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ON THE MODULI SPACE OF POLYGONS IN THE EUCLIDEAN PLANE

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Abstract

We study the topology of moduli spaces of polygons with fixed side lengths in the Euclidean plane. We establish a duality between the spaces of marked Euclidean polygons with fixed side lengths and marked convex Euclidean polygons with prescribed angles.

1. We consider the space $\mathcal{P}_n$ of all polygons with n distinguished vertices in the Euclidean plane $\mathbb{E}^2$ whose sides have nonnegative length allowing all possible degenerations of the polygons except the degeneration of the polygon to a point. Two polygons are identified if there exists an orientation preserving similarity of the complex plane $\mathbb{C} = \mathbb{E}^2$ which sends vertices of one polygon to vertices of another one. We shall denote the edges of the n-gon P by: $e_1, ..., e_n$ and vertices by $v_1, ..., v_n$ so that $\vec{e}_j = v_{j+1} - v_j$. The space $\mathcal{P}_n$ is canonically isomorphic to the complex projective space $P(H)$ where $H \subset \mathbb{C}^n$ is the hyperplane given by

$$H = \{(e_1, ..., e_n) \in \mathbb{C}^n : e_1 + ... + e_n = 0\}.$$

Therefore, the space $\mathcal{P}_n$ can be identified with $\mathbb{C}P^{n-2}$. The length of the edge e_j will be denoted by r_j. We shall assume that all polygons are normalized so that the perimeter is equal to 1.

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Let σ be the conjugation $z \mapsto \bar{z}$, $z \in \mathbb{C}$. This transformation induces the involution $\sigma_* : \mathcal{P}_n \to \mathcal{P}_n$. The set of fixed points of σ_* consists of all "degenerate polygons" P which are contained in a straight line.

The map $Area : \mathcal{P}_n \to \mathbb{R}$ assigns to every normalized polygon its signed area.

Define the projection $\pi : \mathcal{P}_n \to \mathbb{R}^n$ which assigns to each normalized polygon $P \in \mathcal{P}_n$ the n-tuple of its side lengths :

$$\pi(P) = r = (r_1, ..., r_n).$$

Our ultimate goal is to understand the topology of the moduli space of polygons with prescribed sides $M_r = \pi^{-1}(r)$, $r \in \mathbb{R}^n$. We shall find necessary and sufficient conditions for M_r to be connected, and describe the topology of the moduli spaces M_r for $n = 4, 5, 6$.

Theorem 1. *The space M_r is not connected if and only if there are 3 different sides e_i, e_j, e_k in the normalized polygon so that*

$$r_i + r_j > 1/2, \quad r_i + r_k > 1/2, \quad r_k + r_j > 1/2.$$

If e_i, e_j, e_k exist as stated then M_r is diffeomorphic to the disjoint union of two $(n-3)$-dimensional tori.

Remark 1. Roughly speaking Theorem 1 states that the moduli space of a polygon is disconnected if and only if it has "three long sides". To see examples with disconnected moduli spaces (and to motivate our result) start with a triangle– so that the moduli space consists of two points. Then cut off a small neighborhood of one of the vertices to create a quadrilateral which is (Hausdorff) close to the triangle. It is clear that the deformation space of such a quadrilateral has two components (for example use the function $Area$). Continue the process. Thus we obtain an n-gon with three long sides and disconnected moduli space. The fact that the moduli space consists of two $(n-3)$–dimensional tori is clear from Corollary 15: degenerate a short side and use induction.

Theorem 2. *If the moduli space of pentagons M_r is nonsingular, then M_r is a compact oriented surface of the Euler characteristic $2(l(r)-3)$. Here $l(r)$ is the "level of the chamber" which contains the point $r \in \mathbb{R}^5$ (see Section 4, definition 2).*

For example, the deformation space of the regular pentagon is a compact orientable surface of genus 4.

Theorem 3. *If the moduli space of hexagons M_r is nonsingular and connected, then it is either diffeomorphic to a connected sum of k*

copies of $\mathbb{S}^2 \times \mathbb{S}^1$ *and of the product* $\Sigma_g \times \mathbb{S}^1$*, or it is diffeomorphic to a connected sum of* $T^3 \# T^3$ *and of* t *copies of* $\mathbb{S}^2 \times \mathbb{S}^1$*. Here* $k \leq 4$*, the genus* g *of the surface* Σ_g *is not greater than* 4 *and* $t \leq 2$*. If* M_r *is nonsingular but is not connected, then it is diffeomorphic to the disjoint union of two copies of* $T^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$.

We will also show that for $n \geq 5$ the function $sign(Area)$ on M_r fails to distinguish different connected components of M_r (see Section 10).

In Sections 16– 19 we establish a duality τ between the spaces of Euclidean polygons with prescribed side lengths and marked convex polygons with prescribed angles. Unlike the classical duality in the spherical geometry, the correspondence τ is transcendental– it is given by hypergeometric integrals. Another difference is that the duality of polygons in $\mathbb{S}^2$ is *local*: it is enough to know two adjacent vertices of P to find the corresponding vertex of the dual polygon P^*. It is not so in the case of Euclidean polygons: one has to know the whole polygon $P \subset \mathbb{C}$ to calculate a single vertex of the dual polygon $\tau(P)$. As a consequence of this duality we construct a piecewise–geodesic embedding $\mathcal{D}M$ of the moduli space $\bar{M}_r = M_r/\sigma$ into the complex-hyperbolic space $\mathbb{H}_{\mathbb{C}}^{n-3}$ for generic values of r. Points of M_r/σ where the image of $\mathcal{D}M$ fails to be totally–geodesic correspond to polygons in M_r which have parallel edges.

In our forthcoming work we shall discuss the relation of the moduli spaces of polygons with toric varieties, Hodge theory for representations of reflection groups and bending deformations of representations of polygons of groups. Results of Sections 2– 15 of our paper admit straightforward generalization to the case of polygons on the hyperbolic plane and unit sphere. In the last case it would be necessary to restrict consideration to polygons with perimeter not greater than 2π.

Previous results

After we had finished our paper we learned that many of our results were already known. Theorem 1 is proved in [12] and [18], is implicit in [16] and was known to Connelly. Theorem 2 is implicit in [12] and [16]. The result that the deformation space of a regular pentagon is a genus 4 surface was obtained independently in [11], [12], [13] and [16]. Our results concerning the Morse theory of lengths of diagonals (see Lemma

11) are to be found in [10] and [12]. In [16], Walker gives an intrinsic description of the homology classes added after "crossing a wall" in terms of moduli spaces of the original polygon. He also makes an interesting conjecture: suppose that r and r' lie in chambers which are not related by a permutation. Then M_r and $M_{r'}$ have different homology rings. In [1], Banchoff treates the Morse theory of the algebraic area function (the case of regular pentagons was considered in [11]). Some other aspects of moduli spaces of Euclidean and spherical polygons are discussed in [2], [7], [8].

2. First of all we have to understand the image of the projection $\pi : \mathcal{P}_n \to \mathbb{R}^n$. The image of this map is contained in the standard simplex S in $\mathbb{R}^n$, which is equal to the intersection of the hyperplane $\Pi = \{r_1 + ... + r_n = 1\}$ with the first octant:

$$r_1 \geq 0, ..., r_n \geq 0. \tag{1}$$

Any point in the image of the map π must satisfy the "triangle inequalities" :

$$r_j \leq 1/2, \qquad j = 1, ..., n. \tag{2}$$

Lemma 1. *The inequalities (1) and (2) completely describe the domain $D_n = \pi(\mathcal{P}_n)$ in Π.*

Proof. Let us prove this lemma by induction. For $n = 3$ the assertion of Lemma is obviously correct. Now, assume that the assertion is correct for $n-1$. Suppose that a vector $r = (r_1, ..., r_n)$ satisfies the inequalities (1), (2). If the sum of lengths of each pair of adjacent edges $r_i + r_{i+1}$ is greater than 1/2, then $r_1 + ... + r_n > 1$ which contradicts our assumptions. Hence, up to renumeration of vertices we have: $r_{n-1} + r_n \leq 1/2$. Therefore, according to the induction hypothesis, there exists an $n-1$-gon P (with the perimeter 1) which has sides $(r_1, r_2, ..., r_{n-1} + r_n)$. Now, if we consider P as an element of $\mathcal{P}_n$ with one extra vertex on e_{n-1}, then $\pi(P) = r$.

We can think about truncation of S by the inequalities (2) as follows. For each vertex $\overrightarrow{E_j}$ of S (which is a vector of the standard orthonormal basis of $\mathbb{R}^n$) we choose the middle points $(\overrightarrow{E_j} + \overrightarrow{E_i})/2$ on the edges emanating from $\overrightarrow{E_j}$. Now, consider the hyperplane $\Pi_j = \{r_j = 1/2\}$ in Π spanned by these middle points. The intersection of this plane with S is a simplex.

Now, the polyhedron $D_n = \pi(\mathcal{P}_n)$ has faces of two types:

(a) "Tetrahedral" faces which appear as intersections of Π_j and S. These faces are combinatorially equivalent to $n-2$-simplices.

(b) "Octahedral" faces O_j which are contained in the faces $r_j = 0$ of the simplex S.

Thus all octahedral faces of D_n are congruent to the polyhedron D_{n-1}.

Definition 1. A polygon P in $\mathbb{E}^2$ is said to be degenerate if it is contained in a straight line.

