then AC lies in an $O(\epsilon_3)$-neighborhood of $AB \cup BC$. Applying this to our situation, and repeating for J_1, we conclude that $g(\beta)^*$ is $O(\epsilon_3)$-near to $g(\beta)$.

Thus, H may be extended to all the edges of the triangulation of R so that $H(\cdot, 1)$ takes each edge to a geodesic, and each trajectory $H(x \times [0,1])$ is of length $O(\epsilon_1)$. Extending H to the remaining triangular regions is now a simple matter, since the image of each such region by a lift $\tilde{g}$ of g to $\mathbf{H}^3$ lies in an $O(\epsilon_1)$-neighborhood of a totally geodesic triangle.

The simplicial map h which results is hyperbolic because through each vertex passes a segment of one of the γ_i, which is mapped geodesically. It follows that the total angle around such a vertex is at least 2π.

The bound $\mathcal{G}$ on the number of vertices of $\mathcal{T}$ follows from the bound on the number of components of λ, and the fact that all the vertices of $\mathcal{T}$ come from endpoints of leaves ℓ'_j, or isolated components γ_i.

The simplicial hyperbolic surface obtained in Lemma 6.1 may not be *useful* – it may have more than one vertex. The following lemma shows that after a bounded adjustment we may obtain a useful simplicial hyperbolic surface.

Lemma 6.2. *Let $g : F \to N$ be a simplicial hyperbolic surface whose associated triangulation has a closed curve which g maps to a closed geodesic, and at most $\mathcal{G}$ vertices. Then there is a homotopy $H : F \times [0,1] \to N$ such that $H(\cdot, 0) = g$, and $h = H(\cdot, 1)$ is a useful simplicial hyperbolic surface, and such that*

$$H(F \times [0,1]) \subset \mathcal{N}((\mathcal{G} - 1)c, g(F))$$

for an independent constant c.

Proof of 6.2. We first describe the process of collapsing an edge joining two distinct vertices of the associated triangulation of a simplicial pre-hyperbolic surface. Let $g : F \to N$ be a simplicial pre-hyperbolic surface with associated triangulation $\mathcal{T}_1$. Let e be an edge of $\mathcal{T}_1$ with distinct vertices v_1 and v_2. We first construct a new triangulation $\mathcal{T}_2$ from $\mathcal{T}_1$ by squeezing e to a point in F, and then identifying the remaining sides of any triangle which has E as an edge. We now define a map $G : F \times [0,1] \to N$. Define $G(v_2, t)$ such that $G(v_2 \times [0,1]) = g(e)$, $G(v_2, 0) = g(v_2)$ and $G(v_2, 1) = g(v)$. For any vertex $v \neq v_2$ of $\mathcal{T}$ let $G(v,t) = g(v)$. There is a natural continuous extension of G to $G : F \times [0,1] \to N$ such that $G(\cdot, t)$ is a simplicial pre-hyperbolic surface with associated triangulation $\mathcal{T}_1$ for all $t \in [0,1)$, and $G(\cdot, 1)$ is a simplicial pre-hyperbolic surface with associated triangulation $\mathcal{T}_2$. (This is related to the process of dragging a simplicial hyperbolic surface along a path, which is discussed in section 5 of [14].) We may also arrange that $G(x \times [0,1])$ is a geodesic for any $x \in F$.

Our next claim is that there exists a constant $c > 0$ such that $G(F \times [0,1]) \subset \mathcal{N}(c, G(F \times \{0\}))$. Let $\widetilde{G} : \widetilde{F} \times [0,1] \to \mathbf{H}^3$ denote a lifting of G to the universal covers. Let Δ be a face of the lift $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ with vertices $\widetilde{w}_1$, $\widetilde{w}_2$ and $\widetilde{w}_3$. Then $\widetilde{G}(\Delta \times [0,1])$ is a polyhedron spanned by $\widetilde{G}(\Delta \times \{0\})$, $\widetilde{G}(\Delta \times \{1\})$, and the three edges $\widetilde{G}(\widetilde{w}_q \times [0,1])$ $(q = 1,2,3)$. Notice that $\widetilde{G}(\Delta \times \{0\})$ and the $\widetilde{G}(\widetilde{w}_q \times [0,1])$ are all contained in $\widetilde{G}(\widetilde{F} \times \{0\})$. We may write $\widetilde{G}(\Delta \times [0,1])$ as the union of three tetrahedra (any of which may be degenerate), such that the first tetrahedron T_1 has as edges the three edges of $\widetilde{G}(\Delta \times \{0\})$ and $G(\widetilde{w}_1 \times [0,1])$, T_2 shares three edges with T_1 and has $G(\widetilde{w}_2 \times [0,1])$ as an edge, and T_3 shares three edges with T_2 and has $G(\widetilde{w}_3 \times [0,1])$ as an edge. We now recall that there is a constant a such that any tetrahedron in $\mathbf{H}^3$ is contained in an a-neighborhood of any four of its edges. Our claim therefore follows with $c = 3a$.

