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GEOMETRY OF THE ENDS OF THE MODULI SPACE OF ANTI-SELF-DUAL CONNECTIONS

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1. Introduction

Let X_0 be a closed, oriented, C^∞ four-manifold and let $M_{X_0,P}(g_0)$ be the moduli space of g_0-anti-self-dual connections on a principal G bundle P over X_0. The subspace $M^*_{X_0,P}(g_0)$, obtained by excluding the reducible connections is then a finite-dimensional, usually non-compact, C^∞ manifold. The moduli space $M^*_{X_0,P}(g_0)$ is naturally endowed with a metric $\mathbf{g}$ of Weil-Petersson type, called the L^2 metric, and our purpose in this article is to study the geometry of the moduli space ends.

(a) Main results. It has been conjectured by D. Groisser and T. Parker in [13], [14] and by S. K. Donaldson in [5] that the moduli space of anti-self-dual connections, endowed with the L^2 metric, has finite volume and diameter. The goal of this article is to prove this conjecture under the hypotheses described below.

Theorem 1.1. *Let X_0 be a closed, connected, oriented, simply-connected, C^∞ four-manifold with generic metric g_0 and let P be a principal G bundle over X_0 such that either (1) $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$ and $b^+(X_0) = 0$, or (2) $G = \mathrm{SO}(3)$ and $w_2(P) \neq 0$, where $w_2(P)$ is the second Stiefel-Whitney class of P. Then the moduli space $M^*_{X_0,P}(g_0)$ of irreducible g_0-anti-self-dual connections on P has finite volume and diameter with respect to the L^2 metric $\mathbf{g}$ defined by g_0.*

We plan to discuss the case of $G = \mathrm{SU}(2)$ and $b^+(X_0) > 0$ in a subsequent article. Note that when $G = \mathrm{SO}(3)$ and $w_2(P) \neq 0$, the trivial (product) connection Θ does not appear in the Uhlenbeck compactification $\overline{M}^u_{X,P}(g_0)$. By 'diameter' we mean the sum of the diameters

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of the connected components of $M^*_{X_0,P}(g_0)$; the hypotheses imply that $M^*_{X_0,P}(g_0)$ has finitely many path components. In [5] Donaldson conjectured that the L^2-metric completion of the moduli space coincides with the Uhlenbeck compactification [3], [7]. We announce here the following result whose proof is included in [9].

Theorem 1.2. *Under the hypotheses of* Theorem 1.1, *the completion of $M^*_{X_0,P}(g_0)$ with respect to the L^2 metric $\mathbf{g}$ is homeomorphic to the Uhlenbeck compactification $\overline{M}^u_{X_0,P}(g_0)$.*

The requirement that X_0 be simply-connected implies that the moduli space of flat connections consists of a single point representing the product connection over X_0. This assumption simplifies the description of the ends of the moduli spaces $M^*_{X_0,P}(g_0)$, but is not important in the derivation of bounds for the components of $\mathbf{g}$. We assume $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$ in order to appeal to the generic metric theorems of Freed and Uhlenbeck which ensure that the moduli space is a C^∞ manifold; otherwise, the bounds for $\mathbf{g}$ obtained in Chapter 5 hold for any compact Lie group. For the sake of clarity, we assume $G = \mathrm{SU}(2)$ for the remainder of the article and denote $M_{X_0,P}(g_0)$ by $M_{X_0,k}(g_0)$, where $c_2(P) = k \geq 0$ is the second Chern class.

(b) History. The properties of the L^2 metric have been investigated by many authors in recent years, but most extensively by Groisser and Parker. In particular, they have conducted detailed studies of its behaviour at the boundary of certain $k = 1$ moduli spaces. Explicit formulas for the components of $\mathbf{g}$ have been found by Doi, Matsumoto, and Matumoto [2], Groisser and Parker [13], and Habermann [15] when $k = 1$ and X_0 is the four-sphere $\mathbb{S}^4$ with its standard round metric g_1. Groisser conducted a similar study when X_0 is the complex projective space $\overline{\mathbb{C}P}^2$, equipped with the Fubini-Study metric g_{FS} [11]. Their formulas imply that these $k = 1$ moduli spaces have finite $\mathbf{g}$-volume and $\mathbf{g}$-diameter. More generally, Groisser and Parker have established Theorem 1.1 in the special case $k = 1$ [14]. They also obtained C^0 bounds for $\mathbf{g}$ in neighbourhoods of the reducible connections, the 'conical ends', for any $k \geq 1$. In [12], Groisser refined some of the $k = 1$ results obtained in [14]. It is worth recalling that the L^2 metric is not invariant with respect to conformal changes in the metric g_0 on X_0.

The approach of [14] does not appear to readily generalise to the case $k > 1$, since their method relies on Donaldson's collar map which gives a diffeomorphism from the 'bubbling end' of $M^*_{X_0,1}(g_0)$ to the collar $X_0 \times (0, \lambda_0)$. For this reason we adopt a quite different method which uses the gluing techniques of Taubes and Donaldson to construct a system of local coordinate charts covering the 'ends' of the moduli space. We then estimate the components of $\mathbf{g}$ with respect to these coordinates. In the case of the Weil-Petersson metric on Teichmüller space, estimates of this type have been obtained by Masur [16]. In [8], the author proved Theorem 1.1, when $X_0 = \mathbb{S}^4$ and $k = 2$, using the ADHM correspondence [7]. After the present work was submitted, a preprint was received from Peng giving L^2 estimates for the derivatives with respect to moduli parameters of the family of anti-self-dual connections A on the connected sum $X_0 \#_\lambda \mathbb{S}^4$ constructed in §7.2.2 of [7], with $H^2_{A_i} = 0$ [18]. His L^2 estimates are defined with respect to a family of metrics g_λ which are conformally equivalent to g_0 and which pinch the neck of the connected sum as $\lambda \to 0$; away from the neck g_λ coincides with g_0 on X_0 and it converges in C^∞ to the standard round metric on the unit sphere $\mathbb{S}^4$.

(c) Outline and strategy. It remains to summarise the methods used in the proofs of our main results. Let us first recall the definition of the L^2 metric. The tangent space $T_A M^*_{X_0,k}(g_0)$ is identified with the cohomology group $H^1_A = \ker d^{+,g_0}_A / \operatorname{im} d^{*,g_0}_A$. Given tangent vectors $[a]$, $[b]$, the L^2 metric $\mathbf{g}$ is defined by

$$\mathbf{g}_{[A]}([a],[b]) = (\pi_A a, \pi_A b)_{L^2(X_0,g_0)}, \tag{1.1}$$

where $\pi_A = 1 - d_A(d^{*,g_0}_A d_A)^{-1} d^{*,g_0}_A$ is the L^2 orthogonal projection from $L^2\Omega^1(X_0, \operatorname{ad} P)$ to the subspace $\ker d^{*,g_0}_A$. Clearly, $\mathbf{g}([a],[b])$ is bounded above by $\|a\|_{L^2}\|b\|_{L^2}$, and so a reasonable strategy is to seek upper bounds for $\mathbf{g}$ over the moduli space ends. This will suffice for our present application.

(i) Moduli space ends and the bubble tree compactification. Our first task is to describe useful models for the ends of the moduli space of anti-self-dual connections. Let $(A_0, x_1, \ldots, x_{m_0})$ be a point in the stratum $\overline{M}^u_{X_0,k}(g_0) \cap (M_{X_0,k_0}(g_0) \times s^{k-k_0}(X_0))$ of the Uhlenbeck compactification (see §4.1) which lies away from the diagonals of the symmetric product, so that $m_0 = k - k_0$ and each point x_i has multiplicity 1. Then every point $[A] \in M_{X_0,k}(g_0)$ which is close enough to $(A_0, x_1, \ldots, x_{m_0})$ in the Uhlenbeck topology can be shown to lie in a neighbourhood constructible by gluing or 'gluing neighbourhood' [3],

[7]. Thus, suppose $[A_\alpha]$ is a sequence in $M_{X_0,k}(g_0)$ which converges weakly to $(A_0, x_1, \dots, x_{m_0})$. As described in §4.2, the sequence of connections $[A_\alpha]$ produces sequences of local mass centres $x_{i\alpha}$ converging to the points x_i and sequences of local scales $\lambda_{i\alpha}$ converging to zero. Using the scales $\lambda_{i\alpha}$, one now dilates the metric g_0 around the points $x_{i\alpha}$ and produces a sequence of conformally equivalent, C^∞ metrics g_α on a connected sum $X \equiv X_0 \#_{i=1}^{m_0} \mathbb{S}^4$. As the scales $\lambda_{i\alpha}$ tend to zero, the corresponding neck is pinched and the connected-sum metrics g_α converge in C^∞ on compact subsets away from the neck regions to the metric g_0 on X_0 and the standard round metric g_1 (of radius 1) on each copy of $\mathbb{S}^4$. This 'conformal blow-up' procedure gives a sequence of g_α-anti-self-dual connections $[\breve{A}_\alpha]$ which converges *strongly* (in the sense of [7]) to a limit $(A_0, I_1, \dots, I_{m_0})$ over the join $X_0 \vee_{i=1}^{m_0} \mathbb{S}^4$, where the I_i are the standard one-instantons over $X_i = \mathbb{S}^4$ with centre at the north pole n and scale 1. Here, strong convergence means C^∞ convergence on compact sets away from the necks and such that $c_2(A_0) + \sum_{i=1}^{m_0} c_2(I_i) = k$; there are no singular points and there is no curvature loss over the necks. One obtains an open neighbourhood in $\overline{M}^u_{X_0,k_0}(g_0)$ of the boundary point $(A_0, x_1, \dots, x_{m_0})$ by gluing up the limit $(A_0, I_1, \dots, I_{m_0})$.

On the other hand, if the set $Z_0 \equiv (x_1, \dots, x_{m_0})$ lies in the diagonal of the symmetric product $s^{k-k_0}(X_0)$, the limiting behaviour of the sequence $[A_\alpha]$ may be rather more complicated. Suppose $[\breve{A}_\alpha]$ is the corresponding sequence of g_α-anti-self-dual connections over $X = X_0 \#_{i=1}^{m_0} \mathbb{S}^4$ produced by conformal blow-ups. The sequence $\breve{A}_\alpha$ converges in C^∞ on compact subsets of $X_0 \setminus Z_0$ to a g_0-anti-self-dual connection A_0 over X_0, but in general only converges weakly to an Uhlenbeck limit (A_i, Z_i) over the four-spheres $X_i \equiv \mathbb{S}^4$, where $Z_i = (x_{i1}, \dots, x_{im_i})$ is contained in $X_i \setminus \{s\}$ and s is the south pole. If the connection A_i, $i > 0$, is not flat, then the conformal blow-ups may be chosen so that its curvature density is *centred* in the sense of [23]; its mass centre lies at the north pole and has scale (essentially its 'standard deviation') equal to 1 (see §4.2).

Unless all the singular sets Z_i are empty, one can no longer produce an open subset of the moduli space $M_{X_0,k}(g_0)$ simply by gluing up the connections $(A_i)_{i=0}^{m_0}$; because of the nature of the convergence process, some of the required moduli parameters have been lost in the limit.

Instead, the above conformal blow-up process must be iterated. The idea of iterating conformal blow-ups has been suggested by Sacks and Uhlenbeck in the context of harmonic maps of $\mathbb{S}^2$ [19]. Taubes described an iterative scheme of this type which is used to analyse the limiting behaviour of sequences of connections with uniformly bounded Yang-Mills functional and functional gradient tending to zero [23]. Parker and Wolfson described a bubble tree compactification for pseudoholomorphic maps of Riemann surfaces into symplectic manifolds and noted that their method should apply to the case of Yang-Mills connections over four-manifolds [17].

For the problem at hand, by repeatedly applying conformal blow-ups, we obtain a sequence of g_α-anti-self-dual connections $\check{A}_\alpha$ over a large connected sum $X \equiv \#_{I\in\mathcal{I}} X_I$. Here, $\mathcal{I}$ is a set of multi-indices I obtained when the conformal blow-up process is iterated. Thus, $\mathcal{I}$ records the tree structure and if $I = 0$, then X_I is the four-manifold X_0, while if $I \neq 0$, then X_I is a copy of $\mathbb{S}^4$. The construction of the 'conformal blow-up maps' $f_{I\alpha}$ ensures that the blow-up process must be repeated at most k times in order to produce a sequence of connections $[\check{A}_\alpha]$ which converge strongly to a limit $(A_I)_{I\in\mathcal{I}}$ over a join $\vee_{I\in\mathcal{I}} X_I$, where A_0 is a g_0-anti-self-dual connection over X_0 and each A_I, for $I \neq 0$, is a g_1-anti-self-dual connection over $X_I = \mathbb{S}^4$. The sequence of metrics g_α converges in C^∞ on compact subsets away from the neck regions to the metric g_0 on X_0 and the standard round metric g_1 on each sphere X_I. This convergence scheme produces the 'bubble tree compactification' $\overline{M}^{\mathcal{T}}_{X_0,k_0}(g_0)$ and is described in §4.3.

In particular, bubble tree degeneration and gluing are inverse to one another in a natural way. Using the techniques of [7] one can now glue up the bubble tree limits $(A_I)_{I\in\mathcal{I}}$ to form g-anti-self-dual connections A over a connected sum $X \equiv \#_{I\in\mathcal{I}} X_I$, and construct open subsets of the moduli space $M_{X,k}(g)$ by small deformations of the limit data. The gluing procedure gives a collection of conformal maps f_I (from a small ball in a lower level summand X_{I_-} to the complement in the sphere X_I of a small ball around the south pole) defined in exactly the same way as the above conformal blow-up maps $f_{I\alpha}$. Here, g is a C^∞ metric on X, which is conformally equivalent to the old metric g_0 (via the maps f_I) and depends on the choice of gluing sites, frames in the principle SO(4) frame bundle FX_0, scales, and the metric g_0 on X_0; its construction and properties are discussed in §3.5. Similar metrics over connected sums are described in [3] and [24]. Pulling back via the blow-up maps then gives g_0-anti-self-dual connections $\hat{A}$ over X_0 and hence, produces open subsets of the moduli space $M_{X_0,k}(g_0)$.

Generalising the arguments in [3] and [7] and employing the compactness results of §4.3, one then shows that $\overline{M}^u_{X_0,k}(g_0)$ has a finite cover consisting of gluing neighbourhoods $\overline{\mathcal{V}}$. Of course, any precompact open subset of $M_{X_0,k}(g_0)$ is covered by finitely many Kuranishi charts, and these comprise the 'gluing charts' in this case. Moreover, the L^2-metric geometry near the reducible connections, the conical ends, has already been analysed by Groisser and Parker [14], so we may confine our attention to the more troublesome bubbling ends.

(ii) Upper bounds for the components of the L^2 metric. We now outline a method of computing estimates for the L^2 metric $\mathbf{g}$ over the ends of the moduli space. In §§3.3 and 3.4 we apply the techniques of [3] and [7] to first construct approximate gluing maps $\mathcal{J}' : \mathcal{T}^0/\Gamma \to \mathcal{B}^*_{X,k}$, $t \to [A'(t)]$. Here, X is the connected sum $\#_{I\in\mathcal{I}}X_I$ with C^∞ metric g conformally equivalent to g_0 on X_0, and $\mathcal{T}/\Gamma$ is a certain parameter space. If the g-self-dual curvature $F^{+,g}(A')$ is sufficiently small, one can then solve the g-anti-self-dual equation, $F^{+,g}(A'+a) = 0$, or equivalently

$$d^{+,g}_{A'}a + (a \wedge a)^{+,g} = -F^{+,g}(A'), \tag{1.2}$$

for $a \in \Omega^1(X, \operatorname{ad} P)$. This gives a C^∞ family of g-anti-self-dual connections $A \equiv A' + a$ and thus a gluing map $\mathcal{J} : \mathcal{T}^0/\Gamma \to M^*_{X,k}(g)$, $t \to [A(t)]$. The solutions a to Eq. (1.2) are expressed in the form $a = P\xi$, where $\xi \in \Omega^{+,g}(X, \operatorname{ad} P)$ and P is a right inverse to the operator $d^{+,g}_{A'}$ constructed (as in [7]) by patching together right inverses P_I for the operators $d^{+,g_I}_{A_I}$ over the summands X_I. Therefore, Eq. (1.2) takes the shape

$$\xi + (P\xi \wedge P\xi)^{+,g} = -F^{+,g}(A'). \tag{1.3}$$

Following [7], we assemble the framework required for solving Eq. (1.2) in §5.1.

Now the L^2 metric $\mathbf{g}$ depends on the choice of metric g_0, not just the conformal class $[g_0]$. So, using the conformal maps f_I, we pull back the family of g-anti-self-dual connections $A(t) = A'(t) + a(t)$ over X to an equivalent C^∞ family of g_0-anti-self-dual connections $\hat{A}(t) = \hat{A}'(t)+\hat{a}(t)$ over X_0. Hence, we obtain gluing maps $\hat{\mathcal{J}} : \mathcal{T}^0/\Gamma \to M^*_{X_0,k}(g_0)$, $t \to$

$[\hat{A}(t)]$ analogous to those constructed by Taubes. The properties of the gluing maps $\mathcal{J}$ and $\hat{\mathcal{J}}$ are discussed in §5.2.

The problem then is to estimate the differentials $D\hat{\mathcal{J}}$ and this task is comprised of two parts. The first part is to bound the derivatives $\partial A'/\partial t$; this local calculation is the subject of §§3.7 to 3.9 and the main results are summarised in §3.10. The more difficult part is to bound the derivatives of the correction terms, $\partial\hat{a}/\partial t$; this involves bounding the derivatives of global operators such as P and is described in §§5.3 to 5.5. The problem of expressing bounds for derivatives of $\hat{a}(t)$ in terms of bounds for derivatives of $a(t)$ is the subject of §3.5. Some care is required here, since the conformal maps f_I vary with the scale and centre parameters, as does the metric g in Eq. (1.2). The required estimates for the derivatives $\partial a/\partial t$ are then computed in §§5.3 to 5.5 in terms of bounds for $\partial P/\partial t$ and $\partial\xi/\partial t$; the estimates for $\partial\xi/\partial t$ are obtained implicitly from Eq. (1.2). For the special case of a neighbourhood of a point (A_0, A_1) (with $H^2_{A_0} = 0$), L^2 estimates for the derivatives $\partial A/\partial t$ were later obtained independently by Peng using similar methods [18].

It is the estimates for derivatives with respect to the scales λ_I which require the most care. For example, difficulties arise when bounding the derivatives $\partial\hat{A}'/\partial\lambda_I$ because of the dependence on λ_I of the conformal maps f_I and the cutoff functions required to patch the connections A_I together over the connected sum. These derivatives are ill-behaved as $\lambda_I \to 0$, and the necks of the connected sum X are pinched. Problems also occur when one attempts to bound $\partial a/\partial\lambda_I$, since $a = P\xi$ and the construction of P involves cutoff functions with badly behaved derivatives with respect to λ_I as $\lambda_I \to 0$. The final estimates for the differentials $D\hat{\mathcal{J}}$ and the corresponding bounds for the L^2 metric $\mathbf{g}$ are sumarised in §5.6. The constants appearing in the bounds for $\mathbf{g}$ depend only on the gluing neighbourhood. Theorem 1.1 then follows immediately from these estimates.

2. Preliminaries

In this Chapter we establish our notation and define the L^2 metric. Unless stated otherwise, we adhere to the standard conventions of [7]. For further details concerning gauge theory, we refer to [7] or [10] and the references therein, while for details concerning the L^2 metric, we refer to [13], [14].

Let X be a closed, connected, oriented, C^∞ four-manifold with Riemannian metric g and let P be a principal G bundle over X with Lie algebra $\mathfrak{g}$. As noted in the Introduction, we will generally confine our attention in this article to the case $G = \mathrm{SU}(2)$ for the sake of clarity. We let $\Omega^l(P, \mathfrak{g})$ denote the space of C^∞ $\mathfrak{g}$-valued l-forms, let $\operatorname{ad} P = P \times_{\mathrm{Ad}} \mathfrak{g}$ be the adjoint bundle, and let $\Omega^l(X, \operatorname{ad} P)$ be the space of C^∞ $\operatorname{ad} P$-valued l-forms on X. Let $\mathcal{A}_P$ be the affine subspace in $\Omega^1(P, \mathfrak{g})$ of C^∞ connection 1-forms on P. For a connection A on P, we let ∇_A be the corresponding covariant derivative, let d_A be the exterior covariant derivative, and let $F_A \in \Omega^2(X, \operatorname{ad} P)$ denote the curvature.

Let $\mathcal{G}_P$ be the group of C^∞ bundle automorphisms or gauge transformations. Recall that the isotropy group $\Gamma_A \subset \mathcal{G}_P$ of a connection A on P is isomorphic to the centraliser of the holonomy group of A in G, and the centre Z of the bundle structure group G is isomorphic to the centre of $\mathcal{G}_P$. Thus $\Gamma_A \supset Z$ and we let $\mathcal{A}_P^*$ be the dense open subset of connections $A \in \mathcal{A}_P$ with $\Gamma_A = Z$, so that $\mathcal{A}_P^*$ is the space of irreducible connections on P when $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$.

The bundles $\Lambda^l T^*X \otimes \operatorname{ad} P$ have fibre metrics $\langle\ ,\ \rangle$ induced by the Riemannian metric g on X and the inner product on the Lie algebra $\mathfrak{g}$ given by -1 times the Cartan-Killing form; if $\xi_1, \xi_2 \in \mathfrak{g}$, then $\langle \xi_1, \xi_2 \rangle = -\operatorname{tr}(\xi_1 \xi_2)$. In particular, we may define Sobolev spaces $L^p_n\Omega^l(X, \operatorname{ad} P)$ in the usual way and consider the action of the L^2_{n-1} gauge transformations $\mathcal{G}$ on the space of L^2_n connections $\mathcal{A}_P$ (for $n > 2$) with quotient $\mathcal{B}_P = \mathcal{A}_P/\mathcal{G}_P$, omitting the explicit Sobolev notation when no confusion can arise.

The tangent space $T_A\mathcal{A}_P^*$ is equal to $\Omega^1(X, \operatorname{ad} P)$ while the tangent space to the $\mathcal{G}$-orbit through $A \in \mathcal{A}_P^*$ is $\operatorname{im} d_A \subset \Omega^1(X, \operatorname{ad} P)$. This induces an L^2-orthogonal decomposition $T_A\mathcal{A}_P^* = \ker d_A^* \oplus \operatorname{im} d_A$, where $\ker d_A^* \subset \Omega^1(X, \operatorname{ad} P)$. There is an associated horizontal projection operator $\pi_A : T_A\mathcal{A}_P^* \to \ker d_A^*$, with $\pi_A = 1 - d_A G_A^0 d_A^*$, where G_A^0 is the Green's operator for the Laplacian $\Delta_A^0 = d_A^* d_A$. To identify the tangent space $T_{[A]}\mathcal{B}_P^*$, introduce C^∞ paths $A(t)$ in $\mathcal{A}_P^*$ and $u(t)$ in $\mathcal{G}_P$, $u(0) = 1$. If $A^u(t) \equiv u_t(A_t)$, then

$$\frac{dA^u}{dt} = \operatorname{Ad}(u^{-1})\frac{dA}{dt} + d_{A^u}\left(u^{-1}\frac{du}{dt}\right). \tag{2.1}$$

Thus $dA/dt(0)$ defines an element of $\Omega^1(X, \operatorname{ad} P)/\operatorname{im} d_A$ and therefore the tangent space $T_{[A]}\mathcal{B}_P^*$ is given by $\Omega^1(X, \operatorname{ad} P)/\operatorname{im} d_A \simeq \ker d_A^*$.

Let $M_P(g)$ be the moduli space of g-anti-self-dual connections on the G bundle P over X, that is $\{[A] \in \mathcal{B}_P : F^{+,g}(A) = 0\}$, and let $M_P^*(g)$ be the dense open subset $M_P(g) \cap \mathcal{B}_P^*$. If $A(t)$ is a C^∞ path in $\mathcal{A}_P$ satisfying $F^{+,g}(A(t)) = 0$, then $dA/dt(0)$ defines an element of $\ker d_A^*/\operatorname{im} d_A^{+,g}$. The g-anti-self-dual condition $F^{+,g}(A) = 0$ is equivalent to $d_A^{+,g} \circ d_A = 0$, and so we have the elliptic deformation complex

$$\Omega^0(X, \operatorname{ad} P) \xrightarrow{d_A} \Omega^1(X, \operatorname{ad} P) \xrightarrow{d_A^{+,g}} \Omega^{+,g}(X, \operatorname{ad} P) \tag{2.2}$$

with associated cohomology groups H_A^*, where H_A^0 is the Lie algebra of Γ_A, the group $H_A^1 = \ker d_A^{+,g}/\operatorname{im} d_A$ is just the tangent space $T_{[A]}M_P(g)$, and $H_A^2 = \operatorname{coker} d_A^{+,g}$. By Hodge theory there are natural isomorphisms $H_A^0 \simeq \ker \Delta_A^0$, $H_A^1 \simeq \ker d_A^* \cap \ker d_A^{+,g}$, and $H_A^2 \simeq \ker \Delta_A^{+,g}$, where the Laplacian $\Delta_A^{+,g}$ is equal to $d_A^{+,g}(d_A^{+,g})^*$.

If $[A]$ is an irreducible point of $M_P(g)$, then $H_A^0 = 0$, and an irreducible point $[A]$ is regular if $H_A^2 = 0$. The moduli space $M_P(g)$ is regular if all its irreducible points are regular points, and in that case, $M_P^*(g)$ is a C^∞ manifold of dimension

$$\dim M_P(g) = 8k(P) - 3(1 - b_1(X) + b^+(X)), \tag{2.3}$$

with tangent space $T_{[A]}M_P^*(g) = H_A^1$ at the point $[A]$.

According to the Freed-Uhlenbeck theorems, the anti-self-dual moduli spaces $M_P^*(g)$ are smooth manifolds when g is generic. More precisely, if $b^+(X) > 0$, P is any SU(2) or SO(3) bundle P over X, and the metric g on X is generic, then the following hold: (1) $M_P^*(g)$ contains no points $[A]$ with $H_A^2 \neq 0$. (2) If $b^+(X) > 0$ and $l > 0$, then $M_P(g)$ contains no points $[A]$ with $H_A^0 \neq 0$ for any bundle P with $0 < k(P) \leq l$. (3) If $b^+(X) = 0$ and P is non-trivial, then the cohomology groups H_A^2 are zero for all the reducible g-anti-self-dual connections A on P, and a neighbourhood of point $[A] \in M_P(g)$ with $H_A^0 \neq 0$ is homeomorphic to a cone over $\mathbb{C}P^{4k-2}$ and diffeomorphic away from the cone point $[A]$.

It remains to define the L^2 metric. The quotient space $\mathcal{B}_P^*$ inherits a (weak) Riemannian L^2 metric $\mathbf{g}$ by requiring that the projection map for the principal $\mathcal{G}_P/Z$ bundle $\mathcal{A}_P^* \to \mathcal{B}_P^*$ be a Riemannian submersion: if $[a], [b]$ are tangent vectors in $T_{[A]}\mathcal{B}_P^*$, then

$$\mathbf{g}_{[A]}([a],[b]) \equiv \int_X \langle \pi_A a, \pi_A b \rangle \, dV_g, \tag{2.4}$$

and this restricts to give a C^∞ Riemannian metric $\mathbf{g}$ on the moduli space $M_P^*(g)$.

3. Differentials of the approximate gluing maps

Our purpose in this Chapter is to construct the approximate gluing maps $\mathcal{J}' : \mathcal{T}/\Gamma \to \mathcal{B}_{X,k}^*$ and $\hat{\mathcal{J}}' : \mathcal{T}/\Gamma \to \mathcal{B}_{X_0,k}^*$, and to estimate the differentials $D\mathcal{J}'$, and especially $D\hat{\mathcal{J}}'$. The construction of $\mathcal{J}'$ uses the method employed by Donaldson in [3], [7]. The induced maps $\hat{\mathcal{J}}'$ are essentially the approximate gluing maps described by Taubes in [20], [21], [23]. In the former case, we obtain an almost g-anti-self-dual connection A' over a connected sum $X = X_0\#_{I\in\mathcal{I}}\mathbb{S}^4$ with metric g conformally equivalent to g_0 on X_0, while in the latter case we obtain an almost g_0-anti-self-dual connection $\hat{A}'$ over X_0 with its fixed metric g_0. In Chapter 5, we obtain a system of coordinate charts $\hat{\mathcal{J}} : \mathcal{T}/\Gamma \to M_{X_0,k}^*(g_0)$ covering the moduli space by perturbing the maps $\hat{\mathcal{J}}'$ using the techniques of [7] for solving the anti-self-dual equation.

3.1. Preliminary estimates for connections and curvature. We describe some pointwise estimates for local connection one-forms and curvature two-forms. We first consider estimates for connection one-forms in radial gauge on a C^∞ manifold X with C^∞ metric g. Suppose $P \to X$ is a principal G bundle, A is a C^∞ connection on P, and B is an open geodesic ball centred at $x_0 \in X$ with radius $\varrho/2$, where ϱ is the injectivity radius of (X,g). Define a C^∞ local section $\sigma : B \to P$ by parallel transport of a point in the fibre $P|_{x_0}$ along radial geodesics through x_0. If γ is a radial geodesic in B with $\gamma(0) = x_0$ and $\dot\gamma(t) = \xi_t$, then $\sigma^*A(x_0) = 0$ and $\iota_{\xi_t}\sigma^*A(\gamma(t)) = 0$, $t > 0$. If $\phi^{-1} : B \to \mathbb{R}^4$ is a geodesic normal coordinate system centred at x_0, and we define a geodesic γ by $\gamma(t) = \phi(tx)$, $x \in B$, $t \in [0,1]$, then $\gamma^\mu(t) = tx^\mu$, $\dot\gamma = \mathbf{x}$, and $\iota_{\mathbf{x}}\sigma^*A = x^\mu(\sigma^*A)_\mu$. We recall the following estimates for local connection one-forms in radial gauge.

Lemma 3.1. [25 (p. 14)] *Let A be a C^∞ connection on a principal G bundle $P \to X$, where X is a C^∞ manifold with C^∞ metric g, and let B be a geodesic ball of radius $\varrho/2$ centred at $x_0 \in X$, $\sigma : B \to P$ be a local section such that σ^*A is in radial gauge centred at x_0, and $\phi^{-1} : B \to \mathbb{R}^n$ be a geodesic normal coordinate system centred at x_0. If $K = \|F_A\|_{L^\infty(B,g)}$, then $|\phi^*\sigma^*A|_g(x) \le K|x|$, for $|x| < \varrho/2$.*

Let $\mathbb{H}P^1$ be the right quaternionic projective space, with the standard identifications $\mathbb{H} \simeq \mathbb{R}^4$ and $\mathbb{H}P^1 \simeq \mathbb{S}^4$. Coordinate patches for $\mathbb{S}^4$ may then be defined by $U_n = \{[x,y] : y \neq 0\} = \mathbb{S}^4 \setminus \{s\}$ and $U_s = \{[x,y] : x \neq 0\} = \mathbb{S}^4 \setminus \{n\}$ covering the north pole $n = [0,1]$ and south pole $s = [1,0]$, respectively. We let $\phi_n^{-1} : U_n \to \mathbb{R}^4$, $[x,y] \mapsto xy^{-1}$ and $\phi_s^{-1} : U_s \to \mathbb{R}^4$, $[x,y] \mapsto yx^{-1}$ denote the standard local coordinate charts. If g_1 is the standard round metric of radius 1 on $\mathbb{S}^4$, then

$$(3.1) \qquad (\phi_\alpha^* g_1)_{\mu\nu}(x) = h_1^2(x)\delta_{\mu\nu} = \frac{4}{(1+|x|^2)^2}\delta_{\mu\nu}, \qquad x \in \mathbb{R}^4,$$

for $\alpha = n, s$, where the standard flat metric on $\mathbb{R}^4$ is denoted by δ.

Let A be a C^∞ connection on a principal G bundle $P \to \mathbb{S}^4$, where $\mathbb{S}^4$ has its standard metric g_1. We define a system of local sections $\sigma_\alpha : U_\alpha \to P$, $\alpha = n, s$, by parallel transport of points in the fibres $P|_\alpha$ along radial geodesics through the north or south poles. The estimates below follow easily since A is smooth over $\mathbb{S}^4$ with metric g_1:

Lemma 3.2. *Let A be a C^∞ connection on a principal G bundle $P \to \mathbb{S}^4$, where $\mathbb{S}^4$ has metric g_1 and $K = \|F_A\|_{L^\infty(\mathbb{S}^4, g_1)}$. Then, for $\alpha, \beta \in \{n, s\}$,*

$$|\phi_\beta^* F(\sigma_\alpha^* A)|_\delta(x) \leq 4K\frac{1}{(1+|x|^2)^2} \quad \text{for} \begin{cases} x \in \mathbb{R}^4 & \text{if } \alpha = \beta, \\ x \in \mathbb{R}^4 \setminus \{0\} & \text{if } \alpha \neq \beta. \end{cases}$$

Lemma 3.3. *Given the hypotheses of* Lemma 3.2, *if the local connection one-forms $\sigma_\alpha^* A$ are in radial gauge, then $|\phi_\alpha^* \sigma_\alpha^* A|_{g_1}(x) \leq K|x|$, for $x \in \mathbb{R}^4$ and $\alpha = n, s$.*

3.2. Connections over the four-sphere and conformal diffeomorphisms. Recall that the group of conformal diffeomorphisms of $\mathbb{S}^4$ acts on the space $\mathcal{A}_P$ of C^∞ connections on a G bundle P over $\mathbb{S}^4$. The group $\mathbf{D} \times \mathbf{T}$ of dilations and translations of $\mathbb{R}^4$ may be identified with a subgroup of the conformal group of $\mathbb{S}^4$. Hence, in this section we discuss some aspects of the induced action of $\mathbb{R}^+ \times \mathbb{R}^4$ on the space $\mathcal{A}_P$. For related material we refer to [5], [10], [13], [14], and [23].

Let P be a G bundle with C^∞ connection $A \in \Omega^1(P, \mathfrak{g})$ over a C^∞ manifold X and suppose φ_t is a C^∞ one-parameter group of diffeomorphisms of X generating a vector field $\xi \in C^\infty(TX)$. Let $\tilde{\xi} \in C^\infty(TP)$ be the horizontal vector field covering ξ and let $\tilde{\varphi}_t$ be the one-parameter group of diffeomorphisms of P generated by $\tilde{\xi}$. Then $\tilde{\varphi}_t$ commutes with right G multiplication and covers φ. Fixing $\Omega \in \Omega^1(P, \mathfrak{g})$, we obtain a C^∞ one-parameter family of C^∞ one-forms $\tilde{\varphi}_t^*\Omega$ on P with

$$\left.\frac{d\tilde{\varphi}_t^*\Omega}{dt}\right|_{t=0} = \mathcal{L}_{\tilde{\xi}}\Omega, \tag{3.2}$$

where $\mathcal{L}_{\tilde{\xi}}\Omega \in \Omega^1(P, \mathfrak{g})$ denotes the Lie derivative of Ω with respect to $\tilde{\xi}$; in particular, $\tilde{\varphi}_t^*A$ is a C^∞ one-parameter family of C^∞ connection one-forms on P.

Lemma 3.4. *Let P be a G bundle with connection $A \in \Omega^1(P, \mathfrak{g})$ over a manifold X. Given a vector field $\xi \in C^\infty(TX)$, let $\tilde{\xi} \in C^\infty(TP)$ be its horizontal lift. If $F_A \in \Omega^2(P, \mathfrak{g})$ is the curvature of A, then $\mathcal{L}_{\tilde{\xi}}A = \iota_{\tilde{\xi}}F_A$.*

Proof. Since $\tilde{\xi}$ is horizontal, $A(\tilde{\xi}) = 0$ and so for any vector field $\eta \in TP$, we have $(\mathcal{L}_{\tilde{\xi}}A)(\eta) = (\iota_{\tilde{\xi}}dA + d\iota_{\tilde{\xi}}A)(\eta) = dA(\eta, \tilde{\xi})$. But $F_A(\eta, \tilde{\xi}) = dA(\eta, \tilde{\xi}) + \frac{1}{2}[A(\eta), A(\tilde{\xi})]$ and so the result follows.

We also need to consider Lie derivatives of $\operatorname{ad} P$-valued one-forms. Recall that if $\pi : P \to X$ is the bundle projection, there is an injective map $\pi^* : \Omega^1(X, \operatorname{ad} P) \hookrightarrow \Omega^1(P, \mathfrak{g})$. The one-forms Ω in the image of π^* are characterised by the properties (a) $R_u^*\Omega = \operatorname{Ad}(u^{-1})\Omega$, for all $u \in G$, and (b) $\Omega(\eta) = 0$ if $\eta \in TP$ is vertical. Hence, the action of $\tilde{\varphi}_t$ on $\Omega^1(P, \mathfrak{g})$ induces an action on $\Omega^1(X, \operatorname{ad} P) = \Gamma(T^*X \otimes \operatorname{ad} P)$. Thus, if $\omega \in \Omega^1(X, \operatorname{ad} P)$, we obtain a C^∞ one-parameter family of C^∞ $\operatorname{ad} P$-valued one-forms $\tilde{\varphi}_t^*\omega$ on X with

$$\left.\frac{d\tilde{\varphi}_t^*\omega}{dt}\right|_{t=0} = \mathcal{L}_{\tilde{\xi}}\omega, \tag{3.3}$$

where $\mathcal{L}_{\tilde{\xi}}\omega \in \Omega^1(X, \operatorname{ad} P)$ denotes the Lie derivative of ω with respect to $\tilde{\xi}$.

For the purposes of calculation, it is useful to phrase the preceding discussion in terms of local one-forms on X. It is convenient to choose a system of local sections $\sigma_\alpha : U_\alpha \to P$ which are *parallel* with respect to the connection A and vector field ξ, in the sense that $A(\sigma_{\alpha *}\xi) = 0$. For example, one can try to construct σ_α by first choosing a section $\sigma_\alpha|_{V_\alpha}$, where V_α is a submanifold of U_α transverse to the vector field ξ, and then extend by parallel translation along integral curves of ξ to construct a section σ_α over a tubular neighbourhood U_α of V_α. Local sections of this type are described in [10 (pp. 146-147)] and [25 (pp. 14-15)].