The set of "boundary points" $\delta(\mathcal{P}_n) = \pi^{-1}(\partial D_n) \subset \mathcal{P}_n$ consists of:

(a) degenerate polygons which belong to the preimage (under π) of the tetrahedral faces of D_n;

(b) polygons which have one edge of length zero , they belong to the preimage (under π) of the octahedral faces of D_n.

Remark 2. The space $\mathcal{P}_n$ is compact, connected and has the Fubini-Study metric. Thus, each polygon $P \in \mathcal{P}_n$ can be continuously deformed to any polygon $Q \in \mathcal{P}_n$ so that:

the curve $\gamma = P(t)$ in $\mathcal{P}_n$ between P and Q is a geodesic segment in $\mathcal{P}_n$ of length not greater than a certain number $w_n \in \mathbb{R}$, where w_n doesn't depend on P, Q.

3. Now we are interested in points r of D_n such that $\pi^{-1}(r)$ contains a degenerate polygon P. The set Σ of such points is called the "critical locus" of D_n. The fact that the polygon P is degenerate is equivalent to:

$$f(r) = \sum_j (-1)^{u_j} r_j = 0, \tag{3}$$

where r_j are the lengths of the sides of P and $u_j \in \{0, 1\}$. We shall always normalize the functionals f so that the number of odd exponents u_j is not less than the number of even exponents. Therefore, the "critical locus" of D_n is the intersection of D_n with the union of the hyperplanes (3). Intersection of any plane (3) with $int(D_n)$ is called a **wall**. The "index" $ind(W)$ of the wall $W = \{f = 0\}$ is the number of even exponents u_j in the formula (3) for the functional f.

Lemma 2. *If $[P] \in \mathcal{P}_n$ is nondegenerate, then the map π is a submersion at $[P]$.*

Proof. Let $P = [v_1, ..., v_n]$. Then the kernel of the derivative $d\pi_{[P]}$

is isomorphic to the space

$$Z = \{ \xi = (\xi_2, ..., \xi_n) \in (\mathbb{R}^2)^{n-1} : \quad \xi_j \cdot \overrightarrow{e_j} = 0 \ (j = 2, ..., n), \quad \textstyle\sum_{j=2}^n \xi_j = 0\} \tag{5}$$

Since P is nondegenerate, the vectors $\{\overrightarrow{e_j}, j = 2, ..., n\}$ span $\mathbb{R}^2$. Direct calculation shows that the dimension of the space Z is $n-3$. The real dimension of $\mathcal{P}_n$ near $[P]$ is $2n-4$. However, $2n-4 = (n-3)+n-1$, i.e.,

$$\dim Ker(d\pi_{[P]}) + \dim D_n = \dim \mathcal{P}_n, \tag{6}$$

and therefore, π is a submersion near $[P]$.

This lemma can be also proven using the results of [17] and the connection between the space $\pi^{-1}[r]$ and a representation variety of a hyperbolic reflection group (relative to its parabolic subgroups).

4. The components of the space $D_n - \Sigma$ are called the **chambers** of D_n.

Lemma 2 implies that the moduli spaces $\pi^{-1}(r)$ are diffeomorphic for all r in one and the same chamber. In treating the problem of connected components of $\pi^{-1}(r)$ the essential role is played by the **great walls** W_{ij} which are given by (3) where $u_j = u_i = 0$ and all other exponents u_m are equal to 1. Equivalently, $r_i + r_j = 1/2$ on such a wall. The components of the decomposition of D_n by great walls are called **great chambers**. We start with the **negative** great chamber where all the functionals

$$f_{ij} = r_i + r_j - 1/2 \tag{7}$$

are negative. If it exists, this chamber is called the chamber of level 0. For each great chamber we introduce the *multiindex* μ.

Definition 2. The great chamber C has *multiindex* $\mu = \{i_1, j_1\}\&...\&\{i_l, j_l\}$ if it is given by the system of inequalities:

$$f_{i_1, j_1} > 0, ..., f_{i_l, j_l} > 0 \text{ and } , f_{k,s} < 0 \quad \text{if} \quad \{k, s\} \notin \mu \tag{8}$$

The signs & in the multiindex mean that we identify multiindices which are obtained by permutation of the pairs $\{i_k, j_k\}$. The number $l = l(C)$ here is called the **level** of the chamber C. In this case the chamber C will be denoted by

$$C_\mu^n \tag{9}$$

where the index n means that C is a chamber in D_n. Sometimes we shall omit the upper index. If $C' \subset C$ is a subchamber of a great chamber C, then the level $l(C')$ is by definition equal to the level $l(C)$.

5. To begin with we notice that if two couples $\{i,j\}$ and $\{k,m\}$ are presented in the multiindex of the chamber C_μ, then either $i = k$ or $i = m$ or $j = k$ or $j = m$. Otherwise we would have:

$$r_i + r_j > r_k + r_m \text{ and } r_i + r_j < r_k + r_m \tag{10}$$

in this chamber, which is impossible. Therefore, all possible multiindexes are:

(1) level 1- $\{i,j\}$;

(2) level 2 - $\{i,j\}\&\{i,k\}$.

(3) The set of multiindexes of the level 3 consists of 2 classes:

$$\{i,j\}\&\{j,k\}\&\{k,i\} \text{ (class A)} \tag{11}$$

$$\{i,j\}\&\{i,k\}\&\{i,m\} \text{ (class B)} \tag{12}$$

(4) All multiindexes of level $l > 3$ look like:

$$\{j,i_1\}\&\{j,i_2\}...\&\{j,i_l\}. \tag{13}$$

The multiindexes above are said to **admissible** .

Proposition 1. *The great chambers C of level 3 and class (A) are actually chambers of D_n.*

Proof. Let $C = C_\mu$, $\mu = \{i,j\}\&\{j,k\}\&\{k,i\}$. Suppose that some wall W (given by (3)) intersects C. Then $u_i+u_j, u_j+u_k, u_k+u_i$ are odd numbers, otherwise inside C we would have say $r_i + r_j < 1/2$, which is impossible. Then the sum of these 3 odd numbers is again odd. On the other hand, the sum is equal to $2(u_i + u_j + u_k)$ which is even. This contradiction shows that C can not intersect any wall.

Lemma 3. *If $n = 4$, then D_4 has only chambers of level 3, and all admissible multiindexes of level 3 are realized by great chambers in D_4.*

Proof. We prove the existence of chambers applying a procedure that we shall refer to as **regeneration**. First consider any multiindex of class (A). Take a nondegenerate triangle Δ with sides r_i, r_j, r_k, where i,j,k are different, and consider Δ as a degenerate quadrilateral where the side r_m has zero length $(m \neq i,j,k)$. Then the inequalities (A) are satisfied for Δ, and $\pi(\Delta)$ belongs to the face of D_4 given by equation $r_m = 0$. Now, take the chamber of D_4 adjacent to this face. We can find an element of this chamber in the following way.

Take any positive value for r_m, which is less than

$$\min\{1/2 - r_s \ , \ s = i, j, k\}. \tag{14}$$

Then there exists a quadrilateral Q with the sides equal to r_i, r_j, r_k, r_m ordered in some way. The perimeter $p(Q)$ of Q is $1 + r_m > 1$, but we can renormalize the perimeter applying the similarity $Q \mapsto Q/p(Q)$.

All inequalities (A) are satisfied for $Q/p(Q)$, and no other inequalities of the type $f > 0$ can occur. Therefore, we have an element of the chamber with the multiindex $\mu = \{i,j\}\&\{j,k\}\&\{k,i\}$.

Now, consider the case of a multiindex of class (B). Let Δ be a degenerate quadrilateral with nonzero sides r_i, r_j, r_k, r_m such that $r_i = 1/2$. Now again find the adjacent chamber of D_4 by decreasing the length of r_i and rescaling. Thus, we have proved the "existence" part of the Lemma. Suppose that we have a quadrilateral Q with the sides $r_1, ..., r_4$. The set of sides breaks into pairs in 3 different ways. Suppose that $\{i,j\}, \{k,m\}$ is one of these decompositions. Then we have either $r_i + r_j \geq r_m + r_k$ or $r_i + r_j \leq r_m + r_k$. In the first case we have either $f_{i,j} \geq 0$ or $f_{k,m} \geq 0$. Therefore, $\pi(Q)$ is either singular or belongs to a chamber of level 3.

Remark 3. The polyhedron D_4 has the combinatorial type of the regular 3-dimensional octahedron which is split by the walls into 8 chambers. The walls are squares whose vertices are the vertices of the octahedron. The group $\mathbb{Z}_4$ (cyclic permutations of vertices of a quadrilateral) acts on D_4 preserving the chamber structure, and the action is transitive on the set of chambers of a given class.

Lemma 4. *Suppose that $n \geq 5$. Then all kinds (1)-(4) of admissible multiindexes are realized by great chambers in D_n. If $n \geq 6$, then all great chambers are adjacent to the boundary of D_n.*

Proof. We start the induction with the case of pentagons. The negative chamber C_0 is represented by a regular pentagon P_0. Now, for any pair (i,j) consider the pentagon P_{ij} where $r_i = r_j = 1/4$, and other 3 sides have the same length $1/6$. Then each $\pi(P_{ij})$ belongs to the face of C_0 contained in the hyperplane $f_{ij} = 0$. Therefore, the chamber which is adjacent to C_0 along this face has the multiindex $\mu = \{i,j\}$.

