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A MATHEMATICAL THEORY OF QUANTUM COHOMOLOGY

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1. Introduction

Topological σ models, proposed by Witten [25], have become increasingly important in string theory and many of its important applications like quantum cohomology and mirror symmetry. But Witten proposed it based on physical intuition. Until recently, its rigorous mathematical foundation remained to be established. The first step was taken by the first author in [19], where he established the mathematical definition of topological σ model invariant, *k-point correlation function*, for rational curves. One of the main features in [19] is, predicted by Witten, the use of symplectic topology, in particular, of pseudo-holomorphic curves. As Witten pointed out [25], the topological σ model is a $1+1$ topological field theory. A key topological field theory axiom is the composition law. In this paper, we will first define a *mixed invariant* for arbitrary genus, which combines the topological σ-model invariant with the Gromov invariant. Such a mixed invariant is natural in considering the composition law of the Gromov invariant. The main part of this paper is to give a mathematical proof of the composition law of our mixed invariant, which includes the topological σ model invariant. There are many applications of this composition law. The obvious one is to compute any k-point correlation function in terms of 3-point functions. In this paper, we will give three other important applications. The first application is a mathematical proof of the existence of quantum ring structures on cohomology groups of semi-positive symplectic manifolds. The existence of quantum ring structures was first sug-

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gested by the physicist C. Vafa in a different way. Our approach here follows a suggestion of Witten [25]. We will also compute the quantum ring structures for some Kähler manifolds using the composition law we proved. In [23], one can find more examples of applying the theory here to compute the quantum ring structure. The second application is to the mirror symmetry conjecture for algebraic manifolds. The last application is to compute the enumerative geometric invariants such as the degree of the moduli space of rational curves in CP^n of fixed degree. This is a classical and difficult problem in enumerative algebraic geometry. We compute them in terms of recursion formulas. In fact, our method also yields recursion formulas for the Gromov invariants of more general Fano manifolds of Picard number 1, for examples, hypersurfaces or complete intersections. But it seems to be difficult to determine if the Gromov invariants are enumerative invariants. We believe that this is the case if the degree of rational curves is sufficiently large. The Gromov invariants for rational curves are indeed enumerative invariants in case of complex projective spaces, complex Grassmannian manifolds (cf. Lemma 10.1), Del-Pezzo surfaces.

Let us first sketch how to define the mixed invariant using pseudo-holomorphic curves. Its definition is analogous to the definition of the Donaldson polynomial invariants. We refer the readers to section 2 for details. Let (Σ, j) be a Riemann surface with a fixed complex structure j. Let (V, ω) be a semi-positive symplectic manifold, and $A \in H_2(V, Z)$ with $C_1(V)(A) \geq 0$. Choose a generic almost complex structure J on V, tamed by ω. Let ν be an inhomogeneous term defined to be an anti-J-linear section of $\mathrm{Hom}(\pi_1^* T\Sigma, \pi_2^* TV)$ on $\Sigma \times V$, where π_i is the projection from $\Sigma \times V$ to its i-th factor. A (J, ν)-perturbed holomorphic map, or simply, a (J, ν)-map, is a smooth map $f : \Sigma \to V$ satisfying $(\bar{\partial}_J f)(x) = \nu(x, f(x))$. The last equation is an inhomogeneous Cauchy-Riemann equation. The mixed invariant is defined as follows:

Fix a set of marked points $(x_1, \cdots, x_k) \in \Sigma$ $(k + 2g \geq 3)$, where g is the genus of Σ. Let $\alpha_1, \cdots, \alpha_k$, $\beta_1, \cdots, \beta_l$ be integral homology classes in $H_*(V, Z)$ satisfying:

$$\sum_{i=1}^{k}(2n - \deg \alpha_i) + \sum_{j=1}^{l}(2n - 2 - \deg \beta_j) = 2C_1(V)(A) + 2n(1 - g),$$

where g is the genus of Σ. Every integral homology class can be represented by a so called pseudo-manifold. A pseudo-manifold is a singular space P together with a map $F : P \to V$ such that the singularity of P is of codimension 2. Every two such pseudo-manifolds representing the same homology class are the boundary of a pseudo-manifold cobordism in the usual sense. For simplicity, we shall also use A_i, B_j to denote the pseudo-manifolds representing those homology classes α_i, β_j. Then we can choose a generic almost complex structure J and a generic inhomogeneous term ν such that there are only finitely many (J,ν)-perturbed holomorphic maps $f : \Sigma \to V$ satisfying: $f(x_i) \in A_i$ $(1 \leq i \leq k)$, $f(\Sigma) \cap B_j \neq \emptyset$ $(1 \leq j \leq l)$, and $f_*[\Sigma] = A$. For each such f, the set $\{(y_1, \ldots, y_l); f(y_j) \in B_j\}$ is also finite. We define the multiplicity $m(f)$ to be the algebraic sum of the elements of this set with appropriate sign according to its orientation. Then, we define the mixed invariant

$$\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \,|\, \beta_1, \cdots, \beta_l) = \sum m(f).$$

One can prove that this number $\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \,|\, \beta_1, \cdots, \beta_l)$ is independent of the choices of J, ν, marked points $x_1, \cdots, x_k$ in Σ, pseudo-manifolds representing α_i, β_j, and the complex structure on Σ. Furthermore, the number depends only on the semi-positive deformation class of ω. Therefore we obtain a mixed invariant $\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \,|\, \beta_1, \cdots, \beta_l)$, where g is the genus of Σ. This invariant is nothing else but Witten's topological σ-model invariant or k-point correlation function in case $l = 0$. It is also clear that $\Phi_{(A,\omega,0)}(\alpha_1, \alpha_2, \alpha_3 \,|\, \beta_1, \cdots, \beta_l)$ is just the Gromov invariant $\Phi_{(A,\omega,0)}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \cdots, \beta_l)$. For convenience, we would like to extend $\Phi_{(A,\omega,g)}(\alpha_1, \ldots, \alpha_k|\beta_1, \cdots, \beta_l)$ for any α_i, β_j regardless of their degree. We just simply define it to be zero unless

$$\sum_{i=1}^{k}(2n - \deg \alpha_i) + \sum_{j=1}^{l}(2n - 2 - \deg \beta_j) = 2C_1(V)(A) + 2n(1-g).$$

The invariant $\Phi_{(A,\omega,0)}(\alpha_1, \cdots, \alpha_k|\beta_1, \cdots, \beta_l)$ in the case of rational curves was already defined by the first author in [19] for $k = 0$ or $l = 0$. In fact, he only defined the invariant over rational numbers since he assumed that α_i and β_j can be represented by the bordism classes. It is well-known that not every integral homology class can be represented by a bordism class. It was Gang Liu who pointed out to the first author that a pseudo-manifold (5.1) can be used in place of a bordism class in the definition of the invariant. The use of pseudo-manifolds does not cause any extra difficulties. The construction of $\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k|\beta_1, \cdots, \beta_l)$ for higher genus can be processed in a similar fashion. However, for higher genus, one possible difficulty is that a sequence of J-holomorphic curves could degenerate to a lower genus curve. For example, a sequence of elliptic curves can degenerate to a cusp elliptic curve, whose normalization is a holomorphic sphere. Contrary to intuition, the dimension of moduli space will increase when the genus decreases. This is thought to be a major difficulty to compactify the moduli space of pseudo-holomorphic curves of higher genus. A key observation in our work is that the phenomenon we just describe will NOT happen for perturbed J-holomorphic maps for a generic inhomogenous term. In principle, the freedom of choosing inhomogenous terms allows us to show that perturbed J-holomorphic curves only degenerate to stable curves in the sense of Deligne-Mumford. Hence, we have a good control over "bad degenerations" like cusp elliptic curves. Therefore we can use perturbed J-holomorphic maps, instead of J-holomorphic maps, in defining our mixed invariants for higher genus in much the same way as for genus 0. Some applications of special cases of the mixed invariant have been considered by [5], [15], [20], [21].

Let $\mathcal{M}_{g,k}$ be the moduli space of k-point stable smooth curves $(\Sigma_g, x_1, \cdots, x_k)$, i.e., Deligne-Mumford stable curves, where Σ_g is a smooth Riemann surface of genus g. Then $\Phi_{(A,\omega,g)}$ can be considered as a constant function on $\mathcal{M}_{g,k}$. Now we let k-point stable smooth curves degenerate to a singular k-point stable curve C in $\overline{\mathcal{M}}_{g,k}$. The composition law associated to this degeneration is a formula which computes $\Phi_{(A,\omega,g)}$ in terms of the mixed invariants on the components of C. The general formulation of the composition law is rather complicated. We will leave it to section 7. Let us first discuss two special cases, which play an important role in the general theory and applications. Write the Deligne-Mumford stable curve C as $(\Sigma, x_1, \cdots, x_k)$. In the first case, Σ has two components Σ_1, Σ_2 satisfying: (1) Σ_1 and Σ_2 have genus g_1, g_2 $(g_1 + g_2 = g)$ and intersect at a double point P; (2) Σ_1 carries m marked points $x_1, \cdots, x_m$ $(k+2g_2-2 \geq m \geq 2-2g_1)$ and Σ_2 carries the rest of the marked points. Then, the composition law states that $\Phi_{(A,\omega,g)}$ can be calculated by the invariants corresponding to Σ_i and contributions from the double point P. More precisely, we can do as follows: Let $\{H_\sigma\}$ be a basis for the torsion free part of $H_*(V, Z)$, and Δ be the diagonal of $V \times V$. By the Künneth formula, we can write

$$[\Delta] = \sum_{\sigma,\tau} \eta^{\sigma\tau} H_\sigma \times H_\tau,$$

where $\{\eta^{\sigma\tau}\}$ is the intersection matrix of the basis $\{H_\sigma\}$. Thus the composition law for this case is

$$\begin{aligned}
&\Phi_{(A,\omega,g)}\,(\alpha_1,\cdots,\alpha_k \,|\, \beta_1,\cdots,\beta_l)\\
(1.1)\qquad &= \sum_{A=B_1+B_2}\sum_{j=0}^{l}\sum_{\sigma}\sum_{\gamma,\delta}\frac{\epsilon(\sigma)}{j!(l-j)!}\eta^{\gamma\delta}\\
&\cdot\Phi_{(B_1,\omega,g_1)}(\alpha_1,\cdots,\alpha_m,H_\gamma \,|\, \beta_{\sigma(1)},\cdots,\beta_{\sigma(j)})\\
&\cdot\Phi_{(B_2,\omega,g_2)}(\alpha_{m+1},\cdots,\alpha_k,H_\delta \,|\, \beta_{\sigma(j+1)},\cdots,\beta_{\sigma(l)}),
\end{aligned}$$

where σ runs over all permutations of $1,\cdots,l$, and $\epsilon(\sigma)$ is the sign of the permutation induced by σ on odd dimensional β_j's.

The second case is that Σ is a genus $g-1$ curve with a node. Then the composition law for this case is

$$\begin{aligned}
&\Phi_{(A,\omega,g)}\,(\alpha_1,\cdots,\alpha_k \,|\, \beta_1,\cdots,\beta_l)\\
(1.2)\qquad &= \sum_{\gamma,\tau}\eta^{\gamma\tau}\Phi_{(A,\omega,g-1)}(\alpha_1,\cdots,\alpha_k,H_\gamma,H_\tau \,|\, \beta_1,\cdots,\beta_l).
\end{aligned}$$

In the general case, a Deligne-Mumford stable curve C may have many components with complicated intersection pattern, but all intersection points are ordinary double points. Thus the general composition law can be derived from the above formula by induction. We refer the readers to section 7 for more details. Our main theorem is

Theorem A (Theorem 7.2). *The composition law of the mixed invariants holds for any semi-positive symplectic manifold (V,ω).*

Corollary 1.1. *The composition law of the topological σ-model invariants holds for any semi-positive symplectic manifold (V,ω).*

The first application is to establish a quantum ring structure on the cohomology of a semi-positive symplectic manifold. The k-point function $\tilde{\Phi}_{A,\omega}(\alpha_1,\cdots,\alpha_k)$ was defined in [19] and coincides with the mixed invariant $\Phi_{(A,\omega,0)}(\alpha_1,\cdots,\alpha_k|\ )$; it also depends on a homology class A. We can drop this condition by summing the contributions over all possible A. So, we can formally write the k-point function as

$$(1.3)\qquad \tilde{\Phi}_\omega(\alpha_1,\cdots,\alpha_k)(t) = \sum_A \tilde{\Phi}_{A,\omega}(\alpha_1,\cdots,\alpha_k)\, e^{-t\omega(A)}.$$

We define

$$f_{\alpha\beta\gamma} = \tilde{\Phi}_\omega(H_\alpha, H_\beta, H_\gamma)(t) \tag{1.4}$$

and

$$f^\delta_{\alpha\beta} = \eta^{\gamma\delta} f_{\alpha\beta\gamma}. \tag{1.5}$$

Let $\{H^*_\sigma\}$ be the basis of $H^*(V, Z)$ dual to $\{H_\sigma\}$. Then, we define the quantum multiplication

$$H^*_\alpha \times_Q H^*_\beta = \sum_\gamma f^\gamma_{\alpha\beta} H^*_\gamma. \tag{1.6}$$

The associativity is not obvious and is equivalent to the following identity:

$$\begin{aligned}\sum_{\sigma,\tau} \eta^{\sigma\tau} \tilde{\Phi}_\omega(H_\alpha, H_\beta, H_\sigma)\, \tilde{\Phi}_\omega(H_\tau, H_\gamma, H_\delta) \\ = \sum_{\sigma,\tau} \eta^{\sigma\tau} \tilde{\Phi}_\omega(H_\alpha, H_\delta, H_\sigma) \tilde{\Phi}_\omega(H_\tau, H_\tau, H_\delta).\end{aligned} \tag{1.7}$$

This is a consequence of the composition law for 4-point functions, where two different stable degenerations give the two sides of (1.6). Therefore, we have

Theorem B (Theorem 8.1). *The quantum multiplication is associative; consequently, there is a quantum ring structure on the cohomology of a semi-positive symplectic manifold V.*

From Theorem A and Theorem B it follows that

$$H^*_{\alpha_1} \times_Q \cdots \times_Q H^*_{\alpha_k} = \sum_{\gamma,\delta} \eta^{\gamma\delta} \tilde{\Phi}_\omega(H_{\alpha_1}, \cdots, H_{\alpha_k}, H_\gamma) H^*_\delta. \tag{1.8}$$

There is a convergence problem with the series in (1.3). There may be infinitely many homology classes which contribute to the summation in (1.3), such as, in the case of Calabi-Yau 3-folds or CP^2 blown up at 9-points. But for a symplectic manifold with positive first Chern class, the summation in (1.3) is always finite (cf. section 3). However, for a general semi-positive symplectic manifold, the Novikov ring can be used to be the coefficient ring of the quantum cohomology to get around the problem of convergence (cf. section 8). The quantum multiplication on the cohomology with the Novikov coefficient ring is also associative (Theorem 8.4). The idea of using the Novikov ring was first used by Hofer and Salamon in the context of Floer homology [8].

The mirror symmetry conjecture relates the quantum cohomology with the variation of Hodge structures of its mirror (may or may not exist). A crucial step to prove the mirror symmetry conjecture is to construct a family of flat connections on $H_*(V, C)$, which deform the trivial connection. Those flat connections should be different from the Gauss-Manin connections, which come from the variation of Hodge structures. Using the composition law of mixed invariants (not just σ model invariants), we can construct such a family of flat connections.

Let $W = H^*(V, Z) \otimes C$. For simplicity, assume that $H^*(V, Z)$ is torsion-free. Then any w in W is of the form

$$w = \sum_{j=1}^{L} t_j H_j^*,$$

where L is the dimension of W. Write $w_* = \sum_{j=1}^{L} t_j H_j$ as the point in $H_*(V, C)$ corresponding to w. We extend the mixed invariant $\Phi_{(A,\omega,0)}$ to $H^*(V, C)$ by linearity. Following E. Witten [25], we define a generating function

$$\begin{aligned}(1.9) \qquad \Psi_\omega(w) = \sum_{A \in H_2(V,Z)} e^{-\omega(A)} \sum_{\substack{m=3 \\ w_*=w_{1*}=\cdots=w_{m-3*}}}^{\infty} \frac{1}{m!} \\ \cdot \Phi_{(A,\omega,0)}(w_*, w_*, w_* | w_{1*}, \cdots, w_{m-3*}).\end{aligned}$$

This function is a power series in $t_1, \cdots, t_L$. From Theorem A in the case $k = 4$ it follows that Ψ_ω satisfies the WDVV equation:

$$(1.10) \qquad \sum_{\sigma,\tau} \frac{\partial^3 \Psi_\omega}{\partial t_\alpha \partial t_\beta \partial t_\sigma} \eta^{\sigma\tau} \frac{\partial^3 \Psi_\omega}{\partial t_\gamma \partial t_\delta \partial t_\tau} = \sum_{\sigma,\tau} \frac{\partial^3 \Psi_\omega}{\partial t_\alpha \partial t_\gamma \partial t_\sigma} \eta^{\sigma\tau} \frac{\partial^3 \Psi_\omega}{\partial t_\beta \partial t_\delta \partial t_\tau}.$$

If we define $A = \{A_{\alpha\beta}^\gamma\}$ by

$$A_{\alpha\beta}^\gamma = \sum_\tau \eta^{\gamma\tau} \frac{\partial^3 \Psi_\omega}{\partial t_\alpha \partial t_\beta \partial t_\tau},$$

then $\nabla_\epsilon = \nabla_0 + \epsilon A$ defines a family of connections on the tangent bundle TW over W, where ∇_0 is the trivial connnection on W. The

WDVV equation is equivalent to the flatness of the connections ∇_ϵ. Therefore, we have

Theorem C (Theorem 9.1). *∇_ϵ is a flat connection and a deformation of the trivial flat connection ∇_0.*

Finally, we give an application of this theorem to a classical problem in enumerative algebraic geometry. Let $\sigma_{n,d}(j_1, j_2, \cdots, j_s)$ be the number of rational curves of degree d in CP^n intersecting linear subspaces of codimension $j_1, \ldots, j_s$ where $j_i \geq 2$ and $\Sigma(j_i - 1) = (n+1)d + n$. It is a difficult problem in enumerative algebraic geometry to calculate $\sigma_{n,d}(j_1, j_2, \cdots, j_s)$. One of these $\sigma_{n,d}(j_1, j_2, \cdots, j_s)$ has the following interpretation. Given any degree d algebraic curve C in CP^n, its Chow coordinate X_C is a hypersurface in the Grassmannian manifold $G(n-1, n+1)$ and consists of all $(n-2)$-subspaces in CP^n, which have nonempty intersection with C. This Chow coordinate X_C is, unique up to multiplication by constants, defined by a section in $H^0(G(n-1, n+1), \mathcal{O}(d))$, where $\mathcal{O}(1)$ is the positive line bundle generating the Picard group of $G(n-1, n+1)$. Let $N(n,d)+1$ be the dimension of $H^0(G(n-1, n+1), \mathcal{O}(d))$. Then there is a subvariety in $CP^{N(n,d)}$ consisting of Chow coordinates of rational (possibly singular) curves in CP^n. We denote by n_d the degree of this subvariety. Then we have

$$n_d = \sigma_{n,d}(n-2, n-2, \cdots, n-2).$$

Using the symmetry, we may arrange $j_1 \geq j_2 \geq \cdots \geq j_s$. For convenience, we put $\sigma_{n,d}(j_1, j_2, \cdots, j_s) = 0$ if $j_1 > n$ and $\sigma_{n,d}(\cdots, j_{s-1}, 1) = d\sigma_{n,d}(\cdots, j_{s-1})$. Then from Theorem C follows

Theorem D (Theorem 10.4). *The following recursion formula holds*

$$\begin{aligned}\sigma_{n,d}(j_1, j_2, j_3, j_4, \cdots, j_k) = {} & \sigma_{n,d}(j_1, j_2+1, j_3-1, j_4, \cdots, j_k) \\ & + d\,\sigma_{n,d}(j_1+j_3-1, j_2, j_4, \ldots, j_k) \\ & - d\,\sigma_{n,d}(j_1+j_2, j_3-1, j_4, \cdots, j_k), \\ & \bmod (\sigma_{n,1}, \cdots, \sigma_{n,d-1}).\end{aligned}$$

The explicit expression of the lower order terms will be given in section 10.

Corollary 1.2. *Let n_d be the degree of the subvariety of degree d rational curves in the space of all degree d homogeous polynomials over*

CP^2. Then

$$n_d = \frac{1}{2} \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \frac{d_1 d_2(3dd_1d_2 - 2d^2 + 6d_1d_2)(3d-4)!}{(3d_1-1)!(3d_2-1)!} n_{d_1} n_{d_2}.$$

In particular, $n_1 = 1$, $n_2 = 1$, $n_3 = 12$.

This recursion formula of computing n_d for CP^2 was first derived by Kontsevich, based on the composition law predicted by physicists. Its generalization to CP^n in Theorem D was also derived by M. Kontsevich and Y. Manin [10] from some axioms, which were first suggested by physicists and formulated by them. Here we give a different and mathematically rigorous proof. We also generalize our method to Fano manifolds with Picard number one like hypersurfaces and complete intersections. In particular, we give a recursion formula of our invariants for rational curves on those manifolds. We conjecture that our invariants on any Fano manifold are indeed the enumerative geometric invariants in case the degree of rational curves is sufficiently large. A precise formulation of this conjecture will be given in section 10. However, it is not hard to show that our mixed invariants are enumerative on any Del-Pezzo surface.

Let us briefly describe our method of proving the composition law. Let Σ_i be a sequence of genus g Riemann surfaces with k-marked points, and $f_i : \Sigma_i \mapsto V$ be a (J,ν)-perturbed holomorphic curve for each i. In the Deligne-Mumford compactification, Σ_i degenerates to a k-point stable curves $\mathcal{C}$. Geometrically, one can obtain $\mathcal{C}$ by collapsing a disjoint union of simple close curves. Collapsing of each simple closed curve gives rise to a double point. To achieve the degeneration which we need to prove the composition law, we take advantage of inhomogeneous Cauchy-Riemann equations and let the inhomogeneous term degenerate along some prescribed circles in the Deligne-Mumford compactification. By taking a subsequence, we may assume that f_i converges to a limit map f whose domain is the stable curve $\mathcal{C}$ with some bubbles. There might be some bubbles at the double points. Such a situation did not exist in previous compactness theorems. On each component of $\mathcal{C}$, f satisfies an inhomogeneous Cauchy-Riemann equation. On each bubble, f satisfies a homogeneous Cauchy-Riemann equation. Then by counting dimensions, one can show that the space of such f with some bubbles will be of smaller dimension. Thus, in order to compute the mixed invariant in terms of the perturbed holomorphic maps from $\mathcal{C}$, we have to prove that any f from $\mathcal{C}$ can be deformed into a (J,ν)-perturbed holomorphic map from Σ_i into V for sufficiently large i. We also need to show that such a deformation is unique for a given inhomogeneous term ν and has the same orientation as that of f. In general, such a deformation is impossible to achieve. However, due to the freedom in choosing inhomogeneous terms, we can prove the existence of deformation with the required properties. This will be done by using the Implicit Function Theorem.

The paper is organized as follows: The definition of our symplectic invariants will be given in the next section. Then we will prove the transversality and compactness theorems on moduli spaces of perturbed holomorphic maps in sections 3 - 6. They are necessary for both defining our invariants and proving their composition law. We will prove the composition law in section 7. The quantum cohomology will be discussed in section 8, and its application to the mirror symmetry conjecture will be discussed in section 9. In last section, we give applications to enumerative algebraic geometry.

The main results of this paper were announced in [22] with the same title. In a forthcoming paper, we will allow the conformal structure to vary in the definition of the mixed invariants and prove the composition law for them. During the preparation of this paper, we were informed that G. Liu and D. McDuff [11] could also prove some results related to the composition law for 4-point functions of rational curves for monotone symplectic manifolds. We also received a preprint of Kontsevich and Manin [10]. In [10], among other things, they also derived the formula in Theorem D for CP^n by assuming the associativity for the mixed invariants. The first author would like to thank Kontsevich to share with him his elegant idea of deriving the recursion formula for rational curves in CP^2. We would also like to thank Dr. Siebert for his many suggestions towards the improvement of this manuscript.

Very recently, we received a preprint from B. Crauder and R. Miranda [4]. In the preprint, they discussed the quantum cohomology ring for a general rational surface; by a "general" surface they mean one in which all linear systems have the expected dimension.

2. Mixed invariants

In this section, we will construct a mixed invariant for any given genus. The construction is based on two propositions, which we will prove in the following sections. Such a mixed invariant generalizes Gromov's and Witten's invariants defined by the first author in [19] for genus zero. The basic idea in constructing the mixed invariant is similar to that in [19]. The same idea was previously used by Donaldson in defining his celebrated polynomial invariants for 4-manifolds. There are two motivations for our generalization. First of all, it is necessary for our applications to mirror symmetry and enumerative geometry. Secondly, it is needed in the composition law. Let us begin with a brief discussion on the topological idea behind our construction.

Consider the evaluation map

$$ev : \mathrm{Map}(\Sigma, V) \times \Sigma \to V.$$

Given any $\alpha \in H^*(V, Z)$, there are two ways to induce a cohomology class on $\mathrm{Map}(\Sigma, V)$: $\mu(\alpha) = ev^*(\alpha)/[\Sigma]$ or $\tilde{\mu}(\alpha) = ev^*(\alpha)/[pt]$, where "/" is the slant product. The first operation μ descends to the quotient of $\mathrm{Map}(\Sigma, V)$ by the automorphism group G of Σ, which gives rise to the Gromov invariant. The second operation $\tilde{\mu}$ gives rise to Witten's topological σ-model invariant. We combine both μ and $\tilde{\mu}$ in the definition of our mixed invariant.

To define the mixed invariant rigorously, we need to introduce inhomogeneous Cauchy-Riemann equations. Let (V, ω) be a symplectic manifold, Σ be a Riemann surface of genus g, and $A \in H_2(V, Z)$ with $C_1(V)(A) \geq 0$. Let J be an almost complex structure on V. There are two relative tangent bundles over $\Sigma \times V$ with respect to π_i $(i = 1, 2)$, where π_i is the projection from $\Sigma \times V$ to its i-th factor. A section ν of $\mathrm{Hom}(\pi_1^* T\Sigma, \pi_2^* TV)$ is said to be anti-J-linear if for any tangent vector v in $T\Sigma$,

$$\nu(j_\Sigma(v)) = -J(\nu(v)), \tag{2.1}$$

where j_Σ is the almost complex structure on Σ. Usually, we call such a ν an inhomogeneous term.

Definition 2.1. Let ν be an inhomogeneous term. A (J, ν)-perturbed holomorphic map, or simply a (J, ν)-map, is a smooth map $f : \Sigma \to V$

satisfying the inhomogeneous Cauchy-Riemann equation

$$(\bar{\partial}_J f)(x) = \nu(x, f(x)), \tag{2.2}$$

where $\bar{\partial}_J$ denotes the differential operator $d + J \cdot d \cdot j_\Sigma$.

We denote by $\mathcal{M}_A(\Sigma, J, \nu)$ the moduli space of (J, ν)-perturbed holomorphic maps from Σ into V, such that $f_*[\Sigma] = A$. By (4.12) for a generic pair (J, ν), the moduli space $\mathcal{M}_A(\Sigma, J, \nu)$ is smooth and admits a canonical orientation induced by the linearization of the Cauchy-Riemann operator at each (J, ν)-map.

Let $\{\alpha_i\}_{1 \le i \le k}$, $\{\beta_j\}_{1 \le j \le l}$ be integral homology classes of V satisfying

$$\sum_1^k (2n - \deg(\alpha_i)) + \sum_1^l (2n - \deg(\beta_j) - 2) = 2C_1(V)(A) + 2n(1-g). \tag{2.3}$$

Note that by the Index Theorem, the real dimension of $M_A(\Sigma, J, \nu)$ is $2C_1(V)(A) + 2n(1-g)$. We denote by α_i^*, β_j^* the Poincaré duals of α_i, β_j. Intuitively, the mixed invariant should be defined to be

$$\begin{aligned} &\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k | \beta_1, \cdots, \beta_l) \\ &\quad = \mu(\alpha_1^*) \cup \cdots \cup \mu(\alpha_k^*) \cup \tilde{\mu}(\beta_1^*) \cup \cdots \cup \tilde{\mu}(\beta_l^*)[\overline{\mathcal{M}}_A(\Sigma, J, \nu)], \end{aligned}$$

where $\overline{\mathcal{M}}_A(\Sigma, J, \nu)$ is a suitable compactification of $\mathcal{M}_A(\Sigma, J, \nu)$ (cf. section 3). To make it rigorous, we have to use the following construction through intersection theory.

Let $x_1, \cdots, x_k$ be a set of distinct points on Σ, which is a Riemann surface of genus g. One can think of $(\Sigma; x_1, \cdots, x_k)$ as a Riemann surface with k marked points. Then we can define the evaluation map

$$\begin{aligned} e_{(\Sigma,X,J,\nu)} : \mathcal{M}_A(\Sigma, J, \nu) \times (\Sigma)^l &\mapsto V^k \times V^l = V^{k+l}, \\ (f; y_1, \cdots, y_l) &\mapsto (f(x_1), \cdots, f(x_k); f(y_1), \cdots, f(y_l)), \end{aligned} \tag{2.4}$$

where $X = \{x_1, \cdots, x_k\}$ is the set of the marked points. Clearly, $e_{(\Sigma,X,J,\nu)}$ is smooth. On the other hand, every integral homology class can be represented by a so called pseudo-manifold. A dimension d pseudo-manifold (Y, F) is a dimension d stratified space Y together with a continuous map $F : Y \to V$ satisfying: each lower stratum is of codimension at least two, and F is smooth on each stratum. Any two such pseudo-manifolds representing the same homology class are the boundary of a pseudo-manifold cobordism in the usual sense. Now we choose pseudo-manifolds (Y_i, F_i) (Z_j, G_j) representing α_i, β_j, where $i = 1, \cdots, k$ and $j = 1, \cdots, l$. We define

$$F = \prod_i^k F_i \times \prod_j^l G_j : \prod_i^k Y_i \times \prod_j^l Z_j \mapsto V^{k+l}. \tag{2.5}$$

We will denote by P the domain pseudo-manifold of the map F. Clearly the pseudo-manifold (P, F) represents the integral homology class $\prod_i \alpha_i \times \prod_j \beta_j$ in $H_*(V^{k+l}, Z)$. From our assumption on the degrees of α_i and β_j it follows that the images of $e_{(\Sigma,X,J,\nu)}$ and F have complimentary dimensions in V^{k+l}. Moreover, we have

Proposition 2.2. *For a generic almost complex structure J and a generic inhomogeneous term ν, we can choose F (5.4) such that the following hold:*

(i) The maps $e_{(\Sigma,X,J,\nu)}$ and F intersect transversally at finitely many points. More precisely, there are only finitely many $(f; y_1, \cdots, y_l)$ in $\mathcal{M}_A(\Sigma, J, \nu) \times (\Sigma)^l$, and p in P such that $e_{(\Sigma,X,J,\nu)}(f; y_1, \cdots, y_l) = F(p)$, and furthermore, at each such intersection point, p is a smooth point of P, and the image of the tangent space T_pP under F is transversal to the image of the tangent space

$$T_{(f;y_1,\cdots,y_l)}\mathcal{M}_A(\Sigma, J, \nu) \times (\Sigma)^l$$

under the evaluation map. In particular, there are only finitely many (J, ν)-perturbed holomorphic maps $f : \Sigma \to V$ satisfying: $f(x_i) \in \mathrm{Im}(F_i)$ $(1 \leq i \leq k)$, $f(\Sigma) \cap \mathrm{Im}(G_j) \neq \emptyset$ $(1 \leq j \leq l)$, and $f_*[\Sigma] = A$.

(ii) There are no sequences $\{f_s\}_{s\geq 1}$ in $\mathcal{M}_A(\Sigma, J, \nu)$ such that as s goes to infinity, $f_s(x_i)$ converges to a point in $F_i(Y_i)$, and $f_s(\Sigma)$ converges to a subset in V which intersects any $G_j(Z_j)$.

The proof of this proposition needs some results from the following sections, so we will postpone its proof until the end of section 5. In fact, it follows easily from a dimension count that for a generic pair (J, ν), the image of $e_{(\Sigma,X,J,\nu)}$ does not intersect any lower strata of $F(P)$. Here by a lower stratum of $F(P)$, we mean the image of a lower stratum of P under F.

Now we can define our mixed invariant as follows: Fix a pair (J, ν) such that $e_{(\Sigma,X,J,\nu)}$ and $\mathcal{M}_A(\Sigma, J, \nu)$ satisfy all properties described in Proposition 2.2. First we associate a multiplicity $m(f)$ to each f in $\mathcal{M}_A(\Sigma, J, \nu)$. We define $m(f)$ to be zero if either $f(x_i)$ is not in $F_i(Y_i)$

for some i, or $f(\Sigma)$ does not intersect with one of $G_j(Z_j)$. If f is as given in Proposition 2.2, there are finitely many $(y_{s1}, \cdots, y_{sl})$ $(1 \leq s \leq m)$ such that $f(y_{sj}) \in G_j(Z_j)$. We put $\epsilon(f, s)$ to be ± 1; the sign is determined by the orientations of $\mathcal{M}_A(\Sigma, J, \nu) \times (\Sigma)^l$ (**See Remark 4.12**), P, V_{k+l} at $(f; y_{s1}, \cdots, y_{sl})$, etc., and the Jacobians of the maps $e_{(\Sigma,X,J,\nu)}$ and F. Define

$$m(f) = \sum_{s=1}^{m} \epsilon(f, s), \tag{2.6}$$

and finally the mixed invariant

$$\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \,|\, \beta_1, \cdots, \beta_l) = \sum m(f). \tag{2.7}$$

For convenience, we simply define

$$\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \,|\, \beta_1, \cdots, \beta_l) \;=\; 0 \tag{2.8}$$

in case $\sum_1^k (2n - \deg(\alpha_i)) + \sum_1^l (2n - \deg(\beta_j) - 2)$ is not the same as $2C_1(V)(A) + 2n(1 - g)$.

The following proposition assures that $\Phi_{(A,\omega,g)}$ is indeed an invariant, although we chose special representatives in its definition.

Proposition 2.3. *$\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \,|\, \beta_1, \cdots, \beta_l)$ is independent of choices of the J, ν, marked points $x_1, \cdots, x_k$ in Σ, pseudo-manifolds $(Y_i, F_i), (Z_j, G_j)$ representing α_i, β_j, and the conformal structure on Σ. Furthermore, the number depends only on the semi-positive deformation class of ω.*

As before, we will postpone the proof of Proposition 2.3 until section 5. First we collect a few properties of our invariant. These properties can be easily proved by using (2.5), and Propositions 2.2, 2.3.

Proposition 2.4. *Assume that $g = 0$. Then the mixed invariant $\Phi_{(A,\omega,0)}(\alpha_1, \cdots, \alpha_k \,|\, \beta_1, \cdots, \beta_l)$ coincides with the Witten invariant $\tilde{\Phi}_{(A,\omega)}(\alpha_1, \cdots, \alpha_k)$* (cf. [19]) *in case $l = 0$, and with the Gromov invariant $\Phi_{(A,\omega)}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \cdots, \beta_l)$* (cf. [19]) *in case $k = 3$ and $C_1(V)(A) > 0$.*

This follows directly from (2.7) and the definitions of the Witten invariant and the Gromov invariant in [19].

Proposition 2.5. *The mixed invariant*

$$\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \,|\, \beta_1, \cdots, \beta_l)$$

is multilinear on α_i *and* β_j. *Furthermore, we have the following:*

(1) *The invariant* $\Phi_{(A,\omega,g)}$ *is identically zero if the "virtual" dimension* $2C_1(V)(A) + 2n(1-g) < 0$.

(2) $\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l)$ *is zero if one of* β_j *is of degree greater than* $2n-2$.

(3) *If* $k + 2g \geq 4$, α_k *is the fundamental class* $[V]$, *then* $\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l)$ *is equal to* $\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_{k-1} \mid \beta_1, \cdots, \beta_l)$.

(4) $\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l)$ *is equal to* $d\,\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_{l-1})$, *if* β_l *is of degree* $2n-2$ *and* $d = A \cap \beta_l$ *is the intersection number.*

(5) *In case* $A = 0$, $\Phi_{(A,\omega,0)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l)$ *is zero if* $l > 0$ *and the intersection number* $\alpha_1 \cap \cdots \cap \alpha_k$ *if* $l = 0$.

Proof. We only prove the linearity in a special case:

$$\begin{aligned} (2.9) \qquad & \Phi_{(A,\omega,g)}(\alpha_1 + \alpha_1', \cdots, \alpha_k|\beta_1, \cdots, \beta_l) \\ &= \Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k|\beta_1, \cdots, \beta_l) + \Phi_{(A,\omega,g)}(\alpha_1', \cdots, \alpha_k|\beta_1, \cdots, \beta_l). \end{aligned}$$

The proof for other cases is identical.

Let (Y_i, F_i), (Z_i, G_i) be as before. Suppose that (Y_1', F_1') is a pseudo-manifold representing α_1'. We may assume that $F_i(Y_i)$, $G_j(Z_j)$, $F_1'(Y_1')$ are in general position. Then,

$$((Y_1 \cup Y_1') \times \prod_{i=2}^{k} Y_i \times \prod_{j=1}^{l} Z_j, (F_1 \cup F_1') \times \prod_{i=2}^{k} F_i \times \prod_{j=1}^{l} G_j)$$

represents the homology class $(\alpha_1 + \alpha_1') \times \prod_{i=2}^{k} \alpha_i \times \prod_{j=1}^{l} \beta_j$ in $H_*(V^{k+l}, Z)$. Let f be any map in $\mathcal{M}_A(\Sigma, J, \nu)$ satisfying: $f(x_1) \in F_1(Y_1) \cup F_1'(Y_1')$, $f(x_i) \in F_i(Y_i)$ for $i \geq 2$, and $f(\Sigma) \cap G_j(Z_j) \neq \emptyset$. Then f contributes to both $\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k|\beta_1, \cdots, \beta_l)$ and $\Phi_{(A,\omega,g)}(\alpha_1', \cdots, \alpha_k|\beta_1, \cdots, \beta_l)$, hence, (2.9) follows.

Next we prove (1)-(5). (1) is trivial, since the moduli space $\mathcal{M}_A(\Sigma, J, \nu)$ is empty for a generic (J, ν) by the standard Transversality Theorem (cf. [16]). For (2), we may assume that β_l is of degree greater than $2n-2$. If the invariant is nonzero, for any generic (J, ν), there is at least one f in $\mathcal{M}_A(\Sigma, J, \nu)$ such that $f(x_i) \in F_i(Y_i)$ $(1 \leq i \leq k)$ and $f(\Sigma) \cap G_j(Z_j)$ is nonempty $(1 \leq j \leq l-1)$, where (Y_i, F_i), (Z_j, G_j) are pseudo-submanifolds given as above. However, since $\dim(Z_l) > 2n-2$, we have

$$\sum_{i=1}^{k}(2n - \dim(Y_i)) + \sum_{j=1}^{l-1}(2n - 2 - \dim Z_j) > \dim \mathcal{M}_A(\Sigma, J, \nu). \tag{2.10}$$

Therefore, by counting dimensions and using the Sard-Smale Transversality Theorem, one can show that such a f can not possibly exist for a generic (J, ν), so (2) is proved. Similarly, one can prove (3) and (4). For (5), we may assume that $\nu = 0$; then all (J, ν)-maps are constant maps and the moduli space $\mathcal{M}_A(\Sigma, J, \nu)$ is naturally identified with V. A map f in $\mathcal{M}_A(\Sigma, J, \nu)$ satisfying: $f(x_i) \in F_i(Y_i)$ and $f(\Sigma) \cap G_j(Z_j)$ is nonempty, is in one-to-one correspondence with an intersection point in $\prod_i F_i(Y_i) \cap \prod_j G_j(Z_j)$. Hence (5) follows.

