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ACTIONS OF DISCRETE LINEAR GROUPS AND ZIMMER'S CONJECTURE

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Abstract

We consider lattice subgroups of higher-rank semisimple Lie groups acting as automorphisms of principal H-bundles with connection. Under the assumption that the action is not isometric and that certain conditions relating H and the higher rank group are satisfied, we show that such actions are classified by two essentially algebraic examples.

1. Introduction

Let Γ be a lattice subgroup of a higher rank semisimple Lie group G. More precisely, $G = \mathbf{G}(\mathbb{R})^\circ$ is the identity component of the set of real points of $\mathbf{G}$, and $\mathbf{G}$ is a semisimple $\mathbb{R}$-group whose almost simple factors have $\mathbb{R}$-rank at least 2. We assume that the elements of Γ are automorphisms of a principal H-bundle P which is an H-reduction of the frame bundle $F(M)$ of a compact d-dimensional smooth manifold M, and $H \subset SL_d\mathbb{R}$ is the set of real points of a linear algebraic $\mathbb{R}$-group $\mathbf{H}$. Γ acts in this way as a group of diffeomorphisms of M preserving a volume form.

R. Zimmer asked in [18] whether all ergodic, volume preserving actions of lattices in higher rank semisimple Lie groups on compact manifolds can be derived from essentially algebraic building-block actions obtained from homomorphisms of G into other groups. The model actions given in [18] are of three types: (1) a special kind of isometric action; (2) affine actions on compact nilmanifolds and (3) left translations on compact quotients L/Λ where L is a connected Lie group, Λ

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a cocompact lattice, and Γ acts on the quotient via a homomorphism $\pi : \Gamma \to L$.

A new class of actions, not previously considered by Zimmer, was discovered by Katok and Lewis in [10]. Their examples, obtained by a kind of blow-up construction at finite orbits of examples of type (2) above, show that M cannot always be expected to possess an invariant complete locally homogeneous structure. Similar exceptional examples in which more geometric structure is preserved than only a volume form seems, however, very hard to construct and possibly do not exist. Their examples also leave open the possibility that the volume preserving actions of Γ still correspond to those built out of the original examples described by Zimmer in [18] at least on an open invariant set (of full measure).

The results we state below verify Zimmer's conjecture under a number of extra assumptions. The most important one is that the action on the principal bundle P preserves a connection. This is a natural hypothesis since, as first remarked by Zimmer, all measure-preserving smooth actions of Γ do preserve a (Borel) measurable connection (cf. Proposition 1.7), which can often be proved unique and sometimes continuous. (Cf. Theorem 1.8. See also [3].) Moreover, it will be seen that very little regularity of the assumed connection is actually used below. Therefore, hypothesis H2 should be understood as an assumption on *regularity* rather than on *existence*. In this regard we point out that after the present work was completed N. Qian [16] showed that Lemma 3.4 holds true under the assumption $r \geq 1$. Therefore all that follows (Theorem 1.1 in particular) only requires that the assumed invariant connection be C^r for $r \geq 0$.

The main observation we make is that Zimmer's superrigidity theorem can be used to understand the holonomy of an invariant connection, thus giving new connections between curvature and superrigidity. The first example of that interaction was obtained by Zimmer in [19], which shows that if a lattice of $SL_n\mathbb{R}$ ($n \geq 3$) acts smoothly on a compact smooth n-dimensional Riemannian manifold by affine (i.e., connection-preserving) transformations and such that the action is not finite, then the curvature tensor must vanish and the lattice must be commensurable to a conjugate of $SL_n\mathbb{Z}$. (It is also shown there that if $\dim M < n$ and the action is affine for an arbitrary affine connection, then the action is finite.)

The present characterization of actions of Γ seems to be the first which does not exclude the type E2 described below. Moreover, the local Lie methods used here give a more geometric approach to results classifying lattice actions with suitable hyperbolicity assumptions. Thus, for example, one may compare the proof of Theorem 1.8 with the more dynamical methods of [6], [8], [9], [11], [10].

A second assumption made here is that any representation of the algebraic universal cover of $\mathbf{G}$ on $\mathbb{C}^d$ ($d = \dim M$) that arises from a homomorphism of that group into $\mathbf{H}$ is irreducible. (We stress the point that we have fixed a homomorphic imbedding of H into $SL_d\mathbb{R}$ that comes from the definition of P as a reduction of $F(M)$.) For example, we may assume that G and H are locally isomorphic and the imbedding of H into $GL_d\mathbb{C}$ defines an irreducible representation; alternatively, we could assume that $H = SL_d\mathbb{R}$ and that G is such group as $SO_{p,q}(\mathbb{R})$ with $p+q=d$, $p \geq q \geq 2$, or $Sp_n\mathbb{R}$ for $2n = d$, $n \geq 2$, or any other higher rank linear group for which the first nontrivial representation occurs in dimension d. More precisely, we assume the following:

H1. *Given any nontrivial homomorphism $\pi : \tilde{\mathbf{G}} \to \mathbf{H}$, where $\tilde{\mathbf{G}}$ is the algebraic universal cover of $\mathbf{G}$, the representation of $\tilde{\mathbf{G}}$ that π defines on $\mathbb{C}^d$ is irreducible.*

H2. *The action of Γ on M preserves a C^r connection ($r \geq 1$) on P. If the associated connection on TM is torsion-free, it suffices that $r \geq 0$.*

(The recent work of N. Qian noted above allows one to replace H2 with

H2'. *The action preserves a C^r connection for $r \geq 0$.*

In particular, Theorem 1.1 holds true with a $C^r, r \geq 0$, connection, irrespective of its torsion.)

H3. *The action does not preserve a (Borel) measurable Riemannian metric.*

H4. *The smooth measure associated with the Γ-invariant volume form has countably many ergodic components.*

The existence of a C^0 Γ-invariant connection on P and the assumption that the action preserves a measurable Riemannian metric imply that it actually preserves a smooth Riemannian metric ([21]). (We stress that P is a reduction of the frame bundle $F(M)$ and not an abstract principal bundle.)

Condition H3 is implied by a number of natural dynamical assumptions. For example, if we know that the action does not have discrete spectrum, then it cannot preserve a measurable Riemannian metric ([22]). Also the existence of a measurable invariant Riemannian metric forces the characteristic (Lyapunov) exponents of any $\gamma \in \Gamma$ to vanish, as can be easily checked.

Before stating our results, we describe the two main classes of examples that occur here. (Zimmer's list in [18] also includes a special class of isometric actions on Riemannian manifolds, which does not arise under our assumptions. Affine actions on (non-flat) nilmanifolds also do not arise, due to H1.)

E1. *Affine actions on flat manifolds and tori.*

Γ acts on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ via a homomorphism $\pi : \Gamma \to SL_d\mathbb{Z}$. We also allow for a translational component given by a finite homomorphism of Γ into $\mathbb{R}^d$. More generally, a smooth manifold M may admit $\mathbb{T}^d$ as a finite cover, and the action of Γ on M lifts to an affine action (of some finite extension of Γ) on $\mathbb{T}^d$.

E2. *Left translations on compact quotients of non-compact semisimple Lie groups.*

Let L be a connected, noncompact semisimple Lie group, and Λ a cocompact lattice in L. Also consider a homomorphism $\pi : \Gamma \to L$. Then Γ acts on L/Λ by left multiplication through π. More generally, M may admit L/Λ as a finite cover such that the action of Γ on M lifts to an action by left translation (by a finite extension of Γ) on that cover. The manifold can also be a quotient $K\backslash M$ where M is as above, and K is a compact group commuting with the image of the homomorphism π.

We say that an action $a_1 : \Gamma \times M \to M$ is C^s isomorphic to another action $a_2 : \Gamma \times N \to N$ if there exists a C^s diffeomorphism $f : M \to N$ that conjugates the two : $a_2(\gamma, f(m)) = f(a_1(\gamma, m))$.

The main result is the following.