Now starting with our original simplicial hyperbolic surface g, let v^0 be a vertex of the triangulation lying on a closed curve γ that maps to a closed geodesic. We may repeat the above collapsing process $\mathcal{G} - 1$ times, each time collapsing an edge joining v^0 to an adjacent vertex. If we first collapse all such edges which lie on γ and only then collapse the rest, then the resulting simplicial pre-hyperbolic surface h will have an edge mapped to a closed geodesic, and only one vertex, so it will be a useful simplicial hyperbolic surface. Since the homotopy in each step is contained in a c-neighborhood of the previous one, we obtain the desired bound on the whole homotopy.

7. Far out simplicial hyperbolic surfaces

In this section we discuss the existence of end-homotopic useful simplicial hyperbolic surfaces in the ends of the approximating manifolds N_i. Let us begin by recalling some definitions and results.

A hyperbolic 3-manifold N is said to be non-elementary if $\pi_1(N)$ is non-abelian. Recall that the *convex core* $\mathcal{CH}(N)$ of a non-elementary hyperbolic manifold $N = \mathbf{H}^3/\Gamma$ is the smallest convex submanifold whose inclusion is a homotopy equivalence. Alternatively $\mathcal{CH}(N)$ is obtained as the quotient by Γ of the convex hull $\mathcal{CH}(\Lambda_\Gamma)$ of the limit set Λ_Γ. Ahlfors' finiteness theorem [1] implies that if Γ is finitely generated (and has no parabolics), then $\partial\mathcal{CH}(N)$ consists of a finite number of closed hyperbolic surfaces.

An end E of a non-elementary hyperbolic manifold N without cusps is called *geometrically finite* if E has a neighborhood U disjoint from $\mathcal{CH}(N)$. If E is geometrically finite, then we let Π_E denote the boundary of the component $\mathcal{O}_E$ of $N-\mathcal{CH}(N)$ which contains a neighborhood of E. $\mathcal{O}_E$ is homeomorphic to $\Pi_E \times \mathbf{R}$ (see Marden [29] or Epstein-Marden [17].)

A non-elementary hyperbolic 3-manifold N with finitely generated fundamental group is called *geometrically finite* if its convex core has finite volume (see Bowditch [10] for many equivalent definitions.) In particular, a hyperbolic 3-manifold without cusps is geometrically finite if and only if it has finitely generated fundamental group and all of its ends are geometrically finite (see section 1.2 in Bonahon [9]). A geometrically finite hyperbolic 3-manifold without cusps is often called *convex cocompact.*

An end E of N is *geometrically infinite* if it is not geometrically finite. In this case, by Ahlfors' finiteness theorem, there exists an entire neighborhood of E which is contained in the convex core.

Consider first the following simple lemma about convergence of convex cores under geometric limits.

Lemma 7.1. *Let Γ_i be a sequence of torsion-free Kleinian groups converging geometrically to a non-abelian, torsion-free Kleinian group Γ_∞. Let $f_i : V_i \to N_\infty$ be the maps given by Lemma 3.1. Then, possibly after restricting to a subsequence, $f_i(\mathcal{CH}(N_i) \cap V_i)$ converge geometrically to a convex submanifold C containing $\mathcal{CH}(N_\infty)$.*

Proof of 7.1. Since the geometric topology on closed subsets of $\mathbf{H}^3$

is compact, we may restrict to a subsequence so that the convex hulls of the limit sets $\{\mathcal{CH}(\Lambda_{\Gamma_i})\}$ converge geometrically. Let $\hat{\mathcal{C}}$ denote the limit. Let x and y be any two points in $\hat{\mathcal{C}}$, and let $\{x_i \in \mathcal{CH}(\Lambda_{\Gamma_i})\}$ and $\{y_i \in \mathcal{CH}(\Lambda_{\Gamma_i})\}$ be sequences converging to x and y respectively. Since $x_i, y_i \in \mathcal{CH}(\Lambda_{\Gamma_i})$, the geodesic arc $[x_i, y_i]$ joining them lies in $\mathcal{CH}(\Lambda_{\Gamma_i})$. Thus, since $[x, y]$ is the geometric limit of $\{[x_i, y_i]\}$, we see that $[x, y] \subset \hat{\mathcal{C}}$. Therefore, $\hat{\mathcal{C}}$ is convex.

