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FAMILIES OF FLAT MINIMAL TORI IN CP^n

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1. Introduction

A beautiful and quite complete theory has been developed for pseudoholomorphic (also called superminimal or isotropic harmonic) maps from Riemann surfaces into complex projective space (cf. Eells and Wood [7] and Wolfson [15]). All harmonic maps from the Riemann sphere are of this type. Pseudoholomorphic maps are generated from holomorphic maps by a process of taking derivatives. Consequently, methods of algebraic geometry have been very effective in their study. The harmonic sequence of a harmonic map has played an important role in work to establish useful criteria for deciding whether or not a harmonic map is pseudoholomorphic (cf. [15] and Liao [11]).

We know very few examples of harmonic maps which are not pseudoholomorphic. One source of such examples is the family, of dimension $2(n-2)$, of weakly Lagrangian, isometric harmonic maps from the flat complex plane into $\mathbf{C}P^n$. None of these maps is pseudoholomorphic. This family was first described by Kenmotsu [9] and, then Bolton and Woodward [2], all of whom used the term *totally real* instead of *weakly Lagrangian.* We derive a workable criterion for deciding when the image is a torus (that is, when the map is doubly periodic), and we use this to find continuous families of noncongruent flat minimal tori in $\mathbf{C}P^n$ when $n \geq 5$ (as well as in S^{2m+1} when $m \geq 3$).

In Section 3 we state, as our starting point, results of Kenmotsu [8], [9], Bryant [3] and Bolton and Woodward [2]. In Section 4 we analyse the solution space of weakly Lagrangian, isometric harmonic maps $f : \mathbf{C} \to \mathbf{C}P^n$. The analysis is based on the implicit function theorem applied to the most symmetric solution, which we call the

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Clifford solution.

In Section 5 we establish a simple criterion for when such a harmonic map $f : \mathbf{C} \to \mathbf{C}P^n$ is invariant under some lattice $\Lambda \subset \mathbf{C}$; that is, when f descends to a torus $\mathbf{C}/\Lambda$. Surprisingly, the Clifford solution in $\mathbf{C}P^n$ is a torus only when $n = 2, 3$ and 5, but in Theorem A we find an infinite number of distinct tori when $n \geq 3$, while in Thereom B we find continuous families of noncongruent tori when $n \geq 5$. The existence of such families contrasts with rigidity results for the pseudoholomorphic case (cf. Chi [4], Chi and Mo [5] and Chi and Zheng [6]). Also in contrast to the pseudoholomorphic case is the fact that our examples are not, in general, contained in a congruent copy of $\mathbf{R}P^n \subset \mathbf{C}P^n$. However, we do find infinite families, even continuous families if $m \geq 3$, of flat minimal tori in S^{2m+1}.

We exhibit some explicit examples of such families of minimal tori and check that the harmonic sequence is cyclic for some of them, but for most of the examples it is infinite, noncyclic. This answers a question at the end of [15].

In Section 6 we obtain a necessary condition on a torus for there to exist an isometric, weakly Lagrangian harmonic map from it into $\mathbf{C}P^n$ for some n. The condition shows that no such map exists for the generic torus.

2. Harmonic maps from surfaces into $\mathbf{C}P^n$

Consider complex projective space $\mathbf{C}P^n$ with the Fubini-Study metric g, which is the Kähler metric of constant holomorphic sectional curvature equal to four. Let κ denote its Kähler form. The projection mapping

$$\begin{aligned} \mathbf{C}^{n+1} \setminus \{0\} &\to \mathbf{C}P^n \\ z &\mapsto [z] \end{aligned} \tag{2.1}$$

sends a nonzero vector z to the one-dimensional complex subspace $[z]$ containing it. This projection maps the orthogonal complement of the line through z isometrically onto the complex tangent space of $\mathbf{C}P^n$ at $[z]$ (with the metric g).

Let M, ds^2 be a connected metric Riemann surface. Let

$$f : M \to \mathbf{C}P^n \tag{2.2}$$

be a conformal harmonic map, which is equivalent to the assumption that the image of f is a minimal surface in $\mathbf{C}P^n$. For background and details on such maps we refer to [15]. We will assume that all maps (2.2) are full in the sense that the image is not contained in any hyperplane of $\mathbf{C}P^n$.

The map (2.2) is *weakly Lagrangian* if $f^*\kappa = 0$. Kenmotsu in [9] and Bolton and Woodward in [2] call this condition *totally real*. However, Wolfson in [15] used the term *totally real* to mean no complex or anti-complex tangent points. At the referee's suggestion, we use the term *weakly Lagrangian*.

For any simply connected domain $\Omega \subset M$, $f = [v_0]$, where $v_0 : \Omega \to \mathbf{C}^{n+1}$ is smooth and has unit length and is called a unit length lift of f. The harmonic sequence of f is $(f_j)_{j\in\mathbf{Z}}$, where $f = f_0$, and each $f_j : M \to \mathbf{C}P^n$ is a harmonic map. These are defined recursively by

$$f_1 = [v_1], \qquad \text{where} \qquad v_1 = \frac{\partial v_0}{\partial z} - \langle \frac{\partial v_0}{\partial z}, v_0 \rangle v_0,$$

while f_{-1} is defined in the same way with z replaced by $\bar{z}$. This is well defined as long as v_j is not identically zero, in which case the process stops and f is called pseudoholomorphic. In this paper we exclude this case, which means that the harmonic sequence will always extend infinitely in both directions.

Denoting the harmonic sequence by $(f_j)_{j\in\mathbf{Z}}$, we let v_j denote a local unit length lift of f_j to $\mathbf{C}^{n+1}$. By their definition, $v_0 \perp v_1$. It can be shown that f is conformal if and only if also $v_0 \perp v_2$. The harmonic map f is said to have isotropy order at least r if $v_0, \ldots, v_r$ form a unitary set of vectors at each point. Observe that $1 \leq r \leq n$, with $r = n$ if and only if the harmonic sequence is cyclic (cf. [11] for details and additional references).