Next, for any multiindex $\{i,j\}\&\{i,k\}$ we consider a pentagon $P_{\{ij\}\&\{ik\}}$ with the sides:

$$r_i = r_j = r_k = 1/4 \text{ and } r_s = r_q = 1/8. \tag{15}$$

It is clear that only two of the functionals (3) are nonpositive on $\pi(P_{\{ij\}\&\{ik\}})$. Therefore, $\pi(P_{\{ij\}\&\{ik\}})$ belongs to the boundary of a chamber with the multiindex $\mu = \{i,j\}\&\{i,k\}$. The chambers of level 3 with multiindex μ are adjacent to the chambers of octahedral faces of D_5 which have the same multiindex μ. Thus, we have established the existence for all chambers of levels 3, 2 and 1. Finally, let $\mu = \{1,2\}\&\{1,3\}\&\{1,4\}\&\{1,5\}$. Take the open tetrahedral face T_1 of ∂D_5 given by the equation $r_1 = 1/2$. Then all inequalities $r_1 + r_j > 1/2$ are satisfied near T_1. Thus the multiindex of the chamber adjacent to T_1 is equal to μ. Hence we have proved the Lemma in the case $n = 5$.

Suppose now that the assertion of the lemma is proven for $n - 1 \geq 5$, and we want to prove it for n. Let μ be any admissible multiindex which does not contain one number , say n (this is equivalent to the assumption that the level of μ is less than $n - 1$). Now, consider the polyhedron D_{n-1} and use the induction assumption to find a chamber C_μ^{n-1} with the multiindex μ. Then take the chamber $C = C_\mu^n$ of D_n which is adjacent to C_μ^{n-1}. All inequalities (8) corresponding to the multiindex μ are satisfied in C, and no inequality involving n like $f_{i,n} < 0$ can be satisfied in C since otherwise we would have : $r_i = 1/2$ on C_μ^{n-1} which is not the case. Thus, C has the multiindex μ and is adjacent to the boundary of D_n.

The last case to consider is when the multiindex has level $n - 1$:

$$\mu = \{1,2\}\&\{1,3\}...\&\{1,n\}. \tag{16}$$

Take the tetrahedral face F_1 of D_n given by the equation $r_1 = 1/2$, and apply the same arguments as in the case of pentagons. We conclude that the chamber C adjacent to F_1 has multiindex μ.

Lemma 5. *Suppose that we have two chambers C_α, C_β in D_5 where the multiindex β is obtained from α by adding the pair $\{i,j\}$. Then C_α, C_β have a common face at the wall $f_{ij} = 0$.*

Proof. Consider a generic segment $I \subset D_5$ which connects the interiors of these 2 chambers. Suppose that I intersects some wall f_{km}, where $\{k,m\} \neq \{i,j\}$. Then the inequality $f_{km} < 0$ is satisfied in one chamber, and $f_{km} > 0$ is satisfied in another. According to the hypothesis of lemma this implies that $\{k, m\} = \{i, j\}$.

Corollary 5. *Each chamber of level 2 in D_5 is adjacent to three chambers of level 3, one of which has class (A).*

6. Definition 3. A chamber C of D_n is said to have type I if for some (any) point $r \in C$ the moduli space $M_r = \pi^{-1}(r)$ is connected. The chamber C is said to be of type II if the moduli space is not connected.

Lemma 6. *A chamber C in D_n has type I iff for some (any) $r \in C$ a polygon $P \in M_r$ can be deformed to the symmetric polygon $\sigma(P) = \bar{P}$. For any r the moduli space M_r has at most two connected components. These two components are diffeomorphic via the map σ_*.*

Proof. We will prove the statement by induction over the number of vertices in the polygon. If $n = 3$ the assertion is evident. Now, suppose that we have proved the lemma for all $k < n$. Let P, P', P'' be n-gons such that $\pi(P) = \pi(P') = \pi(P'') = r \in C$. There are two numbers r_i, r_{i+1} in r (say $i = 1$) such that $r_i + r_{i+1} < 1/2$. Therefore, we can deform P, P', P'' to polygons where 1-st and 2-nd edges belong to a straight line and their intersection is one point. Thus, we have $n-1$-gons Q, Q', Q'' with the sides : $r_1 + r_2, r_3, ..., r_n$. However, the moduli space of $n-1$-gons consists of not more than 2 components. Thus, two of the $n-1$-gons , say Q, Q' can be deformed one to another. This means that $\pi^{-1}(r)$ consists of not more than 2 connected components.

Consider the diagonal d between 1-st and 3-nd vertices of P. This diagonal splits P into the union of a triangle Δ and an $n-1$-gon Q so that $1/2 \geq r_1 + r_2$. Applying the symmetry σ in d to the polygon P we obtain another polygon $\bar{P}$ so that $Area(P) = -Area(\bar{P})$. We can also apply the symmetry only to Δ to obtain another polygon N; put $\sigma(N) = \bar{N}$.

Claim. *The pairs of polygons P, N and $\bar{P}, \bar{N}$ belong to the same connected components of the moduli space. Each polygon in the moduli space M_r can be deformed either to P or $\bar{P}$.*

Proof. First we apply the induction hypothesis to deform Q to a triangle with the vertices v_3, v_4, v_1 keeping the length of d fixed. Thus, we obtain a quadrilateral Q with the vertices v_1, v_2, v_3, v_4. Now, since $r_1 + r_2 \leq 1/2$, we can deform Q to a triangle Δ' where e_1, e_2 belong to the straight line. We can assume that for all deformations the diagonal d belongs to one and the same line $\ell = \{z \in \mathbb{C} : Im(z) = 0\}$. Let

$P_t = g_t(\Delta')$ denote the deformation of Δ' to P where $g_t : \Delta' \to P$ is a continuous family of combinatorial maps.

However, we can construct in the same way a deformation h_t of the triangle Δ' to the polygon N as follows. If v_j is the vertex of δ' with $1 \leq j \leq 3$ then $h_t(v_j) = \sigma g_t(\sigma)$; if $j > 3$ then $h_t(v_j) = g_t(v_j)$. Thus we obtain a deformation of P to N. Application of the symmetry implies that $\bar{P}, \bar{N}$ belong to the same connected component.

However, the triangle Δ' up to symmetry is determined only by r. So each polygon $P \in M_r$ can be deformed either to Δ' or $\overline{\Delta'}$.

Therefore, if M_r has 2 connected components, no polygon P can be deformed to $\bar{P}$. In this case the involution $\sigma : P \mapsto \bar{P}$ permutes the two components. Hence the lemma is proved.

7. The following is the basic property of the types of chambers.

Lemma 7. *(a) Suppose that the chamber $C = C^n$ is adjacent to a chamber $c = C^{n-1} \subset D_{n-1}$ which has the type X. Then C has the same type X. (b) If a chamber C is adjacent to a tetrahedral face of D_n then C has type I.*

Proof. Consider (a). Let $\pi(Q) \in c \subset D_{n-1}$ and

$$Q = \lim_{t \to 0} P_t, \tag{17}$$

where P_t is a continuous family of n-gons. Suppose that C has type I. Then for each t there exists a curve $P_t(s)$ between P_t and its mirror image $\bar{P}_t$ so that the family of maps

$$[0,1] \ni s \mapsto P_t(s) \tag{18}$$

is uniformly continuous (see Remark 2). Therefore, there exists a limit

$$\lim_{k \to \infty} P_{1/k}(s) = Q(s), \tag{19}$$

so that the continuous curve $Q(s)$ connects Q with its mirror image $\bar{Q}$. Hence the chamber c has type I.

Now, suppose c is a chamber of type I. Let $P \in \mathcal{P}_{n-1}$ be a polygon which belongs to $\pi^{-1}c$. Then we construct a sequence of n-gons $Q_k \in \mathcal{P}_n$ approximating P so that

$$\lim_{k \to \infty} r_n(Q_k) = 0, \tag{20}$$

and the sides e_n, e_{n-1} of Q_k belong to one and the same line. Thus we can consider the union $e_n \cup e_{n-1}$ as a single edge of Q_k, and Q_k become elements P_k of $\mathcal{P}_{n-1}$. However, the chambers of D_{n-1} are open, so $P_k \in c$ for large k. Therefore, each P_k can be deformed to its mirror image $\bar{P}_k$, and Q_k belongs to the chamber of type I.

Hence, we have proved the assertion (a) of the Lemma.

First we prove the assertion (b) for $n = 4$. Let P be a degenerate trapezoid where $r_1 = r_3$, $r_4 = 1/2$. Consider a trapezoid Q sufficiently close P so that for Q we have:

$$r_4 > r_2, r_4 > r_1 + r_2, r_1 = r_3. \tag{21}$$

In this case we first can deform Q to a triangle T where e_1 and e_2 form a single side (Figure 1). Then we deform T to the "butterfly" B where the point of intersection $e_4 \cap e_2$ is the center of symmetry (Figure 1). Finally, we deform B to the triangle $\bar{S}$ which can be deformed to the trapezoid $\overline{Q}$ (Figure 1). This proves that the chamber C has type I.

Suppose now that $n \geq 5$ and P is a degenerate polygon such that $r_1 = 1/2$; then we can organize the edges $e_3, ..., e_n$ to a single edge. As the result we have a degenerate quadrilateral Q and for any Q' sufficiently close to Q, there is a deformation of Q' to $\overline{Q'}$. But Q' can be considered also as an element P' of $\mathcal{P}_n$. Therefore, $\pi(P')$ belongs to the chamber C of the type I. This proves the assertion.