Theorem 1.1. *Let a lattice group Γ of a higher rank semisimple Lie group G act by bundle automorphisms on a principal H-bundle P over a smooth compact manifold M, so that hypothesis H1 through H4 are satisfied. Then, either the Zariski closure of the connection's full holonomy group contains a nontrivial homomorphic image of G, or the action is C^{r+2} isomorphic to either E1 or E2. Here $r \geq 1$ is the same as in H2, where the connection is assumed of class C^r. If the connection is torsion-free and C^r for $r \geq 0$, the action is C^{r+2} isomorphic to an* *example of type E1.*

We can draw a number of immediate corollaries. Notice that the assumptions below are chosen so as to exclude the possibility that the holonomy of the invariant connection be larger than G. For large holonomy groups our techniques do not work.

Corollary 1.2. *Assume the same conditions of the theorem. Assume moreover that the connection in H2 is either Lorentzian or has amenable holonomy (e.g., a Riemannian or conformal connection), not necessarily without torsion, and C^r. Then the action is C^{r+2} isomorphic to either E1 or E2.*

Corollary 1.3. *Assume that a lattice subgroup of a simple higher rank Lie group G acts on a compact d-dimensional pseudo-Riemannian manifold M, so as to preserve a pseudo-Riemannian connection (possibly with torsion). Also assume that the normalized pseudo-Riemannian volume density has countably many ergodic components and that* $\dim G > \frac{1}{2}d(d-1)$. *Then, either the action preserves a Riemannian metric, or it is isomorphic to an affine action on a (Riemannian) flat manifold, as described in example E1.*

Under the conditions of the theorem, the *restricted* holonomy is also either trivial or contains a homomorphic image of G. (We believe that the latter never occurs.) In fact, the following proposition is part of the proof of the theorem.

Proposition 1.4. *Assume the same set-up of the theorem, plus hypothesis H1, H2 and H3. We may relax hypothesis H2 and assume that the action preserves a continuous H-connection, rather than a differentiable one. Let Hol(m) denote either the connection's restricted holonomy group or the connected component of the (either Hausdorff or Zariski) closure of the full holonomy group at m (calculated for a fixed basis of TM_m.) Then Hol(m) is either trivial or contains a nontrivial homomorphic image of G.*

Corollary 1.5. *Assume the conditions of the theorem. Assume moreover that the invariant connection's restricted holonomy does not contain a non-trivial homomorphic image of G and that the fundamental group of M is virtually solvable. Then the action is C^{r+2}isomorphic to an affine action on a flat manifold.*

The following theorem is an example of how it is sometimes possible to rule out the possibility of a large holonomy group, using more detailed information about the action.

Theorem 1.6. *Let a lattice* Γ *in* $SL_d\mathbb{R}$ *act on a compact d-dimensional smooth manifold M so as to preserve a Lipschitz continuous affine connection and an ergodic volume density. Assume moreover that the action does not preserve a Riemannian metric. Then the manifold is diffeomorphic to a torus and the action is* C^∞ *isomorphic to one of type E1.*

A result similar to the above, for a fiber bundle M, was obtained in [4]; part of the proof of Theorem 1.6 is taken from there.

We have so far assumed that the action preserves a connection. It is expected that ergodic actions of higher rank lattices will automatically preserve a connection, perhaps defined only on an open invariant set. In fact, a *measurable* invariant connection always exists and it can sometimes be shown to be unique.

The following remark is due to Zimmer [24]. It is a consequence of Proposition 2.1.

Proposition 1.7. *Let* Γ *be a lattice in a higher rank semisimple Lie group G. Assume that* Γ *acts ergodically on M by diffeomorphisms preserving a Borel probability measure. Then there exists a* Γ*-invariant Borel affine connection on TM. As a special case, let* Γ *be a lattice in* $SL_d\mathbb{R}$, $d \geq 3$, *and* $\dim M = d$. *If the action is not measurably isometric, it must preserve* a unique *Borel connection.*

Our care to assume as little differentiability for the invariant connection as the arguments permit is motivated by the above proposition and the hope that by using dynamical properties of the action such as hyperbolicity, it may be possible to show that an invariant connection has better regularity properties than measurability. The following result illustrates the point. Note that no connection is explicitly mentioned in the statement of the next theorem. The proof provides an alternative, more geometric approach to the problem of rigidity of lattice actions with Anosov elements, which does not depend on Anosov's structural stability theorem and is therefore not limited to the study of *perturbations* of lattice actions. For other results of a similar type, concerning the rigidity of lattice actions under perturbations, see [6], [8], [9], [11], [10].

A smooth diffeomorphism f of a compact smooth manifold M is *Anosov* if TM decomposes continuously as a direct sum $TM = E^+ \oplus E^-$ of invariant subbundles E^+ and E^-, so that the following estimate applies: For some (in fact, any) Riemannian metric $\|\cdot\|$, there exist positive constants $C > 1$, $\epsilon < a < A$, such that for all $x \in M$, for all positive integers n, and for all $v \in E^{\pm}(x)$,

$$\frac{1}{C}\|v\|e^{-nA} \leq \|Tf_x^{\mp n}v\| \leq C\|v\|e^{-na}.$$

The subbundles $E^{\pm}$ are the tangent bundles of C^0 foliations $\mathcal{E}^{\pm}$, the *Anosov foliations* of f, whose leaves are smooth. We say that f satisfies the $\frac{1}{2}$*-pinching condition* if $A < 2a$.

Theorem 1.8. *Let Γ be a lattice in a higher rank semisimple Lie group G. Assume that Γ acts on a compact, d-dimensional smooth manifold M so as to preserve a volume form and so that some element of Γ is a $\frac{1}{2}$-pinched Anosov diffeomorphism. Also assume that the dimension of the first nontrivial representation of the universal cover of G is d. Then the action is C^2 isomorphic to example E1.*

For some groups such as $SL_n\mathbb{R}$, it is possible to show that if the isomorphism is differentiable as above, then it (as well as the connection) is actually C^∞. We refer to [6], [9] and [11] for this aspect of the problem.

2. The holonomy of invariant connections

The main purpose of this section is to prove Proposition 1.4. Let $E \to M$ be a smooth vector bundle of *rank* $= d$ over a compact, connected manifold M, and denote by $p : F(E) \to M$ its bundle of frames. Let Γ be a group of automorphisms of E, and $A : \Gamma \times M \to GL_q(\mathbb{R})$ the cocycle representing the Γ-action with respect to a (Borel) section $\sigma : M \to F(E)$. A reduction of $F(E)$ to an $SL_d(\mathbb{R})$-subbundle corresponds to giving a volume form on E.

Let P denote a principal subbundle of $F(E)$ whose structure group H is contained in $SL_d(\mathbb{R})$. If Γ acts as automorphisms of P, the cocycle A representing the action can be assumed to take values in H.

Let μ be a Borel measure on M. Two sections σ and σ' of P are μ-*equivalent* if there is a Borel function $\phi : M \to H$ such that $\sigma' = \sigma \circ \phi$ μ-almost everywhere. Two cocycles are μ*-equivalent* if they are associated to equivalent sections.

We now suppose that Γ acts by automorphisms of P preserving an ergodic measure μ on M. Then (9.2.1, p.166 [20]) there exists an algebraic $\mathbb{R}$-group $\mathbf{L} \subset \mathbf{H}$ with the following property: For some Borel section of P, the cocycle representing the Γ-action takes values in $\mathbf{L}(\mathbb{R})$ and there is no equivalent cocycle taking values in $\mathbf{L}'(\mathbb{R})$ for a proper $\mathbb{R}$-subgroup $\mathbf{L}'$ of $\mathbf{L}$. $\mathbf{L}(\mathbb{R})$ is unique up to conjugacy in H and its conjugacy class is called the *algebraic hull* of (a cocycle associated to) the Γ-action. The algebraic hull is *compact* if any of its representatives $\mathbf{L}(\mathbb{R})$ is. The algebraic hull for an invariant measure on M can be defined as a map from the ergodic components of that measure into the conjugacy classes of algebraic subgroups of H.

Proposition 2.1 is taken from [12]. The case of a cocompact Γ was first proved by Zimmer in [23]. The general case below combines [23] with [13].