It is also easy to see that $\hat{\mathcal{C}}$ is invariant by the limit group Γ_∞. It therefore projects to a convex submanifold $\mathcal{C} \subset N_\infty$, which contains the convex core $\mathcal{CH}(N_\infty)$. (Remark. This is closely related to the fact that the limit set Λ_Γ is contained in the geometric limit of the limit sets Λ_{Γ_i}. See [27].)

Let $\tilde{f}_i : B_{R_i}(0) \to \mathbf{H}^3$ be the maps given in Lemma 3.1 which descend to f_i. Since $\{\tilde{f}_i\}$ converges to the identity on compact sets, $\{\tilde{f}_i(\mathcal{CH}(\Lambda_{\Gamma_i}) \cap B_{R_i}(0))\}$ converges geometrically to $\hat{\mathcal{C}}$, and the lemma follows.

Let us set some notation for the next proposition, as well as for the proof of the main theorem. Fixing an end E of N_∞, let U_∞ be the (unique) component of $N_\infty - C_\infty$ which is a neighborhood of E. Notice that we may assume that C_∞ was chosen to be contained in the interior of $\mathcal{CH}(N_\infty)$. Let $F = \partial U_\infty$ and let U_i be the component of $N_i - C_i$ bounded by $f_i^{-1}(F)$. Then U_i is a neighborhood of an end E_i, and we say that E_i *corresponds to* E.

Proposition 7.2. *Let M be a compact irreducible 3-manifold with non-abelian fundamental group, and let $\{\rho_i : \pi_1(M) \to Isom_+(\mathbf{H}^3)\}$ be a sequence of purely hyperbolic, discrete, faithful, topologically tame representations converging strongly to a purely hyperbolic representation ρ_∞. Let E be a geometrically infinite end of N_∞ and let E_i be the corresponding end of N_i. Then given R, there exists $n(R)$ such that if $i \geq n(R)$, then N_i contains an end-homotopic useful, simplicial hyperbolic surface (homotopic to the end E_i) with image in $N_i - \mathcal{N}(R, b_i)$.*

Proof of 7.2. If E_i is geometrically infinite, we may apply directly the main theorem of [13], which states that, if N_i is topologically tame, then *every* neighborhood of E_i contains an end-homotopic, useful simplicial hyperbolic surface.

If E_i is geometrically finite, then the convex hull of N_i contains a boundary component corresponding to E_i, which should be far from C_i since the ends E_i are converging to the geometrically infinite end E.

After verifying this we apply the results of section 6 to approximate this boundary component by a nearby useful simplicial hyperbolic surface.

For simplicity let us assume (extracting a subsequence if necessary) that E_i is geometrically finite for all i. Let U_∞ and U_i be neighborhoods of E and E_i as above. Let $\mathcal{O}_i$ denote the component of $N_i - \mathcal{CH}(N_i)$ which contains a neighborhood of E_i, and let $\Pi_i = \partial\mathcal{O}_i$.

Since C_∞ separates the ends of N_∞ and C_∞ is contained in $\mathcal{CH}(N_\infty)$, it follows that U_∞ can meet $N_\infty - \mathcal{CH}(N_\infty)$ only if it contains an entire component of $N_\infty - \mathcal{CH}(N_\infty)$. Since U_∞ is a neighborhood of a geometrically infinite end, this cannot happen, and we conclude that U_∞ is contained in the convex core.

We can now show that the distance of Π_i from b_i grows without bound as $i \to \infty$. If not, let us suppose (possibly restricting again to a subsequence) that $\Pi_i \cap \mathcal{N}(D, b_i) \neq \emptyset$ for some $D > 0$ and all $i > 0$, and obtain a contradiction.

Thus, $\Pi_i \cap V_i \neq \emptyset$ as soon as $R_i > D$. Passing if necessary to a subsequence, the images $\{f_i(\Pi_i \cap V_i)\}$ converge to a subsurface Π_∞ in the boundary of $\mathcal{C}$, the geometric limit of $f_i(\mathcal{CH}(N_i) \cap V_i)$. Since, by Lemma 7.1, $\mathcal{C}$ contains $\mathcal{CH}(N_\infty)$, Π_∞ must be disjoint from $int(\mathcal{CH}(N_\infty))$.

On the other hand, Lemma 7.1 implies that C_i is eventually contained in $\mathcal{CH}(N_i)$, since C_∞ is eventually contained in the interior of $f_i(\mathcal{CH}(N_i))$. Therefore, $\overline{\mathcal{O}}_i$, and hence also Π_i must eventually be contained in U_i. It follows that $\Pi_\infty \subset U_\infty$, but this contradicts the fact that $U_\infty \subset \mathcal{CH}(N_\infty)$. Thus we have established that $d(\Pi_i, b_i) \to \infty$.

