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FLAT FLOW IS MOTION BY CRYSTALLINE CURVATURE FOR CURVES WITH CRYSTALLINE ENERGIES

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Abstract

For interface energies which are crystalline, motion of curves in the plane by crystalline curvature typically coincides with their flat curvature flow.

We study the time evolution of curves $P(t)$ in the plane moving with normal velocities equal to a "weighted mean curvature" derived from an interface energy function Φ which is assumed to be *crystalline* and *even.* Our main result is that if one starts with a *good* polygonal curve $P(0)$, then two different formulations of what such evolution should mean typically coincide. (All terms in italics are defined below.)

The first formulation is that of *motion by crystalline curvature,* in which the $P(t)$'s are computed by integration of a coupled system of ordinary differential equations in the time variable t; each $P(t)$ continues to be a good polygonal curve having no more line segments than $P(0)$ has [10].

The second formulation is that of *flat Φ curvature flow,* in which discrete time approximations are obtained by solving sequences of variational problems in which all possible curves (not just polygonal ones) are in competition; the flow is obtained as a limit of such discrete time approximations [4].

We show below that these two different procedures produce the same curve evolutions, except perhaps in cases in which many edges coalesce into one in the motion by crystalline curvature. A more painstaking

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analysis than the present one would be required in order to understand what happens in such a case; it is conceivable that different flows might result.

Variational ideas have always been at the heart of the study of evolution processes involving crystalline curvature; see, for example [10]. Our present result means that curves which minimize crystalline energy in competition with all conceivable curves will in fact lie in a very restricted class of polyhedral curves. Furthermore, the solution curve at time $t + \Delta t$ typically will be obtained as a slight perturbation of the solution curve at time t, this perturbation leaving normal vectors unchanged.

In the context of smooth elliptic interface energies (instead of the present crystalline ones), flat curvature flows coincide with classical smooth flows so long as the latter exist; this was shown by the authors together with L. Wang in [4]. It seems, therefore, that the variational formulation of curvature driven flows as flat flows is a reasonable theory within which to formulate and analyze motions driven by arbitrary interface energy functions, and that the crystalline methods, at least sometimes, provide a practical method for computing such flows.

The evolutions of curves and surfaces driven by various curvatures have been studied by a number of mathematicians in recent years; the article [13] surveys several different mathematical approaches and gives examples of how and why such evolutions have significance within materials science. Flat Φ curvature flows were first proposed at the 1991 Five Colleges Regional Geometry Institute and were discussed in [3]; the paper [4] contains precise statements and proofs. To the best of our knowledge, formulation of crystalline curvature evolution within the context of coupled ordinary differential equations was first publicly proposed in a paper [9] presented by the second author in a 1988 lecture in Brazil at a meeting in honor of M. do Carmo. The second author also showed a videotape in her 1989 AMS-MAA lecture [8] which demonstrated an early version of her computer program for computing such motion, and gave a mathematical analysis of such motion in her paper [10] and demonstrated a much more elaborate version of the program in [11]. A similar formulation of motion of curves by crystalline curvature was suggested independently in 1989 by S. Angenent and M. Gurtin in [1]. Also in 1989, Roberts [6] investigated a similar motion, purely as a computational approximation to motion by curvature of embedded or immersed closed planar curves (with no proofs about the motion or the nature of the approximation). The motion of a curve by its crystalline curvature is illustrated in Figure 0.

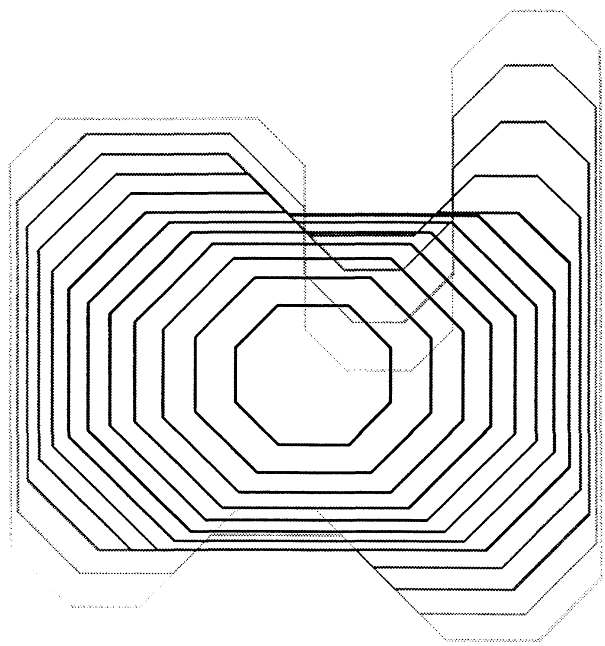

FIGURE 0. This curve evolution was computed using the equations for motion by crystalline curve, with the Wulff shape being a regular octagon. The main theorem of this paper is that the same motion results from flat curvature flow.

The second author has also studied motion of surfaces in space driven by crystalline curvature and set forth a system of differential equations for such motion [9]. In contrast to the case of polyhedral curves in the plane in which the number of line segments can only decrease under crystalline evolution, it often happens for polyhedral surfaces that facets should subdivide. In some situations the amount of subdivision is not bounded, and one is led to formulate such curvature evolution within varifold geometry; see for example the video and text of [12]. The second author conjectures that the crystalline motion of surfaces in space, properly understood, will coincide with flat curvature evolution (as is the case in the plane according to the present paper).

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1. Basic assumptions

1.1. Ambient dimension. We consider motion of curves in the plane.

1.2. Surface energy. We assume that our "surface" energy (here curve energy) integrand $\Phi : \mathbb{R}^2 \to \mathbb{R}^+$ is even and crystalline, that is, that Φ maps non-zero (normal) vectors to positive real numbers and satisfies $\Phi(\lambda v) = \lambda\Phi(v)$ for all real numbers λ and every $v \in \mathbb{R}^2$, and the Wulff shape for Φ

$$\mathcal{W} = \{x : x \bullet v \leq \Phi(v) \ \mathit{for\ every}\ v\}$$

is a centrally symmetric convex set with *polygonal* boundary in the plane, with positive orientation. We denote $\mathcal{W}$'s unit exterior normal vectors by $n_1, n_2, \dots, n_J$ in counter clockwise order, and define $n_{J+1} := n_1$, $n_0 := n_J$.

2. Admissible simple closed curves

Our detailed analysis will be for simple closed polyhedral curves parametrized with coordinates characteristic of the crystalline variational calculus. This analysis can easily be extended to apply to disjoint unions of simple closed polyhedral curves, as stated in 9 below.