Proposition 2.1. *The algebraic hull for an ergodic action of a lattice in a higher rank semisimple Lie group preserving a Borel probability measure is reductive with compact center.*

Let $A : \Gamma \times M \to \mathbf{L}(\mathbb{R})$ be a measurable cocycle into the real points of an $\mathbb{R}$-group $\mathbf{L}$. Denote by $\mathbf{L}^\circ$ the identity component of $\mathbf{L}$. Define on $M' = M \times (\mathbf{L}(\mathbb{R})/\mathbf{L}^\circ(\mathbb{R}))$ the *skew-product action* by Γ:

$$\gamma(x, [g]) = (\gamma(x), [A(\gamma, x)g]).$$

According to [20 (9.2.6)], if μ is an ergodic, Γ-invariant, finite measure on M, the product of μ with the counting measure on (the finite set) $\mathbf{L}(\mathbb{R})/\mathbf{L}^\circ(\mathbb{R})$ is also an ergodic, Γ-invariant, finite measure on M' with respect to the skew-action. Moreover, by the same proposition, the cocycle $A' : \Gamma \times M' \to \mathbf{L}(\mathbb{R})$ on M' defined as $A'(\gamma, (x, [g])) = A(\gamma, x)$ has algebraic hull $\mathbf{L}^\circ(\mathbb{R})$. Let $P' = P \times \mathbf{L}(\mathbb{R})/\mathbf{L}^\circ(\mathbb{R})$.

The following proposition is a consequence of the superrigidity theorem for cocycles ([20]).

Proposition 2.2. *Let Γ be a lattice in a higher rank semisimple Lie group that acts as smooth automorphisms of a principal H-bundle over a compact manifold M, preserving an ergodic probability measure, so that hypothesis H1 is satisfied. Then, the algebraic hull L of the action is either compact or almost simple with center contained in $\{\pm 1\}$. Moreover, if the latter holds, there exists a measurable section of $P' \to M'$ with respect to which the action's cocycle into $L/\{\pm 1\}$ has the form $A(\gamma, m) = \pi(\gamma)$, where $\pi : \mathbf{G} \to \mathbf{L}/\mathbf{Z}$ is a rational homomorphism defined over $\mathbb{R}$ and Z is the center of $\mathbf{L}$.*

Proof. Let $L = \mathbf{L}(\mathbb{R}) \subset H$ be the algebraic hull of a cocycle $\alpha : \Gamma \times M \to H$ representing the action. After passing to an ergodic finite extension M' we may assume that $\mathbf{L}$ is connected. Since $\mathbf{L}$ is reductive, we may write $\mathbf{L} = \mathbf{Z} \cdot \mathbf{S}$, where $\mathbf{Z}$ is the (connected) center, and $\mathbf{S} = \mathbf{S}_1 \cdot \ldots \cdot \mathbf{S}_k$ is semisimple with almost simple factors $\mathbf{S}_i$. Any two distinct factors of $\mathbf{S}$ centralize each other since their intersection is a finite subgroup in $\mathbf{Z}$. If we show that for some i, $\mathbf{S}_i$ is locally isomorphic to a simple factor of $\mathbf{G}$, it will follow from hypothesis H1 and Schur's lemma that $\mathbf{L} = \mathbf{S}_i$ and the center of L is contained in $\{\pm 1\}$. That is in fact true due to the superrigidity theorem for cocycles. To see that, first note that since L is not compact, some $\mathbf{S}_i(\mathbb{R})$ is not compact. Consider the cocycle $\beta = p \circ \alpha$ where $p : L \to (\mathbf{S}_i/\mathbf{Z} \cap \mathbf{S}_i)(\mathbb{R})$. By passing to a finite ergodic extension, we may assume that $\mathbf{S}_i/\mathbf{Z} \cap \mathbf{S}_i$ is connected. Then the cocycle superrigidity theorem applies, and it follows that β is equivalent to the cocycle obtained from a surjective rational homomorphism $\pi : \mathbf{G} \to \mathbf{S}_i/\mathbf{Z} \cap \mathbf{S}_i$ defined over $\mathbb{R}$. Therefore $\mathbf{L}$ is almost simple and the claim follows.

The following version of the Borel density theorem is taken from [25].

Theorem 2.3 [Borel density theorem]. *Let $\mathbf{L}$ be a semisimple algebraic group defined over $\mathbb{R}$ such that no simple factor of $\mathbf{L}(\mathbb{R})$ is compact. Let $\mathbf{V}$ be an algebraic variety defined over $\mathbb{R}$ and suppose that $\mathbf{L}$ acts upon $\mathbf{V}$ regularly over $\mathbb{R}$. Assume moreover that the action of L on $V = \mathbf{V}(\mathbb{R})$ preserves a Borel probability measure μ. Then μ is supported on the set of fixed points of L.*

The next proposition is key to our argument. Our use of it here should be compared with the argument in, say [26], where Zimmer wants to derive information about the stabilizer groups of points for actions of G, whereas our concern is the holonomy group of a Γ-invariant connection.

Proposition 2.4. *Let $T : \mathbf{H} \times \mathbf{V} \to \mathbf{V}$ be a regular action of an algebraic group $\mathbf{H}$ on an algebraic variety $\mathbf{V}$ so that the group, the action and the variety are defined over $\mathbb{R}$. Denote $T_h(v) = T(h, v)$, $H = \mathbf{H}(\mathbb{R})$ and $V = \mathbf{V}(\mathbb{R})$. Let Γ be a lattice in a connected semisimple real group $G = \mathbf{G}(\mathbb{R})$ such that no simple factor of G is compact. Let $\pi : \mathbf{G} \to \mathbf{H}$ be a homomorphism defined over $\mathbb{R}$. Let M be a compact metric space, μ a Borel probability measure on M and suppose that Γ acts ergodically on M as a group of homeomorphisms preserving μ. Let $\varphi : M \to V$ be a Γ-equivariant Borel function, that is, $\varphi \circ \gamma = T_{\pi(\gamma)} \circ \varphi$, μ-almost everywhere. Then $\varphi = \varphi_0$ =constant μ-a.e. and φ_o is a fixed* *point for* $\pi(G)$.

Proof. Denote $\eta = \varphi_*\mu$, the push-forward of μ to V. Then η is preserved by $\pi(\Gamma)$:

$$T_{\pi(\gamma)_*}\eta = (T_{\pi(\gamma)} \circ \varphi)_*\mu = (\varphi \circ \gamma)_*\mu = (\varphi \circ \gamma)_*\mu = \varphi_*\mu = \eta.$$

Now, define

$$\bar{\mu} = \int_{G/\Gamma} T_{\pi(g)_*}\eta \; d\nu([g]),$$

where ν is the G-invariant measure on G/Γ and $[g]$ denotes an element of G/Γ. $\bar{\mu}$ is well defined since η is preserved by $\pi(\Gamma)$. It is also immediate that $\bar{\mu}$ defines a $\pi(G)$-invariant probability measure on V. Borel density theorem now applies and we obtain that $\bar{\mu}$ is supported on the fixed points of $\pi(G)$. In particular, there exists a Γ-invariant subset M' of M of full μ-measure such that φ maps M' into the set of fixed points of $\pi(G)$ in V. Therefore, φ is Γ-invariant on M'. The claim now follows from the assumption that the action of Γ on M is ergodic.

Lemma 2.5. *Assume hypothesis H1; that is, given any nontrivial homomorphism* $\pi : \tilde{\mathbf{G}} \to \mathbf{H} \subset SL_d$ *defined over* $\mathbb{R}$, *for a fixed homomorphic imbedding of* $\mathbf{H}$ *in the special linear group, we assume that the representation of* $\tilde{\mathbf{G}}$ *that* π *defines on* $\mathbb{C}^d$ *is irreducible. If* H' *is a closed subgroup of* H *normalized by* $\pi(G)$, *then* H' *either contains* $\pi(G)$ *or is contained in* $\{\pm I\}$.