We may now approximate Π_i by a simplicial hyperbolic surface. Recall, from Proposition 2.2, that Π_i is homeomorphic to $f_i^{-1}(F)$, and hence to F for all i. Recall also (see [17]) that Π_i is the image of a pleated surface, since it is the boundary of a convex hull. Now let $g_i : F \to N_i$ be the pleated surface with image Π_i and pick $\epsilon > 0$. Lemmas 6.1 and 6.2 guarantee that there exist a useful simplicial hyperbolic surface $h_i : F \to N_i$ and a homotopy $G_i : F \times [0,1] \to N_i$ such that $G_i(\cdot, 0) = g_i$, $G_i(\cdot, 1) = h_i$ and $G_i(F \times [0,1]) \subset \mathcal{N}(K + \epsilon, \Pi_i)$, where K depends only on the genus of F.

If i is sufficiently large that Π_i lies outside $\mathcal{N}(R', b_i)$ for $R' = R + \mathrm{diam}(C_i) + K + \epsilon$, then the homotopy G_i lies outside $\mathcal{N}(R, C_i)$, and in particular the resulting surface is still end-homotopic and does not intersect $\mathcal{N}(R, b_i)$.

8. Proof of the main theorem

In this section we will finally assemble the proof of our main theorem, which we restate here.

Main Theorem. *Let M be a compact irreducible 3-manifold, and let $\{\rho_i : \pi_1(M) \to Isom_+(\mathbf{H}^3)\}$ be a sequence of purely hyperbolic, discrete, faithful, topologically tame representations converging strongly to a purely hyperbolic representation $\rho_\infty : \pi_1(M) \to Isom_+(\mathbf{H}^3)$. Then ρ_∞ is topologically tame. Moreover, for all sufficiently large i there exists a homeomorphism $\Phi_i : N_i \to N_\infty$, such that $(\Phi_i)_* = \rho_\infty \circ \rho_i^{-1}$.*

Proof of Main Theorem. When $\pi_1(M)$ is abelian the groups in question are elementary, and this case is well-understood (see Jørgensen [24]). For the remainder of the proof we assume that $\pi_1(M)$ is not abelian.

Let E be a geometrically infinite end of N_∞, and let U_∞ be the (unique) component of $N_\infty - C_\infty$ which contains a neighborhood of E. Let F be the boundary of U_∞ and let U_i be the component of $N_i - C_i$ bounded by $f_i^{-1}(F)$.

If F is incompressible, then we may lift to the cover N_F of N_∞ associated to $\pi_1(F)$. Notice that U_∞ is isometric to a neighborhood U_F of an end E_F of N_F. Bonahon's theorem [9] guarantees that N_F is topologically tame and hence that there is a neighborhood U' of E_F which is contained in U_F and homeomorphic to $F \times (0, \infty)$. Thus, in this case, E has a neighborhood homeomorphic to $F \times (0, \infty)$.

For the remainder of the proof we will assume that F is compressible in N.

Let δ denote the diameter of C_∞, so that 2δ bounds the diameters of C_i for all i. Recall the function $\mathcal{D}$ from Corollary 5.3, which gives a bound $\mathcal{D}(s)$ on the diameter of an end-homotopic useful simplicial hyperbolic surface that meets $\mathcal{N}(s, b_i)$ for any $i \in \mathbf{Z}_+$. Recall also the functions $L(R)$ and $n_0(R)$ from Lemma 3.4 which bound the diameters of compact regions in N_n in terms of their boundaries. Define inductively a pair of sequences $t(n), u(n) \in \mathbf{R}$ such that:

1. $t(0) = 2\delta + 2$,
2. $u(n) = \mathcal{D}(t(n)) + t(n)$,
3. $t(n+1) = 4L(u(n) + 1) + 1$.

Note that if an end-homotopic simplicial hyperbolic surface meets $\mathcal{N}(t(n), b_i)$ and misses $\mathcal{N}(1+2\delta, b_i)$, then it will be contained, by Corollary 5.3, in $\mathcal{N}(u(n), b_i)$.

We can now begin to construct the product regions X_n referred to in the outline of the proof.

First let us restrict to a subsequence i_n such that $i_n > n_0(u(n)+1)$ and such that $\mathcal{N}(L(u(n)+1), b_{i_n}) \subset V_{i_n}$, where V_i is the subset on which the K_i-bilipschitz map f_i is defined. Additionally, Proposition 7.2 guarantees that we may choose the subsequence so that there exists a useful, end-homotopic, simplicial hyperbolic surface $j_{i_n} : F \to U_{i_n} \subset N_{i_n}$ such that $j_{i_n}(F) \cap \mathcal{N}(t(n), b_{i_n}) = \emptyset$. From here on we reindex the sequence so that we may write N_n for N_{i_n}, j_n for j_{i_n}, etc.