3. Weakly Lagrangian, flat minimal surfaces

We suppose now that M, ds^2 is a connected, simply connected domain in the complex plane $\mathbf{C}$ with the flat metric $ds^2 = 2dz\, d\bar{z}$. Let

$$f : M \to \mathbf{C}P^n$$

be a full, weakly Lagrangian, isometric, harmonic map.

The following result of [9], with a simpler proof given in [2], is the starting point for this paper.

Theorem 3.1. *There is a* $1:1$ *correspondence between the set of unitary congruence classes of weakly Lagrangian, isometric harmonic maps* $f: M \to \mathbf{C}P^n$ *and the set* $\mathcal{H}(n)/\tau$, *where* $\mathcal{H}(n)$ *is the set of solutions to the* 3 *equations*

$$\begin{aligned} r_0 + r_1 \cdots + r_n &= 1, \\ r_0 + \sum_1^n a_j r_j &= 0, \\ r_0 + \sum_1^n a_j^2 r_j &= 0 \end{aligned} \tag{3.1}$$

in the set

$$U = \{(r_0, \dots r_n; \theta_1, \dots, \theta_n) \in \mathbf{R}^{2n+1} : r_k > 0, k = 0, \dots n, \\ 0 < \theta_1 \cdots < \theta_n < 2\pi\},$$

where $a_j = e^{i\theta_j}$, *for* $j = 1, \dots, n$; *and* τ *is the involution on* $\mathcal{H}(n)$ *defined by* $\theta_j \mapsto 2\pi - \theta_j$, *for* $j = 1, \dots, n$.

The correspondence is given by

$$p = (r_0, \dots, r_n; \theta_1, \dots, \theta_n) \in \mathcal{H}(n)$$

corresponds to $f(z) = [v_0(z)]$, *where*

$$v_0(z) = \begin{pmatrix} e^{z-\bar{z}}\xi^0 \\ e^{a_1 z - \bar{a}_1 \bar{z}}\xi^1 \\ \vdots \\ e^{a_n z - \bar{a}_n \bar{z}}\xi^n \end{pmatrix}, \tag{3.2}$$

and $\xi^k = \sqrt{r_k}$, *for* $k = 0, \dots, n$. *Hence,* f *extends to a weakly Lagrangian, isometric harmonic map* $f: \mathbf{C} \to \mathbf{C}P^n$.

Remark. To assure that the map f has isotropy order at least r, for some $3 \le r \le n$, we must add to (3.1) the equations

$$r_0 + \sum_1^n a_j^k r_j = 0, \qquad k = 3, \dots, r. \tag{3.3}$$

Remark. Let $f = [v_0]$, where v_0 is given by (3.2). Let

$$A = \operatorname{diag}(1, a_1, \dots, a_n). \tag{3.4}$$

Then the harmonic sequence of f is given by $f_j = A^j f$, for any $j \in \mathbf{Z}$.

Remark. Maps $v_0 : \mathbf{C} \to S^{2n+1} \subset \mathbf{C}^{n+1}$ given by (3.2) with the first and third equations of (3.1) satisfied are flat minimal immersions. The second equation of (3.1) is the condition that the map also be horizontal with respect to the Hopf projection $S^{2n+1} \to \mathbf{C}P^n$. Thus, the classification of weakly Lagrangian, flat minimal surfaces in $\mathbf{C}P^n$ is equivalent to the classification of flat, horizontal minimal surfaces in S^{2n+1} (cf. [3 (p. 269)]).

The involution $[z] \mapsto \overline{[z]} = [\bar{z}]$ is an isometry of $\mathbf{C}P^n$ whose fixed point set, necessarily a totally geodesic submanifold, is $\mathbf{R}P^n$. For any Riemann surface M, if $f : M \to \mathbf{C}P^n$ is a weakly Lagrangian, pseudoholomorphic map then there exists a unitary transformation $A \in U(n+1)$ such that $Af(M) \subset \mathbf{R}P^n$ (cf. Bolton et al. [1]). This result contrasts sharply with the nonpseudoholomorphic case.

Definition. A map $f : M \to \mathbf{C}P^n$ (or its image $f(M)$) is said to be *absolutely real* if there exists a unitary matrix $Q \in U(n+1)$ such that $Qf(M) \subset \mathbf{R}P^n$.

The following theorem is in [2], and goes back to [8] and [3].

Theorem 3.2. *Let $f : \mathbf{C} \to \mathbf{C}P^n$ be a weakly Lagrangian, isometric harmonic map corresponding to a solution $p = (r_0, \dots, r_n; \theta_1, \dots, \theta_n) \in \mathcal{H}(n)$, so that $f(z) = [v_0(z)]$, where $v_0(z)$ is given by (3.2). Then, f is absolutely real if and only if $n = 2m+1$ is odd, with $m \geq 1$, and*

$$\begin{aligned} r_{m+j+1} &= r_j \\ \theta_{m+1+j} &= \theta_j + \pi, \quad 0 = \theta_0 < \theta_1 < \cdots < \theta_m < \pi, \end{aligned} \tag{3.5}$$

for $j = 0, 1, \dots, m$.

Equations (3.5) state that the components of the lift (3.2) for f satisfy the equations $\bar{v}_0^{m+1+j} = v_0^j$, for $j = 0, \dots, m$. Consequently, for the unitary matrix

$$P = \begin{pmatrix} I_{m+1} & I_{m+1} \\ -iI_{m+1} & iI_{m+1} \end{pmatrix},$$

the map

$$Pv_0(z) = \begin{pmatrix} \operatorname{Re} e^{a_j z - \bar{a}_j \bar{z}} r_j^2 \\ \operatorname{Im} e^{a_j z - \bar{a}_j \bar{z}} r_j^2, \end{pmatrix}$$

where $a_0 = 1$, is an isometric, minimal immersion of $\mathbf{C}$ into S^n.