Proposition 2.6. *Let $(V_1, \omega_1), (V_2, \omega_2)$ be two symplectic manifolds. Let $V = V_1 \times V_2$, and $\omega = \omega_1 \oplus \omega_2$. Then we have*

$$\begin{aligned} &\Phi^V_{(A_1 \otimes A_2, \omega, g)}(\alpha_1 \otimes \alpha_1', \cdots, \alpha_k \otimes \alpha_k' |) \\ &\qquad = \Phi^{V_1}_{(A_1, \omega_1, g)}(\alpha_1, \cdots, \alpha_k |) \Phi^{V_2}_{(A_2, \omega_2, g)}(\alpha_1', \ldots, \alpha_k' |), \end{aligned} \tag{2.11}$$

where Φ^V, Φ^{V_1} and Φ^{V_2} denote the mixed invariants on V, V_1 and V_2, respectively.

This is a straightforward corollary of the definition of the mixed invariants.

Proposition 2.7. *The invariant $\Phi_{(A,\omega,g)}$ is symmetric in the following sense:*

$$\begin{aligned} &\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_j, \beta_{j+1}, \cdots, \beta_l) \\ &\quad = (-1)^{\deg(\alpha_i)\deg(\alpha_{i+1})} \Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_{i+1}, \alpha_i, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l) \\ &\quad = (-1)^{\deg(\beta_j)\deg(\beta_{j+1})} \Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_{j+1}, \beta_j, \cdots, \beta_l), \end{aligned}$$

where $1 \leq i \leq k - 1$ and $1 \leq j \leq l - 1$.

This follows directly from the definition of our mixed invariants.

Remark 2.8 (about relaxing the genericity condition). Usually, it is difficult to check if a particular (J, ν) satisfies the properties stated in Proposition 2.2 and required in the definition of our mixed invariants. This amounts to establish certain vanishing theorems, which do not hold in general, for such a pair (J, ν). However, when we calculate the invariants in applications, we often use a particular (J, ν), such as $(J_0, 0)$, on an algebraic manifold with complex structure J_0. Therefore, we would like to relax the genericity condition used in the definition of our invariants.

In fact, we only need to assume that the singular part of $\overline{\mathcal{M}}_A(\Sigma, J, \nu)$ is of real codimension at least two. Then we can perturb pseudo-manifold representatives to achieve all the properties needed in the definition of our mixed invariants. This is very useful for computing the invariants for algebraic manifolds, since the moduli space involved in these cases is usually a variety, possibly with some singularities. Given any sequence $\{f_s\}$ in $\mathcal{M}_A(\Sigma, J, \nu)$, by taking a subsequence, we may assume that f_s converges to a (J, ν)-map f_∞ from Σ and finitely many holomorphic maps from S^2 into V (cf. Proposition 3.1). Putting all these limits together, we obtain the Gromov-Uhlenbeck compactification $\overline{\mathcal{M}}_A(\Sigma, J, \nu)$ of $\mathcal{M}_A(\Sigma, J, \nu)$.

Definition 2.9. A pair (J, ν) is said to be A-good if the set of $f \in \mathcal{M}_A(\Sigma, J, \nu)$ such that $Coker L_f \neq 0$ is of codimension 2, and the set of curves, which are in the image of maps in $\overline{\mathcal{M}}_A(\Sigma, J, \nu) \backslash \mathcal{M}_A(\Sigma, J, \nu)$, has dimension less than $\dim_R(\mathcal{M}_A(\Sigma, J, \nu)) - 2 - r(\Sigma)$, where $r(\Sigma)$ is the dimension of the automorphism group of Σ, and L_f denotes the linearization of the inhomogeneous Cauchy-Riemann equation at f.

If (J, ν) is A-good, we can define the mixed invariant $\Phi_{(A,\omega,g)}$ by using (J, ν)-maps and generic pseudo-manifold representative (Theorem 5.4).

Remark 2.10 (When can we let $\nu = 0$?). There are two places where we need to use the inhomogeneous term ν, namely, multiple covering maps and $A = 0$, $g > 1$. The second situation is quite subtle. For example, consider the genus-one holomorphic curves in CP^2 with zero homology class. Obviously, they are constant maps, so the moduli space has real dimension 4. But it follows from the Index Theorem that the "virtual" dimension is 0. Therefore, we have to use perturbed holomorphic maps in this case. This phenomenon also affects the invariant for genus-one curves of degree one. It is well-known that there are no holomorphic genus-one curves of degree one. But our invariant is not zero in this case, which can be seen from the composition law that we prove in section 7. What happens here is that the component of degree-zero genus-one curves creates a large component of cusp curves in the Gromov-Uhlenbeck compactification of the moduli space of genus-one degree-one curves. Our invariant depends on the compactification of the moduli space instead of the moduli space itself only. This large component of cusp curves will make nontrivial contributions. When we perturb the Cauchy-Riemann equation by adding ν, we will have solutions for the inhomogeneous Cauchy-Riemann equation even though there are no solutions for the homogeneous one.

There is also a problem with the Gromov-Uhlenbeck compactification $\overline{\mathcal{M}}_A(\Sigma, J, 0)$. A sequence of maps f_n in $\mathcal{M}_A(\Sigma, J, 0)$ can converge to a cusp curve with a bubble, such that the homology class of the bubble is A. Therefore, $\overline{\mathcal{M}}_A(\Sigma, J, 0)$ has a component consisting of J-holomorphic maps from S^2 with homology class A. This DOES happen in algebraic geometry. Note that

$$\begin{aligned}\dim \mathcal{M}_A(\Sigma, J, 0) &= 2C_1(V)(A) + 2n(1-g)\\ &< 2C_1(V)(A) + 2n = \dim \mathcal{M}_A(S^2, J, 0).\end{aligned}$$

Hence, it is more difficult to prove that the boundary $\overline{\mathcal{M}}_A(\Sigma, J, 0)\backslash\mathcal{M}_A(\Sigma, J, 0)$ has smaller dimension in the case of higher genus curves.

To overcome this difficulty, we need a finer compactness theorem than the Gromov-Uhlenbeck compactness theorem (Proposition 3.1). In algebraic geometry, there is a notion of arithmetic genus for holomorphic curves (smooth or singular), which takes into account mulplicities of singularities, order of tangency between different components and some algebraic data. The arithmetic genus is upper-semi-continuous for flat deformations of holomorphic curves. We conjecture that there is a notion of arithmetic genus for J-holomorphic curves, which is lower-semi-continuous with respect to Gromov-Uhlenbeck convergence. If this conjecture is true, then we will be able to show that in the situation we described, the bubbles will develop sufficiently many singularities and give a finer compactification. This will enable us to show that the boundary components of the compactification of $\mathcal{M}_A(\Sigma, J, 0)$ have smaller dimension. Using an estimate of the minimal surface theory, we can affirm this conjecture in the case of genus-one curves.

The moduli space $\mathcal{M}_A(\Sigma, J, 0)$ may contain maps of the form $h \cdot \pi$, where h is a map in $\mathcal{M}_B(\Sigma', J, 0)$ and $\pi : \Sigma \mapsto \Sigma'$ is a branched covering map of degree m. Note that $A = mB$. We will see later that such multiple covering maps do not make any contributions to the mixed invariant $\Phi_{(A,\omega,g)}$ under some numerical condition (cf 2.13). A particularly interesting case is when $\mathcal{M}_A(\Sigma, J, 0)$ contains only maps of the form $h \cdot \pi$ as above. Let $\{\alpha_i\}_{1\leq i\leq k}$, $\{\beta_j\}_{1\leq j\leq l}$ be homology classes of V in (2.3). If we further have

$$\begin{aligned}(2.12)\qquad \sum_{i=1}^{k}(2n-2-\deg(\alpha_i)) + \sum_{j=1}^{l}(2n-2-\deg(\beta_j))\\ > 2C_1(V)(B) + 2n(1-g') - r(g')\end{aligned}$$

where g' is the genus of Σ', $r(0) = 6$, $r(1) = 2$ and $r(g') = 0$ for $g' \geq 2$, we may choose pseudo-submanifolds (X_i, F_i), (Y_j, G_j) representing α_i, β_j, such that no holomorphic map in $\mathcal{M}_B(\Sigma', J, 0)$ intersects with all $F_i(X_i)$, $G_j(Y_j)$. For any small inhomogeneous term ν, since all (J,ν)-maps have their image in the vicinity of the image of maps in $\mathcal{M}_B(\Sigma', J, 0)$, there is no (J,ν)-map which intersects with all $F_i(X_i)$, $G_j(Y_j)$. It follows from the definition of the mixed invariant that

$$\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \,|\, \beta_1, \cdots, \beta_l) \;=\; 0.$$

On the other hand, because of (2.3), (2.12) is equivalent to

$$(2.13)\qquad 2(m-1)C_1(V)(B) + r(g') > 2k + 2n(g-g'),$$

and therefore, we have the following vanishing theorem.

Theorem 2.11. *Let $\mathcal{M}_A(\Sigma, J, 0)$, $\mathcal{M}_B(\Sigma', J, 0)$ be given as above. Assume that* (2.13) *holds. Then $\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k|\beta_1, \cdots, \beta_l)$ vanishes for any homology classes α_i, β_j.*

This theorem is often very useful in computing our invariants.

3. A compactness theorem

In this section, we prove a compactness theorem, which is needed in both defining the mixed invariant and proving the recursion formula; its proof is based on certain estimates of Uhlenbeck and Sacks, Schoen on harmonic maps, and a result in [17]. There have been various compactness theorems, which are all based on Gromov's original idea, for J-holomorphic maps (see [16], [17], [26]). In our situation, we allow degeneration of Riemann surfaces, so we have to analyse bubbling from singular points of the domain.

An admissible curve is a connected Riemann surface, possibly singular, with at most nodes as singularities. Recall that a degeneration of admissible curves is a holomorphic fibration $\pi : S \mapsto \Delta \in \mathbb{C}^m$ satisfying:

(1) S is an $(m+1)$-dimensional complex variety with normal crossings; (2) all fibers of π are admissible. A special class of such degenerations consists of all surface fibrations $\pi : S \mapsto \Delta$ in C with smooth generic fibers.

We denote by J_S the complex structure on S, and J an almost complex structure on a compact symplectic manifold (V, ω). Assume that J is ω-tamed. An inhomogeneous term ν over S is simply a homomorphism from the tangent bundle TS of S into TV satisfying: ν is anti-(J_S, J)-linear, i.e., $J \cdot \nu = -\nu \cdot J_S$. It is easy to show that any inhomogeneous term on the central fiber $\pi^{-1}(0)$ extends over S.

Now we fix a degeneration $\pi : S \mapsto \Delta$ in C^m. Let $\{t_i\}$ be a sequence converging to the origin 0 in Δ as i goes to infinity. We denote $\pi^{-1}(t_i)$ by $\{\Sigma_i\}$, and $\nu|_{\Sigma_i}$ by ν_i. Then Σ_i converges to an admissible curve $\Sigma_\infty = \pi^{-1}(0)$, and ν_i converges to a smooth inhomogeneous term $\nu_\infty = \nu|_{\Sigma_\infty}$ in C^4-topology. Namely, there are continuous maps $\tau_i : \Sigma_i \mapsto \Sigma_\infty$ and compact subsets K_i in Σ_∞ satisfying: (1) $\bigcup K_i = \Sigma_\infty \backslash \{\text{double points}\}$; (2) τ_i restricts to a diffeomorphism from $\tau^{-1}(K_i)$ onto K_i; (3) for each j, both $||j_{\Sigma_\infty} \cdot d\tau_i - d\tau_i \cdot j_{\Sigma_i}||_{C^4(K_j)}$ and $||\nu_\infty \cdot d\tau_i - \nu_i||_{C^4(K_j)}$ converge to zero as i goes to infinity.

Consider

$$(3.1) \quad \mathcal{M}_A(\Sigma_i, J, \nu_i) = \{f : \Sigma_i \to V | df + J \cdot df \cdot j_{\Sigma_i} = \nu_i \, , \, f_* \Sigma_i = A\} \, ,$$

where $A \in H_2(V, Z)$ is a fixed homology class, and j_{Σ_i} is the conformal structure on Σ_i.

Let $f_i : \Sigma_i \mapsto V$ be a smooth map for each i. Then we say that f_i converges to $f_\infty : \Sigma_\infty \mapsto V$ if $||f_\infty \cdot \tau_i - f_i||_{C^3,loc}$ converges to zero as i goes to infinity (cf. [17 (p.386)]), where τ_i is given as above. We want to study the limit of $\mathcal{M}_A(\Sigma_i, J, \nu_i)$. In general, $\lim\limits_{i \to \infty} \mathcal{M}_A(\Sigma_i, J, \nu_i)$ may not be contained in $\mathcal{M}_A(\Sigma_\infty, J, \nu_\infty)$.

Proposition 3.1. *Let f_i be in $\mathcal{M}_A(\Sigma_i, J, \nu_i)$. Then there is a connected curve Σ, which is the union of the smooth resolution $\tilde{\Sigma}_\infty$ of Σ_∞ and finitely many smooth rational curves, such that a subsequence of $\{f_i\}$ converges to a $(J, \tilde{\nu})$-perturbed holomorphic map f on Σ, where the inhomogeneous term $\tilde{\nu}$ coincides with ν_∞ on $\tilde{\Sigma}_\infty$ and vanishes on those rational curves. Moreover, we have $f_*(\Sigma) = A$.*

The rest of this section is devoted to proving this proposition. For reader's convenience, we may sketch the proofs of some known lemmas. We recommend [17] for more details on these lemmas.

Without loss of generality, we may assume that Σ_i is smooth. In the general case, we can replace Σ_i by its desingularization and proceed as we are doing in the following.

First we make a reduction. Put $W = S \times V$. We can introduce an almost complex structure J_W on W as follows: any tangent vector on W is of the form (u, v), where u is in TS and v is in TV, define $J_W(u, v) = (J_S(u), J(v) + \nu(J_S(u)))$. It is easy to check that this is an almost complex structure, tamed by some symplectic form ω_W on W of the form $\omega_S + \omega$, where ω_S is some symplectic form on S. Moreover, if we define $F_i : \Sigma_i \mapsto W$ by $F_i(x) = (x, f_i(x))$, where x is in Σ_i, then F_i is J_W-holomorphic. Therefore, it suffices to show that a subsequence of $\{F_i\}$ converges to a holomorphic map $F : \Sigma \mapsto W$ with $F_*(\Sigma) = F_{i*}(\Sigma_i)$, where Σ is given in Proposition 3.1. Thus we reduce the general case to the case that the inhomogeneous term ν is actually zero. From now on, we assume that each f_i is J-holomorphic. We also fix a J-invariant metric h on V.

Lemma 3.1. *There is a uniform constant C_A, depending only on A, such that for $f_i \in \mathcal{M}_A(\Sigma_i, J, 0)$, we have*

$$\int_{\Sigma_i} |df_i|_\mu^2 d\mu \ < \ C_A, \tag{3.2}$$

where μ is any hermitian metric on Σ_i.

Proof: Fix a $\Sigma = \Sigma_i$ and a map $f = f_i$. Then we have

$$\int_\Sigma f^*\omega = \int_{f(\Sigma)} \omega = \int_A \omega.$$

The last integral is a fixed number.

At each point $x \in V$, choose a local unitary basis $e_1, \cdots, e_n$ of $T_x^{1,0}V$ with respect to h. Then

$$\omega = 2Re\left(\omega_{ij}^{(2,0)}\ e_i^* \wedge e_j^*\right) + \omega_{ij}^{(1,1)} e_i^* \wedge \bar{e_j}^*,$$

where $\{e_i^*\}$ is the dual basis of $\{e_i\}$. Let $\{u_1, u_2\}$ be a local orthonormal basis of (Σ, μ), such that $j_\Sigma u_1 = u_2$, $j_\Sigma u_2 = -u_1$, then

$$df(u_1) + J \cdot df(u_2) = 0,$$

i.e.,

$$f_1^i e_i + f_2^i J(e_i) + f_1^{\bar{i}} \bar{e}_i + f_2^i J(\bar{e}_i) = 0,$$

and therefore

$$f_1^i = -\sqrt{-1} f_2^i, \; f_1^{\bar{i}} = \sqrt{-1} f_2^{\bar{i}},$$

We denote by $\{u_1^*, u_2^*\}$ the dual basis of $\{u_1, u_2\}$. Then

$$\begin{aligned}\omega_{ij}^{(1,1)} f^*(e_i^* \wedge \bar{e_j^*}) &= \omega_{ij}^{(1,1)} (f_1^i f_2^{\bar{j}} - f_2^i f_1^{\bar{j}}) u_1^* \wedge u_2^* \\ &= -\sqrt{-1}\omega_{ij}^{(1,1)} (f_1^i f_1^{\bar{j}} + f_2^i f_2^{\bar{j}}) u_1^* \wedge u_2^*.\end{aligned}$$

We also have

$$\omega_{ij}^{(2,0)} f^*(e_i^* \wedge e_j^*) = \omega_{ij}^{(2,0)} (f_1^i f_2^j - f_2^i f_1^j) u_1^* \wedge u_2^* = 0.$$

Since J is ω-tamed, there is a constant $c > 0$ such that

$$\omega_{ij}^{(1,1)} (f_1^i f_1^{\bar{j}} + f_2^i f_2^{\bar{j}}) \geq c |df|_\mu^2,$$

so that the lemma follows.

Lemma 3.2. *There are $\epsilon_0 > 0$ and $C > 0$, such that for any J-holomorphic map $f : \Sigma \mapsto V$, and any metric μ on Σ with curvature bounded by 1, if $\int_{B_{2r}(x)} |df|_\mu^2 d\mu \leq \epsilon_0$ and the injectivity radius at x is not less than $2r$, where $x \in \Sigma$ and $r > 0$, then we have*

$$\sup_{B_r(x)} |df|_\mu^2 \leq \frac{C}{r^2}, \tag{3.3}$$

where $B_r(x)$ is the geodesic ball centered at x and with radius r. Consequently, $||f||_{C^4(B_r(x))} \leq C_4$ for some constant C_4, which may depend on the C^5-norm of μ.

Proof. This is essentially Theorem 2.3 in [17]. For the reader's convenience, we sketch a proof here. By scaling, we may assume that $r = 1$. Let ρ_0 be the maximum of the function $4\rho^2 \sup_{B_{2-2\rho}(x)} |df|_\mu^2$. Choose x_0 in $B_{2-2\rho_0}(x)$ such that

$$|df|_\mu^2(x_0) = \sup_{B_{2-2\rho_0}(x)} |df|_\mu^2 = e_0;$$

then for any y in $B_{\rho_0}(y)$, $|df|_\mu^2(y) \leq 4e_0$. By switching to the metric $\mu' = e_0\mu$, the ball $D = B_{\rho_0}(x_0)$ has radius $R = \rho_0\sqrt{e_0}$. We claim

$$\sup_{B_1(x)} |df|_{\mu'}^2 \leq 16;$$

otherwise, $R \geq 2$. Since $\sup_D |df|^2_{\mu'} \leq 1$, by the standard Bochner-type formula, one can show,

$$\Delta_{\mu'}(|df|^2_{\mu'}) \leq a|df|^2_{\mu'}$$

for some constant a depending only on V. Therefore, by the Mean-Value Inequality (cf. [7 (Theorem 9.20)]), one can deduce

$$1 = |df|^2_{\mu'}(x_0) \leq C(1+a) \int_{B_1(x_0,\mu')} |df|^2_{\mu'} d\mu',$$

where C is a uniform constant. Using the invariance of the Dirichlet integral under conformal transformations, one can easily see that the above integral is less than ϵ_0. Therefore, we derive a contradiction if ϵ_0 is sufficiently small, and the lemma is proved.

Let μ_i be a sequence of metrics on Σ_i satisfying: (1) The curvature $K(\mu_i)$ is bounded by 1; (2) the injectivity readius $\mathrm{InjRad}(\mu_i) \geq 1$; (3) μ_i converges uniformly to a complete metric μ_∞ on the nonsingular part of Σ_∞ in C^6-topology; (4) the limit metric μ_∞ is cylinder-like near the singular points of Σ_∞. The existence of such metrics μ_i is well-known.

Set $r_m = 2^{-m}$. Define

$$E_{m,i} = \{x \in \Sigma_i | x \in \int_{B_{r_m}(x,\mu_i)} |df_i|^2_{\mu_i} d\mu_i \geq \epsilon_0\},$$

where ϵ_0 is given in Lemma 4.2. Clearly, $E_{m,i}$ is contained in $E_{m',i}$ if $m \geq m'$.

Claim. *For i sufficiently large, each $E_{m,i}$ can be covered by balls $B_{8r_m}(x_{1i},\mu_i), \cdots, B_{8r_m}(x_{li},\mu_i)$, where l is independent of i.*

Proof. We fix i. By the standard covering lemma, we can find $x_1^m, \cdots, x_{k_m}^m$ in $E_{m,i}$ such that $E_{m,i}$ is covered by balls $B_{2r_m}(x_1^m,\mu_i), \cdots, B_{2r_m}(x_{k_m}^m,\mu_i)$, and

$$B_{r_m}(x_\alpha^m,\mu_i) \cap B_{r_m}(x_\beta^m,\mu_i) = \emptyset, \quad \text{if } \alpha \neq \beta.$$

This implies

$$C_A \geq \int_{\Sigma_i} |df_i|^2_{\mu_i} d\mu_i \geq \sum_{\alpha=1}^{k_m} \int_{B_{r_m}(x_\alpha^m,\mu_i)} |df_i|^2_{\mu_i} d\mu_i \geq k_m\epsilon_0,$$

i.e., $k_m \leq \frac{C_A}{\epsilon_0}$, where C_A is the constant in Lemma 4.1. Note that $k_m = 0$ for m sufficiently large.

We will select $x_{1i}, \cdots, x_{li}$ from those points x_α^m $(m \geq 1, 1 \leq \alpha \leq k_m)$ as follows: We say that $\{x_{\alpha_p}^p\}_{m_1 \leq p \leq m_2}$ is a string if $x_{\alpha_p}^p$ is contained in $B_{2r_{p-1}}(x_{\alpha_{p-1}}^{p-1}, \mu_i)$ for all $m_1 < p \leq m_2$. Clearly, for any $m_1 \leq p < q \leq m_2$, $B_{2r_p}(x_{\alpha_p}^p, \mu_i)$ is contained in $B_{6r_p}(x_{\alpha_q}^q, \mu_i)$. Such a string is maximal if it is not contained in another string. The last point in a maximal string is called a maxiaml element. We put all maximal elements together and order them as $x_{1i}, \cdots, x_{li}$. By using Lemma 3.1 and the standard covering lemma, one can show that l is uniformly bounded. Hence the claim is proved.

Without loss of generality, we may assume that for each α, the sequence $\{x_{\alpha i}\}$ converges to a point $x_{\alpha\infty}$. Note that $x_{\alpha\infty}$ can be a singular point of Σ_∞.

By Lemma 3.2, there is a constant C'_m, which depends only on m, such that

$$||f_i||_{C^4(\Sigma_i \backslash N_{r_m}(E_{m,i}))} \leq C'_m,$$

where $N_{r_m}(E_{m,i})$ is the tubular neighborhood $\{x \in \Sigma_i | d(x, E_{m,i}) < 4r_m\}$. By the Ascoli Theorem and taking a subsequence if necessary, we deduce that f_i converges to a J-holomorphic map $\tilde{f}_m$ on $\Sigma_\infty \backslash N_{r_m}(E_{m,\infty} \cap \Sigma'_\infty)$ in C^3- topology, where Σ'_∞ is the nonsingular part of Σ_∞. Since $E_{m,\infty}$ is contained in $E_{m',\infty}$ for $m \geq m'$, we can glue $\tilde{f}_m$ together to obtain a map $\tilde{f}$ on $\Sigma_\infty \backslash E_\infty$. Clearly, f_i converges to $\tilde{f}$ on $\Sigma_\infty \backslash E_\infty$, and $\tilde{f}$ is a J-holomorphic map. Moreover, we have

$$\int_{\Sigma_\infty} |d\tilde{f}|^2_{\mu_\infty} d\mu_\infty < \infty.$$

Lemma 3.3. *Let f be any J-holomorphic map from a punctured disk $D - \{0\}$ in a Riemann surface into V, and let μ be a metric on D. If $\int_D |df|^2_\mu d\mu < \infty$, then f extends to a smooth J-holomorphic map on D.*

This is Theorem 3.6 in [17] (also see [16]). It is essentially the Removable Singularity Theorem of Uhlenbeck. We omitted its proof.

Applying this lemma, we can extend $\tilde{f}$ to be a J-holomorphic map on the smooth resolution $\tilde{\Sigma}_\infty$ of Σ_∞. We still denote this extension by $\tilde{f}$.

We need to examine the behavior of f_i near $x_{1i}, \cdots, x_{li}$. For simplicity, we may assume $l = 1$, say $x_i = x_{\alpha i}$ for each i. The general case can be treated identically. There is a point y_{1i} in $B_{\frac{1}{8}}(x_i,\mu_i)$, such that

$$|df_i|^2_{\mu_i}(y_{1i}) = \sup_{B_1(x_i,\mu_i)} |df_i|^2_{\mu_i} = e_i. \tag{3.4}$$

We may assume that e_i diverges to infinity. We change the metric μ_i on $B_1(x_i,\mu_i)$ as follows: Let η be a cut-off function satisfying: $\eta(t) = t$ for $t \leq 1 + e_i$, $\eta(t) = e_i$ for $t \geq 2 + e_i$, and $|\eta'(t)| \leq 1$. Let ρ be the distance function from y_{1i} of μ_i. Define

$$\mu_i' = \frac{e_i\mu_i}{\eta(1+16e_i\rho^2)}. \tag{3.5}$$

Then

$$|df_i|_{\mu_i'}(y_{1i}) = 1,\ |df_i|^2_{\mu_i'} \leq \eta(1+16e_i\rho^2); \tag{3.6}$$

in particular, $|df_i|_{\mu_i'} \leq 9$ for $\sqrt{e_0}\rho \leq 4$. Since f_i is J-holomorphic, by the standard elliptic estimates, one can easily show

$$\int_{B_1(y_{1i},\mu_i')} |df_i|^2_{\mu_i'} d\mu_i' \geq \delta, \tag{3.7}$$

where δ is a positive number independent of i. Note that μ_i' coincides with μ_i, and $|df_i|_{\mu_i}$ is bounded by a uniform constant in $B_1(x_i,\mu_i)\backslash B_{\frac{1}{2}}(x_i,\mu_i)$. If $|df_i|_{\mu_i'}$ is not uniformly bounded on $B_1(x_i,\mu_i)$, then there is a y_{2i} in $B_{\frac{1}{2}}(x_i,\mu_i)\backslash B_4(y_{1i},\mu_i')$, such that

$$|df_i|^2_{\mu_i}(y_{2i}) = \sup_{B_1(x_i,\mu_i)} |df_i|^2_{\mu_i'} = e_i'.$$

Now e_i' diverges to infinity as i increases. We change μ_i' in a small neighborhood of y_{2i} as we did for μ_i in a neighborhood of y_{1i}. For saving notation, we still denote by μ_i' the changed metric. If $|df_i|_{\mu_i'}$ is not uniformly bounded, we repeat the above arguments and obtain y_{3i}, y_{4i}, so on. Suppose that we have found $y_{1i},\cdots,y_{ki}$. Then

$$B_1(y_{\alpha i},\mu_i') \cap B_1(y_{\beta i},\mu_i') = \emptyset, \quad \text{for} \quad \alpha \neq \beta,$$

$$\int_{B_1(y_{\alpha i},\mu_i')} |df_i|^2_{\mu_i'} d\mu_i' \geq \delta.$$

Since the Dirichlet integral is conformally invariant, from these we deduce that $k \leq \frac{C_A}{\delta}$. Therefore, after repeating the above arguments in finitely many steps, we obatin a metric μ_i' such that $|df_i|_{\mu_i'}$ is uniformly bounded on $B_1(x_i, \mu_i)$. Moreover, by taking a subsequence if necessary, we can find $y_{1i}, \cdots, y_{Li}$, where L is independent of i, such that $|df_i|_{\mu_i'}(y_{\alpha i}) = 1$, each $B_{R_i}(y_{\alpha i}, \mu_i')$ is biholomorphic to a ball in C, and

$$B_{R_i}(y_{\alpha i}, \mu_i') \cap B_{R_i}(y_{\beta i}, \mu_i') = \emptyset, \quad \text{for } \alpha \neq \beta,$$

where $\lim R_i = \infty$. It follows that for each α, $f_i|_{B_{R_i}(y_{\alpha i}, \mu_i')}$ converges to a J-holomorphic map $\tilde{f}_\alpha$ from C into V. By Lemma 3.3, such a $\tilde{f}_\alpha$ extends to be a J-holomorphic map from S^2 into V.

Let us examine the limiting behavior of the restriction of f_i to $B_1(x_i, \mu) \backslash \bigcup_{\alpha=1}^{L} B_{R_i}(y_{\alpha i}, \mu_i')$. Define

$$\text{(3.8)} \quad F_i^\epsilon = \{x \in B_1(x_i, \mu) \backslash \bigcup_{\alpha=1}^{L} B_{R_i}(y_{\alpha i}, \mu_i') \mid \int_{B_1(x, \mu_i')} |df_i|^2_{\mu_i'} d\mu_i' \geq \epsilon\},$$

where ϵ is any fixed number less than ϵ_0 in Lemma 3.2. By Lemma 3.1 and our construction of μ_i', one can show that each F_i^ϵ is strictly contained in a disjoint union of pants or annuli $P_{\beta i}^\epsilon$ $(1 \leq \beta \leq N_i)$ satisfying: (1) the diameter $\operatorname{diam}(P_{\beta i}^\epsilon) \leq r$, where $r > 0$ depends only on ϵ; (2) each boundary component of $P_{\beta i}^\epsilon$ is connected to either $\Sigma_i \backslash B_1(x_i, \mu)$ or one of the balls $B_{R_i}(y_{\alpha i}, \mu_i')$ by a connecting cylinder. Each connecting cylinder $C_{\gamma i}^\epsilon$ is of the form $S^1 \times [0, T_{\gamma i}]$, where $1 \leq \gamma \leq M_i$. Note that $T_{\gamma i}^\epsilon$ can be zero. Since both M_i and N_i are bounded independent of i, by taking a subsequence if necessary, we may assume that they are equal to fixed integers M and N, which are independednt of i and ϵ.

Lemma 3.4. (cf. [17]) *There is a uniform constant c such that for each $1 \leq \gamma \leq M$ and $i \geq 1$,*

$$\text{(3.9)} \qquad \int_{C_{\gamma i}^\epsilon} |df_i|^2_{\mu_i'} d\mu_i' \leq c\,\epsilon.$$

Furthermore, if ρ denotes the distance function from the boundary of the cylinder $C_{\gamma i}^\epsilon$, then there is a uniform constant $\lambda < 1$ such that for any $R > 0$,

$$\text{(3.10)} \qquad \int_{\rho \geq R} |df_i|^2_{\mu_i'} d\mu_i' \leq c\,\epsilon\,\lambda^R$$

Proof. We will always denote by c a uniform constant. We fix a pair γ, i. By Lemma 3.2, $|df_i|^2_{\mu_i'} \leq c\epsilon$. It follows that the boundary of

$f_i(C^\epsilon_{\gamma i})$ consists of two circles Γ_1 and Γ_2 satisfying:

$$\mathrm{Length}(\Gamma_1) \le \sqrt{c\epsilon},\ \mathrm{Length}(\Gamma_2) \le \sqrt{c\epsilon}.$$

Since ϵ is small, Γ_1, Γ_2 span two disks D_1, D_2 in two small balls of radius $2\sqrt{c\epsilon}$. Let S be the closed surface in V obtained by gluing D_1, D_2 to $f_i(C_{\gamma i})$ along Γ_1, Γ_2, respectively. Then S is homologeous to zero in V. This can be easily seen as follow: Cut $C^\epsilon_{\gamma i}$ into cylinders with diameter less than one, and glue disks to boundaries of these cylinders as above; thus we obtain a number of spheres of bounded size in V. By the gradient estimate on f_i, each such sphere is contained in a small ball of radius $2\sqrt{c\epsilon}$, so it is homologeous to zero. Now S is homologeous to the sum of those small spheres, so it is homologically zero, too. Since S bounds a 3-dimensional set, by the Stokes' Theorem, we have

$$0 = \int_S \omega = \int_{C_{\gamma i}} f_i^*\omega + \int_{D_1} \omega - \int_{D_2} \omega.$$

On D_1, we can integrate ω to get an 1-form u_1 such that $\omega = du_1$ and $|u_1| \le 4\sqrt{c\epsilon}$ on D_1. Then by the Stokes' Theorem,

$$\int_{D_1} \omega = \int_{\Gamma_1} u_1 \le \mathrm{Length}(\Gamma_1) \sup_{D_1} |u_1| \le 4c\epsilon.$$

Similarly, one can show

$$\int_{D_2} \omega \le 4c\epsilon.$$

Therefore, we have

$$\int_{C_{\gamma i}} f_i^*\omega \le 8c\epsilon$$

Hence (3.9) follows from this and the ω-tameness of J. In [17], (3.10) was derived from (3.9) by an isoperimetric inequality. One can also derive (3.10) directly. Since a similar estimate will be given in the proof of Lemma 6.10, we omit the proof of (3.10) and refer the readers to Lemma 6.10.

Since f_i is J-holomorphic, for each $1 \le \beta \le N$, by taking a subsequence if necessary, the restriction of f_i to $P^\epsilon_{\beta i}$ converges to a J-holomorphic map $f^\epsilon_{\beta\infty}$ as i goes to zero. We can arrange $P^\epsilon_{\beta i} \subset P^{\epsilon'}_{\beta i}$ for $\epsilon > \epsilon'$. Then for each β, we can glue $f^\epsilon_{\beta\infty}$ together to obtain a J-holomorphic map f'_β from a punctured sphere into V. By Lemma 3.3, this f'_β extends to be a J-holomorphic map from S^2 into V. Note that f'_β can be a trivial map. Since $P^\epsilon_{\gamma i}$ is connected to either $\Sigma_i\backslash B_1(x_i,\mu_i)$, or some $B_{R_i}(y_{\alpha i},\mu'_i)$ by some cylinder $C^\epsilon_{\gamma i}$, using Lemmas 3.2, 3.4, one can easily show the image of f'_β must intersect with either $\tilde{f}(\tilde{\Sigma}_\infty)$ or one of $\tilde{f}_\alpha(S^2)$. We define f to be the sum of J-holomorphic maps $\tilde{f}$, $\tilde{f}_\alpha$ and f'_β. Let Σ be the domain of f. In fact, one can take Σ to be an admissible curve.

Finally, we show that $f_*(\Sigma) = A$. Let ϕ be any closed 2-form on V. Then for any sufficiently small ϵ, by Lemma 3.4 we have

$$\int_{\Sigma_i} f_i^*\phi = \left(\int_{\Sigma_i\backslash B_1(x_i,\mu_i)} + \sum_{\alpha=1}^{L}\int_{B_{R_i}(y_{\alpha i},\mu'_i)} + \sum_{\beta=1}^{N}\int_{P^\epsilon_{\beta i}}\right) f_i^*\phi \;+\; \mathcal{O}(\epsilon).$$

The restriction of f_i to $\Sigma_i\backslash B_1(x_i,\mu_i)\bigcup B_{R_i}(y_{\alpha i},\mu'_i)\bigcup P^\epsilon_{\beta i}$ converges to $f|_{U_\epsilon}$, where U_ϵ is an open subset in Σ such that $\bigcup_{\epsilon>0} U_\epsilon$ is equal to Σ with punctures. Therefore, by letting ϵ go to zero, from the above we deduce

$$\int_\Sigma f^*\phi = \int_{\Sigma_i} f_i^*\phi = \int_A \phi,$$

so that $f_*(\Sigma) = A$. Hence Proposition 3.1 is proved.

4. Transversality of compactification

From Proposition 3.1 it follows that the moduli space $\mathcal{M}_A(\Sigma, J, \nu)$ admits a natural compactification $\overline{\mathcal{M}}_A(\Sigma, J, \nu)$, which contains all geometric limits ("cusp" curves) of sequences in $\mathcal{M}_A(\Sigma, J, \nu)$ (see paragraph 5 for details). Such a compactification is called Gromov-Uhlenbeck compactification. In this section, we prove a structure theorem for $\overline{\mathcal{M}}_A(\Sigma, J, \nu)$ (Theorem 4.2). For the purpose of proving the composition law, we also need to consider the case that Σ is only a stable curve (possibly singular).

Recall that an *l-point genus g stable curve* $C = (\Sigma, x_1, \cdots, x_l)$ is a reduced, connected curve Σ, whose singularities are at most ordinary double points, plus k distinct smooth points x_i in Σ, such that every smooth rational component of C contains at least three points which are either x_i's or double points of Σ. Geometrically, one gets an l-point genus g stable curve by adjoining a set of curves with double points, and adding some marked points to each rational component. We will call a double point of C a distinguished point on the component of Σ on which it lies.

Let Σ be a stable curve and have m components $\Sigma_1, \cdots, \Sigma_m$. An inhomogeneous term ν over Σ is a set of inhomogeneous terms $\nu_1, \cdots, \nu_m$ such that each ν_i is an inhomogeneous term on Σ_i (cf. (2.1)). A map $f : \Sigma \to V$ is (J, ν)-perturbed holomorphic if its restriction $f|_{\Sigma_i}$ to each Σ_i is a (J, ν_i)-perturbed holomorphic map. As before, we denote by $\mathcal{M}_A(\Sigma, J, \nu)$ the moduli space of (J, ν)-perturbed holomorphic maps from Σ into V with $f_*(\Sigma) = A$. Note that $\mathcal{M}_A(\Sigma, J, \nu)$ contains many components $\mathcal{M}_{(A_1, \cdots, A_m)}(\Sigma, J, \nu)$ whose elements are (J, ν)-maps $f : \Sigma \mapsto V$ with $f_*(\Sigma_i) = A_i$ for $1 \leq i \leq m$, where the sum of the A_i is A. The goal of this section is to prove a structure theorem for $\overline{\mathcal{M}_A}(\Sigma, J, \nu)$.

Definition 4.1. A Σ-cusp (J, ν)-map f is a continuous map from Σ' to V, which is smooth at smooth points of Σ', where the domain Σ' of f is obtained by joining a chain of S^2's at some double point of Σ to separate the two components at the double point, and then attaching some trees of S^2's. We call components of Σ *principal components* and others *bubble components*. For each principal component, the restriction of f is (J, ν)-perturbed holomorphic. The restriction of f to a bubble component is J-holomorphic, i.e., $\bar{\partial}_J f = 0$. A Σ-cusp curve is an equivalence class of cusp maps modulo the parametrization groups of bubbles PSL_2C. Because we take into account of constant holomorphic maps, we **allow** the image of a principal component under f to have **zero homology class**. But the image of any bubble component under f will always have nonzero homology class. Furthermore, we will always use **intersection points** to denote the intersection points between the components of the domain.

By our definition, a cusp curve is just a collection of curves whose domains interest according to the intersection pattern sepecified by the homeomorphism type of domain of a cusp map. By Proposition 3.1, we can compactify $\mathcal{M}_A(\Sigma, J, \nu)$ by adding Σ-cusp curves with total homology class A. We will divide the set of cusp curves by some equivalence relation and study the structure of the quotient. There are two cases: (i) Some of the bubble components may be multiple covering maps, and it is well-known that transversality theory fails for multiple covering maps [14]. In this case, we will simply forget the multiplicity and take the reduced map onto its image; (ii) Adjacent or consecutive bubbles have the same image, in this case, we will collapse them into one bubble. Clearly, the resulting curves are still cusp curves with possibly different total homology class. Let $\overline{\mathcal{M}}_A(\Sigma, J, \nu)$ be the quotient of the cusp-curve-compactification of $\mathcal{M}_A(\Sigma, J, \nu)$ by this equivalence relation. Then $\overline{\mathcal{M}}_A(\Sigma, J, \nu) - \mathcal{M}_A(\Sigma, J, \nu)$ is just a union of cusp curves with possibly different total homology class. We will prove the following structure theorem.