2.1. Admissible sequences of normal vectors. By an *admissible sequence of normal vectors* we mean an N-tuple $\nu = (\nu_1, \nu_2, \dots, \nu_N)$

(some N, with the convention $\nu_{N+1} := \nu_1$, $\nu_0 := \nu_N$) such that each ν_i is one of the n_j's and, for each $i = 1, 2, \ldots, N$, if $\nu_i = n_j$ then $\nu_{i+1} \in \{n_{j-1}, n_{j+1}\}$.

2.2. Intersection points, line segments, and associated currents. Associated with each admissible sequence $\nu = (\nu_1, \nu_2, \nu_3, \ldots, \nu_N)$ of normal vectors and each N-tuple $s = (s_1, s_2, s_3, \ldots, s_N)$ (with $s_{N+1} := s_1$, $s_0 := s_N$) of real numbers are the following:

(i) a sequence $p(s) = (p_1(s), p_2(s), p_3(s), \ldots, p_N(s))$ (with again $p_0 := p_N, p_{N+1} := p_1$) of points of intersection in the plane defined by requiring for each i that

$$p_i(s) \bullet \nu_i = s_i, \quad \text{and} \quad p_i(s) \bullet \nu_{i-1} = s_{i-1};$$

(ii) oriented line seqments (integral 1 currents)

$$J_i(s) := [\![p_i(s), p_{i+1}(s)]\!]$$

having lengths

$$\ell_i(s) := |p_{i+1}(s) - p_i(s)|$$

for each i;

(iii) the oriented closed polygonal curve

$$J(s) := \sum_{i=1}^{N} J_i(s).$$

In order to avoid using even more notation that we already do, we use the same symbol for oriented curves, regarded as integral 1-currents, as for the same curves regarded as sets. Thus we will usually say, for example, that $q \in J$ as opposed to $q \in \operatorname{spt} J$. However, for reasons of consistency with [4], we will continue to distinguish between a 2-dimensional region K and its associated positively oriented 2-current $[\![K]\!]$.

2.3. The parameter spaces $\mathcal{S}(\nu)$ and admissible curves. Associated with each admissible sequence $\nu = (\nu_1, \nu_2, \nu_3, \ldots, \nu_N)$ of normal vectors is the open (possibly empty for the moment) subset $\mathcal{S}(\nu)$ of $\mathbb{R}^N$ consisting of those N-tuples $s = (s_1, s_2, s_3, \ldots, s_N \equiv s_0)$ for which the following is true:

(i) $\ell_i \neq 0$ for each i, and $p_{i+1}(s) = p_i(s) + \ell_i(s) * \nu_i$ (the operator $*$ rotates a vector counterclockwise by 90 degrees; for example, $*(0,1) = (-1,0)$);

(ii) $J(s)$ is a simple closed oriented polygonal curve in the plane with multiplicity one, and its interior $K(s)$ (the bounded open set for which the curve is the boundary), when positively oriented to make an integral 2-current $[K]$, satisfies $\partial[K(s)] = J(s)$.

An oriented simple closed curve P is called *admissible* provided there is an admissible sequence ν of normal vectors and an $s \in \mathcal{S}(\nu)$ such that $P = J(s)$. Note that ν and s are unique up to the choice of which line segment is called the first segment.

Condition (i), together with the admissibility of ν, is the requirement that the curve be "good," in the terminology of [10]. Flat Φ curvature flows consider only boundaries of regions, which is why we impose condition (ii) as well. The assumption that the orientation of the curve makes it be the boundary of a bounded region of positive orientation is for simplicity of notation in the following analysis; the results apply equally well without that assumption, as stated in 9 below.

One of the distinctive characteristics of the crystalline variational calculus is parametrization of interfaces in terms of coordinates in spaces $\mathcal{S}(\nu)$.

3. Crystalline curvature, motion by crystalline curvature, the energy $\mathcal{E}$, and flat flows

Suppose ν is an admissible sequence of normal vectors and $s \in \mathcal{S}(\nu)$.

3.1. Surface energy and crystalline curvature. We define

$$\Phi(J(s)) = \sum_i \Phi(\nu_i)\ell_i$$

(see 3.3 for a more general definition).

As shown in [10], a straightforward calculation produces the fact that for each i,

$$\frac{\partial\Phi(J(s))}{\partial s_i} = \sigma_i \Lambda(\nu_i),$$

where

$$\sigma_i = \begin{cases} 1 & \text{if } (\nu_{i-1}, \nu_i, \nu_{i+1}) = (n_{j-1}, n_j, n_{j+1}) \text{ for some } j, \\ -1 & \text{if } (\nu_{i-1}, \nu_i, \nu_{i+1}) = (n_{j+1}, n_j, n_{j-1}) \text{ for some } j, \\ 0 & \text{otherwise ,} \end{cases}$$

and $\Lambda_i(\nu_i)$ is the length of the edge of the Wulff shape $\mathcal{W}$ having normal vector ν_i. Note in particular that given ν, $\Phi(J(s))$ is an affine function of $s \in \mathcal{S}(\nu)$.

A simple calculation also shows

$$\frac{\partial \operatorname{Area}(K(s))}{\partial s_i} = \ell_i(s).$$

Therefore, in analogy with the fact that curvature is the rate of decrease of arc length with respect to area swept out under deformations, the number

$$-\frac{\sigma_i \Lambda(\nu_i)}{\ell_i(s)}$$

is called the *crystalline curvature of $J(s)$ at $J_i(s)$*.

3.2. Motion by crystalline curvature. Suppose ν is an admissible sequence of normal vectors and $s^0 \in \mathcal{S}(\nu)$. By motion by crystalline curvature (for the crystalline even integrand Φ in the plane) starting at the admissible curve $P_0 = J(s^0)$ we mean a function P which assigns to each nonnegative time t an integral 1 current $P(t)$ constructed by the following procedure.