For future reference we note that equations (3.5) reduce (3.1) to the two equations

(3.6) $\sum_0^m r_j = \frac{1}{2}, \quad \sum_0^m a_j^2 r_j = 0.$

4. The space of solutions

Proposition 4.1. *For any $n \geq 2$, a solution to (3.1) is given by $r_j = \frac{1}{n+1}$, for $j = 0, \ldots, n$ and $a_k = e^{\frac{2\pi i k}{n+1}}$, for $k = 1, \ldots, n$. We will call this the Clifford solution to (3.1). It defines the unique (up to congruence) weakly Lagrangian isometric harmonic map $f : \mathbf{C} \to \mathbf{C}P^n$ of isotropy order n.*

Proof. Now the $a_k = e^{\frac{2\pi i k}{n+1}}$ are the $(n+1)$-roots of unity, and a_1 is a primitive $(n+1)$-root of unity, so that the equations (3.1) are satisfied as well as equations (3.3) for $k = 3, \ldots, n$ when $r_0 = r_1 = \cdots = r_n = \frac{1}{n+1}$ (cf. p. 115 of van der Waerden [13]).

To prove uniqueness, let $(r_0, \ldots, r_n; \theta_1 < \cdots < \theta_n) \in \mathcal{H}(n)$. Let $f = [v_0]$ be the corresponding weakly Lagrangian, isometric harmonic map of isotropy order n given by (3.2). Let A be given by (3.4). Then, for any $z \in \mathbf{C}$, the vectors $v_0(z), Av_0(z), \ldots, A^n v_0(z)$ form a unitary set in $\mathbf{C}^{n+1}$. But the same is true for the vectors $Av_0(z), \ldots, A^{n+1}v_0(z)$. It follows that $A^{n+1}v_0(z) = tv_0(z)$, for some $t(z) \in \mathbf{C}$, with $|t(z)| = 1$. Therefore $t(z) = 1$ for every z, since the first entry of $v_0(z)$ is never zero. By the fullness of f, there exist points $z_0, \ldots, z_n \in \mathbf{C}$ such that $v_0(z_0), \ldots, v_0(z_n)$ form a basis of $\mathbf{C}^{n+1}$. Thus, $A^{n+1} = I$, and hence each a_j is an $(n+1)$-root of unity.

Proposition 4.2. *For $n \geq 3$, the Clifford solution is a smooth point of $\mathcal{H}(n)$. In detail, let*

$$F : U \to \mathbf{R} \times \mathbf{C}^2 \cong \mathbf{R} \times \mathbf{R}^4,$$

$$(r_0, r_j; \theta_k) \mapsto \begin{pmatrix} \sum_0^n r_j \\ r_0 + \sum_1^n a_j r_j \\ r_0 + \sum_1^n a_j^2 r_j \end{pmatrix}, \tag{4.1}$$

where $a_j = e^{i\theta_j}$. Let $p_0 \in U$ denote the Clifford solution. Then *rank* $DF_{p_0} = 5$. *Thus* $\mathcal{H}(n) = F^{-1}\{\begin{pmatrix}1\\0\\0\end{pmatrix}\}$ *is a smooth* $2(n-2)$-*dimensional submanifold of* U *in a neighborhood of* p_0.

Proof. If $a_j = e^{i\theta_j}$, then $da_j = ia_j d\theta_j$ and $d(a_j^2) = 2ia_j^2 d\theta_j$. Thus

$$DF = \begin{pmatrix} \sum_0^n dr_j \\ dr_0 + \sum_1^n a_j dr_j + \sum_1^n r_j i a_j d\theta_j \\ dr_0 + \sum_1^n a_j^2 dr_j + \sum_1^n r_j 2ia_j^2 d\theta_j \end{pmatrix}. \tag{4.2}$$

If $n = 3$, then at p_0 we have $a_1 = i$, $a_2 = -1$, $a_3 = -i$ and thus

$$DF_{p_0} = \begin{pmatrix} \sum_0^3 dr_j \\ dr_0 + i dr_1 - dr_2 - i dr_3 - \frac{1}{4}d\theta_1 - \frac{i}{4}d\theta_2 + \frac{1}{4}d\theta_3 \\ dr_0 - dr_1 + dr_2 - dr_3 - \frac{i}{2}d\theta_1 + \frac{i}{2}d\theta_2 - \frac{i}{2}d\theta_3 \end{pmatrix},$$

whose rank is 5 because the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} \tag{4.3}$$

is easily seen to have rank 5.

If $n \geq 4$, put $c_j = \cos\theta_j$ and $s_j = \sin\theta_j$ for $j = 1, \ldots, 4$, where $\theta_j = \frac{2\pi j}{n+1}$ for each j. Then DF_{p_0} has rank 5 if the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & c_1 & c_2 & c_3 & c_4 \\ 0 & s_1 & s_2 & s_3 & s_4 \\ 1 & c_1^2 - s_1^2 & c_2^2 - s_2^2 & c_3^2 - s_3^2 & c_4^2 - s_4^2 \\ 0 & 2c_1 s_1 & 2c_2 s_2 & 2c_3 s_3 & 2c_4 s_4 \end{pmatrix} \tag{4.4}$$

has rank 5. Letting $c = c_1$ and $s = s_1$, one can show that

$$\begin{aligned} \det A = -32 s^2 c\,(1+c)(1-c^4)(1+2c) \\ \cdot(12c^5 + 18c^4 + 7c^3 + c^2 - 3c - 2), \end{aligned} \tag{4.5}$$

of which $c = \cos(\frac{2\pi}{n+1})$ is never a zero, for any $n \geq 4$ (cf. Watkins and Zeitlin [14] on the minimal polynomial over $\mathbf{Q}$ of c).

The system (3.1) consists of five real equations linear in $r_0, \ldots, r_n$. By the Implicit Function Theorem, a neighborhood of the Clifford solution in $\mathcal{H}(n)$ can be expressed as a graph with the first five variables given as functions of the remaining $2n-4$ variables.