Given a system of (A,ξ)-parallel local sections σ_α, we have $\tilde{\xi} = \sigma_{\alpha *}\xi$ and $\varphi_t = \sigma_\alpha^* \tilde{\varphi}_t$ over U_α. Hence, for $\omega \in \Omega^1(X, \operatorname{ad} P)$ we see that $\sigma_\alpha^* \tilde{\varphi}_t^* \omega = \varphi_t^* \sigma_\alpha^* \omega$ and $\sigma_\alpha^* \mathcal{L}_{\tilde{\xi}} \omega = \mathcal{L}_\xi \sigma_\alpha^* \omega$ on U_α, and similarly for $A \in \Omega^1(P, \mathfrak{g})$. Indeed, one can see that the transition functions $\{u_{\alpha\beta}\}$ are constant along the vector field ξ. For if $\sigma_\beta = \sigma_\alpha u_{\alpha\beta}$, then $\sigma_{\beta *}\xi = \sigma_{\alpha *}\xi \cdot u_{\alpha\beta} + \sigma_\alpha \cdot u_{\alpha\beta *}\xi$, which gives $A(\sigma_{\beta *}\xi) = \operatorname{Ad}(u_{\alpha\beta}^{-1}) A(\sigma_{\alpha *}\xi) + A(\sigma_\alpha \cdot u_{\alpha\beta *}\xi)$, and thus $du_{\alpha\beta}(\xi) = 0$, since $A(\sigma_{\alpha *}\xi) = A(\sigma_{\beta *}\xi) = 0$ and $A(\sigma_\alpha \cdot u_{\alpha\beta *}\xi) = u_{\alpha\beta *}\xi$. Here, $\sigma_\alpha \cdot u_{\alpha\beta *}\xi$ is the vector field on $P|_{U_\alpha}$ obtained by differentiating the maps $G \to P$ given by $u \mapsto \sigma_\alpha(x)u$. When computing Lie derivatives of local connection one-forms or $\operatorname{ad} P$-valued one-forms with respect to a vector field ξ, we shall always require that the local sections σ_α be (A,ξ)-parallel.

It is often useful to express $\mathcal{L}_{\tilde{\xi}}\omega$ in terms of covariant derivatives. Suppose X has a C^∞ metric g. We have $\mathcal{L}_\xi \omega = \iota_\xi d\omega + d\iota_\xi \omega$, or in local coordinates, $(\mathcal{L}_\xi \omega)_\mu = \xi^\nu \partial\omega_\mu/\partial x^\nu + \omega_\nu \partial \xi^\nu/\partial x^\mu$. We find that

$$\mathcal{L}_{\tilde{\xi}}\omega = \nabla_\xi^{A,g}\omega + \omega(\nabla^g \xi), \tag{3.4}$$

using normal geodesic coordinates $\{x^\mu\}$ and (A,ξ)-parallel local sections $\{\sigma_\alpha\}$. In the sequel, we omit the "tildes" to indicate lifts of vector fields or diffeomorphisms on the base to the total space of a principal bundle, this being understood from the context. Note that if $\Phi : X \to X$ is a diffeomorphism and $\omega \in \Omega^1(X, \operatorname{ad} P)$, then we have $\mathcal{L}_\xi \Phi^* \omega = \Phi^* \mathcal{L}_{\Phi_* \xi}\omega$.

Let A be a C^∞ connection on a G bundle P over $\mathbb{S}^4$ and let $\omega \in \Omega^1(\mathbb{S}^4, \operatorname{ad} P)$. For any $t \in (-\infty, \infty)$, let δ_t be the dilation of $\mathbb{R}^4$ given by $x \mapsto e^t x$, and for any $p \in \mathbb{R}^4$ let τ_p be the translation of $\mathbb{R}^4$ defined by $\tau_p : x \mapsto x - p$. If δ_t and τ_p again denote the conformal diffeomorphisms of $\mathbb{S}^4$ induced by the chart $x = \phi_n^{-1}$, then the group $\mathbf{C} = \mathrm{SO}(4) \times \mathbf{D} \times \mathbf{T}$ of rotations, dilations, and translations of $\mathbb{R}^4$ is identified with the subgroup in $\operatorname{Conf}(\mathbb{S}^4, g_1)$ of diffeomorphisms which fix the south pole $s \in \mathbb{S}^4$. Setting $\varphi_t = \delta_t$ or τ_{tp}, we see that these diffeomorphisms are generated by the vector fields

$$\mathbf{r} \equiv x^\mu \frac{\partial}{\partial x^\mu} \quad \text{and} \quad -\mathbf{p} \equiv -p^\mu \frac{\partial}{\partial x^\mu}. \tag{3.5}$$

We always choose $p \in \mathbb{R}^4$ with $|p| \leq 1$. We next describe the construction of (A,ξ)-parallel local sections σ_α for $\xi = \mathbf{r}$ or $\mathbf{p}$.

Considering the group of dilations $\mathbf{D}$, let σ_n, σ_s be the local sections formed by choosing points in the fibres $P|_n, P|_s$ and then parallel translating along radial directions from the poles. The transition function u will be constant along the radial directions, $du(\mathbf{r}) = 0$, and the local connection one-forms $\sigma_\alpha^* A$ are in *radial* gauge. On the other hand, considering the group of translations $\mathbf{T}$, suppose first that $\mathbf{p} = \partial/\partial x^4$ and let $\sigma_n|_{\mathbb{S}^3}, \sigma_s|_{\mathbb{S}^3}$ be the local sections formed by parallel translation from the north and south poles of the three-sphere $\mathbb{S}^3 \subset \mathbb{S}^4$ defined by the image of the $x^1x^2x^3$-plane under the map $\phi_n : \mathbb{R}^4 \to \mathbb{S}^4 \setminus \{s\}$. We obtain local sections σ_n, σ_s by parallel translation along the x^4-axis. The transition function u will now be constant along the x^4-axis, so $du(\mathbf{p}) = 0$, and the local connection one-forms $\sigma_\alpha^* A$ are in a *transverse* gauge. By a linear change of coordinates, the same argument applies to arbitrary translations.

For the dilations, we have

$$\left.\frac{d\delta_t^* \omega}{dt}\right|_{t=0} = \mathcal{L}_{\mathbf{r}}\omega = \iota_{\mathbf{r}} d\omega + \omega, \tag{3.6}$$

using $\mathcal{L}_\xi \omega = \iota_\xi d\omega + d\iota_\xi \omega$, or in local coordinates, $(\mathcal{L}_\xi \omega)_\mu = \xi^\nu \partial\omega_\mu/\partial x^\nu + \omega_\nu \partial \xi^\nu/\partial x^\mu$. Similarly, for the translations we have

$$\left.\frac{d\tau_{tp}^* \omega}{dt}\right|_{t=0} = -\mathcal{L}_{\mathbf{p}}\omega = -\iota_{\mathbf{p}} d\omega, \tag{3.7}$$

where $\mathbf{p} = p^\mu \partial/\partial x^\mu$.

For any $\lambda \in (0, \infty)$, let c_λ be the diffeomorphism of $\mathbb{S}^4$ defined by the chart $x = \phi_n^{-1}$ and the dilation c_λ of $\mathbb{R}^4$ given by $x \mapsto x/\lambda$. Then $c_\lambda = \delta_t$ with $t = -\log\lambda$, and so from Eq. (3.6) we have $\frac{\partial}{\partial\lambda} c_\lambda^* \omega = -\frac{1}{\lambda} c_\lambda^* \mathcal{L}_{\mathbf{r}} \omega$. Similarly, for the translations τ_q, $q \in \mathbb{R}^4$, we see that Eq. (3.7) gives $\frac{\partial}{\partial p}\tau_q^* \omega = -\tau_q^* \mathcal{L}_{\mathbf{p}} \omega$, where $\partial/\partial p \equiv p^\mu \partial/\partial q^\mu$ on the left-hand side and using $\tau_{q+tp} = \varphi_{tp} \circ \tau_q$ on the right. Combining these actions, we find that

$$\frac{\partial}{\partial\lambda}\tau_q^* c_\lambda^* \omega = -\frac{1}{\lambda}\tau_q^* c_\lambda^* \mathcal{L}_{\mathbf{r}}\omega \quad \text{and} \quad \frac{\partial}{\partial p}\tau_q^* c_\lambda^* \omega = -\frac{1}{\lambda}\tau_q^* c_\lambda^* \mathcal{L}_{\mathbf{p}}\omega. \tag{3.8}$$

Similarly, considering the action of the dilations c_λ and translations τ_q on connection one-forms, we have

$$\frac{\partial}{\partial\lambda}\tau_q^* c_\lambda^* A = -\frac{1}{\lambda}\tau_q^* c_\lambda^* \iota_{\mathbf{r}} F_A \quad \text{and} \quad \frac{\partial}{\partial p}\tau_q^* c_\lambda^* A = -\frac{1}{\lambda}\tau_q^* c_\lambda^* \iota_{\mathbf{p}} F_A. \tag{3.9}$$

These derivative formulas play a significant role in the sequel.

It is convenient at this point to recall Taubes' definition of a centred connection over the four-sphere [23 (p. 343)]. Let A be a g_1-anti-self-dual connection on a G bundle P with $c_2(P) = k$ over $\mathbb{S}^4$, where $\mathbb{S}^4$ has its standard metric g_1. Pulling back via the chart $x = \phi_n^{-1} : \mathbb{S}^4 \setminus \{s\} \to \mathbb{R}^4$, we obtain a δ-anti-self-dual connection A on a G bundle P over $\mathbb{R}^4$ with its standard metric δ. Let Θ denote the flat connection on the product bundle. Suppose $A \neq \Theta$; then the *mass centre* q and *scale* λ are defined by

$$\begin{aligned} q &= \mathrm{Centre}[A] \equiv \frac{1}{8\pi^2 k}\int_{\mathbb{R}^4} x|F_A|^2\, d^4x, \\ \lambda^2 &= \mathrm{Scale}^2[A] \equiv \frac{1}{8\pi^2 k}\int_{\mathbb{R}^4} |x-q|^2|F_A|^2\, d^4x \end{aligned} \tag{3.10}$$

If $A = \Theta$, we set $\mathrm{Centre}[A] = 0$ and $\mathrm{Scale}[A] = 0$. The connection A is called *centred* if $\mathrm{Centre}[A] = 0$ and $\mathrm{Scale}[A] = 1$. Eq. (3.10) leads to the following *Tchebychev ineqality*:

$$\int_{|x-q|\geq R\lambda} |F_A|^2\, d^4x \leq 8\pi^2 k R^{-2}, \qquad R \geq 1. \tag{3.11}$$

Hence, the ball $B(q, R\lambda)$ contains A-energy greater than or equal to $8\pi^2 k(1 - R^{-2})$.

Setting $f_{\lambda,q} = c_\lambda \circ \tau_q$, we see that $\mathrm{Centre}[(f_{\lambda,q}^{-1})^*A] = 0$ and $\mathrm{Scale}[(f_{\lambda,q}^{-1})^*A] = 1$. Let M_k denote the moduli space of g_1-anti-self-dual connections on the bundle P over $\mathbb{S}^4$ and let M_k^0 denote the *moduli space of centred g_1-anti-self-dual connections*. Note that M_1^0 consists of a single point representing the standard one-instanton over $\mathbb{S}^4$. More generally, the relationship between M_k and M_k^0 is explained below.

Proposition 3.5. *For any $k > 0$, the space M_k^0 is a smooth submanifold of M_k. Moreover, M_k is diffeomorphic to $M_k^0 \times \mathbb{R}^4 \times (0, \infty)$.*

Proof. One argues as in [23 (pp. 343-344)] and [22 (pp. 365-367)]. Given $[A] \in M_k$ with $\mathrm{Centre}[A] = q$ and $\mathrm{Scale}[A] = \lambda$, set $f_{\lambda,q} = c_\lambda \circ \tau_q$. The map $[A] \to ([(f_{\lambda,q}^{-1})^*A], q, \lambda)$ then gives the required diffeomorphism.

3.3. Gluing construction of approximately anti-self-dual connections. We describe the approximate gluing constructions of Donaldson [3], [7], and Taubes [20], [21], [23], adapted to the case of 'bubble trees'. For clarity, we first discuss the construction of approximately anti-self-dual connections over single connected sums. Let X_0 be our closed, smooth four-manifold with metric g_0 and injectivity radius ϱ_0, and let $X_1 = \mathbb{S}^4$ with its standard round metric g_1 of radius 1. Let x_1 be a point in X_0 and let x_{1n}, x_{1s} denote the north and south poles of X_1. Let $P_i \to X_i$ be principal G bundles with $c_2(P_i) = k_i$, $i = 0, 1$. Let FX_0 be the principle SO(4) bundle of oriented, orthonormal frames over X_0.

A choice of frame $v_1 \in FX_0|_{x_1}$ defines a geodesic normal coordinate system $\phi_1^{-1} = \exp_{v_1}^{-1} : B_1(\varrho_0) \to \mathbb{R}^4$. Denote $\phi_{1\alpha} = \phi_\alpha$, $\alpha = s, n$, where $\phi_\alpha^{-1} : U_\alpha = \mathbb{S}^4 \setminus \{\alpha\} \to \mathbb{R}^4$ are the standard coordinate charts on the four-sphere. Let $B_1(r) = B(x_1, r)$ be the open geodesic ball in X_0 with centre x_1 and radius r, and let $B_{1s}(r) = \phi_{1s}(\{x \in \mathbb{R}^4 : |x| < r\})$, an open ball in X_1 with centre x_{1s}. Let $\Omega_1(r, R) = \Omega(x_1, r, R)$ be the open annulus $B_1(R) \setminus \overline{B}_1(r)$ centred at $x_1 \in X_0$, with inner radius r and outer radius R; similarly, let $\Omega_{1s}(r, R) = \Omega(x_{1s}, r, R)$ be the open annulus $B_{1s}(R) \setminus \overline{B}_{1s}(r)$ in X_1.

Let $N > 4$ be a large parameter, to be fixed later, and let $\lambda_1 > 0$ be a small scale parameter such that $\lambda_1^{1/2} N \ll 1$. We define open sets $X_0' = X_0 \setminus \overline{B}_1(N^{-1}\lambda_1^{1/2})$, $X_0'' = X_0 \setminus \overline{B}_1(\frac{1}{2}\lambda_1^{1/2})$, and $X_0''' = X_0 \setminus \overline{B}_1(2N\lambda_1^{1/2})$ — the complements in X_0 of small balls around the point x_1. Likewise, define open sets X_1', X_1'', and X_1''' in the sphere X_1. Let Ω_1 denote the annulus $\Omega_1(N^{-1}\lambda_1^{1/2}, N\lambda_1^{1/2})$ in X_0 and let $\Omega_{1s} = \Omega_{1s}(N^{-1}\lambda_1^{1/2}, N\lambda_1^{1/2})$ be the corresponding annulus in X_1. Let c_1 be the dilation map on $\mathbb{R}^4$ defined by $x \mapsto x/\lambda_1$. Define balls $B_1' = B_1(N\lambda_1^{1/2})$ and $B_1'' = B_1(2\lambda_1^{1/2})$ centred at x_1 in X_0 and a diffeomorphism

$$f_1 = \phi_{1n} \circ c_1 \circ \phi_1^{-1} : B_1' \longrightarrow X_1'. \tag{3.12}$$

Hence, f_1 identifies the small balls B_1' and B_1'' in X_0 with the open sets X_1' and X_1'' in X_1, and restricts to a diffeomorphism $f_1 : \Omega_1 \to \Omega_{1s}$.

We let X be the connected sum $X_0 \#_{f_1} X_1$. In §3.5 we define a smooth metric g on X which closely approximates the metrics g_i on each summand X_i' and such that the map $f_1 : B_1' \to X_1'$ is conformal. Thus, (X, g) is conformally equivalent to (X_0, g_0).

Let A_i be g_i-anti-self-dual connections on the bundles $P_i \to X_i$, $i = 0, 1$. The connections A_0, A_1, together with a choice of points in the fibres $P_0|_{x_1}, P_1|_{x_{1s}}$, define local sections $\sigma_1 : B_1(\varrho_0) \to P_0$ and $\sigma_{1s} : X_1 \setminus \{x_{1n}\} \to P_1$ by parallel transport along radial geodesics through x_1, x_{1s}. Hence, we obtain local trivialisations $P_0|_{B_1} \simeq B_1 \times G$ and $P_1|_{B_{1s}} \simeq B_{1s} \times G$.

Let $b_1 \geq 4N\lambda_1^{1/2}$ be a small parameter, $b_1 < \frac{1}{4}\min\{1, \varrho_0\}$; we will eventually set $b_1 = 4N\lambda_1^{1/2}$. Choose cutoff functions ψ_i on X_i such that $0 \leq \psi_i \leq 1$, with $\psi_0 = 1$ on $X_0 \setminus B_1(b_1)$, $\psi_0 = 0$ on $B_1(b_1/2)$, and similarly for ψ_1 on X_1. We let $A_0' = \psi_0 A_0$ be the C^∞ connection on the bundle $\pi_0 : P_0 \to X_0$ defined by

$$A_0' = \begin{cases} A_0 & \text{on } P_0|_{X_0 \setminus B_1(b_1)}, \\ \pi_0^*(\psi_0 \sigma_1^* A_0) & \text{on } P_1|_{B_1(b_1)}. \end{cases} \tag{3.13}$$

Of course, we have the analogous definition for the C^∞ connection A_1' over X_1, and we obtain almost anti-self-dual connections which are flat on the balls B_1', B_{1s}'.

To construct the cutoff functions ψ_i, choose a C^∞ bump function ζ on $\mathbb{R}^1$ such that $\zeta(t) = 1$ for $t \geq 1$ and $\zeta(t) = 0$ for $t \leq 1/2$. Define a C^∞ cutoff function ψ_b on $\mathbb{R}^4$ by $\psi_b(x) = \zeta(|x|/b)$, for any $b > 0$. Set $\psi_0 = (\phi_1^{-1})^*\psi_{b_1}$ and extend by 1 on $X_0 \setminus B(x_1, b_1)$ and by zero on $B(x_1, b_1/2)$ to give $\psi_0 \in C^\infty(X_0)$; likewise, set $\psi_1 = (\phi_{1s}^{-1})^*\psi_{b_1}$ and extend to give $\varphi_1 \in C^\infty(X_1)$. Each φ_i extends by zero to give a C^∞ cutoff function on the connected sum X.

Choose a G-equivariant isomorphism $\rho_1 \in \mathrm{Gl}_{x_1}$, where $\mathrm{Gl}_{x_1} \equiv \mathrm{Hom}_G(P_0|_{x_1}, P_1|_{x_{1s}}) \simeq G$ is the space of 'gluing parameters'. Using the connections A_i over the small $\frac{1}{2}b_1$-balls, spread out the fibre isomorphism ρ_1 to give a bundle isomorphism $\tilde\rho_1 : P_0|_{\Omega_1} \to P_1|_{\Omega_{1s}}$ covering the diffeomorphism $f_1 : \Omega_0 \to \Omega_1$. Thus, $\sigma_1\tilde\rho_1 = f_1^*\sigma_{1s}$ on Ω_1. We define the smooth connected-sum bundle $P \to X$ with second Chern class $c_2(P) = k = k_0 + k_1$ by setting $P|_{X_0'} = P_0|_{X_0'}$ and $P|_{X_1'} = P_1|_{X_1'}$. Note that the bundle P is defined by transition functions independent of the scale λ_1. We define a smooth connection $A' = A_0' \# A_1'$ on $P \to X$ by setting $A' = A_i'$ on each summand X_i'.

If Γ_{A_i} are the isotropy groups of the connections A_i, and $\Gamma = \Gamma_{A_0} \times \Gamma_{A_1}$, then we recall that the gluing construction gives a bijection between the gauge equivalence classes $[A'(\rho_1)]$ in $\mathcal{B}_{X,k}$ and Gl_{x_1}/Γ [7 (p. 286)].

Using the diffeomorphism $f_1 : B_1' \to X_1'$, we pull back the bundle P over X to a bundle $\hat P$ over X_0, given by $\hat P|_{X_0'} = P_0|_{X_0'}$ and $\hat P|_{B_1'} = f_1^* P_1|_{B_1'}$. We have an induced system of local sections of $\hat P|_{B_0'}$ given near x_1 by $\hat{\sigma}_{1n} = f_1^*\sigma_{1n} : B_1' \to P$, $\hat{\sigma}_{1s} = f_1^*\sigma_{1s} : B_1' \setminus \{x_1\} \to P$, and $\hat{\sigma}_1 = \sigma_1 : \Omega_1(N^{-1}\lambda_1^{1/2}, \varrho_0) \to P$. The corresponding transition functions $\hat{u}_1 = f_1^* u_1 : B_1' \setminus \{x_0\} \to G$ and $\tilde{\rho}_1 : \Omega_1 \to G$ are determined by $\hat{\sigma}_{1s} = \hat{\sigma}_{1n}\hat{u}_1$ on $B_1' \setminus \{x_0\}$ and $\sigma_1\tilde{\rho}_1 = f_1^*\hat{\sigma}_{1s}$ on Ω_1.

On the pull-back bundle $\hat{P} \to X_0$ we define the corresponding smooth pull-back connection $\hat{A}'$ by setting $\hat{A}' = A_0'$ on $\hat{P}|_{X_0'}$ and $\hat{A}' = f_1^* A_1'$ on $\hat{P}|_{B_1'}$. We obtain local connection 1-forms for $\hat{A}'$ over X_0 given by $\hat{\sigma}_{1n}^*\hat{A}' = f_1^*\sigma_{1n}^* A_1'$ on the ball B_1', $\hat{\sigma}_1^*\hat{A}' = \sigma_1^* A_0'$ on the annulus $\Omega_1(N^{-1}\lambda_1^{1/2}, \varrho_0)$, and $\hat{\sigma}_{1s}^* A' = f_1^*\sigma_{1s}^* A_1'$ on the punctured ball $B_1' \setminus \{x_1\}$.

On the annulus Ω_1 we have $\hat{\sigma}_{1s}^*\hat{A}' = \sigma_1^*\hat{A}' = 0$, and since

$$\hat{\sigma}_{1s}^*\hat{A}' = \tilde{\rho}_1^{-1}\hat{\sigma}_1^*\hat{A}_0'\tilde{\rho}_1 + \tilde{\rho}_1^{-1} d\tilde{\rho}_1 \quad \text{on } \Omega_1, \tag{3.14}$$

we see that $d\tilde{\rho}_1 = 0$ on Ω_1 and so $\tilde{\rho}_1$ is constant on Ω_1. The transition function $\hat{u}_1$ on $B_1' \setminus \{x_0\}$ is independent of λ_1, since u_1 on $X_1 \setminus \{x_{1n}, x_{1s}\}$ is constant along geodesics connecting the north and south poles. Thus, the bundle $\hat{P}$ is defined by transition functions which are constant with respect to λ_1.

We now generalise the preceding discussion to give a construction of approximately anti-self-dual connections over multiple connected sums. The description we give here is closely related to Taubes' iterated gluing construction [23 (§4)]. The construction parallels the description of the ends of the bubble tree compactification $M_{X_0,k}(g_0)$ described in Chapter 4.

It is convenient at this point to introduce some terminology. Let $I = (i_1, \dots, i_r)$ denote a multi-index of positive integers. The *length* of I is r; we regard 0 as a multi-index of length zero. Given $I = (i_1, \dots, i_r)$, we let $I_- = (i_1, \dots, i_{r-1})$; we will often denote a multi-index of the form $(i_1, \dots, i_{r+1})$ by I_+ or if we wish to be more specific, by Ij, where $j = i_{r+1} > 0$ or s, n (indicating north or south poles of $\mathbb{S}^4$), with a slight abuse of notation. Let $\mathcal{I}$ be an oriented tree with a finite set of vertices $\{I\}$, including a base vertex 0, and a set of edges $\{(I, I_+)\}$. If $I = (i_1, \dots, i_r)$ and $I = (j_1, \dots, j_t)$, then we say $I < J$ if $r < t$ and $J = (i_1, \dots, i_r, j_{r+1}, \dots, j_t)$. The valence of each vertex I is the number of edges emanating from that vertex. The height of the tree $\mathcal{I}$ is the number of levels — the length of the longest multi-index minus one. With respect to a given vertex I, the edge (I_-, I) is called *incoming*, and the edge (I, I_+) *outgoing*.

The construction of a C^∞, approximately g-anti-self-dual connection A' of second Chern class $k \geq 1$, associated with a tree $\mathcal{I}$, requires the following data:

Data 3.6. *Gluing data for approximately anti-self-dual connections.*

(1) To each *vertex* I, we associate a g_I-anti-self-dual connection A_I on a G bundle $P_I \to X_I$ with $c_2(P_I) = k_I \geq 0$. If $I = 0$, then X_0 is the base four-manifold with metric g_0, while if $I > 0$, then $X_I = \mathbb{S}^4$ with its standard round metric $g_I \equiv g_1$ of radius 1.

(2) To each *edge* (I_-, I), we associate the data $(b_I, \lambda_I, \rho_I, x_I, v_I)$ given by the

  (i) Connection cutoff parameter b_I.
  (ii) Scale parameter λ_I.
  (iii) Bundle gluing parameter $\rho_I \in \mathrm{Gl}_{x_I}$, where $\mathrm{Gl}_{x_I} = \mathrm{Hom}(P_{I_-}|_{x_I}, P_I|_{x_{Is}})$.
  (iv) Centre or gluing site $x_I \in X_{I_-}$.
  (v) Frame $v_I \in FX_0|_{x_I}$ if $I_- = 0$.

(3) Constants b_0, d_0, λ_0, N.

For convenience, if $I_+ = Is$, we denote $b_{Is} = b_I$, $\lambda_{Is} = \lambda_I$, $N_{Is} = N$, and $\rho_{Is} = \rho_I$. We let x_{In}, x_{Is} denote the north and south poles of the spheres $X_I = \mathbb{S}^4$. If $I_- > 0$, then $x_I \equiv \phi_{I_-n}(q_I) \in X_I$, where $q_I \in \mathbb{R}^4$. Define

$$\bar{b} = \max_{I \in \mathcal{I}} b_I \quad \text{and} \quad \bar{\lambda} = \max_{I \in \mathcal{I}} \lambda_I. \tag{3.15}$$

The gluing data should satisfy the following constraints:

Condition 3.7. *Gluing data constraints.*

(1) Scales: $4N\lambda_I^{1/2} \leq b_I < \frac{1}{4}\min\{1, \varrho_0, d_0\}$, $4 < N_0 \leq N$, and $0 < \lambda_I \leq \lambda_0$.

(2) Separation of centres: Suppose $x_I, x_{I'} \in X_{I_-}$.

  (i) If $I_- = 0$, then $\mathrm{dist}_{g_0}(x_I, x_{I'}) > 4(b_I + b_{I'})$.
  (ii) If $I_- > 0$, then $|q_I - q_{I'}| > 4(b_I + b_{I'})$.

(3) Topology: $\sum_{I \in \mathcal{I}} k_I = k$ and $k_I > 0$ for some $I > 0$.

Remark 3.8. Definition 3.6, together with the constraints of Condition 3.7 should be compared with the definition of 'bubble tree ideal' connections in §4.3. The requirements on the scales and separation of centres are in place simply to ensure that the different gluing regions do not interfere with one another.

The gluing procedure now generalises to give a C^∞ family of approximately g-anti-self-dual connections $A' = \#_{I\in\mathcal{I}} A'_I$ on a bundle P over a multiple connected sum $X = \#_{I\in\mathcal{I}} X'_I$. First, consider the definition of coordinate charts, open balls, and annuli in X_0. If $I_- = 0$, let $\phi_I^{-1} = \exp_{v_I}^{-1} : B(x_I, \varrho_0) \to \mathbb{R}^4$ be a geodesic normal coordinate chart defined by a point v_I in the oriented frame bundle fibre $FX_0|_{x_I}$. Let $B_I(r) = B(x_I, r)$ be the open geodesic ball in X_0 with centre x_I and radius r.

Turning to the four-spheres X_I, for any $I > 0$, let $\phi_{I\alpha} = \phi_\alpha$, $\alpha = s, n$ be the standard inverse coordinate charts on X_I. Define open neighbourhoods in X_I by

$$
\begin{aligned}
B_{Is}(r) &= B(x_{Is}, r) = \phi_{Is}\left(\{x \in \mathbb{R}^4 : |x| < r\}\right), \\
B_{I_+}(r) &= B(x_{I_+}, r) = \phi_{In}\left(\{x \in \mathbb{R}^4 : |x - q_{I_+}| < r\}\right).
\end{aligned}
\tag{3.16}
$$

Let $\Omega_I(r, R) = \Omega(x_I, r, R)$ be the open annulus $B_I(R) \setminus \overline{B}_I(r)$ centred at $x_I \in X_{I_-}$, with inner radius r and outer radius R.

Define small balls $B'_I = B(x_I, N\lambda_I^{1/2})$ and annuli $\Omega_I = \Omega(x_I, N\lambda_I^{1/2}, N\lambda_I^{1/2})$ in X_{I_-}, $I > 0$. The open subset X'_{I_-} is the complement in X_{I_-} of the balls $\overline{B}_I(N^{-1}\lambda_I^{1/2})$, the open subset X''_{I_-} is the complement in X_{I_-} of the balls $\overline{B}_I(\frac{1}{2}\lambda_I^{1/2})$, and the open subset X'''_{I_-} is the complement in X_{I_-} of the balls $\overline{B}_I(2N\lambda_I^{1/2})$.

We define identification maps f_I by

$$
f_I = \phi_{In} \circ c_I \circ \phi_I^{-1} : B'_I \longrightarrow X'_I,
\tag{3.17}
$$

where c_I is the dilation $x \to x/\lambda_I$ on $\mathbb{R}^4$. The above maps ϕ_I are local coordinate charts on X_{I_-} given by

$$
\phi_I^{-1} = \begin{cases} \exp_{v_I}^{-1} & \text{if } I_- = 0, \\ \tau_I \circ \phi_{I_-n}^{-1} & \text{if } I_- > 0, \end{cases}
\tag{3.18}
$$

where τ_I is the translation $x \to x - q_I$ on $\mathbb{R}^4$. The charts $\phi_I^{-1} = \exp_{v_I}^{-1}$ may be replaced by $\bar{\phi}_I^{-1} = \tau_{p_I} \circ \exp_{v_I}^{-1}$, $|p_I| \ll \varrho_0$, if we wish to compute derivatives with respect to the centres x_I in X_0. For notational consistency, we let f_0 denote the identity map on X_0.

Using the diffeomorphisms $f_I : \Omega_I \to \Omega_{Is}$ we obtain a connected sum $X = \#_{I\in\mathcal{I}} X'_I$. We again defer to §3.5 for the precise definition of a metric g on X closely approximating the metrics g_I on the summands X'_I and such that the maps $f_I : B'_I \to X'_I$ are conformal. With this choice of metric, the connected sum (X, g) is conformally equivalent to (X_0, g_0).

We have a local section σ_I of P_{I_-} defined by a choice of point in the fibre $P_{I_-}|_{x_I}$ and A_{I_-}-parallel translation from x_I; similarly, we have local sections σ_{In}, σ_{Is} of P_I defined by a choice of points in the fibres $P_I|_{x_{In}}$, $P_I|_{x_{Is}}$ and A_I-parallel translation from x_{In}, x_{Is}. These sections provide local trivialisations $P_{I_-}|_{B_I(\varrho_0)} \simeq B_I(\varrho_0) \times G$ and $P_I|_{X_I\setminus\{x_{In}\}} \simeq X_I \setminus \{x_{In}\} \times G$. Define C^∞ cutoff functions ψ_I on each summand X_I by setting

$$\psi_I \equiv (\phi_{Is}^{-1})^* \psi_{b_I} \prod_{I_+} (\phi_{I_+}^{-1})^* \psi_{I_+} \quad \text{on } X_I, \tag{3.19}$$

where the factor $(\phi_{Is}^{-1})^*\psi_{b_I}$ is omitted when $I = 0$. Note that $\psi_I = 0$ on the balls $B_{Is}(b_I/2)$ and $B_{I_+}(b_{I_+}/2)$ in X_I and smoothly extends by 1 on the complement of the balls $B_{Is}(b_I)$ and $B_{I_+}(b_{I_+})$ in X_I. Lastly, extend each φ_I by zero to give a C^∞ cutoff function on the connected sum X. Setting $A'_{I_-} = \psi_{I_-}A_{I_-}$, $A'_I = \psi_I A_I$, we obtain C^∞ almost anti-self-dual connections A'_{I_-}, A'_I which are flat on the balls $B_I(b_I/2), B_{Is}(b_I/2)$.

The gluing parameter ρ_I provides an isomorphism of the fibres : $P_{I_-}|_{x_I} \simeq P_I|_{x_{Is}}$. Using the connections A_{I_-}, A_I, this identification is extended to give a bundle isomorphism $\tilde\rho_I : P_{I_-}|_{\Omega_I} \to P_I|_{\Omega_{Is}}$ covering f_I. By these identification maps we obtain a connected-sum G bundle $P \to X$ with $c_2(P) = k$ and transition functions which are constant with respect to the scales λ_I. The cutoff connections A'_I on P_I patch together to give a C^∞ connection A' on P. As before, the connection A' on the connected-sum bundle P over X pull back via the maps f_I to give a connection $\hat A'$ on a bundle $\hat P$ over X_0.

Lastly, we record some estimates for the connections A' when restricted to a summand X'_I. For this and later purposes, we define the following Sobolev norms: Let ∇^{g_I} denote the Levi-Civita connection on TX_I defined by the metric g_I, so that if $f \in C^\infty(X_I)$, then

$$\|f\|_{L^p_n(X_I,g_I)} = \sum_{i=0}^{n} \|(\nabla^{g_I})^i f\|_{L^p(X_I,g_I)}, \tag{3.20}$$

for any $1 \le p \le \infty$ and integer $n \ge 0$. Similarly, if $\alpha \in \Omega^l(X_I, \operatorname{ad} P_I)$, then

$$\|\alpha\|_{L^p_n(X_I,A_I,g_I)} = \sum_{i=0}^{n} \|(\nabla^{A_I,g_I})^i \alpha\|_{L^p(X_I,g_I)}. \tag{3.21}$$

It is important to note that these norms will depend only on a set of *fixed* connections, $\{A_I\}_{I\in\mathcal{I}}$, and a set of *fixed* metrics $\{g_I\}_{I\in\mathcal{I}}$.

Recalling that $A'_I = \psi_I A_I$, define one-forms $a_I \in \Omega^1(X_I, \operatorname{ad} P_I)$ by setting $A_I = A'_I + a_I$. Thus

$$a_I = \begin{cases} (1-\psi_I)\sigma^*_{I_+} A_I & \text{on } B_{I_+}(b_{I_+}), \\ 0 & \text{on } X_I \setminus \bigcup_{I_+} B_{I_+}(b_{I_+}). \end{cases}$$

With the aid of bounds for the derivatives of the cutoff functions ψ_J for $C = C(g_J)$ and $J = I_-$ or I,

$$\begin{aligned} &|d\psi_J|_{g_J} \le C b_J^{-1} \quad \text{on } \Omega_J(b_J/2, b_J), \\ &\|d\psi_J\|_{L^4(X_J,g_J)} \le C. \end{aligned} \tag{3.22}$$

Standard arguments then give the following estimates.

Lemma 3.9. *Let $1 \le p < \infty$. Then there exists a constant $C = C(A_I, g_I, p)$ such that*

(a) $\|a_I\|_{L^\infty(X_I,g_I)} \le C\bar{b}$ *and* $\|a_I\|_{L^p(X_I,g_I)} \le C\bar{b}^{4/p+1}$,

(b) $\|F(A'_I)\|_{L^\infty(X_I,g_I)} \le C$ *and* $\|F^{+,g_I}(A'_I)\|_{L^p(X_I,g_I)} \le C\bar{b}^{4/p}$.

3.4. Approximate gluing maps. Adopting a more global perspective, the construction of §3.3 yields a family of 'approximate gluing maps', $\mathcal{J}' : \mathcal{T}/\Gamma \to \mathcal{B}^*_{X,k}$ and $\hat{\mathcal{J}}' : \mathcal{T}/\Gamma \to \mathcal{B}^*_{X_0,k}$, which we describe in this section. We first recall that the standard Kuranishi models give the required parametrisations for neighbourhoods of points $[A_I]$ in $M_{X_I,k_I}(g_I)$. Let A_I be a g_I-anti-self-dual connection over X_I, with isotropy group Γ_{A_I} and $H^2_{A_I} = 0$. For a small enough open neighbourhood T_{A_I} of $0 \in H^1_{A_I}$, we have smooth Γ_{A_I}-equivariant maps

$$\alpha_I : T_{A_I} \longrightarrow \ker d^{*,g_I}_{A_I} \subset \Omega^1(X_I, \operatorname{ad} P_I) \tag{3.23}$$

solving the g_I-anti-self-dual equation $F^{+,g_I}(A_I + \alpha_I(t_I)) = 0$, $t_I \in T_{A_I}$. Setting $A_I(t_I) = A_I + \alpha_I(t_I)$, we obtain a homeomorphism

$$\vartheta_I : T_{A_I}/\Gamma_{A_I} \longrightarrow U_{A_I}, \qquad t_I \longmapsto [A_I(t_I)], \tag{3.24}$$

onto an open neighbourhood U_{A_I} of $[A_I] \in M_{X_I,k_I}$. If A_I is the product connection, Θ, then $\Gamma_{A_I} = \mathrm{SU}(2)$ and so $H^0_{A_I} \neq 0$, while $H^1_{A_I} = 0$. If A_I is a non-trivial reducible connection, then $\Gamma_{A_I} = \mathbb{S}^1$ and $H^0_{A_I} \neq 0$; we have a homeomorphism $\vartheta_I : T_{A_I}/\Gamma_{A_I} \to U_{A_I}$ and a diffeomorphism $\vartheta_I : (T_{A_I} \setminus \{0\})/\Gamma_{A_I} \to U_{A_I} \setminus [A_I]$. Finally, if A_I is irreducible, then $\Gamma_{A_I} = (\pm 1)$ and $H^0_{A_I} = 0$; in this case we have a diffeomorphism $\vartheta_I : T_{A_I}/\Gamma_{A_I} \to U_{A_I}$.

We now dispose of the construction of neighbourhoods of reducible connections in $M_{X_0,k}(g_0)$. Recall that the reducible connections in $M_{X_0,k}(g_0)$ are in one-to-one correspondence with pairs $\{\pm c\}$, where $c \in H^2(X_0, \mathbb{Z})$ satisfies $c^2 = k$. In particular, there are only finitely many and so to describe a neighbourhood of any such reducible connection $[A] \in M_{X_0,k}(g_0)$, we may employ the Kuranishi model $\vartheta_A : T_A/\Gamma_A \to U_A$.