Let $s(t)$ be a solution to the system of ordinary differential equations

$$\frac{ds_i}{dt} = \frac{-\sigma_i \Lambda(\nu_i)}{\ell_i(s)}, \qquad i = 1, \dots, N,$$

with initial condition $s(0) = s^0$, and let $t_1 > 0$ be the supremum of the times t at which all $\ell_i(s(t))$'s are positive. For $0 \leq t < t_1$ we set

$$P(t) = J(s(t)).$$

The second author has analyzed the behavior of $P(t)$ as $t \uparrow t_1$ and has established the following in Proposition 3.1 in [10]:

Let $I \subset \{1, \dots, N\}$ denote the collection of those i's for which $\lim_{t\uparrow t_1} \ell_i(s(t)) = 0$. Then, either $I = \{1, \dots, N\}$, in which case we set $P(t) = 0$ for all $t_1 \leq t < \infty$, or $\sigma_i = 0$ for each i in I. (The intuition behind the proof is that if an edge with σ_i nonzero gets short, then it moves fast, and so avoids getting squeezed to zero length.) The vanishing of a single edge $J_i(s(t))$ (with $\sigma_i = 0$) as $t \uparrow t_1$ results in a "merging" of edge $J_{i-1}(s(t))$ with edge $J_{i+1}(s(t))$ at time $t = t_1$ since the condition $\sigma_i = 0$ implies $\nu_{i-1} = \nu_{i+1}$.

Similarly, if i and $i+2$ belong to I then the edges $J_{i-1}(s(t))$, $J_{i+1}(s(t))$, and $J_{i+3}(s(t))$ all merge to make a single edge at time $t = t_1$, etc.

At time $t = t_1$ one can thus choose a new admissible sequence ν^1 (a subsequence of ν) of normal vectors and a new s^1 in $\mathcal{S}(\nu^1)$ such that $P(t_1) = J(s^1) = \lim_{t\uparrow t_1} P(t)$. One then defines $P(t)$ for times greater than t_1 up until the time that the next edge vanishes by solving the obvious new system of ordinary differential equations. One then defines $P(t)$ for all times in the obvious manner.

It is also part of the analysis of [10] that several different $P(t)$'s so constructed will never meet each other provided the supports of the $P(0)$'s are pairwise disjoint. The analysis of [10] also applies equally well to curves $P(t)$ which are oriented clockwise, i.e.,

$$\partial[K(s)] = -J(s)$$

in 2.3(iii). (Much of [10] applies more generally than considered in this paper. In particular, [10] also considers immersed oriented curves and integrands Φ which are not even).

3.3. The space $\mathcal{K}$, the energy $\mathcal{E}$, and flat Φ curvature flows. We denote by $\mathcal{K}$ the space of integral 2-currents in $\mathbb{R}^2$ of positive orientation associated with bounded Lebesgue-measurable subsets K of $\mathbb{R}^2$ having finite perimeter, and when we write $[K] \in \mathcal{K}$ we mean, in particular, that K is such a set of finite perimeter. Associated with each such K is the rectifiable set ∂K of points p in $\mathbb{R}^2$ at which K has a well defined unit exterior normal vector $n_K(p)$. As currents we can write $\partial[K] = \mathcal{H}^1 \llcorner \partial K \wedge *n_K$ (a rectifiable 1-current is specified by giving a rectifiable set together with a unit tangent vector and a multiplicity at almost every each point in that set; here the set is ∂K, the unit tangent vector is $*n_K$, and the multiplicity is 1).

If $[K] \in \mathcal{K}$, we define

$$\Phi(\partial[K]) := \int_{x\in\partial K} \Phi(n_K(x))\, d\mathcal{H}^1 x;$$

this agrees with our definition in 3.1 if $[K] = [K(s)]$ for some $s \in \mathcal{S}(\nu)$.

Whenever $[K] \in \mathcal{K}$ and $[L] \in \mathcal{K}$ and Δt is a positive number we set

$$\mathcal{E}([K], [L], \Delta t) = \Phi(\partial[L]) + \frac{1}{\Delta t}\int_{p\in K\Delta L} \cdot(p, \partial K)\, d\mathcal{L}^2 p;$$

here

$$K\Delta L = ((K \sim L) \cup (L \sim K)) = K \cap (\mathbb{R}^2 \sim L) \cup (\mathbb{R}^2 \sim K) \cap L$$

denotes the symmetric difference between K and L and $\mathcal{L}^2$ is Lebesgue measure on $\mathbb{R}^2$.

We define *approximate flows* $\partial[K_j(t)]$ for $j = 1, 2, 3, \ldots$ starting at the initial position $\partial[K(0)]$ ($[K(0)] \in \mathcal{K}$) as follows. For each fixed integer j we set $\Delta t = 2^{-j}$ and choose

$$[K_j(\cdot)]\colon \mathbb{R}^+ \to \mathcal{K}$$

by the inductive requirements that

$$[K_j(0)] = [K(0)],$$

and whenever $k = 0, 1, 2, 3, \ldots$, then $[K_j(k\Delta t + \Delta t)]$ is chosen so that

$$\mathcal{E}\Big([K_j(k\Delta t)],\, [K_j(k\Delta t + \Delta t)],\, \Delta t\Big) = \inf_{[L]\in\mathcal{K}} \Big\{\mathcal{E}([K_j(k\Delta t)],\, [L],\, \Delta t)\Big\},$$

and

$$K_j(k\Delta t + \tau) \;= K_j(k\Delta t + \Delta t)$$

for each $0 < \tau \le \Delta t$.

A function $\partial[K(\cdot)]\colon \mathbb{R}^+ \to \partial\mathcal{K}$ is called a flat Φ curvature flow provided

$$\lim_{i\to\infty} M([K(t)] - [K_{j(i)}(t)]) = 0$$

(that is, the area of the symmetric difference of the sets goes to zero) locally uniformly in time t for some approximate flows $\partial[K_j(t)]$ and some subsequence $j(1), j(2), j(3), \ldots$ of $1, 2, 3, \ldots$.