Proof. First note that $\overline{\mathbf{G}} = \pi(\tilde{\mathbf{G}})$ is almost simple. $\mathbf{H}' \cap \overline{\mathbf{G}}$ is a normal subgroup of $\overline{\mathbf{G}}$, hence it either contains $\overline{\mathbf{G}}$ or is finite, contained in the center of $\overline{\mathbf{G}}$. Let us suppose the latter. Let $Lie(\mathbf{H}')$ denote the Lie algebra of $\mathbf{H}'$ and $Lie(\mathbf{Q})$ its radical. We claim that $Lie(\mathbf{Q})$ is zero. For each $g \in \overline{\mathbf{G}}$, $Ad_g(Lie(\mathbf{H}')) = Lie(\mathbf{H}')$ since, by assumption, $\overline{\mathbf{G}}$ normalizes $\mathbf{H}'$. In particular, $\overline{\mathbf{G}}$ normalizes $Lie(\mathbf{Q})$. Since $Lie(\mathbf{Q})$ is a solvable subalgebra of $Lie(SL_d\mathbb{C})$, it has a common eigenvector $v \in \mathbb{C}^d$. Since $\overline{\mathbf{G}}$ normalizes $Lie(\mathbf{Q})$, gv is also a common eigenvector for all $g \in \overline{\mathbf{G}}$. As $\overline{\mathbf{G}}$ is irreducible (hypothesis H1), $Lie(\mathbf{Q})$ is an abelian diagonalizable subalgebra of $Lie(SL_d\mathbb{C})$ and hence is conjugate to a subalgebra of the diagonal algebra. We may assume without loss of generality that $Lie(\mathbf{Q})$ is itself diagonal. Note that, if $A = (a_{ij}) \in Lie(\overline{\mathbf{G}})$ and $D = \text{diagonal}(l_1, \ldots, l_d) \in Lie(\mathbf{Q})$, their bracket is $[A, D] = ((l_i - l_j)a_{ij})$ so that if $l_i \neq l_j$ for some pair of indices i and j and some D, it follows that $a_{ij} = 0$ for all $A \in Lie(\overline{\mathbf{G}})$. Fix some index i and define the maximal set $I = \{i_1, \ldots, i_s\}$ such that if j is not in I, then $l_j \neq l_i$ for some $D \in Lie(\mathbf{Q})$. Since $a_{ij} = 0$ for all i in I and j in the complement of I, the vector space V spanned by e_i for $i \in I$ is invariant under multiplication by matrices in $Lie(\overline{\mathbf{G}})$. Therefore, V is also invariant under $\overline{\mathbf{G}}$, so that $V = \mathbb{C}^d$ due to the irreducibility assumption. It follows that $I = \{1, \ldots, d\}$ and $l_i = l_j$ for all i and j. Since $Lie(\mathbf{Q})$ is a subalgebra of $Lie(SL_d\mathbb{C})$, it must be zero. It follows that $Lie(\mathbf{H}')$ is semisimple. Recall that we are under the assumption that $Lie(\overline{\mathbf{G}}) \cap Lie(\mathbf{H}') = 0$, and that $Lie(\mathbf{H}')$ is normalized by $Lie(\overline{\mathbf{G}})$. Therefore, there is an $\mathbb{R}$-subgroup $\mathbf{G}'$ of $\overline{\mathbf{G}} \cdot \mathbf{H}'$ isomorphic to $\overline{\mathbf{G}}$, which centralizes $\mathbf{H}'$. One can now apply the hypothesis once more and Schur's lemma to conclude that $\mathbf{H}'$ is finite. In fact, as the elements of $\mathbf{H}'$ have determinant 1, $\mathbf{H}'$ is contained in $\{\lambda I : \lambda^d = 1\}$. Hence H' is contained in $\{\pm 1\}$.

We assume now that P admits a Γ-invariant connection and we denote by ∇ the associated affine connection on TM. The connection is assumed continuous in the following sense: If $\sigma : M \to P$ is a C^1 section of P, we may write $(\nabla\sigma)_m = \sigma \circ A_m$, where $m \in M \mapsto A_m$ is a continuous function of M taking values in $Hom(\mathbb{R}^d, Lie(H))$ and $Lie(H)$ is the Lie algebra of H.

Let now σ be a not necessarily continuous section of P. At each point $m \in M$ we may define the holonomy group of ∇ at m, $Hol^\nabla(m) \subset H$, consisting of all $h = h(m, c) \in H$ such that $\Pi_c\sigma(m) = \sigma(m) \circ h$, where $c : [0, 1] \to M$ is a C^1-closed path in M with endpoints at m and Π_c denotes parallel translation along c. Note that $t \in [0, 1] \mapsto \Pi_{c|[0,t]}\sigma(m)$ is a C^1 function. Likewise, denote by $Hol_0^\nabla(m)$ the restricted holonomy group at m, where here the closed curves are homotopically trivial. $Hol_0^\nabla(m)$ is the connected component of the identity in $Hol^\nabla(m)$.

We also assume from now on all the conditions of Proposition 1.4. Since Γ leaves invariant the connection, for every $\gamma \in \Gamma$ and closed path c with endpoints at m we have

$$(T\gamma)_m \circ \Pi_c\sigma(m) = \Pi_{\gamma\circ c}(T\gamma)_m \circ \sigma(m),$$

where σ is a section of P. Let μ be an ergodic component of the invariant smooth measure whose algebraic hull is not compact. (If for some invariant measure the algebraic hull is compact, Γ must preserve a Riemannian metric due to [21].) Denote by $A : \Gamma \times M \to H$ the cocycle describing the action of Γ with respect to σ and h the connection's holonomy, also expressed in terms of σ. Then the above formula, that

$$A(\gamma, m)h(m, c) = h(\gamma(m), \gamma \circ c)A(\gamma, m).$$

As in Proposition 1.4, denote by $Hol(m)$ either the restricted holonomy group or the connected component of the identity of the closure of the full holonomy group at m. Let $Lie(Hol(m))$ denote the Lie algebra of $Hol(m)$. It follows from Proposition 2.2 and the above discussion that $Lie(Hol(\gamma(m))) = T_{\pi(\gamma)}Lie(Hol(m))$, where $T_g h = ghg^{-1}$. (Note that this action is defined on the quotient of the algebraic hull by its center.) Defining $\varphi : M \to V =$ Grassmann variety of vector subspaces of $Lie(H)$, we obtain that $\varphi \circ \gamma = T_{\pi(\gamma)}\varphi$. Proposition 2.4 now implies that φ is almost everywhere equal to some Lie algebra that is normalized by $\pi(\mathbf{G}(\mathbb{R}))$. Proposition 1.4 follows now from the above lemma.

3. Local Lie group structures on M

We assume in this section that the invariant connection's holonomy group is finite. After passing to a finite cover of M (and a finite extension of Γ acting on that cover) we may assume that the connection has trivial holonomy. Therefore there exists a continuous, parallel section σ of P. A parallel section of P linearizes the action, in the sense that its associated cocycle is independent of $m \in M$ and so it defines a homomorphism of Γ into H. If the connection is continuous, this linearizing section is C^1.

Denote $W = \mathbb{R}^d$ and for each $e \in W$ consider the C^1 vector field σe on M. Define a continuous function $C_\sigma : M \to W \otimes \bigwedge^2 W^*$ as follows:

$$C_\sigma(m)(e \wedge e') = \sigma(m)^{-1}[\sigma e, \sigma e']_m,$$

where $[\sigma e, \sigma e']$ denotes Lie bracket of vector fields.