We now describe the approximate gluing maps $\mathcal{J}'$ and $\hat{\mathcal{J}}'$, beginning with the parameter spaces $\mathcal{T}/\Gamma$. First, with the centres $\{x_I\}$ and scales $\{\lambda_I\}$ held fixed, the parameter spaces T_{A_I} and Gl_{x_I} combine to give a C^∞ manifold

$$T \equiv T_{A_0} \times \prod_{I \in \mathcal{I}} (T_{A_I} \times \mathrm{Gl}_{x_I}), \tag{3.25}$$

parametrising a 'small' family of approximately anti-self-dual connections. Then

$$\Gamma \equiv \Gamma_{A_0} \times \prod_{I \in \mathcal{I}} \Gamma_{A_I} \tag{3.26}$$

acts freely on T and T/Γ is a C^∞ manifold. If we allow the centres, now denoted y_I, to move over disjoint balls $B(x_I, r_0) \subset X_{I\_}$ and allow the scales λ_I to vary in the interval $(0, \lambda_0)$, the parameter space of Eq. (3.25) is augmented to give a C^∞ manifold

$$\mathcal{T} \equiv T_{A_0} \times \prod_{I \in \mathcal{I}} (T_{A_I} \times \mathrm{Gl}_{x_I} \times B(x_I, r_0) \times (0, \lambda_0)), \tag{3.27}$$

parametrising a 'large' family of approximately g-anti-self-dual connections. Again, Γ acts freely on $\mathcal{T}$, and $\mathcal{T}/\Gamma$ is a C^∞ manifold. We fix local trivialisations of the frame bundle FX_0 over the balls $B(x_I, r_0)$, and these provide smooth families of geodesic normal coordinate charts on X_0.

We note that the almost anti-self-dual connections A' produced by §3.3 are indeed irreducible:

Lemma 3.10. *Let A' be a connection on the G bundle P over X defined by* Data 3.6 *and* Condition 3.7. *Then A' is irreducible, that is, $H^0_{A'} = 0$, for small enough b_0 and large enough N_0.*

The Lemma follows from Aronszajn's unique continuation principle for solutions to $\Delta_{A'}\eta = 0$ via standard methods, so the proof is omitted. Hence, the approximate gluing construction of §3.3 gives a C^∞ map

$$\mathcal{J}' : \mathcal{T}/\Gamma \longrightarrow \mathcal{B}^*_{X,k}, \qquad t \longmapsto [A'(t)], \tag{3.28}$$

where $\mathcal{B}^*_{X,k}$ has the structure of an L^2_n Hilbert manifold, $n \geq 3$. Moreover, $\mathcal{J}'$ is a C^∞ submersion onto its image; see §5.2. We refer to $\mathcal{J}'$ as an *approximate gluing map* over X and its image $\mathcal{U}' \subset \mathcal{B}^*_{X,k}$ as an *approximate gluing neighbourhood.*

The dimension of the parameter space $\mathcal{T}/\Gamma$ is given by

$$\begin{aligned}(3.29)\ \dim \mathcal{T}/\Gamma \\ &= \dim H^1_{A_0} - \dim H^0_{A_0} + \sum_{I>0}(\dim H^1_{A_I} - \dim H^0_{A_0} + 8),\end{aligned}$$

since each factor $\mathrm{Gl}_{x_I} \times B(x_I, r_0) \times (0, \lambda_0)$ has dimension 8, $\dim H^0_{A_I} = \dim \Gamma_{A_I}$, and $H^2_{A_I} = 0$ for all $I \geq 0$ by hypothesis. Families of centred g_I-anti-self-dual connections $A_I \in M^0_{X_I,k_I}(g_I)$ are parametrised by small balls $T^0_{A_I}$, and thus we obtain a C^∞ parameter space

$$\mathcal{T}^0 \equiv T_{A_0} \times \prod_{I\in\mathcal{I}} \left(T^0_{A_I} \times \mathrm{Gl}_{x_I} \times B(x_I, r_0) \times (0, \lambda_0)\right), \tag{3.30}$$

with C^∞ quotient $\mathcal{T}^0/\Gamma$ of dimension equal to $\dim M_{X,k}(g)$. The map $\mathcal{J}' : \mathcal{T}^0/\Gamma \to \mathcal{B}^*_{X,k}$ is a C^∞ embedding; see §5.2.

Lastly, using the conformal diffeomorphisms f_I, the bundle P over X pulls back to a bundle $\hat{P}$ over X_0. The gluing construction now produces an approximately g_0-anti-self-dual connection $\hat{A}'$ in $\mathcal{B}^*_{X_0,k}$. The map $\mathcal{J}'$ of Eq. (3.28) pulls back to a C^∞ map

$$\hat{\mathcal{J}}' : \mathcal{T}/\Gamma \longrightarrow \mathcal{B}^*_{X_0,k}, \qquad t \longmapsto [\hat{A}'(t)]. \tag{3.31}$$

Again, $\hat{\mathcal{J}}'$ is a C^∞ submersion onto its image and is a C^∞ embedding when the parameter space $\mathcal{T}/\Gamma$ is replaced by the smaller parameter space $\mathcal{T}^0/\Gamma$; see §5.2. As before, the image $\mathcal{V}'$ of $\hat{\mathcal{J}}'$ in $\mathcal{B}^*_{X_0,k}$ is called an approximate gluing neighbourhood.

3.5. Metrics on connected sums. In this section we define a conformal structure $[g]$ on the connected sum $X = \#_{I\in\mathcal{I}}X_I$. This is accomplished by replacing the standard round metric g_I on each spherical summand X_I' by a quasi-conformally equivalent metric $\tilde{g}_I$ so that the identification maps $f_I : B_I' \to X_I'$ are conformal. We then construct a C^∞ metric g on X in the conformal class $[g] = [g_0]$ and compare the resulting L^p norms for the different possible metrics on each summand X_I'. Our construction is modelled on the constructions of Donaldson and Taubes for metrics on connected sums — see [3 (p. 322)], [7 (p. 293)], and [24]. The metric g depends on the choice of fixed base metric g_0, fixed neck width parameter N, scales λ_I, centres x_I, and frames v_I. We also obtain bounds for the derivatives of g with respect to λ_I and x_I.

With respect to a geodesic normal coordinate system $x = \phi_{i_1}^{-1}$ on $B_{i_1}(\varrho_0) \subset X_0$, the covariant components of g_0 satisfy

$$(3.32)\qquad \begin{aligned} &(\phi_{i_1}^* g_0)_{\mu\nu}(0) = \delta_{\mu\nu} \quad\text{and}\quad \frac{\partial(\phi_{i_1}^* g_0)_{\mu\nu}}{\partial x^\alpha}(0) = 0,\\ &|(\phi_{i_1}^* g_0)_{\mu\nu} - \delta_{\mu\nu}|(x) \le c|x|^2 \quad\text{and}\\ &\left|\frac{\partial(\phi_{i_1}^* g_0)_{\mu\nu}}{\partial x^\alpha}\right|(x) \le c|x|, \qquad |x| < \varrho_0/2, \end{aligned}$$

for some constant $c = c(g_0)$. The analogous relations hold for the contravariant components of g_0. We now define a conformal structure $[g]$ on X:

Definition 3.11. The conformal structure $[g]$ on X is defined by the C^∞ metric g_0 on X_0' and a choice of C^∞ metric $\tilde{g}_I$ on each summand X_I', $I > 0$, given by

$$(\phi_{In}^*\tilde{g}_I)_{\mu\nu}(x) \equiv \begin{cases} h_1^2(x)(\phi_I^* g_0)_{\mu\nu}(\lambda_I x) & \text{if } I_- = 0,\\ h_1^2(x)h_1^{-2}(\lambda_I x + q_I)(\phi_I^*\tilde{g}_{I_-})_{\mu\nu}(\lambda_I x) & \text{if } I_- > 0, \end{cases}$$

where $|x| < N\lambda_I^{-1/2}$. For convenience, we let $g_I \equiv g_1$ denote the standard metric on X_I and let $\tilde{g}_0 \equiv g_0$ denote the metric on X_0.

Definition 3.11 provides the following expression for $\tilde{g}_I$:

$$(3.33)\qquad (\phi_{In}^*\tilde{g}_I)_{\mu\nu}(x) = h_1^2(x)(\phi_{i_1}^* g_0)_{\mu\nu}(y(x)), \quad |x| < N\lambda_I^{-1/2},$$

where

$$\begin{aligned} y(x) &= \phi_{i_1}^{-1} \circ f_{i_1}^{-1} \circ \cdots \circ f_I^{-1} \circ \phi_{In}(x) \\ &= \lambda_{i_1}(\lambda_{i_1 i_2}(\cdots(\lambda_{I_-}(\lambda_I x + q_I) + q_{I_-})\cdots) + q_{i_1 i_2}). \end{aligned} \tag{3.34}$$

The map $f_I : B_I' \to X_I'$ is now conformal with respect to the metrics $\tilde{g}_{I_-}$ on $B_I' \subset X_{I_-}'$ and $\tilde{g}_I$ on X_I':

$$(\phi_I^* f_I^* \tilde{g}_I)_{\mu\nu}(x) = \begin{cases} \lambda_I^{-2} h_1^2(x/\lambda_I)(\phi_I^* \tilde{g}_0)_{\mu\nu}(x) & \text{if } I_- = 0, \\ \lambda_I^{-2} h_1^2(x/\lambda_I) h_1^{-2}(x + q_I)(\phi_I^* \tilde{g}_{I_-})_{\mu\nu}(x) & \text{if } I_- > 0, \end{cases}$$

where $|x| < N\lambda_I^{1/2}$. Thus, $f_I^* \tilde{g}_I$ is conformally equivalent to the metric g_{I_-} on Ω_I and so we obtain a conformal structure $[g]$ on $X = \#_{I \in \mathcal{I}} X_I$.

We must verify that $\tilde{g}_I$ is a good approximation to the standard round metric g_I on X_I' for small λ_{i_1}.

Lemma 3.12. *For any $I > 0$, the metric $\tilde{g}_I$ converges to g_I in C^∞ on compact subsets of $X_I \setminus \{x_{Is}\}$ as $\lambda_{i_1} \to 0$. Moreover, we have the following bounds:*

(a) For any integer $l \geq 0$, there is a constant $c = c(g_0, l)$ such that

$$\left| \frac{\partial^l (\phi_{In}^* \tilde{g}_I)_{\mu\nu}}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_l}} - \frac{\partial^l (\phi_{In}^* g_I)_{\mu\nu}}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_l}} \right| \leq c N^2 \lambda_{i_1} h_1^2(x), \qquad |x| < N\lambda_I^{-1/2}.$$

The analogous bounds hold for the contravariant components $(\phi_{In}^* \tilde{g}_I)^{\mu\nu}$, provided $h_1^2(x)$ is replaced by $h_1^{-2}(x)$.

(b) Let $*_{\tilde{g}_I}$ denote the Hodge star operator for $\tilde{g}_I$. Then there is a constant $c = c(g_0)$ such that

$$\| *_{\tilde{g}_I} \zeta - *_{g_I} \zeta \|_{L^\infty(X_I', g_I)} \leq c N^2 \lambda_{i_1} \|\zeta\|_{L^\infty(X_I', g_I)}, \quad \zeta \in \Omega^2(X_I', \operatorname{ad} P_I).$$

Proof. (a) This follows easily from Eq. (3.32) and Definition 3.11. (b) This follows immediately from (a) and the definition of the Hodge star operator.

We will also require bounds for the derivatives of $\tilde{g}_J$ with respect to the scales λ_I and centres x_I. The following estimates will suffice for our application.

Lemma 3.13. *If $0 < I \le J$, there is a constant $c = c(g_0, J)$ such that the following bounds hold:*
(a) For any $|x| < N\lambda_J^{-1/2}$,

$$\left|\frac{\partial(\phi_{Jn}^* \tilde{g}_J)_{\mu\nu}}{\partial\lambda_I}\right|(x) \le \begin{cases} c\lambda_{i_1} h_1^2(x) & \text{if } I < J \text{ and } |I| = 1, \\ c\lambda_{i_1}^2 h_1^2(x) & \text{if } I < J \text{ and } |J| \ge 2, \\ cN^2 h_1^2(x) & \text{if } I = J \text{ and } |J| = 1, \\ c\lambda_{i_1}^2 |x| h_1^2(x) \le cN\lambda_{i_1}^2 \lambda_I^{-1/2} h_1^2(x) & \text{if } I = J \text{ and } |J| \ge 2. \end{cases}$$

(b) If $\partial/\partial p_I \equiv p_I^\alpha \partial/\partial q_I^\alpha$, then for any $|x| < N\lambda_J^{-1/2}$,

$$\left|\frac{\partial(\phi_{Jn}^* \tilde{g}_J)_{\mu\nu}}{\partial p_I}\right|(x) \le \begin{cases} cN\lambda_{i_1}^{1/2} h_1^2(x) & \text{if } I = J \text{ and } |J| = 1, \\ c\lambda_{i_1}^2 h_1^2(x) & \text{if } I \le J \text{ and } |J| \ge 2. \end{cases}$$

The analogous bounds in (a) and (b) hold for the contravariant components of $\tilde{g}_J$, if $h_1^2(x)$ is replaced by $h_1^{-2}(x)$.
(c) For any $\zeta \in \Omega^2(X_J', \operatorname{ad} P_J)$, then

$$\left\|\frac{\partial *_{\tilde{g}_J}}{\partial\lambda_I}\zeta\right\|_{L^\infty(X_J', g_J)} \le \begin{cases} cN\lambda_I^{-1/2}\|\zeta\|_{L^\infty(X_J', g_J)} & \text{if } I = J \text{ and } |J| \ge 2, \\ cN^2\|\zeta\|_{L^\infty(X_J', g_J)} & \text{otherwise,} \end{cases}$$

$$\left\|\frac{\partial *_{\tilde{g}_J}}{\partial p_I}\zeta\right\|_{L^\infty(X_J', g_J)} \le cN\lambda_{i_1}^{1/2}\|\zeta\|_{L^\infty(X_J', g_J)}.$$

Proof. (a) The inequalities follow from Eq. (3.32) and Definition 3.11.

(b) The proof is similar. When $|I| = 1$, we recall that the normal geodesic chart $\phi_{i_1} \equiv \exp_{v_{i_1}}$ is replaced by $\bar{\phi}_{i_1} \equiv \exp_{v_{i_1}} \circ \tau_{q_{i_1}}$ in order to compute the required derivative at $q_{i_1} = 0$ (corresponding to $x_{i_1} = \phi_{i_1}(0)$). The estimates follow immediately from (a) and (b).

We next define an honest C^∞ metric g on X. Consider a neck $\Omega_I = f_I^{-1}(\Omega_{Is})$ labelled by the multi-index I. We replace the metric $\tilde{g}_{I_-}$

on the annulus Ω_I and replace the metric $\tilde{g}_I$ on the annulus Ω_{Is} by conformally equivalent metrics $m_{I_-}\tilde{g}_{I_-}$ and $m_I\tilde{g}_I$ so that

$$m_{I_-}g_{I_-} = f_I^*(m_I g_I) \quad \text{on } \Omega_I. \tag{3.35}$$

Hence, the metrics $m_{I_-}\tilde{g}_{I_-}$ and $m_I\tilde{g}_I$ agree on the neck and patch together to give a C^∞ metric, say g, on a neighbourhood of the neck in the connected sum $X_{I_-}\#X_I$. On the annulus $\Omega_I = \phi_I(\{x \in \mathbb{R}^4 : N^{-1}\lambda_I^{1/2} < |x| < N\lambda_I^{1/2}\})$ we have

(3.36)
$$(\phi_I^* f_I^* \tilde{g}_I)_{\mu\nu}(x) = \begin{cases} 4\lambda_I^2(\lambda_I^2 + |x|^2)^{-2}(\phi_I^*\tilde{g}_0)_{\mu\nu}(x) & \text{if } I_- = 0, \\ 4\lambda_I^2(\lambda_I^2 + |x|^2)^{-2}h_1^{-2}(x+q_I)(\phi_I^*\tilde{g}_{I_-})_{\mu\nu}(x) & \text{if } I_- > 0. \end{cases}$$

By comparing $f_I^*\tilde{g}_I$ and g_{I_-} on Ω_I, a little experimentation reveals that the C^∞ conformal factors m_{I_-} and m_I can be chosen so that

$$\begin{aligned} &\kappa^{-1} \le m_{I_-} \le \kappa N^4 && \text{on } \Omega_I(N^{-1}\lambda_I^{1/2}, N\lambda_I^{1/2}), \\ &\kappa^{-1} \le m_{I_-} \le \kappa && \text{on } \Omega_I(\tfrac{1}{2}\lambda_I^{1/2}, N\lambda_I^{1/2}), \\ &m_{I_-} = 1 && \text{on } \Omega_I(2\lambda_I^{1/2}, 4N\lambda_I^{1/2}), \end{aligned} \tag{3.37}$$

and likewise for m_I on Ω_{Is}, and some constant $\kappa = \kappa(g_0)$. For each summand X_I, we smoothly extend the m_I to X_I' by setting $m_I \equiv 1$ away from the neck regions. This gives a C^∞ metric g on $X = \#_{I\in\mathcal{I}}X_I$ by setting

$$g \equiv m_I\tilde{g}_I \quad \text{on } X_I', \quad \text{for all } I \in \mathcal{I}. \tag{3.38}$$

The construction ensures that each m_I obeys

(3.39)
$$\kappa^{-1} \le m_I \le \kappa N^4 \text{ on } X_I', \ \ \kappa^{-1} \le m_I \le \kappa \ \text{ on } X_I'', \ \text{and} \\ m_I = 1 \ \text{ on } X_I'''.$$

Thus, the metrics $\tilde{g}_I$ and g are equivalent on X_I'' with constants independent of N, and equivalent over X_I' with constants now depending on N.

The Hodge star operator $*_g : \Omega^2(X, \operatorname{ad} P) \to \Omega^2(X, \operatorname{ad} P)$ only depends on the conformal class $[g]$ of g and so over each summand X_I' of X we have $*_g = *_{m_I\tilde{g}_I} = *_{\tilde{g}_I}$. From Lemma 3.13, we obtain:

Lemma 3.14. *There is a constant $c = c(g_0)$ such that for any $\zeta \in \Omega^2(X, \operatorname{ad} P)$, we have*
(a) $\|(\partial *_g/\partial\lambda_I)\zeta\|_{L^\infty(X,g)} \le cN\lambda_I^{-1/2}\|\zeta\|_{L^\infty(X,g)}$,
(b) $\|(\partial *_g/\partial p_I)\zeta\|_{L^\infty(X,g)} \le cN\lambda_{i_1}^{1/2}\|\zeta\|_{L^\infty(X,g)}$.

We will often need to compare L^p norms defined by the different metrics g_I, $\tilde{g}_I$, and g over $X_I' \subset X$. The required 'comparison estimates' given below follow in a straightforward way from Lemma 3.12 and Eq. (3.39), and similar inequalities may be found in [7 (p. 294)].

Lemma 3.15. *For any $I \ge 0$, the following holds.*

(a) If $2 \le p < \infty$ and $4 \le q < \infty$, there is a constant $c = c(g_0, k, p, q)$, $1 \le c < \infty$, such that for any $\omega \in \Omega^1(X_I', \operatorname{ad} P_I)$ and $\zeta \in \Omega^2(X_I', \operatorname{ad} P_I)$, we have

$$\|\omega\|_{L^q(X_I',g)} \le c\|\omega\|_{L^q(X_I',g_I)} \quad \text{and} \quad \|\zeta\|_{L^p(X_I',g)} \le c\|\zeta\|_{L^p(X_I',g_I)},$$
$$\|\omega\|_{L^q(X_I'',g_I)} \le c^{-1}\|\omega\|_{L^q(X_I'',g)} \quad \text{and} \quad \|\zeta\|_{L^p(X_I'',g_I)} \le c^{-1}\|\zeta\|_{L^p(X_I'',g)}.$$

(b) If $1 \le p < \infty$, $n \ge 1$, and b_0 is sufficiently small, there is a constant $c = c(g_0, k, n, N, p)$, $1 \le c < \infty$, such that for any $\alpha \in \Omega^n(X_I', \operatorname{ad} P_I)$, we have

$$c^{-1}\|\alpha\|_{L^p(X_I',g_I)} \le \|\alpha\|_{L^p(X_I',\tilde{g}_I)}, \quad \|\alpha\|_{L^p(X_I',g)} \le c\|\alpha\|_{L^p(X_I',g_I)}.$$

Lastly, having defined the conformal structure $[g]$ of X, we apply the estimates for $d\psi_I$ in Eq. (3.22), the estimates for A_I' and $F^{+,g_I}(A_I')$ in Lemma 3.9, and the estimates for $*_g - *_{g_I}$ in Lemma 3.12 to obtain a bound for the L^p-norm of the g-self-dual curvature $F^{+,g}(A') = \frac{1}{2}(1 + *_g)F(A')$ of the connection A' on the connected sum bundle P over X. Similar estimates have been given by Taubes and Donaldson.

Proposition 3.16. *For $1 \le p < \infty$ and sufficently small b_0, there exists a constant $C = C(g_0, p, \mathcal{T})$ such that for any $t \in \mathcal{T}$ one has $\|F^{+,g}(A')\|_{L^p(X,g)} \le C\bar{b}^{4/p}$.*

3.6. Estimates over connected sums and conformal vector fields. The goal of this section is to obtain L^2 estimates for the derivatives with respect to the scales λ_I and centres x_I of $\operatorname{ad}\hat{P}$-valued one-forms $\hat{\omega}$ over the base manifold X_0 obtained by pulling back $\operatorname{ad} P$-valued one-forms ω over the connected sum X.

Following Taubes [22], [23], let us begin by defining some useful Sobolev norms on $\Omega^1(\mathbb{S}^4, \operatorname{ad} P)$ and examine their behaviour under conformal diffeomorphisms. Suppose A is a C^∞ connection on a G bundle

P over $\mathbb{S}^4$. Let g_1 be the standard round metric on $\mathbb{S}^4$ and let δ be the flat metric on $\mathbb{S}^4 \setminus \{s\}$ obtained via the conformal identification $\phi_n^{-1} : \mathbb{S}^4 \setminus \{s\} \to \mathbb{R}^4$. Let ∇^{A,g_1} denote the covariant derivative on $\Omega^1(\mathbb{S}^4, \operatorname{ad} P)$ defined by the connection A and metric g_1, while $\nabla^{A,\delta}$ denotes the covariant derivative on $T^*(\mathbb{S}^4 \setminus \{s\}) \otimes \operatorname{ad} P$ defined by A and δ. Define an L^2_1 norm on $\Omega^1(\mathbb{S}^4, \operatorname{ad} P)$ by

$$\|\omega\|_{L^2_1(\mathbb{S}^4,A,g_1)} \equiv \|\omega\|_{L^2(\mathbb{S}^4,g_1)} + \|\nabla^{A,g_1}\omega\|_{L^2_1(\mathbb{S}^4,g_1)}. \tag{3.40}$$

Similarly, if ω has compact support in $\mathbb{S}^4 \setminus \{s\}$, define

$$\begin{aligned} |\omega|_A &\equiv \|\nabla^{A,\delta}\omega\|_{L^2(\mathbb{S}^4,\delta)}, \\ \|\omega\|_{L^2_1(\mathbb{S}^4,A,\delta)} &\equiv \|\omega\|_{L^2(\mathbb{S}^4,\delta)} + \|\nabla^{A,\delta}\omega\|_{L^2(\mathbb{S}^4,\delta)}. \end{aligned} \tag{3.41}$$

The properties of $|\cdot|_A$ and $\|\cdot\|_{L^2_1(\mathbb{S}^4,A,\delta)}$ are described by the following result of [22]. Recall that $\mathbf{C} = \mathbf{D}\times\ \mathbf{T}\ \times \mathrm{SO}(4)$ is identified, using $\phi_n : \mathbb{R}^4 \to \mathbb{S}^4 \setminus \{s\}$, with the subgroup of conformal diffeomorphisms of $(\mathbb{S}^4, g_1)$ which fix the south pole.

Lemma 3.17. [22 (Proposition 2.4)] *Given an L^2_1 connection A on a G bundle P over $\mathbb{S}^4$, then the following holds:*

(a) $|\cdot|_A$ *extends to a continuous norm on* $L^2_1\Omega^1(\mathbb{S}^4, \operatorname{ad} P)$.

(b) The norm $|\cdot|_A$ *is* $\mathbf{C}$*-invariant: for any* $f \in \mathbf{C}$, $|f^*\omega|_{f^*A} = |\omega|_A$.

(c) There exists a constant $1 \le z < \infty$, *which is independent of* P, A, f, *and* $\omega \in \Omega^1(\mathbb{S}^4, \operatorname{ad} P)$, *such that*

$$\begin{aligned} z^{-1}\|\omega\|_{L^2_1(\mathbb{S}^4,A,g_1)} &\le |\omega|_A \le z\|\omega\|_{L^2_1(\mathbb{S}^4,A,g_1)}, \\ z^{-1}\|\omega\|_{L^2_1(\mathbb{S}^4,A,g_1)} &\le \|\omega\|_{L^2_1(\mathbb{S}^4,A,\delta)} \le z\|\omega\|_{L^2_1(\mathbb{S}^4,A,g_1)}. \end{aligned}$$

Lemma 3.18. [23 (Lemma 3.1)] *Let A be a C^∞ connection on a G bundle P over $\mathbb{S}^4$ with its standard metric g_1 and let $f : \mathbb{S}^4 \to \mathbb{S}^4$ be a conformal diffeomorphism. Then there exists a constant $1 \le z < \infty$, which is independent of P, A, f, and $\omega \in \Omega^1(\mathbb{S}^4, \operatorname{ad} P)$, with the following significance:*

$$z^{-1}\|\omega\|_{L^2_1(\mathbb{S}^4,A,g_1)} \le \|f^*\omega\|_{L^2_1(\mathbb{S}^4,f^*A,g_1)} \le z\|\omega\|_{L^2_1(\mathbb{S}^4,A,g_1)}.$$

Recall that c_λ denotes both the dilation $x \mapsto x/\lambda$ of $\mathbb{R}^4$ and the conformal diffeomorphism of $(\mathbb{S}^4, g_1)$ induced by ϕ_n. A straightforward application of Hölder's inequality yields the following 'transfer estimates' for the maps c_λ.

Lemma 3.19. *Let $2 \le p \le p_1 \le 4$ and let $\lambda \in (0,1]$. Let U be an open subset of $\mathbb{S}^4 \setminus B(s, N\lambda^{1/2})$ and let P be a G bundle over $\mathbb{S}^4$. Then there is a constant $C = C(N)$ such that the following holds.*
(a) If $\omega \in \Omega^1(U, \operatorname{ad} P)$, then

$$\|c_\lambda^* \omega\|_{L^p(c_\lambda^{-1}(U), g_1)} \le C\lambda^{2/p - 2/p_1} \|\omega\|_{L^{p_1}(U, g_1)}.$$

(b) If $\zeta \in \Omega^2(U, \operatorname{ad} P)$, then $\|c_\lambda^* \zeta\|_{L^2(c_\lambda^{-1}(U), g_1)} \le C\|\zeta\|_{L^2(U, g_1)}$.

We next consider the action of the conformal group on $\Omega^1(\mathbb{S}^4, \operatorname{ad} P)$. Let $f_{\lambda,q}$ denote the lift to $\mathbb{S}^4$, via the chart ϕ_n, of the conformal diffeomorphism $c_\lambda \circ \tau_q$ on $\mathbb{R}^4$. Let P be a G bundle over $\mathbb{S}^4$ and suppose $\omega \in \Omega^1(\mathbb{S}^4, \operatorname{ad} P)$. Then Eq. (3.8) gives

$$\frac{\partial f_{\lambda,q}^* \omega}{\partial \lambda} = -\frac{1}{\lambda} f_{\lambda,q}^* \mathcal{L}_{\mathbf{r}} \omega \quad \text{and} \quad \frac{\partial f_{\lambda,q}^* \omega}{\partial p} = -\frac{1}{\lambda} f_{\lambda,q}^* \mathcal{L}_{\mathbf{p}} \omega, \tag{3.42}$$

where $\partial/\partial p \equiv p^\mu \partial/\partial q^\mu$. It will be convenient to express the above Lie derivatives in terms of covariant derivatives. If A is a C^∞ connection on P, then Eqs. (3.6) and (3.7) imply that

$$\mathcal{L}_{\mathbf{r}} \omega = \omega + \nabla_{\mathbf{r}}^{A,\delta} \omega \quad \text{and} \quad \mathcal{L}_{\mathbf{p}} \omega = \nabla_{\mathbf{p}}^{A,\delta} \omega. \tag{3.43}$$

This leads to the following estimates for the derivatives of $f_{\lambda,q}^* \omega$ with respect to λ and q.

Lemma 3.20. *Let A be a C^∞ connection on a G bundle P over $\mathbb{S}^4$, let $U \subset \mathbb{S}^4 \setminus B(s, N\lambda^{1/2})$ be an open subset, $\omega \in \Omega^1(U, \operatorname{ad} P)$, where ω has compact support in U, and $\partial/\partial p = p^\mu \partial/\partial q^\mu$, $|p| \le 1$. Then there is a constant $C = C(q, N)$ such that the following bounds hold.*
(a) $\|\partial f_{\lambda,q}^* \omega / \partial \lambda\|_{L^2(f_{\lambda,q}^{-1}(U), g_1)} \le C\lambda^{-1/2} \|\omega\|_{L_1^2(U, A, g_1)}$;
(b) $\|\partial f_{\lambda,q}^* \omega / \partial p\|_{L^2(f_{\lambda,q}^{-1}(U), g_1)} \le C\|\omega\|_{L_1^2(U, A, g_1)}$.

Proof. (a) Observe that $U = \phi_n(B(0, N\lambda^{-1/2}))$ and $f_{\lambda,q}^{-1}(U) = \phi_n(B(q, N\lambda^{1/2}))$. From Eqs. (3.42) and (3.43), we have

$$\frac{\partial f_{\lambda,q}^* \omega}{\partial \lambda} = -\lambda^{-1} f_{\lambda,q}^* \mathcal{L}_{\mathbf{r}} \omega,$$
$$f_{\lambda,q}^* \mathcal{L}_{\mathbf{r}} \omega = f_{\lambda,q,*}^{-1} \mathbf{r} \lrcorner f_{\lambda,q}^* \nabla^{A,\delta} \omega + f_{\lambda,q}^* \omega \quad \text{on} \quad f_{\lambda,q}^{-1}(U),$$

where $\mathbf{r} = y^\mu \partial/\partial y^\mu$ and $f_{\lambda,q,*}^{-1} \mathbf{r} = x^\mu \partial/\partial x^\mu$ with respect to the coordinates $y = \phi_n^{-1}$ on U and $x = \tau_q \circ \phi_n^{-1}$ on $f_{\lambda,q}^{-1}(U)$. Since $|f_{\lambda,q,*}^{-1} \mathbf{r}|_{g_1} \le$

$C\lambda^{1/2}$ on $f_{\lambda,q}^{-1}(U)$, Lemma 3.19 implies

$$\begin{aligned}\|f_{\lambda,q}^*\mathcal{L}_{\mathbf{r}}\omega\|_{L^2(f_{\lambda,q}^{-1}(U),g_1)} &\leq \|f_{\lambda,q}^*\omega\|_{L^2(f_{\lambda,q}^{-1}(U),g_1)} \\ &\quad + C\lambda^{1/2}\|f_{\lambda,q}^*\nabla^{A,\delta}\omega\|_{L^2(f_{\lambda,q}^{-1}(U),g_1)} \\ &\leq C\lambda^{1/2}\|\omega\|_{L^4(U,g_1)} + C\lambda^{1/2}\|\nabla^{A,\delta}\omega\|_{L^2(U,g_1)} \\ &= C\lambda^{1/2}\|\omega\|_{L^2_1(U,A,\delta)},\end{aligned}$$

the last step following by conformal invariance. Lemma 3.17 then gives (a).

(b) From Eqs. (3.42) and (3.43), we have

$$\frac{\partial f_{\lambda,q}^*\omega}{\partial p_I} = -\lambda^{-1}f_{\lambda,q}^*\mathcal{L}_{\mathbf{p}}\omega \quad\text{and}\quad f_{\lambda,q}^*\mathcal{L}_{\mathbf{p}}\omega = f_{\lambda,q,*}^{-1}\mathbf{p}\lrcorner f_{\lambda,q}^*\nabla^{A,\delta}\omega \quad\text{on}\quad f_{\lambda,q}^{-1}(U),$$

where $\mathbf{p} = p^\mu\partial/\partial y^\mu$ on U and $f_{\lambda,q,*}^{-1}\mathbf{p} = \lambda p^\mu\partial/\partial x^\mu$ on $f_{\lambda,q}^{-1}(U)$. Since $|f_{\lambda,q,*}^{-1}\mathbf{p}|_{g_1} \leq C\lambda$ on $f_{\lambda,q}^{-1}(U)$, Lemma 3.19 yields

$$\begin{aligned}\|f_{\lambda,q}^*\mathcal{L}_{\mathbf{p}}\omega\|_{L^2(f_{\lambda,q}^{-1}(U),g_1)} &\leq C\lambda\|f_{\lambda,q}^*\nabla^{A,\delta}\omega\|_{L^2(f_{\lambda,q}^{-1}(U),g_1)} \\ &\leq C\lambda\|\nabla^{A,\delta}\omega\|_{L^2(U,\delta)} \leq C\lambda\|\omega\|_{L^2_1(U,A,\delta)}.\end{aligned}$$

Hence (b) follows from Lemma 3.17.

We will frequently need to compute estimates for families of one-forms ω over connected sums X, and to this end, it will be useful to define suitable Sobolev norms which depend only on the fixed connections A_I and, in particular, the fixed metrics g_I on each summand X_I rather than varying metric g on X. Let P be the G bundle over the connected sum $X = \#_{I\in\mathcal{I}}X_I$ defined in §3.3. Then we may view any $\omega \in \Omega^1(X, \operatorname{ad} P)$ as a collection of $\omega_I \in \Omega^1(X_I', \operatorname{ad} P_I)$ which agree over the necks $\Omega_I = f_I^{-1}(\Omega_{Is})$ connecting each pair X_{I_-} and X_I:

$$\sigma_I^*\omega_{I_-} = \operatorname{Ad}(\rho_I^{-1})f_I^*\sigma_{Is}^*\omega_I \quad \text{on } \Omega_I,$$

where $f_I : \Omega_I \to \Omega_{Is}$ is the identification map.

From §3.5, we recall that there is a C^∞ metric g on X which agrees, modulo the conformal factors m_I, with the metrics g_0 on the base X_0' and $\tilde{g}_I \simeq g_1$ on the four-spheres X_I'. Moreover, the L^q norms on $\Omega^1(X_I', \operatorname{ad} P_I)$, $4 \leq q < \infty$, and L^p norms on $\Omega^2(X_I', \operatorname{ad} P_I)$, $2 \leq p < \infty$, compare uniformly when defined with the metrics g_I, $\tilde{g}_I$, or $g = m_I\tilde{g}_I$ on

X_I'. The constants involved in these norm comparisons are independent of the scale parameters λ_J for forms supported on X_I' and independent of both the λ_J and N for forms supported on X_I''. Thus, we may conveniently define L^q norms on $\Omega^1(X, \operatorname{ad} P)$, $4 \le q < \infty$, and L^p norms on $\Omega^2(X, \operatorname{ad} P)$, $2 \le p < \infty$, using the metric g on X.

In Chapter 5, we will need to bound the L_1^2 norms of solutions $\omega \in \Omega^1(X, \operatorname{ad} P)$ to the g-anti-self-dual equation $F^{+,g}(A' + \omega) = 0$ over X. Unfortunately, since the conformal factors m_I have badly behaved derivatives over the neck regions, the norm comparisons described above do not hold for L_n^2 Sobolev norms if $n \ge 1$. Of course, problems of this type are encountered in [3], [7], and [24]. So, given such an $\omega \in \Omega^1(X, \operatorname{ad} P)$, with $\omega = \{\omega_I\}_{I \in \mathcal{I}}$ as above, and $1 \le p < \infty$, we define

$$\|\omega\|_{\mathcal{L}_1^p(X)} \equiv \sum_{I \in \mathcal{I}} \|\omega_I\|_{L_1^p(X_I, A_I, g_I)}, \tag{3.44}$$

by analogy with Eq. (6.25) in [24].

Recall that a one-form $\omega \in \Omega^1(X, \operatorname{ad} P)$ pulls back to a one-form $\hat{\omega} \in \Omega^1(X_0, \operatorname{ad} \hat{P})$ defined by

$$\hat{\omega} = f_0^* \cdots f_J^* \omega \quad \text{on } f_0^{-1} \cdots f_J^{-1}(X_J') \subset X_0, \tag{3.45}$$

for each $J \in \mathcal{I}$. We will need estimates for the derivatives of $\hat{\omega}$ with respect to the scales λ_I and centres x_I. To begin, we need suitable expressions for these derivatives:

Lemma 3.21. *Let $\omega \in \Omega^1(X_J', \operatorname{ad} P_J)$, $0 < I \le J$, and $\partial/\partial p_I = p_I^\mu \partial/\partial q_I^\mu$. Then:*

(a) $\frac{\partial}{\partial \lambda_I} f_0^* \cdots f_J^* \omega = f_0^* \cdots f_J^* \frac{\partial \omega}{\partial \lambda_I}$, *for* $J < I$;

(b) $\frac{\partial}{\partial \lambda_I} f_0^* \cdots f_I^* \omega = -\lambda_I^{-1} f_0^* \cdots f_I^* \mathcal{L}_r \omega$, *for* $J = I$;

(c) $\frac{\partial}{\partial \lambda_I} f_0^* \cdots f_J^* \omega = -\lambda_I^{-1} f_0^* \cdots f_I^* \mathcal{L}_r f_{I_+}^* \cdots f_J^* \omega$, *for* $J > I$;

(d) $\frac{\partial}{\partial p_I} f_0^* \cdots f_J^* \omega = f_0^* \cdots f_J^* \frac{\partial \omega}{\partial p_I}$, *for* $J < I$;

(e) $\frac{\partial}{\partial p_I} f_0^* \cdots f_I^* \omega = -\lambda_I^{-1} f_0^* \cdots f_I^* \mathcal{L}_p \omega$, *for* $J = I$;

(f) $\frac{\partial}{\partial p_I} f_0^* \cdots f_J^* \omega = -\lambda_I^{-1} f_0^* \cdots f_I^* \mathcal{L}_p f_{I_+}^* \cdots f_J^* \omega$, *for* $J > I$.

Remark 3.22. When $I_- = 0$, then $\partial/\partial p_I = p_I^\mu \partial/\partial p_I^\mu$ and $f_I = \phi_{In} \circ c_I \circ \phi_I^{-1}$ is replaced by $\overline{f}_I = \phi_{In} \circ c_I \circ \tau_{p_I} \circ \phi_I^{-1}$ in order to compute the derivative at $p_I = 0$.