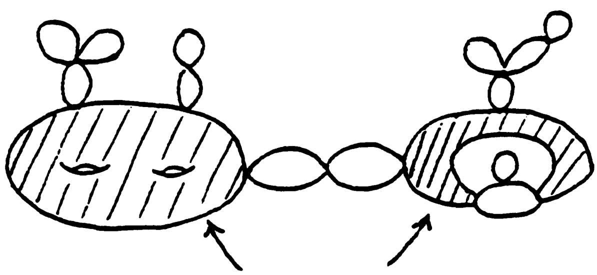


FIGURE 1. DOMAIN OF A CUSP MAP

Theorem 4.2. *Let (V, ω) be a semi-positive symplectic manifold. For a generic (J, ν), $\mathcal{M}_A(\Sigma, J, \nu)$ is a smooth, oriented manifold, and $\overline{\mathcal{M}}_A(\Sigma, J, \nu) - \mathcal{M}_A(\Sigma, J, \nu)$ consists of finitely many pieces (called* strata*), and each stratum is branchedly covered by a smooth manifold of codimension at least 2.*

Remark 4.3. There is a problem whether or not $\overline{\mathcal{M}}_A(\Sigma, J, \nu)$ carries a fundamental class. To prove the existence of a fundamental class, we need the additional property that $\overline{\mathcal{M}}_A(\Sigma, J, \nu) - \mathcal{M}_A(\Sigma, J, \nu)$ has a neighborhood which is a deformation retract to itself. This is much stronger than what we gave in Theorem 4.2. It is a difficult problem to show the existence of a fundamental class, and involves much more analysis. However, we do not need the fundamental class in this paper.

The rest of this section is devoted to the proof of Theorem 4.2. First we shall decompose $\overline{\mathcal{M}}_A(\Sigma, J, \nu) - \mathcal{M}_A(\Sigma, J, \nu)$ into strata. A stratum is the set of cusp curves (possibly with total homology class different from A) satisfying: (1) they have domain of the same homeomorphic type; (2) Each connected component has a fixed homology class. Furthermore, for technical reasons, we need to specify those components, which have the same image even though they may not be adjacent to each other, and their intersection points having the same image. Therefore, the strata of $\overline{\mathcal{M}}_A(\Sigma, J, \nu) - \mathcal{M}_A(\Sigma, J, \nu)$ are indexed by datas: (i) homeomorphism type of the domain of cusp curves with distinguished points at appropriate intersection; (ii) a homology class associated to each component; (iii) a specification of components with the same image and their intersection points with the same iamge. We denote by D a set of those three data. Let $\mathcal{D}_\Sigma$ be the collection of such D's. Note that when we drop the multiplicity from a multiple covering map, we change the homology class. However it is still A-admissible in the following sense:

Definition 4.4. Let D be given as above. We define $[D]$ to be the sum of homology classes of components in (ii). Let $P_1, \cdots, P_m$ be principal components and $B_1, \cdots, B_k$ be bubble components of D. We say that D is callled A-admissible if there are positive integers $b_1, \cdots, b_k$ such that

$$A = \sum_1^m [P_i] + \sum_1^k b_j [B_j], \tag{4.1}$$

where $[P_i]$, $[B_j]$ are the homology classes of P_i, B_j. We say that D is (J, ν)-effective if every principal component can be represented by a (J, ν)-map, and every bubble component can be represented by a J-holomorphic map.

We will always denote by Σ_i the domain of the (J, ν)-map representing P_i. Let $\mathcal{D}^{J,\nu}_{A,\Sigma} \subset \mathcal{D}_\Sigma$ be the set of A-admissible, (J, ν)-effective D.

Lemma 4.5. *The set $\mathcal{D}^{J,\nu}_{A,\Sigma}$ is finite.*

Proof. First we remark that for any given stable curve Σ and the number of principal and bubble components, there are only finitely many homeomorphism types of possible domains of cusp curves. Therefore, it is enough to show that the number of possible homology classes of principal components or bubble components is finite. For each D in $\mathcal{D}^{J,\nu}_{A,\Sigma}$, let $P_1, \cdots, P_m$ be the principal components of D, and $B_1, \cdots, B_k$ be the bubble components of D. Note that m depends only on Σ, so it suffices to bound k uniformly. Assume that P_i, B_j are represented by f_{P_i}, f_{B_j}. We denote by $E(P_i)$ the energy of f_{P_i}, and by

$E(B_j)$ the energy of f_{B_j}. Then by Lemma 3.1, there is a uniform constant c such that $E(P_i) \leq c(\omega(P_i)+1)$ and $E(B_j) \leq c\omega(B_j)$. Since $A = \sum_1^m P_i + \sum_1^k b_j B_j$, we have

$$\sum_1^m E(P_i) + \sum_j^k b_j E(B_j) \leq c(\omega(A) + m).$$

On the other hand, there is a uniform constant $\epsilon > 0$ such that $E(B_j) \geq \epsilon$ for all j. This implies that k is finite. Also, $E(P_i)$, $E(B_j)$ are uniformly bounded from above, so from the Gromov-Uhlenbeck Compactness Theorem (Proposition 3.1) it follows that there are only finitely many possible homology classes, for P_i, B_j. Therefore, $\mathcal{D}^{J,\nu}_{A,\Sigma}$ is finite.

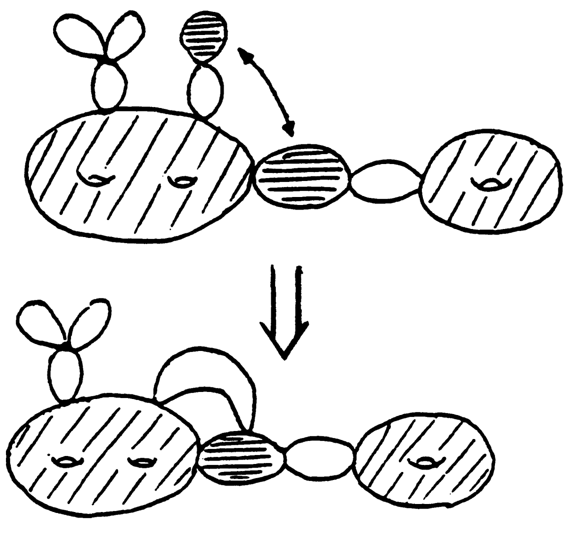


FIGURE 2. CREATE A CYCLE

One can consider $\mathcal{D}^{J,\nu}_{A,\Sigma}$ as the set of indices of strata. For each $D \in \mathcal{D}^{J,\nu}_{A,\Sigma}$, let $\mathcal{M}_\Sigma(D, J, \nu)$ be the space of Σ-cusp curves such that the homeomorphism type of its domain, homology class of each component, and components and their intersection points, which have the same image, are specified by D. Now we make another reduction by identifying the domains of those components which have the same image, and change the homology class accordingly. Furthermore, we identify the corresponding intersection points with the same image. Suppose that the resulting new domain and homology class of each component are specified by $\bar{D}$. This process may destroy the tree structure and creat some cycles in the domain. The total homology class also changes. However, it remains to be A-admissible. The following two diagrams illustrate this process.

Given such D and $\bar{D}$, we can identify $\mathcal{M}_\Sigma(D, J, \nu)$ with the space of (J, ν)-maps whose domain and homology class of each component are specified by $\bar{D}$. Let us denote this space by $\mathcal{M}_\Sigma(\bar{D}, J, \nu)$. Then for each f in $\mathcal{M}_\Sigma(\bar{D}, J, \nu)$, bubble components have different images. As before, let $P_1, \cdots, P_m$ be the principal components, and let $B_1, \cdots, B_k$ be the bubble components. Now we shall construct a smooth branched covering of $M_{A,\Sigma}(\bar{D}, J, \nu)$. Let $\Sigma_{\bar{D}}$ be the domain of maps in the stratum $\mathcal{M}_\Sigma(\bar{D}, J, \nu)$. This is a union of Σ_i and some S^2's. Consider

$$\tilde{\mathcal{M}}_\Sigma(\bar{D}, J, \nu) = \{f : \Sigma_{\bar{D}} \to V \mid f_{P_i} \in \mathcal{M}_{[P_i]}(\Sigma_i, J, \nu_i), \\ f_{B_j} \in \mathcal{M}^*_{[B_j]}(S^2, J, 0),\ \mathrm{Im}(f_{B_j}) \neq \mathrm{Im}(f_{B_{j'}}) \text{ if } j \neq j' \Big\}, \tag{4.2}$$

where $\mathcal{M}^*_{[B_j]}(S^2, J, 0) \subset \mathcal{M}_{[B_j]}(S^2, J, 0)$ is the space of non-multiple covering maps. Sometimes, we drop (J, ν) in case there are no confusions. For each bubble component, there is a parametrization group $G = PSL_2$. Therefore G^k acts on $\tilde{\mathcal{M}}_\Sigma(\bar{D}, J, \nu)$, and $\mathcal{M}_\Sigma(\bar{D}, J, \nu) = \tilde{\mathcal{M}}_\Sigma(\bar{D}, J, \nu)/G^k$. Clearly,

$$\mathcal{M}_\Sigma(\bar{D}, J, \nu) \subset \prod \mathcal{M}_{[P_i]}(\Sigma_i, J, \nu_i) \times \prod \mathcal{M}^*_{[B_j]}(S^2, J, 0)/G. \tag{4.3}$$

But $\mathcal{M}_\Sigma(\bar{D}, J, \nu)$ is not smooth in general. We would like to desingularize $\mathcal{M}_\Sigma(\bar{D}, J, \nu)$. More precisely, we will use an idea in [19] to construct a smooth manifold $\mathcal{N}_\Sigma(\bar{D}, J, \nu)$ and a branched covering map $\pi : \mathcal{N}_\Sigma(\bar{D}, J, \nu) \to \mathcal{M}_\Sigma(\bar{D}, J, \nu)$. Note that $\mathcal{M}_\Sigma(\bar{D}, J, \nu)$ is a proper subset of $\prod \mathcal{M}_{[P_i]}(\Sigma_i, J, \nu_i) \times \prod \mathcal{M}^*_{[B_j]}(S^2, J, 0)$, whose components intersect each other according to the intersection pattern given by D. Let h_i be the number of intersection points on the component P_i. Note that we count a self-intersection point twice. Here, the intersection points between the components are the points in their domain, not in their image. Among them, there are p_i many distinguished points and marked points which are bubbling points. Suppose that they are $z^i_1, \cdots, z^i_{p_i}$. Similarly, let h^j be the number of intersection points on the bubble component B_j. Consider the evaluation map

$$e_{\bar{D}} : \prod \mathcal{M}_{[P_i]}(\Sigma_i, J, \nu_i) \times (\Sigma_i)^{h_i - p_i} \times \prod \mathcal{M}^*_{[B_j]}(S^2, J, 0) \times (S^2)^{h^j} \mapsto \prod V^{h_i} \times \prod V^{h^j} = V^{h_{\bar{D}}},$$

where $h_{\bar{D}} = \sum h_i + \sum h^j$, and $e_{\bar{D}}$ is defined as follows: We first define

$$\begin{gathered} e_{P_i} : \mathcal{M}_{[P_i]}(\Sigma_i, J, \nu_i) \times (\Sigma_i)^{h_i - p_i} \to V^{h_i}, \\ e_{P_i}(f, x_1, \cdots, x_{h_i - p_i}) = (f(z^i_1), \cdots, f(z^i_{p_i}), f(x_1), \cdots, f(x_{h_i - p_i})). \end{gathered} \tag{4.4}$$

For each B_j, we define $e_{B_j} : \mathcal{M}^*_{[B_j]}(S^2, J, 0) \times (S^2)^{h^j} \to V^{h^j}$ by

$$e_{B_j}(f, y_1, \cdots, y_{h^j}) = (f(y_1), \cdots, f(y_{h^j})). \tag{4.5}$$

Then we define $e_{\bar{D}} = \prod e_{P_i} \times \prod e_{B_j}$. Recall that if M, N are submanifolds of X, $M \cap N$ can be reinterpreted as $M \times N \cap \Delta$, where Δ is the diagonal of $X \times X$. This means that we can realize any intersection pattern by constructing a "diagonal" in the product. Let us construct a submanifold $\Delta_{\bar{D}} \subset V^h$ which plays the role of the diagonal. Let $z_1, \cdots, z_{t_{\bar{D}}}$ be all the intersection points. For each z_s, let

$$I_s = \{P_{i_1}, \cdots, P_{i_q}, B_{j_1}, \ldots, B_{j_r}\}$$

be the set of components which intersect at z_s. Now we will construct a product V_s of V such that its diagonal describes the intersection at z_s. This is done as follows: We allocate one or two factors from each of $V^{h_{i_1}}, \cdots, V^{h_{i_q}}$, according to whether or not z_s is a self-intersection point of the corresponding principal component. We allocate one factor from each of $V^{h^{j_1}}, \cdots, V^{h^{j_r}}$. Here V^{h_i} or V^{h^j} are the image of e_{P_i} or e_{B_j}. Then, we take the product of those factors and denote it by V_s. Let Δ_s be the diagonal of V_s. Then the product $\Delta_{\bar{D}} = \Delta_1 \times \cdots \times \Delta_{t_{\bar{D}}} \subset V^{h_{\bar{D}}}$ is the diagonal to realize the intersection pattern between the components of $\bar{D}$. Let π be the natural projection from

$$\prod (\mathcal{M}_{[P_i]}(\Sigma_i, J, \nu_i) \times (\Sigma_i)^{h_i}) \times \prod \left(\mathcal{M}^*_{[B_j]}(S^2, J, 0) \times (S^2)^{h^j}\right)$$

onto

$$\prod \mathcal{M}_{[P_i]}(\Sigma_i, J, \nu_i) \times \prod \mathcal{M}^*_{[B_j]}(S^2, J, 0).$$

Then $\pi(e_{\bar{D}}^{-1}(\Delta_{\bar{D}})) \supset \tilde{\mathcal{M}}_\Sigma(\bar{D}, J, \nu)$. But they may not be equal because we require that bubble components have different image. Moreover, the group $G^{b_{\bar{D}}}$ acts on $e_{\bar{D}}^{-1}(\Delta_{\bar{D}})$ and $\pi^{-1}(\tilde{\mathcal{M}}_\Sigma(\bar{D}, J, \nu))$, where $b_{\bar{D}} = \sum h^j$.

Definition 4.6. We define

$$\tilde{\mathcal{N}}_\Sigma(\bar{D}, J, \nu) = e_{\bar{D}}^{-1}(\Delta_{\bar{D}}) \cap \pi^{-1}(\tilde{\mathcal{M}}_\Sigma(\bar{D}, J, \nu))$$

and

$$\mathcal{N}_\Sigma(\bar{D}, J, \nu) = \tilde{\mathcal{N}}_\Sigma(\bar{D}, J, \nu)/G^{b_{\bar{D}}}. \tag{4.6}$$

Clearly, $\pi : \mathcal{N}_\Sigma(\bar{D}, J, \nu) \to \mathcal{M}_\Sigma(\bar{D}, J, \nu) = \mathcal{M}_\Sigma(D, J, \nu)$ is a branched covering. The fiber over f in $\mathcal{M}_\Sigma(\bar{D}, J, \nu)$ is the set of unordered tuples of intersection points between components of f.

Theorem 4.7. *For a generic (J, ν), $\mathcal{N}_\Sigma(\bar{D}, J, \nu)$ is a smooth manifold of dimension*

$$\begin{aligned}&\sum(2C_1(V)(P_i) + 2n(1 - g_i))\\&\quad + \sum(2C_1(V)(B_j) + 2n - 6) + 2h_{\bar{D}} - 2u_\Sigma - 2s_D - 2n(h_{\bar{D}} - t_{\bar{D}}),\end{aligned}$$

where g_i is the genus of Σ_i, u_Σ is the number of distinguished points, i.e., twice of the number of double points of Σ, s_D is the number of marked points which are bubbling points and $t_{\bar{D}}$ is the number of intersection points of $\bar{D}$. Moreover, for generic (J, ν) and (J', ν'), there is a generic path (J_t, ν_t) connecting (J, ν) and (J', ν') such that $\bigcup_{t\in[0,1]} \mathcal{N}_\Sigma(\bar{D}, J_t, \nu_t) \times \{t\}$ is a smooth manifold of one dimension higher.

By the construction of $\bar{D}$, it is evident that $t_{\bar{D}} \leq t_D$ and $h_{\bar{D}} \leq h_D$. But, $h_{\bar{D}} - t_{\bar{D}} = h_D - t_D$. Therefore, we have

Corollary 4.8. *Under the conditions of Theorem 4.7, the dimension of* $\dim \mathcal{N}_\Sigma(\bar{D}, J, \nu)$ *is less than or equal to*

$$\begin{aligned}&\sum(2C_1(V)(P_i) + 2n(1 - g_i))\\&\quad + \sum(2C_1(V)(B_j) + 2n - 6) + 2h_D - 2u_\Sigma - 2s_D - 2n(h_D - t_D).\end{aligned}$$

Proof of Theorem 4.7. In [13 (Lemma 4.8-4.11)], McDuff proves the theorem for rational curves. The proof for general cases is identical. For the reader's convenience, we give a sketched proof here. Let $\mathcal{J}$ be the space of ω-tamed almost complex structures equipped with Sobolev norm $W^{l,2}$ for a sufficiently large l. It is a Banach manifold. For each $J \in \mathcal{J}$, the inhomogeneous terms are elements of $\prod_1^m \overline{\mathrm{Hom}}_J^c(T\Sigma_i, TV)$, i.e., sections of the bundle of anti-J-linear homomorphisms from TP_i to TV over $P_i \times V$. So our space of inhomogeneous terms is a vector bundle over $\mathcal{J}$ with fiber $\prod_i^m \overline{\mathrm{Hom}}_J^c(T\Sigma_i, TV)$ equipped with the $W^{l,2}$-Sobolev norm.

For each P_i, we have a universal moduli space

$$\mathcal{H}_{P_i} = \{(f, J, \nu_i) \mid f : \Sigma_i \mapsto V,\ \bar\partial_J f = \nu_i\}. \tag{4.7}$$

For each B_j, we also have a universal moduli space

$$\begin{aligned} \mathcal{H}^*_{B_j} = \{(f, J) \mid f : S^2 \to V, \mathrm{Im}(f) = B_j,\ \bar\partial_J f = 0;\\ f \text{ is not a multiple covering}\}. \end{aligned} \tag{4.8}$$

It is well-known (cf. [13], [19]) that those universal moduli spaces are smooth Banach manifolds. We define an evaluation map

$$e_{\bar D} : \prod_i (\mathcal{H}_{P_i} \times (\Sigma_i)^{h_i - p_i}) \times \prod_j (\mathcal{H}^*_{B_j} \times (S^2)^{h^j}) \mapsto V^{h_{\bar D}} \tag{4.9}$$

and the diagonal $\Delta_{\bar D}$ as before. On the other hand, there is a projection map

$$\Theta : \prod_i (\mathcal{H}_{P_i} \times (\Sigma_i)^{h_i - p_i}) \times \prod_j (\mathcal{H}^*_{B_j} \times (S^2)^{h^j}) \to \mathcal{J}^{m+k}. \tag{4.10}$$

Let $\Delta_{\mathcal{J}} \subset (\mathcal{J})^{m+k}$ be the diagonal.

Lemma 4.9. *$\Theta^{-1}(\Delta_{\mathcal{J}})$ is a smooth Banach manifold.*

Proof. There is another way to construct $\Theta^{-1}(\Delta_{\mathcal{J}})$ as follows: Consider the space $\mathcal{P}$ of inhomogeneous terms, which is a vector bundle over $\mathcal{J}$ with fiber $\prod_i \overline{\mathrm{Hom}}_J^c(T\Sigma_i, TV)$. For a fixed $J \in \mathcal{J}$, we can form a Hilbert bundle $\Omega_J^{0,1}(e^*_{P_i}TV)$ or $\Omega_J^{0,1}(e^*_{B_j}TV)$ over $\mathrm{Map}_{[P_i]}(\Sigma_i, V)$ or $\mathrm{Map}^*_{[B_j]}(S^2, V)$ in the same way as one did in [19]. Then by varying J, we obtain a bundle

$$\begin{aligned} &\prod_i \Omega^{0,1}(e^*_{P_i}TV) \times \prod_j \Omega^{0,1}(e^*_{B_j}TV) \\ &\qquad \to \prod_i \mathrm{Map}_{[P_i]}(\Sigma_i, TV) \times \prod_j \mathrm{Map}^*_{[B_j]}(S^2, V) \times \mathcal{P}. \end{aligned}$$

There is a canonical section $\mathcal{S} = (\bar{\partial}_J - \nu_1, \ldots, \bar{\partial}_J - \nu_m, \bar{\partial}_J, \cdots, \bar{\partial}_J)$ of this bundle. Then $\Theta^{-1}(\Delta_{\mathcal{J}})$ is a fiber bundle over $\mathcal{S}^{-1}(0)$ with fiber $\prod(\Sigma_i)^{h_i - p_i} \times (S^2)^{b_{\bar{D}}}$. To show that $\Theta^{-1}(\Delta_{\mathcal{J}})$ is a smooth Banach manifold, we only need to show that $\mathcal{S}$ is transversal to the zero zection.

Let $a = (f_1, \cdots, f_m, C_1, \cdots, C_k, J, \nu) \in \mathcal{S}^{-1}(0)$. We want to show that the differential $\delta\mathcal{S}(a)$ is surjective to the fiber

$$\prod_i \Omega^{0,1}_J(e^*_{P_i}TV) \times \prod_j \Omega^{0,1}_J(e^*_{B_j}TV).$$

As we showed in [19], the differential $\delta\mathcal{S}$ is surjective on each factor $\Omega^{0,1}_J(e^*_{P_i}TV)$ regardless of J. So it suffices to show that $\delta\mathcal{S}$ is surjective on each factor $\Omega^{0,1}_J(e^*_{B_j}TV)$. Let $(\alpha_1, \ldots, \alpha_k)$ be in the cokernel of $\delta\mathcal{S}$. Then α_j satisfies the equation $L^*_f \alpha_j = 0$, where

$$L_f : \Omega^0(f^*TV) \to \Omega^{0,1}(f^*TV)$$

is the linearization of the Cauchy-Riemann operator $\bar{\partial}_J$ at f, which is an elliptic operator. Choose a point $x_j \in S^2$ such that $dC_j(x_j) \neq 0$ and $C_j^{-1}(C_j(x_j)) = \{x_j\}$. We want to show that α_j is identically zero. By the Unique Continuation Theorem for elliptic operators, it suffices to show that α_i vanishes in a neighborhood of x_j. If $k = 1$, this is a classical argument [14]. Let us sketch the argument. We can perturb the almost complex structure in a neighborhood of x_1 such that the image of δS contains all the local sections of $\Omega^{0,1}_J(e^*_{B_j}TV)$ at a neighborhood of x_1, which vanish outside a slightly larger neighborhood of x_1 . Since α_1 is orthogonal to the image of $\delta\mathcal{S}$, this implies that α_1 vanishes in a neighborhood of x_1. In the general case, we can always suppose that $x_1, \cdots, x_k$ are distinct and perform the same argument at a disjoint union of neighborhoods of x_j. Then, the proof of the general case is identical to the case $k = 1$.

Next we claim that $e : \Theta^{-1}(\Delta_{\mathcal{J}}) \to V^h$ is transversal to $\Delta_{\bar{D}}$.

Let $(v_1, \cdots, v_{t_{\bar{D}}})$ be a point in $\Delta_{\bar{D}} = \Delta_1 \times \cdots \times \Delta_{t_{\bar{D}}}$. Let $I_s = \{P_{i_1}, \cdots, P_{i_p}, B_{j_1}, \cdots, B_{j_r}\}$ and

$$(f_1, g_1, x_1, \cdots, f_m, g_m, x_m; C_1, y_1, \cdots, C_k, y_k, J, \nu) \in e^{-1}((v_1, \cdots, v_{t_{\bar{D}}})).$$

Then, $f_{i_1}(x_{i_1}) = \cdots = f_{i_p}(x_{i_p}) = C_{j_1}(y_{j_1}) = \cdots = C_{j_r}(y_{j_r}) = v_s$. We claim that there are only finitely many accumulation points of $\operatorname{Im} C_j \cap \operatorname{Im} C_{j'}$ (see [13 (Lemma 4.4)]). Its proof is standard and can be outlined as follows: The Hartman-Winter Lemma ([9 (Lemma 2.6.1)]) implies that the derivative of C_j or $C_{j'}$ vanishes at most at finitely many points; consequently, $\mathrm{Im}(C_j)$ or $\mathrm{Im}(C_{j'})$ is smooth outside finitely many points. If the claim is false, then there is an accumulation point y' where both $\mathrm{Im}(C_j)$ and $\mathrm{Im}(C_{j'})$ are smooth. Near y', one can choose local coordinates $(z_1, \cdots, z_n)$ of V such that $z_k \cdot C_j = 0$ for all $k \geq 2$. Since $\mathrm{Im}\, C_j \neq \mathrm{Im}\, C_{j'}$, $z_k \cdot C_{j'}$ is not identically zero for some $k \geq 2$. Using the Hartman-Winter Lemma again, one can show that $z_k \cdot C_{j'}$ vanishes only at finitely many points, a contradiction to our assumption on y'. Then the claim follows. Therefore, for each C_{j_s}, we can choose a small disc D_{j_s} around y_{j_s} such that there is a smaller disc D'_{j_s} satisfying that the anulus $D_{j_s} - D'_{j_s}$ does not intersect other bubble components. By the work of McDuff [14], given any tangent vector $X \in T_{v_s}V$, there is a perturbation $C^t_{j_s}$ of C_{j_s} on D_{j_s} such that $C^t_{j_s}$ is still J-holomorphic, $C^0_{j_s} = C_{j_s}$ and $\frac{dC^t_{j_s}}{dt}(y_{j_s})|_{t=o} = X$. We can patch $C^t_{j_s}$ with $C_{j_s}|_{S^2 - D_{j_s}}$ to get $C^t_{j_s}$ defined on S^2 such that $C_{j_s}|_{D_{j_s} - D'_{j_s}}$ does not intersect other bubble components either. Here we also need to perturb the almost complex structure J to J_t in a small neighborhood of $\mathrm{Im}\, C_{j_s}|_{D_{j_s} - D'_{j_s}}$ such that $C^t_{j_s}$ is J_t-holomorphic. Clearly, other bubbles are also J_t holomorphic. But $\frac{dC^t_{j_s}}{dt}(y_{j_s})|_{t=0} = X$. For a principal component f_{i_s}, the argument is even easier. We can just choose an arbitrary perturbation $f^t_{i_s}$ on a small disc D_{i_s} around x_{i_s} (x_{i_s} could be a distinguished point) such that $\frac{df^t_{i_s}}{dt}(x_{i_s})|_{t=0} = X$. Then we patch it with $f_{i_s}|_{\Sigma_i - D_{i_s}}$ to get a globally defined $f^t_{i_s}$ with $\frac{df^t_{i_s}}{dt}(x_{i_s})|_{t=0} = X$. Then we simply perturb ν_{i_s} such that $\nu^t_{i_s} = \bar{\partial}_J f^t_{i_s}$ on the graph of $f^t_{i_s}$. Then $f^t_{i_s}$ satisfies an inhomogeneous Cauchy-Riemann equation with inhomogeneous term $\nu^t_{i_s}$. Applying this argument to each v_s and every point in $e^{-1}_{\bar{D}}(\Delta_{\bar{D}})$, we show that

$$e : \Theta^{-1}(\Delta_{\mathcal{J}}) \to V^h$$

is transversal to $\Delta_{\bar{D}}$. Therefore, $\Theta^{-1}(\Delta_{\mathcal{J}}) \cap e^{-1}(\Delta_{\bar{D}})$ is a smooth Banach manifold. Moreover, we have a Fredholm map

$$\Theta^{-1}(\Delta_{\mathcal{J}}) \cap e^{-1}(\Delta_{\bar{D}}) \to \mathcal{P}.$$

By the Sards-Smale Transversality Theory, for a generic element (J, ν) of $\mathcal{P}$, its preimage $\tilde{\mathcal{N}}_\Sigma(\bar{D}, J, \nu)$ is a smooth manifold. $G^{b_{\bar{D}}}$ acts freely on

$\tilde{\mathcal{N}}_{A,\Sigma}(\bar{D}, J, \nu)$. Therefore, $\mathcal{N}_\Sigma(\bar{D}, J, \nu) = \tilde{\mathcal{N}}_\Sigma(\bar{D}, J, \nu)/G^{b_{\bar{D}}}$ is smooth. A routine counting argument yields the dimension. The proof of second part of theorem is identical.

A special case is that D only has one principal component Σ, where we call D to be *sectional* in [19]. In this case, $\mathcal{M}_\Sigma(D, J, \nu)$ is in the boundary of $\overline{\mathcal{M}}_A(\Sigma, J, \nu)$.

Corollary 4.10. *Suppose that D is sectional. Then for a generic (J, ν), $\mathcal{N}_\Sigma(\bar{D}, J, \nu)$ is a smooth manifold, and for generic (J_t, ν_t), $\bigcup_t \mathcal{N}_\Sigma(\bar{D}, J_t, \nu_t) \times \{t\}$ is a smooth cobordism.*

We need this corollary in the definition of the mixed invariant. Another special case is

Corollary 4.11. *If D has no bubble components at all, then for a generic (J, ν), $\mathcal{M}_A(\Sigma, J, \nu) = \mathcal{N}_\Sigma(\bar{D}, J, \nu)$ is smooth, for generic (J_t, ν_t), $\bigcup_t \mathcal{M}_A(\Sigma, J_t, \nu_t) \times \{t\} = \bigcup_t \mathcal{N}_\Sigma(\bar{D}, J_t, \nu_t) \times \{t\}$ is a smooth cobordism.*

Remark 4.12 (on orientations). Compared to Donaldson gauge theory, the theory of pseudo-holomorphic curves is considerable more complicated in many aspects like compactness. But the orientation problem for pseudo-holomorphic curves is much easier. In fact, there is a canonical orientation over $\mathcal{M}_A(\Sigma, J, \nu)$ [13] [19] for genus 0 case. The argument for higher genus case is completely the same. For the reader's convenience, we sketch the argument here.

First of all, we can view a (J, ν)-map as a holomorphic map to $\Sigma \times V$ (the paragraph before Lemma 3.1). Without loss of generality, we can assume that $\nu = 0$. Recall that the linearization of the Cauchy-Riemann operator $\bar{\partial}_J$ at $f \in \mathcal{M}_A(\Sigma, J, 0)$ is

$$L_f : \Omega^0(f^*TV) \to \Omega^{0,1}(f^*TV).$$

The tangent space $T_f\mathcal{M}_A(\Sigma, J, 0) = KerL_f$.. The determinant line bundle $\det(T\mathcal{M}_A(\Sigma, J, 0)) = \det(L_f)$, which is defined over $\mathrm{Map}_A(\Sigma, V)$. An orientation of $\mathcal{M}_A(\Sigma, J, 0)$ is just a nowhere vanishing section of $\det(T\mathcal{M}_A(\Sigma, J, 0))$ up to a multiple of positive function. We shall omit " up to a multiple of positive function without any confusion" without any confusion. Therefore, to construct a canonical orientation of $\mathcal{M}_A(\Sigma, J, 0)$, it is enough to construct a canonical section of $\det(L_f)$ over whole $\mathrm{Map}_A(\Sigma, V)$. Choose a J-linear connetion ∇ over V. Then, L_f can be written as

$$L_f = \nabla_f + Z_f,$$

where ∇ is the induced J-linear connection over f^*TV, and Z_f is the zero order term. Let

$$L_{f,t} = \nabla_f + tZ_f.$$

Then, $\det(L_{f,t})$ is isomorphic to $\det(L_{f,0})$. Hence, $\det(L_f)$ is isomorphic to $\det(\nabla_f)$. On the other hand, both $ker\nabla_f$ and $coker\nabla_f$ are complex vector spaces. Therefore, there is a canonical section of real determinant line bundle $\det(\nabla_f)$ corresponding to the complex structure. We refer to [19 (Theorem 3.3.1)] for the argument of the independence of this canonical section from the choice of ∇. By the same argument, one can show that $\cup_{t\in[0,1]}\mathcal{M}_A(\Sigma, J_t, \nu_t) \times \{t\}$ carry a canonical orientation.

Next, we compute the codimension of $\mathcal{N}_\Sigma(\bar{D}, J, \nu)$. First we consider an easy case.

Proposition 4.13. *Suppose that Σ is a genus g stable curve. Then*

$$\dim \mathcal{M}_A(\Sigma, J, \nu) = 2C_1(V)(A) + 2n(1-g).$$

Proof. Clearly, $h_\Sigma = u_\Sigma = 2t_\Sigma$ and $s_D = 0$. Hence

$$\dim \mathcal{M}_A(\Sigma, J, \nu) = 2C_1(V)(A) + 2n(1 - \sum g_i) + 2n(m-1) - 2nt_\Sigma.$$

Recall that we can obtain Σ by adjoining s_Σ-many disjoint simple closed loops on a genus g Riemann surface. Collapsing of each loop will correspond to a double point. We shall prove Proposition 4.13 inductively by collapsing the loops one by one. Let Σ_1 be a stable curve. We collapse a circle on some component of Σ_1 and obtain another stable curve Σ_2. Suppose that we collapse a circle γ on a component B of genus $\tilde{g}$. There are two cases. First, we separate the component B into two components B_1, B_2 of genus g_1, g_2. Then $\tilde{g} = g_1 + g_2$, m increases by 1 and s_Σ increases by 1. Therefore, $2C_1(V)(A) + 2n(1-\sum g_i) + 2n(m-1) - 2nt_\Sigma$ remains the same. If collapsing of γ does not separate the component B, then it creates a self intersection on the component B. In this process, the genus drops by 1, m remains the same, but s_Σ increases by 1. Clearly, the dimension formula remains the same.

Proposition 4.14. *Suppose that (V, ω) is a semi-positive symplectic manifold. Then for a generic (J, ν) and a D in $\mathcal{D}^{J,\nu}_{A,\Sigma}$,*

$$\dim \mathcal{N}_\Sigma(\bar{D}, J, \nu) \leq 2C_1(V)(A) + 2n(1-g) - 2k_D - 2s_D, \tag{4.11}$$

where k_D is the number of bubble components of D (not $\bar{D}$).

Proof. By Corollary 4.8, the dimension of $\mathcal{N}_\Sigma(\bar{D}, J, \nu)$ is less than or equal to

$$\begin{aligned}\sum 2C_1(V)([\bar{D}]) &+2n(1-\sum g_i)+2n(m-1)+(2n-6)k_{\bar{D}}\\ &+2h_D-2u_\Sigma-2s_D-2n(h_D-t_D).\end{aligned}$$

For a generic J,

$$2C_1(V)(B_j)+2n-6=\dim \mathcal{M}^*_{[B_j]}(S^2,J,0)/PSL_2\geq 0.$$

If some bubble component B_j happens to be the image of two or more bubble components of D, then

$$\begin{aligned}\dim \mathcal{N}_\Sigma(\bar{D}, J,\nu)\leq &\sum 2C_1(V)([D])+2n(1-\sum g_i)+2n(m-1)\\ &+(2n-6)k_D+2h_D-2u_\Sigma-2n(h_D-t_D).\end{aligned}$$

Since (V,ω) is semi-postive, $C_1(V)(B_j)\geq 0$ for a generic J. Since D is A-admissible, $C_1(V)([D])\leq C_1(V)(A)$. Let

$$\lambda_D=(2n-6)k_D+2h_D-2n(h_D-s_D).$$

By Proposition 4.13, it is enough to show that $\lambda_D\leq\lambda_\Sigma-2k_D=2h_\Sigma-2n(h_\Sigma-t_\Sigma)-2k_D$. We will prove this by induction on k. When $k=0$, it follows from Proposition 4.13. Suppose that it is true for k and D has $k+1$ many bubble components. We consider two cases:

Case 1. There is a bubble component B such that there is only one intersection point on the B-component. In this case, there is a bubble tree in D, and B is at the tip of one branch. Therefore, we can remove a B-component and obtain the domain of a Σ-cusp curve D' with k-many bubble components. Suppose that the intersection point is z. Then, there are two situations. First, there are at least three components intersecting at z.

In this case, after the removal of a B-component, we still have the same number of intersection points. Then,

$$h_D = h_{D'} + 1, \; s_D = s_{D'}.$$

So,

$$\lambda_D \leq \lambda_{D'} + 2n - 6 + 2 - 2n < \lambda_\Sigma - 2(k+1).$$

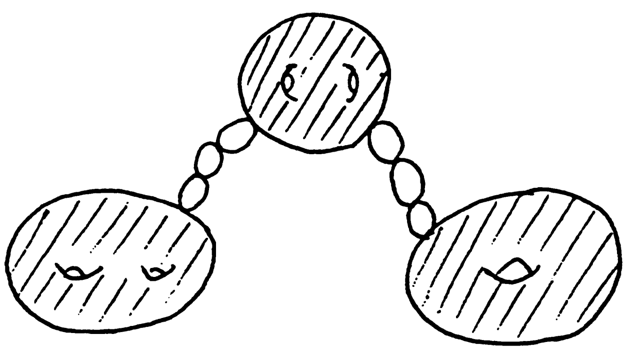
Suppose that there are only two components intersecting at z. After the removal of a B-component, we will have one fewer intersection point on the component intersecting the B-component. Thus,

$$h_D = h_{D'} + 2, \; s_D = s_{D'} + 1.$$

Hence,

$$\lambda_D \leq \lambda_{D'} + 2n - 6 + 4 - 2n \leq \lambda_\Sigma - 2(k+1).$$

Case 2. Every bubble component has at least two intersection points. Then, there is no bubble tree on D, and D is obtained by joining chains of bubbles to separate double points. In this case, every bubble component has exactly two intersection points.



Now we collapse any bubble component, say B, to end up with D', a domain of a Σ-cusp curve with k many bubble components. Clearly,

$$h_D = h_{D'} + 2, \qquad s_D = s_{D'} + 1.$$

Hence,

$$\lambda_D = \lambda_{D'} + 2n - 6 + 4 - 2n \leq \lambda_\Sigma - 2(k+1).$$

Finally, we have

Proof of Theorem 4.2 (Structure Theorem). It follows from Lemma 4.5, Theorem 4.7, Proposition 4.13 and 4.14.

5. Proof of Propositions 2.2, 2.3

After our work on compactness and transversality theory in last two sections, we are ready to prove Propositions 2.2, 2.3, which were used in section 2 to establish the existence of the mixed invariant and its independence from various parameters. Hence, the mixed invariant is a symplectic invariant. We also consider transversality between intersecting components of (J,ν)-map from a stable curve. This is needed in the gluing argument of next section.

In order to prove Propositions 2.2, 2.3, we need an additional transversality result, i.e., transversality of moduli spaces with a pseudo-manifold representative of a homology class in the target space V. As we mentioned in section 1, every homology class can be represented by a pseudo-submanifold. Now it is the time to give a precise definition.

Definition 5.1. A dimension-n finite simplical complex P is called an abstract pseudo-manifold if $P_{reg} = P - P_{n-2}$ ($(n-2)$-skeleton) is an open smooth oriented n-dimensional manifold. P is called an abstract pseudo-manifold with boundary if P_{reg} is a n-dimensional oriented smooth manifold with boundary ∂P_{reg}. Let $\partial P = \overline{\partial P_{reg}}$. Then $\partial P \cap P_{n-2}$ is a subcomplex of dimension less than or equal to $n-3$. A pseudo-submanifold is a pair (P,f), where P is an abstract pseudo-manifold, and $f : P \to V$ is a piece-wise linear map (PL) with respect to some triangulation of V such that f is smooth over P_{reg}. A pseudo-submanifold cobordism between pseudo-submanifolds (P,f), (Q,h) is a pair (K,H) such that K is an abstract pseudo-manifold with boundary with $\partial K = P \cup -Q$, and H is PL with respect to some triangulation of V and smooth over K_{reg} with $H|_{P\cup -Q} = f \cup -h$, where $-$ means the opposite orientation.

Furthermore, we have the following lemma on transversality.

Lemma 5.2. *Let (P,f) be a pseudo-submanifold representative of a homology class α, and $h_i : X_i \to V$ be smooth maps from smooth* *manifolds X_i. Then, there is a small perturbation $\tilde{f} : P \to V$ such that f is transverse to each h_i, i.e., f is transverse to h_i as a PL map and transverse over P_{reg} as a smooth map.*

Proof. This lemma is the consequence of standard transversality results in PL topology [18], [12].