Lemma 3.1. *Let σ be a linearizing section for the connection preserving action of Γ on M, with associated cocycle given by the homomorphism $\pi : \Gamma \to GL_d\mathbb{R}$. Denote $V = W \otimes \bigwedge^2 W^*$. If $A \in V$ and $h \in H$, denote $T_h A = hAh^{-1}$, so that $(T_h A)(e \wedge e') = hA(h^{-1}e \wedge h^{-1}e')$. Then*

$$C_\sigma \circ \gamma = T_{\pi(\gamma)} C_\sigma.$$

Proof. Let σ be a C^1 section of $F(M)$, and γ any diffeomorphism of M. A straightforward computation that uses the fact that diffeomorphisms preserve the Lie bracket of vector fields yields that

$$C_{\gamma_*\sigma} \circ \gamma = C_\sigma.$$

If $\alpha : \Gamma \times M \to H \subset GL_d\mathbb{R}$ is the cocycle of Γ relative to σ and $A_\gamma(m) = \alpha(\gamma, m)$, we define $\theta(e) = \theta_\gamma(e) = A_\gamma^{-1} dA_\gamma(\sigma A_\gamma e)$, for any $e \in \mathbb{R}^d$, so that θ is a continuous function taking values in $Lie(H) \otimes W^*$. Define $\partial : Lie(H) \otimes W^* \to W \otimes \bigwedge^2 W^*$ by $\partial\eta(e \wedge e') = -\eta(e)e' + \eta(e')e$. A somewhat long but still totally straightforward computation shows that if $\sigma' = \sigma A$,

$$C_{\sigma'} = T_A^{-1} C_\sigma - \partial\theta.$$

Since $\gamma_*\sigma = \sigma A_\gamma$, the above two formulas give $C_\sigma \circ \gamma^{-1} = T_{A_\gamma}^{-1} C_\sigma - \partial\theta$. If σ is a linearizing framing for the action, then $\theta_\gamma = 0$ for all $\gamma \in \Gamma$ and the claim follows.

If the invariant volume has countably many ergodic components, it follows from the above and Proposition 2.4 that C_σ is constant. Therefore, C_σ defines the structure of a d-dimensional Lie algebra $Lie(L)$ on W, and for all $\gamma \in \Gamma$, $\pi(\gamma) \in H \subset GL_d\mathbb{R}$ is a Lie algebra automorphism. Also, as Γ is Zariski dense in G, $T_{\pi(g)} C_\sigma = C_\sigma$ for all $g \in G$, so that $\pi(G) \subset H$ is a group of automorphisms of $Lie(L)$. Thus by the hypothesis H1 and the observation that the radical of $Lie(L)$ as well as the subspace $[Lie(L), Lie(L)]$ are $\pi(G)$-invariant subspaces, $Lie(L)$ is either abelian or semisimple. We summarize the above discussion in the next proposition.

Proposition 3.2. *Assume the conditions of Proposition1.4 plus hypothesis H4. Assume moreover that the connected component of the holonomy group's Zariski closure does not contain a homomorphic image of G. Then a finite cover M' of M admits a framing of C^1 vector fields $X_1, \dots, X_d$ spanning a d-dimensional Lie algebra $Lie(L)$ and such that the section of P defined by $\{X_i\}$ linearizes the action of a finite extension of Γ on M'. The Lie algebra $Lie(L)$ is either abelian or semisimple.*

Given a Lie group L, denote by $Aff(L)$ the group of all diffeomorphisms of L that send right-invariant vector fields into right-invariant vector fields. Define on TL an affine connection ∇^L by the property that right-invariant vector fields are parallel. Then ∇^L is flat (in fact, the global holonomy of ∇^L is trivial since L admits a framing by right-invariant vector fields) and its torsion satisfies $T(X,Y) = -[X,Y]$, for right-invariant vector fields X and Y. (In particular, $\nabla^L T = 0$.) Also ∇^L is biinvariant since for all $g \in L$ and X right-invariant, $L_{g_*}X$ is also right-invariant. ($R_{g'}$ and L_g are respectively the right and left translations by g' and g and they commute.) *Aff*(L) coincides with the set of ∇^L-affine diffeomorphisms of L, that is, diffeomorphisms that preserve the connection.

Denote by *Aff*$_R(L)$ the subgroup of *Aff*(L) consisting of those transformations f such that $f_*X = X$ for all right-invariant vector fields. Note that *Aff*$_R(L)$ is exactly the group L acting by right translations. In fact, let $g \in L$ be such that $R_g^{-1} \circ f(e) = e$ where e is the identity in L. Then $R_g^{-1} \circ f$ induces the identity automorphism of the Lie algebra of L and so must correspond to the identity of L.

If Λ is a discrete subgroup of L, we denote *Aff*(L/Λ) the group of affine diffeomorphisms of L/Λ, which are the diffeomorphisms that lift to affine diffeomorphisms of L.

Proposition 3.3. *Let M be a compact, d-dimensional smooth manifold and $\{X_i\}_{i=1}^d$ a framing of C^r vector fields on M, $r \geq 2$, such that $[X_i, X_j] = \sum_{k=1}^d C_{ij}^k X_k$, where C_{ij}^k are constants. If $C_{ij}^k = 0$, we may assume that $r \geq 1$. Assume moreover that $\{X_i\}$ linearizes the action of a group Γ of smooth diffeomorphisms of M. Then there exists a C^{r+1} diffeomorphism $F : M \to L/\Lambda$, where L is a connected, simply connected Lie group, Λ a cocompact lattice in L and Γ is conjugate under F to a subgroup of affine diffeomorphisms of L/Λ.*

The proposition follows from the previous discussion and the elementary lemma below.

Lemma 3.4. *Let M and N be smooth manifolds of dimension d, and $\{X_i\}_{i=1}^d$ and $\{Y_i\}_{i=1}^d$ C^r framings with $r \geq 1 +$ Lipschitz. Assume that $[X_i, X_j] = \sum_{k=1}^d C_{ij}^k X_k$ and $[Y_i, Y_j] = \sum_{k=1}^d C_{ij}^k Y_k$, where C_{ij}^k are constants. Then for every $m \in M$ and $n \in N$ we can find neighborhoods $\mathcal{U}$ of m and $\mathcal{V}$ of n and a C^{r+1} diffeomorphism F between $\mathcal{U}$ and $\mathcal{V}$ such that $F_*X_i = Y_i$. If $C_{ij}^k = 0$ and the vector fields are only Lipschitz continuous, the same is true.*

Proof. Denote by $\varphi_t^{(i)}$ and $\psi_t^{(i)}$ the local flows of X_i and Y_i, respectively. For each m, $t \mapsto \varphi_t^{(i)}(m)$ is C^s for $s = 2 +$ Lipschitz, and $m \mapsto \varphi_t^{(i)}(m)$ is C^u for $u = 1 +$ Lipschitz. The same holds for $\psi_t^{(i)}$. We define the local C^u diffeomorphisms from a neighborhood of $0 \in \mathbb{R}^d$ to a neighborhood of m and n, respectively:

$$\Phi_m : (t_1, \ldots, t_d) \mapsto \varphi_{t_d}^{(d)} \circ \cdots \circ \varphi_{t_1}^{(1)}(m),$$

$$\Psi_n : (t_1, \ldots, t_d) \mapsto \psi_{t_d}^{(d)} \circ \cdots \circ \psi_{t_1}^{(1)}(n).$$

We also define $F = \Psi_n \circ \Phi_m^{-1}$, a C^u diffeomorphism from a neighborhood of m onto a neighborhood of n, taking m into n. In what follows, we assume that $[Y_i, Y_j] = \sum_k f_{ij}^k Y_k$, for Lipschitz continuous functions f_{ij}^k and $[X_i, X_j] = \sum_k f_{ij}^k \circ F X_k$.

Denote by S_i, $i = 0, \ldots, d$, a small submanifold of M containing m, consisting of points of the form $\Phi_m(t_1, \ldots, t_i, 0, \ldots, 0)$. Note that S_d is an open neighborhood of m. It follows from the definition that on S_k, $F_* X_i = Y_i$ for $i \geq k$. We claim that $F_* X_i = Y_i$ over an entire neighborhood of m. The proof will be by induction. Assume that $F_* X_i = Y_i$, $i = 1, \ldots, d$ on S_{l-1} and we show next that the same in true on S_l. It suffices to establish the equality for $i = 1, \ldots, l-1$ since it is automatically satisfied by the other values.