These expressions lead to the following bounds for the derivatives with respect to the scales λ_I and centres x_I of the pull-backs $f_0^* \cdots f_J^*\omega$.

Lemma 3.23. *Let $\omega \in \Omega^1(X'_J, \operatorname{ad} P_J)$, where ω has compact support in X'_J, $U = f_0^{-1} \circ \cdots \circ f_J^{-1}(X'_J) \subset X_0$, $0 < I \le J$, and $\partial/\partial p_I = p_I^\mu \partial/\partial q_I^\mu$ with $|p_I| \le 1$. Then there is a constant $C = C(g_0, N)$ such that the following hold.*

(a) $\left\|\frac{\partial}{\partial\lambda_I} f_0^* \cdots f_J^*\omega\right\|_{L^2(U,g_0)} \le C \left\|\frac{\partial\omega}{\partial\lambda_I}\right\|_{L^2(X'_J,g_J)}$, *for $J < I$;*

(b) $\left\|\frac{\partial}{\partial\lambda_I} f_0^* \cdots f_J^*\omega\right\|_{L^2(U,g_0)} \le C\lambda_I^{-1/2}\|\omega\|_{L^2_1(X'_J,A_J,g_J)}$, *for $J \ge I$;*

(c) $\left\|\frac{\partial}{\partial p_I} f_0^* \cdots f_J^*\omega\right\|_{L^2(U,g_0)} \le C \left\|\frac{\partial\omega}{\partial p_I}\right\|_{L^2(X'_J,g_J)}$, *for $J < I$;*

(d) $\left\|\frac{\partial}{\partial p_I} f_0^* \cdots f_J^*\omega\right\|_{L^2(U,g_0)} \le C\|\omega\|_{L^2_1(X'_J,A_J,g_J)}$, *for $J \ge I$.*

Proof. (a) By repeatedly applying Lemma 3.19, we find that

$$\left\|\frac{\partial}{\partial\lambda_I} f_0^* \cdots f_J^*\omega\right\|_{L^2(U,g_0)} = \left\|f_0^* \cdots f_J^*\frac{\partial\omega}{\partial\lambda_I}\right\|_{L^2(U,g_0)} \le C\left\|\frac{\partial\omega}{\partial\lambda_I}\right\|_{L^2(X'_J,g_J)},$$

as required for (a). For $J = I$ and $U = f_0^{-1} \circ \cdots \circ f_I^{-1}(X'_I) \subset X_0$, Lemmas 3.19 and 3.20 show that

$$\begin{aligned}\left\|\frac{\partial}{\partial\lambda_I} f_0^* \cdots f_I^*\omega\right\|_{L^2(U,g_0)} &= \left\|f_0^* \cdots f_{I_-}^*\frac{\partial f_I^*\omega}{\partial\lambda_I}\right\|_{L^2(U,g_0)} \\ &\le C\left\|\frac{\partial f_I^*\omega}{\partial\lambda_I}\right\|_{L^2(f_I^{-1}(X'_I),g_{I_-})} \\ &\le C\lambda_I^{-1/2}\|\omega\|_{L^2_1(X'_I,A_I,g_I)}.\end{aligned}$$

Let $V = f_{I_+}^{-1} \circ \cdots \circ f_J^{-1}(X'_J) \subset X'_I$, so that $U = f_0^{-1} \circ \cdots \circ f_I^{-1}(V) \subset X_0$. Then for $J > I$, we have

$$\begin{aligned}\left\|\frac{\partial}{\partial\lambda_I} f_0^* \cdots f_J^*\omega\right\|_{L^2(U,g_0)} &= \left\|f_0^* \cdots f_{I_-}^*\frac{\partial}{\partial\lambda_I} f_I^* \cdots f_J^*\omega\right\|_{L^2(U,g_0)} \\ &\le C\left\|\frac{\partial}{\partial\lambda_I} f_I^* f_{I_+}^* \cdots f_J^*\omega\right\|_{L^2(f_I^{-1}(V),g_{I_-})} \\ &\le C\lambda_I^{-1/2}\|f_{I_+}^* \cdots f_J^*\omega\|_{L^2_1(V,f_{I_+}^* \cdots f_J^* A_J,g_I)} \\ &\le C\|\omega\|_{L^2_1(X'_J,A_J,g_J)},\end{aligned}$$

by repeatedly applying Lemma 3.18 in the last step. This gives (b); the proofs of (c) and (d) are similar.

Finally, we obtain our estimate for the derivatives of $\hat{\omega}$ with respect to the scales λ_I and centres x_I.

Proposition 3.24. *There is a constant $C = (g_0, \mathcal{T})$ such that for any $\omega \in \Omega^1(X, \operatorname{ad} P)$ and $t \in \mathcal{T}$, the following bounds hold.*
(a) $\|\partial\hat{\omega}/\partial\lambda_I\|_{L^2(X_0,g_0)} \leq C(\|\partial\omega/\partial\lambda_I\|_{L^2(X,g)} + \lambda_I^{-1/2}\|\omega\|_{\mathcal{L}_1^2(X)})$;
(b) $\|\partial\hat{\omega}/\partial p_I\|_{L^2(X_0,g_0)} \leq C(\|\partial\omega/\partial p_I\|_{L^2(X,g)} + \|\omega\|_{\mathcal{L}_1^2(X)})$.

Proof. By Lemma 3.23 we have

$$\begin{aligned}\left\|\frac{\partial\hat{\omega}}{\partial\lambda_I}\right\|_{L^2(X_0,g_0)} \leq C\sum_{J<I}\left\|\frac{\partial\omega}{\partial\lambda_I}\right\|_{L^2(X'_J,g_J)} \\ +C\lambda_I^{-1/2}\sum_{J\geq I}\|\omega\|_{L_1^2(X'_J,A_J,g_J)},\end{aligned}$$

and so (a) follows from Lemma 3.15. Similarly, Lemma 3.23 gives

$$\begin{aligned}\left\|\frac{\partial\hat{\omega}}{\partial p_I}\right\|_{L^2(X_0,g_0)} \leq C\sum_{J<I}\left\|\frac{\partial\omega}{\partial p_I}\right\|_{L^2(X'_J,g_J)} \\ +C\sum_{J\geq I}\|\omega\|_{L_1^2(X'_J,A_J,g_J)},\end{aligned}$$

and likewise, (b) follows from Lemma 3.15.

3.7. Derivatives with respect to scales and centres. We obtain L^p estimates for the derivatives of the connections A' and $\hat{A}'$ and of the g-self-dual curvature $F^{+,g}(A')$ with respect to the scales λ_I and centres x_I.

Throughout this section we require that $b_J = 4N\lambda_J^{1/2}$ for all J. Let us first record the following bounds for the derivatives of the cutoff functions ψ_J for $J = I_-$ or I:

(3.46)
$$\begin{aligned}\left|\frac{\partial\psi_J}{\partial\lambda_I}\right|_{g_J} &\leq CN^{-1}\lambda_I^{-1}, & \left|\frac{\partial d\psi_J}{\partial\lambda_I}\right|_{g_J} &\leq CN^{-2}\lambda_I^{-3/2} \quad \text{on } X'_J,\\ \left|\frac{\partial\psi_J}{\partial p_I}\right|_{g_J} &\leq CN^{-1}\lambda_I^{-1/2}, & \left|\frac{\partial d\psi_J}{\partial p_I}\right|_{g_J} &\leq CN^{-2}\lambda_I^{-1} \quad \text{on } X'_J,\end{aligned}$$

where $\partial/\partial p_I \equiv p_I^\mu\partial/\partial q_I^\mu$ and $|p_I| \leq 1$. The constant C depends only on g_J. We now begin with the L^p estimates for derivatives of the connections A'.

Proposition 3.25. *Suppose $1 \le p < \infty$ and $I > 0$. Then for sufficiently small λ_0, there is a constant $C = C(g_0, p, \mathcal{T})$ such that for any $t \in \mathcal{T}$,*
(a) $\|\partial A'/\partial\lambda_I\|_{L^p(X,g)} \le C\lambda_I^{2/p-1/2}$,
(b) $\|\partial A'/\partial p_I\|_{L^p(X,g)} \le C\lambda_I^{2/p}$.

Proof. (a) Observe that $\partial A'/\partial\lambda_I$ is non-zero only on the supports of $\partial\psi_{I_-}/\partial\lambda_I$ and $\partial\psi_I/\partial\lambda_I$, given by the annuli $\Omega_I(\frac{1}{2}b_I, b_I)$ in X'_{I_-} and $\Omega_{Is}(\frac{1}{2}b_I, b_I)$ in X'_I.

Step 1. *Estimate of $\partial A'/\partial\lambda_I$ over X'_{I_-}.* Recall that $\psi_{I_-} = 1$ on the complement of the balls $B_I(b_I)$ in X_{I_-}, while $0 < \psi_{I_-} < 1$ on $\Omega_I(\frac{1}{2}b_I, b_I)$, and $\psi_{I_-} = 0$ on $B_I(\frac{1}{2}b_I)$. We have $\sigma_I^* A' = \psi_{I_-}\sigma_I^* A_{I_-}$ on $\Omega_I(\frac{1}{2}b_I, b_I)$ and thus $\sigma_I^*(\partial A'/\partial\lambda_I) = (\partial\psi_{I_-}/\partial\lambda_I)\sigma_I^* A_{I_-}$ on X'_{I_-}. Since $|\partial\psi_I/\partial\lambda_I| \le C\lambda_I^{-1}$ by Eq. (3.46) and $|\sigma_I^* A_{I_-}|_{g_{I_-}} \le C\lambda_I^{1/2}$ on $\Omega_I(\frac{1}{2}b_I, b_I)$ by Lemmas 3.1 and 3.3, we obtain the pointwise bound

$$\left|\frac{\partial A'}{\partial\lambda_I}\right|_{g_{I_-}} \le \begin{cases} C\lambda_I^{-1/2} & \text{on } \Omega_I(\frac{1}{2}b_I, b_I), \\ 0 & \text{on } X'_{I_-} \setminus \Omega_I(\frac{1}{2}b_I, b_I). \end{cases}$$

Hence, we get the integral estimate

$$\int_{X'_{I_-}} \left|\frac{\partial A'}{\partial\lambda_I}\right|_g^p dV_g \le C\lambda_I^{2-p/2}, \tag{3.47}$$

noting that $g = \tilde{g}_{I_-}$ on $X_{I_-} \setminus B_I(\frac{1}{2}b_I)$ and appealing to Lemma 3.12.

Step 2. *Estimate of $\partial A'/\partial\lambda_I$ over X'_I.* A similar argument shows that

$$\int_{X'_I} \left|\frac{\partial A'}{\partial\lambda_I}\right|_g^p dV_g \le C\lambda_I^{2-p/2}, \tag{3.48}$$

and combining the integral bounds from Steps 1 and 2 gives (a). For (b) we use the pointwise estimates $|\partial\psi_J/\partial p_I| \le C\lambda_I^{-1/2}$, $J = I_-, I$. The same argument as in (a) then gives the required bound.

Our next task is to obtain a L^p estimates for the derivatives of the g-self-dual curvature $F^{+,g}(A')$.

Proposition 3.26. *Suppose $1 \le p < 4$ and $I > 0$. Then for sufficiently small λ_0, there exists a constant $C = C(g_0, p, \mathcal{T})$ such that for any $t \in \mathcal{T}$,*
(a) $\|\partial F^{+,g}(A')/\partial\lambda_I\|_{L^p(X,g)} \le C\lambda_I^{2/p-1}$,

(b) $\|\partial F^{+,g}(A')/\partial p_I\|_{L^p(X,g)} \leq C(\lambda_I^{2/p-1/2} + \bar{\lambda}^{1/2p})$.

Proof. (a) We note that $F^{+,g}(A') = F^{+,\tilde{g}_J}(\psi_J A_J)$ on X'_J and so $\partial F^{+,g}(A')/\partial\lambda_I$ is supported on $\cup_{J\geq I_-} X'_J$. It is convenient to obtain estimates separately over the regions X'_{I_-}, X'_I, and X'_J, $J > I$.

Step 1. *Estimate of $\partial F^{+,g}(A')/\partial\lambda_I$ over X'_{I_-}.* On the annulus $\Omega_I(\frac{1}{2}b_I, b_I)$ we have $F^{+,g}(A') = \frac{1}{2}(1 + *_{\tilde{g}_{I_-}})F(\psi_{I_-}A_{I_-})$ and

$$\begin{aligned} F(\psi_{I_-}A_{I_-}) = \psi_{I_-}F(A_{I_-}) &+ d\psi_{I_-} \wedge \sigma_I^* A_{I_-} \\ &+ (\psi_{I_-}^2 - \psi_{I_-})\sigma_I^* A_{I_-} \wedge \sigma_I^* A_{I_-}. \end{aligned}$$

Therefore, we see that

$$\begin{aligned} \frac{\partial F^{+,g}(A')}{\partial\lambda_I} &= \frac{1}{2}(1 + *_{\tilde{g}_{I_-}})\frac{\partial F(\psi_{I_-}A_{I_-})}{\partial\lambda_I}, \\ \frac{\partial F(\psi_{I_-}A_{I_-})}{\partial\lambda_I} &= \frac{\partial\psi_{I_-}}{\partial\lambda_I}F(A_{I_-}) + \frac{\partial d\psi_{I_-}}{\partial\lambda_I} \wedge \sigma_I^* A_{I_-} \\ &\quad + (2\psi_{I_-} - 1)\frac{\partial\psi_{I_-}}{\partial\lambda_I}\sigma_I^* A_{I_-} \wedge \sigma_I^* A_{I_-}. \end{aligned}$$

on X'_{I_-}. The metric $\tilde{g}_{I_-}$ is independent of λ_I, and so applying the pointwise estimates of Lemmas 3.1, 3.3, and Eq. (3.46), we find that

$$\left|\frac{\partial F^{+,g}(A')}{\partial\lambda_I}\right|_{g_{I_-}} \leq \begin{cases} C\lambda_I^{-1} & \text{on } \Omega_I(\frac{1}{2}b_I, b_I), \\ 0 & \text{on } X_{I_-} \setminus \Omega_I(\frac{1}{2}b_I, b_I). \end{cases}$$

Consequently, we obtain

$$\int_{X'_{I_-}} \left|\frac{\partial F^{+,g}(A')}{\partial\lambda_I}\right|_g^p dV_g \leq C\lambda_I^{2-p}, \tag{3.49}$$

where we observe that $g = \tilde{g}_{I_-}$ on $\Omega_I(\frac{1}{2}b_I, b_I)$.

Step 2. *Estimate of $\partial F^{+,g}(A')/\partial\lambda_I$ over X'_I.* We have $F^{+,g}(A') = \frac{1}{2}(1 + *_{\tilde{g}_I})F(\psi_I A_I)$ and $F(\psi_I A_I) = \psi_I F(A_I) + d\psi_I \wedge \sigma_{Is}^* A_I + (\psi_I^2 - \psi_I)\sigma_{Is}^* A_I \wedge \sigma_{Is}^* A_I$ on X'_I. Thus,

$$\begin{aligned} \frac{\partial F^{+,g}(A')}{\partial\lambda_I} &= \frac{1}{2}\frac{\partial *_{\tilde{g}_I}}{\partial\lambda_I}F(\psi_I A_I) + \frac{1}{2}(1 + *_{\tilde{g}_I})\frac{\partial F(\psi_I A_I)}{\partial\lambda_I}, \\ \frac{\partial F(\psi_I A_I)}{\partial\lambda_I} &= \frac{\partial\psi_I}{\partial\lambda_I}F(A_I) + \frac{\partial d\psi_I}{\partial\lambda_I} \wedge \sigma_{Is}^* A_I \\ &\quad + (2\psi_I - 1)\frac{\partial\psi_I}{\partial\lambda_I}\sigma_{Is}^* A_I \wedge \sigma_{Is}^* A_I \end{aligned}$$

on X'_I. Applying the pointwise estimates of Lemmas 3.3, 3.12, 3.13, and Eq. (3.46), we find that

$$\left|\phi^*_{In}\frac{\partial F^{+,g}(A')}{\partial\lambda_I}\right|_{g_I}(x) \le \begin{cases} 0 & \text{on } B_{Is}(\tfrac{1}{2}b_I), \\ C\lambda_I^{-1} & \text{on } \Omega_{Is}(\tfrac{1}{2}b_I, b_I), \\ C|x| & \text{on } X_I \setminus B_{Is}(b_I). \end{cases}$$

Now $g = \tilde{g}_I$ on $X_I \setminus B_{Is}(\frac{1}{2}b_I)$, and so applying the above estimates and Hölder's inequality gives

$$\int_{X'_I}\left|\frac{\partial F^{+,g}(A')}{\partial\lambda_I}\right|_g^p dV_g \le C\lambda_I^{2-p}, \tag{3.50}$$

completing Step 2.

Step 3. *Estimate of $\partial F^{+,g}(A')/\partial\lambda_I$ over X'_J, $J > I$.* We have

$$\frac{\partial F^{+,g}(\psi_J A_J)}{\partial\lambda_I} = \frac{1}{2}\frac{\partial *_{\tilde{g}_J}}{\partial\lambda_I}F(\psi_J A_J) \quad \text{on } X'_J,$$

since $F^{+,g}(A') = \frac{1}{2}(1 + *_{\tilde{g}_J})F(\psi_J A_J)$. The pointwise estimates of Lemmas 3.9, 3.12, and 3.13 show that

$$\left|\phi^*_{Jn}\frac{\partial F^{+,g}(A')}{\partial\lambda_I}\right|_{g_J}(x) \le \begin{cases} 0 & \text{on } B_{Js}(\tfrac{1}{2}b_J), \\ C|x| & \text{on } X_J \setminus B_{Js}(\tfrac{1}{2}b_J). \end{cases}$$

Again, $g = \tilde{g}_J$ on $X_J \setminus B_{Js}(\frac{1}{2}b_J)$, and so

$$\int_{X'_J}\left|\frac{\partial F^{+,g}(A')}{\partial\lambda_I}\right|_g^p dV_g \le C. \tag{3.51}$$

Combining the integral estimates of Steps 1 to 3 then gives (a).

(b) The argument is the same, except that we now use the cutoff function estimates $|\partial\psi_J/\partial p_I| \le C\lambda_I^{-1/2}$, $|\partial d\psi_J/\partial p_I| \le C\lambda_I^{-1}$, $J = I_-, I$, and metric estimates $|\partial\tilde{g}_J/\partial p_I| \le CN\overline{\lambda}^{1/2}$, $J \ge I$.

Lastly, we have L^2 estimates of the derivatives of $\hat{A}'$ with respect to λ_I and x_I.

Proposition 3.27. *Suppose $I > 0$. Then for sufficiently small λ_0, there is a constant $C = C(g_0, \mathcal{T})$ such that for any $t \in \mathcal{T}$,*
(a) $\|\partial\hat{A}'/\partial\lambda_I\|_{L^2(X_0,g_0)} \le C$,

(b) $\|\partial \hat{A}'/\partial p_I\|_{L^2(X_0,g_0)} \le C$.

Proof. (a) The connection one-forms over X_0 having non-zero derivatives with respect to λ_I are given by

$$\hat{A}' = \begin{cases} f_0^* \cdots f_{I_-}^* \psi_{I_-} A_{I_-} & \text{over } f_0^{-1} \circ \cdots \circ f_{I_-}^{-1}(X'_{I_-}) \subset X_0, \\ f_0^* \cdots f_I^* \hat{A}'_I & \text{over } X_I \setminus B_{Is}(N_I^{-1}\lambda_I^{1/2}), \end{cases}$$

where $\hat{A}'_I$ is the C^∞ connection over X_I, $I > 0$, defined by

$$\hat{A}'_I = \begin{cases} f_{I_+}^* \cdots f_J^* \psi_J A_J & \text{over the regions } f_{I_+}^{-1} \circ \cdots \circ f_J^{-1}(X'_J) \subset X_I \\ \psi_I A_I & \text{over the complement of these regions in } X_I. \end{cases}$$

It is convenient to consider the estimates over these different regions of X separately.

Step 1. *Estimate of* $\partial f_0^* \cdots f_{I_-}^* \psi_{I_-} A_{I_-}/\partial \lambda_I$. We have $\hat{A}' = f_0^* \cdots f_{I_-}^* \psi_{I_-} A_{I_-}$, which is supported on $U_1 \equiv f_0^{-1} \circ \cdots \circ f_{I_-}^{-1}(X'_{I_-}) \subset X_0$, and therefore $\frac{\partial}{\partial \lambda_I}\hat{A}' = f_0^* \cdots f_{I_-}^* \frac{\partial}{\partial \lambda_I}\psi_{I_-} A_{I_-}$ on U_1. Lemma 3.19 implies that

$$\left\| f_0^* \cdots f_{I_-}^* \frac{\partial \psi_{I_-} A_{I_-}}{\partial \lambda_I} \right\|_{L^2(U_1,g_0)} \le C \left\| \frac{\partial \psi_{I_-} A_{I_-}}{\partial \lambda_I} \right\|_{L^2(X'_{I_-},g_{I_-})}$$

We have $\sigma_I^* \psi_{I_-} A_{I_-} = \psi_{I_-} \sigma_I^* A_{I_-}$, where the section σ_I is chosen so that $\sigma_I^* A_{I_-}$ is in radial gauge, and so the pointwise estimates of Lemmas 3.1, 3.3, and Eq. (3.46) yield

$$\left| \frac{\partial \psi_{I_-} A_{I_-}}{\partial \lambda_I} \right|_{g_{I_-}} \le \begin{cases} C\lambda_I^{-1/2} & \text{on } \Omega_I(\tfrac{1}{2}b_I, b_I), \\ 0 & \text{on } X_{I_-} \setminus B_I(b_I). \end{cases}$$

Noting that $g = \tilde{g}_{I_-}$ on $X_{I_-} \setminus B_I(b_I)$, we obtain the integral bound

$$\int_{X'_{I_-}} \left| \frac{\partial \psi_{I_-} A_{I_-}}{\partial \lambda_I} \right|_g^2 dV_g \le C\lambda_I,$$

and combining the preceding integral estimates gives $\|\frac{\partial}{\partial \lambda_I}\hat{A}'\|_{L^2(U_1,g_0)} \le C\lambda_I^{1/2}$, completing Step 1.

Step 2. *Estimate of* $\partial f_0^* \cdots f_I^* \psi_I A_I/\partial \lambda_I$. We denote $\hat{A}' = f_0^* \cdots f_I^* \psi_I A_I$, which is supported on $U_2 \equiv f_0^{-1} \circ \cdots \circ f_I^{-1}(X'_I) \subset X_0$, and so $\frac{\partial}{\partial\lambda_I}\hat{A}' = f_0^* \cdots f_{I_-}^* \frac{\partial}{\partial\lambda_I} f_I^*\psi_I A_I$ on U_2. Repeated application of Lemma 3.19 then gives the integral bound

$$\left\| f_0^* \cdots f_{I_-}^* \frac{\partial f_I^*\psi_I A_I}{\partial\lambda_I}\right\|_{L^2(U_2,g_0)} \leq C \left\|\frac{\partial f_I^*\psi_I A_I}{\partial\lambda_I}\right\|_{L^2(X'_{I_-},g_{I_-})}.$$

Recall that Eq. (3.9) implies $\frac{\partial}{\partial\lambda_I} f_I^*\psi_I A_I = -\lambda_I^{-1} f_I^*\iota_{\mathbf{r}} F(\psi_I A_I)$ on B'_I. The curvature $F(\psi_I A_I)$ is supported on $X_I \setminus B_{Is}(\frac{1}{2}b_I)$, and $\frac{\partial}{\partial\lambda_I} f_I^*\psi_I A_I$ is supported on $B_I(\frac{1}{2}N_I^{-1}\lambda_I^{1/2})$. Then,

$$|\phi_{In}^*\iota_{\mathbf{r}} F(\psi_I A_I)|_{\delta_I}(x) \leq K\frac{|x|}{(1+|x|^2)^2},$$

and since $\phi_I^* f_I^*\iota_{\mathbf{r}} F(\psi_I A_I)(x) = \lambda_I^{-1}\phi_{In}^*\iota_{\mathbf{r}} F(\psi_I A_I)(x/\lambda_I)$, we obtain

$$\left|\phi_I^*\frac{\partial f_I^*\psi_I A_I}{\partial\lambda_I}\right|_{\delta_{I_-}}(x) \leq \begin{cases} 4K\dfrac{\lambda_I^2|x|}{(\lambda_I^2+|x|^2)^2} & \text{if } |x| < \frac{1}{2}N_I^{-1}\lambda_I^{1/2}, \\ 0 & \text{if } |x| \geq \frac{1}{2}N_I^{-1}\lambda_I^{1/2}, \end{cases}$$

where $K \equiv \|F(\psi_I A_I)\|_{L^\infty(X_I,g_I)}$ is bounded by a constant C independent of λ_I by Lemma 3.9. But $g = \tilde{g}_{I_-}$ on $B_I(\frac{1}{2}N_I^{-1}\lambda_I^{1/2}) \subset X''_{I_-}$, and moreover, the metrics $\tilde{g}_{I_-}$, g_{I_-}, and δ_{I_-} are equivalent over the ball $B_I(\frac{1}{2}N_I^{-1}\lambda_I^{1/2})$, with constants depending at most on x_I. Thus, we obtain the integral estimate

$$\int_{X'_{I_-}}\left|\frac{\partial f_I^* A'}{\partial\lambda_I}\right|_g^2 dV_g \leq C\lambda_I^2,$$

and so, combining these bounds, we have $\|\frac{\partial}{\partial\lambda_I}\hat{A}'\|_{L^2(U_2,g_0)} \leq C\lambda_I$, completing Step 2.

Step 3. *Estimate of* $\partial f_0^* \cdots f_I^*\hat{A}'_I/\partial\lambda_I$. We have $\hat{A}'_I = f_{I_+}^* \cdots f_J^*\psi_J A_J$ over $V_3 \equiv f_{I_+}^{-1}\circ\cdots\circ f_J^{-1}(X'_J) \subset B'_{I_+} \subset X_I$, with $J > I$. We denote $\hat{A}' = f_0^* \cdots f_I^*\hat{A}'_I$ and observe that $\frac{\partial}{\partial\lambda_I}\hat{A}' = f_0^* \cdots f_{I_-}^* \frac{\partial}{\partial\lambda_I} f_I^*\hat{A}'_I$ over $U_3 \equiv f_0^{-1}\circ\cdots\circ f_I^{-1}(V_3) \subset X_0$. Thus,

$$\begin{aligned}\frac{\partial f_I^*\hat{A}'_I}{\partial\lambda_I} &= -\lambda_I^{-1} f_I^*\iota_{\mathbf{r}} F(\hat{A}'_I) = -\lambda_I^{-1} f_I^*\iota_{\mathbf{r}} F(f_{I_+}^* \cdots f_J^*\psi_J A_J) \\ &= -\lambda_I^{-1}\iota_{f_{I*}^{-1}\mathbf{r}} f_I^* \cdots f_J^* F(\psi_J A_J).\end{aligned}$$

Note that $\partial\hat{A}'_I/\partial\lambda_I$ is supported on $f_I^{-1}(B'_{I_+}) \subset B'_I$.

As $\mathbf{r} = y^\mu\partial/\partial y^\mu$ with respect to $y = \phi_{In}^{-1}$ on $X_I \setminus \{x_{Is}\}$, we have $f_{I*}^{-1}\mathbf{r} = x^\mu\partial/\partial x^\mu$ with respect to $x = \phi_I^{-1}$ on B'_I. If $|y| \le R_0$ on B'_{I_+}, for some constant $0 < R_0 < \infty$ depending at most on x_I, then $|x| \le R_0\lambda_I$ on $f_I^{-1}(B'_{I_+})$. Thus, $|f_{I*}^{-1}\mathbf{r}|_{g_{I_-}} \le R_0\lambda_I$ on $f_I^{-1}(B'_{I_+})$ and so we have the pointwise bound

$$\left|\frac{\partial\hat{A}'_I}{\partial\lambda_I}\right|_{g_{I_-}} \le R_0|f_I^*\cdots f_J^*F(A'_J)|_{g_{I_-}} \qquad \text{on } f_I^{-1}(B'_{I_+}).$$

Therefore, with the aid of repeated applications of Lemma 3.19, we find that

$$\begin{aligned}\left\|f_0^*\cdots f_{I_-}^*\frac{\partial\hat{A}'_I}{\partial\lambda_I}\right\|_{L^2(U_3,g_0)} &\le C\left\|\frac{\partial\hat{A}'_I}{\partial\lambda_I}\right\|_{L^2(f_I^{-1}(V_3),g_{I_-})}\\ &\le C\|f_I^*\cdots f_J^*F(A'_J)\|_{L^2(f_I^{-1}(V_3),g_{I_-})}\\ &\le C\|F(A'_J)\|_{L^2(X'_J,g_J)}.\end{aligned}$$

and since $\|F(A'_J)\|_{L^2(X'_J,g_J)} \le C$, this gives $\left\|\frac{\partial}{\partial\lambda_I}\hat{A}'\right\|_{L^2(U_3,g_0)} \le C$, completing Step 3. Combining the results from Steps 1 to 3 then yields (a). For (b) we use the cutoff function estimate $|\partial\psi_J/\partial p_I| \le C\lambda_I^{-1/2}$, $J = I_-, I$. The vector field $\mathbf{r}$ is replaced by $\mathbf{p} = p_I^\mu\partial/\partial y^\mu$, with respect to the coordinates $y = \phi_{In}^{-1}$. Then, $f_{I*}^{-1}\mathbf{p} = \lambda_I p_I^\mu\partial/\partial x^\mu$ with respect to the coordinates $x = \phi_I^{-1}$ and we have the vector field estimate $|f_{I*}^{-1}\mathbf{p}| \le R_0\lambda_I$ on $f_I^{-1}(B'_{I_+})$. The required bound hence follows by an argument similar to that of (a).

3.8. Derivatives with respect to bundle gluing parameters. The purpose of this section is to obtain estimates for the derivatives of the almost ASD connections A' and $\hat{A}'$ with respect to the bundle gluing parameters $\rho_I \in \mathrm{Gl}_I$, $I > 0$. These estimates may be extracted from [7 (§7.2)] and we include them here for completeness.

Since we wish to differentiate a family of connections $A'(\rho_I)$ on a family of G-bundles $P(\rho_I)$ with respect to the gluing parameters $\rho_I \in \mathrm{Gl}_I$, we first pull this family back to an equivalent family on a *fixed* bundle, say $P(\bar{\rho}_I)$, as described in [7 (p. 296)]. Let $\bar{\rho}_I \in \mathrm{Gl}$ be a given gluing parameter; then points ρ_I in a small neighbourhood of $\bar{\rho}_I$ in Gl_I can be written in the form $\rho = \bar{\rho}_I\exp(v)$, where $v \in V_I \equiv \mathrm{ad}\,P_I|_{x_{Is}} \simeq \mathfrak{g}$.

One regards the fibres of P_{I_-} and P_I as being identified by $\bar\rho_I$ and so v may be considered as a local section of both P_{I_-} and P_I, covariantly constant with respect to the connections A'_{I_-}, A'_I.

We digress in order to construct a set of cutoff functions $\{\gamma_I\}$ on X such that $\sum_{I\in\mathcal{I}}\gamma_I = 1$. These cutoffs will be needed here and again in §5.1 for patching together certain integral operators over the X_I to give an integral operator over X. Choose a bump function $\gamma\in C^\infty(\mathbb{R}^1)$ such that $\gamma(t)=1$ if $t\geq 2$ and $\gamma(t)=0$ if $t\leq\frac12$. Define a cutoff function $\gamma_\lambda\in C^\infty(\mathbb{R}^4)$ by

$$\gamma_\lambda(x)\equiv\gamma(|x|/\lambda^{1/2}),\qquad x\in\mathbb{R}^4. \tag{3.52}$$

Now define C^∞ cutoff functions γ_I on each summand X_I by setting

$$\gamma_I\equiv(\phi_{Is}^{-1})^*(1-\gamma_{\lambda_I})\prod_{I_+}(\phi_{I_+}^{-1})^*\gamma_{\lambda_{I_+}}\quad\text{on } X_I, \tag{3.53}$$

where the factor $(\phi_{Is}^{-1})^*(1-\gamma_{\lambda_I})$ is omitted when $I=0$. Note that $\gamma_I=0$ on the balls $B_{Is}(\frac12\lambda_I^{1/2})$ and $B_{I+}(\frac12\lambda_{I_+}^{1/2})$ in X_I. We extend γ_I to a C^∞ cutoff function on X_I by zero on these balls and by 1 on the complement of the larger balls $B_{Is}(2\lambda_I^{1/2})$ and $B_{I+}(2\lambda_{I_+}^{1/2})$ in X_I; then extend by zero outside $X_I''\subset X$ to give $\gamma_I\in C^\infty(X)$. By construction, we have $\sum_{I\in\mathcal{I}}\gamma_I=1$ on X, with a slight abuse of notation. Indeed, note that f_I maps the annulus $\Omega_I(\frac12\lambda_I,2\lambda_I)$ around the point x_I in X_{I_-} onto the annulus $\Omega_{Is}(\frac12\lambda_I,2\lambda_I)$ around the south pole x_{Is} in X_I. Thus, $f_I^*\gamma_I+\gamma_{I_-}=1$ on each annulus Ω_I. Lastly, note that there is a constant C, depending at most on the metric g_0, such that

$$|d\gamma_I|_{g_I}\leq C\lambda_I^{-1/2}\quad\text{on }\Omega_I,\Omega_{Is},\qquad \|d\gamma_I\|_{L^p(X_I,g_I)}\leq C\bar\lambda^{2/p-1/2}, \tag{3.54}$$

for any $1\leq p<\infty$. Define gauge transformations $u_{I_-}(v)$ on $\operatorname{Aut}P_{I_-}|_{X'_I}$ and $u_I(v)$ on $\operatorname{Aut}P_I|_{X'_I}$ by setting

$$\begin{aligned} u_{I_-}(v) &= \begin{cases}\exp(\gamma_I v) & \text{on } \Omega_I,\\ 1 & \text{on } X'_{I_-}\setminus\Omega_I,\end{cases}\\ u_I(v) &= \begin{cases}\exp(-\gamma_{I_-}v) & \text{on } \Omega_{Is},\\ 1 & \text{on } X'_I\setminus\Omega_{Is}.\end{cases}\end{aligned} \tag{3.55}$$

Note that u_I has a natural extension to a gauge transformation of P_I over all of X_I — equal to $\exp(-v)$ on $B_{Is}(N_I^{-1}\lambda_I^{1/2})$, the ball enclosed by the annulus Ω_{Is}. Similarly for the gauge transformation u_{I_-}. After identifying the bundles and base manifolds over $\Omega = \Omega_I = \Omega_{Is}$, we have $u_{I_-}u_I^{-1} = \exp((\gamma_{I_-} + \gamma_I)v) = \exp(v)$. Hence, relative to the flat connections A'_{I_-}, A'_I, the gauge transformations u_I differ by a constant bundle automorphism over Ω, and so their action on the connection $A'(\bar\rho_I)$ is the same: $u_{I_-}(A'(\bar\rho_I))|_\Omega = u_I(A'(\bar\rho_I))|_\Omega$. Therefore, while the automorphisms u_I do *not* patch together to give a global automorphism of $P(\bar\rho_I)$, their actions on the connection $A'(\bar\rho_I)$ do. Indeed, we can define a connection $A'(\bar\rho_I, v)$ on $P(\bar\rho_I)$ by

$$(3.56)\qquad A'(\bar\rho_I, v) = \begin{cases} u_{I_-}(A'(\bar\rho_I)) & \text{on } X'_{I_-}, \\ u_I(A'(\bar\rho_I)) & \text{on } X'_I. \end{cases}$$

If $\rho_I = \bar\rho_I \exp(v)$, the connections $A'(\bar\rho_I, v)$ and $A'(\rho_I)$ are gauge equivalent [7 (p. 296)]. Thus, as desired, we have an equivalent family of connections $A'(\bar\rho_I, v)$ on the fixed connected sum bundle $P = P(\bar\rho_I)$. Let $L_I \subset \mathrm{Gl}_I$ be a coordinate neighbourhood and suppose $\bar\rho_I \in L_I$. Then

$$(3.57)\qquad \mathfrak{g} \supset B_{\mathfrak{g}} \longrightarrow L_I \subset \mathrm{Gl}_I, \qquad v \longmapsto \rho_I(v) \equiv \bar\rho_I \exp(v)$$

is a coordinate chart centred at $\bar\rho_I$, where $B_{\mathfrak{g}}$ is the unit ball in $\mathfrak{g}$, and there is a C^∞ embedding

$$(3.58)\qquad \mathfrak{g} \supset B_{\mathfrak{g}} \longrightarrow \mathcal{A}^*_{X,P}, \qquad v \longmapsto A'(\bar\rho_I, v).$$

It remains to consider the derivative of the family $A'(\bar\rho_I, v)$ with respect to v.

Recall that if $u = u(s)$ is a one-parameter family of gauge transformations, B is a fixed connection, and $B^u(s)$ is the induced one-parameter family of gauge transformed connections, then $dB^u/ds(0) = d_{B^u}(u^{-1}\dot u(0))$, where $u^{-1}\dot u(0) \in \Omega^0(X, \operatorname{ad} P)$. Although the u_{I_-}, u_I are not globally defined gauge transformations, this differentiation formula still applies to the one-parameter families $u_{I_-}(s) = u_{I_-}(sv)$ and $u_I(s) = u_I(sv)$. Therefore, we have

$$(3.59)\qquad \frac{\partial A'}{\partial v}\bar\rho_I) \equiv \frac{d}{ds}A'(\bar\rho_I, sv)\Big|_{s=0} = \begin{cases} d_{A'}(\gamma_I v) & \text{on } X'_{I_-} \cap \Omega, \\ -d_{A'}(\gamma_{I_-} v) & \text{on } X'_I \cap \Omega, \\ 0 & \text{on } X \setminus \Omega. \end{cases}$$

This leads to the following estimate for the derivative of the family $A'(\rho_I)$ with respect to the gluing parameters ρ_I; a related and more general estimate is given by Lemma 7.2.49 in [7].

Proposition 3.28. *Let $2 \le p < 4$ and suppose that $4 \le q < \infty$ is determined by $1/4+1/q = 1/p$. Then there is a constant $c = c(g_0, p, \mathcal{T})$ such that*

(a) $c|v|\lambda_I^{2/p-1} \le \|\partial A'/\partial v\|_{L^q(X,g)} \le c^{-1}|v|\lambda_I^{2/p-1}$,

(b) $c|v|\lambda_I^{2/p-1/2} \le \|\partial A'/\partial v\|_{L^p(X,g)} \le c^{-1}|v|\lambda_I^{2/p-1/2}$.