Corollary 7. *The chambers of type I in D_4 are the chambers of level 3 and class (B); the chambers of type II in D_4 are the chambers of level 3 and class (A).*

Proof. The chambers of class (B) are adjacent to the "tetrahedral" faces T_j of D_4 given by the equations $r_j = 1/2$. Thus, they have the type I. The chambers of class (A) are adjacent to the "octahedral" faces of D_4 which are the polyhedrons D_3. However, the moduli space for each nondegenerate triangle consists of 2 points. Therefore, chambers of class (A) have type II.

Corollary 8. *Suppose that two chambers of D_4 are adjacent along a wall W given by the equation $f = 0$. Then the chamber contained in the subspace $f < 0$ has type I.*

Proof. This follows from the description of the types of chambers in Corollary 7.

8. Lemma 8. *Let $n \geq 5$. Suppose that two chambers are adjacent along a wall W given by the equation $f = 0$. If W is a great wall (i.e.,*

$f = f_{ij}$), then the chamber contained in the half-space $f < 0$ has type I.

Proof. We prove all assertions of the Lemma by induction. If $n = 4$ then we have already proved all assertions (Corollary 8). Suppose that we have proved the assertion for $n-1$. Let P be a degenerate polygon such that $\pi(P) \in W$. Then there are at least two adjacent edges e_s, e_{s+1} of P such that their indices are different from i, j. Define the new polygon Q with $n-1$ edges obtained from P by considering e_s, e_{s+1} as a single edge. Now, the assertion follows from the induction hypothesis applied to Q.

Lemma 9. *Suppose that two chambers are adjacent along a wall W given by the equation $f = 0$. If W is not a great wall, then both chambers have type I.*

Proof. We prove the assertion of the lemma by induction. If $n \leq 5$ then the lemma is trivially correct. Now suppose that the assertion holds for all $k < n$. If $r = \pi(P) \in W$ does not belong to any other wall, then $P \subset \mathbb{R} = \ell$, and the image of P is a closed interval with the end-points V_1, V_j which are vertices of P. Split P along the diagonal d between V_1, V_j into $L_1 \cup L_2$.

Remark 4. It can not happen that d is a side of P since that would mean that the wall W is a boundary face of D_n.

We suppose that $V_1 < V_j$ on $\mathbb{R}$, and the edge $[V_1, V_2]$ with length r_1 is directed in the positive direction on $\mathbb{R}$. We also assume that r_1 enters equation (3) of W with a positive sign. We call β_1 the number of *backtracks* on L_1 if this is the number of segments of L_1 which enter the equation of W with a negative sign; β_2 , the number of backtracks on L_2 , is the number of edges of L_2 which enter the equation of W with a positive sign.

Now, if $max(\beta_1, \beta_2) \geq 2$ (say $\beta_1 > 1$) then we can apply the induction hypothesis to $L_1 \cup d$ and thus prove that the whole neighborhood of $\pi(P)$ has type I.

Suppose that $\beta_1 = \beta_2 = 1$. Then either we can apply induction or both L_1 and L_2 contain exactly 3 segments. Thus, we have:

$$d > r-1, d > r_3, r_2 < r_3, r_2 < r_1, d + r_2 = r_1 + r_3. \tag{22}$$

Therefore,

$$d + r_2 = r_1 + r_3, d + r - 1 > r_2 + r_3, d + r_3 > r_2 + r_1. \tag{23}$$

This means that if we deform P so that the length of d increases and r_1, r_2, r_3 are fixed, then this deformation of the quadrilateral $L_1 \cup d$ has type I and we are done. Now, we can increase the length of d by increasing negative as well as positive segments of L_2 (there are both). Thus for both deformations of P where $f < 0$ as well as $f > 0$ we end up at a chamber of type I.

Now, assume that L_2 has no backtracks. Then L_2 consists of 2 segments and L_1 consists of 4 segments. Consider the pentagon $P' = L_1 \cup d$. Direct calculation shows that $\pi(P')$ belongs to a common face of the negative chamber and a chamber of the level 1. Therefore, any small deformation of P' (and hence of P) leads to a polygon of the type I.

Finally, we are left with the case where $\beta_1 = \beta_2 = 0$. Consider the chamber adjacent to W where $f > 0$. Then we can organize the chain L_2 into a single segment ; denote the new polygon by P^+. Our deformation of P^+ leads to a chamber of the type I in D_4 (see Lemma 8). Therefore, this deformation of P has the type I. In the case of the chamber where $f < 0$ we repeat the argument above by organizing L_1 into a single segment.

Corollary 9. *In D_n the chambers of level 3 and class (A) have type II, all chambers of the class (B) have type I.*

Proof. If $n > 4$, then each chamber of level 3 with the multiindex μ is adjacent to the boundary face D_{n-1} along a chamber of the same multiindex (Lemma 4). Now the assertion follows by combining Lemma 8 and Corollary 7.

Lemma 10. *In D_n a chamber has type II if and only if it has level 3 and class (A).*

Proof. Suppose that $C = C_\mu$ is a great chamber in D_n with multiindex μ which is not a multiindex of the maximal level n and is not of class (A), level 3. Then, we can always add to this multiindex a pair (i, j) such that the new multiindex ν is still admissible (see the description of the multiindexes in section 5). Therefore, according to Lemma 8 and Lemma 5 the chamber C has type I. If C has maximal level n, then it is adjacent to a tetrahedral face of D_n and according to Lemma 7 we conclude that C again has the type I.

9. So we have proved

Theorem 1. *In the polyhedron D_n the chamber C has type II if and only if C is a great chamber of the level 3 and class (A). All other* *chambers correspond to connected moduli spaces* M_r.

This theorem has the following geometric interpretation. We know that for each nondegenerate triangle T with the sides $r = (r_1, r_2, r_3)$ the moduli space M_r has exactly two components. If we "regenerate" this triangle to a n-gon P which is sufficiently close to T in $\mathcal{P}_n$, then the moduli space M_ρ of P again has exactly two connected components (here $\rho = \pi(P)$). According to Theorem 1 this is essentially the only way that any n-gon Q can have nonconnected moduli space. Namely, if Q has a nonconnected moduli space and $\pi(Q)$ belongs to a chamber C, then Q can be deformed inside $\pi^{-1}(C)$ to a polygon P as above.

10. Example . There is a point $r \in D_5$ which belongs to a chamber of type II such that:

on each component of M_r the function *Area* has negative as well as positive values.

This means (contrary to our original expectations) that for all $n > 4$ the function *Sign* $\circ$ *Area* fails to distinguish connected components of M_r for some values of r. Nevertheless, one can prove that for $n = 4$ the *Area* does distinguish connected components of M_r for any r. We do not know any numerical invariant of polygons that can distinguish components for $n > 4$.

Construction. We start with a convex polygon $P' = ABCD$ such that:

(a) the angle at the vertex C is $\pi/2$;

(b) the side BC has length $|BC| = 1$ and $|AB| = 1 - \epsilon$;

(c) the angle between the diagonal BD and the side AB is $\pi/2$, and the length of BD is equal to d.

The numbers $d, 1 > \epsilon > 0$ will be specified below. See Figure 2.

Then, fold the triangle ABD along the diagonal BD. As the result we have the 4-gon Q'. Let Q'' be the mirror image of Q' with respect to CD. Then the union of two 4-gons Q', Q'' is a pentagon P (we do not consider C as a vertex of P). We denote the vertices of P as: A, B, B', A', D (see Figure 2). We assume that the area of the triangle $BB'D$ is negative, then the triangles $ABD, B'A'D$ have equal positive area.

The conditions on d, ϵ are:

$$d^2(1-\epsilon)^2 > d^2 - 1, \tag{24}$$

$$(1-\epsilon)+1<\sqrt{d^2+(1-\epsilon)^2}. \tag{25}$$

The first condition (24) means that the area of BDB' is less than $2 \cdot Area(ABD)$, so $Area(P) > 0$. The second condition (25) means that $|AD|+|DA'| > |AB|+|BB'|+|B'A'|$. The inequalities (25), (24) also imply that

$$d+|BB'| > |AD|+|AB|. \tag{26}$$

The inequalities (25, 26) are equivalent to:

$$\frac{3-d^2}{2} < \epsilon < \frac{d-\sqrt{d^2-1}}{d}. \tag{27}$$

However, the left side is negative if $d > 2$. The right side of (27) is always positive and less than 1. Therefore, we can find ϵ and d which satisfy both conditions (25, 24).

The condition (26) implies that we can deform P keeping the length of $B'D$ fixed so that the quadrilateral $ADB'B$ becomes an embedded triangle with negative area (the triangle BDB' does not degenerate under this deformation, but the triangle ABD degenerates and changes the sign of area). We continue this deformation so that the triangle $DABB'$ doesn't change the area, but the triangle $DA'B'$ degenerates and changes the sign of area). As the result we have an embedded pentagon Q where all vertices except D belong to a straight line. The area of this pentagon is negative.

Finally, we can normalize the perimeter of the polygon P by applying a similarity. If the image $\pi(P)$ is the vector r, then direct calculation shows that the point r belongs to the chamber of class (A), level 3. Therefore, the moduli space M_r is not connected, but the function $Area$ has both negative and positive values in one component of M_r.