We write $F_* X_i = \sum_{j=1}^d h_{ij} Y_j$ for $i = 1, \ldots, l-1$, where h_{ij} are Lipschitz continous functions. It will suffice to show that $Y_l h_{ij} = 0$ almost everywhere, since in that case $h_{ij}(\psi_t^{(l)}(p)) = h_{ij}(p) = \delta_{ij}$, for $p \in F(S_{l-1})$ and $1 \leq i \leq l-1$ and $1 \leq j \leq d$, so that $F_* X_i = Y_i$ on S_l. Almost everywhere, we have the identity: $F_*[X_l, X_i] = [F_* X_l, F_* X_i]$. We can write this identity in terms of the $(l-1) \times d$ matrix $H = (h_{ij})$ and the $d \times d$ matrix $F = (f_{ij})$, where $f_{ij} = f_{li}^j$, as follows. Denote by f_1 the upper left $(l-1) \times (l-1)$ block of f and by f_2 the lower $(d-l+1) \times d$ block of f. Then H must satisfy the system of ordinary differential equations: $Y_l H = f_1 H - H f + f_2$ with initial value $H(t_l = 0) = (\delta_{\alpha\beta})$. It can be checked that the constant matrix $H_0 = (\delta_{\alpha\beta})$ is a solution of this initial value problem, so that the claim follows from the uniqueness of the solution.

From the above it is seen that M admits a C^{r+2} (L, L)-structure according to definition in [17], which must be complete, due to Proposition 3.6 [17]. Therefore between the universal cover of M and L we obtain a diffeomorphism that maps the lift of X_i to a right-invariant vector field on L. The deck transformations of $\tilde{M}$ are conjugate under this diffeomorphism to a group Λ of affine automorphisms of L that preserve the vector fields X_i. The proposition now follows.

The conclusions up to now are summarized below.

Proposition 3.5. *We assume here all the conditions of Theorem 1.1 and that G does not embed into the Zariski closure of the full holonomy group. Then there exist a connected, simply connected Lie group L and a cocompact lattice Λ of L, such that some finite cover M' of M is C^{r+2} diffeomorphic to L/Λ and the diffeomorphism conjugates the finite extension of Γ acting on M' to a group of affine automorphisms of L/Λ. L is either semisimple or the additive group $\mathbb{R}^d$.*

It should be noted that the actions of type E2 do admit connections with large holonomy group (as well as connection without holonomy). When the holonomy group is large (i.e., contains a nontrivial homomorphisc image of G) the techniques of our paper do not work.

4. The affine automorphisms of L/Λ

In this section we conclude the proof of Theorem 1.1. The main point left is to determine the group of affine automorphisms of the compact quotient L/Λ referred to in Proposition 3.5.

Let L be a connected Lie group, and σ a smooth section of the frame bundle $F(L)$ such that for each $e \in \mathbb{R}^d$, $d = \dim L$, $m \mapsto \sigma(m)e$ is a right-invariant vector field. Consider the homomorphism

$$a : \mathit{Aff}(L) \to \mathit{Aut}(\mathit{Lie}(L)) \subset GL_d\mathbb{R},$$

such that $Tf_m\sigma(m) = \sigma(f(m))a(f)$. (Here, we identify the Lie algebra $\mathit{Lie}(L)$ of L with $\mathbb{R}^d$ equipped with bracket $[e, e'] = \sigma^{-1}[\sigma e, \sigma e'].$) The kernel of a is the group $\mathit{Aff}_R(L) = \{R_g : g \in L\} \cong L$. Therefore $\dim \mathit{Aff}(L) \leq d + \dim(\mathit{Aut}(\mathit{Lie}(L)))$.

If L is semisimple, $\mathit{Aut}(\mathit{Lie}(L))$ contains $\mathit{Ad}(L)$ as a normal subgroup of finite index ([15]). Note that $\mathit{Ad}(L)$ is contained in the image of a, so $\dim \mathit{Aff}(L) = 2d$. Moreover, as L is connected and $\mathit{Aut}(\mathit{Lie}(L))$ has finitely many connected components, $\mathit{Aff}(L)$ also has finitely many components. The component of the identity, $\mathit{Aff}(L)^\circ$, can be identified with $(L \times L)/Z(L)$, where $Z(L)$ is the center of L, and the inclusion into $\mathit{Aff}(L)$ is given as follows: $[g_1, g_2] \in (L \times L)/Z(L) \mapsto f$ such that $f(m) = g_1 m g_2^{-1}$.

Proposition 4.1. *Let L be a connected, semisimple Lie group, and Λ a cocompact lattice. Then L, viewed as a subgroup of $\mathit{Aff}(L/\Lambda)$ of left translations, constitutes a subgroup of finite index in $\mathit{Aff}(L/\Lambda)$.*

It will help to first set some notation. The lattice Λ can be regarded as a subgroup of $\mathit{Aff}(L)^\circ$ under the correspondence $\lambda \mapsto f_\lambda$, where $f_\lambda(m) = m\lambda^{-1}$. We also define the subroup $\mathit{Aff}^*(L) = \{f \in \mathit{Aff}(L) : \forall m \in L, \forall \lambda \in \Lambda, \exists \lambda' \in \Lambda \text{ such that } f(m\lambda) = f(m)\lambda'\}$. $\mathit{Aff}^*(L)$ is the set of lifts to L of elements of $\mathit{Aff}(L/\Lambda)$, and contains the deck transformations $f_\lambda \in \lambda$ as a normal subgroup. Then $\mathit{Aff}^*(L)/\Lambda$ is isomorphic to $\mathit{Aff}(L/\Lambda)$. Define $A = \mathit{Aff}^*(L) \cap \mathit{Aff}(L)^\circ$, a normal subgroup in $\mathit{Aff}^*(L)$ which contains Λ as a normal subgroup. Note that $(\mathit{Aff}^*(L)/\Lambda)/(A/\Lambda) \cong \mathit{Aff}^*(L)/A \cong (\mathit{Aff}^*(L)\mathit{Aff}(L)^\circ)/\mathit{Aff}(L)^\circ \subset \mathit{Aff}(L)/\mathit{Aff}(L)^\circ$, which is finite. Therefore A/Λ is a normal subgroup of finite index in $\mathit{Aff}^*(L)/\Lambda \cong \mathit{Aff}(L/\Lambda)$.

Lemma 4.2. *Let Λ be a lattice in a connected semisimple Lie group L. Then $Z_L(\Lambda) = \{g \in L : g\lambda g^{-1} = \lambda, \forall \lambda \in \Lambda\}$ is contained in the center of L.*

Proof. This is immediate from the fact that $(\Lambda Z(L))/Z(L)$ is Zariski dense in $L/Z(L)$.

Lemma 4.3. *There is an injective map*

$$L\backslash A/\Lambda \to \mathit{Aut}(\Lambda)/\Lambda,$$

where Λ, on the right-hand side, is identified with a subgroup of inner automorphisms and, on the left-hand side, with a subgroup of right translations.

Proof. From the definition of $\mathit{Aff}^*(L)$, if $f \in A$ we have $f(m\lambda) = f(m)\theta(\lambda)$, where θ is an automorphism of Λ. Moreover $(L_g \circ f)(m\lambda) = (L_g \circ f)(m)\theta(\lambda)$, so L is in the kernel of the homomorphism $f \in A \mapsto \theta \in \mathit{Aut}(\Lambda)$, and the map $L\backslash A/\Lambda \to \mathit{Aut}(\Lambda)/\Lambda$ is well defined. Let f_1 and f_2 be two elements of A associated to the same θ. From the definition of θ we obtain that $f_1(m)^{-1}f_1(m\lambda) = f_2(m)^{-1}f_2(m\lambda)$, so $gf_1(\lambda) = f_2(\lambda)$ for all $\lambda \in \Lambda$, where $g = f_2(e)f_1(e)^{-1}$. Recall that $\mathit{Aff}(L)^\circ$ is isomorphic with $(L \times L)/Z(L)$, so that $f_i(m) = g_i m {g'}_i^{-1}$. If we write $u = g_2^{-1}g_1 = {g'}_2^{-1}g'_1$, then for all $\lambda \in \Lambda$, $u\lambda u^{-1} = \lambda$. But from the previous lemma we know that the set of all such u is contained in the center of L, so that $[g_1, g_2] = [g'_1, g'_2]$, which shows that the map is injective.