Proof. Note that $\gamma_{I\_} + \gamma_I = 1$ on Ω and so $d_{A'}(\gamma_I v) = -d_{A'}(\gamma_{I\_} v)$ on Ω. Moreover, $d_{A'}(\gamma_I v) = d\gamma_I \otimes v$ on Ω, and so we have $\|d_{A'}(\gamma_I v)\|_{L^q(X,g)} = |v| \cdot \|d\gamma_I\|_{L^q(X,g)}$. From Eq. (3.54) there is a constant $c > 0$ independent of λ_I such that

$$c|v|\lambda_I^{2/q-1/2} \le \left\|\frac{\partial A'}{\partial v}\right\|_{L^q(X,g)} \le c^{-1}|v|\lambda_I^{2/q-1/2},$$

since $\|\partial A'/\partial v\|_{L^q(X,g)} = \|d_{A'}(\gamma_I v)\|_{L^q(X,g)}$. Then (a) follows since $2/q - 1/2 = 2/p - 1$, and likewise for (b).

Using the conformal maps f_J, we pull back the family $A' = A'(\bar\rho_I, v)$ on the fixed bundle P over X to a family $\hat A'(\bar\rho_I, v)$ on the fixed bundle $\hat P$ over X_0.

Proposition 3.29. *If $2 \le p < 4$, there is a constant $C = C(g_0, p, \mathcal{T})$ such that for any $t \in \mathcal{T}$, $\|\partial \hat A'/\partial v\|_{L^p(X_0,g_0)} \le C\lambda_I^{2/p-1/2}$.*

Proof. Since $\partial A'/\partial v = 0$ outside the annulus $\Omega_{Is} \subset X_I'$, Proposition 3.28 gives

$$\left\|\frac{\partial A'}{\partial v}\right\|_{L^p(X_I',g_I)} \le C\lambda_I^{2/p-1/2}.$$

But $\hat A' = f_0^* \cdots f_I^* A'$ on $U \equiv f_0^{-1} \circ \cdots \circ f_I^{-1}(X_I') \subset X_0$, and so Lemma 3.19 yields

$$\left\|f_0 \cdots f_I^* \frac{\partial A'}{\partial v}\right\|_{L^p(U,g_0)} \le C \left\|\frac{\partial A'}{\partial v}\right\|_{L^p(X_I',g_I)}$$

Combining these estimates gives the desired bound.

3.9. Derivatives with respect to lower moduli. In this section we obtain L^p estimates for the derivatives of the connections A', $\hat A'$, and the self-dual curvature $F^{+,g}(A')$ with respect to the 'lower moduli parameters' t_I. The bundle P_I carrying the family of connections

$\{A_I(t_I)\}_{t_I\in T_{A_I}}$ can be assumed to be fixed with respect to the parameters $t \in T_{A_I}$ since the space T_{A_I} — an open ball in $H^1_{A_I}$ centred at 0 — is contractible. However, the local sections $\sigma_{I+}(t_I)$ are defined by the connections $A_I(t_I)$ (together a choice of point in $P_I|_{x_{I_+}}$) and will vary with t_I. Thus, the bundle gluing maps for the connected sum bundle P, defined by $\sigma_{I_+}(t_I) \mapsto \sigma_{I_+s}\tilde{\rho}_{I_+}(t_I)$ (suppressing the identification map $f_I : \Omega_{I_+} \to \Omega_{I_+s}$), will in general vary with t_I. We may suppose that the remaining parameters are fixed and thus we obtain a family of connections $A'(t_I)$ on a family of bundles $P(t_I)$. The difficulty, of course, is that unless we have a family of connections defined on a fixed bundle, we cannot define the derivative $\partial A'/\partial t_I$. Problems such as these are discussed in [4 (p. 423)]. For our purposes, we note the bundles are all isomorphic and as T_{A_I} is contractible, the connections $A'(t_I)$ could be pulled back by bundle isomorphisms $h_I \in \mathrm{Hom}(P(0), P(t_I))$ to an equivalent family $h_I^* A'(t_I)$ on the *fixed* bundle $P(0)$, and then we could define

$$\frac{\partial A'}{\partial t_I} \equiv \frac{\partial h_I^* A'}{\partial t_I}. \tag{3.60}$$

Since any two such families $h_I(t_I)$ of bundle isomorphisms would differ by a family of automorphisms of the fixed bundle $P(0)$, by using (3.60), $\partial A'/\partial t_I$ would give a well-defined tangent vector to $\mathcal{B}^*_{P(0)}$ at $[A'(t_I)]$. Naturally, the analogous remarks apply to the family of connections $\hat{A}'(t_I)$ on the bundles $\hat{P}(t_I)$.

In our case, a family of isomorphisms $h_J(t_I) : P_J(0) \to P_J(t_I)$ may be described quite explicitly, in a manner similar to that of §3.8, and these will give a gauge equivalent family of connections $h_I^* A'(t_I)$, $\hat{h}_I^* \hat{A}'(t_I)$ on fixed bundles $P(0)$, $\hat{P}(0)$ respectively, although just as in §3.8, the isomorphisms $h_J(t_I)$ will *not* patch together to give a global isomorphism of $P(0)$ with $P(t_I)$ or $\hat{P}(0)$ with $\hat{P}(t_I)$. Nonetheless Eq. (3.60) still makes sense and this allows us to estimate the length of the tangent vector $\partial A'/\partial t_I$ in terms of derivatives of the local connection one-forms, as desired. Let $h_I(t_I) : P_I(0) \to P_I(t_I)$ be a family of bundle isomorphisms represented locally by $\sigma_{I_+}(0) \mapsto \sigma_{I_+}(t)\theta_{I_+}(t_I)$. Then $h_I^* A_I(t_I)$ is an equivalent family on the fixed bundle $P_I(0)$, with

$$\begin{aligned}\sigma_{I_+}(0)^* h_I(t_I)^* A_I(t_I) &= \theta_{I_+}(t_I)^{-1}\sigma_{I_+}(t_I)^* A_I(t_I)\theta_{I_+}(t_I)\\ &\quad + \theta_{I_+}(t_I)^{-1} d\theta_{I_+}(t_I).\end{aligned}$$

Note that while the local connection one-forms $\sigma_{I_+}(t_I)^*A_I(t_I)$ are in radial gauge, this will *not* in general be the case for the one-forms $\sigma_{I_+}(0)^*h_I(t_I)^*A_I(t_I)$. We next consider the variation in the bundle gluing maps $\tilde{\rho}_{I_+}(t_I)$ induced by the variation in $\sigma_{I_+}(t_I)$ with t_I. Over X_I, we replace $\theta_{I_+}(t_I)$ above by $\theta_{I_+}(t_I)\exp(\gamma_{I_+}v_I(t_I))$ and over X_{I_+}, define $h_I(t_I)$ by right multiplication with $\exp(\gamma_I v_I(t_I))$. Recalling the notation of §3.8, $v_I : T_{A_I} \to \mathfrak{g}$ is a smooth map with $v(0) = 0$ defined (for small enough T_{A_I}) by the identity $\rho_{I_+}(t_I) = \rho_{I_+}(0)\exp(v_I(t_I))$. Lastly, for $J \neq I, I_+$, we set $h_J(t_I) = 1$. *Then, for the remainder of this article, we require that the derivatives $\partial A'/\partial t_I$ be defined by (3.60).*

This understood, we obtain the following estimates for the derivatives with respect to the parameters t_I of the connections A' and $\hat{A}'$ and for the g-self-dual curvature $F^{+,g}(A')$. The proofs are straightforward, following the pattern in §3.7, and so are omitted.

Proposition 3.30. *Let $1 \leq p < \infty$. For sufficiently small b_0, there exists a constant $C = C(g_0, p, \mathcal{T})$ such that for any $t \in \mathcal{T}$,*
(a) $\|\partial A'/\partial t_I - \partial A_I/\partial t_I\|_{L^p(X_I'',g_I)} \leq C\lambda_I^{2/p}$,
(b) $\|\partial A'/\partial t_I\|_{L^p(X,g)} \leq C$.

Proposition 3.31. *Let $1 \leq p < \infty$. For sufficiently small b_0, there is a constant $C = C(g_0, p, \mathcal{T})$ such that for any $t \in \mathcal{T}$,*

$$\|\partial F^{+,g}(A')/\partial t_I\|_{L^p(X,g)} \leq C\bar{\lambda}^{2/p-1/2}.$$

Proposition 3.32. *For sufficiently small b_0, there is a constant $C = C(g_0, N, \mathcal{T})$ such that for any $t \in \mathcal{T}$, $\|\partial\hat{A}'/\partial t_I\|_{L^2(X_0,g_0)} \leq C$.*

Proof. Let $U \equiv f_0^{-1}\cdots f_I^{-1}(X_I') \subset X_0$ and note that $\frac{\partial}{\partial t_I}\hat{A}' = \frac{\partial}{\partial t_I}f_0^*\cdots f_I^*\psi_I A_I$, which is equal to $f_0^*\cdots f_I^*\frac{\partial}{\partial t_I}\psi_I A_I$ on U and is zero elsewhere. Now

$$\left\|f_0^*\cdots f_I^*\frac{\partial\psi_I A_I}{\partial t_I}\right\|_{L^2(U,g_0)} \leq C\left\|\frac{\partial\psi_I A_I}{\partial t_I}\right\|_{L^2(X_I',g_I)}$$

by Lemma 3.19 and so the result follows.

3.10. Differentials of the approximate gluing maps. We close this Chapter by summarising the results of the preceding sections and record our bounds for the differentials of the approximate gluing maps $\mathcal{J}'$ (which follow by combining Propositions 3.25, 3.30, and 3.28)) and $\hat{\mathcal{J}}'$ (which follow by combining Propositions 3.27, 3.32, and 3.29).

Theorem 3.33. *Let $\mathcal{J}' : \mathcal{T} \to \mathcal{B}^*_{X,k}$ be the approximate gluing map $t \mapsto [A'(t)]$. Assume $b_I = 4N_I\lambda_I^{1/2}$ for all I. Then for sufficiently small λ_0 and any $t \in \mathcal{T}$, there is a constant $C = C(g_0, \mathcal{T})$ such that the following estimates hold.*

(a) $\|D\mathcal{J}'(\partial/\partial t_I^\alpha)\|_{L^2(X,g)} \le C$,

(b) $\|D\mathcal{J}'(\partial/\partial \rho_I^\beta)\|_{L^2(X,g)} \le C\lambda_I^{1/2}$,

(c) $\|D\mathcal{J}'(\partial/\partial x_I^\mu)\|_{L^2(X,g)} \le C$,

(d) $\|D\mathcal{J}'(\partial/\partial \lambda_I)\|_{L^2(X,g)} \le C$.

Theorem 3.34. *Let $\hat{\mathcal{J}}' : \mathcal{T} \to \mathcal{B}^*_{X_0,k}$ be the approximate gluing map $t \mapsto [\hat{A}'(t)]$. Let $b_I = 4N_I\lambda_I^{1/2}$ for all I. Then for sufficiently small λ_0 and any $t \in \mathcal{T}$, there is a constant $C = C(g_0, \mathcal{T})$ such that the following estimates hold.*

(a) $\|D\hat{\mathcal{J}}'(\partial/\partial t_I^\alpha)\|_{L^2(X_0,g_0)} \le C$,

(b) $\|D\hat{\mathcal{J}}'(\partial/\partial \rho_I^\beta)\|_{L^2(X_0,g_0)} \le C\lambda_I^{1/2}$,

(c) $\|D\hat{\mathcal{J}}'(\partial/\partial x_I^\mu)\|_{L^2(X_0,g_0)} \le C$,

(d) $\|D\hat{\mathcal{J}}'(\partial/\partial \lambda_I)\|_{L^2(X_0,g_0)} \le C$.

4. Bubble tree compactification of the moduli space of anti-self-dual connections

In order to describe the ends of the moduli space $M_{X_0,k}(g_0)$ one customarily appeals to the Uhlenbeck compactification $\overline{M}^u_{X_0,k}(g_0)$. This allows one to give quite explicit descriptions of the parts of the ends away from the diagonals in the symmetric products $M_{X_0,k}(g_0) \times s^l(X_0)$ appearing in the compactification, as for example in [3 (§V)] and [7 (§8.2)]. These examples consider ideal boundary points of the form $(A_0, x_1, \ldots, x_l)$, where the x_i are distinct points with multiplicity 1, and A_0 is a g_0-anti-self-dual connection over X_0. Open neighbourhoods of $(A_0, x_1, \ldots, x_l)$ in $\overline{M}^u_{X_0,k}(g_0)$ are then constructed by gluing standard one-instantons onto A_0.

In order to construct open neighbourhoods of ideal boundary points corresponding to the diagonals of $\overline{M}^u_{X_0,k}(g_0)$ we must employ the iterated gluing construction of Chapters 3 and 5. This strategy is mentioned briefly in [7 (§8.2)]. The construction gives a homeomorphism $\hat{\mathcal{J}} : \mathcal{T}^0/\Gamma \to \overline{\mathcal{V}}$, where $\overline{\mathcal{V}}$ is an open neighbourhood of a boundary point in $\overline{M}^u_{X_0,k}(g_0)$ — a 'gluing neighbourhood'. In order to use this procedure to describe the ends of $\overline{M}^u_{X_0,k}(g_0)$, we need to show that $\overline{M}^u_{X_0,k}(g_0)$ is covered by finitely many such gluing neighborhoods. In particular, we need to show that any point in $M_{X_0,k}(g_0)$ which is sufficiently close to the ideal boundary (with respect to the Uhlenbeck topology) lies in the image of a gluing map $\hat{\mathcal{J}}$. This is accomplished in two steps:

Step 1. We show that any sequence $\{A_\alpha\}$ of g_0-anti-self-dual connections over X_0 converging *weakly* to a limit $(A_0, x_1, \dots, x_{m_0})$ determines a sequence of metrics $\{g_\alpha\}$ and a sequence $\{\check{A}_\alpha\}$ of g_α-anti-self-dual connections over a connected sum $X \equiv \#_{I\in\mathcal{I}} X'_{I\alpha}$ which converges *strongly* to a limit $(A_I)_{I\in\mathcal{I}}$, in the sense of [7 (§7.3)]. Here, (X, g_α) is conformally equivalent to (X_0, g_0) for all α, and is defined exactly as in §3.3 and §3.5.

Step 2. We apply an analogue of Theorem 7.3.2 [7] to show that the new sequence $\{\check{A}_\alpha\}$ is D_q-convergent, $q \geq 4$, in the sense of [7 (§7.3)]. The appropriate analogue of Theorem 7.2.62 [7] then shows that the points $[A_\alpha] \in M_{X,k}(g_\alpha)$ lie in the image of some $\mathcal{J}$ for sufficiently large α. Consequently, the points $[A_\alpha] \in M_{X_0,k}(g_0)$ lie in the image of the corresponding map $\hat{\mathcal{J}}$, for some parameter space $\mathcal{T}^0/\Gamma$. The choice of parameter space $\mathcal{T}^0/\Gamma$ is essentially determined by $(A_I)_{I\in\mathcal{I}}$, which we call the *strong* or *bubble tree limit* of the sequence $\{A_\alpha\}$. . In this Chapter we discuss Step 1 and describe the bubble tree compactification of the moduli space of anti-self-dual SU(2) connections — the extension to the general case of compact, semi-simple Lie groups being straightforward. Step 2 is discussed in §§5.1 and 5.2 after the necessary analytical framework has been established. Throughout this Chapter, we suppose only that X_0 is a closed, oriented, simply-connected C^∞ four-manifold, g_0 is a C^∞ metric, and $G = \mathrm{SU}(2)$.

4.1. Uhlenbeck compactification. We recall the definition of the Uhlenbeck compactification [7] and describe some of the related convergence results we will need for our description of the bubble tree compactification.

Definition 4.1. An *Uhlenbeck ideal g_0-anti-self-dual connection* on a G bundle P over X_0 with $c_2(P) = k \geq 0$ is a pair (A_0, Z_0), where A_0 is a g_0-anti-self-dual connection on a G bundle P_0 over X_0 with $c_2(P_0) = k_0 \geq 0$ and $Z_0 = \{x_i\}_{i=1}^{m_0}$ is a (possibly empty) set of points in X_0 with multiplicities $k_i \geq 1$, for $i = 1, \dots, m_0$, such that $\sum_{i=0}^{m_0} k_i = k$.

The curvature density of (A_0, Z_0) is defined to be the Borel measure

$$\mu(A_0, Z_0) = |F(A_0)|^2_{g_0} + 8\pi^2\delta_{Z_0}, \tag{4.1}$$

where $\delta_{Z_0} \equiv \sum_{i=1}^{m_0} k_i\delta_{x_i}$, so that the total mass of $\mu(A_0, Z_0)$ is $8\pi^2 k$. Setting $l = k_1 + \cdots + k_m$ and repeating points according to their multiplicity, one obtains an element $(x_1, \dots, x_l)$ of the symmetric product $s^l(X_0)$.

Definition 4.2. Let $\{A_\alpha\}_{\alpha=1}^\infty$, be a sequence of g_0-anti-self-dual connections on a G bundle P over X_0 with $c_2(P) = k \geq 0$ and let (A_0, Z_0) be an ideal g_0-anti-self-dual connection on P. Then the sequence $\{A_\alpha\}$ *converges weakly* to (A_0, Z_0) if the following hold:

(a) The sequence $\{\mu_\alpha\}_{\alpha=1}^\infty$ converges to $\mu(A_0, Z_0)$ in the weak-* topology on measures.

(b) There is a sequence of C^∞ bundle maps $\gamma_\alpha : P_0|_{X_0\setminus Z_0} \to P|_{X_0\setminus Z_0}$ such that $\gamma_\alpha^* A_\alpha$ converges in C^∞ on compact subsets of $X_0 \setminus Z_0$ to the connection A_0. Equivalently, require that for any integer $n \geq 1$, there is a sequence of L^2_{n+1} bundle maps γ_α such that $\gamma_\alpha^* A_\alpha$ converges in $L^2_{n,\mathrm{loc}}$ on $X_0 \setminus Z_0$ to A_0.

Via the natural extension of Definition 4.2 to sequences of ideal connections, the set of all Uhlenbeck ideal g_0-anti-self-dual connections of fixed second Chern class k,

$$IM_{X_0,k}(g_0) \equiv \coprod_{l=0}^{k} (M_{X_0,k-l}(g_0) \times s^l(X_0)),$$

is endowed with a metrisable topology. Let $\overline{M}^u_{X_0,k}(g_0)$ be the closure of $M_{X_0,k}(g_0)$ in $IM_{X_0,k}(g_0)$. According to [7 (Theorem 4.4.4)], any infinite sequence in $M_{X_0,k}(g_0)$ has a weakly convergent subsequence with limit point in $\overline{M}^u_{X_0,k}(g_0)$, and in particular, the latter space is compact [7 (Theorem 4.4.3)].

For our description of the bubble tree compactification, we will need the following minor extension of the convergence result in Theorem 4.4.4 [7] and its cousin, Proposition 9.4.2 [7], which allows for a sequence of metrics $\{g_\alpha\}$ converging to g_0 in C^∞. The proof employs standard arguments well described in [7 (§4.4)] and is left to the reader.

Proposition 4.3. *Let $\{U_\alpha\}_{\alpha=1}^\infty$ be an exhaustion of the punctured manifold $X_0 \setminus \{p\}$ by an increasing sequence $\{U_\alpha\}_{\alpha=1}^\infty$ of precompact* *open sets, so that* $U_1 \Subset U_2 \Subset \cdots \Subset X_0 \setminus \{p\}$ *and* $\cup_{\alpha=1}^{\infty} U_\alpha = X_0 \setminus \{p\}$. *Let* $\{g_\alpha\}_{\alpha=1}^{\infty}$ *be a sequence of metrics on the subsets* U_α *converging in* C^r $(r \geq 3)$ *on compact subsets of* $X_0 \setminus \{p\}$ *to a* C^r *metric* g_0 *on* X_0. *Let* P *be a* G *bundle over* $X_0 \setminus \{p\}$ *and let* $\{A_\alpha\}_{\alpha=1}^{\infty}$ *be a sequence of* g_α*-anti-self-dual connections on the restrictions* $P|_{U_\alpha}$. *If there is a constant* $M < \infty$ *such that*

$$\int_{U_\alpha} |F(A_\alpha)|_{g_\alpha}^2 \, dV_{g_\alpha} \leq M \quad \textit{for all } \alpha,$$

then there is a set of points $Z_0 = \{x_i\}_{i=1}^{m_0} \subset X_0$ *and a* g_0*-anti-self-dual connection* A_0 *on a* G *bundle* P_0 *over* X_0 *such that a subsequence* $\{A_\alpha\}_{\alpha=1}^{\infty}$ *converges weakly to* (A_0, Z_0).

The mass of the Uhlenbeck limit (A_0, Z_0) in Proposition 4.3 is $8\pi^2$ times an integer and may be computed from the weakly convergent sequence $\{A_\alpha\}_{\alpha=1}^{\infty}$ by

$$\lim_{n\to\infty} \lim_{\alpha\to\infty} \int_{V_n} |F(A_\alpha)|_{g_\alpha}^2 \, dV_{g_\alpha}, \tag{4.2}$$

where $\{V_n\}_{n=1}^{\infty}$ is any exhaustion of $X_0 \setminus \{p\}$ by an increasing sequence of precompact open subsets.

4.2. Conformal blow-ups. Given a sequence of g_0-anti-self-dual connections on a G bundle P over X_0 with curvature densities concentrating near a set of 'singular points' in X_0, we define associated sequences of mass centres and scales. In a manner analogous to Chapter 3, we then obtain sequences of 'conformal blow-up maps' $f_{I\alpha}$ (defined exactly as in §3.3) which resolve these singularities in a sense that will be made precise below and in §4.3. As will become evident, the process of applying conformal blow-ups may need to be iterated before the singularities are completely 'resolved'.

Let us commence by defining the first level conformal blow-ups. Suppose $\{A_\alpha\}_{\alpha=1}^{\infty}$ is a sequence of g_0-anti-self-dual connections over X_0 with weak limit (A_0, Z_0). Let us consider the behaviour of the sequence $\{A_\alpha\}_{\alpha=1}^{\infty}$ in $M_{X_0,k}(g_0)$ near the singular set $Z_0 = \{x_i\}_{i=1}^{m_0}$ in more detail. If the point x_i has multiplicity k_i, then

$$\lim_{r\to\infty} \lim_{\alpha\to\infty} \int_{B(x_i,r)} |F(A_\alpha)|_{g_0}^2 \, dV_{g_0} = 8\pi^2 k_i. \tag{4.3}$$

Choose constants d_0, r_0 such that

$$(4.4)\qquad 0 < d_0 \le \min_{i \ne j} \operatorname{dist}_{g_0}(x_i, x_j), \quad 0 < r_0 < \frac{1}{4}\min\{1, \varrho_0, d_0\}.$$

We next define mass centres and scales of g_0-anti-self-dual connections restricted to the fixed ball $B(x_i, r_0) \subset X_0$ by appropriately modifying the previous definitions of mass centres and scales of §3.2 for g_1-anti-self-dual connections over $\mathbb{S}^4$. First, note that

$$(4.5)\qquad \lim_{\alpha\to\infty} \int_{B(x_i,r_0)} \left(|F(A_\alpha)|_{g_0}^2 - |F(A_0)|_{g_0}^2\right) dV_{g_0} = 8\pi^2 k_i.$$

Choose a frame v_i in $FX_0|_{x_i}$ and let $q = \phi_{x_i}^{-1}$ be the associated geodesic normal coordinate chart. For each i, define a sequence of *mass centres* $\{x_{i\alpha}\}_{\alpha=1}^\infty$ in $B(x_i, r_0)$ by $x_{i\alpha} \equiv \phi_{x_i}(q_{i\alpha})$, where $q_{i\alpha} = \operatorname{Centre}[A_\alpha|_{B(x_i,r_0)}] \in \mathbb{R}^4$ and

$$(4.6)\qquad \begin{aligned}&\operatorname{Centre}[A_\alpha|_{B(x_i,r_0)}]\\ &\quad\equiv \frac{1}{8\pi^2 k_i}\int_{B(x_i,r_0)} q\left(|F(A_\alpha)|_{g_0}^2 - |F(A_0)|_{g_0}^2\right) dV_{g_0}.\end{aligned}$$

Define a sequence of *scales* $\{\lambda_{i\alpha}\}_{\alpha=1}^\infty$ in $(0,\infty)$ by setting $\lambda_{i\alpha} = \operatorname{Scale}[A_\alpha|_{B(x_i,r_0)}]$, where

$$(4.7)\qquad \begin{aligned}&\operatorname{Scale}^2[A_\alpha|_{B(x_i,r_0)}]\\ &\quad\equiv \frac{1}{8\pi^2 k_i}\int_{B(x_i,r_0)} |q - q_{i\alpha}|^2\left(|F(A_\alpha)|_{g_0}^2 - |F(A_0)|_{g_0}^2\right) dV_{g_0}.\end{aligned}$$

As in §3.2., Eq. (4.7) leads to a *Tchebychev inequality*:

$$(4.8)\qquad \begin{aligned}\int_{B(x_i,r_0)\setminus B(x_{i\alpha},R\lambda_{i\alpha})} &\left(|F(A_\alpha)|_{g_0}^2 - |F(A_0)|_{g_0}^2\right) dV_{g_0}\\ &\le 8\pi^2 k_i R^{-2}, \quad R \ge 1.\end{aligned}$$

Hence, if $R \gg 1$ and α is sufficiently large, the balls $B(x_{i\alpha}, R\lambda_{i\alpha})$ contain most of the $8\pi^2 k_i$ quantity of A_α-energy bubbling off at $x_{i\alpha}$.

Remark 4.4. Other choices of scale function are possible. For example, we might have chosen $\lambda_{i\alpha}$ to be the radius of the ball centred at $x_{i\alpha}$ containing A_α-energy $8\pi^2(k_i - \frac{1}{2})$. A cutoff function is required in order to regularise this definition.

Thus, we obtain a sequence of scales $\{\lambda_{i\alpha}\}_{\alpha=1}^\infty$ associated to the sequences of mass centres $\{x_{i\alpha}\}_{\alpha=1}^\infty$ and connections $\{A_\alpha\}_{\alpha=1}^\infty$. Moreover, Eq. (4.3) implies that the sequence $x_{i\alpha}$ converges to x_i and that the sequence of scales $\lambda_{i\alpha}$ converges to zero. Choose a sequence of frames $v_{i\alpha} \in FX_0|_{x_{i\alpha}}$ converging to the frame $v_i \in FX_0|_{x_i}$ and let $\phi_{x_{i\alpha}}^{-1}$ be the corresponding geodesic normal coordinate charts. Let $f_{x_{i\alpha}} \equiv \phi_{in} \circ c_{\lambda_{i\alpha}} \circ \phi_{x_{i\alpha}}^{-1}$, where $c_{\lambda_{i\alpha}}$ is the dilation of $\mathbb{R}^4$ given by $x \mapsto x/\lambda_{i\alpha}$, and let $\tilde{g}_{i\alpha}$ be the approximately round metric on $X'_{i\alpha}$ defined as in §3.5. Let $P_{i\alpha} = (f_{x_{i\alpha}}^{-1})^* P$ be the induced G bundle over $X'_{i\alpha}$ and $A_{i\alpha} = (f_{x_{i\alpha}}^{-1})^* A_\alpha$ be the induced $\tilde{g}_{i\alpha}$-anti-self-dual connection on $P_{i\alpha}$. We call the maps $f_{x_{i\alpha}}$ *conformal blow-ups.*

We obtain a sequence of open subsets $X'_{i\alpha}$ which exhaust $X_i \setminus \{x_{is}\}$, a sequence of metrics $\{\tilde{g}_{i\alpha}\}_{\alpha=1}^\infty$, and a sequence of $\tilde{g}_{i\alpha}$-anti-self-dual connections $\{A_{i\alpha}\}_{\alpha=1}^\infty$ over the $X'_{i\alpha}$. The sequence $\{\tilde{g}_{i\alpha}\}_{\alpha=1}^\infty$ converges in C^∞ on compact subsets of $X_i \setminus \{x_{is}\}$ to the standard round metric g_i on $X_i \equiv \mathbb{S}^4$. Let $\{g_\alpha\}_{\alpha=1}^\infty$ be the sequence of C^∞ metrics, defined as in §3.5, on the connected sum $X \equiv \#_{i=0}^{m_0} X'_{i\alpha}$, defined as in §3.3, and let $\{\breve{A}_\alpha\}_{\alpha=1}^\infty$ be the induced sequence of g_α-anti-self-dual connections over X. We call the connected sums (X, g_α) *conformal blow-ups* of (X_0, g_0).

There is a uniform upper bound on the L^2 norms $\|F(A_{i\alpha})\|_{L^2(X'_{i\alpha}, \tilde{g}_{i\alpha})}$ since

$$\text{(4.9)} \qquad \begin{aligned} \int_{X'_{i\alpha}} |F(A_{i\alpha})|^2_{\tilde{g}_{i\alpha}}\, dV_{\tilde{g}_{i\alpha}} &= \int_{B(x_{i\alpha}, N\lambda_{i\alpha}^{1/2})} |F(A_\alpha)|^2_{g_0}\, dV_{g_0} \\ &\leq 8\pi^2(k_i + 1/2), \end{aligned}$$

for sufficiently large α by Eq. (4.3), while Eqs. (4.5) and (4.8) give a lower bound

$$\text{(4.10)} \qquad \begin{aligned} \int_{X'_{i\alpha}} |F(A_{i\alpha})|^2_{\tilde{g}_{i\alpha}}\, dV_{\tilde{g}_{i\alpha}} &= \int_{B(x_{i\alpha}, N\lambda_{i\alpha}^{1/2})} |F(A_\alpha)|^2_{g_0}\, dV_{g_0} \\ &\geq 8\pi^2(k_i - 1/2). \end{aligned}$$

Proposition 4.3 provides a subsequence $\{A_{i\alpha}\}_{\alpha=1}^\infty$ which converges weakly to an ideal g_i-anti-self-dual connection (A_i, Z_i) over X_i, where $Z_i = \{x_{ij}\}_{j=1}^{m_i}$. The energy bound of Eq. (4.8) ensures that $Z_i \subset X_i \setminus \{x_{is}\}$.

Let $\mu_i = \mu(A_i, Z_i)$ be the associated singular measure on X_i, and note that its mass may be computed by

$$\int_{X_i} d\mu_i = \lim_{R\to\infty} \lim_{\alpha\to\infty} \int_{B(x_{in},R)} |F(A_{i\alpha})|^2_{\tilde{g}_{i\alpha}}\, dV_{\tilde{g}_{i\alpha}}.$$

Since this must be $8\pi^2$ times an integer, Eqs. (4.9) and (4.10) imply that μ_i has mass $8\pi^2 k_i$, where $k_i = \sum_{j=0}^{m_i} k_{ij}$, A_i is a g_i-anti-self-dual connection on a bundle P_i over X_i with $c_2(P_i) = k_{i0}$, and each point x_{ij} has multiplicity k_{ij}.

Remark 4.5. It is not strictly necessary that we construct a sequence of honest metrics g_α over the connected sums $X = \#_{i=0}^{m_0} X'_{i\alpha}$ above; a sequence of conformal structures $[g_\alpha]$ constructed as in §3.5 would suffice and this would eliminate the need for the choice of conformal factors over the necks. In any case, the actual limits obtained are independent of such choices.

The above conformal blow-up construction produces a sequence of $\tilde{g}_{x_{i\alpha}}$-anti-self-dual connections $A_{x_{i\alpha}}$ on increasing subsets $X'_{i\alpha}$ of the four-sphere X_i with weak g_i-anti-self-dual limit (A_i, Z_i). With the inverse process of gluing in mind, we describe a modified choice of conformal blow-ups which yield *centred limits* $(\tilde{A}_i, \tilde{Z}_i)$. First, a technical lemma concerning the variation of geodesic normal coordinate charts with their coordinate centres is required. The proof uses Taylor's theorem and is left to the reader.

Lemma 4.6. *Let X_0 be a closed C^∞ n-manifold with metric g_0 and injectivity radius ϱ_0. Let $x_0 \in X$ and $x = \exp_{v_0}^{-1}$ be the geodesic normal coordinate chart on $B(x_0, \varrho_0)$ defined by a choice of frame $v_0 \in FX|_{x_0}$. Suppose $x_1 \in B(x_0, \varrho_0/4)$ and $p = \exp_{v_0}^{-1}(x_1)$, so that $\operatorname{dist}_{g_0}(x_1, x_0) = |p|$. We now define two coordinate charts on $B(x_1, \varrho_0/2)$:*

(a) Let $v_1 \in FX|_{x_1}$ be the frame obtained by parallel translating v_0 along the geodesic joining x_0 to x_1, and let $w = \exp_{v_1}^{-1}$ on $B(x_1, \varrho_0/2)$.

(b) Let τ_p be the translation on $\mathbb{R}^n$ given by $q \mapsto q-p$, and let $\bar{w} = \tau_p \circ \exp_{v_0}^{-1}$ on $B(x_1, \varrho_0/2)$. Then the coordinates $\bar{w}$ converge to w in C^∞ on $B(x_0, \varrho_0/4)$ as $p \to 0$: $|\bar{w}^\mu - w^\mu| = O(|w||p|)$, $|\partial\bar{w}^\mu/\partial w^\alpha - \delta^\mu_\alpha| = O(p)$, and for all $m \geq 2$, $\partial^m \bar{w}^\mu/\partial w^{\alpha_1} \cdots \partial w^{\alpha_m} = O(p)$.

Next, we define the mass centre and scale of a positive Borel measure

μ on $\mathbb{R}^4$ by

$$
\begin{aligned}
(4.11)\qquad p &= \mathrm{Centre}[\mu] \equiv \int_{\mathbb{R}^4} x\, d\mu, \\
\lambda^2 &= \mathrm{Scale}^2[\mu] \equiv \int_{\mathbb{R}^4} |x-p|^2\, d\mu.
\end{aligned}
$$

Let Θ be the product connection over X_i. The proof of the following lemma describes how to choose conformal blow-ups which produce centred limits.

Lemma 4.7. *Let $\{A_\alpha\}$ be a sequence of g_0-anti-self-dual connections over X_0 with weak limit (A_0, Z_0), where $Z_0 = \{x_i\}_{i=1}^{m_0}$ is non-empty. Choose r_0 as in* Eq. (4.4). *Then for each $x_i \in Z_0$, the sequence $\{A_\alpha\}$ determines a sequence of points $\{w_{i\alpha}\}$ converging to x_i, a sequence of frames $v_{i\alpha} \in FX_0|_{w_{i\alpha}}$ converging to a frame $v_i \in FX_0|_{x_i}$, and a sequence of scales $\{\kappa_{i\alpha}\}$ converging to zero such that the following holds. Fix $N > 4$, let $f_{w_{i\alpha}}$ be the corresponding sequences of conformal blow-ups, and let $A_{w_{i\alpha}}$ be the induced sequence of $\tilde{g}_{w_{i\alpha}}$-anti-self-dual connections with weak g_i-anti-self-dual limit $(\tilde{A}_i, \tilde{Z}_i)$ over the four-sphere X_i. The limit $(\tilde{A}_i, \tilde{Z}_i)$ has the following properties:*

(a) If $\tilde{A}_i \neq \Theta$, then $\tilde{A}_i$ is centred;

(b) If $\tilde{A}_i = \Theta$, then the corresponding singular measure $\tilde{\mu}_i$ is centred.

Proof. (a) We begin by defining, exactly as before, a sequence of points $\{x_{i\alpha}\}$ converging to x_i, a sequence of frames $v_{i\alpha} \in FX_0|_{x_{i\alpha}}$ converging to a frame $v_i \in FX_0|_{x_i}$, and a sequence of scales $\{\lambda_{i\alpha}\}$ converging to zero. Let $f_{x_{i\alpha}}$ be the corresponding sequences of conformal blow-ups and let $A_{x_{i\alpha}}$ be the induced sequence of $\tilde{g}_{x_{i\alpha}}$-anti-self-dual connections with weak g_i-anti-self-dual limit (A_i, Z_i) over X_i. Suppose Center$[A_i] = p_i$ and Scale$[A_i] = \nu_i$.

Case 1. *Z_i is empty.* Recall that $f_{x_{i\alpha}} = \phi_{in} \circ c_{\lambda_{i\alpha}} \circ \phi_{x_{i\alpha}}^{-1}$, $A_{x_{i\alpha}} = (f_{x_{i\alpha}}^{-1})^* A_\alpha$, and $\tilde{g}_{x_{i\alpha}} = \lambda_{i\alpha}^{-2} (f_{x_{i\alpha}}^{-1})^* g_0$. Define $h_i = \phi_{in} \circ c_{\nu_i} \circ \tau_{p_i} \circ \phi_{in}^{-1}$ and set $\bar{f}_{w_{i\alpha}} = h_i \circ f_{x_{i\alpha}}$. Then

$$\bar{f}_{w_{i\alpha}} = \phi_{in} \circ c_{\lambda_{i\alpha}\nu_i} \circ \tau_{p_i\lambda_{i\alpha}} \circ \phi_{x_{i\alpha}}^{-1} = \phi_{in} \circ c_{\kappa_{i\alpha}} \circ \bar{\phi}_{w_{i\alpha}}^{-1},$$

where $w_{i\alpha} \equiv \phi_{x_{i\alpha}}(p_i\lambda_{i\alpha})$, $\kappa_{i\alpha} \equiv \lambda_{i\alpha}\nu_i$, and $\bar{\phi}_{w_{i\alpha}} \equiv \phi_{x_{i\alpha}} \circ \tau_{p_i\lambda_{i\alpha}}^{-1}$. Thus, $\bar{f}_{w_{i\alpha}}$ provides a diffeomorphism from the small ball $B(w_{i\alpha}, N\kappa_{i\alpha}^{1/2})$ in X_0 to the open subset $B(x_{in}, N\kappa_{i\alpha}^{-1/2})$ of X_i. The sequence of points $\{w_{i\alpha}\}$ converges to x_i and the sequence of scales $\{\kappa_{i\alpha}\}$ converges to zero. As in §3.5, define a sequence of metrics on the increasing subsets $B(x_{in}, N\kappa_{i\alpha}^{-1/2})$ by $\bar{g}_{w_{i\alpha}} \equiv \kappa_{i\alpha}^{-2}h_1^2(\bar{f}_{w_{i\alpha}}^{-1})^*g_0$. Then $\bar{g}_{w_{i\alpha}}$ converges to the standard metric g_i in C^∞ on compact subsets of $X_i \setminus \{x_{is}\}$. Define a sequence of $\bar{g}_{w_{i\alpha}}$-anti-self-dual connections over the balls $B(x_{in}, N\kappa_{i\alpha}^{-1/2})$ by $\bar{A}_{w_{i\alpha}} \equiv (\bar{f}_{w_{i\alpha}}^{-1})^*A_\alpha$, and observe that $\bar{A}_{w_{i\alpha}} = (h_i^{-1})^*A_{x_{i\alpha}}$. The sequence $\{\bar{A}_{w_{i\alpha}}\}$ converges to the centred connection $(h_i^{-1})^*A_i$ in C^∞ on compact subsets of $X_i \setminus \{x_{is}\}$.