When $n \geq 4$, this can be done explicitly. Namely, writing out the real and imaginary parts of (3.1) we obtain

$$A \begin{pmatrix} r_0 \\ \vdots \\ r_4 \end{pmatrix} + B \begin{pmatrix} r_5 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{4.6}$$

where $A = A(\theta_1, \ldots, \theta_4)$ is given by (4.4), for arbitrary $\theta_1 < \cdots < \theta_4$, and $B = B(\theta_5, \ldots, \theta_n)$ is the $5 \times (n-4)$ matrix given by (3.1). In Proposition 4.2 we proved that A is nonsingular at the Clifford solution. Whenever A is nonsingular, the solution space of (3.1) is given as the graph

$$\begin{pmatrix} r_0 \\ \vdots \\ r_4 \end{pmatrix} = A^{-1}(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - B \begin{pmatrix} r_5 \\ \vdots \\ r_n \end{pmatrix}). \tag{4.7}$$

When $n = 3$, the equations (3.1) are not linear in the first five variables. However, one has the following 1-parameter family of explicit solutions. The technique works for any $n \geq 3$ to give an explicit m-parameter family of solutions, where $m = [\frac{n}{2}]$.

Proposition 4.3. *Let $n = 3$. A smooth curve in $\mathcal{H}(3)$ is given by $\theta_1 = \theta$, $\theta_3 = -\theta_1$, $\theta_2 = \frac{2\pi 2}{4} = \pi$, $r_3 = r_1$, and*

$$\begin{aligned} r_0 &= \frac{1-2c}{4(1-c)}, \\ r_1 &= \frac{1}{4(1-c^2)}, \\ r_2 &= \frac{1+2c}{4(1+c)}, \end{aligned} \tag{4.8}$$

where $c = \cos\theta$, for any $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$.

Proof. If $\theta_1 = \theta$, $\theta_3 = -\theta$, $\theta_2 = \pi$ and $r_3 = r_1$, then the five equations of (3.1) defining $\mathcal{H}(3)$ reduce to

$$\begin{aligned} r_0 + 2r_1 + r_2 &= 1, \\ r_0 + c2r_1 - r_2 &= 0, \\ r_0 + (2c^2-1)2r_1 + r_2 &= 0, \end{aligned} \tag{4.9}$$

where $c = \cos\theta$. Solving for r_0, r_1 and r_2 in terms of c, we find that the only positive solutions are given by (4.8).

5. Tori

We consider now when the solutions $f : \mathbf{C} \to \mathbf{C}P^n$ found in §4 are actually tori. That is, we look for criteria on the solutions that insure that there exists a lattice $\Lambda \subset \mathbf{C}$ such that $f(z+\lambda) = f(z)$ for every $z \in \mathbf{C}$ and $\lambda \in \Lambda$, for in that case f induces a full, weakly Lagrangian, harmonic map $\tilde{f} : T \to \mathbf{C}P^n$ of the same isotropy order as f, where $T = \mathbf{C}/\Lambda$ is a torus with the flat metric induced from $\mathbf{C}$.

Let $f(z) = [v_0(z)]$, where $v_0(z)$ is given by (3.2) by the solution

$$p = (r_0, \dots, r_n, \theta_1, \dots, \theta_n) \in \mathcal{H}(n).$$

Recall that $\xi^k = \sqrt{r_k}$. For points $z, w \in \mathbf{C}$, we have $f(w) = f(z)$ if and only if $v_0(w) = bv_0(z)$ for some $b \in S^1 \subset \mathbf{C}$, which in terms of the components of v_0 is equivalent to

$$\begin{aligned} e^{w-\bar{w}} &= be^{z-\bar{z}}, \\ e^{a_j w - \bar{a}_j \bar{w}} &= be^{a_j z - \bar{a}_j \bar{z}}, \quad j = 1, \dots, n. \end{aligned} \tag{5.1}$$

Put $z = x+iy$, $w = u+iv$, $a_j = c_j + is_j$ (that is, $c_j = \cos\theta_j$ and $s_j = \sin\theta_j$). Using the first equation to eliminate b from the remaining n equations, we reduce (5.1) to the equivalent system

$$s_j(u-x) + (c_j - 1)(v-y) = m_j\pi, \quad \text{for some } m_j \in \mathbf{Z}, \tag{5.2}$$

j=1, ... ,n. Let

$$\begin{aligned} \Lambda_f = \{z = x+iy \in \mathbf{C} : s_j x + {} & (c_j - 1)y \equiv 0 \\ & \bmod (\pi), j = 1, \dots, n\}, \end{aligned} \tag{5.3}$$

a free abelian subgroup of $\mathbf{C}$. Here, $t \equiv 0 \bmod (\pi)$ means that $t = m\pi$ for some integer m.

We have shown that

$$f(w) = f(z) \Leftrightarrow w - z \in \Lambda_f. \tag{5.4}$$

Proposition 5.1. *$f : \mathbf{C} \to \mathbf{C}P^n$ descends to a torus $T = \mathbf{C}/\Lambda$ if and only if $\operatorname{rank} \Lambda_f = 2$.*

Proof. If $\operatorname{rank} \Lambda_f = 2$, then $\mathbf{C}/\Lambda_f$ is a torus to which f descends, by (5.4). Conversely, if f descends to the torus T, then $f(w) = f(z)$ for $w - z \in \Lambda$, and thus $\Lambda \subset \Lambda_f$ by (5.4), and hence $\operatorname{rank} \Lambda_f = \operatorname{rank} \Lambda = 2$.

Proposition 5.2. *$\operatorname{rank} \Lambda_f = 2$ if and only if*

$$\dim_{\mathbf{Q}} \operatorname{span}_{\mathbf{Q}}\{(s_1, c_1 - 1), \ldots, (s_n, c_n - 1)\} = 2, \tag{5.5}$$

where $\mathbf{Q}$ is the field of rational numbers. Since any two of the vectors in (5.5), say $(s_1, c_1 - 1)$, $(s_2, c_2 - 1)$, are linearly independent over $\mathbf{R}$, it follows that (5.5) holds if and only if

$$(s_j, c_j - 1) \begin{pmatrix} s_1 & c_1 - 1 \\ s_2 & c_2 - 1 \end{pmatrix}^{-1} \in \mathbf{Q}^2, \tag{5.6}$$

for $j = 1, \ldots, n$.