11. Now, we consider the global topology of the moduli spaces M_r for nonsingular r. Since the projection π is a trivial fibration near M_r, if the fiber M_r is not orientable, then for some small neighborhood U of r the priemage $\pi^{-1}(U) = \mathbb{R}^{n-1} \times M_r$ is nonorientable as well. Therefore, the space $\mathcal{P}_n$ is not orientable which contradicts the fact that $\mathcal{P}_n = \mathbb{C}P^{n-2}$. This contradiction shows that all regular fibers M_r are orientable. For $r \in \Sigma$ the moduli space M_r is always singular (the singularities are isolated and correspond to degenerate polygons). These singularities are always quadratic. Each singular point P is an isolated fixed point of the involution $\sigma_* : M_r \to M_r$. If a point P separates its neighborhood U in M_r, then σ_* interchanges the two connected components of $U - P$, and $\pi(P)$ belongs to a great wall. We shall discuss the singularities in detail in our forthcoming paper.

12. Our next problem is to understand how the topology of the moduli space M_r changes when the parameter r "crosses a wall" in D_n. We recall that the space $\mathcal{P}_n$ is a hyperplane section in the space of "free linkages" $\tilde{\mathcal{P}}_n = \mathbb{C}P^{n-1}$. We will think about the space $\tilde{\mathcal{P}}_n$ as the configuration space of ordered $n+1$-tuples $(v_0, ..., v_n)$ in $\mathbb{C}^{n+1}$ modulo the diagonal action of the group of similarities $Aff(\mathbb{C})$. The edges of the free linkage $z = (v_0, ..., v_n)$ are the vectors $e_j = v_j - v_{j-1}$. We denote by $\tilde{\pi}$ the projection

$$\tilde{\pi} : z = (e_1, ..., e_n) \mapsto (r_1 = |e_1|, ..., r_n = |e_n|) \tag{28}$$

for any point $z \in \tilde{\mathcal{P}}_n$ with the normalized perimeter:

$$|e_1| + ... + |e_n| = 1. \tag{29}$$

Then $F_r = \tilde{\pi}^{-1}(r)$ is the space of free linkages with fixed side lengths modulo the action of the group of similarities $Aff(\mathbb{C})$. We consider the smooth function $h : F_r \to \mathbb{R}$ given by

$$h(e_1, ..., e_n) = |e_1 + ... + e_n|^2 = d^2(v_0, v_n). \tag{30}$$

We let E_r be the complement $F_r \setminus h^{-1}(0)$. In what follows we will use $\mathbb{S}^1(r_i)$ to denote the circle in $\mathbb{C}$ with center at 0 and radius r_i. Let $N_r = \mathbb{S}^1(r_1) \times ... \times \mathbb{S}^1(r_n)$. Then F_r is the quotient of N_r by $SO(2)$ acting diagonally. We next let $X_r \subset N_r$ be the smooth submanifold defined by

$$X_r = \{ (e_1, ..., e_n) \in N_r : v_n \in \mathbb{R}_+, \quad \text{i.e., } Im(v_n) = 0, Re(v_n) > 0\} \tag{31}$$

Then X_r is a slice for the projection $N_r \to F_r$ over E_r. Suppose that $\Lambda \in E_r$ is a degenerate configuration, i.e., all points of Λ belong to a straight line L.

Define $f = f(\Lambda)$ to be the number of edges e_i that point in the direction $\overrightarrow{v_0 v_n} = \overrightarrow{d}$ (f is the number of "forwardtracks"), and $b = b(\Lambda)$

to be the number of edges e_j that point toward the direction $-\vec{d}$ (the number of "backtracks"). We shall assume that the line L which contains Λ is $\mathbb{R} = \{z : Im(z) = 0\} \subset \mathbb{C}$, $v_0 = 0$ and $v_n \in \mathbb{R}_+$.

Lemma 11. *(i) The function h is a Morse function on E_r. (ii) The Hessian of h at any degenerate configuration Λ has $b(\Lambda)$ positive eigenvalues and $f(\Lambda) - 1$ negative eigenvalues.*

Proof. We shall identify the slice $X = X_r$ with E_r. We define two functions g, k on N_r by

$$k(e_1, ..., e_n) = \sum_{j=1}^{n} Re(e_j), \tag{32}$$

$$g(e_1, ..., e_n) = \sum_{j=1}^{n} Im(e_j). \tag{33}$$

We observe that $X_r = g^{-1}(0) \cap k^{-1}(R_+)$. If $\Lambda \in X_r$ then g is a smooth function near Λ. We note that $h|_X = k^2$, and since $k > 0$ on X (by definition) we may replace h by k. We first study the critical point behavior of k on N_r. Let (r, θ) be the polar coordinates in $\mathbb{C}$. Hence

$$k((r_1, \theta_1), ..., (r_n, \theta_n)) = \sum_{j=1}^{n} r_j \cos\theta_j. \tag{34}$$

It is immediate that $\Lambda^* \in N_r$ is a critical point of k if and only if Λ^* is degenerate (i.e., contained in the real line $\mathbb{R}$). We shall use the vector-fields $\frac{\partial}{\partial\theta_1}, ..., \frac{\partial}{\partial\theta_n}$ to define coordinates on the tangent bundle of X. Then the matrix representation of the Hessian $D^2k|_{\Lambda^*}$ of the function k is

$$\begin{bmatrix} -\epsilon_1 r_1 & 0 & \dots & 0 \\ 0 & -\epsilon_2 r_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\epsilon_n r_n \end{bmatrix}, \tag{35}$$

where $\epsilon_j = 1$ if e_j is a forward track and $\epsilon_j = -1$ otherwise. Hence the Hessian $D^2k|_{\Lambda^*}$ has signature (b, f).

We claim that critical points of $k|_X$ coincide with the critical points of k which lie on X. Indeed, $k|_X$ has a critical point at Λ if and only if there exists $\lambda \in \mathbb{R}$ such that $dk|_\Lambda = \lambda dg|_\Lambda$. Hence if $(\theta_1, ..., \theta_n)$ are coordinates of Λ, then we have $\lambda = \tan\theta_j$ for all $j = 1, ..., n$. Therefore either $\theta_j = \theta_i$ or $\theta_j = \pm\pi + \theta_i$. This means that Λ is contained in $\mathbb{R}$ and $\lambda = 0$. Thus the claim follows.

We now investigate the Hessian of $k|_X$ at a critical point Λ^*. We put $B = (D^2k)|_{\Lambda^*}$. Since the submanifold $X \subset N_r$ is defined by the equation

$$\sum_{j=1}^{n} Im(e_j) = \sum_{j=1}^{n} r_j \sin\theta_j = 0, \tag{36}$$

we conclude that $T_{\Lambda^*}(X) \subset T_{\Lambda^*}(N_r)$ is defined by the equation

$$\sum_{j=1}^{n} \epsilon_j r_j d\theta_j = 0. \tag{37}$$

Hence using the basis $\frac{\partial}{\partial\theta_1}, ..., \frac{\partial}{\partial\theta_n}$ we find that

$$T_{\Lambda^*}(X) = \{(c_1, ..., c_n) \in \mathbb{R}^n : \sum_{j=1}^{n} \epsilon_j r_j c_j = 0\}. \tag{38}$$

Therefore, $T_{\Lambda^*}(X)$ is the orthogonal complement of the vector $\Delta = (1, ..., 1)$ for the quadratic form B. However,

$$B(\Delta, \Delta) = -\sum_{j=1}^{n} \epsilon_j r_j = -d(v_0, v_n). \tag{39}$$

Since $B(\Delta, \Delta) < 0$ we find that the signature of the restriction of B on $T_{\Lambda^*}(X)$ is $(b, f-1)$.

12. We restrict ourselves now to the cases $n = 4, n = 5, n = 6$.

Suppose that $n = 4$ and $r \in C_\mu$ which is a chamber of class (A). Then the moduli space M_r is a union of two disjoint smooth circles. If C_μ has class (B), then M_r is a smooth circle. For each $r \in \Sigma(D_n)$ the configuration space M_r is connected and has singularities. If r is a point of a wall W in $int(D_4)$ which does not belong to any other wall, then M_r contains a unique degenerate polygon. Therefore M_r is a bouquet of two circles, and the only singular point is the point of intersection of two circles. Suppose now that r belongs to the intersection of exactly 2 walls. Then M_r has two singular points. In this case (up to renumeration) we have either $r_1 = r_3, r_2 = r_4$ or $r_1 = r_2, r_3 = r_4$.

Neither of these two points can separate M_r. Therefore, the moduli space is the union of 2 circles identified at two different points. Finally, we have the case of the triple intersection point (rhombus). In this case we have 3 singular points on M_r; neither of them separates M_r, and thus M_r is a cycle of 3 circles , any pair of which has a common point. See Figure 3 below.

13. Now, assume that $n = 5$.

Suppose that we have either (a) two adjacent chambers $C = C_\mu, C' = C_\nu$ so that the level of C_μ is l, the level of C_ν is $l+1$, or (b) a chamber C adjacent to a tetrahedral boundary face of D_5. Without loss of generality we can assume that in the case (a) $\{i, 5\}$ is in ν but not in μ, in the case (b) the chamber C is adjacent to the face W given by the equation $r_5 = 1/2$ and we put $C' = W$ in this case. Denote by $W = \{f_{i,5} = 0\}$ the common wall of C, C'. Consider a smooth path $\gamma(t)$ so that $\gamma(a) \in C$, $\gamma(b) \in C'$ and along this path the lengths r_j $(j = 1, ..., 4)$ are constant and for $r = \gamma(t)$ we have: $r_5 = t$. We can assume that $0 < a < b < \infty$. Let $s = \gamma^{-1}(W)$. Then $q = \gamma(s)$. We consider q to be generic if it does not belong to any wall different from W.