It remains to replace the chart $\bar{w} \equiv \bar{\phi}_{w_{i\alpha}}^{-1}$ on $B(w_{i\alpha}, \varrho_0/2)$ by a geodesic normal coordinate chart $w \equiv \phi_{w_{i\alpha}}^{-1}$. Choose a frame $v'_{i\alpha} \in FX_0|_{w_{i\alpha}}$ by parallel translating the frame $v_{i\alpha} \in FX_0|_{x_{i\alpha}}$ along the geodesic connecting $x_{i\alpha}$ and $w_{i\alpha}$, noting that $\operatorname{dist}_{g_0}(x_{i\alpha}, w_{i\alpha}) = |p_i|\lambda_{i\alpha}$. Thus, as $\lambda \to \infty$, the coordinate chart $\bar{w}$ converges in C^∞ on $B(x_i, \varrho/4)$ to the geodesic normal coordinate chart w in the sense of Lemma 4.6. Define a new sequence of conformal blow-up maps by setting $f_{w_{i\alpha}} = \phi_{in} \circ c_{\kappa_{i\alpha}} \circ \phi_{w_{i\alpha}}^{-1}$, and define corresponding sequences of connections and metrics on the balls $B(x_{in}, N\kappa_{i\alpha}^{-1/2})$ by $A_{w_{i\alpha}} = (f_{w_{i\alpha}}^{-1})^*A_\alpha$ and $\tilde{g}_{w_{i\alpha}} = \kappa_{i\alpha}^{-2}h_1^2(f_{w_{i\alpha}}^{-1})^*g_0$. Lemma 4.6 implies that the sequences $\{\tilde{g}_{w_{i\alpha}}\}$ and $\{A_{w_{i\alpha}}\}$ converge in C^∞ on compact subsets of $X_i \setminus \{x_{is}\}$ to the metric g_i and centred g_i-anti-self-dual connection $\tilde{A}_i \equiv (h_i^{-1})^*A_i$. This completes the proof of (a) in Case 1.

Case 2. *Z_i is non-empty.* The proof is similar to that of Case 1. Let $\tilde{Z}_i = h_i^{-1}(Z_i)$. Then the sequences $\{\bar{A}_{w_{i\alpha}}\}$ and $\{A_{w_{i\alpha}}\}$ converge in C^∞ on compact subsets of $X_i \setminus (\tilde{Z}_i \cup \{x_{is}\})$ to the centred connection $(h_i^{-1})^*A_i$.

(b) One sets $\operatorname{Centre}[\mu_i] = p_i$, $\operatorname{Scale}[\mu_i] = \nu_i$, and essentially repeats the proof of Part (a) for the sequence of measures $\mu_{x_{i\alpha}} \equiv |F(A_{x_{i\alpha}})|_{g_0}^2$.

Remark 4.8. In the sequel, we require that the conformal blow-up maps be chosen as in Lemma 4.7. However, to conserve notation, we will relabel the points $w_{i\alpha}$ and scales $\kappa_{i\alpha}$ by $x_{i\alpha}$ and $\lambda_{i\alpha}$, respectively, and the limit $(\tilde{A}_i, \tilde{Z}_i)$ by (A_i, Z_i).

A technical point that we have not addressed above is that, just as in [17], the weak limit of the sequence $\{A_{i\alpha}\}$ apparently depends on certain choices of parameters in the conformal blow-up construction:

(1) Neck width parameter N. This was only included in this Chapter for the sake of consistency with the gluing construction of Chapters 3 and 5: we could just as well have set $N = 2$, say.

(2) Radius r_0. Following [17], the dependency is removed by letting

$r_0 \to 0$. The conformal blow-up process gives a sequence of points $\{x_{i\alpha}(r_0)\}$, scales $\{\lambda_{i\alpha}(r_0)\}$, blow-up maps $\{f_{x_{i\alpha}}(r_0)\}$, metrics $\{\tilde{g}_{i\alpha}(r_0)\}$, and connections $\{A_{i\alpha}(r_0)\}$. The sequence of connections $\{A_{i\alpha}(r_0)\}$ converges to an ideal g_i-anti-self-dual limit $(A_i(r_0), Z_i(r_0))$, for any fixed $r_0 > 0$. We now let $r_0 \to 0$ and by a standard diagonal argument, we obtain a weakly convergent subsequence $\{A_{i\alpha}(r_0)\}$ with weak limit (A_i, Z_i), say.

(3) Frames $v_{i\alpha}$ and v_i. The construction is SO(4) equivariant: Rotating the frames $v_{i\alpha} \in FX|_{x_{i\alpha}}$ and $v_i \in FX_{x_i}$ by elements of SO(4) induces an $SO(4)$ action on the connections $A_{i\alpha}$ and A_i as described in §3.2.

There is one final issue which will be important in our later discussion of alternative modes of convergence for sequences of anti-self-dual connections: We must exclude the possibilty that curvature is lost over the necks Ω_i arising in the conformal blow-up process described above. Of course, the curvature can only bubble off with masses equal to an integer multiple of $8\pi^2$, so it suffices to show that we can choose the neck parameters to ensure that the curvature masses over the necks are strictly less than $8\pi^2$. So, consider again the sequence $\{A_\alpha\}_{\alpha=1}^\infty$ of g_0-anti-self-dual connections over X_0 with weak limit (A_0, Z_0), where $Z_0 = \{x_i\}_{i=1}^{m_0}$, and let $\{A_{i\alpha}\}_{\alpha=1}^\infty$ be the corresponding sequences of $\tilde{g}_{i\alpha}$-anti-self-dual connections over $X'_{i\alpha}$ having weak limits (A_i, Z_i), where $Z_i = \{x_{ij}\}_{j=1}^{m_j}$. Let $\{\lambda_{i\alpha}\}_{\alpha=1}^\infty$ be the sequence of scales associated to the sequence of connections $\{A_\alpha\}_{\alpha=1}^\infty$ and the singular point $x_i \in Z_0$. Given this situation, standard arguments yield the following curvature estimates near x_i:

Lemma 4.9. *Given $\varepsilon > 0$, there exist positive constants R_0, r_1, and α_0 with the following significance. For large enough R_0, small enough r_1 and large enough α_0, then $R_0\lambda_{i\alpha} < r_1$ for any $\alpha \geq \alpha_0$ and the following hold.*

(a) $\left| \|F(A_{i\alpha})\|^2_{L^2(B(x_i,R_0),\tilde{g}_{i\alpha})} - 8\pi^2 k_i \right| < \varepsilon^2$,

(b) $\left| \|F(A_\alpha)\|^2_{L^2(B(x_i,R_0\lambda_{i\alpha}),g_0)} - 8\pi^2 k_i \right| < \varepsilon^2$,

(c) $\|F(A_{i\alpha})\|_{L^2(\Omega(x_i,R_0,r_1\lambda_{i\alpha}^{-1}),\tilde{g}_{i\alpha})} < \varepsilon$,

(d) $\|F(A_\alpha)\|_{L^2(\Omega(x_i,R_0\lambda_{i\alpha},r_1),g_0)} < \varepsilon$.

Thus, we have the following curvature estimate which ensures that in the limit there is no 'curvature loss' over the necks Ω_i. (In particular, if $A_{i\alpha}$ converges weakly to (A_i, Z_i), then the singular set $Z_i \subset X_i$ does not contain the south pole x_{is}.)

Corollary 4.10. *Given $\varepsilon > 0$ and $N > 4$, there is an $\alpha_0 > 0$ with the following significance. If $\Omega_{i\alpha} \equiv \Omega(x_{i\alpha}, N^{-1}\lambda_{i\alpha}^{1/2}, N\lambda_{i\alpha}^{1/2})$ and, $B'_{i\alpha} \equiv B(x_{i\alpha}, N\lambda_{i\alpha}^{1/2})$, then for any $\alpha \geq \alpha_0$, we have*
(a) $\|F(A_\alpha)\|_{L^2(\Omega_{i\alpha}, g_0)} < \varepsilon$, *and*
(b) $|\|F(A_\alpha)\|^2_{L^2(B'_{i\alpha}, g_0)} - 8\pi^2 k_i| < \varepsilon$.

Lastly, we note that the conformal blow-up process may of course be iterated if the singular sets Z_i are non-empty. In the next section we show that after repeating the conformal blow-up process at most k times, we obtain a sequence of g_α-anti-self-dual connections $\{\check{A}_\alpha\}$ which is strongly convergent. Indeed, given the weakly convergent sequence $\{A_{i\alpha}\}_{\alpha=1}^\infty$ over the $X'_{i\alpha}$ near a point x_{ij} with multiplicity k_{ij} in the singular set $Z_i \subset X_i$, the second-level process differs from the first-level only in minor technical details: We define sequences of centres $x_{ij\alpha} = \phi_{x_{ij}}(q_{ij\alpha})$ converging to x_{ij} and scales $\lambda_{ij\alpha}$ converging to zero, now using the metrics $\tilde{g}_{i\alpha}$ and a coordinate chart $\phi_{x_{ij}}$ on X_i given by $\phi_{x_{ij}} = \phi_{in} \circ \tau_{q_{ij}}^{-1}$ where $\phi_{in}(q_{ij}) = x_{ij}$. The blow-up maps are defined using coordinate charts on X_i given by $\phi_{x_{ij\alpha}} = \phi_{in} \circ \tau_{q_{ij\alpha}}^{-1}$ and setting $f_{x_{ij\alpha}} = \phi_{ijn} \circ c_{\lambda_{ij\alpha}} \circ \phi_{x_{ij\alpha}}^{-1}$. We then proceed exactly as before and similarly for all higher-level blow-ups.

4.3. Bubble tree compactification. By analogy with the arguments of [23 (§5)] and [17], we define a bubble tree compactification for the moduli space $M_{X_0,k}(g_0)$ of anti-self-dual connections. First, we need an appropriate notion of an 'ideal connection':

Definition 4.11. A *bubble tree ideal g_0-anti-self-dual connection A of second Chern class k* over X_0 is determined by the following data.

(a) An oriented tree $\mathcal{I}$ with a finite set of vertices $\{I\}$, including a base vertex 0, and a set of edges $\{(I_-, I)\}$. Each vertex I is labelled with an integer $k_I \geq 0$ such that:

  (i) $\sum_{I\in\mathcal{I}} k_I = k$,
  (ii) if $I > 0$ is a terminal vertex, then $k_I > 0$,
  (iii) there are at most k terminal vertices, excluding the base vertex.

(b) A $(2m-1)$-tuple $(A_I, x_I)_{I\in\mathcal{I}}$, where m is the number of vertices in $\mathcal{I}$.

(c) If $I = 0$, then A_0 is a g_0-anti-self-dual connection on a G bundle

P_0 over X_0 with $c_2(P_0) = k_0 \geq 0$.

(d) If $I > 0$, then

(i) A_I is either the product connection Θ or a centred g_I-anti-self-dual connection on a G bundle P_I over the sphere $X_I \equiv \mathbb{S}^4$ with $c_2(P_I) = k_I$, where g_I is the standard round metric,

(ii) x_I is a point in X_0 if $I_- = 0$ and a point in $X_{I_-} \setminus \{x_{Is}\}$ if $I_- > 0$.

(e) If $I > 0$ and $A_I = \Theta$, then there are at least 2 outgoing edges emanating from that vertex.

Definition 4.11 should be compared with the construction of approximately anti-self-dual connections in §3.3. The ideal connection $(A_I, x_I)_{I\in\mathcal{I}}$ is often written as $(A_I)_{I\in\mathcal{I}}$. Heuristically, we may view an ideal g_0-anti-self-dual connection $A = (A_I)_{I\in\mathcal{I}}$ as a 'connection' over the join $\vee_{I\in\mathcal{I}} X_I$, where each sphere X_I is attached to the lower level X_{I_-} by identifying the south pole x_{Is} with the point $x_I \in X_{I_-}$. Let $Z_{I_-} \subset X_{I_-}$ denote the set of 'attachment points' x_I in X_0, if $I_- = 0$, or points x_I in $X_{I_-} \backslash \{x_{I_s}\}$, if $I_- > 0$. Let m_I be the number of points in Z_I, i.e., the number of outgoing edges emanating from vertex I.

Second, we need an appropriate notion of convergence. Let $X \equiv \#_{I\in\mathcal{I}} X_I$ be the connected sum defined in §3.3 by a set of scales $\{\lambda_{I\alpha}\}_{I\in\mathcal{I}}$, with $\overline{\lambda}_\alpha \to 0$ as $\alpha \to \infty$, and a fixed neck parameter N. Similarly, if $\{g_\alpha\}$ is the corresponding sequence of C^∞ metrics on X defined in §3.5, then g_α converges to g_I in C^∞ on compact subsets of $X_I \backslash (Z_I \cup \{x_{I_s}\})$ for each $I \geq 0$. As in [7 (§7.3.1)], we consider the following modes of convergence for sequences of anti-self-dual connections over X.

Definition 4.12. Let $\{A_\alpha\}_{\alpha=1}^\infty$ be a sequence of g_α-anti-self-dual connections on a fixed bundle P with $c_2(P) = k$ over the connected sum $X = \#_{I\in\mathcal{I}} X_I$.

(a) Let $Y \in s^k(X)$ be a multiset in $\cup_{I\in\mathcal{I}} X_I \setminus (Z_I \cup \{x_{Is}\})$. The sequence $\{A_\alpha\}$ *converges weakly* to $((A_I, x_I)_{I\in\mathcal{I}}, Y)$, if the gauge equivalence classes $[A_\alpha]$ converge in C^∞ to $([A_I])_{I\in\mathcal{I}}$ over compact subsets of $\cup_{I\in\mathcal{I}} X_I \setminus (Z_I \cup \{x_{Is}\} \cup Y)$, and if the curvature densities converge, then

$$|F(A_\alpha)|^2_{g_\alpha} \longrightarrow \sum_{I\in\mathcal{I}} |F(A_I)|^2_{g_I} + 8\pi^2 \delta_Y,$$

over compact subsets of $\cup_{I\in\mathcal{I}}X_I \setminus (Z_I \cup \{x_{Is}\})$.

(b) The sequence $\{A_\alpha\}$ *converges strongly* to the limit $(A_I, x_I)_{I\in\mathcal{I}}$ if it converges weakly to $(A_I, x_I)_{I\in\mathcal{I}}$ (with no singular set Y) and $\sum_{I\in\mathcal{I}} c_2(P_I) = c_2(P)$. Here, the A_I are g_I-anti-self-dual connections on G bundles P_I over X_I with $c_2(P_I) = k_I$.

We let $BM_{X_0,k}(g_0)$ denote the set bubble tree ideal g_0-anti-self-dual connection over X_0 of total second Chern class k. Thus, each point of $BM_{X_0,k}(g_0)$ is represented by a $(2m-1)$-tuple $(A_I, x_I)_{I\in\mathcal{I}}$, with m being the total number of vertices of the tree $\mathcal{I}$.

Definition 4.13. We say that a sequence $\{A_\alpha\}_{\alpha=1}^\infty$ of g_0-anti-self-dual connections on a G bundle P over X_0 with $c_2(P) = k$ *converges strongly to a bubble tree ideal g_0-anti-self-dual connection* $(x_I, A_I)_{I\in\mathcal{I}}$ in $BM_{X_0,k}(g_0)$ if there exist sequences of conformal blow-ups $\{f_{I\alpha}\}_{I\in\mathcal{J}}$ with the following property. Let $\{g_\alpha\}$ be the induced sequence of C^∞ metrics in the conformal class $[g_0]$ on the connected sum $X = \#_{I\in\mathcal{I}}X_I$, and let $\{\check{A}_\alpha\}$ denote the induced sequence of g_α-anti-self-dual connections over X. Then we require that the sequence of metrics $\{g_\alpha\}$ converges in C^∞ on compact sets of $X_I \setminus (Z_I \cup \{x_{Is}\})$ to the metric g_I, $I \geq 0$, and that the sequence of connections $\{\check{A}_\alpha\}$ converges strongly to the ideal g_0-anti-self-dual connection $(A_I, x_I)_{I\in\mathcal{I}}$.

This definition of convergence extends to the space of bubble tree ideal connections $BM_{X_0,k}(g_0)$, which is then endowed with a second countable Haussdorf topology. Define the *bubble tree compactification* $\overline{M}^{\tau}_{X_0,k}(g_0)$ to be the closure of $M_{X_0,k}(g_0)$ in $BM_{X_0,k}(g_0)$.

Theorem 4.14 *The space $\overline{M}^{\tau}_{X_0,k}(g_0)$ is compact.*

The result follows from the special case below.

Theorem 4.15. *Any infinite sequence in $M_{X_0,k}(g_0)$ has a strongly convergent subsequence with limit point in $\overline{M}^{\tau}_{X_0,k}(g_0)$.*

Proof. The argument is similar to the proof of Proposition 5.3 in [23]. Fix a G bundle P over X_0 with $c_2(P) = k > 0$ and let $\{A_\alpha\}_{\alpha=1}^\infty$ be a sequence of g_0-anti-self-dual connections on P. The main point is to repeatedly apply conformal blow-ups $f_{I\alpha}$ until we obtain a sequence of induced metrics g_α over a connected sum X, with (X, g_α) conformally equivalent to (X_0, g_0), and a sequence of induced g_α-anti-self-dual connections over X, denoted by $\{\check{A}_\alpha\}$, which is strongly convergent. We adopt the convention below that subsequences are immediately relabelled.

Step 1. There is a subsequence $\{A_\alpha\}$ which converges weakly to an ideal g_0-anti-self-dual connection (A_0, Z_0), with $Z_0 = \{x_i\}_{i=1}^{m_0}$ corresponding to a point in the symmetric product $s^k(X_0)$. If Z_0 is empty then we are done, so assume that $m_0 \geq 1$. Let k_i be the multiplicity of x_i and note that $0 < k_i \leq k$. For each i and large enough α, the connection A_α determines a set of mass centers $\{x_{i\alpha}\}_{i=1}^{m_0}$, with $x_{i\alpha} \to x_i$, and a set of scales $\{\lambda_{i\alpha}\}_{i=1}^{m_0}$ with $\lambda_{i\alpha} \to 0$ as $\alpha \to \infty$. Fix a neck width parameter $N > 4$, choose a sequence of frames $v_{i\alpha} \in FX_0|_{x_{i\alpha}}$ converging to a frame $v_i \in FX_0|_{x_i}$, and let $\{f_{i\alpha}\}_{i=1}^{m_0}$ be the conformal blow-up maps defined by these centres, frames, scales, and parameter N. If $X = \#_{i=0}^{m_0} X'_{i\alpha}$, then (X, g_α) is the conformal blow-up of (X_0, g_0) determined by the maps $f_{i\alpha}$. Let P now denote the induced G bundle over X, let $\check{A}_\alpha$ denote the induced g_α-anti-self-dual connection over X, and let $A_{i\alpha}$ be the restriction of $\check{A}_\alpha$ to the open subset $X'_{i\alpha}$.

The sequence $[A_{i\alpha}]$ has a weakly convergent subsequence, again denoted $[A_{i\alpha}]$, with weak limit (A_i, Z_i), where Z_i corresponds to a point in $s^{k_i}(X_i)$. Corollary 4.10 implies that no mass is lost over the neck Ω_i. Hence, if Z_i is empty for $i = 1 \ldots m_i$, then we have $\sum_{i=0}^{m_i} k_i = k$, the sequence $[A_{i\alpha}]$ converges strongly to $[A_i]$, and we proceed to the Final Step. Otherwise, Z_i is non-empty for some $i > 0$ and we proceed to Step 2.

Step 2. For some $i > 0$, Step 1 produces a non-empty singular set $Z_i = \{x_{ij}\}_{j=1}^{m_i}$. Let k_{ij} be the multiplicity of the point x_{ij}, let $c_2(A_i) = k_{i0}$, and note that $\sum_{j=0}^{m_i} k_{ij} = k_i > 0$. Let μ_i be the singular measure associated with (A_i, Z_i). We now consider two cases, depending on whether or not A_i is the flat product connection Θ over X_i.

Case (a). $A_i = \Theta$. Since $\mathrm{Scale}[\mu_i] = 1$, the diameter of the set Z_i must be positive and so this case can only occur if $m_i > 1$. Let k_{ij} be the multiplicity of the point x_{ij} and note that as $m_i > 1$ we must have $\max_j k_{ij} \leq k - 1$.

Case (b). $A_i \neq \Theta$. Therefore, $k_{i0} = c_2(A_i) > 0$ and so we again must have $\max_j k_{ij} \leq k - 1$, since $\sum_{j=0}^{m_i} k_{ij} = k_i \leq k$.

For large enough α, the connection $A_{i\alpha}$ determines a set of mass centres $\{x_{ij\alpha}\}_{i=1}^{m_i}$, with $x_{ij\alpha} \to x_{ij}$, and a set of scales $\{\lambda_{ij\alpha}\}_{i=1}^{m_i}$, with $\lambda_{ij\alpha} \to 0$ as $\alpha \to \infty$. Let $\{f_{ij\alpha}\}_{j=1}^{m_i}$ be the conformal blow-up maps defined by these centres, scales, and parameter N. Let P denote the induced G bundle over the new connected sum $X = \#_{i=0}^{m_0} X'_{i\alpha} \#_{j=1}^{m_i} X'_{ij\alpha}$, let $\check{A}_\alpha$ denote the induced g_α-anti-self-dual connection over X, and let $\{A_{ij\alpha}\}$ be the induced sequence of g_α-anti-self-dual connections over the open subsets $X'_{ij\alpha}$ of the spheres X_{ij}.

The sequence $[A_{ij\alpha}]$ has a weakly convergent subsequence with weak limit (A_{ij}, Z_{ij}) and no loss of mass over the necks $\Omega_{ij\alpha}$. If Z_{ij} is empty for $j = 1, \ldots, m_i$, then we have $\sum_{j=0}^{m_i} k_{ij} = k_i$, the sequence $[A_{ij\alpha}]$ converges strongly to $[A_{ij}]$, and the blow-up process terminates at the vertices A_{ij}. Otherwise, Z_{ij} is non-empty for some j and we proceed to Step 3.

Step l. $3 \le l \le k$. For some multi-index I of length $|I| = l - 1$, Step $l - 1$ produces a non-empty singular set $Z_I = \{x_{Ij}\}_{j=1}^{m_I}$ contained in the sphere X_I. The sequence $[A_{I\alpha}]$ has a weak limit (A_I, Z_I), where Z_I corresponds to a point in $s^{k_I}(X_I)$. Let k_{Ij} be the multiplicity of the point x_{Ij}, let $c_2(A_I) = k_{I0}$, and note that $\sum_{j=0}^{m_I} k_{Ij} = k_I > 0$. Let μ_I be the singular measure associated with (A_I, Z_I).

Case (a). $A_I = \Theta$. Since $\mathrm{Scale}[\mu_I] = 1$, the diameter of the set Z_I must be positive. Hence, $m_I > 1$ and so we have

$$(4.12) \qquad \max_j k_{Ij} \le k - l + 1, \qquad |Ij| = l, \quad 1 \le l \le k.$$

Case (b). $A_I \ne \Theta$. Therefore, $k_{I0} = c_2(A_I) > 0$, and Eq. (4.12) again holds, since $\sum_{j=0}^{m_I} k_{Ij} = k_I \le k$.

Eq. (4.12) implies that the conformal blow-up process terminates completely after at most k steps.

For large enough α, the connection $A_{I\alpha}$ determines a set of mass centres $\{x_{Ij\alpha}\}_{j=1}^{m_I}$ in $X_I \setminus \{x_{Is}\}$, with $x_{Ij\alpha} \to x_{Ij}$, and a set of scales $\{\lambda_{Ij\alpha}\}_{j=1}^{m_I}$, with $\lambda_{Ij\alpha} \to 0$ as $\alpha \to \infty$. Let $\{f_{Ij\alpha}\}_{j=1}^{m_I}$ be the conformal blow-up maps defined by these centres, scales, and parameter N. Let P denote the induced G bundle over the connected sum $X = \#_I X'_{I\alpha} \#_{j=1}^{m_i} X'_{Ij\alpha}$, let $\check{A}_\alpha$ denote the induced g_α-anti-self-dual connection over X, and let $\{A_{Ij\alpha}\}$ be the induced sequence of g_α-anti-self-dual connections over the open subsets $X'_{Ij\alpha}$ of the spheres X_{Ij}.

The sequence $[A_{Ij\alpha}]$ has a weakly convergent subsequence with weak limit (A_{Ij}, Z_{Ij}) and no loss of mass over the necks $\Omega_{Ij\alpha}$. If Z_{Ij} is empty for $j = 1, \ldots, m_I$, then we have $\sum_{j=0}^{m_I} k_{Ij} = k_I$, the sequence $[A_{Ij\alpha}]$ converges strongly to $[A_{Ij}]$, the blow-up process terminates at the vertices A_{Ij}, and we proceed to Step $l + 1$.

Final Step. After performing at most k conformal blow-ups, we obtain a sequence of g_α-anti-self-dual connections $\{\check{A}_\alpha\}$ over a connected sum $X = \#_{I\in\mathcal{I}} X'_{I\alpha}$. The sequence $\{\check{A}_\alpha\}$ converges strongly to a bubble tree limit $(A_I, x_I)_{I\in\mathcal{I}}$, since the singular points have all been blown up and there has been no mass loss over the necks $\Omega_{I\alpha}$.

Plainly, the compactification $\overline{M}^{\tau}_{X_0,k}(g_0)$ is 'larger' than the Uhlenbeck compactification $\overline{M}^{u}_{X_0,k}(g_0)$. Indeed, there is an obvious surjective map

$$\pi : \overline{M}^{\tau}_{X_0,k}(g_0) \longrightarrow \overline{M}^{u}_{X_0,k}(g_0) \tag{4.13}$$

obtained by sending a bubble tree ideal connection $(A_I, x_I)_{I\in\mathcal{I}}$ to the corresponding Uhlenbeck ideal connection $(A_0, x_1, \dots, x_{m_0})$. The multiplicity of $x_i \in X_0$ is the sum of the second Chern classes of the anti-self-dual connections A_I attached to the subtree lying above the vertex i.

Corollary 4.16. *The map $\pi : \overline{M}^{\tau}_{X_0,k}(g_0) \to \overline{M}^{u}_{X_0,k}(g_0)$ is continuous.*

4.4. D_q convergence and strong convergence. We will need one further notion of convergence in order to show that every point of the moduli space $M_{X,k}(g)$ lies in the image of the gluing map $\mathcal{J}$ constructed in Chapter 5. Let P be a G bundle over a closed manifold X with metric g. Following [7 (§7.2.4)], fix $4 \le q < \infty$ and let D_q be the distance function on the space $\mathcal{B}_{X,P}$ given by

$$D_q([A],[B]) = \inf_{u\in\mathcal{G}} \|A - u(B)\|_{L_q(X,g)}. \tag{4.14}$$

We recall the following definition of Donaldson and Kronheimer.

Definition 4.17. [7 (p. 308)] *Let $\{\lambda_{I\alpha}\}_{\alpha=1}^\infty$, for each $I > 0$, be sequences of scales satisfying $\overline{\lambda}_\alpha \to 0$, where $\overline{\lambda}_\alpha = \max_I \lambda_{I\alpha}$, and let $\{A_\alpha\}_{\alpha=1}^\infty$ be a sequence of connections on a fixed G bundle $P \to X$, where $X \equiv \#_{I\in\mathcal{I}} X'_I$ and $X_I = \mathbb{S}^4$ if $I > 0$. The connected sum X has a sequence of metrics $\{g_\alpha\}_{\alpha=1}^\infty$ defined by the sequence of scales $\{\lambda_{I\alpha}\}_{\alpha=1}^\infty$, a sequence of points $\{x_{I\alpha}\}_{\alpha=1}^\infty$, where the $x_{I\alpha}$ converge with respect to the fixed metric g_I to a point $x_I \in X_{I_-}$, and a neck width parameter N. Assume that the connections A_α are g_α-ASD with respect to the sequence of metrics $\{g_\alpha\}_{\alpha=1}^\infty$ on X. Then the sequence $\{A_\alpha\}_{\alpha=1}^\infty$ is D_q-convergent to $(A_I, x_I)_{I\in\mathcal{I}}$ if $D_q([A_\alpha|X''_I], [A_I|X''_I]) \to 0$ as $\alpha \to \infty$.*

D_q convergence is called 'L^q convergence' in [7]. The result below explains the relationship between strong convergence and D_q convergence.

Theorem 4.18. [7 (p. 309)] *Let $\{A_\alpha\}_{\alpha=1}^\infty$ be a sequence of connections on a bundle $P \to X$ which are ASD with respect to the sequence of metrics $\{g_\alpha\}_{\alpha=1}^\infty$ determined by the sequences of scales $\{\lambda_{I\alpha}\}$, where $\overline{\lambda}_\alpha \to 0$. Then the sequence $\{A_\alpha\}_{\alpha=1}^\infty$ is strongly convergent if and only if it is D_q-convergent.*

5. Differentials of the gluing maps

In this Chapter we obtain L^2 estimates for the differentials of the gluing maps $\hat{\mathcal{J}} : \mathcal{T}/\Gamma \to M^*_{X_0,k}$. These give C^0 bounds for the components of the L^2 metric $\mathbf{g}$ on the bubbling ends of $M^*_{X_0,k}(g_0)$ and allow us to complete the proofs of Theorems 1.1 and 1.2. In particular, *for the remainder of the article, the hypotheses of Theorem 1.1 are assumed to be in effect.*

5.1. Construction of the gluing maps. In this section we construct the gluing maps $\mathcal{J} : \mathcal{T}/\Gamma \to M^*_{X,k}(g)$ and $\hat{\mathcal{J}} : \mathcal{T}/\Gamma \to M^*_{X_0,k}(g_0)$, and set up the analytical framework required for the later sections. Our first task is to construct a right inverse to the linear operator $d^{+,g}_{A'}$ and so we choose suitable Sobolev spaces L^q, L^p_1 and for the remainder of this Chapter, we fix

$$2 \le p < 4 \quad \text{and} \quad 4 \le q < \infty \quad \text{so that} \quad 1/4 + 1/q = 1/p. \tag{5.1}$$

By hypothesis, $H^2_{A_I} = 0$ for all I and thus the operators $d^{+,g_I}_{A_I}$ have right inverses P_I. More explicitly, if $\Delta^{+,g_I}_{A_I}$ is the Laplacian $d^{+,g_I}_{A_I}(d^{+,g_I}_{A_I})^*$ and $G^{+,g_I}_{A_I}$ is the corresponding Green's operator, we may set $P_I = (d^{+,g_I}_{A_I})^* G^{+,g_I}_{A_I}$. A standard application of the Calderon-Zygmund theory and the Sobolev inequalities gives the following bounds.

Lemma 5.1. *Assume $H^2_{A_I} = 0$. Then the operators $P_I : L^p \to L^p_1$ and $P_I : L^p \to L^q$ are bounded and there are constants $C_i = C_i(A_I, g_I, p)$, $i = 1, 2$, such that for all $\xi \in L^p\Omega^2(X_I, \operatorname{ad} P_I)$,*

$$\|P_I\xi\|_{L^q(X_I,A_I,g_I)} \le C_1\|P_I\xi\|_{L^p_1(X_I,g_I)} \le C_2\|\xi\|_{L^p(X_I,g_I)}.$$

We next define the C^∞ cutoff functions to be used in the construction of a right parametrix Q for $d^{+,g}_{A'}$ by patching together the operators P_I over X.

Lemma 5.2. [7 (Lemma 7.2.10)] *For any $\lambda > 0$ and $N > 4$, there exist a C^∞ function $\beta_{\lambda,N}$ on $\mathbb{R}^4$ and a constant K independent of λ, N, such that $\beta_{\lambda,N}(x) = 1$ for $|x| \geq \frac{1}{2}\lambda^{1/2}$, $\beta_{\lambda,N}(x) = 0$ for $|x| \leq N^{-1}\lambda^{1/2}$, and $\|d\beta_{\lambda,N}\|_{L^4(\mathbb{R}^4,\delta)} \leq K(\log N)^{-3/4}$.*

Define C^∞ cutoff functions β_I on each X_I by setting

$$\beta_I \equiv (\phi_{Is}^{-1})^*\beta_{\lambda_I,N} \prod_{I_+} (\phi_{I_+}^{-1})^*\beta_{\lambda_{I_+},N} \quad \text{on } X_I, \tag{5.2}$$

where the factor $(\phi_{Is}^{-1})^*\beta_{\lambda_I,N}$ is omitted when $I = 0$. Here, the cutoff functions comprising β_I have been extended so that $\beta_I = 1$ on the complement in X_I of the balls $B_{Is}(\frac{1}{2}\lambda_I^{1/2})$ and $B_{I+}(\frac{1}{2}\lambda_{I_+}^{1/2})$. Also, $\beta_I = 0$ on the balls $B_{Is}(N^{-1}\lambda_I^{1/2})$ and $B_{I+}(N^{-1}\lambda_{I_+}^{1/2})$ in X_I; thus, we may extend β_I by zero to give $\beta_I \in C^\infty(X)$. The L^4 estimate of Lemma 5.2 implies that

$$\|d\beta_I\|_{L^4(X_I,g_I)} \leq cK(\log N)^{-3/4}, \tag{5.3}$$

for some $c = c(g_0, k)$. For the cutoff functions $\{\gamma_I\}$ defined by Eqs. (3.52) and (3.53), we recall that $\sum_I \gamma_I = 1$ on X. Note also that $\beta_I = 1$ on the support of γ_I.

Define operators $Q_I : L_1^p\Omega^{+,g_I}(X_I, \operatorname{ad} P_I) \to L^p\Omega^1(X_I, \operatorname{ad} P_I)$ by setting $Q_I = \beta_I P_I \gamma_I$. Define a right parametrix $Q : \mathcal{L}_1^p\Omega^{+,g}(X, \operatorname{ad} P) \to L^p\Omega^1(X, \operatorname{ad} P)$ for the operator $d_{A'}^{+,g}$ by $Q = \sum_I Q_I$. The error operator $R : L^p\Omega^{+,g}(X, \operatorname{ad} P) \to L^p\Omega^{+,g}(X, \operatorname{ad} P)$ is then given by

$$d_{A'}^{+,g} Q = 1 + R. \tag{5.4}$$

Lemmas 3.15 and 5.1 thus yield the following estimates for the operators Q_I and Q.

Lemma 5.3. *There are constants $C_i = C_i(g_0, p, \mathcal{T})$, $i = 1, 2$, such that for any $t \in \mathcal{T}$, the following bounds hold.*

(a) For any $\xi \in L^p\Omega^{+,g_I}(X_I, \operatorname{ad} P_I)$,

$$\|Q_I\xi\|_{L^q(X_I,g_I)} \leq C_1\|Q_I\xi\|_{L_1^p(X_I,A_I,g_I)} \leq C_2\|\xi\|_{L^p(X_I,g_I)}.$$

(b) For any $\xi \in L^p\Omega^{+,g}(X, \operatorname{ad} P)$,

$$\|Q\xi\|_{L^q(X,g)} \leq C_1\|Q\xi\|_{\mathcal{L}_1^p(X)} \leq C_2\|\xi\|_{L^p(X,g)}.$$

Next, there is an analogue of Lemma 7.2.14 [7] (see also [7 (p. 294)]), giving an L^p bound for the operator R. The proof follows easily from Lemmas 3.9, 3.12, and 5.1, and Eq. (5.3). In [7] it is assumed that the metrics g_I are flat in small neighbourhoods of the points x_I, but this restriction is easily removed by using Lemma 3.12.

Lemma 5.4. *There is a constant $\varepsilon = \varepsilon(\bar{b}, N, p)$, with $\varepsilon \to 0$ as $N \to \infty$ and $\bar{b} \to 0$ such that for any $t \in \mathcal{T}$ and $\xi \in L^p\Omega^{+,g}(X, \operatorname{ad} P)$, $\|R\xi\|_{L^p(X,g)} \leq \varepsilon\|\xi\|_{L^p(X,g)}$.*

Thus, *for the remainder of this article*, we choose $N_0 > 4$ large enough and $b_0 \leq 1$ small enough so that $\varepsilon(\bar{b}, N, p) \leq 2/3$ for all $\bar{b} \leq b_0$ and $N \geq N_0$, and fix $N = N_0$ and $b_I = 4N\lambda_I^{1/2}$ for all $I \in \mathcal{I}$. We now construct a right inverse P for $d_{A'}^{+,g}$. Lemma 5.4 yields the (L^p, L^p) operator norm bounds $\|R\| \leq 2/3$ and $\|(1+R)^{-1}\| \leq 3$. Since $Q_I = \beta_I P_I \gamma_I$, we have the (L^p, L^q) operator norm bound $\|Q_I\| \leq C_I$, say, giving the (L^p, L^q) operator norm bound $\|Q\| \leq C \equiv \sum_I C_I$. In summary, there is the following version of Proposition 7.2.35 [7].

Proposition 5.5. *There are constants N_0 and b_0 such that for any $N \geq N_0$, $\bar{b} \leq b_0$, and $t \in \mathcal{T}$, the operator $P \equiv Q(1+R)^{-1}$: $\mathcal{L}_1^p\Omega^{+,g}(X, \operatorname{ad} P) \to L^p\Omega^1(X, \operatorname{ad} P)$ is a right inverse to $d_{A'}^{+,g}$ and there are constants $C_i = C_i(g_0, p, \mathcal{T})$, $i = 1, 2$ such that for any $\xi \in L^p\Omega^{+,g}(X, \operatorname{ad} P)$,*

$$\|P\xi\|_{L^q(X,g)} \leq C_1\|P\xi\|_{\mathcal{L}_1^p(X)} \leq C_2\|\xi\|_{L^p(X,g)}.$$

We next construct families of solutions to the full non-linear anti-self-dual equation over connected sums. For each $t \in \mathcal{T}$ we seek a solution $A(t) = A'(t) + a(t)$ to $F^{+,g}(A' + a) = 0$, or equivalently

$$d_{A'}^{+,g} a + (a \wedge a)^{+,g} = -F^{+,g}(A'), \tag{5.5}$$

where $a \in \Omega^1(X, \operatorname{ad} P)$. If $a = P\xi$, with $\xi(t) \in \Omega^{+,g}(X, \operatorname{ad} P)$, then this equation becomes

$$\xi + (P\xi \wedge P\xi)^{+,g} = -F^{+,g}(A'). \tag{5.6}$$

With the aid of Lemma 7.2.23 [7 (p. 290)] (an application of the Contraction Mapping Theorem to Eq. (5.6)) and Proposition 5.5, one easily obtains the following version of Theorem 7.2.24 [7].