We shall call (5.6) the *Torus Criterion.*

Proof. The first statement is elementary. For the second statement, we see that (5.5) holds if and only if

$$(s_j, c_j - 1) = p_j(s_1, c_1 - 1) + q_j(s_2, c_2 - 1) = (p_j, q_j) \begin{pmatrix} s_1 & c_1 - 1 \\ s_2 & c_2 - 1 \end{pmatrix},$$

for all j, for some $p_j, q_j \in \mathbf{Q}$. Since $0 < \theta_1 < \theta_2 < 2\pi$, it follows that $(s_1, c_1 - 1)$, $(s_2, c_2 - 1)$ are linearly independent over $\mathbf{R}$, since any line through the origin meets the circle $x^2 + (y + 1)^2 = 1$ in at most one other point. Thus, (5.6) follows.

Corollary. *If f is absolutely real, and Λ_f satisfies the Torus Criterion, then the lift of f to S^n is a flat minimal torus which is a $1:1$, $2:1$ or $4:1$ covering of the image of f in $\mathbf{C}P^n$.*

Proof. If $z = x + iy$ and $c_k + is_k = e^{i\theta_k}$ for $k = 0, \ldots, n$, then by Theorem 3.2 we have $n = 2m + 1$, $0 = \theta_0 < \theta_1 < \cdots < \theta_m < \pi$ and

$$v_0(z) = \begin{pmatrix} e^{2i(c_j y + s_j x)} r_j \\ e^{-2i(c_j y + s_j x)} r_j \end{pmatrix},$$

where $0 \le j \le m$. If $\lambda = u + iv \in \Lambda_f$, then $s_j u + (c_j - 1)v = m_j \pi$, for some $m_j \in \mathbf{Z}$, for each j, and therefore

$$v_0(z+\lambda) = \begin{pmatrix} e^{2i(c_j y + s_j x)} e^{2iv} r_j \\ e^{-2i(c_j y + s_j x)} e^{-2iv} r_j \end{pmatrix}.$$

Thus, $f(z+\lambda) = f(z)$ for all $z \in \mathbf{C}$ requires that $e^{2iv} = e^{-2iv}$, so that $v = \frac{\pi k}{2}$ for some $k \in \mathbf{Z}$. Taking for λ two generators of Λ_f over $\mathbf{Z}$, we find the two corresponding integers k_1 and k_2 are either both even, one is odd the other one even, or both odd, which correspond to the cases, respectively, of the lift of f to S^n covering the image of f one, two or four times.

(5.7) **Remark.** If, for a point in $\mathcal{H}(n)$, we have $(c_j, s_j) \in \mathbf{Q}^2$ for $j = 1, \ldots, n$, then Λ_f has rank 2, since (5.6) is satisfied.

Proposition 5.3. *The Clifford solution is a torus (i.e., descends to a torus) if and only if* $n = 2, 3$ *or* 5.

Proof. When $n = 2$ the Clifford solution is the well known Clifford torus in $\mathbf{C}P^2$ (cf. Remark 6.4). We assume that $n \ge 3$.

Let $c_j = \cos \frac{2\pi j}{n+1}$ and $s_j = \sin \frac{2\pi j}{n+1}$, for $j = 1, \ldots, n$. Then $c_2 = 2c_1^2 - 1$ and $s_2 = 2c_1 s_1$, so that

$$K = \begin{pmatrix} s_1 & c_1 - 1 \\ s_2 & c_2 - 1 \end{pmatrix}^{-1} = \frac{1}{2s_1(c_1 - 1)} \begin{pmatrix} 2(c_1^2 - 1) & 1 - c_1 \\ -2s_1 c_1 & s_1 \end{pmatrix}. \tag{5.8}$$

Applying the Torus Criterion (5.6) to $(s_n, c_n - 1) = (-s_1, c_1 - 1)$, since for the Clifford solution $a_n = \bar{a}_1$, we have

$$(s_n, c_n - 1) \begin{pmatrix} s_1 & c_1 - 1 \\ s_2 & c_2 - 1 \end{pmatrix}^{-1} = (-2c_1 - 1, 1),$$

which is in $\mathbf{Q}^2$ if and only if $c_1 \in \mathbf{Q}$.

Now $c = \cos \frac{2\pi}{n+1} \in \mathbf{Q}$ if and only if $n + 1 = 1, 2, 3, 4, 6$. In fact, it is certainly rational for these cases, and to see that it is not rational for any other values of n, we may suppose that $n + 1 \ge 3$ so that $\zeta = e^{\frac{2\pi i}{n+1}}$ is not real. Thus the minimal polynomial of ζ over $\mathbf{R}$ is $x^2 - 2c + 1$. Hence, $c \in \mathbf{Q}$ if and only if the minimal polynomial, $\Phi_{n+1}(x)$, of ζ over $\mathbf{Q}$ is $x^2 - 2c + 1$. But the degree of Φ_{n+1} is known to be $\varphi(n+1)$ (the Euler φ-function), which is easily verified to be 2 if and only if $n + 1 = 3, 4, 6$. (Cf. [13] and [14]).

It remains to verify that the Clifford solution is a torus when $n = 3$ and 5. In fact, the above calculation verifies the Torus Criterion when $n = 3$. As for the $n = 5$ case, we would then have $(s_3, c_3 - 1) = (0, -2)$ and $(s_4, c_4 - 1) = (-s_1, -c_1 - 1)$. Thus, for K given by (5.8), we have

$$(0, -2)K = \frac{1}{c_1 - 1}(2c_1, -1),$$
$$(s_4, c_4 - 1)K = (\frac{c_1 + 1}{c_1 - 1}, -\frac{1}{c_1 - 1}),$$

both of which are in $\mathbf{Q}^2$, since $c_1 \in \mathbf{Q}$, which verifies the Torus Condition when $n = 5$.

As we have seen, the Clifford torus is, up to congruence, the only weakly Lagrangian, isometric harmonic map $f : \mathbf{C} \to \mathbf{C}P^2$. In the next two results we show that there are countably many flat, minimal tori in $\mathbf{C}P^3$, $\mathbf{C}P^4$ and S^5; and that there are continuous families of such tori in $\mathbf{C}P^n$ for $n \geq 5$ and in S^{2m+1} for $m \geq 3$.