Remark 5. In case (a) we may choose a degenerate polygon $P = [v_1, ..., v_5] \in \pi^{-1}(q)$ such that $v_0 = 0, v_5 \in \mathbb{R}_+$. Consider the free degenerate linkage $\Lambda = (v_1, ..., v_5)$. Then for every generic q the linkage Λ has 1 backtrack. (Otherwise either we have case (b) and $b(\Lambda) = 0$, or $b(\Lambda) = 2$ and $r_l + r_k + r_5 = 1/2$, for some l, k such that $1 \leq l < k < 5$.)

Lemma 12. *For a generic choice of $q = \gamma(s)$ there exists an open neighborhood U of q on γ so that:*

(1) $M_U = \pi^{-1}(U)$ is a smooth 3-manifold and

(2) the function r_5 is a Morse function on M_U with a nondegenerate critical value q of signature $(1, 2)$.

Proof. Put $\rho = (r_1, ..., r_4)$. Then M_U is an open subset of the moduli space of free linkages F_ρ. Thus, by Lemma 11, the manifold M_U is smooth and r_5 is a Morse function. If Λ is a degenerate linkage in F_ρ then, by Remark 5, $b(\Lambda) = 1$, $f(\Lambda) = 3$. The lemma hence follows from Lemma 11.

Now, we fix a path γ with a generic choice of q.

The space $M_q = M_{\gamma(s)}$ is a singular surface with unique singular point $[P]$ corresponding to a degenerate pentagon P. Then the fact that r_5 is a Morse function near $[P]$ means that there is a small neighborhood

V of $[P]$ in M_γ so that in the case (a) after some change of coordinates in V the function $h = r_5 - s$ can be written as:

$$h(x, y, z) = -x^2 - y^2 + z^2. \tag{40}$$

In the coordinates (40) we have:

$$M_r \subset \{h < 0\}, M_{r'} \subset \{h > 0\}, \tag{41}$$

since r_5 increases as we go from C to C'.

In the case (b) the function $h = r_5 - s$ can be written as:

$$h(x, y, z) = -x^2 - y^2 - z^2, \tag{42}$$

where $h > 0$ on M_r for each $r \notin W$. Therefore, in the case (a) the Morse surgery from M_r to $M_{r'}$ is equivalent to removing a handle : we first pinch a simple loop ℓ on M_r to obtain M_q , and then remove the point of intersection to obtain $M_{r'}$. If ℓ is homologically trivial, then $M_{r'}$ is not connected (analogously to Lemma 8). This can happen only if C' is the chamber of the level 3, Class (A). In such a case the two components of $M_{r'}$ are diffeomorphic; the diffeomorphism is given by the mirror reflection map $P \mapsto \bar{P}$. In other cases the loop ℓ is homotopically nontrivial. In the case (b) the Morse surgery is attaching a 0-handle to the connected surface M_r, and therefore, in such a case M_r is a sphere for all $r \in C$.

Essentially the same is true for any n-gon. Suppose that W is a wall of index $ind(W)$ in D_n given by the equation $f_{ij} = 0$ where $f_{ij} = r_i + r + j - 1/2$ (as in formula (7)). Let C, C' be chambers adjacent to W along an open subset V which does not intersect any other walls, so that f_{ij} is negative on C. Let $r \in C, p \in C'$. Then M_r is obtained from M_p by a Morse surgery of the index $ind(W) - 2$. In particular, if W is a great wall and $ind(W) = 2$, then M_r is obtained from M_p either by connected sum of two connected components of M_p or "self-connected sum" of a single component of M_p (attaching of a zero handle).

14. Now we can prove a theorem about the topology of M_r for all chambers in D_5.

Theorem 2. *If $r \in C$ where the chamber C has the level l, then the Euler characteristic $\chi(M_r)$ is given by*

$$\chi(M_r) = 2(l - 3). \tag{43}$$

Proof. We recall that all configuration spaces M_r are orientable and begin a level by level consideration.

Level 4. Suppose that C has level 4. Apply Lemma 12 to a generic path γ so that $\gamma(b) \in W = \{r_5 = 1/2\}$. The preimage $r_5^{-1}(1/2) \subset M_U$ consists of a single point which is a degenerate pentagon. Therefore, Lemma 11 implies that M_r is a sphere for all $r \in \pi(\gamma)$. Thus, M_r is a sphere for each $r \in C$.

Level 3, Class (B). Each chamber C of level 3 class (B) is adjacent to some chamber C' of level 4. Then for each $r \in C$ the connected surface M_r is obtained from the sphere by attaching a handle, and thus M_r is a torus.

Level 2. Each chamber C of level two is adjacent to some chamber C' of level 3, class (B). Then, for each $r \in C$ the connected surface M_r is obtained from the torus by attaching a handle, and thus M_r is a surface of genus 2.

Level 3, Class (A). On the other hand, each chamber C' of level 3, class (A) is adjacent to some chamber C of level 2. Thus, for each $r' \in C'$ the surface $M_{r'}$ is obtained from the surface of genus 2 by a surgery along a homologically trivial loop ℓ. The loop ℓ can not be homotopically trivial, otherwise $M_{r'}$ would be the union of two non-homeomorphic surfaces (a sphere and a surface of genus 2), which is impossible by Lemma 6. The only possible case is that $M_{r'}$ is the union of two tori.

We repeat the same arguments as above to prove that if the level of C is 1, then M_r is a surface of genus 3; if C has level 0, the M_r is a surface of genus 4. This concludes the proof of Theorem 2.

15. Our next problem is to describe the topology of the moduli spaces of hexagons. Let $r \in D_n$; denote by M_r' the space of n-gons in $\mathbb{C}$ with the side lengths $r = (r_1, ..., r_n)$ modulo translations. We recall that N_r is the torus

$$\{e = (e_1, ..., e_n) \in \mathbb{C}^n : |e_j| = r_j, j = 1, ..., n\}. \tag{44}$$

We define the momentum map $\sigma : N_r \to \mathbb{C}$ by

$$\sigma(e_1, ..., e_n) = e_1 + ... + e_n. \tag{45}$$

Then M_r' is canonically isomorphic to $\sigma^{-1}(0)$ by sending $(v_1, ..., v_n)$ to $(v_2 - v_1, ..., v_1 - v_n)$. It is easily seen that 0 is a regular value of σ

provided r is not on a wall of D_n. As in Lemma 11 we define two functions k, g on N_r by

$$k(e_1, ..., e_n) = \sum_{j=1}^{n} Re(e_j), \tag{46}$$

$$g(e_1, ..., e_n) = \sum_{j=1}^{n} Im(e_j). \tag{47}$$

We let $Y \subset N_r$ be the zero level set of g. Then Y is smooth. We wish to describe the level sets of $h = k|_Y$ near zero. Let $\hat{r} = (r_1, ..., r_n, \epsilon) \in D_{n+1}$.

Lemma 13. *(i)* $h^{-1}(0) = M'_r$;

(ii) for $\epsilon > 0$, $h^{-1}(\epsilon) = M_{\hat{r}}$ *is the space of* $n+1$*-gons with side lengths* $(r_1, ..., r_n, \epsilon)$ *modulo the action of* $Aff(\mathbb{C})$.

Proof. We observe that $|h| = r_{n+1}|_Y$ where $r_{n+1} = \sqrt{k^2 + g^2}$, since $g \equiv 0$ on Y. If $(e_1, ..., e_n) \in h^{-1}(\epsilon)$, then we put $e_{n+1} = -(e_1 + ... + e_n)$ so that $|e_{n+1}| = \epsilon$. Thus, we obtain an $n+1$-gon with side lengths $(r_1, ..., r_n, \epsilon)$ so that $v_{n+1} = \epsilon \in \mathbb{C}$. The set of these $n+1$-gons is a cross-section to the action of $Isom_+(\mathbb{C})$ on $M'_{\hat{r}}$ and the lemma follows.

Lemma 14. *Zero is not a critical value of the function* h.

Proof. It suffices to show that $dk|_Q$ and $dg|_Q$ are linearly independent (over $\mathbb{R}$) functionals on $T_Q(N_r)$ for $Q \in k^{-1}(0)$. Since $\sigma = k + \sqrt{-1}g$ the lemma follows from the preceding observation that 0 is a regular value for q.

As a consequence of the previous two lemmas we obtain the following.

Corollary 15. *Suppose that* $r = (r_1, ..., r_n) \in int(D_n)$ *does not lie on any wall, and let* $\hat{r} = (r_1, ..., r_n, \epsilon)$. *Then for sufficiently small positive* ϵ *the space* $M_{\hat{r}}$ *is diffeomorphic to* $M_r \times \mathbb{S}^1$.

Proof. Clearly we have $M'_r \cong M_r \times \mathbb{S}^1$. On the other hand, Lemma 14 implies that the fibers $h^{-1}(\epsilon)$ are diffeomorphic to $h^{-1}(0)$ for small positive ϵ. Therefore, by Lemma 13 we conclude that $M_{\hat{r}}$ is diffeomorphic to $M_r \times \mathbb{S}^1$.

Consider now the cell structure of D_6. The walls in D_6 are either great walls or walls of index 3. All hyperplanes defining the walls of index 3 contain the center of D_6 which is the point $\mathcal{O} = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$ represented by the regular hexagon. Therefore, each component of the decomposition of D_6 by walls of index 3 is a cone with center at $\mathcal{O}$ over a tetrahedral face of ∂D_6 or over a chamber of an octahedral face. We know that crossing a great wall (which decreases the level of a great chamber) results in an index zero surgery on the moduli space M_r. So, it is enough to consider the topology of the space M_r for r sufficiently close to chambers of the octahedral faces of D_6.