Theorem 5.6. *For sufficiently small $\lambda_0 < 1$, sufficiently large $N_0 > 4$, and sufficiently small T_{A_I}, $I \in \mathcal{I}$, the following holds. For any $t \in \mathcal{T}$, there exists an L^p_1 g-anti-self-dual connection $A(t) = A'(t) + a(t)$ over X, with $a(t) = P\xi(t)$. There are positive constants $C_i = C_i(g_0, p, \mathcal{T})$, $i = 1, 2, 3$, such that*

$$\|a\|_{L^q(X,g)} \le C_1\|\xi\|_{L^p(X,g)} \le C_2\|F^{+,g}(A')\|_{L^p(X,g)} \le C_3\bar{b}^{4/p}.$$

We pull back the g-anti-self-dual connections A on $P \to X$ via the conformal maps f_I to give g_0-anti-self-dual connections $\hat{A} = \hat{A}' + \hat{a}$ on $\hat{P} \to X_0$, where $\hat{A}$ is defined by

$$\hat{A} = f_0^* \cdots f_I^* A \quad \text{over } f_0^{-1} \cdots f_I^{-1}(X_I'), \tag{5.7}$$

and similarly for $\hat{A}'$ and $\hat{a}$. In particular, $\hat{A} = \hat{A}' + \hat{a}$ is a solution to the g_0-anti-self-dual equation $F^{+,g_0}(\hat{A}' + \hat{a}) = 0$ over X_0, or explicitly

$$d^{+,g}_{\hat{A}'}\hat{a} + (\hat{a} \wedge \hat{a})^{+,g_0} = -F^{+,g_0}(\hat{A}'), \tag{5.8}$$

where $\hat{a} \in \Omega^1(X_0, \operatorname{ad}\hat{P})$. Standard arguments show that the anti-self-dual connections A and $\hat{A}$ are actually C^∞ and that they are smooth points of the moduli spaces $M_{X,k}(g)$ and $M_{X_0,k}(g_0)$ [7]:

Lemma 5.7. *Let A be the g-anti-self-dual connection over X produced by* Theorem 5.6 *and let $\hat{A}$ be the corresponding g_0-anti-self-dual connection over X_0. Then the following hold:*

(a) The connections A and $\hat{A}$ are C^∞,

(b) $H^0_A = 0$ and $H^0_{\hat{A}} = 0$, for small enough b_0 and large enough N_0,

(c) $H^2_A = 0$ and $H^2_{\hat{A}} = 0$.

From §4.4, we recall that D_q is the distance function on $\mathcal{B}_{X,k}$ given by $D_q([A],[B]) = \inf_{u \in \mathcal{G}} \|A - u(B)\|_{L_q(X,g)}$. In particular, we have the following version of Theorem 7.2.62 [7] (compare also Theorem 4.53 [3]).

Theorem 5.8. *Let A_I be g_I-anti-self-dual connections on G bundles P_I over manifolds X_I, $I \in \mathcal{I}$. If $I = 0$, then X_0 is a closed, oriented, C^∞ four-manifold with generic C^∞ metric g_0 and negative definite intersection form. If $I > 0$, then $X_I = \mathbb{S}^4$ with standard round metric g_1 of radius 1. Let $X = \#_{I \in \mathcal{I}} X_I$, the connected sum four-manifold with*

C^∞ metric g (conformally equivalent to g_0) determined by the choice of points $\{x_I\}$, frames $\{v_I\}$, scales $\{\lambda\}$, and neck width parameter N. Let P be the connected sum bundle over X, where $c_2(P) = k \geq 1$. Let $\overline{\lambda} = max_{I\in\mathcal{I}}\lambda_I$. Let T_{A_1} be open balls centred at $0 \in H^1_{A_I}, I \in \mathcal{I}$ $\Gamma = \prod_{I\in\mathcal{I}}\Gamma_{A_I}$ and $T = T_{A_0} \times \prod_{I\in\mathcal{I}}(T_{A_I} \times \mathrm{Gl}_{x_I})$, as in Eqs. (3.25) *and* (3.26). *Then, for sufficiently small $\lambda_0 < 1$, sufficiently large $N_0 > 4$, and sufficiently small T_{A_I}, $I \in \mathcal{I}$, the following holds. There is a C^∞ homeomorphism onto an open subset:*

$$\mathcal{J} : T/\Gamma \longrightarrow U \subset M^*_{X,P}(g), \qquad t \longmapsto [A(t)],$$

where $A(t) = A'(t) + a(t)$, $a(t) = P\xi(t)$, and $\xi(t)$ are as in Theorem 5.6. *For any $\nu > 0$ and $4 \leq q < \infty$, the manifold T and constant $\lambda_0(\nu)$ can be chosen so that, for all $\overline{\lambda} < \lambda_0(\nu)$, $U = \{[A] \in M^*_{X,P}(g) : D_q([A|_{X_I''}], [A_I]) < \nu\}$, for all $I \in \mathcal{I}$.*

Proof. This is a straightforward generalisation of Theorem 7.2.62 [7] to the case of multiple connected sums (see [7 (§7.2.8)]) and a restriction to the case where $G = \mathrm{SU}(2)$ and $b^+(X_0) = 0$. The metric g_0 is not required to be flat in small neighourhoods of the gluing sites $x_I \in X_0$. Lemma 5.7 implies that the image of $\mathcal{J}$ lies in the dense open subset $M^*_{X,P}(g) \subset M_{X,P}(g)$. The fact that $\mathcal{J}$ is C^∞ is a calculation of the type that appears many times in §§5.3, 5.4, and 5.5. See also Appendix A [22] and Remark 4.24 [3].

We refer to $\mathcal{J}$ as a *gluing map* over the connected sum and its image $U \subset M^*_{X,k}(g)$ as a *gluing neighbourhood.* Moreover, $\mathcal{J}$ extends to a C^∞ gluing map on the larger parameter spaces $\mathcal{T}$ and $\mathcal{T}^0$ of Eqs. (3.27) and (3.30). Further properties of these maps are described in the next section. Lastly, for the original metric g_0 on the base four-manifold X_0, Theorem 5.8 takes the following form.

Corollary 5.9. *Given the hypotheses of* Theorem 5.8, *there is a homeomorphism onto an open subset*

$$\hat{\mathcal{J}} : T/\Gamma \longrightarrow V \subset M^*_{X_0,\hat{P}}(g_0), \qquad t \longmapsto [\hat{A}(t)],$$

where $V \subset M^*_{X_0,\hat{P}}(g_0)$ is obtained by pulling back the subset $U \subset M^*_{X,P}(g)$ of Theorem 5.8.

Again, $\hat{\mathcal{J}}$ extends to a C^∞ map on the larger parameter spaces $\mathcal{T}$ and $\mathcal{T}^0$, and additional properties of $\hat{\mathcal{J}}$ are discussed in the next section.

5.2. Structure of the compactified moduli spaces. The bubbling ends of $\overline{M}^u_{X_0,k}(g_0)$ away from the diagonals are described in [7 (§8.2)]. We extend this description to neighbourhoods of points in the diagonals of the Uhlenbeck compactification. For related constructions and some further details, we refer to the papers of Taubes and Donaldson.

The proposition below is the basic result we require in order to parametrise neighbourhoods covering the ends of $M^*_{X,k}(g)$ away from the reducible connections. See also [3 (§IV)], and [22 (p. 529)] for various special cases of the following statements. The following proof is similar to the arguments used in the proof of Theorem 4.53 [3 (p. 316 & p. 325)].

Proposition 5.10. *Given the hypotheses of* Theorem 5.8, *the following hold:*

(a) *The approximate gluing map $\mathcal{J}' : T/\Gamma \to \mathcal{B}^*_{X,k}$ is a C^∞ embedding.*

(b) *The gluing map $\mathcal{J} : T/\Gamma \to U \subset M^*_{X,k}(g)$ is a diffeomorphism onto an open subset.*

(c) *The extended gluing map $\mathcal{J} : \mathcal{T}/\Gamma \to \mathcal{U} \subset M^*_{X,k}(g)$ is a C^∞ submersion onto an open subset.*

(d) *The extended gluing map $\mathcal{J} : \mathcal{T}^0/\Gamma \to \mathcal{U}^0 \subset M^*_{X,k}(g)$ is a diffeomorphism onto an open subset.*

Proof. (a) The proof is essentially the same as the argument required for (b) and so is omitted. (b) From Theorem 5.8, $\mathcal{J}$ is a C^∞ homeomorphism, and so it is enough to show that $\mathcal{J}$ is also an immersion, since T/Γ has dimension equal to that of $M^*_{X,k}(g)$. From the proof of Theorem 5.8, there is a C^∞ Γ-equivariant gluing map $\tilde{\mathcal{J}} : T \to \mathcal{A}^*_{X,k}$, $t \mapsto A(t)$. So, we first show that $\tilde{\mathcal{J}}$ is an immersion and then conclude that the induced map on quotients is a diffeomorphism. The constant λ_0 may be chosen as small as desired and in (a) and (b), the λ_I and x_I may be held fixed.

Step 1. *Definition of restriction maps.* Choose cutoff functions ψ_I, as in §3.3, which are zero on the balls $B_{Is}(b_I/2)$, $B_{I_+}(b_{I_+}/2)$ and equal to 1 on the complement in X_I of the slightly larger balls $B_{Is}(b_I)$, $B_{I_+}(b_{I_+})$. Define a map $\pi_{X_I} : L^2\Omega^1(X, \operatorname{ad} P) \to L^2\Omega^1(X_I, \operatorname{ad} P_I)$ by left multiplication with ψ_I, so that

$$\|\omega - \pi_{X_I}\omega\|_{L^2(X_I,g_I)} = O(\overline{\lambda}), \qquad \omega \in \Omega^1(X_I, g_I), \tag{5.9}$$

since ψ_I is equal to 1 on the complement of a set in X_I of g_I-volume $O(\overline{\lambda}^2)$. Next, for $I > 0$, choose a cutoff function, which is zero outside the annulus $\Omega_{Is} = \Omega(x_{Is}, N^{-1}\lambda_I^{1/2}, N\lambda_I^{1/2})$ in X_I, and is equal to 1 on the slightly smaller annulus $\Omega(x_{Is}, \frac{1}{2}\lambda_I^{1/2}, 2\lambda_I^{1/2})$ containing the supports of the derivatives of the cutoff functions γ_{I_-}, γ_I. Define a map $\pi_{\Omega_I} : L^2\Omega^1(X, \operatorname{ad} P) \to L^2\Omega^1(\Omega_{Is}, \operatorname{ad} P_I)$ by left multiplication with this cutoff function. Lastly, let $\Pi = \pi_0 \oplus_{I>0} (\pi_{X_I} \oplus \pi_{\Omega_I})$ be the induced map

$$\begin{aligned} L^2\Omega^1(X, \operatorname{ad} P) \longrightarrow\ & L^2\Omega^1(X_0, \operatorname{ad} P_0) \\ & \textstyle\bigoplus_{I>0} \left(L^2\Omega^1(X_I, \operatorname{ad} P_I) \oplus L^2\Omega^1(\Omega_{Is}, \operatorname{ad} P_I)\right). \end{aligned}$$

Step 2. *Partial derivatives with respect to lower moduli parameters.* We have C^∞ Γ_{A_I}-equivariant maps $\tilde{\vartheta}_I : T_{A_I} \to \mathcal{A}^*_{X_I,P_I}$, $t_I \mapsto A_I(t_I)$ given by the Kuranishi model. Let v be a tangent vector to T_{A_I}, i.e., suppose $[v] \in H^1_{A_I}$. Then Eq. (5.9) and the estimates of §5.4 give the following bounds for the differentials with respect to the lower moduli parameters:

$$\|\pi_{X_I} D\mathcal{J}(v) - D\tilde{\vartheta}_I(v)\|_{L^2(X_I,g_I)} = O(\overline{\lambda}^{1/2}). \tag{5.10}$$

The map $\tilde{\vartheta}_I$ is an immersion and so the range of $D\tilde{\vartheta}_I$ has dimension equal to $\dim H^1_{A_I}$. For small enough $\overline{\lambda}$, Eq. (5.10) implies that the range of $\pi_{X_I} D\mathcal{J}$ also has dimension equal to $\dim H^1_{A_I}$

Step 3. *Partial derivatives with respect to gluing parameters.* Let v be a tangent vector to Gl_I. The estimates of §5.5 give the following bounds for the differentials with respect to the gluing parameters:

$$\|\pi_{\Omega_I} D\mathcal{J}(v) - D\mathcal{J}'(v)\|_{L^4(X_I,g_I)} = O(\overline{\lambda}^2), \tag{5.11}$$

recalling that $D\mathcal{J}'(v)$ is supported on $\Omega_I(\frac{1}{2}\lambda_I^{1/2}, 2\lambda_I^{1/2})$. But from Proposition 3.28 we have

$$\|D\mathcal{J}'(v)\|_{L^4(X_I,g_I)} \geq c|v|, \tag{5.12}$$

for some constant $c > 0$ independent of $\overline{\lambda}$. In particular, the range of $\pi_{\Omega_I} D\mathcal{J}'$ has dimension equal to $\dim \mathrm{Gl}_I$. So, for sufficiently small

$\overline{\lambda}$, Eqs. (5.11) and (5.12) imply that the range of $\pi_{\Omega_I} D\mathcal{I}$ also has dimension equal to $\dim \mathrm{Gl}_I$.

Step 4. *The quotient map.* Combining these observations, we find that the range of $\Pi D\mathcal{J}$ has dimension equal to $\dim H_{A_0} + \sum_{I>0}(\dim H^1_{A_I} + \dim \mathrm{Gl}_I) = \dim T$, so that $\ker \Pi D\mathcal{J} = 0$ and $\mathcal{J}$ is an immersion. From Theorem 5.8, the open subset $\tilde{U} \equiv \tilde{\mathcal{J}}(T)$ in $\mathcal{A}^*_{X,k}$ projects to an open subset $U \equiv \mathcal{J}(T)$ in $M^*_{X,k}(g)$ and composing $\tilde{\mathcal{J}}$ with the projection $\mathcal{A}^*_{X,k} \to \mathcal{A}^*_{X,k}/\mathcal{G}$, we obtain a submersion $\mathcal{I} : T \to M^*_{X,k}(g)$. The group Γ acts freely on $T, \tilde{\mathcal{J}}$ is Γ-equivariant, $\dim T/\Gamma = \dim M^*_{X,k}(g)$, and the gluing map descends to a diffeomorphism $\mathcal{I} : T/\Gamma \to M^*_{X,k}(g)$, as required. (c) This follows from (b). For the derivatives with respect to λ_I or x_I, the cutoff functions required to define Π should be replaced by cutoffs with similar supports and which are *fixed* with respect to small variations in the scales and centres. (d) This is similar to the proof of (c) and uses Proposition 3.5.

In order to parametrise neighbourhoods of boundary points in $\overline{M}^u_{X_0,k}(g_0)$, we use the following corollary to Proposition 5.10.

Corollary 5.11. *Given the hypotheses of* Theorem 5.8, *the following hold:*

(a) The approximate gluing map $\hat{\mathcal{J}}' : T/\Gamma \to \mathcal{B}^*_{X_0,k}$ is a C^∞ embedding,

(b) The gluing map $\hat{\mathcal{J}} : T/\Gamma \to V \subset M^*_{X_0,k}(g_0)$ is a diffeomorphism onto an open subset,

(c) The extended gluing map $\hat{\mathcal{J}} : \mathcal{T}/\Gamma \to \mathcal{V} \subset M^*_{X_0,k}(g_0)$ is a C^∞ submersion onto an open subset,

(d) The extended gluing map $\hat{\mathcal{J}} : \mathcal{T}^0/\Gamma \to \mathcal{V}^0 \subset M^*_{X_0,k}(g_0)$ is a diffeomorphism onto an open subset.

Taken together, Theorems 7.3.2 and 7.2.62 in [7] imply that if A is any g-anti-self-dual connection on a fixed G bundle P over the connected sum X and the necks Ω are all sufficiently pinched (so that $\overline{\lambda}$ is small), then $[A]$ lies in the image of the gluing map. The corresponding statement in our application is given below.

Theorem 5.12. *Given the hypotheses of* Theorem 5.8, *then the following holds. Let $\{A_\alpha\}_{\alpha=1}^\infty$ be a sequence of connections on a G bundle P over the connected sum $X = \#_{I\in\mathcal{I}} X_I$ which are anti-self-dual with respect to the sequence of metrics $\{g_\alpha\}_{\alpha=1}^\infty$ determined by the sequences of scales $\{\lambda_{I\alpha}\}$ with $\overline{\lambda}_\alpha \to 0$, a fixed neck width parameter N, sequences* *of points $\{x_{I\alpha}\}$ converging to $\{x_I\}$, and frames in $FX_0|_{x_{I\alpha}}$ converging to frames in $FX_0|_{x_I}$. Suppose the sequence $\{A_\alpha\}_{\alpha=1}^\infty$ is strongly convergent to $(A_I)_{I\in\mathcal{I}}$, where A_I is a g_I-anti-self-dual connection over each summand X_I. For α_0 sufficiently large, there exists a gluing neighbourhood $\mathcal{U}$ such that $[A_\alpha]\in\mathcal{U}$, for all $\alpha\geq\alpha_0$.*

Proof. See [7 (§7.3.1)]. Theorem 4.18 implies that the sequence $\{A_\alpha\}$ is D_q convergent (for any $4\leq q<\infty$) to $(A_I)_{I\in\mathcal{I}}$. So, Theorem 5.8 implies that the points $[A_\alpha]$ are contained in a gluing neighbourhood $\mathcal{U}$, for all $\alpha\geq\alpha_0$ if α_0 is sufficiently large.

Recall that $\mathrm{Gl}_{x_I}=\mathrm{SU}(2)\simeq\mathbb{S}^3$, a copy of the standard three-sphere, and let $\overline{\mathrm{Gl}}_{x_I}$ be the closure of $\mathrm{Gl}_{x_I}\times(0,\lambda_0)$ in the cone $(\mathrm{Gl}_{x_I}\times[0,\lambda_0))/\sim$, where $(\rho,0)\sim(\rho',0)$ if $\rho,\rho'\in\mathrm{Gl}_{x_I}$. Then, by analogy with [7 (§8.2)] and [3 (§V)], we set

$$\overline{\mathcal{T}}\equiv T_{A_0}\times\prod_{I\in\mathcal{I}}\left(T_{A_I}\times B(x_I,r_0)\times\overline{\mathrm{Gl}}_{x_I}\right),\tag{5.13}$$

and likewise, define $\overline{\mathcal{T}}^0$. It is also convenient to define

$$\partial\mathcal{T}\equiv\{t_\infty=(t_I,y_I,\rho_I,\lambda_I)_{I\in\mathcal{I}}\in\overline{\mathcal{T}}:\lambda_I=0\text{ for some }I\},\tag{5.14}$$

where the 4-tuple $(t_I,y_I,\rho_I,\lambda_I)$ above is replaced by t^0, if $I=0$. The space $\partial\mathcal{T}^0$ is defined similarly. Moreover, the gluing map $\mathcal{J}$ has a natural definition on the boundary $\partial\mathcal{T}$. Suppose $t_\infty\in\partial\mathcal{T}$ and let $(\lambda_1,\dots,\lambda_c)$ denote the corresponding scales in Eq. (5.14) which have been set equal to zero. By cutting the edges with $\lambda_i=0$, we may view the tree $\mathcal{I}$ as a union of subtrees $\cup_{i=1}^c\mathcal{I}^i$. If $t_\infty\in\partial\mathcal{T}$, we write $t_\infty=(t^1,\dots,t^c)$, with $t^i\in\mathcal{T}^i$, and set

$$\mathcal{J}(t_\infty)\equiv(\mathcal{J}(t^1),\dots,\mathcal{J}(t^c)),\qquad t_\infty\in\mathcal{T},\tag{5.15}$$

where each $\mathcal{J}(t^i)$ is an anti-self-dual connection over a connected sum $Y_i=\#_{I\in\mathcal{I}^i}X_I$, say, and $X=\#_{I\in\mathcal{I}}X_I=\#_{i=1}^cY_i$. The relationship between the gluing maps $\mathcal{J}$ and $\mathcal{J}^i$ is explained by the continuity result below, which we just state in the special cases $X=X_0\#X_1\#X_2$, for the sake of clarity. The argument required for this case carries over with no significant change to the more general cases just described.

Proposition 5.13. *Let $X=X_0\#X_1\#X_2$, let $Y=X_0\#X_1$, and let $Y''=Y\setminus B(x_1,\frac{1}{2}\lambda_1^{1/2})$. Assume that the hypotheses of* Theorem 5.8

hold and let $\mathcal{J}_X$, $\mathcal{J}_Y$ be the gluing maps over the connected sums X and Y, respectively. Then there is an $\varepsilon = \varepsilon(q) > 0$ and a constant $C = C(g_0, q, \mathcal{T})$ such that $\|\mathcal{J}_X(t)|_{Y''} - \mathcal{J}_Y\|_{L^q(Y,g)} \leq C\lambda_1^{\varepsilon_0}$.

The proof is similar to that of Proposition 7.2.64 [7] and the arguments in §5.3, and so is omitted. It now follows that $\mathcal{J}$ extends continuously to $\overline{\mathcal{T}}$.

Proposition 5.14. *Assume that the hypotheses of Theorem 5.8 hold. Let $\{t_\alpha\}_{\alpha=1}^\infty$ be a sequence in $\mathcal{T}$ which converges to $t_\infty \in \partial\mathcal{T}$. Then the sequence $\{\mathcal{J}(t_\alpha)\}_{\alpha=1}^\infty$ converges strongly to $\mathcal{J}(t_\infty)$.*

Proof. Let $\{\lambda_i\}_{i=1}^c$ denote the scales, determined by t_∞, which have been set equal to zero in Eq. (5.14). The points $t_\alpha \in \mathcal{T}$ are then naturally written as $t_\alpha = (t_\alpha^1, \dots, t_\alpha^c)$, with the sequences t_α^i converging to $t^i \in \mathcal{T}^i$, say. According to Proposition 5.13, the sequence $\mathcal{J}(t_\alpha)$ is then D_q convergent to $(\mathcal{J}(t^1), \dots, \mathcal{J}(t^c))$ and hence, strongly convergent by Theorem 5.18.

It remains to show that $M_{X_0,k}(g_0)$ has a finite cover consisting of gluing neighbourhoods. Of course, away from the bubbling ends, the moduli space is covered by the standard Kuranishi charts. In addition, the geometry of these charts around the reducible connections has already been analysed in [14], so our focus here is on the bubbling ends. Given any Uhlenbeck boundary point $(A_0, x_1, \dots, x_l) \in \overline{M}^u_{X_0,k}(g_0)$, where $c_2(A_0) = k - l$ and each x_i has multiplicity 1, Theorem 8.2.3 [7] provides an open neighbourhood $\overline{\mathcal{V}}$ of $(A_0, x_1, \dots, x_l)$ in $\overline{M}^u_{X_0,k}(g_0)$, a parameter space $\mathcal{T}^0/\Gamma$, and a gluing map $\hat{\mathcal{J}}$ giving a homeomorphism of $\mathcal{T}^0/\Gamma$ with $\mathcal{V} = \overline{\mathcal{V}} \cap M^*_{X_0,k}(g_0)$. Theorem 8.2.4 in [7] states that this gluing map extends to a homeomorphism $\hat{\mathcal{J}} : \overline{\mathcal{T}}^0/\Gamma \to \overline{\mathcal{V}}$. Thus, away from the diagonals, the ends of $\overline{M}^u_{X_0,k}(g_0)$ are covered by gluing neighbourhoods. The generalisations below provide a covering of the ends of $\overline{M}^u_{X_0,k}(g_0)$ which includes the diagonals.

Theorem 5.15. *Let $(A_0, x_1, \dots, x_{m_0})$ be a boundary point in $\overline{M}^u_{X_0,k}(g_0)$. Under the hypotheses of Theorem 5.8, there exist neighbourhoods $\overline{\mathcal{V}} \subset \overline{M}^u_{X_0,k}(g_0)$ of $(A_0, x_1, \dots, x_{m_0})$ and a parameter space $\mathcal{T}^0$ such that if $\mathcal{V} = \overline{\mathcal{V}} \cap M^*_{X_0,k}(g_0)$, then the gluing map $\hat{\mathcal{J}} : \mathcal{T}^0/\Gamma \to \mathcal{V}$ is a diffeomorphism.*

Proof. Suppose $\{[A_\alpha]\}_{\alpha=1}^\infty$ is a sequence in $M_{X_0,k}(g_0)$, converging weakly to the Uhlenbeck limit $(A_0, x_1, \dots, x_{m_0})$. Let $\{[\check{A}_\alpha]\}_{\alpha=1}^\infty$ be the corresponding strongly convergent sequence in $M_{X,k}(g_\alpha)$ with the bubble tree limit $(A_I, x_I)_{I\in\mathcal{I}}$. Then Theorem 5.12 produces a gluing neighbourhood $\mathcal{J}(\mathcal{T}^0/\Gamma) = \mathcal{U} \subset M_{X,k}(g_\alpha)$ and an α_0 such that $[\check{A}_\alpha] \in \mathcal{U}$ for all $\alpha \geq \alpha_0$. Let $\mathcal{V}$ be the corresponding neighbourhood in $M_{X_0,k}(g_0)$. Then the conclusions follow from Corollary 5.11.

Theorem 5.16. *Given the hypotheses of Theorem 5.15, the gluing map $\hat{\mathcal{J}}$ extends to a homeomorphism of $\overline{\mathcal{T}}^0/\Gamma$ with a neighbourhood $\overline{\mathcal{V}}$ of $(A_0, x_1, \ldots, x_{m_0})$ in $\overline{M}^u_{X_0,k}(g_0)$.*

Proof. This follows from Proposition 5.14 and Theorem 5.16.

Remark 5.17. So, every boundary point in $\overline{M}^u_{X_0,k}(g_0)$ has a neighbourhood constructible by gluing. Plainly, the same statement holds for boundary points in $\overline{M}^\tau_{X_0,k}(g_0)$.

5.3. Derivatives with respect to scales and centres. The main purpose of this section is to obtain L^2 estimates for the partial derivatives of the family of g_0-anti-self-dual connections $\hat{A}$ with respect to the scales λ_I and centres x_I.

Unless noted otherwise, *throughout this section and for the remainder of this article*, we assume that p and q are Sobolev exponents satisfying the strict inequalities $2 < p < 4$ and $4 < q < \infty$, where q is determined by $1/p = 1/4 + 1/q$. The constant $\lambda_0 > 0$ is assumed small and may be decreased as needed. We use $C = C(g_0, p, \mathcal{T})$ to denote constants which are independent of the points $t = (t_I, \rho_I, x_I, \lambda_I) \in \mathcal{T}$. As usual, we abbreviate the derivative with respect to the centre parameters, $p_I^\mu \partial/\partial q_I^\mu$ (where $|p_I| \leq 1$) by $\partial/\partial p_I$.

Denoting $\eta \equiv -F^{+g}(A')$ in Eq. (5.6), we have the following preliminary estimate for the derivatives of $\hat{a}$ with respect to the parameters λ_I and x_I.

Lemma 5.18. *Let ξ and $a = P\xi$ be as in Theorem 5.8, and assume that the conditions of that theorem hold. Then, for small enough $\lambda_0 > 0$, there is a constant $C = C(g_0, \mathcal{T})$ such that for any $t \in \mathcal{T}$,*

(a) $$\left\|\frac{\partial \hat{a}}{\partial \lambda_I}\right\|_{L^2(X_0,g_0)} \leq C\left(\left\|\frac{\partial P}{\partial \lambda_I}\xi\right\|_{L^2(X,g)} + \left\|\frac{\partial \xi}{\partial \lambda_I}\right\|_{L^2(X,g)} + \bar{\lambda}\lambda_I^{-1/2}\right),$$

(b) $$\left\|\frac{\partial \hat{a}}{\partial p_I}\right\|_{L^2(X_0,g_0)} \leq C\left(\left\|\frac{\partial P}{\partial p_I}\xi\right\|_{L^2(X,g)} + \left\|\frac{\partial \xi}{\partial p_I}\right\|_{L^2(X,g)} + \bar{\lambda}\right).$$

Proof. From Proposition 3.24, we have

$$\left\|\frac{\partial \hat{a}}{\partial \lambda_I}\right\|_{L^2(X_0,g_0)} \leq C\left\|\frac{\partial a}{\partial \lambda_I}\right\|_{L^2(X,g)} + C\lambda_I^{-1/2}\|a\|_{\mathcal{L}^2_1(X)},$$

where $a = P\xi$ and $\partial(P\xi)/\partial\lambda_I = (\partial P/\partial\lambda_I)\xi + P(\partial\xi/\partial\lambda_I)$. The estimates of Proposition 5.5 and Theorem 5.6 then give (a). The proof of (b) is similar.

We now differentiate the g-anti-self-dual equation and obtain a priori estimates for the partial derivatives of ξ with respect to λ_I and x_I.

Lemma 5.19. *Let ξ be as in* Theorem 5.8 *and assume that the conditions of that theorem hold. Then, for small enough $\lambda_0 > 0$, there is a constant $C = C(g_0, \mathcal{T})$ such that for any $t \in \mathcal{T}$,*

(a) $\|\partial\xi/\partial\lambda_I\|_{L^2(X,g)} \le C\left(1 + \bar{\lambda}^2\lambda_I^{-1/2} + \bar{\lambda}\|(\partial P/\partial\lambda_I)\xi\|_{L^4(X,g)}\right)$,

(b) $\|\partial\xi/\partial p_I\|_{L^2(X,g)} \le C\left(1 + \bar{\lambda}\|(\partial P/\partial p_I)\xi\|_{L^4(X,g)}\right)$.

Proof. Differentiating Eq. (5.6) with respect to λ_I gives

$$\begin{aligned}\frac{\partial\xi}{\partial\lambda_I} = \frac{\partial\eta}{\partial\lambda_I} - \frac{\partial *_g}{\partial\lambda_I}(P\xi\wedge P\xi) \\ - \left(\frac{\partial P\xi}{\partial\lambda_I}\wedge P\xi\right)^{+,g} - \left(P\xi\wedge\frac{\partial P\xi}{\partial\lambda_I}\right)^{+,g}.\end{aligned}$$

The estimates of Lemma 3.14 and Proposition 5.5 imply that

$$\begin{aligned}\left\|\frac{\partial\xi}{\partial\lambda_I}\right\|_{L^2} \le \left\|\frac{\partial\eta}{\partial\lambda_I}\right\|_{L^2} + C\|\xi\|_{L^2}^2\lambda_I^{-1/2} \\ + C\|\xi\|_{L^2}\left(\left\|\frac{\partial\xi}{\partial\lambda_I}\right\|_{L^2} + \left\|\frac{\partial P}{\partial\lambda_I}\xi\right\|_{L^4}\right).\end{aligned}$$

Proposition 3.26 and Theorem 5.6 yield $\|\partial\eta/\partial\lambda_I\|_{L^2} \le C$, and $\|\xi\|_{L^2} \le C\bar{\lambda}$ respectively. Thus, for λ_0 small enough, we may assume $C\|\xi\|_{L^2} \le 1/2$. Part (a) then follows by combining the above estimates and rearrangment, and the proof of (b) is similar.

To complete our task, we need an estimate for the derivatives of P with respect to λ_I and x_I. Before proceeding, we first record some bounds for the derivatives of the cutoff functions β_I and γ_I. Suppose $1 \le p < \infty$. From the definition of β_I there is a constant $C = C(g_I, N, p)$ such that

$$\begin{aligned}|d\beta_I|_{g_I} &\le C\lambda_I^{-1/2} \text{ on } \Omega_I, \Omega_{Is}, \\ \|d\beta_I\|_{L^p(X_I,g_I)} &\le C\lambda^{2/p-1/2}.\end{aligned} \tag{5.16}$$

Second, for the derivatives of β_J with respect to λ_I, one has

(5.17)
$$\begin{aligned}
\left|\frac{\partial\beta_J}{\partial\lambda_I}\right|_{L^\infty(X_J,g_J)} &\le C\lambda_I^{-1}, & \left|\frac{\partial d\beta_J}{\partial\lambda_I}\right|_{L^\infty(X_J,g_J)} &\le C\lambda_I^{-3/2},\\
\left\|\frac{\partial\beta_J}{\partial\lambda_I}\right\|_{L^p(X_J,g_J)} &\le \lambda_I^{2/p-1}, & \left\|\frac{\partial d\beta_J}{\partial\lambda_I}\right\|_{L^p(X_J,g_J)} &\le C\lambda_I^{2/p-3/2},
\end{aligned}$$

for $J = I_-$ or I, these derivatives being zero otherwise. Third, for the derivatives of β_J with respect to x_I, one has

(5.18)
$$\begin{aligned}
\left\|\frac{\partial\beta_J}{\partial p_I}\right\|_{L^\infty(X_J,g_J)} &\le C\lambda_I^{-1/2}, & \left\|\frac{\partial d\beta_J}{\partial p_I}\right\|_{L^\infty(X_J,g_J)} &\le C\lambda_I^{-1},\\
\left\|\frac{\partial\beta_J}{\partial p_I}\right\|_{L^p(X_J,g_J)} &\le \lambda_I^{2/p-1/2}, & \left\|\frac{\partial d\beta_J}{\partial p_I}\right\|_{L^p(X_J,g_J)} &\le C\lambda_I^{2/p-1},
\end{aligned}$$

for $J = I_-$ or I, these derivatives being zero otherwise. The cutoff functions γ_J also satisfy the bounds of Eqs. (5.16), (5.17) and (5.18).

Proposition 5.20. *For any $0 < \delta < \frac{1}{2}$ and $2 < p < 4$ defined by $p = 4/(1+2\delta)$, and small enough λ_0, there is a constant $C = C(\delta, g_0, \mathcal{T})$ such that for any $t \in \mathcal{T}$ and $\xi \in L^p\Omega^{+,g}(X, \operatorname{ad} P)$,*

(a) $\|(\partial P/\partial\lambda_I)\xi\|_{L^4(X,g)} \le C\lambda_I^{-1/2-\delta}\|\xi\|_{L^p(X,g)}$,

(b) $\|(\partial P/\partial p_I)\xi\|_{L^4(X,g)} \le C\lambda_I^{-\delta}\|\xi\|_{L^p(X,g)}$.

Proof. (a) As $P = Q(1+R)^{-1}$, we first obtain operator bounds for $\partial Q/\partial\lambda_I$, $\partial R/\partial\lambda_I$, and then deduce an operator bound for $\partial P/\partial\lambda_I$.

Step 1. *Estimate for $\partial Q/\partial\lambda_I$.* Recall that $Q\xi = \sum_J Q_J\xi$, where $Q_J = \beta_J P_J \gamma_J$ is independent of λ_I for $J \ne I_-, I$, and so

$$\frac{\partial Q}{\partial\lambda_I} = \frac{\partial Q_{I_-}}{\partial\lambda_I} + \frac{\partial Q_I}{\partial\lambda_I},$$

where

$$\frac{\partial Q_I}{\partial\lambda_I} = \frac{\partial\beta_I}{\partial\lambda_I}P_I\gamma_I + \beta_I P_I\frac{\partial c_I}{\partial\lambda_I},$$

with the analogous expression for $\partial Q_{I_-}/\partial\lambda_I$. Choose $4 < q, q_1 < \infty$ and $2 < p, p_1 < 4$ by setting

(5.19)
$$\begin{aligned}
p &= 4/(1+2\delta) \quad\text{and}\quad q = 4/(1-2\delta),\\
1/p &= 1/4 + 1/q_1 \quad\text{and}\quad 1/2 = 1/p_1 + 1/q_1,
\end{aligned}$$

and observe that $1/4 = 1/q + 1/q_1$ and $1/2 = 1/p + 1/q$, while $2/p = 1/2 + \delta$ and $2/q = 1/2 - \delta$. Applying Hölder's inequality, the operator bounds for P_I of Lemma 5.1, and the fact that $\|\partial\beta_I/\partial\lambda_I\|_{L^q}$ and $\|\partial\gamma_I/\partial\lambda_I\|_{L^q}$ are bounded by $C\lambda_I^{2/q-1}$ from Eq. (5.17), we find

$$\left\|\frac{\partial Q_I}{\partial\lambda_I}\xi\right\|_{L^4} \le C\left\|\frac{\partial\beta_I}{\partial\lambda_I}\right\|_{L^q}\|\xi\|_{L^p} + C\left\|\frac{\partial\gamma_I}{\partial\lambda_I}\right\|_{L^q}\|\xi\|_{L^p} \le C\lambda_I^{2/q-1}\|\xi\|_{L^p}.$$

Combining the above estimate with the analogous bound for the $\partial Q_{I_-}/\partial\lambda_I$ term, we see that

$$\left\|\frac{\partial Q}{\partial\lambda_I}\xi\right\|_{L^4} \le C\lambda_I^{2/q-1}\|\xi\|_{L^p}, \tag{5.20}$$

completing Step 1.