Let $\{(U_i, L_i)\}_{i=1}^c$, $\{(W_j, M_j)\}_{j=1}^d$ be pseudo-submanifolds. Let $X = \{x_1, \dots, x_c\}$ be a set of marked points on Σ. We consider the evaluation map

$$(5.1)\quad \begin{aligned} e_{(\Sigma,X,J,\nu)} \times \prod L_i \times \prod M_j : \mathcal{M}_A(\Sigma, J, \nu) \times (\Sigma)^d \times \prod_i U_i \times \prod_j W_j \\ \longrightarrow V^{c+d} \times V^{c+d}, \\ e_{(\Sigma,X,J,\nu)}(f, y_1, \dots, y_d) = (f(x_1), \dots, f(x_c), f(y_1), \dots, f(y_d)). \end{aligned}$$

Let Δ_X be the diagonal $\{(z_1, \cdots, z_{c+d}, z_1, \cdots, z_{c+d})\} \subset V^{c+d} \times V^{c+d}$. Write $U = \prod_i U_i$, $W = \prod_j W_j$ and $L = \prod L_i$, $M = \prod M_j$. Then, $(f, \cdots) \in (e_{(\Sigma,X,J,\nu)} \times L \times M)^{-1}(\Delta_X)$ if and only if $f(x_i) \in \mathrm{Im}(L_i)$, $f(\Sigma) \cap \mathrm{Im}(M_j) \neq \emptyset$. For simplicity, we write $e^{-1}_{(\Sigma,X,J,\nu)}(L \times M)$ for $(e_{(\Sigma,X,J,\nu)} \times L \times M)^{-1}(\Delta_X)$. Similarly, if $\{(J_t, \nu_t)\}$ is a path, we can define

$$(e_{(\Sigma,X,\{J_t,\nu_t\})}) \times L \times M : \bigcup_t \mathcal{M}_A(\Sigma, J_t, \nu_t) \times \{t\} \times (\Sigma)^d \times U \times W \longrightarrow V^{c+d} \times V^{c+d}.$$

Put

$$e^{-1}_{(\Sigma,X\{J_t,\nu_t\})}(L \times M) = ((e_{(\Sigma,X,\{J_t,\nu_t\})}) \times L \times M)^{-1}(\Delta_X).$$

Theorem 5.3. *For a generic (J, ν), we can choose L, M such that $e^{-1}_{(\Sigma,X,J,\nu)}(L \times M)$ is a smooth manifold. For a generic path $\{J_t, \nu_t\}$, we can choose L, M such that $e^{-1}_{(\Sigma,X,\{J_t,\nu_t\})}(L \times M)$ is a smooth cobordant.*

This theorem obviously follows from Lemma 5.2.

Let $C = (\Sigma, x_1, \cdots, x_c)$ be a stable curve, and $X = \{x_1, \cdots, x_c\}$ is the set of marked points. Now we want to consider the set of $f \in \mathcal{M}_\Sigma(D, J, \nu) = \mathcal{M}_\Sigma(\bar{D}, J, \nu)$, where D, $\bar{D}$ are as in section 4, such that $f(x_i) \in \mathrm{Im}(L_i)$ and $f(\Sigma) \cap \mathrm{Im}(M_j) \neq \emptyset$. As we showed in last section, $\mathcal{N}_\Sigma(\bar{D}, J, \nu)$ is a smooth manifold for a generic (J, ν). Therefore, we shall use $\mathcal{N}_\Sigma(\bar{D}, J, \nu)$ instead of $\mathcal{M}_\Sigma(D, J, \nu)$ to formularize transversaltity theorem. To do this, we also need an additional data which is the intersection pattern T of components of f with $\mathrm{Im}(M_j)$. Without loss of generality, suppose that $\mathrm{Im}(M_1), \cdots, \mathrm{Im}(M_p)$ intersect principal components and the rest intersect bubble components. Furthermore, suppose that M_s $(1 \leq s \leq p)$ intersects P_{i_s}-component and M_s $(s > p)$ intersects B_{j_s}-component. If M_i intersects more than two components, we simply choose one of them. Then we can define

$$e_{X,T} \times L \times M : \tilde{\mathcal{N}}_C(\bar{D}, J, \nu) \times \prod_{s=1}^{p} \Sigma_{i_s} \times_{G^{d-p}} (S^2)^{d-p} \times U \times W \longrightarrow V^{c+d} \times V^{c+d}.$$

Note that $G = PSL_2 C$. Let $\Delta_T \subset V^{c+d} \times V^{c+d}$ be the "diagonal" corresponding to the intersection pattern T (cf. section 4). Define

$$e^{-1}_{(D,J,\nu,X',T)}(L \times M) = (e_{X,T} \times L \times M)^{-1}(\Delta_T).$$

Then, $f(x_i) \in \mathrm{Im}(L_i), and \mathrm{Im}(f) \cap \mathrm{Im}(M_j) \neq \emptyset$ implies that

$$e^{-1}_{(D,J,\nu,X',T)}(L \times M) \neq \emptyset.$$

Similarly, we can define $e^{-1}_{(D,\{J_t,\nu_t\},X,T)}(L, M)$.

Theorem 5.4. *For a generic (J, ν), we can choose L, M such that $e^{-1}_{(D,J,\nu,X',T)}(L \times M)$ is a smooth manifold of dimension*

$$\begin{aligned}&\dim \mathcal{N}_\Sigma(D, J, \nu) + 2d - \mathrm{codim}(L) - \mathrm{codim}(M)\\ &\quad \leq 2C_1(V)(A) + 2n(1-g) - 2 - 2s_D + 2d - \mathrm{codim}(L) - \mathrm{codim}(M),\end{aligned}$$

where s_D is the number of marked points which are bubbling points. For a generic path, for any $D \in \mathcal{D}^{J,\nu}_{A,\Sigma}$ and X', T we can choose L, M such that $\{J_t, \nu_t\}$, $e^{-1}_{(D,\{J_t,\nu_t\},X',T)}(L \times M)$ is a smooth cobordant of one dimensional higher.

The proof follows from Lemma 5.2

Definition 5.5. When Theorem 5.4 holds, we say that $\mathcal{N}_\Sigma(D, J, \nu)$ (resp. $\bigcup_t \mathcal{N}_\Sigma(D, J_t, \nu_t)$) is transversal to $L \times M$ for X, T.

Proof of Proposition 2.2. We will adopt the notation in section 2. We fix a generic (J, ν). First we prove Proposition 2.2,(ii). If Proposition 2.2,(ii) fails, by the compactness theorem (Proposition 3.1), there is a sectional cusp curve f in $\mathcal{M}_\Sigma(D, J, \nu)$ for some $D \in \mathcal{D}^{J,\nu}_{A,\Sigma}$ satisfying: (1) $f_j(\Sigma) \cap \mathrm{Im}(G_j) \neq \emptyset$ $(1 \leq j \leq l)$; (2) for each marked point x_i

$(1 \le i \le k)$, either $f_i(x_i) \in \mathrm{Im}(F_i)$, or a bubble occurs at x_i, in this case, $f(x_i)$ may not be in $\mathrm{Im}(F_i)$, but $\mathrm{Im}(F_i)$ will intersect the bubble tree coming out of x_i. Note that $\mathrm{Im}(G_j)$ may intersect a bubble tree instead of the principal component $f(\Sigma)$. Therefore, we should view that f has few marked points and the number of homology classes corresponding to unmarked part of mixed invariant increases. Let $X' \subset X$ be a subset of marked points which are not a bubbling point. Suppose that X' has p many points, say $x_1, \cdots, x_p$. Then $l+k-p$ is the number of homology classes which intersect with $\mathrm{Im}(f)$. Let $U_i \subset Y_i$ and $W_j \subset Z_j$ be the smooth submanifolds (possibly non-compact) such that $f(x_i) \in F_i(U_i)$ for $1 \le i \le p$, $\mathrm{Im}(f) \cap F_i(U_i) \ne \emptyset$ for $i > p$, and $\mathrm{Im}(f) \cap G_j(Z_j) \ne \emptyset$. Suppose that f intersects those manifolds in the intersection pattern T. As we discussed before, it implies

$$e^{-1}_{(D,J,\nu,X',T)}(F|_{\prod_i U_i \times \prod_j W_j}) \ne \emptyset.$$

On the other hand, by Theorem 5.4, if (J,ν) is generic, we can choose F such that $e^{-1}_{(D,J,\nu,X',T)}(F)$ is a smooth manifold of dimension

$$\begin{aligned}
&\dim \mathcal{N}_\Sigma(D,J,\nu) + 2(l+k-p)) - \mathrm{codim}(\prod_i U_i) - \mathrm{codim}(\prod_j W_j) \\
&\le 2C_1(V)(A) + 2n(1-g) - 2 - 2(k-p) + 2(l+k-p) - \mathrm{codim}(P) \\
&\le -2.
\end{aligned}$$

Hence, $e^{-1}_{(D,J,\nu,X',T)}(F|_{\prod_i U_i \times \prod_j W_j}) = \emptyset$. This is a contradiction. So (ii) is proved. Moreover, if f is in $\mathcal{M}_A(\Sigma,J,\nu)$, $f(x_i) \in F_i(U_i)$ and $\mathrm{Im}(f) \cap G_j(Z_j) \ne \emptyset$ for all i, j, then the above arguments also show that U_i is an open stratum of Y_i, and W_j is an open stratum of Z_j. Thus Proposition 2.2 follows from Theorem 5.4, since Σ is compact.

The same arguments as above also yield the following generalization of Proposition 2.2.

Proposition 5.6. *Let $C = (\Sigma, x_1, \cdots, x_k)$ be a k-point genus g stable curve, and (Y_i, F_i), (Z_j, G_j) be in* (2.5) *satisfying* (2.3). *Then all properties stated in Proposition 2.2 still hold.*

Let (P,F) be as in (2.5). For simplicity, we denote by $|e^{-1}_{(\Sigma,X,J,\nu)}(F)|$ the algebraic sum of elements in $e^{-1}_{(\Sigma,X,J,\nu)}(F)$, where the sign of a f in $e^{-1}_{(\Sigma,X,J,\nu)}(F)$ is assigned according to the orientations of $\mathcal{M}_A(\Sigma,J,\nu) \times (\Sigma)^l$, P, V_{k+l} at $(f; y_{s1}, \cdots, y_{sl})$, etc., and the Jacobians of the maps

$e_{(\Sigma,X,J,\nu)}$ and F. Now we divide the proof of Proposition 2.3 into a series of lemmas:

Lemma 5.7. *Let (J',ν') be another generic pair. Then*

$$|e^{-1}_{(\Sigma,X,J,\nu)}(F)| = |e^{-1}_{(X,l,J',\nu')}(F)|.$$

Proof. Choose a generic path (J_t,ν_t) connecting (J,ν) to (J',ν'), such that

$$\mathcal{M}_A(\Sigma,\{J_t,\nu_t\}) = \bigcup_t \mathcal{M}_A(\Sigma,J_t,\nu_t)\times\{t\}$$

is a smooth, oriented cobordism between $\mathcal{M}_A(\Sigma,J,\nu)$ and $\mathcal{M}_A(\Sigma,J',\nu')$. Consider an evaluation map

$$e_{(\Sigma,X,\{J_t,\nu_t\})}: \mathcal{M}_A(\Sigma,\{J_t,\nu_t\})\times(\Sigma)^l \to V^{k+l}$$
$$e_{(\Sigma,X,\{J_t,\nu_t\})}(f,x_1,\cdots,x_k,y_1,\cdots,y_l) = (f(x_1),\cdots,f(y_l)).$$

By Theorem 5.4, we may assume that $e_{(\Sigma,X,\{J_t,\nu_t\})}$ is transversal to F, and $\mathcal{N}_\Sigma(D,\{J_t,\nu_t\})$ is transversal to F for any X', T, where $X'\subset X$, and

$$\mathcal{N}_\Sigma(\bar{D},\{J_t,\nu_t\}) = \bigcup_t \mathcal{N}_{(\bar{D},J_t,\nu_t)}\times\{t\}$$

is a smooth, oriented cobordism for a generic path $\{J_t,\nu_t\}$. A dimension counting shows that $e^{-1}_{(D,\{J_t,\nu_t\},X',T)}(F) = \emptyset$ for any X', T. Since the singular set of P is of codimension at least 2, $\mathrm{Im}(e_{(\Sigma,X,\{J_t,\nu_t\})})$ does not intersect the restriction of F to lower stratum of P. Therefore, $e^{-1}_{(\Sigma,X,\{J_t,\nu_t\})}(F)$ is an oriented, smooth compact 1-manifold. Clearly, the boundary

$$\partial(e^{-1}_{(\Sigma,X,\{J_t,\nu_t\})}(F)) = e^{-1}_{(\Sigma,X,J,\nu)}(F)\bigcup -e^{-1}_{(\Sigma,X,J',\nu')}(F),$$

where "$-$" means opposite orientation. Then the lemma follows.

Lemma 5.8. $|e^{-1}_{(\Sigma,X,J,\nu)}(F)|$ *is independent of the complex structure on Σ.*

Proof. The proof is exactly the same as that of Lemma 5.7.

Lemma 5.9. $|e^{-1}_{(\Sigma,X,J,\nu)}(F)|$ *is independent of the representative P,F.*

Proof. Suppose that (P',F') be another representative. There is a cobordism (Q,H) such that

$$\partial(Q) = P\bigcup -P',\ H|_{\partial(Q)} = F\bigcup -F'.$$

We can choose a generic (J,ν) and F such that $e_{(\Sigma,X,J,\nu)}$ and $\mathcal{N}_\Sigma(\bar{D},J,\nu)$ is transverse to F,F' and H for any $D,X'\subset X,T$. Then, a dimension counting argument will show that

$$(e^{-1}_{(\bar{D},J,\nu,X',T)})^{-1}(Q)=\emptyset.$$

Furthermore, the same argument as before will prove that $e^{-1}(\Sigma,X,J,\nu)(Q)$ is an oriented, smooth, compact 1-manifold with boundary

$$e^{-1}(\Sigma,X,J,\nu)(F)\bigcup -e^{-1}(\Sigma,X,J,\nu)(F').$$

Hence, $|e^{-1}_{(\Sigma,X,J,\nu)}(F)|=|e^{-1}_{(\Sigma,X,J,\nu)}(F')|$.

Lemma 5.10. *$|e^{-1}_{(\Sigma,X,J,\nu)}(F)|$ is independent of the set X of marked points.*

Proof. Let X' be another set of marked points. Choose a diffeomorphism $\phi:\Sigma\to\Sigma$ such that ϕ is isotopic to the identity and map X to X'. Let $\Sigma'=\phi^*\Sigma$. Then, $|e^{-1}_{(\Sigma,X',J,\nu)}(F)|=|e^{-1}_{(\Sigma',X,J,\nu)}(F)|$. Therefore, this lemma follows from Lemma 5.8.

Lemma 5.11. *$\Phi_{(A,\omega,g)}(\alpha_1,\cdots,\alpha_k|\beta_1,\cdots,\beta_l)$ is independent of semi-positive symplectic deformation.*

Proof. This is obvious, since (J,ν) does not depend on the symplectic form ω as long as it is ω-tamed, and the tamedness is an open condition.

Next, we prove a technical result about the transversality of component of cusp curves, which is important in the gluing argument of next section. In this case, there are no bubble components. Then we have a (J,ν)-perturbed holomorphic map from Σ into V such that its components intersect each other at distinguished points. We would like to show that the subset where two components intersect nontransversally is of codimension 2. Without loss of generality, we can assume that P_1 intersects P_2, and z_1^1,z_1^2 are distinguished points on P_1,P_2 corresponding to the intersection. If it is a self-intersection, we just let $P_1=P_2$. Then we can define

$$e_{z_1,z_2}:\mathcal{M}_A(\Sigma,J,\nu)\to \mathrm{Hom}(T_{z_1}\Sigma_{g_1},TV)\oplus\mathrm{Hom}(T_{z_2}\Sigma_{g_2},TV),$$

where $e_{z_1,z_2}=df_1(z_1)\oplus df_2(z_2)$ for $f_1\in\mathcal{M}_{[P_1]}(\Sigma_1,J,\nu_1)$ and $f_2\in\mathcal{M}_{[P_2]}(\Sigma_2,J,\nu_2)$.

A generic element in $\mathrm{Hom}(T_{z_1}\Sigma_{g_1},TV)\oplus\mathrm{Hom}(T_{z_2}\Sigma_{g_2},TV)$, which is a smooth fibration over V and of dimension $10n$, has maximal rank 4.

If f_1, f_2 does not intersect tranversally, then its image will have lower rank. The set of homomorphisms of lower ranks is a union of smooth submanifolds consisting of homomorphisms of rank 0 (which is the zero section), 1, 2, 3. Let us denote them by R_1, R_1, R_2, R_3, each of which is a fibration over V. They have dimensions $2n, 4n+3, 6n+2, 8n+1$. Thus their codimensions are $8n, 6n-3, 4n-2, 2n-1$.

Theorem 5.12. *For a generic (J,ν), e_{z_1,z_2} is transversal to R_0, R_1, R_2, R_3. Hence, $e^{-1}_{z_1,z_2}(R_i)$ for $i \leq 3$ have codimension at least $2n-1$. For generic (J,ν) and (J,ν'), there is a path $\{J,\nu_t\}$ connecting (J,ν) and (J,ν') such that*

$$e^t_{z_1,z_2} : \bigcup_t \mathcal{M}_A(\Sigma, J, \nu_t) \times \{t\} \to \mathrm{Hom}(T_{z_1}\Sigma_1, TV) \oplus \mathrm{Hom}(T_{z_2}\Sigma_2, TV)$$

is transversal to R_i. Hence $(e^t_{z_1,z_2})^{-1}(R_i)$ is a smooth cobordism of one dimension higher.

Proof. First of all, we can define

$$\begin{aligned} E_{z_1,z_2} : \mathcal{H}_{P_1}\ (\Sigma_1, J, \nu_1) \times \cdots \times \mathcal{H}_{P_m}(\Sigma_2, J, \nu_k) \\ \to \mathrm{Hom}(T_{z_1}\Sigma_1, TV) \oplus \mathrm{Hom}(T_{z_2}\Sigma_2, TV) \end{aligned}$$

in the same way as we did in the proof of Lemma 4.9. We claim that E_{z_1,z_2} is a submersion onto its image. Let $a \in \mathrm{Hom}(T_{z_1}\Sigma_1, TV) \oplus \mathrm{Hom}(T_{z_2}\Sigma_2, T_{v_0}V)$ be in the image of E_{z_1,z_2} and $a = a_1 \oplus a_2$, where $a_i : T_{z_1}\Sigma_{g_1} \to T_{v_0}V$. Locally, we can always choose f^t_i on a small neighborhood $D(z_i)$ of z_i such that

$$f^0_i = f_i, f^t_i(z_i) = v_0, \frac{d(df^t_i(z_i))}{dt}|_{t=0} = A_i.$$

We can patch f^t_i with $f_i|_{\Sigma_{g_i} - D(z_i)}$ to get a globally defined f^t_i. Then $\prod(f^t_i, \nu_t) \in \prod \mathcal{H}_{P_i}(\Sigma_i, J, \nu_i)$ and $\frac{d}{dt}(e_{z_1,z_2}(\prod(f^t_i, \nu_t)))|_{t=0} = a$. It implies that E_{z_1,z_2} is transversal to R_0, R_1, R_2, R_3, whose preimage is a Banach submanifold. By the Sards-Smale theory, for a generic ν, $e^{-1}_{z_1,z_2}(R_i)$ are smooth manifolds of codimension $8n, 6n-3, 4n-2, 2n-1$. The proof of the second part is the exactly the same.

6. Gluing J-holomorphic maps

In this section, We will apply the Implicit Function Theorem to study the deformation theory of perturbed holomorphic maps from a singular curve. For this purpose, we have to estimate spectrum of certain linear elliptic operators. Unlike the case of Floer homology, the lowest eigenvalue is not uniformly bounded away from zero. This causes difficulties in proving injectivity and, in particular, surjectivity of the deformation map. In this section, we call k-point stable curves Deligne-Mumford stable curves (cf. section 4). As before, we denote by J a generic, ω-tamed almost complex structure on V, and by ν a generic inhomogeneous term on $\Sigma \times V$.

We recall a degeneration of stable curves is a holomorphic fibration $\pi : S \mapsto \Delta \subset C$ with sections $\sigma_1, \cdots, \sigma_k$ satisfying: (1) for $t \in \Delta$ and $t \neq 0$, the fiber $\Sigma_t = \pi^{-1}(t)$ is smooth; (2) for each t, $(\Sigma_t; \sigma_1(t), \cdots, \sigma_k(t))$ is a Deligne-Mumford stable curve. We fix an inhomogeneous term ν on $S \times V$. This is simply a smooth anti-(J, J_S)-linear section of the vector bundle $\mathrm{Hom}(TS, TV)$ over $S \times V$. Note that any inhomogeneous term on $\Sigma_0 \times V$ can be extented to $S \times V$.

Consider the moduli space of (J, ν_t)-perturbed holomorphic maps

$$(6.1) \quad \mathcal{M}_A(\Sigma_t, J, \nu_t) = \{f : \Sigma_t \to V \,|\, df + J \cdot df \cdot j_{\Sigma_t} = \nu_t \,,\; f_* \Sigma_t = A\} \,,$$

where ν_t is the restriction of ν to Σ_t, and A is a fixed homology class in $H_2(V, Z)$.

By the discussions in section 4, the moduli space $\mathcal{M}_A(\Sigma_0, J, \nu_0)$ is a smooth manifold. From the Riemann-Roch Theorem it follows that for a generic (J, ν), we have

$$\dim \mathcal{M}_A(\Sigma_t, J, \nu_t) = 2C_1(V) \cdot A + 2n(1 - g(\Sigma_t)),$$

where $2n$ is the real dimension of V, and $g(\Sigma_t)$ is the genus of Σ_t. By the Transversality Theorem in section 3, we may choose a pair (J, ν) such that $\mathcal{M}_A(\Sigma_0, J, \nu)$ is smooth, i.e., any f in $\mathcal{M}_A(\Sigma_0, J, \nu)$ is a regular (J, ν_0)-perturbed holomorphic map.

Theorem 6.1. *Let f_0 be any map in $\mathcal{M}_A(\Sigma_0, J, \nu_0)$. Then there is a continuous family of injective maps T_t from $\mathcal{W}$ into $\mathcal{M}_A(\Sigma_t, J, \nu_t)$, where t is small and $\mathcal{W}$ is a neighborhood of f_0 in $\mathcal{M}_A(\Sigma_0, J, \nu_0)$, such that* (1) *for any f in $\mathcal{W}$, as t goes to zero, $T_t(f)$ converges to f in C^0-topology on Σ_0 and in C^3-topology outside the singular set of Σ_0;* (2) *there are $\epsilon, \delta > 0$ satisfying : if f' is in $\mathcal{M}_A(\Sigma_t, J, \nu_t)$ and*

$$d_V(f'(x), f_0(y)) \leq \epsilon, \quad \textit{whenever } x \in \Sigma_t,$$

where d_V, d_S are the distance functions of a J-invariant metric h_V on V and a J_S-invariant metric h_S on S, then f' is in $T_t(\mathcal{W})$. Moreover, for each t, T_t is an orientation-preserving smooth map from $\mathcal{W}$ into $\mathcal{M}_A(\Sigma_t, J, \nu_t)$.

Before we prove this theorem, we give one corollary of it. Let $\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_l$ be pseudo-submanifolds in V representing integral homology classes. Each α_i or β_j is the image of a simplicial complex under a piecewise smooth map. We denote by α_i' or β_j' the regular part of α_i or β_j. We assume

$$\sum_{i=1}^{k}(2n - \dim \alpha_i) + \sum_{j=1}^{l}(2n - 2 - \dim \beta_j) = 2C_1(V) \cdot A + 2n(1 - g(\Sigma_t)).$$

Define the evaluation map

$$ev_t : \mathcal{M}_A(\Sigma_t, J, \nu_t) \times \Sigma_t^l \mapsto V^k \times V^l,$$

$$ev_t(f; y_1, \cdots, y_l) = (f(\sigma_1(t)), \cdots, f(\sigma_k(t)), f(y_1), \cdots, f(y_l)).$$

Note that $\mathcal{M}_A(\Sigma_t, J, \nu_t)$ has a canonical orientation (cf. section 1). If the image $\mathrm{Im}(ev_t)$ intersects the product $\prod_{i=1}^k \alpha_i' \times \prod_{j=1}^l \beta_j'$ transversally at a point $(f; y_1, \cdots, y_l)$, then we can assign a sign to $(f; y_1, \cdots, y_l)$ by using the orientations on $\mathcal{M}_A(\Sigma_t, J, \nu_t) \times \Sigma_t^l$, $V^k \times V^l$ and $\prod_{i=1}^k \alpha_i' \times \prod_{j=1}^l \beta_j'$.

By the discussions in sections 3-5, we have known that for a generic (J, ν), the map ev_0 intersects $\prod_{i=1}^k \alpha_i' \times \prod_{j=1}^l \beta_j'$ transversally at finitely many points in $\mathcal{M}_A(\Sigma_0, J, \nu_0) \times \Sigma_0^l$. Let $(f_0; y_{01}, \cdots, y_{0l})$ be one such intersection point.

Corollary 6.1. *Let (J, ν) be generic and $(f_0, y_{01}, \cdots, y_{0l})$ be as above. Then there are $\epsilon, \delta > 0$ such that for t sufficiently small, there is a unique point $(f_t; y_{t1}, \cdots, y_{tl})$ in the space $\mathcal{M}_A(\Sigma_t, J, \nu_t) \times \Sigma_t^l$ satisfying: (1) $d_S(y_{tj}, y_{0j}) \leq \epsilon$, where $1 \leq j \leq l$; (2) $f_t(\sigma_i(t)) \in \alpha_i'$, where $1 \leq i \leq k$; (3) $d_V(f_t(x), f_0(y)) \leq \epsilon$ whenever $d_S(x, y) \leq \delta$. Moreover, the sign associated to $(f_t; y_{t1}, \cdots, y_{tl})$ is the same as the sign associated to $(f_0; y_{01}, \cdots, y_{0l})$.*

The rest of this section is devoted to the proof of Theorem 6.1.

First we make a reduction as we did in section 3. Put $W = S \times V$. As in section 3, one can define a tamed almost complex structure J_W on W as follows: any tangent vector on W is of the form (u, v), where u

is in TS and v is in TV, and $J_W(u,v) = (J_S(u), J(v)+\nu(J_S(u)))$. Then any map f_t in $\mathcal{M}_A(\Sigma_t, J, \nu_t)$ can be converted into a J_W-holomorphic map F_t into by assigning x in Σ_t to $(x, f_t(x))$ in W. Let π_m be the projection from W onto its m^{th} factor. Then $\pi_1 \cdot F_t$ is an identity. Conversely, if we have a J_W-holomorphic map F_t into W such that $\pi_1 \cdot F_t$ is a biholomorphism, then $f_t = \pi_2 \cdot F_t \cdot (\pi_1 \cdot F_t)^{-1}$ is a (J, ν_t)-perturbed holomorphic map f_t. On the other hand, if F_0 comes from a (J, ν_0)-perturbed holomorphic map f_0, and F_t is a deformation of F_0, then $\pi_1 \cdot F_t$ is indeed a biholomorphism. Therefore, we may assume that $\nu \equiv 0$ and J is generic. Note that although W may be noncompact, the objects under study lie in a compact region.

In the following proof of Theorem 6.1, we will always use C to denote a uniform constant independent of t and f near f_0. The actual value of C may vary in different places.

Let f be a J-holomorphic map from Σ_0 into V and is very close to f_0. We will first construct an approximated J-holomorphic map from Σ_t into V for each small t. Let p be any double point of Σ_0, and U_p be a small neighborhood. We may assume that U_p is in a coordinate chart. Choose local coordinates z_{1p}, z_{2p} of S near p such that

$$U_p \cap \Sigma_t = \{z_{1p} z_{2p} = t \,|\, |z_{1p}|, |z_{2p}| < 1\}. \tag{6.2}$$

There is a coordinate system $(y_1, \dots, y_{2n})$ of V near $f(p)$, such that

$$\begin{aligned} J(\tfrac{\partial}{\partial y_i}) &= \tfrac{\partial}{\partial y_{n+i}} + \mathcal{O}(|y|), \\ J(\tfrac{\partial}{\partial y_{n+i}}) &= -\tfrac{\partial}{\partial y_i} + \mathcal{O}(|y|), \quad i = 1, 2, \dots, n, \end{aligned} \tag{6.3}$$

where $|y| = \sqrt{\sum_{i=1}^{2n} |y_i|^2}$. There are two connected components in $U_p \cap \Sigma_0$:

$$U_{p1} = \{z_{2p} = 0 || z_{1p}| < 1\}, \text{ and } U_{p2} = \{z_{1p} = 0 || z_{2p}| < 1\}.$$

Let f_{p1}, f_{p2} be the restrictions of f to U_{p1}, U_{p2}. Then we have the following expansions:

$$f_{pi}(z_{ip}) = \tilde{f}_{pi}(z_{ip}) + \text{ higher order terms },\tag{6.4}$$

where $\tilde{f}_{pi}$ is a homogeneous polynomial in z_{ip}. We identify a neighborhood of $f(p)$ in V with an open subset in C^n by putting $w_i =$

$y_i + \sqrt{-1}y_{n+i}$ $(i = 1, 2, \ldots n)$. By Theorem 5.10, we can assume that f_{p1} and f_{p2} intersect at p transversally. Then by choosing $y_1, \cdots, y_{2n}$ properly, we have

$$f_{p1}(z_{1p}) = (z_{1p}, 0, 0, \cdots, 0) + \mathcal{O}(|z_{1p}|^2) \in C^n, \tag{6.5}$$
$$f_{p2}(z_{2p}) = (0, z_{2p}, 0, \cdots, 0) + \mathcal{O}(|z_{2p}|^2) \in C^n.$$

By changing local coordinates $y_1, \ldots y_{2n}$, we may further assume

$$f_{p1}(z_{1p}) = (z_{1p}, 0, 0, \cdots, 0), \tag{6.6}$$
$$f_{p2}(z_{2p}) = (0, z_{2p}, 0, \cdots, 0).$$

We can construct an approximated J-holomorphic map $\tilde{f}_t : \Sigma_t \mapsto V$ for each t small as follows: Let ϕ_t be a smooth family of diffeomorphisms from Σ_0', where Σ_0' is the nonsingular part of Σ_0, into Σ_t, such that $\phi_0 = \text{id}$ and

$$\|\phi_t - id\|_{C^5(\Sigma_0 \setminus U')} \le C_{U'}|t| \tag{6.7}$$

for any small neighborhood U' of the singular set $\text{Sing}(\Sigma_0)$ in Σ_0, where $C_{U'}$ is a constant depending only on U', and the norm is taken with respect to the fixed metric h_S on S. Note that in (6.7), both ϕ and Id are considered as maps into S. Then for any p in $\text{Sing}(\Sigma_0)$, we have

$$\|\tilde{f}|_{\Sigma_t \cap U_p} - \tilde{f}|_{\Sigma_0 \cap U_p} \cdot \phi_t^{-1}\|_{C^4(\Sigma_t \cap \{\frac{1}{2} \le |z_{ip}| \le 1\,,\, i=1 \text{ or } 2\})} \le C|t|, \tag{6.8}$$

where $\tilde{f}$ is the map: $(z_{1p}, z_{2p}) \in U_p \to (Z_{1p}, z_{2p}, 0, \ldots, 0) \in V$. In this section, C always denotes a constant independent of t. Since

$$\tilde{f}|_{U_{p1}} = f_{p1}\,, \qquad \tilde{f}|_{U_{p2}} = f_{p2}, \tag{6.9}$$

by (6.8), there is a homotopy F_t on

$$\Sigma_t \cap \left(\bigcup_{p \in \text{Sing}(\Sigma_0)} \{\frac{1}{2} \le |z_{ip}| \le 1\,,\, i = 1 \text{ or } 2\} \right)$$

satisfying:

$$\|F_t - \tilde{f}|_{\Sigma_0 \cap U_p} \cdot \phi_t^{-1}\|_{C^4(\Sigma_t \cap \{\frac{1}{2} \le |z_{ip}| \le 1, i=1 \text{ or } 2\})} \le C|t|, \tag{6.10}$$

$$F_t = \tilde{f}|_{\Sigma_t \cap U_p} \quad \text{on } \Sigma_t \cap \{\frac{1}{2} \leq |z_{ip}| \leq \frac{7}{10}, i = 1 \text{ or } 2\}, \tag{6.11}$$

$$F_t = \tilde{f}|_{\Sigma_0 \cap U_p} \cdot \phi_t^{-1} \quad \text{on } \Sigma_t \cap \{\frac{9}{10} \leq |z_{ip}| \leq 1, i = 1 \text{ or } 2\}. \tag{6.12}$$

We define $\tilde{f}_t(x) = f \cdot \phi_t^{-1}(x)$ for X outside $\bigcup_{p \in \mathrm{Sing}(\Sigma_0)} U_p$, and

$$\tilde{f}_t(z_{1p}, z_{2p}) = \begin{cases} h_t(z_{1p}, z_{2p}) & \text{if } \frac{1}{2} \leq |z_{1p}| \leq 1, \\ (z_{1p}, z_{2p}, 0, \dots, 0) & \text{if } |z_{1p}| \leq \frac{1}{2}, |z_{2p}| \leq \frac{1}{2}, \\ h_t(z_{1p}, z_{2p}) & \text{if } \frac{1}{2} \leq |z_{2p}| \leq 1. \end{cases} \tag{6.13}$$

Then $\tilde{f}_t$ is a well-defined map on Σ_t and close to a J-holomorphic map on Σ_0. To examine the asymptotic behavior of $\tilde{f}_t$ more closely, we need to introduce metrics on Σ_t. Let μ be a Kähler metric on S which is flat in each U_p. If z_{1p}, z_{2p} are the local coordinates chosen above, then for t small,

$$\mu = (1 + \frac{|t|^2}{|z_{ip}|^4})|dz_{ip}|^2 \Sigma_t \cap U_p, \ i = 1, 2. \tag{6.14}$$

Let ρ be a smooth function on $S \setminus \mathrm{Sing}(\Sigma_0)$ satisfying:

$$\begin{aligned} &0 \leq |\rho| \leq 3, \\ &\rho|_{U_p}(z_{1p}, z_{2p}) = \sqrt{|z_{1p}|^2 + |z_{2p}|^2}. \end{aligned} \tag{6.15}$$

Clearly, $\rho^2(z_{1p}, z_{2p}) = |z_{ip}|^2(1 + \frac{|t|^2}{|z_{ip}|^4})$ on $\Sigma_t \cap U_p$ for $i = 1, 2$. Define

$$\mu_c = \rho^{-2} \mu. \tag{6.16}$$

Then for t small, the metric $\mu_c|_{\Sigma_t}$ is cylinder-like near each p. The following lemma can be easily proved by using (6.7)-(6.16).

Lemma 6.1. *We denote by D the covariant derivature of μ, and by D_c the covariant derivative of μ_c. Then for $1 \leq k \leq 5$,*

$$|D^k \tilde{f}_t|_\mu(x) \leq C_k \left(1 + \frac{|t|}{\rho(x)^{k+1}}\right), \tag{6.17}$$

or

$$|D_c^k \tilde{f}_t|_{\mu_c}(x) \le C_k' \left(\rho(x) + \frac{|t|}{\rho(x)}\right), \tag{6.18}$$

where $|\cdot|_\mu, |\cdot|_{\mu_c}$ *denote the norms with respect to* μ*,* μ_c*, and* C_k, C_k' *are constants depending only on the integer* $k > 0$.

Let J_0 be the standard complex structure on C^n. Near each double point p, $\tilde{f}_t$ is J_0-holomorphic on $\Sigma_t \cap \{|z_{ip}| < \frac{1}{2}, i = 1, 2\}$, i.e.,

$$d\tilde{f}_t + J_0 \cdot d\tilde{f}_t \cdot j_t \equiv 0 \text{ on } \Sigma_t \cap \{|z_{ip}| < \frac{1}{2}, i = 1, 2\},$$

where $j_t = j_{\Sigma_t}$. Put

$$v_t(x) = (d\tilde{f}_t + J \cdot d\tilde{f}_t \cdot j_t)(x). \tag{6.19}$$

Then in $\Sigma_t \cap \{|z_{ip}| < \frac{1}{2}, i = 1, 2\}$,

$$v_t(x) = (J - J_0) \cdot d\tilde{f}_t \cdot j_t(x). \tag{6.20}$$

Since $\tilde{f}$ is J-holomorphic on $\Sigma_0 \cap \{|z_{ip}| < \frac{1}{2},\ i = 1, 2\}$, we have

$$J(z_{1p}, 0, \cdots, 0)(\frac{\partial}{\partial z_{1p}}) = J_0(z_{1p}, 0, \cdots, 0)(\frac{\partial}{\partial z_{1p}})$$

and

$$J(0, z_{2p}, \cdots, 0)(\frac{\partial}{\partial z_{2p}}) = J(0, z_{2p}, \cdots, 0)(\frac{\partial}{\partial z_{2p}}).$$

Thus we can derive from last lemma,

Lemma 6.2. *For* $1 \le k \le 4$*, there are constants* $C_k, C_k' > 0$ *such that*

$$|D^k v_t|_\mu(x) \le C_k \frac{|t|}{\rho(x)^{k+1}} \tag{6.21}$$

or

$$|D_c^k v_t|_{\mu_c}(x) \le C_k'|t|. \tag{6.22}$$

We want to perturb $\tilde{f}_t$ into a J-holomorphic map from Σ_t into V. Let *exp* be the exponential map of the Hermitian metric h_V on V. Let

f_t be a map from Σ_t into V. If this map is sufficiently close to $\tilde{f}_t$, then we can write

$$f_t(x) = \exp_{\tilde{f}_t(x)}(u(\tilde{f}_t(x))), \tag{6.23}$$

where u is a vector field of $\tilde{f}_t^*TV$ on Σ_t. We need to find a vector field u such that f_t is J-holomorphic.

Let ∇ be the Levi-Civita connection of h_V. Since J may not be integrable, ∇ needs not to be J-linear, but we can construct a J-linear connection ∇^J from it as follows:

$$\nabla^J X = \frac{1}{2}(\nabla X - J\nabla(JX)), \; X \in TV. \tag{6.24}$$

Obviously, $\nabla^J(JX) = J(\nabla^J X)$, i.e., $\nabla^J J = 0$. For any vector field u of $\tilde{f}_t^*TV$ on Σ_t, we denote by $\pi_t(u,x)$ the parallel transport from $T_{f_t(x)}V$ to $T_{\tilde{f}_t(x)}V$ with respect to ∇^J along the path $\{ \exp_{\tilde{f}_t(x)}(s\, u(\tilde{f}_t(x)))\}_{0\le s\le 1}$, where f_t is defined in (6.23). Since ∇^J is J-linear, we have

$$J(\tilde{f}_t(x)) \cdot \pi_t(u,x) = \pi_t(u,x) \cdot J(f_t(x)). \tag{6.25}$$

Let $\wedge^{0,1}(\tilde{f}_t^*TV)$ be the bundle over Σ_t of all anti-(J,j_t)-linear homomorphisms from $T\Sigma_t$ into $\tilde{f}_t^*TV$, i.e.,

$$\begin{aligned} &\wedge^{0,1}(\tilde{f}_t^*TV)_x \\ &\quad = \left\{\theta \in \mathrm{Hom}(T_x\Sigma_t, (\tilde{f}_t^*TV)_x) \mid J(\tilde{f}_t(x)) \cdot \theta = -\theta \cdot j_t(x)\right\}. \end{aligned} \tag{6.26}$$

We will use $\Gamma(\wedge^{0,1}(\tilde{f}_t^*TV))$ and $\Gamma(\tilde{f}_t^*TV)$ to denote the spaces of the sections of $\wedge^{0,1}(\tilde{f}_t^*TV)$ and of $\tilde{f}_t^*TV$ over Σ_t. Define

$$\Phi_t : \Gamma(\tilde{f}_t^*TV) \mapsto \Gamma(\wedge^{0,1}(\tilde{f}_t^*TV)),$$

$$u \to \pi_t(u,\cdot) \cdot (df_t + J \cdot df_t \cdot j_t). \tag{6.27}$$

Note that the image of Φ_t is indeed in $\Gamma(\wedge^{0,1}(f_t^*TV))$, since for any $w \in T_x\Sigma_t$,

$$\begin{aligned} \Phi_t(u)(j_t(w)) &= \pi_*(u,x) \cdot (df_t \cdot j_t(w) + +J(f_t(x))(f_t(j_t^2(w)))) \\ &= -J(\tilde{f}_t(x)) \cdot \pi_t(u,x) \cdot (J(f_t(x)) \cdot df_t \cdot j_t(w) + df_t(w) \\ &= -J(\tilde{f}_t(x)) \cdot \Phi_t(u)(w). \end{aligned}$$

We put

$$L_t(\sigma) = D\Phi_t(0)(\sigma). \tag{6.28}$$

Lemma 6.3. *For any $\sigma \in \Gamma(\tilde{f}_t^* TV), w \in \Gamma(T\Sigma_t)$, we have*

$$L_t(\sigma)(w) = \nabla_w \sigma + J\nabla_{j_t w}\sigma + \tfrac{1}{2}\left\{(\nabla_\sigma J)(j_t w) - J(\nabla_\sigma J)(w)\right\}. \tag{6.29}$$

Here we identify $\Gamma(\tilde{f}_t^* TV)$ with $\Gamma(\tilde{f}_t(\Sigma_t), TV)$, so we may consider ∇ a covariant derivative on $\Gamma(\tilde{f}_t^* TV)$.