3.4. A first variation equation for $\mathcal{E}$ minimizers and the resulting ordinary differential equation for Φ curvature flow. Suppose ν is an admissible sequence of normal vectors and $s^* \in \mathcal{S}(\nu)$, so that $J(s^*)$ is an admissible polyhedral curve bounding the region $K(s^*)$. Since $\mathcal{S}(\nu)$ is open there will exist $\rho > 0$ such that the ball $\mathbb{B}^N(s^*, \rho)$ lies inside $\mathcal{S}(\nu)$. We will show in 6 that for every small enough Δt there is some $s^L \in \mathcal{S}(\nu)$ so that $[J(s^L)]$ is an $\mathcal{E}$ minimizer, namely,

$$\mathcal{E}\Big([K(s^*)],\, [K(s^L)],\, \Delta t\Big) = \inf_{[H]\in\mathcal{K}} \Big\{\mathcal{E}([K(s^*)],\, [H],\, \Delta t)\Big\}.$$

Since it is a minimizer in the larger class, it is also a minimizer within the smaller class of admissible curves. For each i we conclude that

$$\frac{\partial}{\partial s_i}\mathcal{E}\Big([K(s^*)],[K(s^L)],\Delta t\Big)=0.$$

We compute this partial derivative (in particular, we use the calculation mentioned in 3.1) and obtain that for a suitable C (uniformly bounded for s near s^*),

$$\begin{aligned}-\sigma_i\Lambda(\nu_i) &= \frac{1}{\Delta t} sign(s_i^L - s_i^*)\\ &\quad \cdot \lim_{h\to 0}\frac{1}{h}\int_{x\in K(s^L)\Delta K(s_1^L,\ldots,s_i^L+h,\ldots,s_N^L)} \cdot \operatorname{dist}(x,\, \partial K(s^*))\; d\mathcal{L}^2 x\\ &= \frac{1}{\Delta t} sign(s_i^L - s_i^*)\int_{x\in J_i(s^L)} \operatorname{dist}(x,\, \partial K(s^*))\; d\mathcal{H}^1 x\\ &= \frac{1}{\Delta t}\left[\ell_i(s^*)(s_i^L - s_i^*) \pm C|s^L - s^*|^2\right]\\ &= \ell_i(s^*)\frac{\Delta s_i}{\Delta t}\left(1 \pm C\frac{|\Delta s|}{\ell_i(s^*)}\right),\end{aligned}$$

Which can be rearranged to

$$\frac{\Delta s_i}{\Delta t}\left(1 \pm C\frac{|\Delta s|}{\ell_i(s^*)}\right) = \frac{-\sigma_i\Lambda(\nu_i)}{\ell_i(s^*)}.$$

Suppose then that $\partial[K^*(t)]$ is a flat Φ curvature flow starting at some $\partial[K(s^0)] \in \mathcal{S}(\nu)$. Then a straightforward adaptation of Euler's method for constructing solutions to ordinary differential equations (based on the fact, which we will show in 5.1, that $|\Delta s| \downarrow 0$ as $\Delta t \downarrow 0$ locally uniformly for s^* in $\mathcal{S}(\nu)$) guarantees for short times t that we can write

$$[K^*(t)] = [K(s(t))],$$

where $s(t)$ satisfies the system of ordinary differential equations

$$\frac{ds_i}{dt} = -\frac{\sigma_i\Lambda(\nu_i)}{\ell_i(s)}, \qquad i = 1,\ldots,N,$$

with initial condition $s(0) = s^0$.

Therefore the flat Φ curvature flow will be the same as the motion by crystalline curvature, provided we do indeed prove that given an admissible $J(s^*)$ then for every small enough Δt there is some $s^L \in \mathcal{S}(\nu)$ so that $J(s^L)$ is an $\mathcal{E}$ minimizer.

A review of the construction of flat Φ curvature flows together with the uniformity of the estimates in 5 will establish that

$$[K^*(t)] = [K(s(t))]$$

up until the time at which one of the $\ell_i(s)$'s vanishes. We will show in 7 that the flat Φ curvature flow makes the same merging transformation that motion by crystalline curvature does, at the time when one and only one $\ell_i(s)$ approaches zero, thereby enabling us in general to conclude that the flat Φ curvature flow is the same as the motion by crystalline curvature up until the time the entire curve shrinks to a point (Theorem 8) (beyond that time, both are identically zero).

4. Regular and inverse corners

By a *regular corner* we mean a current of the form

$$[p - \alpha *n_j, p] + [p, p + \beta *n_{j+1}],$$

and by an *inverse corner* we mean a current of the form

$$[p - \beta *n_{j+1}, p] + [p, p + \alpha *n_j],$$

each associated with some point p in $\mathbb{R}^2$, some $j \in \{1, 2, \dots, J\}$ and some nonnegative numbers α and β. (Recall that $*$ rotates a normal vector by 90 degrees to make it a tangent vector.) See Figure 1.

Since each regular corner is a translation of part of a tangent cone to the boundary of the Wulff shape for Φ, and each inverse corner is part of a tangent cone to the boundary of the central inversion of the Wulff shape (which is a negatively oriented region) we infer (as in [7]) that every regular and inverse corner is absolutely Φ minimizing. By this we mean that if T is any integral 1 current such that

$$\partial T = [p + \beta *n_{j+1}] - [p - \alpha *n_j]$$

or

$$\partial T = [p + \alpha *n_j] - [p - \beta *n_{j+1}],$$

then $\Phi(T) \geq \alpha\Phi(n_j) + \beta\Phi(n_{j+1})$, the integral of Φ over the corresponding regular or inverse corner.

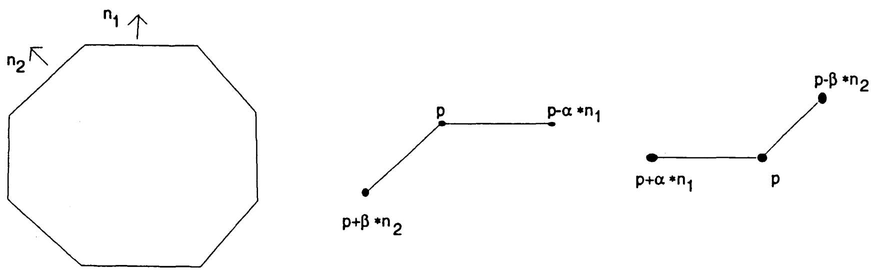

FIGURE 1(A). Sample Wulff shape (b) A regular corner (c) An inverse corner.

The purpose of the α and β is to ensure that the corners, when used as barriers, are long enough to extend into regions where we know that no curve is present, and short enough not to extend into other regions of space where other portions of the curve may again exist.