We shall use $j_n(F)$ to obtain two end-homotopic embeddings $g_n : F \to N_n$ and $h_n : F \to N_n$ such that, for $n > 0$,

$$h_n(F) \cap \mathcal{N}(t(n)-1, b_n) = \emptyset$$

and

$$g_n(F) \subset \mathcal{N}(u(0), b_n),$$

and such that the region between them is a product $F \times [0,1]$, and is contained in $\mathcal{N}(L(u(n)+1), b_n)$; see Figure 2.

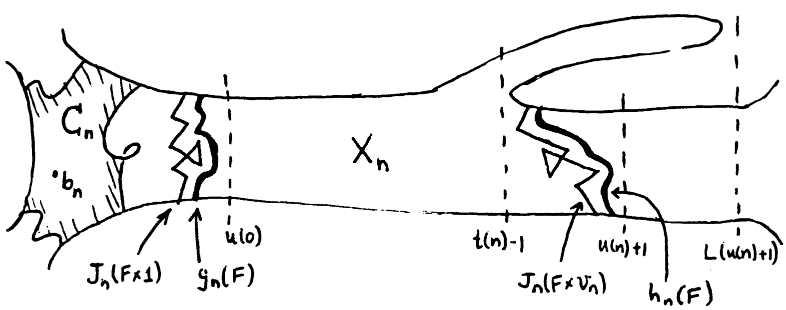

FIGURE 2

We first need to pull down $j_n(F)$ to a surface near the basepoint b_n. Proposition 4.5 guarantees that there exists a homotopy

$$J_n : F \times [0,1] \to N_n, \quad \text{such that } \begin{cases} J_n(x,0) = j_n(x), \\ J_n(F \times \{1\}) \cap cl(\mathcal{N}(1, C_n)) \neq \emptyset, \end{cases}$$

and so that for all $s \in [0,1]$, $J_n(\cdot, s)$ is an end-homotopic simplicial hyperbolic surface, and $J_n(F \times [0,1]) \cap \mathcal{N}(1, C_n) = \emptyset$.

To obtain a surface near the boundary of $\mathcal{N}(t(n), b_n)$, let

$$v_n = \sup\{s \in [0,1] \mid J_n(F \times \{s\}) \cap \mathcal{N}(t(n), b_n) = \emptyset\}.$$

Then $J_n(F \times \{v_n\})$ meets the boundary of $\mathcal{N}(t(n), b_n)$, but not its interior. Corollary 5.3 guarantees that $J_n(F \times \{v_n\}) \subset cl(\mathcal{N}(u(n), b_n))$.

Lemma 2.6 assures us that there exists an end-homotopic embedding $h_n : F \to N_n$ homotopic to j_n, such that

$$h_n(F) \subset \mathcal{N}(1, J_n(F \times \{v_n\})).$$

Similarly, there is an end-homotopic embedding $g_n : F \to N_n$ homotopic to j_n such that $g_n(F) \subset \mathcal{N}(1, J_n(F \times \{1\}))$. Note also that, since $J_n(F \times \{1\})$ meets $cl(\mathcal{N}(1, C_n))$, its diameter is bounded by $\mathcal{D}(1+2\delta)$, and we may conclude that $g_n(F) \subset \mathcal{N}(u(0), b_n)$ since $u(0) = 2 + 2\delta + \mathcal{D}(2 + 2\delta)$. Thus g_n and h_n have the desired properties.

Lemma 2.1 implies that the region X_n between $g_n(F)$ and $h_n(F)$ is homeomorphic to $F \times [0,1]$. Since $\partial X_n \subset \mathcal{N}(u(n) + 1, b_n)$, Lemma 3.4 guarantees that $X_n \subset \mathcal{N}(L(u(n) + 1), b_n)$, and in particular $X_n \subset V_n$.

Let $Y_n = f_n(X_n)$, $\partial_0 Y_n = f_n(g_n(F))$ and $\partial_1 Y_n = f_n(h_n(F))$. Notice that, since f_n is 2-bilipschitz,

$$\partial_0 Y_n \subset \mathcal{N}(2u(0), b_\infty),$$

$$Y_n \subset \mathcal{N}(2L(u(n) + 1), b_\infty)$$

and

$$\partial_1 Y_n \cap \mathcal{N}((t(n) - 1)/2, b_\infty) = \emptyset,$$

where for the last assertion we must apply Lemma 3.2.

By property (3) of the sequences $t(n)$ and $u(n)$, we have

$$\frac{t(n+1) - 1}{2} = 2L(u(n) + 1)$$

for all n. This implies that $\partial_1 Y_{n+1}$ is disjoint from Y_n. We now claim that $\partial_1 Y_n$ is contained in Y_{n+1}.