Suppose that $C' = C'_\mu$ is a chamber of the octahedral face of ∂D_6 given by the equation $r_s = 0$; let $Cone_{\mathcal{O}}(C')$ be the adjacent component of the decomposition of D_6 by walls of index 3. Since all functionals f_{ij} (see (7)) are negative at the point $\mathcal{O}$, the wall $\{f_{ij} = 0\}$ intersects $Cone_{\mathcal{O}}(C')$ if and only if the functional f_{ij} is positive on C'; in particular the index s is not present in the multiindex μ. This means that the number of great walls intersecting $Cone_{\mathcal{O}}(C')$ is equal to the level $l(\mu)$.

(a) First we consider the case where C' has type I. Then for $r' \in C'$ the moduli space of pentagons $M_{r'}$ is an oriented compact surface Σ_g of genus $g \leq 4$. If C is a chamber in $Cone_{\mathcal{O}}(C')$ adjacent to C', then for each $r \in C$ the moduli space of hexagons M_r is diffeomorphic to $\mathbb{S}^1 \times \Sigma_g$ (according to Corollary 15). Therefore, for $r \in \hat{C} \subset Cone_{\mathcal{O}}(C')$ the moduli space M_r is a connected sum of of $\mathbb{S}^1 \times \Sigma_g$ and $k = l(\mu) - l(\hat{C}) \leq l(C') \leq 4$ copies of $\mathbb{S}^2 \times \mathbb{S}^1$, where $l(\hat{C})$ is the level of $\hat{C}$.

(b) Now we consider the case where C' has type II. If C is a chamber in $Cone_{\mathcal{O}}(C')$ adjacent to C', then for each $r \in C$ the moduli space of hexagons M_r is diffeomorphic to the disjoint union of two copies of $T^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. Suppose that $\hat{C}$ is a chamber in $Cone_{\mathcal{O}}(C')$ which is adjacent to C. Then, according to Lemma 8, the chamber $\hat{C}$ must have type I; therefore, for each $r \in \hat{C}$ the moduli space M_r is diffeomorphic to the connected sum $T^3 \# T^3$. The chamber C' has level 3. Hence, for all other chambers $\tilde{C}$ in $Cone_{\mathcal{O}}(C')$ the moduli spaces M_r are diffeomorphic to connected sums of $T^3 \# T^3$ and t copies of $\mathbb{S}^2 \times \mathbb{S}^1$, where t is equal to $3 - l(\tilde{C}) \leq 2$.

(c) The last case is when C' is a tetrahedral face of ∂D_6. Then 5 great walls intersect $Cone_{\mathcal{O}}(C')$. If C is a chamber of D_6 adjacent to C', then the moduli space M_r is diffeomorphic to $\mathbb{S}^3$ for all $r \in C$. If $\hat{C}$ is any chamber in $Cone_{\mathcal{O}}(C')$ and $r \in \hat{C}$, then M_r is diffeomorphic to the connected sum of $5 - l(\hat{C})$ copies of $\mathbb{S}^2 \times \mathbb{S}^1$.

Thus we have proved

Theorem 3. *If the moduli space of hexagons M_r is nonsingular and* *connected, then it is either diffeomorphic to a connected sum of k copies of $\mathbb{S}^2 \times \mathbb{S}^1$ and of the product $\Sigma_g \times \mathbb{S}^1$, or it is diffeomorphic to connected sum of $T^3 \# T^3$ and $t \leq 2$ copies of $\mathbb{S}^2 \times \mathbb{S}^1$. Here r belongs to a chamber C, the genus g of the surface Σ_g is not greater than 4 and $k \leq 4$. If M_r is nonsingular but is not connected, then it is homeomorphic to the disjoint union of two copies of $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$.*

16. Stable measures on the unit circle.

Let D be the unit disc in the complex plane centered at zero. Denote by $\mathbb{S}^1$ the boundary of D. We identify the group of conformal automorphisms $Conf(D)$ of the disc D with the connected component $SO(2,1)_0$ of $SO(2,1)$.

The following definition is motivated by Mumford's notion of stable points for actions of semi-simple Lie groups on algebraic varieties.

Definition 4. A probability measure μ on $\mathbb{S}^1$ is said to be *stable* if the mass of any atom of μ is less than $1/2$. A measure is said to be *semi-stable* if the mass of any atom of μ is not greater than $1/2$. We recall that the center of mass of a probability measure μ on $\mathbb{S}^1$ is equal to

$$B(\mu) = \int_{\mathbb{S}^1} z d\mu(z). \tag{48}$$

Theorem 4. *For each stable measure μ on $\mathbb{S}^1$ there exists a conformal transformation $\gamma \in Conf(D)$ such that $B(\gamma_* \mu) = 0$. The element γ is unique up to the postcomposition $g \circ \gamma$ where g belongs to the orthogonal group $SO(2)$ which is the stabilizer of 0 in $Conf(D)$.*

Proof. Douady and Earle in [5] define the conformal center of mass $C(\mu) \in D$ for any stable probability measure μ on $\mathbb{S}^1$. The point $C(\mu)$ has the following properties:

(a) For any $\gamma \in Conf(D)$

$$C(\gamma_* \mu) = \gamma(C(\mu)). \tag{49}$$

(b) $B(\mu) = 0$ if and only if $C(\mu) = 0$.

Now Theorem 4 follows from transitivity of the action of $Conf(D)$ on D.

17. Consider a stable atomic probability measure μ on $\mathbb{S}^1$ which is equal to

$$\sum_{j=1}^{k} \mu_j \delta_{z_j}, \tag{50}$$

where δ_z is the Dirac measure concentrated at the point $z \in \mathbb{S}^1$. We assume that all the points z_j are distinct. There exists a unique (up to multiplication by $\zeta \in \mathbb{S}^1$) Abelian differential $\phi_\mu \in \Omega^1(\overline{\mathbb{C}} - \{z_1, ..., z_k\})$ such that:

(1) the singularity of ϕ_μ at z_j is

$$(z - z_j)^{-2\mu_j} dz, \tag{51}$$

(2)

$$\iint_D |\phi_\mu|^2 = 1. \tag{52}$$

Explicitly the differential ϕ_μ is given by the following formula:

$$\omega_\mu = \frac{dz}{\prod_{j=1}^k (z - z_j)^{2\mu_j}}, \tag{53}$$

$$\phi_\mu = \omega_\mu \Big(\iint_D |\omega_\mu|^2 \Big)^{-1/2}. \tag{54}$$

Stability of the measure μ implies that the integral $\iint_D |\omega_\mu|$ is finite and therefore the Christoffel-Schwarz map

$$f_\mu : z \in D \mapsto \int_0^z \phi_\mu \tag{55}$$

is continuous in $D \cup \mathbb{S}^1$ and has bounded image. It is well known that the map f_μ is univalent and its image is a convex polygon $P_\mu = f_\mu(D)$ in $\mathbb{C}$ with the interior angles $\alpha_j = \pi(1 - 2\mu_j)$ at the vertices $w_j = f_\mu(z_j)$. The differential ϕ_μ defines a Riemannian metric

$$ds_\mu = |\phi_\mu|, \tag{56}$$

which has area 1 so that the map $f_\mu : (D, ds_\mu) \to \mathbb{C}$ is an isometric embedding. Suppose that $\gamma \in Conf(D), \nu = \gamma_*(\mu)$. Then ϕ_ν is proportional to $\gamma_* \phi_\mu$. Therefore the metrics ds_ν and $\gamma_* ds_\mu$ are isometric. This implies that there exists an orientation-preserving Euclidean isometry $\rho(\gamma)$ such that

$$\rho(\gamma) \circ f_\mu = f_\nu \circ \gamma, \tag{57}$$

$$\rho(\gamma)(P_\mu) = P_\nu. \tag{58}$$

18. Duality between the moduli spaces of Euclidean polygons with fixed angles and Euclidean polygons with fixed side lengths.

Fix a vector $r \in D_n - \Sigma$, i.e., r does not belong to any wall in the polyhedron D_n. Each polygon Z in the moduli space M_r corresponds to the collection of edges $(e_1, ..., e_n) \in (\mathbb{C}^*)^n$. The normalized vectors $z_j = e_j/r_j$ belong to the unit circle $\mathbb{S}^1$. The polygon Z is defined up to Euclidean isometry; therefore the vector $\overrightarrow{z} = (z_1, ..., z_n)$ is defined up to rotation around zero. Since the polygon Z is closed we conclude that

$$\sum_{j=1}^{n} r_j z_j = 0. \tag{59}$$

This means that we get a diffeomorphism

$$\epsilon : M_r \to \mathcal{M}_r = \{\overrightarrow{z} \in (\mathbb{S}^1)^n : \sum_{j=1}^{n} r_j z_j = 0\}/SO(2). \tag{60}$$

The vector $\overrightarrow{z}$ and collection of numbers $r = (r_1, ..., r_n)$ define a measure $\mu = \mu(\overrightarrow{z}, r)$ by the formula

$$\mu = \sum_{j=1}^{n} r_j \delta_{z_j}. \tag{61}$$

This measure has total mass 1 and $B(\mu) = 0$. The assumption $r \in D_n - \Sigma$ implies that μ is a stable atomic measure on $\mathbb{S}^1$. However the number of points in the support of μ can be less than n. Denote by $\{w_1, ..., w_k\}$ the support of μ assuming that all the points w_j are distinct. Let μ_j be the mass of μ in the point w_j. Then we use the formulas (54), (55) to define the Abelian differential ϕ_μ and univalent holomorphic map $f_\mu : D \to \mathbb{C}$. The vector $\overrightarrow{z}$ is defined up to the action of the group $SO(2)$; therefore the map f_μ and the k-gon $P_\mu = f_\mu(D)$ are defined up to rotation in $\mathbb{C}$. The map $Z \mapsto \overrightarrow{z} \mapsto \mu \mapsto P_\mu$ is not injective. To deal with this problem we define *marked* convex k-gons (P, I, b).