In order to prove Proposition 4.1, it suffices to show that, if $\mathcal{A} = \{a \in \mathit{Aut}(\Lambda) : a(\lambda) = g\lambda g^{-1}\}$ (these are automorphisms coming from $\mathit{Aff}(L)^\circ$, under the correspondence $f \mapsto \theta$ defined in the above proof), then the inner automorphisms by elements of Λ constitute a subgroup of finite index in $\mathcal{A}$. Note that $\mathcal{A}$ can be identified with a closed subgroup of $L/Z(L)$, containing the image in $L/Z(L)$ of the lattice Λ. The connected component of $\mathcal{A}$ must centralize the image of Λ in $L/Z(L)$ so, by Lemma 4.2, the group $\mathcal{A}$ is discrete. Therefore the image of Λ in $\mathcal{A}$ must be a subgroup of finite index, proving the claim.

The proof of Theorem 1.1 now follows from the above results. Observe that, when L is abelian, M is covered by a flat torus, and the affine diffeomorphisms of M lift to a subgroup of $SL_d\mathbb{Z} \ltimes \mathbb{T}^d$. The action's translation component defines a finite subgroup of $\mathbb{T}^d$, according to [6].

If the invariant connection is Riemannian of Lorentzian, or of any other type for which the structure group has real rank less than 2, its holonomy group cannot contain a Lie group of higher $\mathbb{R}$-split rank, so Corollary 1.2 follows immediately from the main theorem.

If the connection is pseudo-Riemannian, its holonomy group is contained in the orthogonal group of appropriate signature, which has dimension $\frac{1}{2}d(d-1)$. Therefore, Corollary 1.3 is also a direct consequence of the theorem. Note that the pseudo-Riemannian volume density is automatically Γ-invariant if the manifold is compact.

Corollary 1.5 follows from Proposition 1.4, that is used to conclude that the connection is flat, and the main theorem, plus the remark that the Zariski closure of a virtually solvable subgroup of SL_d (the image of the fundamental group under the holonomy representation) must also be virtually solvable, and connot contain a connected simple Lie group.

For the proof of Theorem 1.6, one first shows that the curvature tensor of the Lipschitz connection vanishes, by following an argument used for example in [1]. This is done by noting that a nonzero invariant measurable tensor field would impose certain relations among the Lyapunov exponents of elements of Γ. These exponents are in turn related to the eigenvalues of semisimple elements of Γ for the representation of Γ obtained from the superrigidity theorem for cocycles. But these eigenvalues are easily seen to violate those relations. (See for example [1].) Once the curvature tensor is shown to vanish (almost everywhere), we know just as for smooth connections, that the restricted holonomy group must be trivial. We give a proof of this fact below since we could not find it in the literature. Next step is to show that the global holonomy is small enough so that it cannot contain a homomorphic image of G. We refer to [4] for the details. We only note here that the global holonomy can be seen to be abelian, and one uses in an essential way the hyperbolic properties of the dynamics of the action of a maximal diagonalizable abelian subgroup of Γ. The crucial remark to make is the following: The fastest contracting directions associated to a connection-preserving (nonuniformly) hyperbolic diffeomorphism of M must be parallel when transported along the local stable manifolds.

Proposition 4.4. *Let E be a smooth vector bundle over a d-dimensional manifold M, equipped with a Lipschitz continuous connection ∇. Suppose that the curvature tensor of ∇, which is defined on a set of full measure, vanishes identically on that set. Then the restricted holonomy group of ∇ is trivial.*

Proof. Let $\{X_i = \frac{\partial}{\partial t_i}\}_{i=1}^n$ be smooth coordinate vector fields, and $\{\eta_i\}_{i=1}^d$ a smooth local framing on E. Denote $\nabla_{X_i}\eta_j = \sum_{k=1}^d A^{(i)}{}_{kj}\eta_k$, where $A^{(i)}$ is a Lipschitz function from $\mathbb{R}^n$ into the set of $d \times d$ real matrices. The condition that the curvature of ∇ is zero almost everywhere corresponds to $[\nabla_{X_i}, \nabla_{X_j}] = 0$ almost everywhere, where translates in coordinates to

$$\frac{\partial A^{(i)}}{\partial t_j} - \frac{\partial A^{(j)}}{\partial t_i} - [A^{(i)}, A^{(j)}] = 0 \tag{1}$$

almost everywhere.

Define a continuous local section of E as follows. Given $u_0 \in E_{p_0}$, let $u(t_1, \ldots, t_n)$ be such that $u_0 = u(0, \ldots, 0)$, $\nabla_{X_n}u = 0$, $\nabla_{X_{n-1}}u|_{\{t_n=0\}} = 0$, ..., $\nabla_{X_k}u|_{\{t_n=\cdots=t_{k+1}=0\}} = 0$, ..., $\nabla_{X_1}u|_{\{t_n=\cdots=t_2=0\}} = 0$. Then Theorem 8.1, p.109 [5] $u(t_1, \ldots, t_n)$ is a Lipschitz continuous function in $(t_1, \ldots, t_n)$, and $u(t_1, \ldots, t_k, 0, \ldots, 0)$ is C^1 in t_k for $k = 1, \ldots, n$. Set $f(t_1, \ldots, t_k) = u_r(t_1, \ldots, t_k, 0, \ldots, 0)$. Then f is a C^1 function in t_k, Lipschitz continuous in $(t_1, \ldots, t_k)$ and (as $\frac{\partial u}{\partial t_k} = -A^{(k)}u$ is Lipschitz continuous) $\frac{\partial f}{\partial t_k}(t_1, \ldots, t_k)$ is also Lipschitz continuous in $(t_1, \ldots, t_k)$.

Define $\overline{D}_i f(t) = \overline{\lim}_{h\to 0}\frac{1}{h}(u(t+he_i) - u(t))$, where $e_i = (0, \ldots, 1, \ldots, 0)$, with 1 at the ith position. We claim that $(\overline{D}_i f)(t_1, \ldots, t_k)$ is a bounded function in $(t_1, \ldots, t_k)$ and Lipschitz continuous in t_k. We have

$$\begin{aligned}(\overline{D}_i f)(t_1, \ldots, t_k^*) &- (\overline{D}_i f)(t_1, \ldots, t_k) \\ &= \overline{\lim}_{h\to 0} g(h, t_1, \ldots, t_{k-1}, t_k^*) - \overline{\lim}_{h\to 0} g(h, t_1, \ldots, t_{k-1}, t_k),\end{aligned}$$

where

$$g(h, t_1, \dots, t_{k-1}, t_k) = \frac{1}{h}[f(t_1, \dots, t_i + h, \dots, t_k) - f(t_1, \dots, t_i, \dots, t_k)].$$

Denoting

$$a(h) = g(h, t_1, \dots, t_{k-1}, t_k^*) \text{ and } b(h) = -g(h, t_1, \dots, t_{k-1}, t_k),$$

and by the elementary properties

$$\underline{\lim}_{h\to 0}[a(h) + b(h)] \leq \overline{\lim}_{h\to 0}a(h) + \underline{\lim}_{h\to 0} \leq \overline{\lim}_{h\to 0}[a(h) + b(h)]$$

and

$$-\overline{\lim}_{h\to 0}|a(h) + b(h)| = \underline{\lim}_{h\to 0}(-|a(h) + b(h)|) \leq \underline{\lim}_{h\to 0}[a(h) + b(h)],$$

and the mean value theorem

$$a(h) + b(h) = \frac{\partial g}{\partial t_k}(h, t_1, \dots, t_{k-1}, \tau)(t_k^* - t_k),$$

we obtain the claim with the same Lipschitz constant as for $\frac{\partial f}{\partial t_k}$. We also have that $D_k(\overline{D}_i f) - D_i(D_k f)$ is almost everywhere 0. We define now $\eta(t_1, \dots, t_k) = u(t_1, \dots, t_k, 0, \dots, 0)$ and make use of the induction assumption that $\eta(t_1, \dots, t_{k-1}, 0)$ is C^1 and satisfies $\nabla_{X_i}\eta|_{t_k=0}$ for $i = 1, \dots, k-1$. Denoting $\overline{\nabla}_{X_i}\eta = \overline{D}_{t_i}\eta + A_{(i)}\eta$, we get from equation (1) that