5. Minimizers near admissible polyhedral curves are oriented simple closed curves

Suppose ν is an admissible sequence of normal vectors, and $s^* \in \mathcal{S}(\nu)$ (so that $J(s^*) \equiv \partial K(s^*)$ is an admissible polyhedral curve). Suppose that for small Δt's, $[L_{\Delta t}]$ is an $\mathcal{E}$ minimizer, namely,

$$\mathcal{E}\Big([K(s^*)],\, [L_{\Delta t}],\, \Delta t\Big) = \inf_{[H]\in\mathcal{K}} \Big\{\mathcal{E}([K(s^*)],\, [H],\, \Delta t)\Big\}.$$

5.1. Closeness of minimizers. The fact that $J(s^*)$ is admissible guarantees the existence of $S > 0$ with the following property: Let W_S be the scaled Wulff set $\{Sx : x \in \mathcal{W}\}$. Suppose $p \in J(s^*)$. Then there will exist q_1 and q_2 such that

$$\{q_1\} + W_S \subset \mathrm{Closure}K(s^*), \quad p \in \{q_1\} + \partial W_S,$$

$$\{q_2\} + W_S \subset \mathrm{Closure}(\mathbb{R}^2 \sim K(s^*)), \quad p \in \{q_2\} + \partial W_S.$$

We assert the existence of constants $C^0 < \infty$ and $\delta > 0$ (depending on $K(s^*)$), with $C^0\delta < \frac{1}{2}S \min_j \Phi(n_j)$, such that if $0 < \Delta t \le \delta$ then

$$(*) \qquad \partial L_{\Delta t} \subset V(s^*, C^0\Delta t) := \left\{x \colon \operatorname{dist}(x, \partial K(s^*)) < C^0\Delta t\right\}.$$

The details of the argument establishing $(*)$ follow closely the proof of conclusions (1) and (2) of Theorem 5.4 of [4] based on the estimates of Proposition 5.3 of [4]. The numbers C^0, δ, and S can be chosen to depend continuously on s^* but are awkward to describe analytically.

5.2. The curve $\partial[L_{\Delta t}]$ is indecomposable if Δt is small. It follows from [5 (4.2.25)] that each $\partial[L_{\Delta t}]$ can be written as the (possibly countably infinite) sum of integral 1-cycles which are indecomposable. Clearly, at least one of these cycles, call it $T_{\Delta t}$, will be homologous, as a current, with $\partial[K(s^*)]$ in $V(s^*, C^0\Delta t)$. One infers from our remarks about corners above that, provided Δt is small, $\Phi(T_{\Delta t})$ will very nearly equal $\Phi(\partial[K(s^*)])$ ($\Phi(T_{\Delta t})$ will not exceed $\Phi(\partial[K(s^*)])$ because $L_{\Delta t}$ is obtained by $\mathcal{E}$ minimization). Hence, $\partial[L_{\Delta t}] - T_{\Delta t}$ is an integral 1-cycle having very small mass (provided Δt is very small). A short argument based on the isoperimetric inequality shows that this 1-cycle is in fact not present ($\partial[L_{\Delta t}] - T_{\Delta t} = 0$), since possible savings in bulk energy cannot match the cost in surface energy (recall that $\partial L_{\Delta t} \subset V(s^*, C_0\Delta t)$). Since $T_{\Delta t}$ is indecomposable, it is in fact an oriented simple closed curve.

6. Proposition. $\mathcal{E}$ minimizers near admissible curves with long sides are admissible curves

Suppose ν is an admissible sequence of normal vectors and $s^* \in \mathcal{S}(\nu)$, so that $J(s^*) = \partial K(s^*)$ is an admissible oriented polyhedral curve. For $\Delta t < \min\{\delta, (8C_0\nu_i \bullet \nu_{i+1})^{-1}\ell_i, (4C_0\nu_i \bullet \nu_{i-1})^{-1}\ell_i\}$ suppose that $T_{\Delta t} = \partial L_{\Delta t}$ is an $\mathcal{E}$ minimizer, namely,

$$\mathcal{E}\Big([K(s^*)], [L_{\Delta t}], \Delta t\Big) = \inf_{[H]\in\mathcal{K}} \Big\{\mathcal{E}([K(s^*)], [H], \Delta t)\Big\}.$$

Then $T = J(s^L)$ for some $s^L \in \mathcal{S}(\nu)$.

Proof. We abbreviate $T = T_{\Delta t}$ and $L = L_{\Delta t}$. The arguments we use in this section are based on the use of corners as barriers. To simplify our terminology we will describe the construction of s^L in the region near J_2 while making specific assumptions on ν and the l_i's; the ideas apply in general. We therefore suppose for purposes of exposition that $n_1 = (2^{-\frac{1}{2}}, 2^{-\frac{1}{2}})$, $n_2 = (0, 1)$, $n_3 = (-2^{-\frac{1}{2}}, 2^{-\frac{1}{2}})$ are normal vectors of consecutive facets of $\partial\mathcal{W}$. We further assume that $C_0\delta \geq 1/8$.

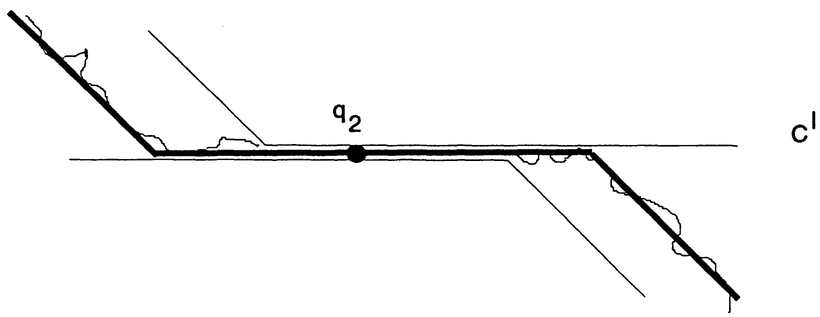

FIGURE 2(A). section 6, first case.

First case (see Figure 2a). Suppose, for example,

$$J_1(s^*) = [(2, -1), (1, 0)], \quad J_2(s^*) = [(1, 0), (-1, 0)],$$
$$J_3(s^*) = [(-1, 0), (-2, 1)],$$

so that the first vertex of $J_2(s^*)$ is regular and the second vertex is inverse. By assumption, the J_i are long compared to $1/8$; this enables us to use our corner barriers. Consider the inverse corner

$$C^I := [(3/2, 0), (-1/2, 0)] + [(-1/2, 0), (-1, 1/2)].$$

Since it is absolutely Φ minimizing, if any part of T inside $(-2, 2) \times (-1, 1)$ were above C^I, then one could replace it with part of C^I without increasing Φ energy while strictly decreasing the bulk energy integral over $K(s^*)L$. Thus C^I is a barrier from above to T. This fact together with an analogous argument from below enables one to conclude that

$$\partial[L] \llcorner (-1/2, 1/2) \times (-1, 1) = [(1/2, 0), (-1/2, 0)].$$

We set $q_2 = (0, 0)$ and $s_2^L = s_2^* = 0$. The general statement is that if one vertex of J_i is regular and the next is inverse, then the part of T

near J_i in fact *is* J_i, with the same s_i^*, except perhaps near the ends of J_i.