Proof. Define a map $\hat{f} : [0,1] \times \Sigma_t \mapsto V$ by

$$\hat{f}(s,x) = \exp_{\tilde{f}_t(x)}(s\,\sigma(\tilde{f}_t(x))).$$

Then $\hat{f}(0,x) = \tilde{f}_t(x)$ and $\frac{\partial \hat{f}}{\partial s}(0,x) = \sigma(\tilde{f}_t(x))$. Using the fact that $\pi_t(s\sigma, x)$ is the parallel transport of ∇^J along the path $\{\hat{f}(s',x)\}_{0\le s'\le s}$, we can compute

$$\begin{aligned} L_t(\sigma)(w) &= \frac{\partial \Phi_t(s\sigma)}{\partial s}|_{s=0}(w) \\ &= \lim_{s\to 0^+} \frac{1}{s}\{\pi_t(s\sigma,\cdot)(d\hat{f}(s,\cdot) + Jd\hat{f}(s,\cdot)j_t\} \\ &= \nabla^J_\sigma(\hat{f}_*(w)) + J\nabla^J_\sigma(\hat{f}_*(j_t w)). \end{aligned} \tag{6.30}$$

Since $[\sigma, \hat{f}_*(w)] = [\hat{f}_*(\frac{\partial}{\partial s}), \hat{f}_*(w)] = \hat{f}_*[\frac{\partial}{\partial s}, w] = 0$, we derive

$$\begin{aligned} \nabla^J_\sigma \hat{f}_*(w) &= \frac{1}{2}(\nabla_\sigma \hat{f}_*(w) - J\nabla_\sigma(J\hat{f}_*(w))) \\ &= \frac{1}{2}(\nabla_{\tilde{f}_{t*}(w)}\sigma - J(\nabla_\sigma J)(\tilde{f}_{t*}(w)) - J^2\nabla_\sigma \hat{f}_*(w)) \\ &= \nabla_w \sigma - \frac{1}{2}J(\nabla_\sigma J)(w). \end{aligned} \tag{6.31}$$

Similarly,

$$\nabla^J_\sigma(\hat{f}_*(j_t w)) = \nabla_{j_t w}\sigma - \frac{1}{2}J(\nabla_\sigma J)(j_t w).$$

Thus (6.29) follows from this, (6.30) and (6.31).

Let $L_t^* : \Gamma(\wedge^{0,1}(\tilde{f}_t^*TV)) \to \Gamma(\tilde{f}_t^*TV)$ be the adjoint of L_t with respect to the Hermitian metric h_V on V and the metric μ_c on Σ_t.

Lemma 6.4. *Assume that $\langle\cdot,\cdot\rangle$ is the induced metric on $\tilde{f}_t^*TV$ by h_V. For any section ξ in $\Gamma(\wedge^{0,1}(\tilde{f}_t^*TV))$,*

$$L_t^*(\xi) = -2\nabla_e(\xi(e)) - 2\nabla_{j_te}(\xi(j_te)) + \langle\xi(e), (\nabla J)(j_te)\rangle^* \\ -\langle\xi(j_te), (\nabla J)(e)\rangle^*,$$

where $\{e, j_te\}$ is any local unitary basis of $T\Sigma_t$ with respect to μ_c, and $\langle X, \nabla Y\rangle^*$ is the vector field in $\tilde{f}_t^*TV$ defined by

$$\langle\langle X, \nabla Y\rangle^*, Z\rangle = \langle X, \nabla_Z Y\rangle,$$

where $X, Y, Z \in \tilde{f}_t^*TV$.

Proof. Fix a local unitary basis $\{e, j_te\}$ of $T\Sigma_t$, for convenience, we write $e_1 = e, e_2 = j_te$. Let $\{e_1^*, e_2^*\}$ be the dual basis of $\{e_1, e_2\}$,. Then $e_2^* = -j_te_1^*$. Write

$$\xi = \xi_1 e_1^* + \xi_2 e_2^*\,,\ \xi_i \in \tilde{f}_t^*TV,\ i = 1, 2.$$

Since ξ is anti-(J, j_t)-linear, we have

$$\xi_2 = -J\xi_1,$$

so

$$\xi = \xi_1 e_1^* - (J\xi_1)e_2^*.$$

Let σ be a section of $\tilde{f}_t^*TV$ with support in a small open subset. We denote by $\langle\cdot,\cdot\rangle_{\mu_c}$ the induced metric on $\wedge^{0,1}(\tilde{f}_t^*TV)$ by h_V and μ_c. Then

$$\begin{aligned}\int_{\Sigma_t}\langle L_t^*\xi, \sigma\rangle d\mu_c &= \int_{\Sigma_t}\langle\xi, L_t(\sigma)\rangle_{\mu_c} d\mu_c \\ &= \int_{\Sigma_t}(\langle\xi_1, L_t(\sigma)(e_1)\rangle + \langle\xi_2, L_t(\sigma)(e_2)\rangle)d\mu_c \\ &= 2\int_{\Sigma_t}\langle\xi_1, \nabla_e\sigma + J\nabla_{j_te}\sigma + \frac{1}{2}(\nabla_\sigma J)(j_te) - \frac{1}{2}J(\nabla_\sigma J)(e)\rangle d\mu_c \\ &= -2\sum_{i=1}^{2}\int_{\Sigma_t}(\langle\nabla_{e_i}\xi_i, \sigma\rangle - \frac{1}{2}\langle\xi_i, (\nabla_\sigma J)(j_te_i)\rangle)d\mu_c.\end{aligned}$$

Thus the lemma follows.

Remark. If ξ is a section in $\Gamma(\wedge^{0,1}(\tilde{f}_t^*TV))$ and $r(\rho)$ is a positive function in ρ, then

$$
\begin{aligned}
&\int_{\Sigma_t} r(\rho)^2 \langle \sum_{i=1}^{2} \nabla^J_{e_i}\xi_i, \sum_{i=1}^{2} \nabla^J_{e_i}\xi_i\rangle d\mu_c \\
&= \int_{\Sigma_t} r(\rho)^2 \langle \nabla^J_{e_1}\xi_1 + \nabla^J_{e_2}\xi_2 \,, \nabla^J_{e_1}\xi_1 + \nabla^J_{e_2}\xi_2\rangle d\mu_c \\
&= \int_{\Sigma_t} r(\rho)^2 \langle J\nabla^J_{e_1}\xi_2 - J\nabla^J_{e_2}\xi_1 \,, J\nabla^J_{e_1}\xi_2 - J\nabla^J_{e_2}\xi_1\rangle d\mu_c \\
&= \int_{\Sigma_t} r(\rho)^2 \langle \nabla^J_{e_1}\xi_2 - \nabla^J_{e_2}\xi_1 \,, \nabla^J_{e_1}\xi_2 - \nabla^J_{e_2}\xi_1\rangle d\mu_c \\
&= \int_{\Sigma_t} r(\rho)^2 \left(\langle \nabla^J_{e_1}\xi_2 \,, \nabla^J_{e_1}\xi_2\rangle + \langle \nabla^J_{e_2}\xi_1 \,, \nabla^J_{e_2}\xi_1\rangle - 2\langle \nabla^J_{e_2}\xi_1 \,, \nabla^J_{e_1}\xi_2\rangle\right) d\mu_c \\
&= \int_{\Sigma_t} r(\rho)^2 \left(\langle \nabla^J_{e_1}\xi_2 \,, \nabla^J_{e_1}\xi_2\rangle + \langle \nabla^J_{e_2}\xi_1 \,, \nabla^J_{e_2}\xi_1\rangle - 2\langle \nabla^J_{e_1}\xi_1 \,, \nabla^J_{e_2}\xi_2\rangle\right. \\
&\qquad \left. - 2\langle \xi_1, (\nabla^J_{e_2}\nabla^J_{e_1} - \nabla^J_{e_1}\nabla^J_{e_2})\xi_2\rangle\right) d\mu_c \\
&\qquad +4\int_{\Sigma_t} r(\rho) \left(\nabla_{e_1} r \,\langle \xi_1 \,, \nabla^J_{e_2}\xi_2\rangle - \nabla_{e_2} r \,\langle \xi_1 \,, \nabla^J_{e_1}\xi_2\rangle\right) d\mu_c,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\int_{\Sigma_t} r(\rho)^2 \langle \sum_{i=1}^{2} \nabla^J_{e_i}\xi_i \,, \sum_{i=1}^{2} \nabla^J_{e_i}\xi_i\rangle d\mu_c \\
&= \frac{1}{2}\int_{\Sigma_t} r(\rho)^2 \left(|\nabla^J\xi|^2 - 2\langle \xi_1, [\nabla^J_{e_2}, \nabla^J_{e_1}]\xi_2\rangle\right) d\mu_c \\
&\quad +2\int_{\Sigma_t} r(\rho) \left(\nabla_{e_1} r\langle \xi_1 \,, \nabla^J_{e_2}\xi_2\rangle - \nabla_{e_2} r\langle \xi_1 \,, \nabla^J_{e_1}\xi_2\rangle\right) d\mu_c.
\end{aligned}
$$

Both $\nabla - \nabla^J$ and $[\nabla^J_{e_2} \,, \nabla^J_{e_1}]$ are zero-order operators, the coefficients of which depend only on the curvature tensor of h_V, μ_c, the almost complex structure J and the gradient of the map $\tilde{f}_t$. Therefore,

(6.32)

$$
\begin{aligned}
\int_{\Sigma_t} r(\rho)^2 |L_t^*\xi|^2 d\mu_c \geq & \int_{\Sigma_t} r(\rho)^2 |\nabla\xi|^2_{\mu_c} d\mu_c \\
& -C\int_{\Sigma_t} \left(r(\rho)^2\, |d\tilde{f}_t|^2_{\mu_c} |\xi|^2_{\mu_c} + |r'(\rho)|^2 |\xi|^2_{\mu_c}\right) d\mu_c.
\end{aligned}
$$

We will apply the Implicit Function Theorem to construct the map T_t. First we need to estimate the lower bound of the spectrum of $\Box_t = L_t L_t^*$.

Lemma 6.5. *Let D be a disk in $\mathbf{C}$ and μ_c be the cylindrical metric* *on* $D\backslash\{0\}$*, i.e.,* $\mu_c = \frac{|dz|^2}{|z|^2}$*. Suppose that* $f_D : D \to V$ *is a* J*-holomorphic map, and* ξ *is a* C^2*-smooth section of* $\wedge^{0,1}(f_D^* TV)$ *over* $D\backslash\{0\}$ *satisfying:*

$$L_0^* \xi = 0 \quad on \quad D\backslash\{0\}, \tag{6.33}$$

where L_0^* *be the adjoint of* L_0 *with respect to* μ_c*, and*

$$\int_{D\backslash\{0\}} \left(|\nabla \xi|^2_{\mu_c} + |z|^2 |\xi|^2 \right) d\mu_c < \infty. \tag{6.34}$$

Then the limit $\lim_{z\to 0} \xi(z)$ *exists and is a vector in* $T_{f_D(0)}V$*. Such a limit is called residue of* ξ *at* $z = 0$*.*

Proof. Without loss of generality, we may identify $f_D^* TV$ with $D \times \mathbb{R}^n$. The almost complex structure J on V becomes a family of complex structures $J(z)$ on $\mathbb{R}^n$, parametrized by z in D. Put $\tau = -\log |z|$. Then

$$\mu_c \;=\; d\tau^2 + d\theta^2.$$

By Lemma 6.4,

$$L_0^* \xi = -2\frac{\partial \xi^1}{\partial \tau} - 2J\frac{\partial \xi^1}{\partial \theta} + \mathcal{O}(e^{-\tau}|\xi|) = 0, \tag{6.35}$$

where $\xi = \xi^1 d\tau + \xi^2 d\theta$, $\xi^2 = -J\xi^1$. Thus

$$\frac{\partial}{\partial \tau}\left(\frac{1}{2\pi}\int_0^{2\pi} \xi d\theta \right) = \mathcal{O}(e^{-\tau}|\xi|). \tag{6.36}$$

Put $\tilde{\xi} = \xi - \frac{1}{2\pi}\int_0^{2\pi} \xi d\theta$. By the Poincaré inequality on the unit circle, we have

$$\int_0^{2\pi} |\nabla \xi|^2_{\mu_c} d\theta \geq \int_0^{2\pi} |\frac{\partial}{\partial \theta}\tilde{\xi}|^2_{\mu_c} d\theta \geq \int_0^{2\pi} |\tilde{\xi}|^2_{\mu_c} d\theta,$$

From (6.35) and (6.36) it follows that

$$L_0^* \tilde{\xi} = \mathcal{O}(e^{-\tau} | \int_0^{2\pi} \xi^1 d\theta|).$$

By the standard elliptic estimates, one can show

$$\begin{aligned} \sup_{0\leq\theta\leq 2\pi} |\tilde{\xi}|^2(\tau,\theta) &\leq C \int_{|\tau'-\tau|\leq 1}\int_0^{2\pi} \left(|\tilde{\xi}|^2 + e^{-2\tau}|\xi|^2 \right) d\tau' d\theta \\ &\leq C \int_{|\tau'-\tau|\leq 1}\int_0^{2\pi} (|\nabla \xi|^2 + e^{-2\tau}|\xi|^2) \, d\tau' d\theta, \end{aligned} \tag{6.37}$$

where C denotes a uniform constant. Together with (6.34), this implies that $\tilde{\xi}(\tau,\theta)$ converges to zero as τ goes to infinity. Integrating equation (6.36), we obtain that

$$\frac{1}{2\pi}\int_0^{2\pi}\xi(\tau,\theta)d\theta - \frac{1}{2\pi}\int_0^{2\pi}\xi(\tau_0,\theta)d\theta = \mathcal{O}(\int_{\tau_0}^{\tau} e^{-\tau'}|\xi|(\tau',\theta)d\tau'),$$

so that $\frac{1}{2\pi}\int_0^{2\pi}\xi d\theta$ converges to a vector in $T_{f_D(0)}V$. Hence the lemma is proved.

Recall that L_0 is the linearization of the Cauchy-Riemann equation at f_0, and L_0^* is the adjoint of L_0 on $\Sigma_0\setminus\mathrm{Sing}(\Sigma_0)$, where $\mathrm{Sing}(\Sigma_0)$ is the set of nodes in Σ_0. We denote by $\mathrm{Ker}(L_0^*)$ the set of those sections ξ of $\Lambda^{0,1}f_0^*TV$ over $\Sigma_0\setminus\mathrm{Sing}(\Sigma_0)$ satisfying:

$$\int_{\Sigma_0}\left(|\nabla\xi|^2_{\mu_c}+\rho^2|\xi|^2\right)d\mu_c<\infty, \tag{6.38}$$

and for every node p in $\mathrm{Sing}(\Sigma_0)$, if U_p is any small neighborhood of p with two irreducible components U_{p1}, U_{p2}, then

$$\lim_{\substack{z\to p\\ z\in U_{p1}}}\xi(z) \;+\; \lim_{\substack{z\to p\\ z\in U_{p2}}}\xi(z) \;=\; 0, \tag{6.39}$$

namely, the two residues of ξ at p sum up to zero.

Proposition 6.1. *For a generic pair* (J,ν), $\mathrm{Ker}(L_0^*)$ *is trivial.*

Proof. The proof is simply an application of the Sard-Smale Transversality Theorem. We will outline a proof here. Let Σ_{0i} ($1\le i\le k$) be the irreducible components of Σ_0, and l_i be the number of nodes in Σ_{0i}. Then $\sum_{i=1}^k l_i$ is twice of the number l of nodes in Σ_0. Without loss of generality, we may assume that $C_1(V)(f_{0*}(\Sigma_{0i}))+n(1-g_i)-l_in$ is nonnegative for $i\le k'$ and negative for $i>k'$, where $1\le k'\le k$ and g_i is the genus of Σ_{0i}. Put $\Sigma_{0i}^0=\Sigma_{0i}\setminus\mathrm{Sing}(\Sigma_0)$, and $L_{0i}^*=L_0^*|_{\Sigma_{0i}^0}$. Define two spaces

$$\begin{aligned}
&\mathcal{H}^1_\rho(\Sigma_{0i}^0,\Lambda^{0,1}f_0^*TV)\\
&\quad = \{\xi\in\Gamma(\Lambda^{0,1}(f_0^*TV))\mid \int_{\Sigma_{0i}^0}\left(|\nabla\xi|^2_{\mu_c}+\rho^2|\xi|^2\right)d\mu_c<\infty\},\\
&\mathcal{L}^2(\Sigma_{0i}^0,f_0^*TV) \;=\; \{u\in\Gamma(f_0^*TV)\mid \int_{\Sigma_{0i}^0}|u|^2d\mu_c<\infty\}.
\end{aligned}$$

Then L_{0i}^* is a Fredholm map from $\mathcal{H}^1_\rho(\Sigma_{0i}^0,\Lambda^{0,1}f_0^*TV)$ into $\mathcal{L}^2(\Sigma_{0i}^0,f_0^*TV)$. By Lemma 6.5, one can compute the index of L_{0i}^*:

$$\mathrm{Ind}(L_{0i}^*) \;=\; -2C_1(V)(f_{0*}(\Sigma_{0i}))-2n(1-g_i)+2l_in. \tag{6.40}$$

Therefore, $\mathrm{Ind}(L_{0i}^*) \leq 0$ for $i \leq k'$ and $\mathrm{Ind}(L_{0i}^*) > 0$ for $i > k'$. Using the Sard-Smale Transversality Theorem, one can show that for a generic (J, ν), $\mathrm{Ker}(L_{0i}^*)$ is trivial for $i \leq k'$ and of real dimension $\mathrm{Ind}(L_{0i}^*)$ for $i > k'$. It follows

$$\begin{aligned} \dim\left(\prod_{i=1}^{k} \mathrm{Ker}(L_{0i}^*)\right) \\ = \sum_{i>k'} (-2C_1(V)(f_{0*}(\Sigma_{0i})) - 2n(1-g_i) + 2l_i n)\,. \end{aligned} \tag{6.41}$$

On the other hand, since $\mathcal{M}_A(\Sigma_0, J, \nu)$ is nonempty for a generic (J, ν), we have

$$\sum_{i>k'} (2C_1(V)(f_{0*}(\Sigma_{0i})) + 2n(1-g_i) - l_i' n) \geq 0, \tag{6.42}$$

where l_i' is the number of nodes in Σ_{0i}, which are not in any Σ_{0j} for $j \leq k'$. Clearly,

$$\sum_{i>k'} (l_i - l_i') = \sum_{i \leq k'} l_i.$$

Therefore, from (6.41) and (6.42) we deduce

$$\dim\left(\prod_{i=1}^{k} \mathrm{Ker}(L_{0i}^*)\right) \leq \sum_{i>k'} (2l_i - l_i')n = 2ln. \tag{6.43}$$

Given each node p, let $\Sigma_{0i(p)}$ and $\Sigma_{0i'(p)}$ be the two components of Σ_0 containing p. We assume that $i(p) < i'(p)$. Define a residue map

$$\mathrm{Res} : \prod_{i=1}^{k} \mathrm{Ker}(L_{0i}^*) \mapsto \prod_{p \in \mathrm{Sing}(\Sigma_0)} T_{f_0(p)}V \times T_{f_0(p)}V$$

by $\mathrm{Res}(\xi) = \prod_p (\lim_{z\to p} \xi|_{\Sigma_{0i(p)}}, \lim_{z \to p} \xi|_{\Sigma_{0i'(p)}})$. Then Res is a continuous linear map. We also define a diagonal

$$\Delta = \prod_{p \in \mathrm{Sing}(\Sigma_0)} \{(v, -v) | v \in T_{f_0(p)}V\} \prod_{p \in \mathrm{Sing}(\Sigma_0)} T_{f_0(p)}V \times T_{f_0(p)}V.$$

Clearly, Δ is a linear subspace of real dimension $2ln$, and $\mathrm{Ker}(L_0^*) =$

$\mathrm{Res}^{-1}(\Delta)$. For a generic (J,ν), the map Res is transversal to Δ, so

$$\begin{aligned}\dim \operatorname{Ker}(L_0^*) \le \dim\left(\prod_{i=1}^{k}\operatorname{Ker}(L_{0i}^*)\right) + \dim\Delta \\ - \dim\left(\prod_{p\in \mathrm{Sing}(\Sigma_0)} T_{f_0(p)}V \times T_{f_0(p)}V\right) \le 0.\end{aligned}$$

Hence the proposition is proved.

From now on, we will fix a generic (J,ν) such that $\operatorname{Ker}(L_0^*)$ is trivial, its existence is assured by Proposition 6.1.

Lemma 6.6. *There is a constant $c > 0$, independent of t, such that for t sufficiently small, the first eigenvalue $\lambda_1(\Box_t)$ of $\Box_t$ is bounded from below by $\frac{c}{(\log t)^2}$.*

Proof. We prove it by contradiction. We will always use C, c to denote uniform positive constants. Suppose that the lemma is not true. Then without loss of generality, we may assume that $(\log t)^2\lambda_1(\Box_t)$ converges to zero as t tends to zero. Let ξ_t be the eigenfunction of $\lambda_1(\Box_t)$ satisfying

$$\sup_{\Sigma_t} |\xi_t|_{\mu_c} = 1. \tag{6.44}$$

Then

$$\int_{\Sigma_t} |\xi_t|^2_{\mu_c} d\mu_c \le C(-\log|t|) \tag{6.45}$$

and

$$\int_{\Sigma_t} \rho^2 \, |\xi_t|^2_{\mu_c} d\mu_c \le C, \tag{6.46}$$

where ρ is the function defined in (6.15). Using $\Box_t\xi_t = \lambda_1(\Box_t)\xi_t$, we deduce from (6.45),

$$(\log|t|)^2 \int_{\Sigma_t} |L_t^*\xi_t|^2 d\mu_c = (\log|t|)^2\lambda_1(\Box_t) \to 0 \quad \text{as} \quad t\to 0. \tag{6.47}$$

It follows from the definition of $\tilde{f}_t$ that

$$\max\{|d\tilde{f}_t|_{\mu_c}(x),\ |\nabla d\tilde{f}_t|_{\mu_c}(x)\}. \tag{6.48}$$

Therefore, replacing $|d\tilde{f}_t|_{\mu_c}$ by $C\rho(x)$ in (6.32) and using (6.46), (6.47), we obtain

$$\int_{\Sigma_t} |\nabla \xi_t|^2_{\mu_c} d\mu_c \le C. \tag{6.49}$$

By using the standard elliptic estimates, from (6.44) and (6.49) one can deduce

$$\sup_{\Sigma_t} |\nabla^i \xi_t|_{\mu_c} \le C \quad \text{for } i = 0, 1, 2, 3. \tag{6.50}$$

Thus by taking a subsequence if necessary, we may assume that ξ_t converges to a smooth section ξ_0 of $\wedge^{0,1}(f^*TV)$ over $\Sigma_0 \backslash \mathrm{Sing}(\Sigma_0)$, satisfying $L_0^*\xi = 0$ and

$$\int_{\Sigma_0} \left(|\nabla \xi_0|^2_{\mu_c} + \rho^2 |\xi_0|^2_{\mu_c} \right) d\mu_c < \infty.$$

In order to derive a contradiction, we will show that ξ_0 is a nonzero section in $\mathrm{Ker}(L_0^*)$. Note that at this moment, we do not even know if ξ_0 is nonzero. Let p be a given node of Σ_0, and $U_{p\delta} = \{|z_{1p}| \le \delta, |z_{2p}| \le \delta\}$ for $\delta > 0$. On $\Sigma_t \cap U_{p\frac{1}{2}}$, we can choose $z = z_{1p}$ as a local coordinate; then $2|t| \le |z| \le \frac{1}{2}$, and

$$\mu_c = (|z|^2 + \frac{|t|^2}{|z|^2})^{-1}(1 + \frac{|t|^2}{|z|^4})|dz|^2 = d\tau^2 + d\theta^2,$$

where $\tau = -\log|z| + \frac{1}{2}\log|t|$.

Without loss of generality, we may assume that $\tilde{f}_t^* TV$ is a trivial bundle over $\Sigma_t \cap U_{p\frac{1}{2}}$. We define

$$\xi_{t,ave}(\tau) = \frac{1}{2\pi} \int_0^{2\pi} \xi_t(\tau, \theta) d\theta. \tag{6.51}$$

Then for any $s > 0$,

$$\begin{aligned} \lim_{t\to 0} \xi_{t,ave}(\frac{1}{2}\log|t| + s) &= \int_{|z|=e^{-s}} \xi_0(z, 0) d\theta, \\ \lim_{t\to 0} \xi_{t,ave}(-\frac{1}{2}\log|t| - s) &= -\int_{|z|=e^{-s}} \xi_0(0, z) d\theta. \end{aligned} \tag{6.52}$$

Put $e_1 = \frac{\partial}{\partial \tau}$, $e_2 = \frac{\partial}{\partial \theta}$ on $\Sigma_t \cap U_{p\frac{1}{2}}$. Then by Lemmas 6.3, 6.4,

$$(6.53)\quad L_t L_t^* \xi_t = -2\frac{\partial^2}{\partial \tau^2}\xi_t - 2\frac{\partial^2}{\partial \theta^2}\xi_t + \mathcal{O}(|\nabla d\tilde{f}_t|_{\mu_c}|\xi_t|_{\mu_c} + |d\tilde{f}_t|_{\mu_c}|\nabla \xi_t|_{\mu_c}),$$

where $\mathcal{O}(A)$ denotes a quantity bounded by $C\,A$. By (6.15) and (6.48), we have

$$\max\{|d\tilde{f}_t|^2_{\mu_c}, |\nabla d\tilde{f}_t|^2_{\mu_c}\}(x) \leq C|t|(e^{2\tau} + e^{-2\tau}) \text{ on } \Sigma_t \cap U_{p\frac{1}{2}},$$

where $x = (\tau, \theta)$. So from (6.53) and (6.50) it follows

$$L_t L_t^* \xi_t = -2\frac{\partial^2}{\partial \tau^2}\xi_t - 2\frac{\partial}{\partial \theta}\xi_t + \mathcal{O}(|t|(e^{2\tau} + e^{-2\tau})) = \lambda_1(\square_t)\xi_t.$$

Integrating this over $\theta \in [0, 2\pi]$, yields

$$(6.54)\qquad \frac{\partial^2}{\partial \tau^2}\xi_{t,ave} + \lambda_t^2 \xi_{t,ave} = a_t,$$

where $\lambda_t = \sqrt{\frac{\lambda_1(\square_t)}{2}}$ and $a_t(\tau) = \mathcal{O}(|t|(e^{2\tau} + e^{-2\tau}))$. It is an easy exercise in ODE to show that the solution of (6.54) is of the form

$$(6.55)\qquad \xi_{t,ave} = \{\alpha_t^i \sin(\pm\lambda_t \tau + \beta_t^i)\} + \int_0^\tau a_t(\tau)\frac{\sin(\lambda_t(\tau - s))}{\lambda_t}ds,$$

where α_t^i, β_t^i are constants, $|\beta_t^i| \leq \frac{\pi}{2}$ and $i = 1, \cdots, 2n$. Since $|\tau| \leq -\frac{1}{2}\log|t|$ and $\lambda_1(\square_t)(\log|t|)^2 \to 0$ as $t \to 0$, we have

$$\frac{|\sin(\lambda_t(\tau - s))|}{\lambda_t} \leq |\tau - s|,$$

and therefore

$$(6.56)\qquad \left|\int_0^\tau a_t(s)\frac{\sin(\lambda_t(\tau - s))}{\lambda_t}ds\right| \leq C|t|e^{2|\tau|}$$

and

$$(6.57)\qquad \left|\frac{\partial}{\partial \tau}\int_0^\tau a_t(s)\frac{\sin(\lambda_t(\tau - s))}{\lambda_t}ds\right| \leq C|t|e^{2|\tau|}.$$

From (6.44), (6.56) it follows that $|\alpha_t^i \sin(\pm\lambda_t \tau + \beta_t^i)| \leq C$ for all i, in particular, $|\alpha_t^i \sin \beta_t^i| \leq C$ for all i. By taking a subsequence if necessary, we may assume that for each i, $\alpha_t^i \sin \beta_t^i$ converges to γ^i as t tends to zero.

Claim. *For each i between 1 and $2n$, $\alpha_t^i \sin(\pm\lambda_t \tau + \beta_t^i)$ converges uniformly to γ^i as t tends to zero, where $|\tau| \le -\frac{1}{2}\log|t| - \log 2$.*

Proof. There are two cases: (1) $|\beta_t^i| \ge c > 0$ for all t; (2) $\beta_{t_j}^i$ converges to zero as t_j tends to zero, where $\{t_j\}$ is a subsequence of $\{t\}$. In the first case, $|\alpha_t^i| \le C$. Since $|\tau| \le -\log|t|$, by our assumption on $\lambda_1(\Box_t)$, $\lambda_t\,\tau$ converges uniformly to zero, so $\alpha_t^i \sin(\pm\lambda_t\tau + \beta_t^i)$ converges uniformly to γ^i as t tends to zero.

Let us consider the second case. For simplicity, we assume that $\{t_j\} = \{t\}$. Put

$$\tilde{\xi}_t = \xi_t - \xi_{t,ave}.$$

Then

$$\int_0^{2\pi} \tilde{\xi}_t(\tau,\theta)d\theta = 0.$$

Using this and Lemma 6.4, we deduce

$$\begin{aligned}&\int_0^{2\pi} |L_t^*\xi_t|^2 d\theta \\ &\quad = \int_0^{2\pi} \left(|L_t^*\xi_{t,ave}|^2 + |L_t^*\tilde{\xi}_t|^2 + \mathcal{O}(|t|(e^{2\tau} + e^{-2\tau})|\xi_t|^2_{\mu_c})\right) d\theta,\end{aligned} \tag{6.58}$$

and therefore, in consequence of (6.44),

$$\int_{|\tau| \le -\frac{1}{4}\log|t|} |L_t^*\xi_{t,ave}|^2 d\tau \le C\left(-\lambda_t^2 \log|t| + \sqrt{|t|}\right) \tag{6.59}$$

On the other hand, by Lemma 6.4,

$$L_t^*\xi_{t,ave} = -2\frac{\partial}{\partial\tau}\xi_{t,ave} + \mathcal{O}(|t|(e^{2\tau} + e^{-2\tau})|\xi_t|_{\mu_c}),$$

which together with (6.56), (6.57) and (6.59) yields

$$\begin{aligned}&(\alpha_t^i\lambda_t)^2 \int_{|\tau| \le -\frac{1}{4}\log|t|} |\cos(\pm\lambda_t\tau + \beta_t^i)|^2 d\tau \\ &\quad = \int_{|\tau| \le -\frac{1}{4}\log|t|} |\frac{\partial}{\partial\tau}\left(\alpha_t^i \sin(\pm\lambda_t\tau + \beta_t^i)\right)|^2 d\tau \\ &\quad \le C\left(-\lambda_t^2\log|t| + \sqrt{|t|}\right).\end{aligned} \tag{6.60}$$

Since both $\lambda_t \log|t|$ and β_t^i converge to zero, (6.60) implies that $\alpha_t^i \lambda_t \log|t|$ converges uniformly to zero as t tends to zero, so $\alpha_t^i \sin(\pm\lambda_t\tau + \beta_t^i)$ converges uniformly to γ^i. Hence the claim is proved.

By using (6.52), (6.55) and (6.56), we can derive from the above claim that for any $s > 0$,

$$\int_{|z|=e^{-s}} \xi_0(z,0)d\theta = \{\gamma^i\} + O(e^{-s}) = \int_{|z|=e^{-s}} \xi_0(0,z)d\theta,$$

in particular, the two residues of ξ_0 at p sum up to zero. To prove that ξ_0 is nonzero, we choose a node p for each t such that

$$\sup_{\Sigma_t \cap U_{p\frac{1}{2}}} |\xi_t| \geq c > 0. \tag{6.61}$$

We may assume that p is independent of t. From (6.58) it follows

$$\begin{aligned} &\int_{\Sigma_t \cap U_{p\frac{1}{2}}} |L_t^* \tilde{\xi}_t|_{\mu_c}^2 d\mu_c \\ &\leq C\left(\int_{\Sigma_t \cap U_{p\frac{1}{2}}} \rho^2(\tau,\theta)|\xi_t|_{\mu_c}^2 d\mu_c - \lambda_1(\Box_t)\log|t|\right), \end{aligned} \tag{6.62}$$

where $\tilde{\xi}_t = \xi_t - \xi_{t,ave}$. By (6.32) and the Poincaré inequality on S^1, we derive from (6.62),

$$\begin{aligned} &\int_{\Sigma_t \cap U_{p\frac{1}{2}}} |\tilde{\xi}_t|_{\mu_c}^2 d\mu_c \\ &\quad\leq \int_{\Sigma_t \cap U_{p\frac{1}{2}}} |\nabla \tilde{\xi}_t|_{\mu_c}^2 d\mu_c \\ &\quad\leq \int_{\Sigma_t \cap U_{p\frac{1}{2}}} |L_t^* \tilde{\xi}_t|_{\mu_c}^2 d\mu_c + C\int_{\Sigma_t \cap (U_{p\frac{1}{2}} \setminus U_{p\frac{1}{4}})} |\tilde{\xi}_t|_{\mu_c}^2 d\mu_c \\ &\quad\leq C\left(\int_{\Sigma_t \cap U_{p\frac{1}{2}}} \rho^2(\tau,\theta)|\xi_t|_{\mu_c}^2 d\mu_c - \lambda_1(\Box_t)\log|t|\right. \\ &\qquad\qquad \left. + \int_{\Sigma_t \cap (U_{p\frac{1}{2}} \setminus U_{p\frac{1}{4}})} |\tilde{\xi}_t|_{\mu_c}^2 d\mu_c\right). \end{aligned} \tag{6.63}$$

If $\xi_0 = 0$, then by (6.44), (6.63) and the above claim, the integral $\int_0^{2\pi}\int_{|\tau-\tau_0|\leq 1} |\xi_t|_{\mu_c}^2 d\tau d\theta$ converges to zero as t tends to zero, where $|\tau_0| \leq$

$-\frac{1}{2}\log|t| - \log 2$. Applying the Mean-Value Inequality to $L_t L_t^* \xi_t = \lambda_1(\Box_t)\xi_t$, we deduce that ξ_t converges to zero on $U_{p\frac{1}{2}}$, which contradicts to (6.61). Therefore, we obtain an nonzero section ξ_0 in $\mathrm{Ker}(L_0^*)$. This is impossible! Hence the lemma is proved.

We denote by Φ_t the map from $\Gamma(\tilde{f}_t^* TV)$ into $\Gamma(\wedge^{0,1}(\tilde{f}_t^* TV))$ defined in (6.27). Define

$$\Psi_t(\xi) = \Phi_t(L_t^* \xi). \tag{6.64}$$

To find a J-holomorphic map f_t of the form (6.64), it suffices to show that $\Psi_t(\xi)$ has a zero ξ. Let v_t be given in (6.19). Then we can expand $\Psi_t(\xi)$ in ξ as follows:

$$\begin{aligned} &\Psi_t(f, 0) = v_t, \\ &\Psi_t(f, \xi) = v_t + L_t L_t^*(\xi) + H_t(\xi), \end{aligned} \tag{6.65}$$

where $H_t(u, \lambda\xi) = \mathcal{O}(\lambda^2)$ is the higher order term in ξ. We denote by $\|\cdot\|_{k,\frac{1}{2}}$ the $C^{k,\frac{1}{2}}$ - Hölder norm on either $\Gamma(\wedge^{0,1}(\tilde{f}_t^* TV))$ or $\Gamma(\tilde{f}_t^* TV)$ by using the metrics h_V on V and μ_c on Σ_t, where k is any nonnegative integer. Then we have

$$\begin{aligned} &\|H_t(\xi_1) - H_t(\xi_2)\|_{0,\frac{1}{2}} \\ &\quad \le C\Big(\|L_t^*(\xi_1)\|_{0,\frac{1}{2}} \\ &\qquad\qquad \|L_t^*(\xi_1 - \xi_2)\|_{1,\frac{1}{2}} + \|L_t^*(\xi_2)\|_{1,\frac{1}{2}} \|L_t^*(\xi_1 - \xi_2)\|_{0,\frac{1}{2}}\ ) \\ &\quad \le C\Big(\|\xi_1\|_{1,\frac{1}{2}} \|\xi_1 - \xi_2\|_{2,\frac{1}{2}} + \|\xi_2\|_{2,\frac{1}{2}} \|\xi_1 - \xi_2\|_{1,\frac{1}{2}}\Big), \end{aligned} \tag{6.66}$$

and therefore, in consequence of Lemmas 6.1, 6.2,

$$\|\tilde{f}_t\|_{5,\frac{1}{2}} \le C, \tag{6.67}$$

$$\|v_t\|_{4,\frac{1}{2}} \le C|t|. \tag{6.68}$$

We define

$$\Gamma^{k,\frac{1}{2}}(\wedge^{0,1}(\tilde{f}_t^* TV)) = \{\xi \in \Gamma(\wedge^{0,1}(\tilde{f}_t^* TV)) \mid \|\xi\|_{k,\frac{1}{2}} < \infty\}, \tag{6.69}$$

where $k = 0, 1, 2, \ldots$. Clearly, from (6.64) - (6.69) it follows that

$$\Psi_t : \Gamma^{2,\frac{1}{2}}(\wedge^{0,1}(\tilde{f}_t^* TV)) \to \Gamma^{0,\frac{1}{2}}(\wedge^{0,1}(\tilde{f}_t^* TV)).$$

We recall that $\Box_t$ is the linear operator $L_t L_t^*$ on $\Gamma(\wedge^{0,1}(\tilde{f}_t^* TV))$.

Lemma 6.7. *Let ξ be in $\Gamma^{2,\frac{1}{2}}(\tilde{f}_t^* TV)$ and ζ be in $\Gamma^{0,\frac{1}{2}}(\tilde{f}_t^* TV)$. Assume*

$$\Box_t \xi = \zeta \quad on \quad \Sigma_t. \tag{6.70}$$

Then we have

$$\|\xi\|_{2,\frac{1}{2}} \le C(-\log|t|)^{\frac{3}{2}} \|\zeta\|_{0,\frac{1}{2}}. \tag{6.71}$$

Proof. First we remark

$$\text{Vol}\,(\Sigma_t, \mu_c) = C(-\log|t|), \tag{6.72}$$

which implies

$$\int_{\Sigma_t} |\zeta|^2_{\mu_c} d\mu_c \le C\|\zeta\|^2_{0,0}(-\log|t|). \tag{6.73}$$

Multiplying both sides of the equation in (6.70) by ξ, integrating by parts and using (6.73), we deduce

$$\begin{aligned}\int_{\Sigma_t} |L_t^*\xi|^2 d\mu_c &= \int_{\Sigma_t} \langle \xi, \zeta\rangle_{\mu_c} d\mu_c \\ &\le C\|\zeta\|_{0,\frac{1}{2}}(-\log|t|)^{1/2}\Big(\int_{\Sigma_t} |\xi|^2_{\mu_c} d\mu_c\Big)^{1/2}.\end{aligned}$$

Together with Lemma 6.6, this implies

$$\int_{\Sigma_t} |\xi|^2_{\mu_c} d\mu_c \le C\|\zeta\|^2_{0,\frac{1}{2}}(-\log|t|)^3. \tag{6.74}$$

Then (6.71) follows from (6.74) and the standard elliptic estimates (cf. [7]).