Theorem A. *Let $n \geq 3$. There exists a countably infinite number of noncongruent weakly Lagrangian, isometric harmonic maps $f : \mathbf{C} \to \mathbf{C}P^n$ whose image is a torus.*

Proof. For the case $n = 3$, consider the 1-parameter family of solutions given by Proposition 4.3. For any θ for which $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, the corresponding solution has $(s_1, c_1 - 1) = (s, c - 1)$, $(s_2, c_2 - 1) = (0, -2)$ and $(s_3, c_3 - 1) = (-s, c - 1)$, where $s = \sin\theta$ and $c = \cos\theta$. Applying the Torus Criterion, we have

$$(-s, c - 1)\begin{pmatrix} s & c - 1 \\ 0 & -2 \end{pmatrix}^{-1} = (-1, 1 - c),$$

which is in $\mathbf{Q}^2$ if and only if $c \in \mathbf{Q}$. There is a dense set of θ in the interval $(\frac{\pi}{3}, \frac{2\pi}{3})$ for which $c = \cos\theta$ is rational.

For the case $n = 4$, (4.7) becomes

$$\begin{pmatrix} r_0 \\ \vdots \\ r_4 \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.9}$$

where A is given by (4.4). By Proposition 4.2 and the Implicit Function Theorem we know that there exists an $\epsilon > 0$ such that for any choice of $\theta_1, \dots, \theta_4$ for which $|\theta_j - \frac{2\pi j}{5}| < \epsilon$, there is a unique solution of (3.1) in $\mathcal{H}(4)$ given by (5.9). Using a rational parametrization of the circle, we can see that there is a dense set of values in these intervals at which $\cos\theta_j$ and $\sin\theta_j$ are rational for $j = 1, \dots, n$. Consequently, for each of these solutions, the Torus Criterion is satisfied and the solution is a torus.

The case $n \geq 5$ has a similar proof. The result in this case is subsumed in the following theorem.

Remarks. One can show that of the tori in $\mathbf{C}P^3$ constructed in the above proof, the Clifford torus has the least area. It is known (cf. Lawson [10]) that the Clifford torus is the unique flat minimal torus in S^3. However, using Theorem 3.2 one can modify the proof of Theorem A to prove that there exists at least a countably infinite number of noncongruent flat minimal tori in S^{2m+1} for any $m \geq 2$. We give the details here only for the case $m = 2$, as we will show in Corollary B below that there are continuous families of such tori when $m \geq 3$.

Corollary A. *There exists a countably infinite family of noncongruent flat minimal tori in S^5.*

Proof. In this case (3.6) becomes the three equations

$$A \begin{pmatrix} r_0 \\ r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, \tag{5.10}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & c_1^2 - s_1^2 & c_2^2 - s_2^2 \\ 0 & 2c_1 s_1 & 2c_2 s_2 \end{pmatrix},$$

where $c_j + is_j = e^{i\theta_j}$, for $j = 0, 1, 2$. By Proposition 4.1 and the Torus Criterion, we know that for $c_1 + is_1$ and $c_2 + is_2$ rational and sufficiently close to $e^{i\frac{\pi}{3}}$ and $e^{i\frac{2\pi}{3}}$, respectively, on the unit circle, the matrix A will be invertible and (5.10) will give positive solutions for r_0, r_1 and r_2.

By the Corollary to Proposition 5.2, this solution gives an absolutely real flat minimal torus in $\mathbf{C}P^5$ whose lift to S^5 is a flat minimal torus.

Theorem B. *For $n \geq 4$ there is a dense set of points in a neighborhood of the Clifford solution in $\mathcal{H}(n)$ which are tori. Furthermore, there exist continuous families (of $n-4$ parameters) of solutions which are tori in $\mathbf{C}P^n$, for $n > 4$.*

Proof. The proof of the first statement is similar to that of the case $n = 4$ given above. For the second statement, choose $\theta_1, \ldots, \theta_n$, such that each θ_j is sufficiently close to $\frac{2\pi j}{n+1}$ and such that $\cos\theta_j$ and $\sin\theta_j$ are rational, for $j = 1, \ldots, n$. Then each of $r_5, \ldots, r_n$ can vary freely in an open interval about $\frac{1}{n+1}$, and each choice of these parameters determines a solution of (3.1), by (4.7). For all of these solutions, the values of θ_j, $j = 1, \ldots, n$, remain fixed. Thus, for every solution the Torus Criterion is satisfied, since $\cos\theta_j$ and $\sin\theta_j$ are all rational.

Corollary B. *There exist continuous families of noncongruent flat minimal tori in S^{2m+1}, for any $m \geq 3$.*

Proof. Apply the method of the proof of Theorem B to the equations (3.6) in order to obtain a continuous family of absolutely real flat minimal tori in $\mathbf{C}P^{2m+1}$. By the Corollary to Proposition 5.2, each of these tori lifts to a flat minimal torus in S^{2m+1}.

(5.11) **Example.** We illustrate the proof of Theorem B for the case of $\mathbf{C}P^5$. This example shows that the condition of rationality for all $\cos\theta_j$ and $\sin\theta_j$, although sufficient for making all solutions tori, is not necessary. The Clifford solution is a torus in $\mathbf{C}P^5$. Let $\theta_j = \frac{2\pi j}{6}$, for $j = 1, \ldots, 5$. Then the matrices A and B of (4.6) are given by

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & -\frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

We find from (4.7) that for each t satisfying $0 < t < \frac{1}{3}$, a smooth curve in $\mathcal{H}(5)$ is given by $\theta_j = \frac{2\pi j}{6}$, for $j = 1, \ldots, 5$, $r_5 = t$ and

$$\begin{pmatrix} r_0 \\ \vdots \\ r_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} - t \\ t \\ \frac{1}{3} - t \\ t \\ \frac{1}{3} - t \end{pmatrix}.$$

Since the Torus Criterion is satisfied for the Clifford solution in $\mathbf{C}P^5$, it must be satisfied for the solution given by each value of the parameter t, for $0 < t < \frac{1}{3}$. Thus we have a 1-parameter family of flat, weakly

Lagrangian, minimal tori in $\mathbf{C}P^5$, all noncongruent. Since $r_0 \neq r_3$ except when $t = \frac{1}{6}$, it follows from Theorem 3.2 that the Clifford solution is the only member of this continuous family which is absolutely real.