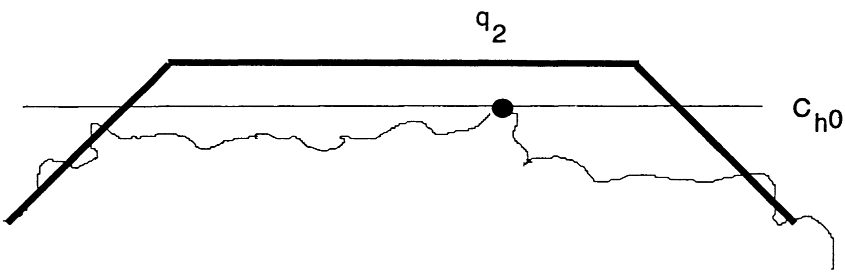

FIGURE 2(B). section 6, second case.

Second case (see Figure 2b). Suppose, for example,

$$J_1(s^*) = [(2,\,-1),\,(1,\,0)],\quad J_2(s^*) = [(1,\,0),\,(-1,\,0)],$$

$$J_3(s^*) = [(-1,\,0),\,(-2,\,-1)],$$

so that both endpoints of $J_2(s^*)$ are regular corners. Suppose also that Δt is small enough and that the geometry of the rest of T is such that

$$T \cap [-2,\,2] \times [-1,\,1] \subset \{x\colon \operatorname{dist}(x,\, J_1(s^*) \cup J_2(s^*) \cup J_3(s^*)) < 1/8\}.$$

(This is not necessary, but since J is a single curve, we can isolate pieces of the curve, and this is equivalent.) The point again is that lengths are long compared to 1/8, so that we can use our corner barriers. For nonpositive numbers h we now consider the corners (straight lines)

$$C_h := [(3/2,\,h),\,(-3/2,\,h)],$$

which are absolutely Φ minimizing. If any part of T inside $(-2,\,2) \times (-1,\,1)$ were above C_h for $h = 0$, then one could replace it with part of C_0 without increasing Φ energy while strictly decreasing the bulk energy integral over $K(s^*)\Delta L$. We conclude that C_0 is a barrier from above to T. We now make h more negative until C_h first makes contact with T, say at $h = h^0$. We denote a first point of contact by q_2. Note that the barriers associated with $J_1(s^*)$ and $J_3(s^*)$ restrict the position of q_2 to lie within the closure of $K(s^*)$ as indicated in Figure 2b. We set $s_2^L = s_2^* + h^0 = h^0$. Similar definitions hold for the case of both endpoints being inverse corners.

Completion of Proof. The key observation is that consecutive q_i's as constructed above must be connected by T and that the unique cheapest way to do this (counting both interface and bulk energies) is by using the regular and inverse corners paralleling $\partial[K(s^*)]$, since interface energy is absolutely minimized by such a connection, and since any other connecting curve respecting the barriers above and of equal or greater interface energy must produce greater bulk energy.

7. The structure of $\mathcal{E}$ minimizers near admissible polyhedral curves with not more than one short side

As in 6 we suppose that ν is an admissible sequence of normal vectors and $s^* \in \mathcal{S}(\nu)$ so that $\partial[K(s^*)]$ is an admissible polyhedral curve. Suppose that for a small $\Delta t < \delta$, $T_{\Delta t} = \partial[L_{\Delta t}]$ is an $\mathcal{E}$ minimizer, namely,

$$\mathcal{E}\Big([K(s^*)], [L_{\Delta t}], \Delta t\Big) = \inf_{[H]\in\mathcal{K}} \Big\{\mathcal{E}([K(s^*)], [H], \Delta t)\Big\}.$$

Again we abbreviate $T = T_{\Delta t}$ and $L = L_{\Delta t}$, with $T = \partial[L]$. The arguments we use in this section are also based on the use of corners as barriers. We need to assume that Δt is sufficiently small compared to the $\ell_i(s^*)$'s with one $\ell_i(s^*)$ excepted and also to the constraints of the global geometry of $J(s^*)$. Assuming that Δt is small enough to justify our arguments (as indicated below), we will analyze the possible structures of T. Because of the results in [10] cited in 3.2 above, we need only consider the situation in which the short side, say $J_3(s^*)$, has as its endpoints one regular and one inverse corner, i.e., $\sigma_3 = 0$; this implies that $\nu_2 = \nu_4$. One possibility is that there will be s^L in $\mathcal{S}(\nu)$ near s^* such that $L = K(s^L)$. The other possibility is that the short edge will disappear and the two adjacent edges will merge into a new single edge. If this happen we make corresponding changes in ν and s.

The three cases we must consider depend on the behavior of the next corners out (i.e., corners between $J_4(s^*)$ and $J_5(s^*)$ and between $J_1(s^*)$ and $J_2(s^*)$). If one is regular and one is inverse, then $\nu_1 = \nu_5$. We suppose that Δt is small enough and that the geometry of the rest of

T is such that

$$T \cap [-2,\ 2] \times [-1,\ 1] \subset \{x\colon \operatorname{dist}(x,\ J_1(s^*) \cup J_2(s^*) \cup J_3(s^*)) < 1/8\}.$$

The point is that the side lengths $\ell_1(s^*)$, $\ell_2(s^*)$, $\ell_4(s^*)$, $\ell_5(s^*)$ are assumed long compared to $1/8$ while the side length $\ell_3(s^*)$ can be short. In our first case, we assume one endpoint of $J_3(s^*)$ is regular and the other is inverse and that $\nu_3 = \nu_1 = \nu_5$. In our second case, we assume one endpoint is regular and the other is inverse but that $\nu_3 \neq \nu_1 = \nu_5$. In our third case, we assume that both endpoints are regular or both points are inverse. It seems useful to describe the ideas in a concrete context rather than introduce general terminology at this point. We therefore suppose for purposes of exposition that $n_1 = (2^{-\frac{1}{2}},\ 2^{-\frac{1}{2}})$, $n_2 = (0,\ 1)$, $n_3 = (-2^{-\frac{1}{2}},\ 2^{-\frac{1}{2}})$ are normal vectors of consecutive facets of $\partial\mathcal{W}$, that $\delta = 1/8$, and that $0 < \epsilon < 1/8$ in what follows.