Proposition 6.2. *There is a $t_0 > 0$ such that for any $0 < |t| < t_0$, there is a unique $\xi \in \Gamma^{2,\frac{1}{2}}(\wedge^{0,1}\tilde{f}_t^* TV)$ satisfying*

$$\|\xi\|_{2,\frac{1}{2}} \le \sqrt{|t|} \tag{6.75}$$

and

$$\Psi_t(\xi) = 0 \quad on \quad \Sigma_t, \tag{6.76}$$

i.e., $f_t = \exp_{\tilde{f}_t}(L_t^*\xi)$ is a J-holomorphic map from Σ_t into V.

Proof. Let $B_{\sqrt{|t|}}(0)$ be the ball in $\Gamma^{2,\frac{1}{2}}(\wedge^{0,1}\tilde{f}_t^*TV)$ with radius $\sqrt{|t|}$ and the center at the origin. Then $\Psi_t(\xi) = 0$ for some $\xi \in B_{\sqrt{|t|}}(0)$ is equivalent to

$$\xi = \Box_t^{-1}(-v_t + H_t(\xi)), \tag{6.77}$$

where $\Box_t^{-1}$ is the inverse of $\Box_t$, which is from $\Gamma^{0,\frac{1}{2}}(\wedge^{0,1}\tilde{f}_t^*TV)$ into $\Gamma^{2,\frac{1}{2}}(\wedge^{0,1}\tilde{f}_t^*TV)$. By Lemma 6.2, 6.7, there is a constant $C > 0$ such that

$$\begin{aligned}
&\| \Box_t^{-1} v_t\|_{2,\frac{1}{2}} \leq C(-\log|t|)^{\frac{3}{2}}\|v_t\|_{0,\frac{1}{2}} \leq C|t|(-\log|t|)^{\frac{3}{2}},\\
&\| \Box_t^{-1} (H_t(\xi_1) - H_t(\xi_2))\|_{2,\frac{1}{2}} \leq C(-\log|t|)^{\frac{3}{2}}\|H_t(\xi_1) - H_t(\xi_2)\|_{0,\frac{1}{2}}\\
&\qquad \leq C(-\log|t|)^{\frac{3}{2}}\|\xi_1 - \xi_2\|_{2,\frac{1}{2}}^2 \leq C\sqrt{|t|}(-\log|t|)^{\frac{3}{2}}\|\xi_1 - \xi_2\|_{2,\frac{1}{2}}.
\end{aligned}$$

Thus the proposition follows from the Implicit Function Theorem.

Assume that f_0 is a smooth point of $\mathcal{M}_A(\Sigma_0, J, 0)$. By the Transversality Theorem in section 3, the tangent space of $\mathcal{M}_A(\Sigma_0, J, 0)$ at f_0 is naturally identified with the kernel $\mathrm{Ker}(L_0)$, where L_0 is defined in (6.28), i.e., a tangent vector at f_0 is a continiuous section u in $\Gamma(f_0^*TV)$ over Σ_0, such that $L_0 u = 0$ on Σ_0'. This implies that there is a local diffeomorphism from a neighborhood of 0 in $\mathrm{Ker}(L_0)$ into $\mathcal{M}_A(\Sigma_0, J, 0)$. We choose $\mathcal{W}$ so small that it is contained in the image of such a diffeomorphism. We may assume that for any f and f' in $\mathcal{W}$,

$$||f - f'||_{C^4(\Sigma_0)} \leq C\,||f - f'||_{C^0(\Sigma_0)}. \tag{6.78}$$

Given any f in $\mathcal{W}$, there is a unique section $\tilde{u}_{tf}$ in $\Gamma(\tilde{f}_{0t}^*TV)$ such that

$$\tilde{f}_t(x) = exp_{\tilde{f}_{0t}(x)}(\tilde{u}_{tf}(\tilde{f}_{0t}(x)))\ x \in \Sigma_t. \tag{6.79}$$

It follows from a straightforward computation:

$$|\nabla\tilde{u}_{tf}|_{\mu_c}(x) \leq C\rho^2(x), \tag{6.80}$$

and for $f, f' \in \mathcal{W}$,

$$||\tilde{u}_{tf} - \tilde{u}_{tf'}||_{C^4(\Sigma_t)} \leq C||f - f'||_{C^0(\Sigma_0)}. \tag{6.81}$$

We define $T_t : \mathcal{W} \mapsto \mathcal{M}_A(\Sigma_t, J, 0)$ by assigning f_t in Proposition 6.2 to each f in $\mathcal{W}$. Clearly, $T_t(f)$ converges to f as t goes to zero. It is easy to see that T_t is smooth. We want to examine the invertibility of the differential of T_t at any point in $\mathcal{W}$. For simplicity, we do it at f_0.

Lemma 6.8. *Let $\pi_t(u,x)$ be the parallel transport along the path $\{exp_{\tilde{f}_{0t}(x)}(su(\tilde{f}_{0t}(x)))\}_{0\leq s\leq 1}$. Then there is a uniform constant $C>0$ such that*

$$||v_{0t} - \pi(\tilde{u}_{tf}, \cdot)v_t||_{C^0} \leq C\sqrt{|t|}||f - f_0||_{C^0(\Sigma_0)}. \tag{6.82}$$

Proof. Choose a diffeomorphism ϕ from a neighborhood of $\tilde{f}_{0t}(\Sigma_t)$ onto a neighborhood of $\tilde{f}_t(\Sigma_t)$ satisfying: (1) $||\phi - \mathrm{Id}\,||_{C^4} \leq C||f_0 - f||_{C^5}$; (2) for each node p in Σ_0, let U_p, z_{1p}, z_{2p} and $w_1, \cdots, w_n$ be as in (6.2), (6.6); then

$$f_0(z_{1p}, z_{2p}) = \phi^{-1} \cdot f(z_{1p}, z_{2p}) = (z_{1p}, z_{2p}, 0, \cdots, 0) \ \text{ in } \ U_p.$$

If z in Σ_t is far enough away from $\mathrm{Sing}(\Sigma_0)$, then by the definition of $\tilde{f}_{0t}$ and $\tilde{f}_t$, we have

$$\tilde{f}_t(z) = f(\Phi_t^{-1}(z)), \ \tilde{f}_{0t}(z) = f_0(\Phi_t^{-1}(z)),$$

where Φ_t is the diffeomorphism as in (6.10). Since both f and f_0 are J-holomorphic, we have

$$v_{0t}(z) = df_0 + J \cdot df_0 \cdot j_t = J \cdot df_0 \cdot (\Phi_{t*}^{-1} j_t - j_0 \Phi_{t*}^{-1})$$

and

$$v_t(z) = J \cdot df \cdot (\Phi_{t*}^{-1} j_t - j_0 \Phi_{t*}^{-1}).$$

It follows

$$\begin{aligned}&|v_{0t}(z) - \phi_*^{-1} v_t(z)| \\ &\quad = |(J \cdot df_0 - \phi_*^{-1} J \cdot df)(\Phi_{t*}^{-1} j_t - j_0 \Phi_{t*}^{-1})|(z) \leq C|t|||f - f_0||_{C^1}.\end{aligned}$$

If z is in U_p, where p is a node of Σ_0, and U_p is a small neighborhood of p, then

$$\phi \cdot \tilde{f}_t(z_{1p}, z_{2p}) = \tilde{f}_{0t}(z_{1p}, z_{2p}) = (z_{1p}, z_{2p}, 0, \cdots, 0),$$

implies

$$\phi_*^{-1} v_t = (\phi_*^{-1} \cdot J \cdot \phi_* - J_0) \cdot d(\phi^{-1} \cdot \tilde{f}_t) \cdot j_t.$$

Since $\phi^{-1} \cdot f$ is $\phi_*^{-1} \cdot J \cdot \phi_*$-holomorphic, we have

$$\phi_*^{-1} \cdot J \cdot \phi_*(z_{1p}, 0) = J_0 = \phi_*^{-1} \cdot J \cdot \phi_*(0, z_{2p}).$$

We may assume that $|z_{1p}| \geq |z_{2p}|$ at z. Then

$$\phi_*^{-1} v_t(z) = (\phi_*^{-1} \cdot J \cdot \phi_*(z_{1p}, z_{2p}) - \phi_*^{-1} \cdot J \cdot \phi_*(z_{1p}, 0)) \cdot d\tilde{f}_{0t} \cdot j_t.$$

On the other hand, we also have

$$v_{0t}(z) = (J(z_{1p}, z_{2p}) - J(z_{1p}, 0)) \cdot d\tilde{f}_{0t} \cdot j_t.$$

Therefore, by the Mean-Value Theorem, we deduce

$$\begin{aligned} &|v_{0t}(z) - \phi_*^{-1} v_t(z)|_{\mu_c} \\ &\quad \leq \sup_{0 \leq \epsilon \leq 1} |\nabla(J - \phi_*^{-1} \cdot J \cdot \phi_*)(z_{1p}, \epsilon z_{2p})||z_{2p}||d\tilde{f}_{0t}|(z) \\ &\quad \leq C\sqrt{|t|}\, ||f - f_0||_{C^0(\Sigma_0)}. \end{aligned}$$

Moreover, use of the definition of $\tilde{f}_t$ and $\tilde{f}_{0t}$ in (6.13) leads to

$$||v_{0t} - \phi_*^{-1} v_t||_{C^0} \leq C\sqrt{|t|}\, ||f - f_0||_{C^0(\Sigma_0)}. \tag{6.83}$$

Hence the lemma follows from (6.83), (6.68) and the fact that $||\pi_t(\tilde{u}_{tf}, \cdot)^{-1} \cdot \phi_*^{-1} - \mathrm{Id}\,||_{C^2} \leq C||f - f_0||_{C^0}$.

Let ξ_1 and ξ_2 be the sections in Proposition 6.2 such that $T_t(f_0) = exp_{\tilde{f}_{0t}}(L_{0t}^*\xi_1)$ and $T_t(f) = exp_{\tilde{f}_t}(L_t^*\xi_2)$. In consequence of the fact that $T_t(f)$, $T_t(f_0)$ are J-holomorphic, we have

$$0 = v_{0t} + L_{0t}L_{0t}^*(\xi_1) + H_{0t}(\xi_1)\ \tilde{f}_{0t}^* TV), \tag{6.84}$$

$$0 = \pi_t(\tilde{u}_{tf}, \cdot)\, (v_t + L_t L_t^*(\xi_2) + H_t(\xi_2)) \ \in\ \wedge^{0,1}(\tilde{f}_{0t}^* TV), \tag{6.85}$$

where v_t, v_{0t} are defined in either (6.19) or (6.20). By Proposition 6.2, we obtain $||\xi_1||_{2,\frac{1}{2}}$, $||\xi_2||_{2,\frac{1}{2}} \leq \sqrt{|t|}$, and therefore

$$||(\pi_t(\tilde{u}_{tf}, \cdot) L_t L_t^* - L_{0t}L_{0t}^* \pi_t(\tilde{u}_{tf}, \cdot))(\xi_2)||_{0,\frac{1}{2}} \leq C\sqrt{|t|}||f - f_0||_{C^0}, \tag{6.86}$$

and

$$(6.87) \quad ||\pi_t(\tilde{u}_{tf},\cdot)H_t(\xi_2) - H_{0t}(\pi_t(\tilde{u}_{tf},\cdot)\xi_2)||_{0,\frac{1}{2}} \le C\sqrt{|t|}||f-f_0||_{C^0}.$$

Substracting (6.84) from (6.85), yields

$$(6.88) \quad \begin{aligned} &\pi_t(\tilde{u}_{tf},\cdot)v_t - v_{0t} \\ &\quad = \Box_t(\xi_1 - \pi_t(\tilde{u}_{tf},\cdot)\xi_2) + H_{0t}(\xi_1) - H_{0t}(\pi_t(\tilde{u}_{tf},\cdot)\xi_2) + B. \end{aligned}$$

From (6.86) and (6.87) it follows that

$$(6.89) \quad ||B||_{0,\frac{1}{2}} \le C\sqrt{|t|}||f-f_0||_{C^0}.$$

Applying Lemma 6.7 to (6.88) and using Lemma 6.8 and (6.89), we deduce

$$||\xi_1 - \pi_t(\tilde{u}_{tf},\cdot)\xi_2||_{2,\frac{1}{2}} \le C\sqrt{|t|}(-\log|t|)^{\frac{3}{2}}||f-f_0||_{C^0}.$$

Therefore, **the map T_t is injective near f_0**. Moreover, if D_fT_t denotes the derivative of T_t at f_0, then for any u in $\mathrm{Ker}(L_0)$, we have

$$(6.90) \quad (1 - C|t|^{\frac{1}{4}})||u||_{0,0} \le ||D_fT_t(u)||_{0,0} \le (1 + C|t|^{\frac{1}{4}})||u||_{0,0}.$$

We denote by $\tilde{L}_t$ the linearization of the Cauchy-Riemann equation at f_{0t}. Note that $\tilde{L}_0 = L_0$.

Lemma 6.9. *Let* $\mathrm{Ker}(\tilde{L}_t)$ *be the set of all solutions for* $\tilde{L}_t u = 0$. *Then* $\mathrm{Ker}(\tilde{L}_t)$ *converges uniformly to* $\mathrm{Ker}(L_0)$ *as t goes to zero. In particular, the dimension of* $\mathrm{Ker}(\tilde{L}_t)$ *stays as a constant for t sufficiently small.*

Proof. We just sketch a proof here. Let $u_{t1},\cdots,u_{tl}$ be an orthonormal set of $\mathrm{Ker}(L_t)$ with respect to the inner product induced by $\mu|_{\Sigma_t}$, i.e.,

$$\int_{\Sigma_t}\langle u_{ti}, u_{tj}\rangle d\mu \ = \ \delta_{ij}.$$

Since the Sobolev inequality holds uniformly for the metrics $\mu|_{\Sigma_t}$, by the standard iteration, one can show that (1) there is a uniform bound on $||u_{ti}||_{0,0}$; (2) for any $\epsilon > 0$, there is a $\delta_0 > 0$ such that

$$(6.92) \quad \begin{aligned} &\sup\{|u_{ti}(x_1) - u_{ti}(x_2)| \,|\, x_\alpha \in \Sigma_t, d_S(x_\alpha, \mathrm{Sing}(\Sigma_0)) \\ &\qquad \le \delta_0, \ \alpha = 1,2\} \le \epsilon. \end{aligned}$$

Therefore, by taking a subsequence if necessary, u_{ti} converges to u_{0i} on Σ_0. Moreover,

$$\int_{\Sigma_0} \langle u_{0i}, u_{0j} \rangle d\mu \;=\; \delta_{ij},$$

i.e., $\{u_{ti}\}$ form an orthonormal basis of a subspace in $\mathrm{Ker}(L_0)$. Thus the lemma follows from (6.90).

Corollary 6.2. *The derivative $D_f T_t$ is an isomorphism between* $\mathrm{Ker}(L_0)$ *and* $\mathrm{Ker}(\tilde{L}_t)$ *satisfying* (6.90).

Remark. The orientation of $\mathcal{M}_A(\Sigma_t, J, \nu_t)$ at $f_t = T_t(f)$ is given by the canonical orientation on the kernel $\mathrm{Ker}(\overline{\partial}_{f_t})$, provided that the cokernel of $\overline{\partial}_{f_t}$ is trivial, where $\overline{\partial}_{f_t}$ denotes the first derivative part of $\tilde{L}_t$ at f_t, i.e., $\tilde{L}_t - \overline{\partial}_{f_t}$ is an zero-order operator. Note that $\overline{\partial}_{f_t}$ induces a natural holomorphic structure on $\mathrm{Ker}\,\overline{\partial}_{f_t}$, which is isomorphic to $\mathrm{Ker}(\tilde{L}_t)$. Since J is generic, we may assume that the cokernel of $\overline{\partial}_{f_t}$ is trivial for any t small or zero. Then by using the same arguments in the proof of Lemma 6.10, one can show that the canonical orientation on $\mathrm{Ker}(\overline{\partial}_{f_t})$ is preserved when t tends to zero. It follows that T_t is orientation-preserving.

It remains to show (2) in Theorem 6.1. Let f' be as given in Theorem 6.1. For any f in $\mathcal{W}$, let f_t be $T_t(f)$. Then there is a unique vector field u_f such that

$$f'(x) = exp_{f_t(x)}(u_f(f_t(x))), \tag{6.93}$$

Furthermore, we have $||u_f||_{0,0} \leq \epsilon'$, where ϵ' is small and depends only on $\mathcal{W}$ and ϵ in Theorem 6.1. We want to show f' coincides with one of f_t in Proposition 6.2.

Lemma 6.10. *Let ρ be the function in* (6.15), *and F be either f' or one of f_t. Then there is a uniform constant $\lambda < 1$ such that*

$$\int_{-\frac{1}{2}\log|t| \geq -\log\rho \geq R} |dF|^2_{\mu_c} d\mu_c \leq 4\lambda^R. \tag{6.94}$$

Consequently, for some uniform $\beta_0 > 0$,

$$|dF|^2_{\mu_c}(x) \leq C\rho(x)^{8\beta_0}. \tag{6.95}$$

Proof. Since F is J-holomorphic, (6.95) follows from (6.94) and the standard elliptic estimates. Therefore, it suffices to show (6.94). We will always use C to denote a uniform constant.

Let ω_h be the Kähler form of the J-invariant metric h_V on V. Then

$$|dF|^2_{\mu_c} = F^*\omega_h|_{\Sigma_t}.$$

Choose a cut-off function η_R such that $\eta_R(t) = 1$ for $t \geq R+1$, $\eta_R(t) = 0$ for $t \leq R$, and $|\eta'_R| \leq 1$. By the assumptions on f' in Theorem 6.1 or construction of f_t, for $\rho(x)$ sufficiently small, the image $F(x)$ lies in a small coordinate chart, say a δ-ball, of V. Therefore, we can deduce

$$\begin{aligned}\int_{-\log\rho \geq R+1} |dF|^2_{\mu_c} d\mu_c &\leq \int_{\Sigma_t} \eta_R(-\log\rho) F^*\omega_h \\ &\leq C\int_{\Sigma_t} |\nabla\eta_R(-\log\rho)|_{\mu_c} |dF|_{\mu_c} d_V(F,\bar{y}) d\mu_c \\ &\quad + \int_{\Sigma_t} \eta_R(-\log\rho)|dF|^2_{\mu_c} d_V(F,\bar{y}) d\mu_c \\ &\leq C\int_{R\leq -\log\rho\leq R+1} d_V(F,\bar{y})|dF|_{\mu_c} d\mu_c + \delta\int_{-\log\rho\geq R+1} |dF|^2_{\mu_c} d\mu_c,\end{aligned}$$

where $\bar{y}$ is a point in V. By the well-known Poincaré inequatlity, we can choose $\bar{y}$ such that

$$\int_{R\leq -\log\rho\leq R+1} d_V(F,\bar{y})^2 d\mu_c \leq 4\int_{R\leq -\log\rho\leq R+1} |dF|^2_{\mu_c} d\mu_c;$$

for instance, we can take $\bar{y}$ to be the average of f_t over the region $\{x | R \leq -\log\rho(x) \leq R+1\}$. Therefore, we have

$$\int_{-\log\rho\geq R+1} |dF|^2_{\mu_c} d\mu_c \leq C\int_{R\leq -\log\rho\leq R+1} |dF|^2_{\mu_c} d\mu_c,$$

and consequently, for $\lambda = \frac{C}{C+1}$,

$$\int_{-\log\rho\geq R+1} |dF|^2_{\mu_c} d\mu_c \leq \lambda\int_{-\log\rho\geq R} |dF|^2_{\mu_c} d\mu_c.$$

Hence the lemma follows from a standard iteration and the fact that $\int_{\Sigma_t} |dF|^2_{\mu_c} d\mu_c = \omega_h(A)$.

Lemma 6.11. *If ϵ, $|t|$ are sufficiently small, then there is f in $\mathcal{W}$ such that*

(6.96) $$||u_f||_{1,0} \leq C|t|^{\beta_0},$$

where β_0 is given in (6.95).

Proof. By cutting Σ_t along the loops in $\{x \in \Sigma_t | \rho(x) = \sqrt{2|t|}\}$ and gluing disks to the boundary components of the resulting surface, we obtain finitely many surfaces Σ_{ti} $(1 \leq i \leq l)$. Putting $\tilde{\Sigma}_t$ to be the disjoint union of those surfaces, we can naturally embed $\Sigma_t \backslash \{x \in \Sigma_t | \rho(x) = \sqrt{2|t|}\}$ into $\tilde{\Sigma}_t$ as a submanifold. Then we can extend the conformal structure j_t on Σ_t to be a natural conformal structure $\tilde{j}$ on $\tilde{\Sigma}_t$.

From Lemma 6.10 It follows that

$$|df'|_{\mu_c}(x) \leq C|t|^{2\beta_0}, \tag{6.97}$$

whenever $\sqrt{2|t|} \leq \rho(x) \leq 10\sqrt{|t|}$. Therefore, we can extend f' to be a map $\tilde{f}$ from $\tilde{\Sigma}_t$ into V satisfying:

$$||\tilde{v}||_{0,\frac{1}{2}} \leq C|t|^{2\beta_0}, \tag{6.98}$$

where $\tilde{v} = d\tilde{f} + J \cdot d\tilde{f} \cdot \tilde{j}$. We denote by $\tilde{L}$ the linearization of the Cauchy-Riemann equation at $\tilde{f}$, and by $\tilde{L}^*$ its adjoint. Then by the same arguments as in the proof of Lemma 6.6, one can show that the first eigenvalue of $\tilde{L}\tilde{L}^*$ is not less than $c(-\log|t|)^{-2}$, where c is independent of f and t. Thus, by applying the Implicit Function Theorem (cf. the proof of Proposition 6.2), we can find an ξ in $\Gamma(\wedge^{0,1}\tilde{f}^*TV)$ such that $\tilde{f}_h = exp_{\tilde{f}}(\tilde{L}^*\xi)$ is J-holomorphic. Moreover, if t is sufficiently small, than we have

$$||\xi||_{2,\frac{1}{2}} \leq C|t|^{\frac{3\beta_0}{2}}.$$

Clearly, it follows that the distance between $\mathrm{Im}(f')$ and $\mathrm{Im}(\tilde{f}_h)$ is less than $C|t|^{\frac{3\beta_0}{2}}$. The map $\tilde{f}_h$ may not be in $\mathcal{M}_A(\Sigma_0, J, 0)$. However, using (6.97) and the fact that $\mathcal{M}_A(\Sigma_0, J, 0)$ is smooth at f_0, one can show that $\tilde{f}_h$ lies in a $|t|^{\beta_0}$-neighborhood of some f_t in $\mathcal{W}$, as long as both $|t|$ and ϵ are sufficiently small. Thus the lemma follows.

Let $f_t = T_t(f)$ be given by Lemma 6.11. Without loss of generality, we may assume that $f = f_0$. Let ξ be the unique solution: $\tilde{L}_t\tilde{L}_t^*\xi = \tilde{L}_t u_f$. Multiplying this equation by ξ and integrating by parts, we obtain

$$\int_{\Sigma_t} |\tilde{L}_t^*\xi|^2_{\mu_c} d\mu_c \leq C\sqrt{\int_{\Sigma_t} |\tilde{L}_t u_f|^2_{\mu_c} d\mu_c \int_{\Sigma_t} |\xi|^2_{\mu_c} d\mu_c}. \tag{6.99}$$

By the same argument as that in the proof of Lemma 6.6, one can show

$$\int_{\Sigma_t} |\xi|^2_{\mu_c} d\mu_c \le C(\log|t|)^2 \int_{\Sigma_t} |\tilde{L}_t^*\xi|^2 d\mu_c, \tag{6.100}$$

which together with (6.96) and (6.99) implies that for t sufficiently small,

$$\int_{\Sigma_t} |\xi|^2_{\mu_c} d\mu_c \le C|t|^{\frac{\beta_0}{2}}. \tag{6.101}$$

Since $\tilde{L}_t\tilde{L}_t^*\xi = \tilde{L}_t u_f = \mathcal{O}(||u_f||^2_{1,0})$, we have

$$||\xi||_{1,\frac{1}{2}} \le C|t|^{\frac{\beta_0}{2}},$$

and consequently,

$$||u_f - \tilde{L}_t^*\xi||_{1,\frac{1}{2}} \le C|t|^{\frac{\beta_0}{2}}.$$

Now we want to find a new f_1 in $\mathcal{W}$, which is very close to $f = f_0$, such that $u_{f_1} = \tilde{L}_t^*\xi_1$ for some ξ_1.

Using the equation $f' = exp_{f_1}(u_{f_1}) = exp_f(u_f)$, we can define a map S_t from a neighborhood of f_0 into $\mathrm{Ker}(\tilde{L}_t)$ at f_0:

$$S_t(f_1) = \pi(f_0, f_1)(u_{f_1} - \tilde{L}_t^*\xi_1), \quad \text{where } \tilde{L}_t u_{f_1} = \tilde{L}_t\tilde{L}_t^*\xi_1,$$

$\pi(f_1)$ denoting an isomorphism from the kernel of $\tilde{L}_t$ at f_1 onto that at f_0, which depends smoothly on f_1. Clearly, S_t is a smooth map. By the same arguments as in the proof of (6.90), one can show

$$(1 - C|t|^{\frac{\beta_0}{4}})||u||_{0,0} \le ||D_f S_t(u)||_{0,0} \le (1 + C|t|^{\frac{\beta_0}{4}})||u||_{0,0}.$$

Then by the Implicit Function Theorem, there is a f_1 such that $S_t(f_1) = 0$ and $||u_{f_1}||_{1,\frac{1}{2}} \le C|t|^{\frac{\beta_0}{8}}$. For simplicity, we may assume that f_1 coincides with f. Then $u_f = \tilde{L}_t^*\xi$. Since both f' and f_t are J-holomorphic, from (6.65) it follows

$$\Box_t\xi = -H_t(\xi). \tag{6.102}$$

Multiplying (6.102) by ξ and integrating by parts, one can deduce

$$\int_{\Sigma_t} |\tilde{L}_t^*\xi|^2 d\mu_c \le C||\tilde{L}_t^*\xi||_{0,0} \int_{\Sigma_t} |\xi|^2_{\mu_c} d\mu_c. \tag{6.103}$$

However, $||\tilde{L}_t^*\xi||_{0,0} \leq C|t|^{\frac{\beta_0}{8}}$, so for t sufficiently small, (6.103) is impossible unless $u_f = \tilde{L}_t^*\xi = 0$, i.e., $f' = f_t$. Hence Theorem 6.1 is proved.

7. Composition law

In this section, we prove the composition law for our mixed invariants (Theorem A). A special case would be the composition law for topological σ-model invariants. The composition law was predicted by physicists based on physical intuitions. Our proof will be based on degenerating stable curves in the sense of Deligne-Mumford.

Let $C = (\Sigma, x_1, \cdots, x_k)$ be a k-point genus g stable curve, and $\mathcal{M}_A(\Sigma, J, \nu)$ be the moduli space of all (J,ν)-maps from Σ into V with the total homology class A. By Corollary 4.11, for a generic (J,ν), $\mathcal{M}_A(\Sigma, J, \nu)$ is a smooth manifold, which may have many components. Let $\Sigma_1, \cdots, \Sigma_m$ be connected components of Σ, and $A_1, \cdots, A_m$ be homology classes with $A = \sum_{i=1}^m A_i$. As before, $\mathcal{M}_{(A_1,\cdots,A_m)}(\Sigma, J, \nu)$ be those (J,ν)-maps f with $f_*(\Sigma_i) = A_i$. We will use this moduli space to construct a generalizd invariant. The construction is exactly the same as that of the mixed invariant in section 2. Let us outline this construction as follows: Suppose that $\alpha_1, \cdots, \alpha_k$, $\beta_1, \cdots, \beta_l$ are integral homology classes satisfying

$$\begin{aligned}(7.1)\qquad \sum(2n - \deg(\alpha_i)) + \sum(2n - \deg(\beta_j) - 2) \\ = \dim \mathcal{M}_{(A_1,\cdots,A_m)}(\Sigma, J, \nu).\end{aligned}$$

They are represented by pseudo-manifolds (Y_i, F_i), (Z_j, G_j) as in section 2. Let $e_{(C,J,\nu)}$ and F be defined in (2.4), (2.5) with $\mathcal{M}_A(\Sigma, J, \nu)$ replaced by $\mathcal{M}_{(A_1,\cdots,A_m)}(\Sigma, J, \nu)$. Then we can define an invariant $\Phi_{(A_1,\cdots,A_m,\omega,C)}$ as follows: Fix a pair (J,ν) such that $e_{(C,J,\nu)}$ and $\mathcal{M}_A(\Sigma, J, \nu)$ satisfy all properties described in Proposition 5.6. We first associate a multiplicity $m(f)$ to each f in $e^{-1}_{(C,J,\nu)}(\Delta)$, and then define $m(f)$ to be zero if either $f(x_i)$ is not in $F_i(Y_i)$ for some i, or $f(\Sigma)$ does not intersect one of $G_j(Z_j)$. If f is as given in Propositions 5.1, 5.2, then there are finitely many $(y_{s1}, \cdots, y_{sl})$ $(1 \leq s \leq m)$ such that $f(y_{sj}) \in G_j(Z_j)$ and each y_{sj} is a smooth point of Σ. We put $\epsilon(f,s)$ to be ± 1; the sign is determined by the orientations of $\mathcal{M}_{(A_1,\cdots,A_m)}(\Sigma, J, \nu) \times (\Sigma)^l$, P, V_{k+l} at $(f; y_{s1}, \cdots, y_{sl})$, etc., and the

Jacobians of the maps $e_{(C,J,\nu)}$ and F, where P is the domain of F. Define

$$m(f) = \sum_{s=1}^{m} \epsilon(f,s), \tag{7.2}$$

and finally,

$$\Phi_{(A_1,\cdots,A_m,\omega,C)}(\alpha_1,\cdots,\alpha_k \mid \beta_1,\cdots,\beta_l) = \sum m(f). \tag{7.3}$$

For convenience, we simply define

$$\Phi_{(A_1,\cdots,A_m,\omega,C)}(\alpha_1,\cdots,\alpha_k \mid \beta_1,\cdots,\beta_l) \;=\; 0, \tag{7.4}$$

in case (7.1) does not hold.

As before, one can show that this is independent of the choices of J, ν, the k-point, genus g curve C and the pseudo-manifold representatives of α_i, β_j.

Remark 7.1. One can define a more refined invariant by specifying components of Σ, which intersect with β_j. Let q be a map from $\{1,\ldots,l\}$ into $\{1,\ldots,m\}$. Then we consider

$$\begin{aligned} &e_{C,q} : \mathcal{M}_A(C,J,\nu) \times \prod_j \Sigma_{q(j)} \longrightarrow V^{k+l}, \\ &e_{C,q}(f_1,\cdots,f_m,y_{q(1)},\cdots,y_{q(l)}) \\ &\qquad = (f_{p(1)}(x_1),\ldots,f_{p(k)}(x_k); f_{q(1)}(y_{q(1)}),\cdots,f_{q(l)}(y_{q(l)})). \end{aligned} \tag{7.5}$$

Choose a generic (J,ν) such that $e_{C,q}$ is transversal to F. On the other hand,

$$\overline{\mathcal{M}}_{(A_1,\cdots,A_m)}(\Sigma,J,\nu) \subset \bigcup_{D\in\mathcal{D}^{J,\nu}_{A,\Sigma}} \mathcal{M}_\Sigma(D,J,\nu).$$

By the same arguments in the proof of Proposition 2.2, one can show that for a generic (J,ν), $e_{C,q}^{-1}(\mathrm{Im}(F))$ does not intersect the boundary of $\mathcal{M}_{(A_1,\cdots,A_m)}(\Sigma,J,\nu)$. Therefore, one can define an invariant

$$\Phi_{(A_1,\cdots,A_m,\omega,C,q)}(\alpha_1,\cdots,\alpha_k|\beta_1,\cdots,\beta_l)$$

by counting points in $e_{C,q}^{-1}(\mathrm{Im}(F))$ with sign. Moreover, if Q denotes the set of all maps q, then

$$\begin{aligned} &\Phi_{(A_1,\cdots,A_m,\omega,C)}(\alpha_1,\cdots,\alpha_k|\beta_1,\cdots,\beta_l) \\ &\qquad = \sum_{q\in Q} \Phi_{(A_1,\cdots,A_m,\omega,C,q)}(\alpha_1,\cdots,\alpha_k|\beta_1,\cdots,\beta_l). \end{aligned} \tag{7.6}$$

We can sum all $\Phi_{(A_1,\cdots,A_m,\omega,g)}$ to obtain an invariant

$$\Phi_{(A,\omega,\mathcal{C})} = \sum_{A=A_1+\cdots A_m} \Phi_{(A_1,\cdots,A_m,\omega,\mathcal{C})}. \tag{7.7}$$

Theorem 7.2 (the composition law). *Suppose that (V,ω) is a semi-positive, symplectic, compact manifold and, $\mathcal{C}$ is a k-point genus g stable curve. Then for any integral homology classes $\alpha_1,\cdots,\alpha_k$, $\beta_1,\cdots,\beta_l$, we have*

$$\Phi_{(A,\omega,g)}(\alpha_1,\cdots\alpha_k \mid \beta_1,\cdots,\beta_l) = \Phi_{(A,\omega,\mathcal{C})}(\alpha_1,\cdots\alpha_k \mid \beta_1,\cdots,\beta_l). \tag{7.8}$$

Proof. Fix a degeneration $\pi : S \to \Delta$ of k-point genus g smooth stable curves, such that the central fiber is $\mathcal{C}$, and other fibers are smooth genus g Riemann surfaces with k marked points, where Δ denotes the unit disk in C^1. For example, in case $g = 0$, we can take S to be the blow-up of $S^2 \times \Delta$ at a point in $S^2 \times \{0\}$. Let ν be an inhomogeneous term on S (cf. section 3), whose restriction on $\mathcal{C}$ is an inhomogeneous term ν_0 on $\mathcal{C}$ (cf. section 4, paragraph 3). Let us use ν_t to denote $\nu|_{\Sigma_t}$, where Σ_t denotes the fiber over t. Let $X_t = (x_1^t,\cdots,x_k^t)$ be the k disjoint sections of S, whose restriction to each fiber gives the marked points. Suppose that for $t \neq 0$, $(f_t, y_1^t,\cdots,y_l^t)$ is in $e^{-1}_{(X_t,\Sigma_t,J,\nu_t)}(\mathrm{Im}(F))$, where F is defined in (2.5). Then $f_t(x_i^t) \in \mathrm{Im}(F_i)$, $f_t(y_j^t) \in \mathrm{Im}(G_j)$. Using Proposition 3.1 and taking a subsequence if necessary, we may assume that f_t converges to f_0 in $\overline{\mathcal{M}}_A(\Sigma_0, J, \nu_0)$ as t tends to 0. By Propositions 5.3, 5.4, if (J,ν_0) is generic, than f_0 is actually in $\mathcal{M}_A(\Sigma_0,J,\nu_0)$. Let y_j^0 be the limit of y_j^t. Clearly, $f_0(x_i^0) \in \mathrm{Im}(F_i)$ and $f_0(y_j^0) \in \mathrm{Im}(G_j) \neq \emptyset$. Therefore, $(f_0, y_1^0,\cdots,y_l^0) \in e^{-1}_{(X_0,\Sigma_0,J,\nu_0)}(\mathrm{Im}(F))$. Thus the theorem follows from Corollary 6.1.

Next, using an idea of Witten, i.e., decomposing the diagonal class by the Küneth formula, we shall prove that $\Phi_{(A,\omega,\mathcal{C})}(\alpha_1,\cdots\alpha_k|\beta_1,\cdots,\beta_l)$ **can be explicitly calculated in terms of the mixed invariant of each component of $\mathcal{C}$ and contributions from double points.** There are two kinds of double points on a stable curve, i.e., intersection of two different components or self-intersection of one component. We shall give formulas for computing $\Phi_{(A,\omega,\mathcal{C})}$ in two special cases corresponding to these two types of intersections. The formula in general case can be derived inductively from these two special cases.

Case 1. Suppose that $\mathcal{C} = (\Sigma_0, x_1, \cdots, x_k)$ has exact two components Σ_{01}, Σ_{02} of genus g_1, g_2 satisfying: (1) Σ_{01} and Σ_{02} intersect at a double point p; (2) Σ_{01} carries m marked points $x_1, \cdots, x_m$ and Σ_{02} carries the rest of marked points. Then $\Phi_{(A,\omega,\mathcal{C})}$ can be calculated as follows:

Let $\{H_\sigma\}$ be a basis for the torsion free part of $H_*(V, Z)$. Consider the diagonal $\Delta \subset V \times V$. By the Künneth Formula, the homology class of the diagonal is given by

$$[\Delta] = \sum_{\gamma,\tau} \eta^{\gamma\tau} H_\gamma \otimes H_\tau,$$

where $\eta_{\gamma\tau} = H_\gamma \cap H_\tau$, and $\{\eta^{\gamma\tau}\}$ is the inverse of the intersection matrix $\{\eta_{\gamma\tau}\}$. Let q be the map: $q(1) = \cdots = q(l') = 1$, $q(l'+1) = \cdots = q(l) = 2$.

Theorem 7.3. *Let A_1, A_2 be two homology classes. Then we have*

$$\begin{aligned}\Phi_{(A_1,A_2,\omega,\mathcal{C},q)} &= \sum_{\gamma,\delta} \eta^{\gamma\delta} \Phi_{(A_1,\omega,g_1)}(\alpha_1, \cdots, \alpha_m, H_\gamma | \beta_1, \cdots, \beta_{l'}) \\ &\qquad \cdot \Phi_{(A_2,\omega,g_2)}(\alpha_{m+1}, \cdots, \alpha_k, H_\delta | \beta_{l'+1}, \cdots, \beta_l),\end{aligned}$$

and consequently,

$$\begin{aligned}\Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l) &= \sum_{A=A_1+A_2} \sum_{j=0}^{l} \sum_{\sigma} \sum_{\gamma,\delta} \frac{\epsilon(\sigma)}{j!(l-j)!} \eta^{\gamma\delta} \\ &\cdot \Phi_{(A_1,\omega,g_1)}(\alpha_1, \cdots, \alpha_m, H_\gamma \mid \beta_{\sigma(1)}, \cdots, \beta_{\sigma(j)}) \\ &\cdot \Phi_{(A_2,\omega,g_2)}(\alpha_{m+1}, \cdots, \alpha_k, H_\delta \mid \beta_{\sigma(j+1)}, \cdots, \beta_{\sigma(l)}),\end{aligned} \tag{7.9}$$

where σ runs over all permutations of $1, \cdots, l$, and $\epsilon(\sigma)$ is the sign of the permutation induced by σ on odd dimensional β_j.

Proof. The invariant $\Phi_{(A_1,A_2,\omega,\mathcal{C},q)}(\alpha_1, \cdots \alpha_k | \beta_1, \cdots, \beta_l)$ can be redefined as follows: Suppose that $z \in \Sigma_{01}$, $z' \in \Sigma_{02}$ are the intersection p. We define an evaluation map

$$ev : \mathcal{M}_{A_1}(\Sigma_{01}, J, \nu_1) \times (\Sigma_{g_1})^{l'} \times \mathcal{M}_{A_2}(\Sigma_{02}, J, \nu_2) \times (\Sigma_{g_2})^{l-l'} \to V^{k+l+2} \tag{7.10}$$

to be

$$\begin{aligned}&e_{C_1} \times e_{C_2}(f_1, y_1, \cdots, y_{l'} f_2, y_{l'+1}, \cdots, y_l) \\ &= (f_1(x_1), \cdots, f_1(x_m), f_1(y_1), \cdots, f_1(y_{l'}), f_1(z), f_2(z'), f_2(x_{m+1}), \\ &\qquad \cdots, f_2(x_k), f_2(y_{l'+1}), \cdots, f_2(y_l)).\end{aligned}$$

Let α be a homology class in $H_*(V^{k+2}, Z)$ satisfying:

$$
\begin{aligned}
(7.11) \qquad & 2n(k+l+2) - \deg(\alpha) \\
&= \dim \mathcal{M}_{A_1}(\Sigma_{01}, J, \nu_1) + \dim \mathcal{M}_{A_2}(\Sigma_{02}, J, \nu_2) + 2l \\
&= 2C_1(V)(A_1 + A_2) + 2n(1 - (g_1 + g_2)) + 2n + 2l,
\end{aligned}
$$

For generic J, ν_1, ν_2, by the same arguments in defining the mixed invariant, one can define an integral invariant $\Phi_{A_1,A_2}(\alpha)$ by counting the number of elements in $ev^{-1}(\mathrm{Im}(F))$ with sign according to orientations of the domain of ev, V^{k+l+2}, etc., where $F : Y \to V^{k+l+2}$ is a pseudo-manifold representing α. This invariant $\Phi_{A_1,A_2}(\alpha)$ is independent of J, ν_1, ν_2, the complex structure of $\mathcal{C}$, the pseudo-manifold representative F of α. Moreover, like the mixed invariants in Proposition 2.5, $\Phi_{(A_1,A_2)}$ is linear on α.