In N_n, $g_n(F)$ is homologous to the boundary F_n of U_n. The two surfaces bound a submanifold which must be contained in $\mathcal{N}(L(u(0)), b_n)$, by Lemma 3.4. Since this is contained in V_n (by our choice of subsequence and the monotonicity of L and the $u(n)$'s), we conclude that

$\partial_0 Y_n$ is homologous to the boundary F of U_∞. Thus the surfaces $\partial_1 Y_n$ are also homologous to F, and we denote by P_n the region bounded by $\partial_1 Y_n$ and F. It must be that $P_n \subset P_{n+1}$ or $P_{n+1} \subset P_n$, since both regions are contained in U_∞. But $\partial_1 Y_{n+1}$ is not in Y_n, and certainly not in $P_n - Y_n$, which is contained in $\mathcal{N}(2L(u(0)), b_\infty)$. Thus $\partial_1 Y_n$ is contained in Y_{n+1}, and is homologous in Y_{n+1} to $\partial_1 Y_{n+1}$.

Let Δ_{n+1} be the subset of Y_{n+1} bounded by $\partial_1 Y_n$ and $\partial_1 Y_{n+1}$. Theorem 2.4 assures us that $\partial_1 Y_n$ is isotopic to the one-half level surface in the product structure on Y_{n+1}, and therefore that Δ_{n+1} is homeomorphic to $F \times [0,1]$. Thus if we set

$$Z_\infty = Y_1 \cup \bigcup_{n=2}^{\infty} \Delta_n,$$

we have Z_∞ homeomorphic to $F \times [0,\infty)$. Since $\{\partial\Delta_n\}$ eventually leaves every compact set, the same is true for $\{\Delta_n\}$. Therefore Z_∞ has compact boundary (namely $\partial_0 Y_1$) while its interior is unbounded in U_∞, which implies that it must contain an entire neighborhood of the end E.

We have shown that every geometrically infinite end E of N_∞ has a neighborhood homeomorphic to $F_E \times (0,\infty)$ where F_E is some closed surface. Since it is well-known (see for example [17]) that every geometrically finite end E of N_∞ has a neighborhood with such a product structure, we see that N_∞ is topologically tame.

It remains to find the homeomorphism Φ_i. Let $\phi_i : C_i \to C_\infty$ be the restriction of f_i. Recall, from the proof of Proposition 3.3, that, for large enough i, $(\phi_i)_* = \rho_\infty \circ \rho_i^{-1}$. Theorem 2 of McCullough-Miller-Swarup guarantees that, since C_i is a compact core for N_i for $i = 1, \ldots, \infty$, there exists a homeomorphism $h_i : int(C_i) \to N_i$ which induces the identity on π_1. Thus for large enough i, $\Phi_i = h_\infty \circ \phi_i \circ h_i^{-1}$ is the desired homeomorphism. This completes the proof of the main theorem.

Remark. One can use the same argument to handle the case where F is incompressible as we did when F is compressible, and hence avoid using Bonahon's theorem. However, one would have to prove an analogue of Proposition 4.5 which guarantees that given a useful end-homotopic simplicial hyperbolic surface $h : F \to N_i$ there exists a homotopy of end-homotopic simplicial hyperbolic surfaces terminating in an end-homotopic simplicial hyperbolic surface which intersects a bounded neighborhood of C_i. (In the incompressible case the bound cannot necessarily be taken to be 1.) To accomplish this one first fixes a simple curve γ on F and observes that the geodesic representative of $f_i^{-1}(\gamma)$ always lies within a bounded neighborhood of $f_i^{-1}(F)$. One then constructs a triangulation $\mathcal{T}'$ with only one vertex and γ as an edge. Lemmas 4.2 and 4.3 then provide a homotopy of simplicial hyperbolic surfaces ending in a simplicial hyperbolic surface which has the geodesic representative of $f_i^{-1}(\gamma)$ in its image. (If this homotopy hits C_i, it can be truncated at that point, so that it remains end-homotopic.) This homotopy gives the desired analogue of Proposition 4.5.

9. Applications

This section contains applications of our main theorem to algebraic limits, boundaries of quasiconformal deformation spaces, and variation of spectral data.

Corollary A. *Let M be a compact irreducible 3-manifold, and let $\{\rho_i : \pi_1(M) \to Isom_+(\mathbf{H}^3)\}$ be a sequence of purely hyperbolic topologically tame discrete faithful representations of $\pi_1(M)$ converging algebraically to $\rho_\infty : \pi_1(M) \to Isom_+(\mathbf{H}^3)$.*

1. *If $\rho_\infty(\pi_1(M))$ has nontrivial domain of discontinuity, then ρ_∞ is the strong limit of $\{\rho_i\}$, and $\rho_\infty(\pi_1(M))$ is topologically tame.*
2. *If M is not homotopy equivalent to a compression body, then ρ_∞ is the strong limit of $\{\rho_i\}$, and $\rho_\infty(\pi_1(M))$ is topologically tame.*
3. *The limit set of $\rho_\infty(\pi_1(M))$ either has measure zero or is the entire 2-sphere.*