$$\nabla_{X_k}(\overline{\nabla}_{X_i}\eta) = \nabla_{X_i}(\nabla_{X_k}\eta) = 0,$$

almost everywhere. Writing $\mathcal{V} = \overline{\nabla}_{X_i}\eta$, we obtain from the above that $\mathcal{V}$ satisfies the differential equation $D_{t_k}\mathcal{V} + A_{(k)}\mathcal{V} = 0$ with initial condition $\mathcal{V}(t_1, \dots, t_{k-1}, 0) = 0$ for all $t_1, \dots, t_{k-1}$. It follows that $\mathcal{V}(t_1, \dots, t_k) = 0$ almost everywhere, so that $\nabla_{X_i}\eta = 0$ for almost all $(t_1, \dots, t_k)$ and $i = 1, \dots, k$. In particular, η is C^1 with Lipschitz derivatives. By induction, u is C^1 with Lipschitz derivatives in $t_1, \dots, t_n$ and parallel: $\nabla_{X_i}u = 0$ for all i. In this way, we have built a parallel local section of E with arbitrary value at $(0, \dots, 0)$. The proposition now follows.

5. Existence of invariant connections

In this section we prove Proposition 1.7 and discuss the existence of invariant connections under the hypothesis of Theorems 1.8 and 1.9. Let $p : E \to M$ be a C^r vector bundle over M. Let $J_1(E)$ denote the vector bundle over M consisting of first jets of germs of differentiable sections of E. The following short sequence of vector bundles is exact:

$$0 \longrightarrow T^*M \otimes E \xrightarrow{\ i\ } J_1(E) \xrightarrow{\ \pi\ } E \longrightarrow 0.$$

A connection on E can be described as a splitting of this exact sequence: $\sigma : E \to J_1(E)$, $\pi \circ \sigma = Id_E$. Such splitting defines a covariant derivative map $\nabla : \Gamma^1(E) \to \Gamma^0(T^*M \otimes E)$, where $\Gamma^r(E)$ denotes the space of C^r-sections of E, as follows: For $X \in \Gamma^1(E)$, and denoting $j_1X \in \Gamma^0(J_1(E))$ the first jet of X, set: $\nabla X = (Id - \sigma \circ \pi) j_1 X$.

We now assume that $(\bar{f}, f)$ is a C^r automorphism of $p : E \to M$ (so that $p \circ \bar{f} = f \circ p$), from which we obtain automorphisms of $J_1(E)$ and $T^*M \otimes E$ as follows: For $j_1X(x) \in J_1(E)_x$, $\bar{f} \cdot j_1X(x) = j_1(\bar{f}X \circ f^{-1})(f(x))$ and for $\alpha \otimes X \in T^*M_x \otimes E_x$, $\bar{f}(\alpha \otimes X) = \bar{f}\alpha \otimes \bar{f}X$, where $\bar{f}\alpha = \alpha \circ Tf^{-1}|_{TM_{f(x)}}$. With these definitions, $\bar{f}$ becomes an automorphism of the above exact sequence.

An $\bar{f}$-invariant C^r-connection $\nabla : \Gamma^{r+1}(E) \to \Gamma^r(T^*M \otimes E)$ can be described as a C^r-splitting σ of the last diagram, for which $\sigma \circ \bar{f}|_E = \bar{f}|_{J_1(E)} \circ \sigma$. Equivalently, it can be described as an $\bar{f}$-equivariant C^r subbundle in $J_1(E)$, complementary to $i(T^*M \otimes E)$.

The existence of Γ-invariant connections, where Γ is a lattice in a higher rank Lie group, can now be derived by taking a bounded section of the frame bundle of $J_1(E)$ that is adapted to the vertical subbundle, and by applying Proposition 2.1 to the cocycle obtained from it. The fact that the algebraic hull is reductive implies the existence of a measurable Γ-invariant complement to the vertical subbundle.

For nonisometric actions of lattices in $SL_d\mathbb{R}$ on d-dimensional manifolds, the uniqueness claimed in Proposition 1.7 is due to the fact that the difference of two invariant connections is an invariant tensor field which, if not zero, has to impose relations among the eigenvalues of elements of Γ, which do not exist. For the details of this type of argument we refer to [2]. These remarks prove Proposition 1.7.

The following proposition is taken from [2].

Proposition 5.1. *Let f be a $\frac{1}{2}$-pinched Anosov diffeomorphism of a* *compact manifold. Then f preserves a continuous, torsion-free, affine connection ∇. The connection is unique in the class of measurable invariant connections, and with respect to it the stable and unstable Anosov distributions E^+ and E^- are parallel.*

The proof of Theorem 1.8 follows from applying the main theorem to the continuous connection obtained from the above proposition. Note that the uniqueness and Proposition 1.7 are needed so that we are certain that the connection is preserved by all of Γ. The fact that the stable and unstable distributions are parallel shows that the holonomy of the continuous connection is contained in $GL_n\mathbb{R} \times GL_{d-n}\mathbb{R}$, $n \neq 0, d$, so that it cannot contain an image of G.

References

[1] R. Feres, *Connection-preserving actions of lattices is $SL_n(\mathbb{R})$*, Israel J. Math. **79** (1992) 1-21.

[2] _____, *The invariant connection of a $\frac{1}{2}$-pinched Anosov diffeomorphism and rigidity*, to appear in the Pacific J. Math.

[3] _____, *Actions of lattices of the special linear group on vector bundles*, preprint, 1994.

[4] E. Goetze, *Connection preserving actions on fiber bundles,* To appear in J. Differential Geom.

[5] P. Hartman. *Ordinary differential equations*, Wiley New York, 1964.

[6] S. Hurder, *Rigidity for Anosov actions of higher rank lattices* Ann. of Math. **135** (1992) 361-410.

[7] _____, *Affine Anosov actions*, preprint.

[8] _____, *A survey of rigidity theory for Anosov actions*, preprint.

[9] A. Katok, J. Lewis & R. Zimmer. *Cocycle superrigidity and rigidity for lattices on tori*, preprint.

[10] A. Katok & J. Lewis. *Global rigidity results for lattice actions on tori and new examples of volume-preserving actions*, preprint.

[11] _____, *Local rigidity for certain groups of toral automorphisms*, preprint.

[12] J. Lewis, preprint.

[13] A. Lubotzky, S. Mozes & M. S. Raghunathan, *Cyclic subgroups of exponential and metrics on discrete groups*, preprint.

[14] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer, Berlin, 1991.

[15] S. Murakami, *On the automorphisms of a real semisimple Lie algebra*, J. Math. Soc. Japan, **4** (1952).

[16] N. Qian, *Tangential flatness and global rigidity of higher rank lattice actions*, preprint, 1994.

[17] W. Thurston, *The geometry and topology of 3-manifolds*, Lecture notes, Princeton.

[18] R. Zimmer, *Actions of semisimple groups and discrete subgroups*, Proc. Internat. Congr. Math., Berkeley, 1986, 1247-1258.

[19] _____, *On connection-preserving actions of discrete linear groups*, Ergodic Theory Dynamical Systems **6** (1986) 639-644.

[20] _____, *Ergodic theory and semisimple groups*, Monographs in Math., Birkhäuser, Boston, 1984.

[21] _____,*Lattices in semisimple groups and invariant geometric structures*, Discrete groups in geometry and analysis, (Ed. R.Howe), Birkhäuser, Boston, 1987, 152-210.

[22] _____, *Spectrum, entropy, and geometric structures for smooth actions of Kazhdan groups*, Israel J. Math. **75** (1991), 65-80.

[23] _____, *On the algebraic hull of an automorphism group of a principal bundle*, Comment. Math. Helv. **65** (1990) 375-387.

[24] _____, Personal communication.

[25] _____, *Automorphism Groups and Fundamental Groups of Geometric Manifolds*, Proc. Sympos. Pure Math. **54** (1993) Part 3.

[26] _____, *Semisimple automorphism groups of G-structures*, J. Differential Geom. **19** (1984) 117-123.

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