Each of these tori has the same area as that of the Clifford torus in $\mathbf{C}P^5$, namely $\frac{4\pi^2}{\sqrt{3}}$. In fact, for every t the solutions have the same angles θ_j as the Clifford solution, and thus Λ_f is the same for each of these solutions. It is an elementary exercise to check that for the Clifford solution

$$\begin{aligned}\Lambda_f &= \mathrm{Span}_{\mathbf{Z}}\{(\frac{1}{\sqrt{3}}+i)\pi, (\frac{1}{\sqrt{3}}-i)\pi\} \\ &= \{\frac{2\pi m}{\sqrt{3}} + \pi n(\frac{1}{\sqrt{3}}+i) : m, n \in \mathbf{Z}\}.\end{aligned}$$

Note further that Λ_f being the same for each t means that this is a 1-parameter family of noncongruent minimal immersions of the same torus, $\mathbf{C}/\Lambda_f$, into $\mathbf{C}P^5$. Since the metric in $\mathbf{C}$ is $2(dx^2+dy^2)$, the area of $\mathbf{C}/\Lambda_f$ is

$$2|\frac{2\pi}{\sqrt{3}} \wedge (\frac{1}{\sqrt{3}}+i)\pi| = \frac{4\pi^2}{\sqrt{3}}.$$

Another interesting feature of this 1-parameter family of solutions is that except for $t = \frac{1}{6}$ (that is, the Clifford solution, which has isotropy order 5) the isotropy order is 2, but not 3. In fact, if we put $a_j = \cos\theta_j + i\sin\theta_j$, for $j = 1, \ldots, 5$, then the solution has isotropy order at least 3 if and only if

$$\begin{aligned}0 = r_0 + \sum_1^5 a_j^3 r_j &= (\frac{1}{3}-t)(1+a_2^3+a_4^3) + t(a_1^3+a_3^3+a_5^3) \\ &= 3(\frac{1}{3}-t) - 3t = 1-6t.\end{aligned}$$

(5.12) **Example.** We illustrate the proof of Corollary B for the case of S^7. The Clifford solution for $n = 7$ has $\theta_j = \frac{2\pi j}{8}$, for $j = 0, 1, \ldots, 7$. Using the rational parametrization of the circle

$$t \mapsto \frac{1-t^2}{1+t^2} + i\frac{2t}{1+t^2},$$

we choose $t_1 = \frac{2}{5}$, $t_2 = 1$ and $t_3 = \frac{5}{2}$ so that

$$b_1 = \frac{21}{29} + i\frac{20}{29}, \quad b_2 = i, \quad b_3 = \frac{-21}{29} + i\frac{20}{29}$$

are close to $a_1 = \frac{1+i}{\sqrt{2}}$, $a_2 = i$ and $a_3 = \frac{-1+i}{\sqrt{2}}$, respectively. Following the requirements of Theorem 3.2 we take $b_4 = -b_0 = -1$, $b_5 = -b_1$, $b_6 = -b_2$ and $b_7 = -b_3$. Then equations (3.6) become

$$\begin{aligned} &r_0 + r_1 + r_2 + r_3 = \frac{1}{2}, \\ &r_0 + \frac{41}{29^2} r_1 - r_2 + \frac{41}{29^2} r_3 = 0, \\ &\frac{840}{29^2} r_1 - \frac{840}{29^2} r_3 = 0. \end{aligned} \tag{5.13}$$

Taking $r_2 = r$, the solutions of (5.13) are

$$r_3 = r_1 = \frac{841}{800}\left(\frac{1}{4} - r\right), \quad r_0 = \frac{-41 + 1764r}{1600},$$

which are all positive when $\frac{41}{1764} < r < \frac{1}{4}$. By Theorem 3.2 and the Torus Criterion, these solutions give a continuous family of noncongruent flat minimal tori in S^7.

(5.14) **Example.** The harmonic sequence is cylic, of period 6, for each of the weakly Lagrangian flat minimal tori found in Example (5.11). In fact, if $A = \operatorname{diag}(1, a_1, \ldots, a_5)$, where $a_j = e^{\frac{2\pi i j}{6}}$, then the harmonic sequence for any one of these tori, say f, is $(A^k f)_{k \in \mathbf{Z}}$, which is cyclic because $A^6 = I$.

A cyclic harmonic sequence seems to be the exception for weakly Lagrangian flat minimal tori in $\mathbf{C}P^n$. For consider that in many of our examples we have taken $a_j = c_j + i s_j$ to be rational in the sense that $c_j, s_j \in \mathbf{Q}$ in order to insure that the Torus Criterion is satisfied. For such tori the harmonic sequence is cyclic if and only if $A^k = I$ for some positive integer k, where $A = \operatorname{diag}(1, a_1, \ldots, a_n)$, which requires that $a_j^k = 1$ for each j. But this never happens if $c_j s_j \neq 0$, by the following Lemma 5.4. Hence for all of our examples for which the a_j are all rational (except for the case $a_j = e^{\frac{2\pi i j}{4}}, j = 0, \ldots, 3$) the harmonic sequence is infinite, noncyclic.

Lemma 5.4. *Let $\zeta = p + iq \in S^1$ where $p, q \in \mathbf{Q}$. If $\zeta^n = 1$ for some positive integer n, then $\zeta = \pm 1$ or $\zeta = \pm i$.*

Proof. Suppose that $\zeta^n = 1$ and that n is the smallest positive integer for which this is true. Then $\zeta = e^{\frac{2\pi i j}{n}}$ for some positive integer j which is relatively prime to n (by the minimality of n). Thus, there is a positive integer $k < n$ such that $kj \equiv 1 \mod n$. If $\eta = e^{\frac{2\pi i}{n}}$, then $\zeta^k = \eta$, and consequently, $\eta = a + ib$, where $a, b \in \mathbf{Q}$. But we have already shown in the proof of Proposition 5.3 that if the real part of $\eta = e^{\frac{2\pi i}{n}}$ is rational, then n must be one of $1, 2, 3, 4$ or 6. Now n cannot be 3 or 6, because $b \in \mathbf{Q}$, and the other values of n show that $\eta \in \{\pm 1, i\}$. Hence, $\zeta = \eta^j \in \{\pm 1, \pm i\}$.