Proof of Corollary A. Part 1 follows directly from our main theorem and the following theorem:

Theorem 9.1. (Anderson-Canary [2]) *Let M be a compact irreducible 3-manifold. Suppose that $\{\rho_i : \pi_1(M) \to Isom_+(\mathbf{H}^3)\}$ is a sequence of purely hyperbolic discrete faithful representations, which converges algebraically to $\rho_\infty : \pi_1(M) \to Isom_+(\mathbf{H}^3)$. If ρ_∞ is purely hyperbolic, and $\rho_\infty(\pi_1(M))$ has non-trivial domain of discontinuity, then $\{\rho_i\}$ converges strongly to ρ_∞.*

Part 2 follows directly from our main theorem, part 1 and the following result:

Theorem 9.2. (Anderson-Canary [2]) *Suppose that $\{\rho_i : \pi_1(M) \to$*

$Isom_+(\mathbf{H}^3)\}$ is a sequence of discrete faithful representations which converges algebraically to $\rho_\infty : \pi_1(M) \to Isom_+(\mathbf{H}^3)$. If $\rho_\infty(\pi_1(M))$ has empty domain of discontinuity and M is not homotopy equivalent to a compression body, then $\{\rho_i\}$ converges strongly to ρ_∞.

Part 3 now follows by observing that if $\rho_\infty(\pi_1(M))$ has non-trivial domain of discontinuity, then it is topologically tame. Hence by Theorem 1.1 its limit set has measure zero. If, alternatively, $\rho_\infty(\pi_1(M))$ has empty domain of discontinuity, then the limit set is the entire sphere.

The general form of Ahlfors' measure conjecture includes the assertion that a finitely generated Kleinian group whose limit set is S^2 acts ergodically on S^2. Thus Corollary A does not give a complete answer to the general form of the conjecture for purely hyperbolic algebraic limits of topologically tame Kleinian groups, in the case where M is homotopy equivalent to a compression body. q.e.d.

Let Γ be a convex cocompact (torsion-free) Kleinian group and let $QC(\Gamma)$ be its quasiconformal deformation space (see Bers [5] for a discussion of quasiconformal deformation spaces of Kleinian groups). One may show, as in Corollary 1.5 in McMullen [32], that purely hyperbolic representations form a dense G_δ in $\partial QC(\Gamma)$. Thus, Corollary B is a nearly immediate consequence of Corollary A.

Corollary B. *Let Γ be a convex cocompact (torsion-free) Kleinian group and let $QC(\Gamma)$ denote its quasiconformal deformation space. If $\mathbf{H}^3/\Gamma$ is not homotopy equivalent to a compression body, then there is a dense G_δ in $\partial QC(\Gamma)$ consisting entirely of topologically tame, geometrically infinite hyperbolic 3-manifolds.*

Proof of Corollary B. We observed above that there exists a dense G_δ of purely hyperbolic representations in the boundary of $QC(\Gamma)$. Part 2 of Corollary A then assures that each of these representations is topologically tame. If any of these representations were geometrically finite, then (see Marden [29]) it would have an open neighborhood consisting entirely of quasiconformally conjugate convex cocompact representations, which would contradict its presence on the boundary of the quasiconformal deformation space. q.e.d.

We note that one could extend Corollary B to all convex cocompact (torsion-free) Kleinian groups other than Schottky groups if one had a positive solution to the following conjecture.

Conjecture. *Let Γ be a convex cocompact Kleinian group with at least two inequivalent components of its domain of discontinuity. Then* *the set of purely hyperbolic representations with non-empty domain of discontinuity intersects $\partial QC(\Gamma)$ in a dense G_δ.*

The following notation will allow us to give a concise restatement of our main theorem and part 2 of Corollary A. Let $D(M)$ denote the set of all discrete faithful representations of $\pi_1(M)$ into $Isom_+(\mathbf{H}^3)$. Let $AD(M)$ and $GD(M)$ denote the set $D(M)$ with the topology of algebraic and strong convergence respectively. Now let $H(M) = D(M)/Isom_+(\mathbf{H}^3)$ where $Isom_+(\mathbf{H}^3)$ acts by conjugation. Let $AH(M)$ denote $H(M)$ with the quotient topology inherited from $AD(M)$; this is called the algebraic topology. Let $GH(M)$ denote the set $H(M)$ with the topology inherited from $GD(M)$; this is called the strong topology. We let $TT(M)$ denote the set of topologically tame representations in $H(M)$. Let $H_{ph}(M)$ denote the set of purely hyperbolic representations and let $TT_{ph}(M) = TT(M) \cap H_{ph}(M)$. Let $GH_{ph}(M)$ denote $H_{ph}(M)$ with the topology inherited as a subset of $GH(M)$, and let $AH_{ph}(M)$ denote $H_{ph}(M)$ with the topology inherited as a subset of $AH(M)$. We can then state:

Corollary C. *If M is a compact irreducible 3-manifold, then $TT_{ph}(M)$ is a closed subset of $GH_{ph}(M)$. If M is not homotopy equivalent to a compression body, then $TT_{ph}(M)$ is a closed subset of $AH_{ph}(M)$.*

We can also consider the variation of spectral data on deformation spaces of hyperbolic manifolds. Define a function $\lambda_0 : D(M) \to \mathbf{R}$ by letting $\lambda_0(\rho)$ denote $\inf spec(-\Delta)$, where Δ is the Laplacian acting on $L^2(\mathbf{H}^3/\rho(\pi_1(M)))$. This function descends to a function $\lambda_0 : H(M) \to \mathbf{R}$. It is conjectured that λ_0 is a continuous function on $GH(M)$, although it is known not to be continuous on $AH(M)$ (see below).

It is a corollary of our techniques that the algebraic and strong topologies agree when restricted to $TT_{ph}(M)$. Thus we are able to apply the results of Gehring-Vaisala, Marden and Canary to obtain:

Corollary D. *If M is a compact 3-manifold, then λ_0 is a continuous function on $TT_{ph}(M)$ in the algebraic topology.*

Proof of Corollary D. Let $CC(M)$ denote the set of convex cocompact representations of $\pi_1(M)$. Marden [29] showed that $CC(M)$ is an open subset of $AH(M)$. In particular, if $\rho \in CC(M)$, and $\{\rho_i\}$ converges to ρ, then for large-enough i there exists a K_i-quasiconformal conjugacy of ρ_i to ρ, where K_i converges to 1. Thus (see Gehring-Vaisala [19]) the Hausdorff dimension $D(\rho_i)$ of the limit set of $\rho_i(\pi_1(M))$ converges to $D(\rho)$. Sullivan [37] showed that if $\tau \in CC(M)$, then

$\lambda_0(\tau) = D(\tau)(2 - D(\tau))$ unless $D(\tau) < 1$ in which case $\lambda_0(\tau) = 1$. Thus, $\lambda_0(\rho_i)$ converges to $\lambda_0(\rho)$, and we conclude that λ_0 is continuous at every point in $CC(M)$.

If $\rho \in TT_{ph}(M) - CC(M)$, then part 3 of Theorem 1.1 implies that $\lambda_0(\rho) = 0$. Let $\{\rho_i\}$ be a sequence in $TT_{ph}(M)$ converging algebraically to ρ. We may conclude that $\{\rho_i\}$ converges strongly to ρ. If $\rho(\pi_1(M))$ has non-trivial domain of discontinuity, then this follows from Theorem 9.2. If $\rho(\pi_1(M))$ has empty domain of discontinuity, then it follows from Theorem 9.2 of [14].

Let $N_i = \mathbf{H}^3/\rho_i(\pi_1(M))$. Since in this case the convex core of $\mathbf{H}^3/\rho(\pi_1(M))$ has infinite volume, it is an immediate corollary of Lemma 7.1 that the volume of the convex core $\mathcal{CH}(N_i)$ goes to infinity. Theorem A in [12] states that

$$\lambda_0(\rho_i) \leq \frac{4\pi|\chi(\partial\mathcal{CH}(N_i))|}{vol(\mathcal{CH}(N_i))}.$$

It follows that $\lambda_0(\rho_i)$ converges to 0. q.e.d.

It is well-known that λ_0 is not continuous in the algebraic topology on all of $AH(M)$. We will outline one example. If ρ is a maximal cusp in the boundary of a Bers slice, then ρ is geometrically finite, hence $\lambda_0(\rho) \neq 0$ (see Lax-Phillips [28].) On the other hand (see [32]), degenerate groups are dense in the boundary of a Bers slice, and if ρ' is degenerate, then $\lambda_0(\rho') = 0$ (see [12].) Thus, λ_0 cannot be continuous in the algebraic topology on the boundary of a Bers slice. In fact, McMullen (see again [32]) proved that maximal cusps are also dense in the boundary of a Bers slice (in the algebraic topology). Therefore λ_0 is discontinuous at a dense set of points in the boundary of the slice.

One similarly conjectures that Hausdorff dimension of the limit set is a continuous function on $GH(M)$. Bishop and Jones [6] have recently announced a proof that the Hausdorff dimension of any finitely generated geometrically infinite Kleinian group is 2. This suffices to show that Hausdorff dimension is a continuous function on $TT_{ph}(M)$.

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