Step 2. *Estimate for $\partial R/\partial\lambda_I$.* We have $R = d_{A'}^{+,g}Q - 1$ on X, and so differentiating with respect to λ_I gives

$$\frac{\partial R}{\partial\lambda_I}\xi = \frac{\partial *_g}{\partial\lambda_I}d_{A'}Q\xi + \left[\frac{\partial A'}{\partial\lambda_I}, Q\xi\right]^{+,g} + d_{A'}^{+,g}\frac{\partial Q}{\partial\lambda_I}\xi.$$

Using our L^∞ bound for $\partial *_g/\partial\lambda_I$ of Lemma 3.14, the L^4 bound for $\partial A'/\partial\lambda_I$ of Proposition 3.25, and the operator norm bounds for Q of Lemma 5.3, we obtain

$$\left\|\frac{\partial R}{\partial\lambda_I}\xi\right\|_{L^2} \le C\lambda_I^{-1/2}\|d_{A'}Q\xi\|_{L^2} + C\|\xi\|_{L^2} + \left\|d_{A'}^{+,g}\frac{\partial Q}{\partial\lambda_I}\xi\right\|_{L^2}. \tag{5.21}$$

For the $d_{A'}Q$ term above, noting that $d_{A'}Q\xi = \sum_J d_{A'_J}Q_I\xi$ and writing $A'_J = A_J + a_J$ over X'_J lead to

$$d_{A'_J}Q_J\xi = d\beta_J \wedge P_J\gamma_J\xi + \beta_J d_{A_J}P_J\gamma_J\xi + \beta_J[a_J, P_J\gamma_J\xi].$$

By the bounds $\|d\beta_J\|_{L^4} \le C$ of Eq. (5.16), $\|a_J\|_{L^4} \le C\overline{\lambda}$ of Lemma 3.9, Hölder's inequality, and the operator bounds for P_J of Lemma 5.1, we find that

$$\|d_{A'}Q\xi\|_{L^2} \le C\|\xi\|_{L^2}. \tag{5.22}$$

For the $d_{A'}^{+,g}\partial Q/\partial\lambda_I$ term, note that

$$d_{A'}^{+,g}\frac{\partial Q}{\partial\lambda_I} = d_{A'_{I_-}}^{+,g}\frac{\partial Q_{I_-}}{\partial\lambda_I} + d_{A'_I}^{+,g}\frac{\partial Q_I}{\partial\lambda_I}.$$

We use $d_{A_I}^{+,g_I} P_I = 1$ and $\beta_I = 1$ on $\operatorname{supp}\gamma_I$ to get

$$\begin{aligned} d_{A_I'}^{+,g} \frac{\partial Q_I}{\partial \lambda_I}\xi = & \left(d\frac{\partial \beta_I}{\partial \lambda_I} \wedge P_I \gamma_I \xi \right)^{+,g} + \frac{\partial \beta_I}{\partial \lambda_I} [a_I, P_I \gamma_I \xi]^{+,g} \\ & + \left(d\beta_I \wedge P_I \frac{\partial \gamma_I}{\partial \lambda_I}\xi \right)^{+,g} + \frac{\partial \gamma_I}{\partial \lambda_I}\xi + \beta_I \left[a_I, P_I \frac{\partial \gamma_I}{\partial \lambda_I}\xi \right]^{+,g} \\ & + \frac{1}{2}\frac{\partial \beta_I}{\partial \lambda_I} (*_g - *_{g_I}) d_{A_I} P_I \gamma_I \xi \\ & + \frac{1}{2}\beta_I (*_g - *_{g_I}) d_{A_I} P_I \frac{\partial \gamma_I}{\partial \lambda_I}\xi, \end{aligned}$$

with the analogous expression for $d_{A_{I_-}'}^{+,g}\, \partial Q_{I_-}/\partial\lambda_I$. From Lemmas 3.14 and 5.1, it follows that

$$\begin{aligned} \left\| d_{A_I'}^{+,g} \frac{\partial Q_I}{\partial \lambda_I}\xi \right\|_{L^2} \leq & C \left\| d\frac{\partial \beta_I}{\partial \lambda_I} \right\|_{L^{p_1}} \|\xi\|_{L^p} + C \left\| \frac{\partial \beta_I}{\partial \lambda_I} \right\|_{L^q} \|a_I\|_{L^4} \|\xi\|_{L^p} \\ & + C\|d\beta_I\|_{L^4} \left\| \frac{\partial \gamma_I}{\partial \lambda_I} \right\|_{L^q} \|\xi\|_{L^p} + \left\| \frac{\partial \gamma_I}{\partial \lambda_I} \right\|_{L^q} \|\xi\|_{L^p} \\ & + C\|a_I\|_{L^4} \left\| \frac{\partial \gamma_I}{\partial \lambda_I} \right\|_{L^q} \|\xi\|_{L^p} \\ & + C\bar{\lambda} \left\| \frac{\partial \beta_I}{\partial \lambda_I} \right\|_{L^q} \|\xi\|_{L^p} + C\bar{\lambda} \left\| \frac{\partial \gamma_I}{\partial \lambda_I} \right\|_{L^q} \|\xi\|_{L^p}. \end{aligned}$$

Now $\|a_I\|_{L^4} \leq C\bar{\lambda}$ by Lemma 3.9, and because of Eq. (5.17), $\|\partial\beta_I/\partial\lambda_I\|_{L^q}$ and $\|\partial\gamma_I/\partial\lambda_I\|_{L^q}$ are bounded by $C\lambda_I^{2/q-1}$. Hence,

$$\left\| d_{A_I'}^{+,g} \frac{\partial Q_I}{\partial \lambda_I}\xi \right\|_{L^2} \leq C\lambda_I^{2/q-1} \|\xi\|_{L^p},$$

with the analogous bound for the $d_{A_{I_-}'}^{+,g}\, \partial Q_{I_-}/\partial\lambda_I$ term. Therefore,

$$\left\| d_{A'}^{+,g} \frac{\partial Q}{\partial \lambda_I}\xi \right\|_{L^2} \leq C\lambda_I^{2/q-1} \|\xi\|_{L^p}. \tag{5.23}$$

Combining the above inequalities and noting that $\|\xi\|_{L^2(X,g)} \leq C\|\xi\|_{L^p(X,g)}$, yield

$$\left\| \frac{\partial R}{\partial \lambda_I}\xi \right\|_{L^4(X,g)} \leq C\lambda_I^{2/q-1} \|\xi\|_{L^p}, \tag{5.24}$$

which completes Step 2.

Step 3. *Estimate for $\partial P/\partial\lambda_I$.* Differentiating $P = Q(1+R)^{-1}$ with respect to λ_I gives

$$\frac{\partial P}{\partial\lambda_I} = \frac{\partial Q}{\partial\lambda_I}(1+R)^{-1} - Q(1+R)^{-1}\frac{\partial R}{\partial\lambda_I}(1+R)^{-1},$$

and thus applying the bounds from Steps 1 and 2, we have

$$\left\|\frac{\partial P}{\partial\lambda_I}]\xi\right\|_{L^4(X,g)} \le C\lambda_I^{2/q-1}\|\xi\|_{L^p(X,g)},$$

which yields (a) since $2/q - 1 = -1/2 - \delta$. For (b), the strategy of (a) shows that $\|(\partial Q/\partial p_I)\xi\|_{L^4}$ and $\|(\partial R/\partial p_I)\xi\|_{L^4(X,g)}$ are bounded by $C\lambda_I^{2/q-1/2}\|\xi\|_{L^p}$, leading to

$$\left\|\frac{\partial P}{\partial p_I}\xi\right\|_{L^4(X,g)} \le C\lambda_I^{2/q-1/2}\|\xi\|_{L^p}, \tag{5.25}$$

and so (b) follows.

As is readily verified, Lemma 5.19 and Proposition 5.20 then provide the following estimates for the derivatives of ξ and a with respect to λ_I and x_I:

Corollary 5.21. *Let ξ and $a = P\xi$ be as in Theorem 5.8 and assume that the conditions of that theorem hold. Then, for small enough $\lambda_0 > 0$, there is a constant $C = C(\delta, g_0, \mathcal{T})$ such that for any $t \in \mathcal{T}$,*

(a) $\|\partial\xi/\partial\lambda_I\|_{L^2(X,g)} \le C(1 + \overline{\lambda}^{3/2+\delta}\lambda_I^{-1/2-\delta})$,

(b) $\|\partial\xi/\partial p_I\|_{L^2(X,g)} \le C(1 + \overline{\lambda}^{3/2+\delta}\lambda_I^{-\delta})$,

(c) $\|\partial a/\partial\lambda_I\|_{L^2(X,g)} \le C(1 + \overline{\lambda}^{1/2+\delta}\lambda_I^{-1/2-\delta})$,

(d) $\|\partial a/\partial p_I\|_{L^2(X,g)} \le C(1 + \overline{\lambda}^{1/2+\delta}\lambda_I^{-\delta})$.

With bounds for the derivatives of ξ and P with respect to λ_I and x_I at hand, we obtain our final estimates for the derivatives of the anti-self-dual connections A and $\hat{A}$. Since $A = A' + a$, combining Proposition 3.25 and Corollary 5.21 gives

Corollary 5.22. *Assume that the conditions of Theorem 5.8 hold. Then, for any $0 < \delta < 1/2$ and small enough $\lambda_0 > 0$, there is a constant $C = C(\delta, g_0, \mathcal{T})$ such that for any $t \in \mathcal{T}$, the following bounds hold:*

(a) $\|\partial A/\partial\lambda_I\|_{L^2(X,g)} \le C(1 + \overline{\lambda}^{1/2+\delta}\lambda_I^{-1/2-\delta})$,

(b) $\|\partial A/\partial p_I\|_{L^2(X,g)} \le C(1 + \overline{\lambda}^{1/2+\delta}\lambda_I^{-\delta})$.

Theorem 5.23. *Assume that the conditions of* Theorem 5.8 *hold. Then, for any* $0 < \delta < 1/2$ *and small enough* $\lambda_0 > 0$, *there is a constant* $C = C(\delta, g_0, \mathcal{T})$ *such that for any* $t \in \mathcal{T}$, *the following bounds hold:*

(a) $\|\partial\hat{a}/\partial\lambda_I\|_{L^2(X_0,g_0)} \le C(1 + \bar{\lambda}^{1/2+\delta}\lambda_I^{-1/2-\delta})$,

(b) $\|\partial\hat{A}/\partial\lambda_I\|_{L^2(X_0,g_0)} \le C(1 + \bar{\lambda}^{1/2+\delta}\lambda_I^{-1/2-\delta})$,

(c) $\|\partial\hat{a}/\partial p_I\|_{L^2(X_0,g_0)} \le C(1 + \bar{\lambda}^{1/2+\delta}\lambda_I^{-\delta})$,

(d) $\|\partial\hat{A}/\partial p_I\|_{L^2(X_0,g_0)} \le C(1 + \bar{\lambda}^{1/2+\delta}\lambda_I^{-\delta})$.

Proof. Using the bound $\|\xi\|_{L^p} \le C\bar{\lambda}^{2/p}$ of Theorem 5.6, the equality $2/p = 1/2 + \delta$, the L^2 estimate for $\partial\hat{a}/\partial\lambda_I$ in Lemma 5.18, the L^2 estimate for $\partial\xi/\partial\lambda_I$ in Corollary 5.21, and the operator estimate for $\partial P/\partial\lambda_I$ in Proposition 5.20, we obtain

$$\left\|\frac{\partial\hat{a}}{\partial\lambda_I}\right\|_{L^2(X_0,g_0)} \le C\left(\bar{\lambda}^{1/2+\delta}\lambda_I^{-1/2-\delta} + 1 + \bar{\lambda}^{3/2+\delta}\lambda_I^{-1/2-\delta} + \bar{\lambda}^{1/2+\delta}\lambda_I^{-1/2}\right),$$

which yields (a). Then (b) follows from (a) and the estimate $\|\partial\hat{A}'/\partial\lambda_I\|_{L^2(X_0,g_0)} \le C$ of Proposition 3.27. The proofs of (c) and (d) are similar.

5.4. Derivatives with respect to lower moduli. In this section we obtain estimates for the derivatives of the family of g_0-anti-self-dual connections $\hat{A}$ with respect to the lower moduli parameters $t_I \in T_{A_I}$. Just as in §5.3, the strategy is to use the g-anti-self-dual equation of Eq. (5.6), together with its derivatives with respect to the t_I parameters, to first obtain estimates for the derivatives of a and ξ, and then the required derivatives of $\hat{a}$ and $\hat{A}'$. The Sobolev exponents p, q are fixed so that $2 \le p < 4$ and $4 \le q < \infty$, where q is determined by $1/p = 1/4 + 1/q$. We have the following preliminary estimates for the derivatives of ξ and a.

Lemma 5.24. *Let* ξ *and* $a = P\xi$ *be as in Theorem 5.8, and assume that the conditions of that theorem hold. Then, for small enough* $\lambda_0 > 0$, *there is a constant* $C = C(g_0, p, \mathcal{T})$ *such that for any* $t \in \mathcal{T}$,

(a) $\|\partial a/\partial t_I\|_{L^p(X,g)} \le C\|\partial\xi/\partial t_I\|_{L^p(X,g)} + \|(\partial P/\partial t_I)\xi\|_{L^p(X,g)}$,

(b) $\|\partial\xi/\partial t_I\|_{L^p(X,g)} \le C\left(\bar{\lambda}^{2/p-1/2} + \bar{\lambda}^{2/p}\|(\partial P/\partial t_I)\xi\|_{L^4(X,g)}\right)$.

Proof. Differentiating Eq. (5.6) with respect to t_I gives

$$\frac{\partial \xi}{\partial t_I} = \frac{\partial \eta}{\partial t_I} - \left(\frac{\partial P\xi}{\partial t_I} \wedge P\xi\right)^{+,g} - \left(P\xi \wedge \frac{\partial P\xi}{\partial t_I}\right)^{+,g},$$
$$\frac{\partial P\xi}{\partial t_I} = \frac{\partial P}{\partial t_I}\xi + P\frac{\partial \xi}{\partial t_I}.$$

The proofs of (a) and (b) are then similar to those of Lemmas 5.18 and 5.19.

Thus, an operator estimate for $\partial P/\partial t_I$ is required. As $P = Q(1+R)^{-1}$, we have

$$\frac{\partial P}{\partial t_I} = \frac{\partial Q}{\partial t_I}(1+R)^{-1} - Q(1+R)^{-1}\frac{\partial R}{\partial t_I}(1+R)^{-1}. \tag{5.26}$$

We recall that $P_I = d^{*,g_I}_{A_I} G^{+,g_I}_{A_I}$. Differentiating with respect to t_I, we obtain

$$\frac{\partial P_I}{\partial t_I} = \frac{\partial d^{*,g_I}_{A_I}}{\partial t_I} G^{+,g_I}_{A_I} - d^{*,g_I}_{A_I} G^{+,g_I}_{A_I} \frac{\partial \Delta^{+,g_I}_{A_I}}{\partial t_I} G^{+,g_I}_{A_I}.$$

The derivatives of $d^{+,g_I}_{A_I}$ and $d^{*,g_I}_{A_I}$ with respect to t_I are given by

$$\frac{\partial d^{+,g_I}_{A_I}}{\partial t_I}\omega = \left[\frac{\partial A_I}{\partial t_I}, \omega\right]^{+,g_I} = \left[\frac{\partial A_I}{\partial t_I}, \cdot\right]^{+,g_I} \omega,$$
$$\frac{\partial d^{*,g_I}_{A_I}}{\partial t_I}\xi = -* \left[\frac{\partial A_I}{\partial t_I}, *\xi\right] = \left[\frac{\partial A_I}{\partial t_I}, \cdot\right]^{*} \xi,$$

for any $\omega \in \Omega^1(X_I, \operatorname{ad} P_I)$ and $\xi \in \Omega^{+,g_I}(X_I, \operatorname{ad} P_I)$. Therefore,

$$\frac{\partial \Delta^{+,g_I}_{A_I}}{\partial t_I} = \left[\frac{\partial A_I}{\partial t_I}, \cdot\right]^{+,g_I} d^{*,g_I}_{A_I} + d^{+,g_I}_{A_I} \left[\frac{\partial A_I}{\partial t_I}, \cdot\right]^{*},$$

and so we find that

$$\frac{\partial P_I}{\partial t_I} = (1 - P_I d^{+,g_I}_{A_I}) \left[\frac{\partial A_I}{\partial t_I}, \cdot\right]^{*} G^{+,g_I}_{A_I} - P_I \left[\frac{\partial A_I}{\partial t_I}, \cdot\right]^{+,g_I} P_I. \tag{5.27}$$

Note that $1 - P_I d^{+,g_I}_{A_I}$ is a bounded (L^q, L^q) operator on $\Omega^{+,g_I}(X_I, \operatorname{ad} P_I)$ by the Calderon-Zygmund theory.

Lemma 5.25. *There is a constant $C = C(g_0, p, \mathcal{T})$ such that for any $t \in \mathcal{T}$,*

(a) $\|(\partial P_I/\partial t_I)\xi\|_{L^q(X_I, g_I)} \le C\|\xi\|_{L^p(X_I, g_I)}$, *for* $\xi \in L^p\Omega^{+,g_I}(X_I, g_I)$,

(b) $\|(\partial Q/\partial t_I)\xi\|_{L^q(X,g)} \le C\|\xi\|_{L^p(X,g)}$, *for* $\xi \in L^p\Omega^{+,g}(X,g)$.

Proof. Since $1 - P_I d_{A_I}^{+,g_I}$ is bounded on $L^q(X_I, g_I)$, Eq. (5.27) and the Hölder inequalities show that

$$\left\|\frac{\partial P_I}{\partial t_I}\xi\right\|_{L^q} \le \left\|\frac{\partial A_I}{\partial t_I}\right\|_{L^\infty} \|G_{A_I}^{+,g_I}\xi\|_{L^q} + C\left\|\frac{\partial A_I}{\partial t_I}\right\|_{L^4} \|P_I\xi\|_{L^q}.$$

But $G_{A_I}^{+,g_I}$ and P_I are bounded (L^p, L^q) operators and noting that the family $A_I(t_I)$ is smoothly parametrised by $t_I \in T_{A_I}$, we obtain (a). Since $Q_I = \beta_I P_I \gamma_I$ and $Q = \sum_I Q_I$, inequality (b) follows.

It remains to estimate the derivative of R with respect to t_I.

Lemma 5.26. *There is a constant* $C = C(g_0, p, \mathcal{T})$ *such that for any* $t \in \mathcal{T}$ *and* $\xi \in L^p\Omega^{+,g}(X,g)$, *we have* $\|(\partial R/\partial t_I)\xi\|_{L^p(X,g)} \le C\|\xi\|_{L^p(X,g)}$.

Proof. We recall that $R = d_{A'}^{+,g}Q - 1$ over X and $R = d_{A_I'}^{+,g}Q_I - 1$ over X_I. Writing $A_I' = A_I + a_I$, we find that

$$\begin{aligned} R = {} & d\beta_I \wedge P_I\gamma_I + \beta_I d_{A_I}^{+,g_I} P_I\gamma_I + \beta_I[a_I, \cdot\,]^{+,g} P_I\gamma_I \\ & + \frac{1}{2}(*_g - *_{g_I})\beta_I d_{A_I} P_I - 1. \end{aligned}$$

Noting that $d_{A_I}^{+,g_I}P_I = 1$ and differentiating with respect to t_I, we have

$$\begin{aligned} \frac{\partial R}{\partial t_I}\xi = {} & d\beta_I \wedge \frac{\partial P_I}{\partial t_I}\gamma_I\xi + \beta_I\left[\frac{\partial a_I}{\partial t_I}, P_I\gamma_I\xi\right]^{+,g} \\ & + \beta_I\left[a_I, \frac{\partial P_I}{\partial t_I}\gamma_I\xi\right]^{+,g} \\ & + \frac{1}{2}(*_g - *_{g_I})\beta_I\left[\frac{\partial A_I}{\partial t_I}, P_I\gamma_I\xi\right] \\ & + \frac{1}{2}(*_g - *_{g_I})\beta_I d_{A_I}\frac{\partial P_I}{\partial t_I}\gamma_I\xi, \end{aligned} \tag{5.28}$$

and therefore

$$\begin{aligned} \left\|\frac{\partial R}{\partial t_I}\xi\right\|_{L^p} \le {} & C\|d\beta_I\|_{L^4}\left\|\frac{\partial P_I}{\partial t_I}\gamma_I\xi\right\|_{L^q} + C\left\|\frac{\partial a_I}{\partial t_I}\right\|_{L^4}\|P_I\gamma_I\xi\|_{L^q} \\ & + C\|a_I\|_{L^4}\left\|\frac{\partial P_I}{\partial t_I}\gamma_I\xi\right\|_{L^q} + C\overline{\lambda}\left\|\frac{\partial A_I}{\partial t_I}\right\|_{L^4}\|P_I\gamma_I\xi\|_{L^q} \\ & + C\overline{\lambda}\left\|d_{A_I}\frac{\partial P_I}{\partial t_I}\gamma_I\xi\right\|_{L^p}, \end{aligned} \tag{5.29}$$

where $a_I = (\psi_I - 1)\sigma_I^* A_I$ and $\partial a_I/\partial t_I = (\psi_I - 1)\sigma_I^* \partial A_I/\partial t_I$. Aside from the self-dual projection and factor β_I, the last term on the right-hand side of Eq. (5.28) is given by

$$d_{A_I}\frac{\partial P_I}{\partial t_I}\gamma_I\xi = -d_{A_I} * \left[\frac{\partial A_I}{\partial t_I}, G_{A_I}^{+,g_I}\gamma_I\xi\right] + d_{A_I}P_I d_{A_I}^{+,g_I} * \left[\frac{\partial A_I}{\partial t_I}, G_{A_I}^{+,g_I}\gamma_I\xi\right]$$
$$-d_{A_I}P_I\left[\frac{\partial A_I}{\partial t_I}, P_I\gamma_I\xi\right]^{+,g}$$

Since P_I is a bounded operator from L^p to L_1^p, using the bounded inclusion $L_1^p \to L^q$ we see that

$$\left\|d_{A_I}\frac{\partial P_I}{\partial t_I}\gamma_I\xi\right\|_{L^p} \le C\left\|d_{A_I} * \left[\frac{\partial A_I}{\partial t_I}, G_{A_I}^{+,g_I}\gamma_I\xi\right]\right\|_{L^p} + \left\|\left[\frac{\partial A_I}{\partial t_I}, P_I\gamma_I\xi\right]\right\|_{L^p}$$
$$\le C\left\|\frac{\partial A_I}{\partial t_I}\right\|_{L^\infty}\|G_{A_I}^{+,g_I}\gamma_I\xi\|_{L^p}$$
$$+\left\|\nabla^{A_I}\frac{\partial A_I}{\partial t_I}\right\|_{L^\infty}\|G_{A_I}^{+,g_I}\gamma_I\xi\|_{L^p}$$
$$+C\left\|\frac{\partial A_I}{\partial t_I}\right\|_{L^\infty}\|\nabla^{A_I}G_{A_I}^{+,g_I}\gamma_I\xi\|_{L^p}$$
$$+\left\|\frac{\partial A_I}{\partial t_I}\right\|_{L^4}\|P_I\gamma_I\xi\|_{L^q}.$$

Since the family $A_I(t)$ is smoothly parametrised by $t_I \in T_{A_I}$, and $G_{A_I}^{+,g_I}$ is a bounded operator from L^p to L_2^p, we have

$$\left\|d_{A_I}\frac{\partial P_I}{\partial t_I}\gamma_I\xi\right\|_{L^p} \le C\|\xi\|_{L^p}. \tag{5.30}$$

Eqs. (5.29), (5.30) and Lemma 5.25 then yield the required bound for $\partial R/\partial t_I$.

Thus, Eq. (5.26), together with Lemmas 5.25 and 5.26, provides an estimate for the derivative of P with respect to t_I:

Proposition 5.27. *There is a constant $C = C(g_0, p, \mathcal{T})$ such that for any $t \in \mathcal{T}$ and $\xi \in L^p\Omega^{+,g}(X, \operatorname{ad} P)$, we have $\|\partial P/\partial t_I\xi\|_{L^q(X,g)} \le C\|\xi\|_{L^p(X,g)}$.*

This leads to our final estimates for the derivatives of ξ and a with respect to t_I.

Corollary 5.28. *Let ξ and $a = P\xi$ be as in Theorem 5.8, and assume that the conditions of that theorem hold. Then, for small enough $\lambda_0 > 0$, there is a constant $C = C(g_0, p, \mathcal{T})$ such that for any $t \in \mathcal{T}$,*

(a) $\|\partial\xi/\partial t_I\|_{L^p(X,g)} \le C\overline{\lambda}^{2/p-1/2}$,

(b) $\|\partial a/\partial t_I\|_{L^p(X,g)} \le C\overline{\lambda}^{2/p-1/2}$.

Proof. Inequality (a) follows from Lemma 5.24 and Proposition 5.27, since $\|\xi\|_{L^p} \le C\overline{\lambda}^{2/p}$ by Theorem 5.6. Inequality (b) then follows from (a) and Lemma 5.24.

By combining Proposition 3.30 and Corollary 5.28 we obtain an estimate for the derivatives of the connections $A = A' + a$ over X:

Corollary 5.29. *Assume that the conditions of Theorem 5.8 hold. Then, for any $2 \le p < 4$ and sufficiently small $\lambda_0 > 0$, there is a constant $C = C(g_0, p, \mathcal{T})$ such that for any $t \in \mathcal{T}$,*

(a) $\|\partial A/\partial t_I - \partial A_I/\partial t_I\|_{L^p(X_I'', g_I)} \le C\overline{\lambda}^{2/p-1/2}$,

(b) $\|\partial A/\partial t_I\|_{L^p(X,g)} \le C$.

We now come to the main result of this section.

Theorem 5.30. *Assume that the conditions of Theorem 5.8 hold. Then, for any $2 \le p < 4$ and sufficiently small $\lambda_0 > 0$, there is a constant $C = C(g_0, p, \mathcal{T})$ such that for any $t \in \mathcal{T}$,*

(a) $\|\partial\hat{a}/\partial t_I\|_{L^p(X_0,g_0)} \le C\overline{\lambda}^{2/p-1/2}$

(b) $\|\partial\hat{A}/\partial t_I\|_{L^p(X_0,g_0)} \le C$.

Proof. Let $U \equiv f_0^{-1} \cdots f_I^{-1}(X_I') \subset X_0$ and note that $\partial\hat{a}/\partial t_I = \sum_I f_0^* \cdots f_I^* \partial a/\partial t_I$ on U. Lemma 3.19 gives

$$\left\|\frac{\partial\hat{a}}{\partial t_I}\right\|_{L^p(X_0,g_0)} \le C\sum_I \left\|f_0^* \cdots f_I^* \frac{\partial a}{\partial t_I}\right\|_{L^p(X_0,g_0)} \le C\left\|\frac{\partial a}{\partial t_I}\right\|_{L^p(X,g)},$$

and so Part (a) follows from Corollary 5.28. Part (b) follows from (a), and the estimate $\|\partial\hat{A}'/\partial t_I\|_{L^p(X_0,g_0)} \le C$ of Proposition 3.32.

5.5. Derivatives with respect to bundle gluing parameters. We obtain estimates for the partial derivatives of the family of g_0-anti-self-dual connections $\hat{A}(t)$ with respect to the bundle gluing parameters $\rho_I \in \mathrm{Gl}_I$. The Sobolev exponents p, q are fixed so that $2 \le p < 4$, with $4 \le q < \infty$ determined by $1/4 + 1/q = 1/p$. We first recall the estimate of Donaldson and Kronheimer for the derivative of $a = P\xi$ with respect to the gluing parameters ρ_I. As described in §3.8, we work with an equivalent family of g-anti-self-dual connections $A = A' + a$ on a fixed bundle P. Thus, considering only the gluing parameters, we have a diffeomorphism $B_{\mathfrak{g}} \ni v \to A(\bar\rho_I, v) \in \mathcal{A}^*_{X,P}$ (where $B_{\mathfrak{g}}$ is the unit ball in $\mathfrak{g}$), giving a family of C^∞ connections on a fixed bundle $P = P(\bar\rho_I)$, as in Eq. (3.58). Here, $B_{\mathfrak{g}} \ni v \to \rho_I(v) = \bar\rho_I \exp(v) \in \mathrm{Gl}_I$ is a coordinate chart centred at $\bar\rho_I \in \mathrm{Gl}_I$, as in Eq. (3.57). This understood, one has the following bounds.

Proposition 5.31. [7 (p. 303)] *Let a be as in Theorem 5.8, and assume that the conditions of that theorem hold. Then, for small enough $\lambda_0 > 0$, there is a constant $C = C(g_0, p, \mathcal{T})$ such that for any $t \in \mathcal{T}$, $\|\partial a/\partial v\|_{L^q(X,g)} \le C\overline{\lambda}^{2/p+1}$*

Proof. The proof in [7] deals only with single connected sums $X = X_0 \# X_1$, but the argument adapts without significant change to the general case of multiple connected sums $\#_{I\in\mathcal{I}} X_I$. Likewise, the assumptions in [7] that $\Gamma_I = 1$ and $H^0_{A_I} = H^1_{A_I} = 0$, for all I, do not affect the relevant estimates.

Corollary 5.32. *Let A be as in Theorem 5.8, and assume that the conditions of that theorem hold. Then, for small enough $\lambda_0 > 0$, there is a constant $C = C(g_0, p, \mathcal{T})$ such that for any $t \in \mathcal{T}$, $\|\partial A/\partial v\|_{L^p(X,g)} \le C\overline{\lambda}^{2/p-1/2}$.*

Proof. Combine Propositions 3.28 and 5.31.

Moreover, we have the following estimates for the derivatives of the g_0-anti-self-dual connections $\hat A = \hat A' + \hat a$ on the fixed bundle $\hat P$ over X_0.

Theorem 5.33. *Assume that the conditions of Theorem 5.8 hold. Then, for small enough $\lambda_0 > 0$, there is a constant $C = C(g_0, p, \mathcal{T})$ such that for any $t \in \mathcal{T}$,*

(a) $\|\partial \hat a/\partial v\|_{L^p(X,g)} \le C\overline{\lambda}^{2/p+1}$,

(b) $\|\partial \hat A/\partial v\|_{L^p(X,g)} \le C\overline{\lambda}^{2/p-1/2}$.

Proof. Since $\hat a = f_0^* \cdots f_I^* a$ on $U \equiv f_0^{-1} \circ \cdots \circ f_I^{-1}(X_I')$, Lemma 3.19 gives

$$\left\| f_0^* \cdots f_I^* \frac{\partial a}{\partial v} \right\|_{L^p(U,g_0)} \le C \left\| \frac{\partial a}{\partial v} \right\|_{L^p(X_I', g_I)},$$

and Proposition 5.31 gives (a). Similarly, (b) follows from (a) and Proposition 3.29.

5.6. Differentials of the gluing maps and final arguments. We summarise the results of the preceding sections and record our bounds for the differentials of the approximate gluing maps $\mathcal{J}$ and $\hat{\mathcal{J}}$. The estimates for $D\hat{\mathcal{J}}$ then give bounds for the diagonal (and so all) components of the L^2 metric $\mathbf{g}$ and completes the proof of Theorem 1.1. Combining these metric bounds with results of Donaldson in [5] then completes the proof of Theorem 1.2. The following two theorems summarise the estimates obtained in §§5.3 to 5.5, the first following from Corollaries 5.22, 5.29, and 5.32 and the second from Theorems 5.23, 5.30, and 5.33.

Theorem 5.34. *Let $\mathcal{J} : \mathcal{T}/\Gamma \to M^*_{X,k}(g)$ be a gluing map and assume that the conditions of Theorem 5.8 hold. Then for sufficiently small $\lambda_0 > 0$ and any $t \in \mathcal{T}$, there exists a constant $C = C(g_0, \mathcal{T})$ such that the following bounds hold:*

(a) $\|D\mathcal{J}(\partial/\partial t_I^\alpha)\|_{L^2(X,g)} \le C$,

(b) $\|D\mathcal{J}(\partial/\partial \rho_I^\beta)\|_{L^2(X,g)} \le C\overline{\lambda}^{1/2}$,

(c) $\|D\mathcal{J}(\partial/\partial x_I^\mu)\|_{L^2(X,g)} \le C(1 + \overline{\lambda}^{1/2+\delta}\lambda_I^{-\delta})$,

(d) $\|D\mathcal{J}(\partial/\partial \lambda_I)\|_{L^2(X,g)} \le C(1 + \overline{\lambda}^{1/2+\delta}\lambda_I^{-1/2-\delta})$.

Theorem 5.35. *Let $\hat{\mathcal{J}} : \mathcal{T}/\Gamma \to M^*_{X_0,k}(g_0)$ be a gluing map and assume that the conditions of Theorem 5.8 hold. Then for any $0 < \delta < 1/2$, sufficiently small $\lambda_0 > 0$ and any $t \in \mathcal{T}$, there exists a constant $C = C(\delta, g_0, \mathcal{T})$ such that the following bounds hold:*

(a) $\|D\hat{\mathcal{J}}(\partial/\partial t_I^\alpha)\|_{L^2(X_0,g_0)} \le C$,

(b) $\|D\hat{\mathcal{J}}(\partial/\partial \rho_I^\beta)\|_{L^2(X_0,g_0)} \le C\overline{\lambda}^{1/2}$,

(c) $\|D\hat{\mathcal{J}}(\partial/\partial x_I^\mu)\|_{L^2(X_0,g_0)} \le C(1 + \overline{\lambda}^{1/2+\delta}\lambda_I^{-\delta})$,

(d) $\|D\hat{\mathcal{J}}(\partial/\partial \lambda_I)\|_{L^2(X_0,g_0)} \le C(1 + \overline{\lambda}^{1/2+\delta}\lambda_I^{-1/2-\delta})$.

It remains to reinterpret the bounds of Theorem 5.35 in terms of the corresponding bounds for the diagonal components of the L^2 metric $\mathbf{g}$.

Corollary 5.36. *Under the hypotheses of* Theorem 5.35, *the following bounds hold:*

(a) $\mathbf{g}(\partial/\partial t_I^\alpha, \partial/\partial t_I^\alpha) \le C$,

(b) $\mathbf{g}(\partial/\partial \rho_I^\beta, \partial/\partial \rho_I^\beta) \le C\overline{\lambda}$,

(c) $\mathbf{g}(\partial/\partial x_I^\mu, \partial/\partial x_I^\mu) \le C(1 + \overline{\lambda}^{1+2\delta}\lambda_I^{-2\delta})$,

(d) $\mathbf{g}(\partial/\partial \lambda_I, \partial/\partial \lambda_I) \le C(1 + \overline{\lambda}^{1+2\delta}\lambda_I^{-1-2\delta})$.

Recall that the $\mathbf{g}$-length of a path $(s_0, s_1) \ni s \to A(s) \in M^*_{X_0,k}(g_0)$ is computed by

$$\int_{s_0}^{s_1} \sqrt{\mathbf{g}}\left(\frac{\partial A}{\partial s}, \frac{\partial A}{\partial s}\right) ds \leq \int_{s_0}^{s_1} \left\|\frac{\partial A}{\partial s}\right\|_{L^2(X_0,g_0)} ds.$$

The proofs of our main results are now essentially complete.

Proof of Theorem 1.1. Since $0 < \delta < 1/2$, the bounds of Theorem imply that the gluing neighbourhoods $\mathcal{V} = \hat{\mathcal{J}}(\mathcal{T}^0/\Gamma)$ have finite $\mathbf{g}$-volume and $\mathbf{g}$-diameter. Therefore, the bubbling ends of $M^*_{X_0,k}(g_0)$ have finite $\mathbf{g}$-volume and $\mathbf{g}$-diameter since the entire moduli space is covered by finitely many such neighbourhoods. Away from the Uhlenbeck boundary, gluing neighbourhoods consist simply of C^∞ Kuranishi charts. The conical ends corresponding to Kuranishi charts around the reducible connections have finite $\mathbf{g}$-volume and $\mathbf{g}$-diameter by Theorem 1 [14].

Next we consider the relationship between the metric completion and the Uhlenbeck compactification of the anti-self-dual moduli space. Let d_2 be the distance function on $M^*_{X_0,k}(g_0)$ defined by the L^2 metric $\mathbf{g}$. Thus, if $[A]$, $[B]$ are two points in $M^*_{X_0,k}(g_0)$, then $d_2([A],[B])$ is the infimum over all $\mathbf{g}$-lengths of paths in $M^*_{X_0,k}(g_0)$ joining $[A]$, $[B]$. If the two points lie in different path components of the moduli space, then set $d_2([A],[B]) = \infty$. Since $b^+(X_0) = 0$, the moduli space has at most finitely many path components; we say that $M^*_{X_0,k}(g_0)$ has *finite* $\mathbf{g}$-*diameter* if the sum of the $\mathbf{g}$-diameters of the connected components is finite. In [5], Donaldson constructs two other distance functions, D_2 and D_2^ε, for any fixed $\varepsilon > 0$. First, given points $[A]$, $[B]$ in $\mathcal{B}^*_{X_0,k}$, set

$$D_2([A],[B]) \equiv \inf_{u \in \mathcal{G}} \|A - u^*B\|_{L^2(X_0,g_0)}.$$

Lemma 2 [5] (or Lemma 4.2.4 [7]) shows that D_2 is a well-defined distance function on $\mathcal{B}^*_{X_0,k}$. Moreover, Lemma 1 [5] implies that $D_2([A],[B])$ is equal to the distance function defined in the usual way by the L^2 metric on $\mathcal{B}^*_{X_0,k}$ as the infimum over $\mathbf{g}$-lengths of paths in $\mathcal{B}^*_{X_0,k}$ joining $[A]$ and $[B]$. One then obtains a second distance function on $M^*_{X_0,k}(g_0)$ by restriction. Define an ε-neighbourhood of $M^*_{X_0,k}(g_0)$ in $\mathcal{B}^*_{X_0,k}$ by

$$\mathcal{B}^{*,\varepsilon}_{X_0,k} \equiv \{[A] \in \mathcal{B}^*_{X_0,k} : \|F_A^{+,g_0}\|_{L^2(X_0,g_0)} < \varepsilon\}.$$

Then $D_2^\varepsilon([A],[B])$ is defined as infimum of the $\mathbf{g}$-lengths of paths in $\mathcal{B}^{*,\varepsilon}_{X_0,k}$ joining two points $[A]$ and $[B]$ in $\mathcal{B}^{*,\varepsilon}_{X_0,k}$. One now obtains a third distance function on $M^*_{X_0,k}(g_0)$ by restriction. The three distance functions d_2, D_2, and D_2^ε on $M^*_{X_0,k}(g_0)$ are related by

$$D_2([A],[B]) \leq D_2^\varepsilon([A],[B]) \leq d_2([A],[B]), \tag{5.31}$$

for all $[A],[B] \in M^*_{X_0,k}(g_0)$. To show that the d_2-completion of $M^*_{X_0,k}(g_0)$ is homeomorphic to the Uhlenbeck compactification $\overline{M}^u_{X_0,k}(g_0)$, it is enough to prove that a sequence $[A^\alpha]$ in $M^*_{X_0,k}(g_0)$ is d_2-Cauchy if and only if it is convergent in the Uhlenbeck topology. For the metric D_2^ε, one has

Theorem 5.37. [5 (Theorem 4)] *For any $\varepsilon > 0$, the D_2^ε-completion of $M^*_{X_0,k}(g_0)$ is homeomorphic to $\overline{M}^u_{X_0,k}(g_0)$.*

Thus Donaldson's result gives part of the proof of Theorem 1.2: Suppose a sequence $[A^\alpha]$ in $M^*_{X_0,k}(g_0)$ is d_2-Cauchy. According to Eq. (5.31), it must also be D_2^ε-Cauchy and so is convergent in the Uhlenbeck topology by Theorem 5.37 or simply by Proposition 6 [5]. The proof of the reverse direction, namely that a sequence $[A^\alpha]$ which is convergent in the Uhlenbeck topology is also d_2-Cauchy, is included in [9].

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