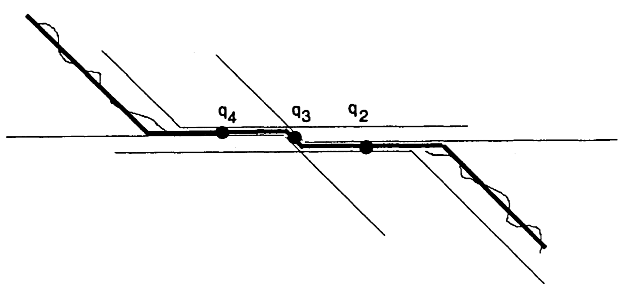

FIGURE 3(A). section 7, first case.

First case (see figure 3a). Suppose

$$J_1(s^*) = [(2,\ -1),\ (1+\epsilon,\ -\epsilon)], \quad J_2(s^*) = [(1+\epsilon,\ -\epsilon),\ (\epsilon,\ -\epsilon)],$$

$$J_3(s^*) = [(\epsilon,\ -\epsilon),\ (-\epsilon,\ \epsilon)], \quad J_4(s^*) = [(-\epsilon,\ \epsilon),\ (-1-\epsilon,\ \epsilon)],$$

$$J_5(s^*) = [(-1-\epsilon,\ \epsilon),\ (-2,\ 1)].$$

The inverse corners

$$[(5/4, \epsilon),\ (-3/4, \epsilon)] + [(-3/4, \epsilon),\ (-1, 1/4+\epsilon)],$$

$$[(5/4, -\epsilon),\ (\epsilon, -\epsilon)] + [(\epsilon, -\epsilon),\ (-1/4+\epsilon, 1/4-\epsilon)]$$

give barriers to T from above while the regular corners

$$[(1,-1/4-\epsilon),\,(3/4,-\epsilon)]+[(3/4,-\epsilon),\,(-5/4,-\epsilon)],$$
$$[(1/4-\epsilon,-1/4+\epsilon),\,(-\epsilon,\epsilon)]+[(-\epsilon,\epsilon),\,(-5/4,\epsilon)]$$

give barriers to T from below. We infer

$$\begin{aligned}&T\llcorner\ (-3/4,\,3/4)\times(-1,\,1)\\&\quad=[(3/4,\,-\epsilon),\,(\epsilon,\,-\epsilon)]+[(\epsilon,\,-\epsilon)\,(-\epsilon,\,\epsilon)]+[(-\epsilon,\,\epsilon),\,(-3/4,\,\epsilon)].\end{aligned}$$

We therefore take $q_2=(1/2,\,-\epsilon)$, $q_3=(0,\,0)$, $q_4=(-1/2,\,\epsilon)$, $s_2^L=s_2^*$, $s_3^L=s_3^*$, $s_4^L=s_4^*$. The specification of the other s_i^L's is done as in section 6 since the other edges are all long. We argue as in section 6 that $T=J(s^L)$.

Second case. Suppose

$$J_1(s^*)=[(2,\,-1),\,(1-\epsilon,\,\epsilon)],\quad J_2(s^*)=[(1-\epsilon,\,\epsilon),\,(\epsilon,\,\epsilon)],$$
$$J_3(s^*)=[(\epsilon,\,\epsilon),\,(-\epsilon,\,-\epsilon)],\quad J_4(s^*)=[(-\epsilon,\,-\epsilon),\,(-1+\epsilon,\,-\epsilon)],$$
$$J_5(s^*)=[(-1+\epsilon,\,-\epsilon),\,(-2,\,1)].$$

For $-1/8<h<1/8$ we consider the inverse and regular corners

$$C_h^I:=[(5/4,h),(-3/4,h)]+[(-3/4,h),(-5/4,1/2+h)],$$
$$C_h^R:=[(5/4,h),(3/4,h)]+[(3/4,h),(-5/4,h)].$$

As in section 6, C_ϵ^I provides a barrier from above for T, and $C_{-\epsilon}^R$ provides a barrier from below. We set $h^0=\inf\{h\colon C_h^I\cap T=\emptyset\}$ and $h_0=\sup\{h\colon C_h^R\cap T=\emptyset\}$, and choose $q_2\in C_{h^0}^I\cap T$, $q_4\in C_{h_0}^R\cap T$. In view of our remarks about barriers above, we infer $\epsilon\geq h^0\geq h_0\geq-\epsilon$ so that we can choose $q_3\in J_3(s^*)\cap C_{h^0}^I$. We now consider two subcases.

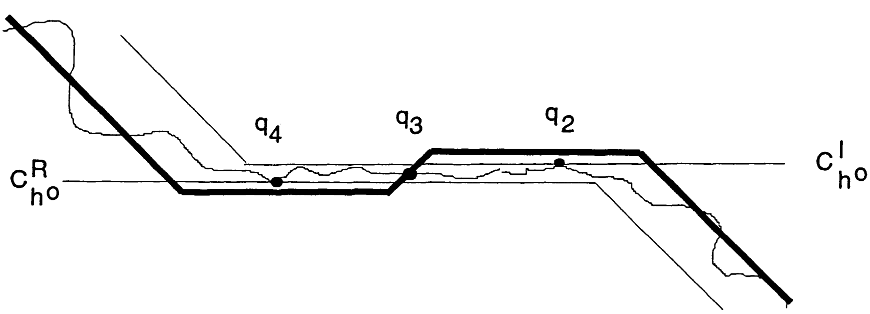

FIGURE 3(B). section 7, first subcase of second case.

First subcase (see Figure 3b). If $h_0 < h^0$, we set $s_4^L = h^0$, $s_3^L = s_3^* = 0$, $s_2^L = h^0$. The specification of the other s_i^L's is done as in section 6. We argue as in section 6 that $T = J(s^L)$.

Second subcase. If $h_0 = h^0$, then we could have chosen q_2 and q_4 so that $q_2 = q_3 = q_4$. We note

$$T \llcorner (-3/4,\ 3/4) \times (-1,\ 1) = [(3/4,\ h^0),\ (-3/4,\ h^0)].$$

The specification of the s_i^L's for $i \notin \{2,\ 3,\ 4\}$ is done as in section 6 since those side lengths are long. We now choose a new index set by setting $\nu^1 = (\nu_1,\ \nu_2,\ \nu_5,\ \dots,\ \nu_N)$ and a new $s^1 = (s_1^0,\ h^0,\ s_5^0,\ \dots,\ s_N^0) \in \mathcal{S}(\nu^1)$. Using $\{q_1, q_3, q_5, \dots\}$ we argue as in section 6 that $T = J(s^1)$.