6. Necessary conditions on a given torus

We derive next a necessary condition on a given torus $T = \mathbf{C}/\Lambda$ that it be realizable as a flat, weakly Lagrangian minimal submanifold of $\mathbf{C}P^n$ for some $n > 2$. It will be evident that the generic torus does not satisfy this condition.

Definition. An *integral line* in $\mathbf{R}P^2$ is a line defined by a linear homogeneous equation with integer coefficients.

As there are only a countable number of such lines in $\mathbf{R}P^2$, a dense set of points does not lie on any integral line.

Proposition 6.1. *Let $T = \mathbf{C}/\Lambda$ and let $\Lambda^* \subset \mathbf{C}$ denote the lattice dual to Λ, that is,*

$$\Lambda^* = \{\alpha \in \mathbf{C} : Re\, \alpha\bar{v} \in \mathbf{Z}\pi, \forall v \in \Lambda\}.$$

Assume that Λ^* is generated over $\mathbf{Z}$ by $\alpha, \beta \in \mathbf{C}$. If there exists a weakly Lagrangian, isometric harmonic map $f : \mathbf{C}/\Lambda \to \mathbf{C}P^n$ for some $n \geq 3$, then the point

$$[p] = \begin{bmatrix} |\alpha|^2 \\ Re\, \alpha\bar{\beta} \\ |\beta|^2 \end{bmatrix} \tag{6.1}$$

lies on an integral line in $\mathbf{R}P^2$.

Proof. Here Re and Im denote the real and imaginary parts, respectively, of a complex number. A weakly Lagrangian, isometric harmonic map $f : \mathbf{C} \to \mathbf{C}P^n$ descends to T, if and only if (after a possible rotation of $\mathbf{C}$) $\Lambda \subset \Lambda_f$ if and only if $\Lambda_f^* \subset \Lambda^*$. Using the notation of this section, we have from $\Lambda_f^* \subset \Lambda^*$ that

$$s_j + i(c_j - 1) = m_j\alpha + n_j\beta,$$

for some $m_j, n_j \in \mathbf{Z}$, for $j = 1, \ldots, n$. Thus,

$$1 = s_j^2 + c_j^2 = |m_j\alpha + n_j\beta + i|^2$$

for each j implies that

$$m_j^2|\alpha|^2 + 2m_jn_j\mathrm{Re}\,\alpha\bar{\beta} + n_j^2|\beta|^2 = -2m_j\mathrm{Im}\,\alpha - 2n_j\mathrm{Im}\,\beta, \tag{6.2}$$

for $j = 1, \ldots, n$. Write the first three equations of (6.2) in matrix form as

$$Qp = -2(\mathrm{Im}\,\alpha)\vec{m} - 2(\mathrm{Im}\,\beta)\vec{n}, \tag{6.3}$$

where

$$\vec{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \qquad \vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix},$$

and

$$Q = \begin{pmatrix} m_1^2 & 2m_1n_1 & n_1^2 \\ m_2^2 & 2m_2n_2 & n_2^2 \\ m_3^2 & 2m_3n_3 & n_3^2 \end{pmatrix}$$

is nonsingular, since it is an homogenized Vandermonde matrix with determinant $2(m_2n_1 - m_1n_2)(m_3n_1 - m_1n_3)(m_2n_3 - m_3n_2)$, which is nonzero since any pair of the vectors $s_j + i(c_j - 1)$ is linearly independent over $\mathbf{R}$.

If we define $\vec{v}$ by the vector cross product, $\vec{v} = \vec{m} \times \vec{n} \in \mathbf{Z}^3$, then

$${}^t\vec{v}Qp = 0$$

is a nontrivial integral equation satisfied by $[p]$.

Example. Let r be any positive irrational number. Let $\alpha = \sqrt{r}$, let $\beta = i$, let Λ^* be the lattice generated by α and β, and let Λ be the dual lattice. Then there is no weakly Lagrangian, isometric harmonic map $f : T = \mathbf{C}/\Lambda \to \mathbf{C}P^n$ for any $n \geq 2$. In fact, in this case the point (6.1) is

$$\begin{bmatrix} r \\ 0 \\ 1 \end{bmatrix},$$

which cannot lie on an integral line in $\mathbf{R}P^2$.

(6.4) **Remarks.** The unique weakly Lagrangian isometric harmonic map $f : \mathbf{C} \to \mathbf{C}P^2$, is given by

$$f(x + iy) = \begin{bmatrix} e^{2iy} \\ e^{i(-y+\sqrt{3}x)} \\ e^{-i(y+\sqrt{3}x)} \end{bmatrix}.$$

In Ludden et al. [12] and [15] the Clifford torus in $\mathbf{C}P^2$ is the name given to the flat, weakly Lagrangian minimal surface given by the Hopf projection of $S^1 \times S^1 \times S^1 \subset S^5 \subset \mathbf{C}^3$. Clearly, $f(\mathbf{C})$ is this surface.

The classical Clifford torus $S^1 \times S^1 \subset S^3$ is given, for our metric $2dzd\bar{z}$ on $\mathbf{C}$, by

$$f(x,y) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos 2x \\ \sin 2x \\ \cos 2y \\ \sin 2y \end{pmatrix}.$$

Our Clifford solution $g : \mathbf{C} \to \mathbf{C}P^3$ is

$$g(x+iy) = \begin{bmatrix} e^{2iy} \\ e^{2ix} \\ e^{-2iy} \\ e^{-2ix} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & i \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 1 & -i & 0 & 0 \end{pmatrix} f(x,y).$$

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