Now we take α to be the homology class

$$(7.12) \quad \alpha_1 \otimes \cdots \otimes \alpha_m \otimes \beta_1 \cdots \otimes \beta_{l'} \otimes [\Delta] \otimes \alpha_{m+1} \cdots \otimes \alpha_k \otimes \beta_{l'+1} \cdots \otimes \beta_l.$$

Suppose that $F_i : Y_i \to V$, $G_j : Z_j \to V$ are pseudo-manifold representatives of α_i, β_j. Then

$$
\begin{aligned}
F = \prod_1^m F_i \times \prod_1^{l'} G_j \times \Delta \times \prod_{m+1}^{k} F_i \times \prod_{l'+1}^{l} G_j : \\
\prod_1^m Y_i \times \prod_1^{l'} Z_j \times V \times \prod_{m+1}^{k} Y_i \times \prod_{l'+1}^{l} Z_j \to V^{k+l+2}
\end{aligned}
$$

is a pseudo-manifold representative of α. Using the representative F, one can easily see

$$(7.13) \qquad \Phi_{A_1,A_2}(\alpha) = \Phi_{(A_1,A_2,\omega,\mathcal{C},q)}(\alpha_1, \cdots, \alpha_k | \beta_1, \cdots, \beta_l).$$

On the other hand, applying the Küneth formula to $[\Delta]$ as Witten suggested in [25], we have

$$\alpha = \sum_{\gamma,\delta} \eta^{\gamma\delta} \alpha_1 \otimes \cdots \otimes \alpha_m \otimes \beta_1 \cdots \otimes \beta_{l'} \otimes H_\gamma \otimes H_\delta \otimes \alpha_{m+1} \cdots \otimes \alpha_k \otimes \beta_{l'+1} \cdots \otimes \beta_l.$$

Therefore,

$$
\begin{aligned}
\Phi_{(A_1,A_2)}\ (\alpha) \\
&= \sum_{\gamma,\delta} \eta^{\gamma\delta} \Phi_{(A_1,A_2)}(\alpha_1 \otimes \cdots \otimes \alpha_m \otimes H_\gamma \\
&\qquad \otimes H_\delta \otimes \alpha_{m+1} \cdots \otimes \alpha_k \otimes \beta_{l'+1} \cdots \otimes \beta_l).
\end{aligned}
$$

But from the definition it follows trivially that

$$\begin{aligned}\Phi_{(A_1,A_2)}(\alpha_1 \otimes \cdots \otimes \alpha_m \otimes \beta_1 \cdots \otimes \beta_{l'} \otimes H_\gamma \otimes H_\delta \otimes \alpha_{m+1} \\ \cdots \otimes \alpha_k \otimes \beta_{l'+1} \cdots \otimes \beta_l) \\ = \Phi_{(A_1,\omega,g_1)}(\alpha_1, \cdots, \alpha_m, H_\gamma | \beta_1, \cdots, \beta_{l'}) \\ \cdot \Phi_{(A_2,\omega,g_2)}(H_\delta, \alpha_{m+1}, \cdots, \alpha_k | \beta_{l'+1}, \cdots, \beta_l).\end{aligned}$$

This finishes the proof.

Remark 7.4. In case $g_1 = g_2$, there is another intepretation of $\Phi_{(A_1,A_2,\omega,\mathcal{C},q)}$ in terms of the mixed invariant of $U = V \times V$ with symplectic form $\Omega = \pi_1^*\omega + \pi_2^*\omega$, where $\pi_i : U \mapsto V$ is the projection onto the i^{th}-factor. More precisely, if $\Phi^U_{(B,\Omega,g_1)}$ denotes the mixed invariant of (U, Ω), where $B = A_1 \otimes 1 + 1 \otimes A_2$, then

$$\begin{aligned}&\Phi_{(A_1,A_2,\omega,\mathcal{C},q)}(\alpha_1, \cdots, \alpha_k|) \\ &\quad = \Phi^U_{(B,\Omega,g_1)}(\pi_1^{-1}(\alpha_1), \cdots, \pi_1^{-1}(\alpha_m), \pi_2^{-1}(\alpha_{m+1}), \cdots, \pi_2^{-1}(\alpha_k), \Delta|).\end{aligned}$$

Then one can use this to give another proof of Theorem 7.3 in this special case.

Case 2. $\mathcal{C} = (\Sigma_0, x_1, \cdots, x_k)$, and Σ_0 is a genus-$(g-1)$ curve with k marked points $x_1, \cdots, x_k$ and a self-intersection point. Then we have

Theorem 7.5.

$$\begin{aligned}&\Phi_{(A,\omega,g)}\,(\alpha_1, \cdots, \alpha_k | \beta_1, \cdots, \beta_l) \\ &\quad = \sum_{\gamma,\delta} \eta^{\gamma\delta} \Phi_{(A,\omega,g-1)}(\alpha_1, \cdots, \alpha_k, H_\gamma, H_\delta | \beta_1, \cdots, \beta_l).\end{aligned}$$

Proof. The proof is similar to that of Theorem 7.3. Let z_1, z_2 be distinguished points on the normalization Σ' of Σ_0; i.e., Σ' is a genus-$(g-1)$ curve, and there is a holomorphic map $\pi : \Sigma' \to \Sigma_0$ such that $\pi(z_1) = \pi(z_2) = p$. Combining the marked points with distinguished points, we can define an evaluation map

$$ev : \mathcal{M}_A(\Sigma', J, \nu) \times (\Sigma')^l \to V^{k+l+2}$$

by

$$ev(f, y_1, \cdots, y_l) = (f(x_1), \cdots, f(x_k), f(z_1), f(z_2), f(y_1), \cdots, f(y_l)).$$

In the same way as before, for any homology class α of V^{k+l+2} satisfying:

$$2n(k + l + 2) - \deg(\alpha) = \dim \mathcal{M}_A(\Sigma', J, \nu) + 2l,$$

one can define an invariant $\Phi_A(\alpha)$ by counting the number of elements in $ev^{-1}(\mathrm{Im}(F))$ with sign, where $F : Y \to V^{k+l+2}$ is a pseudo-manifold representative of α. Moreover, one can show that $\Phi_A(\alpha)$ is independent of the choices of J, ν, complex structures of $\mathcal{C}$ as well as pseudo-manifold representative F of α. Let $\alpha = \otimes_i \alpha_i \otimes [\Delta] \otimes_j \beta_j$. Using its pseudo-manifold representative

$$\prod_i F_i \times \Delta \times \prod_j G_j : \prod_i Y_i \times V \times \prod_j Z_j \to V^{k+l+2},$$

one can show

$$\Phi_A(\alpha) = \Phi_{(A,\omega,\mathcal{C})}(\otimes_i \alpha_i \otimes [\Delta] \otimes_j \beta_j).$$

On the other hand, using the Künneth formula as above, we deduce

$$\alpha = \sum_{\gamma,\delta} \eta^{\gamma\delta} \otimes_i \alpha_i \otimes H_\gamma \otimes H_\delta \otimes_j \beta_j,$$

which implies

$$\Phi_A(\alpha) = \sum_{\gamma,\delta} \eta^{\gamma\delta} \Phi_A(\otimes_i \alpha_i \otimes H_\gamma \otimes H_\delta \otimes_j \beta_j).$$

Since

$$\Phi_A(\otimes_i \alpha_i \otimes H_\gamma \otimes H_\delta \otimes_j \beta_j) = \Phi_{A,\omega,g-1}(\alpha_1, \cdots, \alpha_k, H_\gamma, H_\delta | \beta_1, \cdots, \beta_l),$$

the proof is complete.

8. The quantum cohomology ring

In this section, we will establish a quantum ring structure on the cohomology of a semi-positive symplectic manifold. The key point is the associativity. This will be proved by using the composition law of last section and certain algebraic arguments in [25]. We will also compute the quantum rings for some simple algebraic manifolds.

We put

$$\tilde{\Phi}_{(A,\omega)}(\alpha_1, \cdots, \alpha_k) = \Phi_{(A,\omega,0)}(\alpha_1, \cdots, \alpha_k \mid), \tag{8.1}$$

where $\alpha_1, \cdots, \alpha_k$ are integral homolgy classes. This is the Witten's k-point correlation function (cf. [19]), and depends on a homology class A. We can also drop this dependence by summing contributions over all possible A. So we can formally write the k-point function

$$\tilde{\Phi}_\omega(\alpha_1, \ldots, \alpha_k)(t) = \sum_A \tilde{\Phi}_{(A,\omega)}(\alpha_1, \ldots, \alpha_k)\, e^{-t\omega(A)}, \tag{8.2}$$

where t is a parameter. There may be infinitely many A which contribute the summation in (8.2). Calabi-Yau 3-folds or CP^2 blow-up at 9-points are such examples. Therefore, there is a problem on the convergence of the series in (8.2), which we will not address here. However, for a symplectic manifold (V, ω) with first Chern class $C_1(V) \geq c\omega$ for some $c > 0$, the Gromov-Uhlenbeck Compactness Theorem (cf. Proposition 3.1) implies that there are only finitely many nonzero terms in (8.2) for each fixed set of homology classes $\alpha_1, \cdots, \alpha_k$.

We define quantum multiplication by $\alpha^* \times_Q \beta^*$ for α^*, β^* in $H^*(V, R)$ by the condition that

$$(\alpha^* \times_Q \beta^*)(\gamma) = \tilde{\Phi}_\omega(\alpha, \beta, \gamma), \tag{8.3}$$

where cycles α , β, γ are the Poincaré duals of cocycles $\alpha^*, \beta^*, \gamma^*$. Such a quantum multiplication is defined over the integer ring Z in the following sense: if α^*, β^* are in $H^*(V, Z)$, then the evaluation of $\alpha^* \times_Q \beta^*$ at $t = 0$ lies in $H^*(V, Z)$. From Proposition 2.7 It follows that

$$\alpha^* \times_Q \beta^* \;=\; (-1)^{\deg(\alpha^*)\deg(\beta^*)} \beta^* \times_Q \alpha^*. \tag{8.4}$$

It is also useful to write the quantum multiplication in terms of a basis of $H_*(V, Z)$ as follows: Choose a basis $\{H_\sigma\}$ of $H_*(V, Z)$ modulo torsions. Let $\{\eta_{\gamma\sigma}\}$ be the intersection matrix associated with the basis, i.e., $\eta_{\gamma\sigma} = H_\gamma \cap H_\sigma$. Note that $\eta_{\gamma\sigma} = 0$ if the degrees of H_γ and H_σ do not sum up to the dimension of V. We define

$$f_{\alpha\beta\gamma}(t) = \tilde{\Phi}_\omega(H_\alpha, H_\beta, H_\gamma)(t) \tag{8.5}$$

and

$$f^\delta_{\alpha\beta}(t) = \eta^{\gamma\delta} f_{\alpha\beta\gamma}(t), \tag{8.6}$$

where $\{\eta^{\gamma\sigma}\}$ is the inverse matrix of $\{\eta_{\gamma\sigma}\}$.

Let $\{H_\sigma^*\}$ be the basis of $H^*(V,Z)$, which is dual to $\{H_\sigma\}$. Then the quantum multiplication in terms of $\{H_\sigma^*\}$ is given by

$$H_\alpha^* \times_Q H_\beta^* = \sum_\gamma f_{\alpha\beta}^\gamma(t) H_\gamma^*. \tag{8.7}$$

which implies

$$\begin{aligned}\left(H_\alpha^* \times_Q H_\beta^*\right) \times_Q H_\gamma^*(t) &= \sum_{\delta,\sigma} f_{\alpha\beta}^\delta(t) f_{\delta\gamma}^\sigma(t) H_\sigma^* \\ &= \sum_\sigma \left(\sum_\delta f_{\alpha\beta}^\delta(t) f_{\delta\gamma}^\sigma(t)\right) H_\sigma^*.\end{aligned} \tag{8.8}$$

Similarly,

$$H_\alpha^* \times_Q \left(H_\beta^* \times_Q H_\gamma^*\right)(t) = \sum_\sigma \left(\sum_\delta f_{\beta\gamma}^\delta(t) f_{\alpha\delta}^\sigma(t)\right) H_\sigma^*. \tag{8.9}$$

Therefore, as Witten observed in [25], the quantum multiplication is associative if and only if the following equations hold:

$$f_{\alpha\beta}^\delta(t) f_{\delta\gamma}^\sigma(t) = f_{\beta\gamma}^\delta(t) f_{\alpha\delta}^\sigma(t). \tag{8.10}$$

Using (8.2), (8.5), (8.6), one can easily see that (8.10) is equivalent to the following identities

(8.11)
$$\begin{aligned}&\sum_{A=A_1+A_2} \sum_{\sigma,\tau} \eta^{\sigma\tau} \tilde{\Phi}_{(A_1,\omega)}(H_\alpha, H_\beta, H_\tau) \tilde{\Phi}_{(A_2,\omega)}(H_\sigma, H_\gamma, H_\delta) \\ &\quad = \sum_{A=A_1+A_2} \sum_{\sigma,\tau} \eta^{\sigma\tau} \tilde{\Phi}_{(A_1,\omega)}(H_\beta, H_\gamma, H_\tau) \tilde{\Phi}_{(A_2,\omega)}(H_\alpha, H_\sigma, H_\delta),\end{aligned}$$

where A, $H_\alpha, H_\beta, H_\gamma, H_\delta$ are given. By the symmetry of the Witten invariants (cf. Proposition 2.7), we have

$$\tilde{\Phi}_{(A_2,\omega)}(H_\alpha, H_\sigma, H_\delta) = (-1)^{\deg(H_\alpha)\deg(H_\sigma)} \tilde{\Phi}_{(A_2,\omega)}(H_\sigma, H_\alpha, H_\delta). \tag{8.12}$$

On the other hand, $\tilde{\Phi}_{(A_2,\omega)}(H_\sigma, H_\alpha, H_\delta)$ is zero unless the sum of degrees of H_α, H_σ and H_δ is an even integer. Therefore, from (8.12) it follows that (8.11) is the same as

$$\begin{aligned}&\sum_{A=A_1+A_2} \sum_{\sigma,\tau} \eta^{\sigma\tau} \tilde{\Phi}_{(A_1,\omega)}(H_\alpha, H_\beta, H_\tau) \tilde{\Phi}_{(A_2,\omega)}(H_\sigma, H_\gamma, H_\delta) \\ &\quad = (-1)^{\deg(H_\alpha)(\deg(H_\alpha)+\deg(H_\delta))} \\ &\quad \times \sum_{A=A_1+A_2} \sum_{\sigma,\tau} \eta^{\sigma\tau} \tilde{\Phi}_{(A_1,\omega)}(H_\beta, H_\gamma, H_\tau) \tilde{\Phi}_{(A_2,\omega)}(H_\sigma, H_\alpha, H_\delta).\end{aligned} \tag{8.13}$$

In the case of Calabi-Yau 3-folds, both sides of (8.13) are equal to zero, so the quantum multiplication is automatically associative. In general, (8.13) is not obvious. By the composition law in last section, the left side of (8.13) is the same as $\tilde{\Phi}_{(A,\omega)}(H_\alpha, H_\beta, H_\gamma, H_\delta)$, while the right side is equal to

$$(8.14) \qquad (-1)^{\deg(H_\alpha)(\deg(H_\alpha)+\deg(H_\delta))}\tilde{\Phi}_{(A,\omega)}(H_\beta, H_\gamma, H_\alpha, H_\delta).$$

Thus (8.13) is a direct consequence of the symmetry of the 4-point function, and the invariant in (8.14) is zero unless degrees of H_β, H_γ, H_α, H_δ sum up to an even integer. Therefore, we have

Theorem 8.1. *The quantum multiplication is associative; consequently, there is a quantum ring structure on the cohomology of any semi-positive symplectic manifold V.*

From the composition law in last section it follows that

$$(8.15) \qquad H^*_{\alpha_1} \times_Q \cdots \times_Q H^*_{\alpha_k} = \sum_{\gamma,\delta} \eta^{\gamma\delta}\tilde{\Phi}_\omega(H_{\alpha_1}, \cdots, H_{\alpha_k}, H_\gamma)(t)H^*_\delta.$$

We should remark that both sides of (8.10) are infinite sum, whose convergence in general remains to be checked. However, the above equation is well posed as a sequence of equations involving only finite sums.

We will call the cohomology $H^*(V, R)$ with the quantum multiplication the quantum cohomology of V, where V is a semi-positive symplectic manifold. By Proposition 2.6 we hence have

Proposition 8.2. *The quantum cohomology of the product of two semi-positive symplectic manifolds is the product of quantum cohomologies of the two manifolds.*

We observe that the quantum product $\alpha \times_Q \beta$ does not preserve the grading of the ordinary cup product. The quantum product may be the sum of several cohomology classes of different degree. This can be seen from (8.4). However, one can check that it always decreases the degree by $2C_1(V)(A)$ for some second homology class A, as long as $\tilde{\Phi}_A \neq 0$. Assume that V is a compact symplectic manifold with positive first Chern class $C_1(V)$. I.e., for any nonconstant J-holomorphic map $f : \Sigma \mapsto V$, $f^*(C_1(V))(\Sigma) > 0$; this is particularly true if $C_1(V) \geq c\omega$ for some positive constant c. Let α^*, β^* be any two cohomology classes, and γ be a homology class. Then

$$\alpha^* \times_Q \beta^*(\gamma) \;=\; \tilde{\Phi}_\omega(\alpha, \beta, \gamma) \neq 0$$

only if

$$\deg(\gamma) \leq \deg(\alpha^*) + \deg(\beta^*), \tag{8.16}$$

where α, β are the Poincaré duals of α^*, β^*. This is because $\tilde{\Phi}_{(A,\omega,0)}(\alpha, \beta, \gamma)$ is nonzero only if

$$\deg(\alpha^*) + \deg(\beta^*) + 2n - \deg(\gamma) = C_1(V)(A) + 2n \geq 2n.$$

If $\tilde{\Phi}_{(A,\omega,0)}(\alpha, \beta, \gamma)$ is nonzero, then there is at least J-holomorphic map f from S^2 into V with $f_*(S^2) = A$. It follows that $C_1(V)(A) > 0$, if A is not the zero class. Consequently, in case the equality holds in (8.16), we have

$$\alpha^* \times_Q \beta^*(\gamma) \ = \ \tilde{\Phi}_{(0,\omega)}(\alpha, \beta, \gamma) \tag{8.17}$$

Hence we can deduce the following from Proposition 2.5,(5).

Proposition 8.3. *Let V be a symplectic manifold with positive first Chern class, and α^*, β^* be two cohomology classes in $H^*(V, Z)$. Then*

$$\alpha \times_Q \beta = \alpha \cup \beta + \text{ terms of lower degree }. \tag{8.18}$$

As we mentioned before, there is a problem with the convergence of the power series in (8.2), if (V, ω) is no longer a symplectic manifold with positive first Chern class, such as Calabi-Yau manifolds. It was conjectured that the power series in (8.2) is convergent whenever t is sufficiently large. But we can use the Novikov ring to avoid this problem of convergence. The use of the Novikov ring is extensively discussed in [16]. Here we will only give a brief discussion on formulation of the quantum cohomology ring in terms of the Novikov ring.

The symplectic form ω induces a homomorphism, still denoted by ω, from $H_2(V, Z)$ into R. Fix such a homomorphism, we can define a Novikov ring Λ_ω as follows: Each $A \in H_2(V, Z)$ induces a homomorphism

$$p(A) : H^2(V, C/Z) = H^2(V, C)/H^2(V, Z) \to C^* \tag{8.19}$$

by $p(A)(a) = e^{2\pi\sqrt{-1}a(A)}$. Clearly, $p(A_1 + A_2) = p(A_1)p(A_2)$. For simplicity, we will denote $p(A)$ by $e^{2\pi\sqrt{-1}A}$. Then the Novikov ring Λ_ω consists of the Fourier series of the form

$$\lambda = \sum_A \lambda_A e^{2\pi\sqrt{-1}A}, \tag{8.20}$$

where A runs over all integral homology classes in $H_2(V, Z)$, and the coefficients λ_A satisfy

$$\{A \mid \lambda_A \neq 0, \omega(A) \leq c\} \;<\; \infty \tag{8.21}$$

for any $c > 0$. If $\mu = \sum_A \mu_A e^{2\pi\sqrt{-1}A}$ be another element of Λ_ω, we define the multiplication by

$$\lambda \star \mu = \sum_{A,A'} \lambda_A \mu_{A'} e^{2\pi\sqrt{-1}(A+A')}. \tag{8.22}$$

It is straightforward to check that $\lambda\star\mu$ satisfies the finiteness condition (8.21). The Novikov ring also carries a natural grading defined by $\deg(e^{2\pi\sqrt{-1}A}) = 2C_1(V)(A)$. We will use Λ_j to denote all elements of degree j in Λ_ω. Note that if we choose a basis of $H_2(V, Z)$ and expand $e^{2\pi\sqrt{-1}A}$ over this basis, we can write λ as a power series.

The quantum cohomology ring can be now defined to be

$$H^*_Q(V) = H^*(V, R) \otimes \Lambda_\omega \cong \operatorname{Hom}(H_*(V), \Lambda_\omega). \tag{8.23}$$

This can be graded by

$$H^k_Q(V) = \bigoplus_{j=0}^{2n} \operatorname{Hom}(H_j(V), \Lambda_{k-j}). \tag{8.24}$$

Now we define a new 3-point function

$$\tilde{\Phi}_{\Lambda_\omega}(\alpha, \beta, \gamma) = \sum_A \tilde{\Phi}_A(\alpha, \beta, \gamma) e^{2\pi\sqrt{-1}A}. \tag{8.25}$$

From the Gromov-Uhlenbeck compactness theorem it follows that $\tilde{\Phi}_{\Lambda_\omega}(\alpha, \beta, \gamma) \in \Lambda_\omega$. Furthermore,

$$\deg(\tilde{\Phi}_{\Lambda_\omega}(\alpha, \beta, \gamma)) = 4n - \deg(\alpha) - \deg(\beta) - \deg(\gamma). \tag{8.26}$$

For any α^*, β^* in $H^*(V, R)$, their quantum multiplication $\alpha^* \star_Q \beta^*$ is defined by the condition

$$(\alpha^* \star_Q \beta^*)(\gamma) = \tilde{\Phi}_{\Lambda_\omega}(\alpha, \beta, \gamma), \tag{8.27}$$

where α, β are Poincaré duals of α^*, β^*, and γ is any homology class. Then we extend this quantum multiplication to any two elements of $H^*_Q(V)$ by linearity.

One advantage of this new definition is that the quantum multiplication preserves the grading, i.e.,

$$\deg(\alpha^* \star_Q \beta^*) \;=\; \deg(\alpha^*) deg(\beta^*). \tag{8.28}$$

Again, the same argument as before will show

Theorem 8.4. *The quantum multiplication $\alpha^* \star_Q \beta^*$ is associative; consequently, there is an associative quantum ring structure on the cohomology $H^*_Q(V)$ of any semi-positive symplectic manifold V with coefficient ring Λ_ω.*

Example 8.4. Let V be a $K3$-surface. It is well known that for a generic complex structure on a $K3$-surface, there are no non-constant holomorphic curves. Therefore $\tilde{\Phi}_\omega = \tilde{\Phi}_{(0,\omega)}$, and consequently, the quantum cohomology ring is the same as the ordinary cohomology ring. In fact, the same is true for any hyperkähler manifolds.

Example 8.5. Let V be the complex projective space CP^n with the Fubini-Study form ω as its symplectic form. This is a semi-positive symplectic manifold. Now let us compute the 3-point function of CP^n. Note that for any three homology classes α, β, γ, the sum of their codimensions $6n - \deg(\alpha) - \deg(\beta) - \deg(\gamma) \leq 6n$. Let ℓ be the generator of $H_2(CP^n, Z)$ represented by a complex line. Since $\dim \mathcal{M}_{d\ell}(S^2, J_0, 0) = 2d(n+1) + 2n$, where J_0 is the standard complex structure on CP^n, only $\mathcal{M}_{d\ell}(S^2, J_0, 0)$ for $d = 0, 1$ will give any nontrivial contributions to the 3-point function. In these cases, the moduli space $\mathcal{M}_{d\ell}(S^2, J_0, 0)$ is automatically compact and smooth, so we can use it to calculate the 3-point function.

First we compute $\tilde{\Phi}_{(\ell,\omega)}(pt, pt, H)$, where H is a generic hyperplane in CP^n. Note that the moduli space $\mathcal{M}_\ell(S^2, J_0, 0)$ consists of holomorphic maps from S^2 into CP^n such that its image is a line in CP^n. An elementary fact about lines is that there is a unique line through any two points in CP^n. Fixed two distinct points x_1, x_2 in CP^n and a hyperplane H not through x_1 and x_2. Let L be the unique line passing through x_1, x_2, and intersect H at x_3. Fixed three points, say $0, 1, \infty \in S^2$. By parametrizing this line properly, we can find a holomorphic map f from S^2 onto L, such that $f(0) = x_1$, $f(1) = x_2$, $f(\infty) = x_3$. Such a f is unique since any element of PSL_2 is uniquely determined by three points. Therefore, $\tilde{\Phi}_{(\ell,\omega)}(pt, pt, H) = 1$ and

$$\tilde{\Phi}_\omega(pt, pt, H)(t) = e^{-t}.$$

We now compute the quantum cohomology ring of CP^n. Let x be the Poincaré dual of H. Then x generates $H^*(CP^n, Z)$. We will use x^k to denote the ordinary k^{th}-power of x, and x^k_Q to denote k^{th} quantum power of x. Note that $C_1(CP^n) = (n+1)x$. For $k \leq n$, $\tilde{\Phi}_{(0,\omega)}(x, \cdots, x)$ is the only nonzero term in (8.15) with $\alpha_i = x$. Hence, $x^k_Q = x^k$. Similarly, $\tilde{\Phi}_{(\ell,\omega)}$ is the only term which contributes to the quantum product x^{n+1}_Q. By counting degrees, one can easily show x^{n+1}_Q is of the degree zero, i.e., x^{n+1}_Q is a number. This number is equal to $\tilde{\Phi}_{(\ell,\omega)}(pt, H, pt) = e^{-t}$. It follows $x^{n+1}_Q = e^{-t}$. Higher powers of x can be computed by this and the associativity. Therefore, the quantum cohomology ring of CP^n is the quotient of the polynomial ring $R[x]$ by the ideal generated by $x^{n+1} - e^{-t}$.

Example 8.6. Let V be the surface obtained by blowing up CP^2 at one point. Then $H^*(V, Z)$ is generated by x, y, where x is the pull-back of a line in CP^2, and y is the exceptional divisor from the blowing-up. The relations are $x^2 + y^2 = 0$, $xy = 0$, i.e., the ordinary ring is isomorphic to the quotient of $R[x, y]$ by the ideal generated by $x^2 + y^2$, xy. Note that for simplicity, we will identify a cohomology class with its Poincaré dual. We choose ω to be the first Chern class $C_1(V)$, which is positive. Then $\omega(x) = 3$ and $\omega(y) = 1$. There are two homology classes $2y$, $x - y$ with $C_1(V)(2y) = C_1(V)(x - y) = 2$, two classes $3y$, x with $C_1(V)(x) = C_1(V)(3y) = 3$ and three classes $A = 4y$, or $2(x - y)$, or $x + y$ with $C_1(V)(A) = 4$. For dimensional reasons, these are only homology classes contributing to the 3-point function $\tilde{\Phi}_\omega$. Let J_0 be the complex structure on V, and J be a generic almost complex structure near J_0. It is clear that $\mathcal{M}_{2y}(S^2, J_0, 0)$ consists of only double-branched covering maps onto y. Therefore, all maps in $\mathcal{M}_{2y}(S^2, J, \nu)$ have their image near the exceptional divisor y, where ν is a small inhomogeneous term. The expected dimension of $\mathcal{M}_{2y}(S^2, J, \nu)$ is 8. It implies that $\tilde{\Phi}_{(2y,\omega)}(\alpha, \beta, \gamma)$ is zero unless one of the cycles, say α, has its degree less than 2, so one can choose a pseudo-submanifold representative of α which does not intersect y at all. If (J, ν) is sufficiently close to $(J_0, 0)$, the pseudo-submanifold does not intersect any (J, ν)-maps in $\mathcal{M}_{2y}(S^2, J, \nu)$. It follows that $\tilde{\Phi}_{(2y,\omega)}(\alpha, \beta, \gamma)$ is identically zero. Similarly, $\mathcal{M}_{3y}(S^2, J_0, 0)$, $\mathcal{M}_{4y}(S^2, J_0, 0)$ and $\mathcal{M}_{2(x-y)}(S^2, J_0, 0)$ contain only multiple covering maps. Then one can show by the same arguments that

$$\tilde{\Phi}_{(3y,\omega)} = \tilde{\Phi}_{(4y,\omega)} = \tilde{\Phi}_{(2(x-y),\omega)} = 0.$$

We claim that the 3-point function $\tilde{\Phi}_{(x+y,\omega)} = 0$ as well. We observe that $\mathcal{M}_{x+y}(S^2, J_0, 0) = \emptyset$. But there are cusp curves of the form $L + y$, where L is a line. The space of such cusp curves is of real dimension 8, which is four less than the expected dimension of $\mathcal{M}_{x+y}(S^2, J_0, 0)$. So the complex structure J_0 is $(x + y)$-good and $\tilde{\Phi}_{(x+y,\omega)} = 0$.

On the other hand, the moduli spaces $\mathcal{M}_y(S^2, J_0, 0)$, $\mathcal{M}_{x-y}(S^2, J_0, 0)$, $\mathcal{M}_x(S^2, J_0, 0)$ are all smooth and quite simple. More precisely, $\mathcal{M}_y(S^2, J_0, 0)/PSL_2 = \{y\}$, $\mathcal{M}_{x-y}(S^2, J_0, 0)/PSL_2$ is the space of lines in CP^2 passing through the blowing up point, and $\mathcal{M}_x(S^2, J_0, 0)$ is the space of lines of CP^2. It is not hard to compute contributions to the 3-point function from these moduli spaces. We summarize the results as follows:

$$\begin{aligned} x_Q^2 &= x^2 + e^{-2t}, \\ y_Q^2 &= y^2 + e^{-t}y + e^{-2t}, \\ x \times_Q y &= x \times y + e^{-2t} = e^{-2t}. \end{aligned} \tag{8.29}$$

It follows that the quantum cohomology ring of V is the quotient of the polynomial ring $R[x, y]$ by an ideal generated by $x^2 + y^2 - e^{-t}y - 2e^{-2t}$ and $xy - e^{-2t}$.

9. Mirror symmetry conjecture

The Mirror Symmetry Conjecture relates rational curves on an algebraic manifold with the variation of Hodge structures of its mirror manifold. An crucial step in solving this conjecture is to construct a family of flat connections on $H_*(V, C)$ which deform the trivial connection. These flat connections are different from the Gauss-Manin connections, which come from the variation of Hodge structures. In this section, we will use our mixed invariants and their composition law to construct such a family of flat connection.

Let $W = H^*(V, Z) \otimes C = H^*(V, C)$. As in last section, we choose a basis $\{H_\alpha\}$ of $H^*(V, Z)$ modulo torsions, and let $\{H_\alpha^*\}$ be its dual basis of $H^*(V, Z)$. Any point w in W can be written as

$$(9.1) \qquad w = \sum_{j=1}^{L} t_j H_j^*,$$

where L is the dimension of W. One can regard $t_1, \cdots, t_L$ as the coordinates of w in W. We denote by w_* the corresponding point $\sum_{j=1}^{L} t_j H_j$ in $H_*(V, C)$.

By Proposition 2.5, the mixed invariant $\Phi_{(A,\omega,g)}$ is multilinear, so we can extend $\Phi_{(A,\omega,g)}$ to be a multilinear function on $H_*(V, C)$ in an obvious way. Following E. Witten [25], we define a generating function

$$(9.2) \qquad \begin{aligned} \Psi_\omega(w) = \sum_{A \in H_2(V,Z)} e^{-\omega(A)} \sum_{\substack{m=3 \\ w_* = w_{1*} = \cdots = w_{m-3*}}}^{\infty} \frac{1}{m!} \\ \cdot \Phi_{(A,\omega,0)}(w_*, w_*, w_* | w_{1*}, \cdots, w_{m-3*}). \end{aligned}$$

This function is a power series in $t_1, \cdots, t_L$. We define a connection ∇_ϵ for any number ϵ on the tangent bundle TW over W as follows: For each tangent vector $v = v^\alpha \frac{\partial}{\partial t_\alpha}$ in TW,

$$(9.3) \qquad \nabla_\epsilon v \;=\; \sum_{\alpha,\beta} \left(\frac{\partial v^\alpha}{\partial t_\beta} + \epsilon \sum_{\tau,\gamma} \eta^{\alpha\gamma} \frac{\partial^3 \Psi_\omega}{\partial t_\gamma \partial t_\beta \partial t_\tau} v^\tau \right) \frac{\partial}{\partial t^\alpha} dt_\beta,$$

where $\{\eta^{\alpha\gamma}\}$ is the inverse of the intersection matrix associated with the basis $\{H_\alpha\}$ (cf. (8.5), (8.6)). Obviously, ∇_0 is the trivial connection on W. The following is just Theorem C in the introduction.

Theorem 9.1. *The connection ∇_ϵ is flat and a deformation of the trivial flat connection on W.*

Proof. The flateness of ∇_ϵ means

$$(9.4) \qquad \nabla_\epsilon(\nabla_\epsilon v) \;=\; 0$$

for any v in TW. Using the definition (9.3) of ∇_ϵ, one can easily show that for $\epsilon \neq 0$, (9.4) is equivalent to the so called WDVV equation

$$(9.5) \qquad \sum_{\sigma,\tau} \frac{\partial^3 \Psi_\omega}{\partial t_\alpha \partial t_\beta \partial t_\sigma} \eta^{\sigma\tau} \frac{\partial^3 \Psi_\omega}{\partial t_\gamma \partial t_\delta \partial t_\tau} = \sum_{\sigma,\tau} \frac{\partial^3 \Psi_\omega}{\partial t_\alpha \partial t_\gamma \partial t_\sigma} \eta^{\sigma\tau} \frac{\partial^3 \Psi_\omega}{\partial t_\beta \partial t_\delta \partial t_\tau}.$$

Both sides of (9.5) are power series in $t_1, \cdots, t_L$. We observe that there is a canonical splitting of W, i.e., $W = W_e + W_o$, where W_e consists of all cohomology classes of even degrees, and W_o consists of all cohomology classes of odd degrees; consequently, any w in W is of the form $w_e + w_o$ and $w_* = w_{e*} + w_{o*}$. Since the moduli space of (J, ν)-maps is always even-dimensional, we have

$$\Phi_{(A,\omega,g)}(w_{o*}, w_{e*}, w_{e*} \,|w_{e*}, \cdots, w_{e*}) = 0.$$

Thus from Proposition 2.7 it follows that

$$\begin{aligned}(9.6)\qquad \Phi_{(A,\omega,g)}&(w_*, w_*, w_* \,|\, w_*, \cdots, w_*)\\ &= \Phi_{(A,\omega,g)}(w_{e*}, w_{e*}, w_{e*} \,|w_{e*}, \cdots, w_{e*}).\end{aligned}$$

Hence, the connection ∇_ϵ is flat on TW_o and determined by its restriction to TW_e. For simplicity, we may assume that $W_o = \{0\}$. By denoting Ψ_ω by $\Psi_\omega(t_1, \cdots, t_L)$, we have the following expansion:

$$\begin{aligned}(9.7)\qquad \Psi_\omega(t_1, \cdots, t_L) &= \sum_A e^{-\omega(A)} \sum_{m=3}^{\infty} \sum_{1\le\alpha_1,\cdots,\alpha_m\le L} \frac{1}{m!}\\ &\cdot\Phi_{(A,\omega,0)}(H_{\alpha_1}, H_{\alpha_2}, H_{\alpha_3} | H_{\alpha_4}, \cdots, H_{\alpha_m})\, t_{\alpha_1} \cdots t_{\alpha_m}.\end{aligned}$$

Taking the third derivative on both sides of (9.7), we obtain

$$\begin{aligned}(9.8)\qquad \frac{\partial^3\Psi_\omega}{\partial t_\alpha \partial t_\beta \partial t_\sigma}(t_1, \cdots, t_L) &= \sum_A e^{-\omega(A)} \sum_{m=0}^{\infty} \sum_{1\le\alpha_1,\cdots,\alpha_m\le L} \frac{1}{m!}\\ &\cdot\Phi_{(A,\omega,0)}(H_\alpha, H_\beta, H_\sigma | H_{\alpha_1}, \cdots, H_{\alpha_m})\, t_{\alpha_1} \cdots t_{\alpha_m}\end{aligned}$$

Substituting the third derivatives of Ψ_ω by the power series in (9.8) and equaling the coefficients can easily show that (9.5) is equivalent to

(9.9)

$$\begin{aligned}&\sum_{A=B_1+B_2} \sum_{j=0}^{l} \sum_{p} \sum_{\sigma,\tau} \frac{1}{j!(l-j)!} \eta^{\sigma\tau}\, \Phi_{(B_1,\omega,0)}(H_\alpha, H_\beta, H_\sigma \,|\, H_{\alpha_{p(1)}}, \cdots, H_{\alpha_{p(j)}})\\ &\qquad\qquad \cdot\Phi_{(B_2,\omega,0)}(H_\gamma, H_\delta, H_\tau \,|\, H_{\alpha_{p(j+1)}}, \cdots, H_{\alpha_{p(l)}})\\ =\ &\sum_{A=B_1+B_2} \sum_{j=0}^{l} \sum_{p} \sum_{\sigma,\tau} \frac{1}{j!(l-j)!} \eta^{\sigma\tau}\, \Phi_{(B_1,\omega,0)}(H_\alpha, H_\gamma, H_\sigma \,|\, H_{\alpha_{p(1)}}, \cdots, H_{\alpha_{p(j)}})\\ &\qquad\qquad \cdot\Phi_{(B_2,\omega,0)}(H_\beta, H_\delta, H_\tau \,|\, H_{\alpha_{p(j+1)}}, \cdots, H_{\alpha_{p(l)}})\end{aligned}$$

for all α, β, γ, δ, l and A, where p runs over all permutations on $\{1, \cdots, l\}$. By the composition law in section 7 (cf. (1.1)), both sides of (9.9) are equal to the mixed invariant

$$\Phi_{(A,\omega,0)}(H_\alpha, H_\beta, H_\gamma, H_\delta \,|\, H_{\alpha_1}, \cdots, H_{\alpha_l}).$$

Hence the theorem is proved.

Remark 9.2. (1) The generating function Ψ_ω has a scaling property: For any complex number s,

$$\begin{aligned}\Psi_{\omega+2C_1(V)s}(\, e^{(\deg(H_1^*)-2)s}t_1, \cdots, e^{(\deg(H_L^*)-2)s}t_L) \\ = e^{(2n-6)s}\Psi_\omega(t_1, \cdots, t_L).\end{aligned} \tag{9.10}$$

Which follows easily from (9.7) and the fact that

$$\Phi_{(A,\omega,0)}(H_{\alpha_1}, H_{\alpha_2}, H_{\alpha_3}|H_{\alpha_4}, \cdots, H_{\alpha_m})$$

vanishes unless

$$\sum_{i=1}^{m}(\deg(H_{\alpha_i}^* - 2) = 2C_1(V)(A) - 6.$$

Consequently, the connection ∇_ϵ admits a scaling property. In case V has vanishing first Chern class, this scaling property makes TW into a Hodge bundle. Note that ∇_ϵ preserves the inner product on TW by the cup product.