Third case. Suppose, for small numbers ϵ,

$$\begin{aligned}
&J_1(s^*) = [(2,\ -1),\ (1-\epsilon,\ \epsilon)], \quad J_2(s^*) = [(1+-\epsilon,\ \epsilon),\ (\epsilon,\ \epsilon)],\\
&J_3(s^*) = [(\epsilon,\ \epsilon),\ (-\epsilon,\ -\epsilon)], \quad J_4(s^*) = [(-\epsilon,\ -\epsilon),\ (-1-\epsilon,\ -\epsilon)],\\
&J_5(s^*) = [(-1-\epsilon,\ -\epsilon),\ (-2,\ -1)].
\end{aligned}$$

We consider the inverse corners (actually, corner and line segment)

$$C^I := [(1/4, 1/4), (-\epsilon, -\epsilon)] + [(-\epsilon, -\epsilon) + (-5/4, -\epsilon)],$$

$$C_h^I := [(5/4, h), (-5/4, h)],$$

for $-1/8 \le h \le \epsilon$. Both C^I and C_ϵ^I are barriers to T from above as in section 6. We make the values of h more negative (starting at $h = \epsilon$) until C_h^I makes first contact with T, say at $h = h_0$. We consider two subcases.

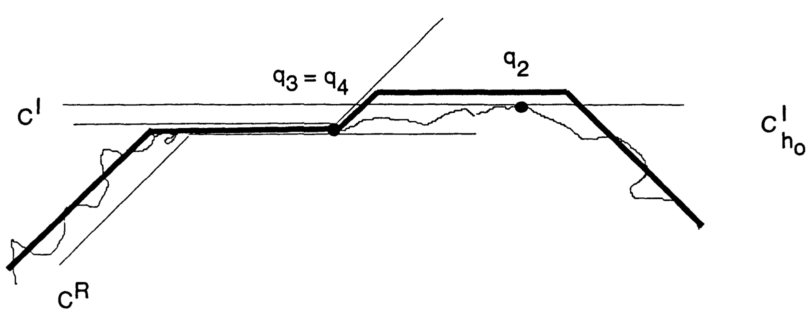

FIGURE 3(C). section 7, first subcase of third case.

First subcase (see Figure 3c). If $h_0 > -\epsilon$, we denote a point of first contact by q_2. We can now restrict our attention to the part of the curve beyond q_2, and employ another barrier

$$C^R := [(1/2, -\epsilon), (-3/4 - \epsilon, -\epsilon)] + [(-3/4 - \epsilon, -\epsilon) + (-5/4, -1/2)].$$

We can thus set $q_3 = q_4 = (-\epsilon,\ -\epsilon)$. We also set $s_2^L = h_0$, $s_3^L = s_3^* = 0$, $s_4^L = s_4^*$. The other q_i's and s_i^L's are determined as in section 6 since those side lengths are long. We argue as in section 6 that $T = J(s^L)$.

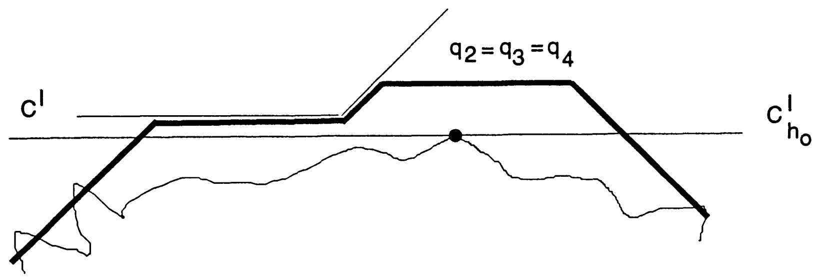

FIGURE 3(D). section 7, second subcase of third case.

Second subcase (see figure 3d). If $h_0 \leq -\epsilon$, then we choose $q_2 = q_3 = q_4$ as a point of contact and argue as in section 6 that

$$\partial[L] \llcorner (-3/4,\ 3/4) \times (-1,\ 1) = [(3/4,\ h_0),\ (-3/4,\ h_0)].$$

The specification of the s_i^L's for $i \notin \{2,\ 3,\ 4\}$ is done as in section 6 since those side lengths are long. We now choose a new index set by setting $\nu^1 = (\nu_1,\ \nu_2,\ \nu_5,\ \dots,\ \nu_N)$ and a new $s^1 = (s_1^0,\ h_0,\ s_5^0,\ \dots,\ s_N^0) \in \mathcal{S}(\nu_1)$. Using $\{q_1, q_2, q_5, \dots\}$ we argue as in section 6 that $T = J(s^1)$.

8. Theorem (Flat crystalline flows starting with admissible curves in the plane remain admissible curves provided edges vanish at distinct times)

Suppose the interface energy function Φ is crystalline and even, and ν is an admissible sequence of normal vectors, and s^0 belongs to $\mathcal{S}(\nu)$. Let $P(t)$ denote the motion by crystalline curvature for Φ with initial condition $P(0) = J(s^0)$. We assume that, prior to the final vanishing of

$P(t)$, at most one of the edges of $P(t)$ vanishes at any particular time. Then $P(t)$ is the unique flat Φ curvature flow beginning with $P(0)$.

Proof. In view of the remarks in 3.4 above and Proposition 6, the main thing to check is that the transitions which occur when a line segment vanishes in the crystalline motion correspond to the behavior of the flat flow. This follows from the estimates given in section 7 and the definitions of flat Φ curvature flows. Details are left to the reader.

9. Theorem (Crystalline even Φ curvature flows of collections of admissible polyhedral curves in the plane)

Suppose the interface energy function Φ is crystalline and even. Let $P_1, P_2, \ldots, P_M$ be admissible curves in the plane whose supports are pairwise disjoint, and suppose $P(0)$ is a 1 current equal to a sum of the P_i's, each with multiplicity plus one or minus one. Denote by $P(t)$ the motion by crystalline curvature with initial condition $P(0)$. We assume that, prior to the final vanishing of each $P_i(t)$, at most one of the edges of $P_i(t)$ vanishes at any particular time. Finally, suppose that $P(0) = \partial[K]$ for some $[K] \in \mathcal{K}$. Then $P(t)$ is the unique flat Φ curvature flow beginning with $P(0)$.

Proof. A change in the orientation of an admissible curve affects the proof only in notation. Also, for an even integrand Φ, polygonal curves moving by crystalline curvature are barriers for each other [10]. Therefore we can apply the results of Theorem 8 in this more general context.

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