Definition 5. A *marked* convex k-gon (P, I, b) is a triple where P is a convex k-gon of area 1 in $\mathbb{C}$, I is a partition of the set $\{1, ..., n\}$

into k nonempty disjoint subsets $\{I_1, ..., I_k\}$, and b is a bijection from I onto the set of vertices of P. This triple must satisfy the following condition. Let

$$\mu_j = \sum_{i \in I_j} r_i. \tag{62}$$

Then the angle of P at the vertex $b(I_j)$ is equal to $\pi(1 - 2\mu_j)$.

Remark 6. Note that in general the cyclic order of vertices on the boundary of P has nothing to do with the bijection b even if $n = k$.

We divide the space of marked k-gons by the action of $Isom_+(\mathbb{E}^2)$ and denote the quotient by $E_k = E_k(r)$. The union

$$E = E(r) = \cup_{k=3}^{n} E_k(r) \tag{63}$$

is the space of (congruence classes) of marked convex polygons in $\mathbb{C}$ which have prescribed angles. Thus, if $[\overrightarrow{z}] \in \mathcal{M}_r$ then we define the corresponding marked polygon $(P, I, b) = \psi([\overrightarrow{z}]) \in E(r)$ as follows. The (congruence class) of the polygon P is given by the image of the Christoffel-Schwarz map f_μ as above. The map from $(z_1, ..., z_n)$ to the support of the measure μ defines the partition I of the set $\{1, ..., n\}$ into preimages of atoms of μ. The projection $(z_1, ..., z_n) \mapsto supp(\mu)$ also indices a bijection b from I to the set of vertices of P. It is easy to see that the composition $\tau = \psi \circ \epsilon : M_r \to E(r)$ is injective.

To prove that τ is surjective take any marked k-gon (P, I, h). All angles of P are less than π. There exists a Riemann mapping $f : D \to P$. This map extends to a homeomorphism $D \cup \mathbb{S}^1 \to clP$. To define the corresponding measure μ we take the preimage of the set of vertices of P to be the support of μ. If v_j is a vertex of P with the angle α_j, then the mass of μ at the corresponding atom w_j is equal to

$$\mu_j = (\pi - \alpha_j)/2.$$

It follows that μ is a stable measure. Thus, Theorem 4 implies that there exists a Moebius transformation $\gamma \in Conf(D)$ such that the center of mass of $\nu = \gamma_*(\mu)$ is zero. The polygon P_ν is in the same isometry class as $P = P_\mu$ (see Section 17). Therefore $P_\nu \cong P$ belongs to the image of the map τ. Hence the map

$$\tau : M_r \to E(r) \tag{64}$$

is a bijection. The space of marked polygons has a natural topology. A sequence $[P_j, I_j, b_j] \in E(r)$ is convergent to $[P, I, b]$ if the following conditions are satisfied:

(0) The partitions I_j do not change for large j.

(1) For some choice of representatives $(P_j, I_j, b_j) \in [P_j, I_j, b_j]$, $(P, I, b) \in [P, I, b]$ the polygons P_j are Hausdorff convergent to P.

(2) The convergence of P_j to P leads to collision of some of vertices of P_j. This defines a new partition I_∞ of $\{1, ..., n\}$ and a bijection $b_\infty : \{1, ..., n\} \to I_\infty$. We require $I_\infty = I$ and $b_\infty = b$.

It is easy to see that $E(r)$ is compact and the map $\tau : M_r \to E(r)$ is continuous. Therefore τ is a homeomorphism. The space $E_n(r)$ is an open dense subset in $E(r)$. It is the space of (marked) convex n-gons with fixed angles α_j. If all the numbers r_j are different, then the marking of these polygons is given by the map $j \mapsto \alpha_j$.

19. The space of marked polygons with fixed angles and the moduli spaces of Deligne–Mostow, Bavard–Ghys, Kojima–Yamashita and Thurston.

Deligne–Mostow in [6] and Thurston in [15] consider the moduli space $M(\alpha)$ of flat metrics on the sphere $\mathbb{S}^2$ with the fixed angles $2\alpha_j < 2\pi$ around singular points $w_1, ..., w_n$ (see [15]). Equivalently $M(\alpha)$ is the moduli space of configurations of marked n-tuples of points on $\mathbb{C}P(1)$ (see [6]). The space $M(\alpha)$ is an incomplete complex-hyperbolic manifold of the complex dimension $n-3$. This space contains several totally-geodesic real submanifolds H_s of real dimension $n-3$. The points in H_s are doubles of convex polygons P with the fixed angles α_j so that the cyclic ordering of vertices of P is the same as the cyclic ordering σ of $(\alpha_1, ..., \alpha_n)$. It was proven in [14] and [3] that H_s are convex real hyperbolic polyhedra. These papers also describe the geometry of H_s in terms of the collection $(\alpha_1, ..., \alpha_n)$. The polyhedra H_s are compact if and only if $r \notin \Sigma$, where $\alpha_j = \pi(1 - 2r_j)$. The boundary points of H_s correspond to k-gons ($k < n$) obtained by collision of some of vertices of $P \in H_s$. From the point of view of [6] the points of H_s correspond to configurations of n distinct points on $\mathbb{S}^1$ with the fixed cyclic ordering. The map from this space to $\mathbb{H}_{\mathbb{C}}^{n-3}$ can be described as follows. Let $[\overrightarrow{z}]$ be any element in $\mathcal{M}_r$. It corresponds to a configuration of n points in $\mathbb{S}^1$ so that some of these points can coincide. Define the vector

$$(65)\qquad \xi = (\int_0^{z_2} \phi_\mu - \int_0^{z_1} \phi_\mu, ..., \int_0^{z_{n-1}} \phi_\mu - \int_0^{z_{n-2}} \phi_\mu) \in \mathbb{C}^{n-2}.$$

The space $\mathbb{C}^{n-2}$ has a Hermitian form h of signature $(1, n-3)$ whose value at ξ is equal to $1 = Area(P_\mu)^2$. The image of ξ after the projectivization $\mathbb{C}^{n-2} - 0 \to \mathbb{C}P^{n-3}$ belongs to a the complex-hyperbolic space $\mathbb{H}^{n-3}_{\mathbb{C}}$. This is the Deligne-Mostow map

$$(66)\qquad DM : \mathcal{M}_r = M_r \to \mathbb{H}^{n-3}_{\mathbb{C}}.$$

It is easy to see that this map is continuous and projects to $\bar{M}_r = M_r/\sigma$ to an injective map $\mathcal{D}M : \bar{M}_r \to \mathbb{H}^{n-3}_{\mathbb{C}}$. The image of DM is an $(n-3)$–dimensional compact polyhedral submanifold S_r without boundary. S_r splits into the union of real hyperbolic polyhedra H_s which are adjacent along boundary faces. For example, the component H_1 where the cyclic ordering of points $z_1, ..., z_n$ on $\mathbb{S}^1$ is the same as $(1, ..., n)$ corresponds to the set of convex polygons in M_r, which have a fixed orientation.

Thus we have proved the following.

Theorem 4. *Let $r \in int(D_n) - \Sigma$. Then the manifold M_r has a natural tiling by hyperbolic polyhedra. Each tile H_s is a moduli space of marked convex n-gons with fixed interior angles $\pi(1-2r_1), ..., \pi(1-2r_n)$ occuring in some fixed order. Each tile is a convex hyperbolic $(n-3)$-dimensional polyhedron which is an orthoscheme (by [3]). The quotient $\bar{M}_r = M_r/\sigma$ admits an embedding $\mathcal{D}M$ into $\mathbb{H}^{n-3}_{\mathbb{C}}$ such that each tile is isometrically embedded in a totally real totally geodesic subspace. For example the deformation space of the regular pentagon is tiled by 24 regular right angled hyperbolic pentagons inducing a hyperbolic structure on the genus 4 surface.*

The duality τ can be generalized to the case of $r \in \Sigma$. In this case we have to take into account also nice semi-stable measures μ on $\mathbb{S}^1$: those which have exactly two atoms. Corresponding polygons P in M_r are degenerate. If μ is a nice semi-stable measure, which is not stable, then the Abelian differential ω_μ (see (53)) has infinite L^2 norm, and therefore we can not find the normalized differential ϕ_μ. Nevertheless the Deligne-Mostow map DM still makes sense, and the point $DM(P)$ belongs to the ideal boundary of the complex–hyperbolic space. This is an isolated ideal boundary point of $DM(M_r)$ (a "cusp"). For any

$r \in \Sigma$ the corresponding polygons in $E_n(r)$ have parallel sides. The number of pairs of such sides is equal to the number of walls in D_n which contain r. This is the number of cusps of the space $DM(M_r)$.

The above construction can be generalized to cover the whole moduli space $M(\alpha)$. In this case instead of M_r one has to consider the space of polygons in $\mathbb{E}^3$ with fixed side lengths.

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