(2) The function Ψ_ω depends only on even degree cohomology classes, as shown above. It is possible to modify the definition to include contributions of odd degree cohomology classes. The simplest modification is to replace $\frac{1}{m!}$ in (9.7) by $\frac{\epsilon(\{\alpha_i\})}{m!}$, where $\epsilon(\{\alpha_i\})$ is $\pm$ depending on the induced ordering on odd degree cohomology classes in $\{H_\alpha\}$ by $\{\alpha_i\}$. Since such a genaralization is straightforward, we leave the details to the readers.

We may assume that the basis $\{H_\alpha\}$ is chosen, such that (1) there is an $L' \le L$, and $\deg(H_\alpha)$ is even or odd according to whether $\alpha \le L'$ or $\alpha > L'$; (2) $\deg(H_\alpha) \le \deg(H_\beta)$ whenever $\alpha < \beta \le L'$. In particular, the degree of $H_{L'}$ is $2n$. We further assume that $\deg(H_\alpha)$ is $2n-2$ for $N < \alpha < L'$.

Proposition 9.3. *Let $\{H_\alpha\}$ be chosen as above. Then we have*

$$\begin{aligned}\frac{\partial^3\Psi_\omega}{\partial t_\alpha \partial t_\beta \partial t_\gamma}(t_1, \cdots, t_L) = (H_\alpha \cap H_\beta \cap H_\gamma) + \sum_{A\neq 0}\sum_{\alpha_i \le N}\frac{1}{m!} \\ \cdot \Phi_{(A,\omega,0)}(H_\alpha, H_\beta, H_\gamma|H_{\alpha_1}, \cdots, H_{\alpha_m})\, t_{\alpha_1}\cdots t_{\alpha_m}\, e^{-\omega(A) + \sum_{\alpha=N+1}^{L'-1} t_\alpha H_\alpha^*(A)},\end{aligned} \tag{9.11}$$

where $(H_\alpha \cap H_\beta \cap H_\gamma)$ denotes the intersection number of H_α, H_β, H_γ, and ω^* is the dual of ω.

Proof. If $A = 0$, by Proposition 2.5.(5), the invariant $\Phi_{(A,\omega,0)}(H_\alpha, H_\beta, H_\gamma | H_{\alpha_1}, \cdots, H_{\alpha_m})$ is zero or $(H_\alpha \cap H_\beta \cap H_\gamma)$ according to $m > 0$ or $m = 0$. Therefore, we obtain the first term in (9.11) from the expression (9.7) of Ψ_ω.

Now we assume that $A \neq 0$. If one of H_{α_i} is of the degree $2n$, then by Proposition 2.5.(3), we have

$$\Phi_{(A,\omega,0)}(H_\alpha, H_\beta, H_\gamma | H_{\alpha_1}, \cdots, H_{\alpha_m}) = 0.$$

Thus (9.11) follows from (9.7) and Proposition 2.5.(4).

Remark 9.4. There is also a problem on convergence of the power series in defining Ψ_ω. In general, it is unknown if the series in (9.7) is convergent for any $t_1, \cdots t_L$, for instance, for Calabi-Yau manifolds. In case V is a symplectic manifold with positive first Chern class, one should be able to prove that the series in (9.11) is convergent when $|t_1|, \cdots, |t_{L'}|$ are small and ω is sufficiently large. This is because the composition law in section 7 implies a recursion formula for $\Phi_{(A,\omega,0)}(H_{\alpha_1}, H_{\alpha_2}, H_{\alpha_3} | H_{\alpha_4} \cdots, H_{\alpha_m})$ (cf. section 10). But we will not discuss this convergence problem here.

Example 9.5. Let V be an irreducible complex surface with positive first Chern class. Then V can be obtained by blowing up CP^2 at generic s points ($s \leq 8$). Choose a basis $\{H_i\}_{1 \leq i \leq s+2}$ satisfying: H_1 is a point, H_2 is the pull-back of a line on CP^2 and H_i is an exceptional divisor from the blowing-ups for each i between 3 and $s+1$. On the other hand, the first Chern class $C_1(V)$ is $3H_2^* - \sum_{i=3}^{s+1} H_i^*$, if there is a nonconstant holomorphic curve with homology class $A = \sum_{i=2}^{s+1} d_i H_i$, then $3d_2 - d_3 - \cdots - d_{s+1} > 0$. We will identify A with $(d_2, \cdots, d_{s+1})$. Thus

$$\begin{aligned} \Psi_\omega(t_1, \cdots, t_{s+2}) &= \frac{1}{2}(t_1 t_{s+2}^2 + t_{s+2}(t_2^2 - t_3^2 - \cdots - t_{s+1}^2)) \\ &+ \sum_{3d_2 - d_3 - \cdots - d_{s+1} > 0} n(d_2, \cdots, d_{s+1}) \frac{t_1^{3d_2 - d_3 - \cdots - d_{s+1} - 1}}{(3d_2 - d_3 - \cdots - d_{s+1} - 1)!} \\ &\cdot e^{-\omega((d_2, \cdots, d_{s+1})) + d_2 t_2 - \sum_{i=3}^{s+1} d_i t_i}, \end{aligned} \tag{9.12}$$

where $n(d_2, \cdots, d_{s+1})$ denotes the number of rational curves in V with homology class $(d_2, \cdots, d_{s+1})$ and through generic $3d_2 - d_3 - \cdots - d_{s+1} - 1$ points.

In case $V = CP^2$, this series is

$$\Psi_\omega(t_1, t_2, t_3) = \frac{1}{2}(t_1 t_3^2 + t_3 t_2^2) + \sum_{d \geq 1} \frac{n_d\, t_1^{3d-1}}{(3d-1)!} e^{d(-\omega(H_2)+t_2)}, \tag{9.13}$$

where n_d is the number of rational curves in CP^2 of degree d.

Example 9.6. Let V be a Calabi-Yau 3-dimensional manifold. Since $C_1(V)$ is zero, the "virtual" dimension of any moduli space $\mathcal{M}_A(S^2, J, \nu)$ is 6. It follows that if $A \neq 0$, the invariant

$$\Phi_{(A,\omega,0)}(H_\alpha, H_\beta, H_\gamma | H_{\alpha_1}, \cdots, H_{\alpha_m})$$

vanishes unless each H_{α_i} is a divisor.

We denote by n_A the number of J-holomorphic rational curves in V with homology class A. Then as Aspinwall and Morrison show heuristically in [1], one expects

$$\Phi_{(A,\omega,0)}(H_\alpha, H_\beta, H_\gamma) \;=\; \sum_{B|A} n_B\, (H_\alpha \cap A)(H_\beta \cap A)(H_\gamma \cap A), \tag{9.14}$$

where $B|A$ means that A is divisible by B, i.e., $A = kB$ for some integer k. Thus from (9.11) one can deduce

$$\frac{\partial^3 \Psi_\omega}{\partial t_\alpha \partial t_\beta \partial t_\gamma}(t_1, \cdots, t_L) = (H_\alpha \cap H_\beta \cap H_\gamma) \tag{9.15}$$

$$+ \sum_{A \neq 0} n_A\, (H_\alpha \cap A)(H_\beta \cap A)(H_\gamma \cap A)\, \frac{e^{-\omega(A) + \sum_{\alpha=N+1}^{L'} t_\alpha H_\alpha^*(A)}}{1 - e^{-\omega(A) + \sum_{\alpha=N+1}^{L'} t_\alpha H_\alpha^*(A)}}.$$

It is still an open problem how to prove (9.15) rigorously, namely, carry out mathematically the computation in [1] for the mixed invariants.

In particular, if V is a quintic hypersurface in CP^4, we have

$$\frac{\partial^3 \Psi_\omega}{\partial t_3^3}(t_1, t_2, t_3, t_4) = 5 + \sum_{d \geq 1} \frac{n_d\, d^3\, e^{-d\omega(H_2)+dt_3}}{1 - e^{-d\omega(H_2)+dt_3}}, \tag{9.16}$$

where n_d is the number of irreducible rational curves of degree d. Note that $n_1 = 2875$ and $n_2 = 609250$. It is still an unsolved problem how to compute all n_d mathematically.

10. An application to enumerative geometry

In this section, we give some applications of our main theorem to some well-known problems in enumerative algebraic geometry.

Let V be an algebraic manifold, A be a homology class in $H_2(V, Z)$, and J be the complex structure of V. Then, $\mathcal{M}^*_A(S^2, J, 0)$ is the moduli space of non-multiple cover holomorphic maps f from S^2 to V such that $f_*([S^2]) = A$. The linear group $G = PGL_2$ acts on $\mathcal{M}_A(S^2, J, 0)$ by changing the parametrization. Then, the quotient of $\mathcal{M}^*_A(S^2, J, 0)/G$ is the moduli space of rational curves in V with fixed homology class A. By the Riemann-Roch Theorem, the virtual complex dimension of $\mathcal{M}^*_A(S^2, J, 0)/G$ is $C_1(V)(A) + n - 3$, where n is the complex dimension of V. Let us recall the definition of the counting function σ_A. We will only define it under some nondegenerate conditions. Given generic algebraic subvarieties $Z_1, \cdots, Z_k$ $(k \geq 3)$ satisfying

$$\sum_{i=1}^{k} n - 1 - \dim_C Z_i = C_1(V)(A) + n - 3. \tag{9.1}$$

If there are only finitely many rational curves in $\mathcal{M}^*_A(S^2, J, 0)/G$, which intersect each Z_i $(i = 1, \cdots, k)$ transversally at some smooth points, each such a curve is a smooth point of $\mathcal{M}^*_A(S^2, J, 0)/G$, and there is no sequence of rational curves with homology A which converge to a curve (possibly singular and reducible) intersecting all Z_i, then we can define $\sigma_A(Z_1, \cdots, Z_k)$ to be the number of such rational curves in $\mathcal{M}^*_A(S^2, J, 0)/G$. In general, the function σ_A has to be defined in terms of the Euler class of certain "bundles". Note that the number $\sigma_A(Z_1, \cdots, Z_k)$ depends only on the homology classes of $Z_1, \cdots, Z_k$.

First we give some examples of algebraic manifolds which satisfy the above nondegenerate conditions.

Lemma 10.1. *If V is the Grassmannian manifold $G(r, m)$ consisting of all r-subspaces in C^m, then the counting function $\sigma_A(Z_1, \ldots, Z_k)$ is well defined as above for Schubert cycles $Z_1, \ldots, Z_k$ and equal to the Gromov invariant $\Phi_{(A,\omega)}([Z_1], \cdots, [Z_k])$ (cf. Proposition 2.4), where $[Z_1], \cdots, [Z_k]$ are homology classes of $Z_1, \cdots, Z_k$.*

Proof. We just give a sketched proof for reader's convenience, since many arguments in the proof are the same as those in section 4. It is well known that the tangent bundle $TV = TG(m, r)$ is semi-positive in Nakano's sense (cf. [6]), so $(f^*TV)^* \otimes K$ is negative, where E^* denotes the dual bundle of E, and K is the canonical bundle of CP^1. From the standard vanishing theorem it follows that the obstruction group $H^1(CP^1, f^*TV)$, which is equal to $H^0(CP^1, (f^*TV)^* \otimes K)$, vanishes. It follows that $\mathcal{M}_A^*(S^2, J, 0)$ and hence $\mathcal{M}_A^*(S^2, J, 0)/G$ are smooth.

Consider the evaluation map

$$\begin{aligned} \mathrm{ev} : \tilde{\mathcal{M}}_A^*(S^2, J, 0) \times_G (CP^1)^k &\mapsto V^k \\ (f; x_1, \cdots, x_k) &\mapsto (f(x_1), \cdots, f(x_k)). \end{aligned}$$

Let $Z_1, \cdots, Z_k$ be generic algebraic smooth subvarieties in V. Since the counting function σ_A, if exists, depends only on the homology classes of $Z_1, \cdots, Z_k$, we may perturb $Z_1, \cdots, Z_k$ if necessary. It is easy to show that the image $\mathrm{Im(ev)}$ is an irreducible open subvariety in V^k of dimension $C_1(V)(A) + n + k - 3$. Let $\overline{\mathrm{Im(ev)}}$ be the compactification of $\mathrm{Im(ev)}$ under the Hausdorff topology, and Z be the subvariety $\prod_{i=1}^k Z_i$ in V^k. Then $\dim Z = \sum_{i=1}^k \dim Z_i$.

If the boundary $\overline{\mathrm{Im(ev)}} \setminus \mathrm{Im(ev)}$ has complex Hausdorff dimension less than $C_1(V)(A) + n + k - 3$, then we can perturb Schubert cycles Z_i such that Z is away from $\overline{\mathrm{Im(ev)}} \setminus \mathrm{Im(ev)}$, i.e., $\mathrm{Im(ev)}$ intersects Z in V^k at finitely many points away from its boundary. On the other hand, since each Z_i can be deformed in any directions at a specified point, we may perturb $Z_1, \cdots, Z_k$ such that all intersections are transversal and occur at smooth points of $\mathcal{M}_A^*(S^2, J, 0)/G$.

It remains to show that $\overline{\mathrm{Im(ev)}} \setminus \mathrm{Im(ev)}$ has complex Hausdorff dimension less than $C_1(V)(A) + n + k - 3$. By the Gromov-Uhlenbeck compactification theorem (Proposition 3.1), $\overline{\mathrm{Im(ev)}} \setminus \mathrm{Im(ev)}$ is the image of

$$\overline{\mathcal{M}_A^*(S^2, J, 0)/G} \setminus \mathcal{M}_A^*(S^2, J, 0)/G$$

under the evaluation map. Therefore, it is enough to show that $\overline{\mathcal{M}_A^*(S^2, J, 0)/G} \setminus \mathcal{M}_A^*(S^2, J, 0)/G$ is of complex codimension 1. As we show in section 4 (4.7), one way to prove this is to show that the evaluation map set up in Theorem 4.7 is transverse to appropriate diagonal. Then, the dimension acounting argument in Proposition 4.14 will imply that $\overline{\mathcal{M}_A^*(S^2, J, 0)/G} \setminus \mathcal{M}_A^*(S^2, J, 0)/G$ is of complex codimension 1. We proved this type of transverslity in Theorem 4.7 by perturbing the almost complex structure. Here, we have to work on the fixed complex structure. Therefore, we need a different proof. In the following, we adopt notation in section 4.

By Proposition 3.1, the limit of a sequence of $\mathcal{M}^*_A(S^2, J, 0)$ is either a multiple curve or a cusp curve (cf. section 4.1). Only difference is that the principal component, which is also a S^2, satisfies a homogeneous equation and should be treated as a bubble by dividing reparametrization group G. If the principal component is a constant map, we just collapse the principal component, and the resulting map is still a map from a tree of S^2. As we did in section 4, we drop the multiplicity of a multiple map and consider its reduced map to its image. For the set of cusp curves, we replace the component of multiple curves and identify the consective components with the same image as we did in section 4. Then the quotient space is the Gromov-Ulenbeck compactification $\overline{\mathcal{M}^*_A(S^2, J, 0)/G}$. Note that the quotient of the set of multiple cover curves is a union of $\mathcal{M}^*_B(S^2, J, 0)/G$ such that there is an integer $m > 1$ with $A = mB$. Since $C_1(A) > 0$, we have

$$\dim(\mathcal{M}^*_B(S^2, J, 0)/G) = C_1(V)(B) + n - 3 < C_1(V)(A) + n - 3.$$

Hence it is enough to show that the set of cusp curves with more than two components is of complex codimension 1. As we showed in the section 4, we can decompose $\overline{\mathcal{M}^*_A(S^2, J, 0)/G} \backslash \mathcal{M}^*_A(S^2, J, 0)/G$ into the union of $\mathcal{M}_{S^2}(D, J, 0)$ for $D \in \mathcal{D}^{J,0}_{A,S^2}$. Here, we can suppose that D has more than two components. Then, it is enough to show that $\mathcal{M}_{S^2}(D, J, 0)$ is of complex codimension 1. Recall that $\mathcal{M}_{S^2}(D, J, 0)$ consists of J-holomorphic map f from Σ_D into $G(r, m)$ such that every component of f is a non-multiple map, there is no consective bubbles having the same image. Furthermore, the bubbles which have the same image are sepecified by D. Here, we drop the last condition and denote the first two condition by D^o. Suppose that the resulting moduli space is $\mathcal{M}_{S^2}(D^o, J, 0)$. Then, $\mathcal{M}_{S^2}(D, J, 0) \subset \mathcal{M}_{S^2}(D^o, J, 0)$, and it is enough to show that $\mathcal{M}_{S^2}(D^o, J, 0)$ is of complex codimension 1. Moreover, we can construct $\mathcal{N}_{S^2}(D^o, J, 0)$ in the same way as we did in section 4. Now we claim that $\mathcal{N}_{S^2}(D^o, J, 0)$ is smooth for the standard complex structure J. Then it is easy to show that $\mathcal{N}_{S^2}(D^o, J, 0)$ is of codimension 1 by a dimension counting argument (cf. Proposition 4.14).

Recall that the domain Σ_{D^o} of maps in $\mathcal{M}_{S^2}(D^o, J, 0)$ is a tree. We can assign a level to each component, starting from zero level at principal component. For each intersection point z, among the components intersecting at z, there is a base component, which has the lowest level, said k. Others are one level higher. In fact, one can imagine them as several subtrees growing out of the base component at z. Therefore, each of those non-base components is the base for a subtree, whose components have the levels higher than $k+1$.

Now we order the intersection points $z_1, \cdots, z_{t_{D^o}}$ in such a way that $l(z_1) \geq l(z_2) \geq \cdots \geq l(z_{t_{D^o}})$, where $l(z_i)$ denotes the level of z_i. For each z_i, suppose that $B, B_1, \ldots, B_{s_i}$ are the components of D^o intersecting at z_i, where B is the base component. As we mentioned previously, each B_i is the base of a subtree, say D_{B_i}. Let $\Delta_i \subset V^{s_i+1}$ be the diagonal corresponding to the intersection pattern of D^o at z_i, and $\Delta_{D^o} = \Delta_1 \times \Delta_2 \times \cdots \times \Delta_{t_{D^o}}$. Here, we suppose that the first factor of V^{s_i+1} is in the image of e_B, and $(i+1)$-th factor is in the image of e_{B_i}, where e_B, e_{B_i} are evaluation maps from the moduli spaces of holomorphic maps corresponding to B, B_i. Let us write V^{s_i+1} as $V_B \times V^{s_i}$ to indicate this order. Let $u = (C(x), C_1(x_1), \cdots, C_{s_i}(x_{s_i}))$ in Δ_i, where $C \in \mathcal{M}^*_B(S^2, J, 0), C_i \in \mathcal{M}_{B_i}(S^2, J, 0)$. Note that $T_u V^{s_i+1} \cong T_u\Delta_1 \oplus T_u V^{s_i}$, i.e., we skip the factor V_B. Since $V = G(r,m)$ is homogeneous, given $(v_1, \cdots, v_{s_i}) \in T_u V^s$, we can find automorphisms ϕ_i^t of $G(r,m)$ such that

$$\phi_i^0 = Id, \ \frac{d}{dt}(\phi_1^t, \cdots, \phi_{s_i}^t)(u)|_{t=0} = (v_1, \cdots, v_{s_i}).$$

Now we use ϕ_i^t to move the whole subtree D_{B_i}. Recall that e_{D^o} is the evaluation map from $\mathcal{M}_{S^2}(D^o, J, 0)$ into a product of V. The derivative of e_{D^o} is a vector of the form

$$(X, 0, v_1, \cdots v_{s_i}, 0, 0, \cdots, 0),$$

where X is a tangent vector of $\prod_{j=1}^{i-1} V_j^{s_j+1}$. Hence, we can prove that e_{D^o} is transversal to Δ_{D^o} by an induction on the level order of $z_1, \cdots, z_{t_D^o}$. Here we need to use the fact that there is no cycle in the intersection pattern D^o, in order to start the induction at the top branch of each subtree. Then we finish the proof.

Remark 10.2. (1) It is possible to generalize this lemma to genus-one curves in $G(r,m)$ by using semi-positivity of the tangent bundle. It is much more difficult to prove this in the case of higher genus curves.

(2) An analog of Lemma 10.1 can be proved for complex homogeneous manifolds with positive first Chern class by the same argument as in the proof of Lemma 10.1.

In case $V = CP^n$, any A in $H_2(V, Z)$ is of the form $d\ell$, where ℓ is a rational curve of degree one. We write $\mathcal{M}_A^*(S^2, J, 0)/G$ as $\mathcal{M}_d$, i.e., $\mathcal{M}_d$ is the moduli space of rational curves in CP^n of degree d. We denote by W_i $(2 \leq i \leq n)$ the subvariety consisting of all those curves in $\mathcal{M}_d$ which intersect a fixed linear subspace in CP^n of codimension $i > 1$. In case $d = 1$, $\mathcal{M}_d$ is the Grassmannian $G(n+1, 2)$, and those W_i are just the classical schubert cycles generating the homology ring of $G(n+1, 2)$. It is a classical problem in enumerative algebraic geometry to compute the intersection number $W_{j_1} \cap \cdots \cap W_{j_k}$ for $\sum(j_i - 1) = (n+1)d + n - 3$. Clearly, this intersection number is just the counting number $\sigma_{d\ell}(H^{j_1}, \cdots, H^{j_k})$, where we denote by H^j a linear subspace in CP^n of codimension j. For simplicity, we denote this intersection number by $\sigma_{n,d}(j_1, \cdots, j_k)$. One interesting special case is $\sigma_{n,d}(2, 2, \cdots, 2)$. It can be interpreted as follows: Given any degree d algebraic curve C in CP^n, its Chow coordinate X_C is a hypersurface in the Grassmannian manifold $G(n-1, n+1)$ and consists of all $(n-2)$-subspaces in CP^n which have nonempty intersection with C. This Chow coordinate X_C is, unique up to multiplication by constants, defined by a section in $H^0(G(n-1, n+1), \mathcal{O}(d))$, where $\mathcal{O}(1)$ is the positive line bundle generating the Picard group of $G(n-1, n+1)$. Let $N(n, d) + 1$ be the dimension of $H^0(G(n-1, n+1), \mathcal{O}(d))$. Then there is a subvariety in $CP^{N(n,d)}$ consisting of Chow coordinates of rational (possibly singular) curves in CP^n. We denote by n_d the degree of this subvariety. Then n_d is just $\sigma_{n,d}(2, 2, \cdots, 2)$. Note that $n_1 = 1$. It has been a difficult problem to compute n_d for higher degree.

The following is simply a corollary of Lemma 10.1 and Proposition 2.4.

Corollary 10.3. *Let V be CP^n. Then we have*

$$\sigma_{n,d}(j_1, \cdots, j_k) = \Phi_{d\ell}(H^{j_1}, H^{j_2}, H^{j_3} | H^{j_4}, \cdots, H^{j_n}), \tag{10.2}$$

where ℓ denotes the homology class of the line in CP^n.

We will use composition law to derive a recursion formula for $\sigma_{n,d}(j_1, \cdots, j_k)$. Note that $\sigma_{n,d}$ is a symmetric function. For convenience, we define $\sigma_{n,d}(j_1, \cdots, j_k)$ to be zero if either some $j_i > n$ or

$\sum_{i=1}^{k}(j_i-1)$ is not equal to $(n+1)d+n-3$, which is the dimension of $\mathcal{M}_d$. We also allow some $j_i=1$. Clearly, we have

$$\sigma_{n,d}(j_1,\cdots,j_{k-1},1)=d\sigma_{n,d}(j_1,\cdots,j_{k-1}).$$

By using the symmetry, we may arrange $j_1,\cdots,j_k$ to be a non-increasing sequence, i.e., $j_1\geq j_2\geq\cdots\geq j_k$. We also need to introduce a partial ordering on the sequences of non-increasing integers $I_{n,d}=\{j_1,j_2,\cdots j_k\}$ such that $2\leq j_i\leq n$ and $\sum_i j_i=(n+1)d+n-3+k$. Let $I'_{n,d}=\{j'_1,\cdots,j'_m\}$ be another such a sequence. We say $I_{n,d}\prec I'_{n,d}$ if and only if there is an i such that $j_1=j'_1,\cdots,j_i=j'_i$ and $j_{i+1}<j'_{i+1}$. Clearly, the maximal element is of the form $n,n,\cdots,n,s$, where $s\leq n$.

Theorem 10.4. *Assume that $j_1\geq j_2\geq\cdots\geq j_k\geq 2$. Then we have the following formula:*

$$\begin{aligned}
&\sigma_{n,d}(j_1,j_2,j_3,j_4,\cdots,j_k)\\
&\quad=\sigma_{n,d}(j_1,j_2+1,j_3-1,j_4,\cdots,j_k)\\
&\qquad+d\left(\sigma_{n,d}(j_1+j_3-1,j_2,j_4,\cdots,j_k)\right.\\
&\qquad-\left.\sigma_{n,d}(j_1+j_2,j_3-1,j_4,\cdots,j_k)\right)\\
&\qquad+\sum_{d_1=1}^{d-1}\sum_{l=0}^{k-3}\sum_{\sigma}\sum_{i=1}^{n}\frac{d-d_1}{l!(k-3-l)!}\\
&\quad\cdot(\sigma_{n,d_1}(j_1,j_3-1,i,j_{\sigma(4)},\cdots,j_{\sigma(l)})\cdot\sigma_{d-d_1}(j_2,n-i,j_{\sigma(l+1)},\cdots,j_{\sigma(k)})\\
&\quad-\sigma_{n,d_1}(j_1,j_2,i,j_{\sigma(4)},\cdots,j_{\sigma(l)})\cdot\sigma_{d-d_1}(j_3-1,n-i,j_{\sigma(l+1)},\cdots,j_{\sigma(k)})).
\end{aligned}\tag{10.3}$$

Furthermore, there is a recursion formula for the intersection numbers $\sigma_{n,d}$.

Proof. This is a straightforward corollary of the associativity (9.9). The associativity is proved by using two different degenerations of a Riemann sphere with 4-marked points. For reader's convenience, we will apply Theorem A in case $k=4$ to rederive (9.9) for CP^n. First we degenerate a Riemann sphere with 4-marked points to a union of two smooth rational curves such that one component has marked points x_1,x_2 and another component has marked points x_3,x_4. Then, by Theorem A,

$$\begin{aligned}
\Phi_{d\ell}(H^{j_1},H^{j_2},H^{j_3-1},H\,|H^{j_4},\cdots,H^{j_k})=&\sum_{d_1=0}^{d}\sum_{l=0}^{k-3}\sum_{\sigma}\sum_{i=1}^{n}\frac{1}{l!(k-3-l)!}\\
&\cdot\Phi_{d_1\ell}(H^{j_1},H^{j_2},H^{i}|H^{j_{\sigma(4)}},\cdots,H^{j_{\sigma(l)}})\\
&\cdot\Phi_{(d-d_1)\ell}(H^{j_3-1},H,H^{n-i}|H^{j_{\sigma(l+1)}},\cdots,H^{j_{\sigma(k)}}),
\end{aligned}$$

where σ runs over all permutations of $4, \cdots, k$. Note that when $0 < d_1 < d$, all the terms involve only $\sigma_{n,1}, \cdots, \sigma_{n,d-1}$. When $d_1 = 0$, we get a term

$$\Phi_{d\ell}(H^{j_3-1}, H, H^{j_1+j_2}|H^{j_4}, \cdots, H^{j_k}) = d\sigma_{n,d}(j_1 + j_2, j_3 - 1, j_4, \cdots, j_k).$$

When $d_1 = d$, we get a term

$$\Phi_{d\ell}(H^{j_1}, H^{j_2}, H^{j_3}|H^{j_4}, \cdots, H^{j_k}) = \sigma_{n,d}(j_1, j_2, j_3, j_4, \cdots, j_k).$$

Hence,

$$\begin{aligned}
\Phi_{d\ell}\,(H^{j_1}, H^{j_2}, H^{j_3-1}, H|H^{j_4}, \cdots, H^{j_k}) \;&=\; \sigma_{n,d}(j_1, j_2, j_3, j_4, \cdots, j_k)\\
+\;\; d\sigma_{n,d}(j_1 + j_2, j_3 - 1, j_4, \cdots, j_k) &+ \sum_{l=0}^{k-3}\sum_{\sigma}\sum_{i=1}^{n} \frac{1}{l!(k-3-l)!}\\
&\cdot\Phi_{d_1\ell}(H^{j_1}, H^{j_2}, H^{i}|H^{j_{\sigma(4)}}, \cdots, H^{j_{\sigma(l)}})\\
&\cdot\Phi_{(d-d_1)\ell}(H^{j_3-1}, H, H^{n-i}|H^{j_{\sigma(l+1)}}, \cdots, H^{j_{\sigma(k)}}).
\end{aligned}$$

Notice that

$$\begin{aligned}
&\Phi_{d-d_1}(H^{j_3-1}, H, H^{n-i}|H^{j_{\sigma(l+1)}}, \cdots, H^{j_{\sigma(k)}})\\
&\qquad = (d-d_1)\,\sigma_{n,d-d_1}(j_3 - 1, n - i, j_{\sigma(l+1)}, \cdots, j_{\sigma(k)}).
\end{aligned}$$

On the other hand, we let the Riemann sphere with 4-marked points degenerate to a union of two smooth rational curves with two marked points x_1, x_3 in one component and other two x_2, x_4 in another component. By a similar argument as above, we can show

$$\begin{aligned}
\Phi_{d\ell}\,(H^{j_1}, H^{j_2}, H^{j_3-1}, H|H^{j_4}, \cdots, H^{j_k}) &= \sigma_{n,d}(j_1, j_2 + 1, j_3 - 1, j_4, \cdots, j_k)\\
+\;\; d\Phi_d(H^{j_1+j_3-1}, H^{j_2}, H^{j_4}, \cdots, H^{j_k}) &+ \sum_{l=0}^{k-3}\sum_{\sigma}\sum_{i=1}^{n} \frac{1}{l!(k-3-l)!}\\
&\cdot\Phi_{d_1\ell}(H^{j_1}, H^{j_3-1}, H^{i}|H^{j_{\sigma(4)}}, \cdots, H^{j_{\sigma(l)}})\\
&\cdot\Phi_{(d-d_1)\ell}(H^{j_2}, H^{n-i}, H|H^{j_{\sigma(l+1)}}, \cdots, H^{j_{\sigma(k)}}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sigma_{n,d}(j_1, j_2, j_3, j_4, \cdots, j_k) \;=\; \sigma_{n,d}(j_1, j_2 + 1, j_3 - 1, j_4, \cdots, j_k)\\
&+ d\,(\sigma_{n,d}(j_1 + j_3 - 1, j_2, j_4, \cdots, j_k) - \sigma_{n,d}(j_1 + j_2, j_3 - 1, j_4, \cdots, j_k))\\
&+ \sum_{d_1=1}^{d-1}\sum_{l=0}^{k-3}\sum_{\sigma}\sum_{i=1}^{n} \frac{d-d_1}{l!(k-3-l)!}\\
&\quad\cdot(\sigma_{n,d_1}(j_1, j_3 - 1, i, j_{\sigma(4)}, \cdots, j_{\sigma(l)})\sigma_{d-d_1}(j_2, n - i, j_{\sigma(l+1)}, \cdots, j_{\sigma(k)})\\
&- \sigma_{n,d_1}(j_1, j_2, i, j_{\sigma(4)}, \cdots, j_{\sigma(l)})\sigma_{d-d_1}(j_3 - 1, n - i, j_{\sigma(l+1)}, \cdots, j_{\sigma(k)})).
\end{aligned}$$

Obviously, $\{j_1, j_2, j_3, j_4, \cdots, j_k\} \prec \{j_1 + j_3 - 1, j_2, j_4, \cdots, j_k\}$, $\{j_1, j_2 + 1, j_3 - 1, j_4, \cdots, j_k\}$ and $\{j_1 + j_2, j_3, j_4, \cdots, j_k\}$. Moreover, if $j_1 = j_2 = n$, then $j_1 + j_2 > n$, $j_2 + 1 > n$ and $j_1 + j_3 - 1 > n$; consequently, on the right side of the above identity, all the terms involving $\sigma_{n,d}$ vanish. Therefore, $\sigma_{n,d}(j_1, \cdots, j_k)$ can be expressed by a recursion formula in terms of $\sigma_{n,1}, \cdots, \sigma_{n,d-1}$. This finishes the proof of Theorem 10.4.

Corollary 10.5. *All the degree n_d can be computed.*

In the case $n = 2$, from Theorem 10.2 and easy computations it follows that

$$n_d = \frac{1}{2} \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \frac{d_1 d_2 (3 d d_1 d_2 - 2d^2 + 6 d_1 d_2)(3d-4)!}{(3d_1 - 1)!(3d_2 - 1)!} n_{d_1} n_{d_2}. \tag{10.4}$$

Since $n_1 = 1$, we deduce from this recursion formula that $n_2 = 1$, $n_3 = 12$. Such a recursion formula of computing n_d for CP^2 was first derived by Kontsevich, using the composition law previously predicted by physicists and now proved in our paper.

Recursion formulas for other manifolds. We can generalize the above method to compute the Gromov invariants for Fano manifolds, i.e., algebraic manifolds with positive first Chern class. Let V be a Fano manifold. Recall that for any A in $H_2(V, Z)$, the Gromov invariant $\Phi_A(\alpha_1, \alpha_2, \cdots, \alpha_k)$ $(k \geq 3)$ coincides with the mixed invariant $\Phi_A(\alpha_1, \alpha_2, \alpha_3 | \alpha_4, \cdots, \alpha_k)$, where α_i are homology classes of V. It is non-trivial to derive the recursion formulas for computing the Gromov invariants of a general Fano manifold because its cohomology group could be very complicated. To illustrate the power of our composition law, we compute the Gromov invariants of odd dimensional Fano manifolds V, whose cohomology groups $H^{2i}(V, Z) = Z$. By the Lefschetz Hyperplane Theorem, Fano hypersurfaces or Fano complete intersections are examples of such manifolds V. Without loss of generality, we may assume that the dimension is not less than three. Let H be the positive generator of $H^2(V, Z)$. Then $H^{2i}(V, R)$ is generated by H^i. Since $H_2(V, Z) = Z$, A is of the form $d\ell$, where ℓ is the positive generator of $H_2(V, Z)$. We will simply write Φ_A as Φ_d.

We would like to derive a recursion formula for $\Phi_d(H^{j_1}, \cdots, H^{j_k})$, where

$$\sum_{i=1}^{k}(j_i - 1) = dc_1(V)(\ell) + n - 3.$$

As before, we will order $j_1, \cdots, j_s$ in such a way that $j_1 \geq \cdots \geq j_s$.

Let Δ be the diagonal in $V \times V$. Then

$$[\Delta] = p^{-1} \sum_{i=0}^{n} H^i \otimes H^{n-i} \oplus \sum(\text{odd degree}) \otimes (\text{ odd degree}),$$

where p is the degree of the manifold V, i.e., $p point = H^n$. We will apply the composition law for $\Phi_d(H^{j_1}, H^{j_2}, H^{j_3-1}, H \mid H^{j_4}, \cdots, H^{j_k})$. In case the degree of α is odd, we have for dimension reason

$$\Phi_d(H^i, H^k, \alpha | H^{j_4}, \cdots, H^{j_k}) = 0.$$

It follows that odd deg homology classes will not appear in the composition law of

$$\Phi_d(H^{j_1}, H^{j_2}, H^{j_3-1}, H | H^{j_4}, \cdots, H^{j_k}).$$

Suppose that q is the intersection number of H and ℓ. Then we have the following formula

$$\begin{aligned}
p\,\Phi_d(H^{j_1}, \cdots, H^{j_k}) = {} & \Phi_d(H^{j_1}, H^{j_2+1}, H^{j_3-1}, H^{j_4}, \cdots, H^{j_k}) \\
& + dq\,(\Phi_d(H^{j_1+j_3-1}, H^{j_2}, H^{j_4}, \cdots, H^{j_k}) \\
& - \Phi_d(H^{j_1+j_2}, H^{j_3-1}, H^{j_4}, \cdots, H^{j_k}) \\
& + \sum_{d_1=1}^{d-1} \sum_{l=0}^{k-3} \sum_{\sigma} \sum_{i=1}^{n} \frac{(d-d_1)q}{l!(k-3-l)!} \\
& \cdot(\Phi_{d_1}(H^{j_1}, H^{j_2}, H^i, H^{j_{\sigma(4)}}, \cdots, H^{j_{\sigma(l)}}) \\
& \cdot\Phi_{d-d_1}(H^{j_3-1}, H^{n-i}, H^{j_{\sigma(l+1)}}, \cdots, H^{j_{\sigma(k)}}) \\
& - \Phi_{d_1}(H^{j_1}, H^{j_3-1}, H^i, H^{j_{\sigma(4)}}, \cdots, H^{j_{\sigma(l)}}) \\
& \cdot\Phi_{d-d_1}(H^{j_2}, H^{n-i}, H^{j_{\sigma(l+1)}}, \cdots, H^{j_{\sigma(k)}})).
\end{aligned} \tag{10.5}$$

An analogous formula of (10.5) can be derived for any Fano manifolds, but the formula will be much more complicated. Moreover, it is not clear how to deduce from such a formula that the higher degree Gromov invariants can be computed in terms of lower degree Gromov invariants.

When the tangent bundle of V is not semi-positive, it is a difficult problem to determine if the Gromov invariant is the same as the corresponding counting function in enumerative algebraic geometry. It is easy to see that they indeed coincide if the counting function is well-defined in the sense described in the second paragraph of this section. In general, the moduli space of rational curves may behave badly and the counting function may not be well-defined. However, we propose

Conjecture 10.6. *For a Fano manifold V, if $C_1(V)(A) >> 0$, the counting function σ_A is well-defined in the classical sense and coincides with the Gromov invariant Φ_A.*

In general, one can compactify $\mathcal{M}^*_A(S^2, J, 0)/G$ in the Hilbert scheme by one-dimensional subschemes of V. Let us denote this compactification by $\overline{\mathcal{M}^*_A(S^2, J, 0)/G}^H$ and the Gromov-Ulenbeck compactification by $\overline{\mathcal{M}^*_A(S^2, J, 0)/G}^{GU}$. We are compelled to make the following conjecture:

Conjecture 10.7. *There is a continuous map from some normalization of the Hilbert scheme compactification $\overline{\mathcal{M}^*_A(S^2, J, 0)/G}^H$ to Gromov-Ulenbeck compactification $\overline{\mathcal{M}^*_A(S^2, J, 0)/G}^{GU}$ whose restriction on*
$\mathcal{M}^*_A(S^2, J, 0)/G$ is identity map.

If this conjecture is true, one only has to check that $\mathcal{M}^*_A(S^2, J, 0)/G$ has correct dimension. Then $\overline{\mathcal{M}^*_A(S^2, J, 0)/G}^{GU} \backslash \mathcal{M}^*_A(S^2, J, 0)/G$ is automatically of complex codimension 1 and one can solve the Conjecture 10.6.

We also defined the Gromov invariants for higher genue curves in section 2. The composition law in section 8 implies that these invariants for higher genus curves can be computed in terms of the invariant for rational curves. In case V is the Grassmannian manifold $G(r, m)$, our composition law gives rise to beautiful formulas, conjectured by physicists Vafa and Intriligate, for computing these invariants, as shown in [23]. The problem is to determine when the Gromov invariants are enumerative geometric invariants. A partial answer to this was given in [3]. In [3], they use a compactification in the Grothendieck Quot Schemes, which depends on the sepecific feature of $G(r, m)$. We also refer the readers to paragraph 2, 3 in Remark 2.10 on counting higher genus curves.

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Notes added in the proof:

The results of this paper were fist lectured by the second author in the seminars at MIT and Harvard, early December, 1993. The full paper was circulated in May, 1994. After we submitted our paper, we received a book by D. Mcduff and D. Salamon in September, 1994 and a preprint by G Liu in October, 1994. In these papers, the authors gave a new proof of the formula (1.1) for monotone manifolds in the case that $g = 0, \quad k = 4, \quad l = 0$.

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