J. DIFFERENTIAL GEOMETRY
41 (1995) 449–477

A NEW METHOD OF CONSTRUCTING SCALAR-FLAT KÄHLER SURFACES

JONGSU KIM & MASSIMILIANO PONTECORVO

Abstract

Building on the work of Donaldson-Friedman [6] we present a geometric way of constructing anti-self-dual hermitian metrics on compact complex surfaces, which is based on the relative complex deformations of singular 3-folds with divisors. Some of the consequences are that under a mild condition, fully described by LeBrun-Singer in [14], any blow-up of a scalar-flat Kähler surface admits scalar-flat Kähler metrics; this is used to prove that in any versal deformation of nonminimal ruled surfaces of genus $g \geq 2$, there exists an open dense set of scalar-flat Kähler surfaces. Related results have been obtained by LeBrun-Singer in [14].

1. Introduction

On an oriented Riemannian 4-manifold (M^4, h) the Hodge star operator defines a linear involution on differential 2-forms $\star : \Lambda^2(M) \longrightarrow \Lambda^2(M)$. As a consequence the bundle of 2-forms splits as a direct sum of the ± 1-eigenbundles $\Lambda^2(M) = \Lambda^2_+ \oplus \Lambda^2_-$. We may consider the curvature operator $\mathscr{R}$ of h as an endomorphism of Λ^2 and decompose it according to this splitting. One then defines the metric h to be anti-self-dual if in the induced decomposition of the Weyl tensor $\mathscr{W} = \mathscr{W}_+ + \mathscr{W}_-$, which is the conformally invariant piece of $\mathscr{R}$, we have $\mathscr{W}_+ = 0$ [3]. A 4-manifold admitting such a metric is usually called anti-self-dual. More specifically we will be interested in compact complex surfaces with hermitian metrics which are also anti-self-dual. In the complex case we have another decomposition of the (complexified) 2-forms: $\Lambda^2 = \Lambda^{0,2} \oplus \Lambda^{1,1} \oplus \Lambda^{2,0}$ and the interplay with the splitting defined by $\star$ implies that a Kähler surface is anti-self-dual if and only if the scalar curvature is 0 (we will use the terminology "scalar-flat" from now on) [14].

Received February 16, 1994. The first author was partially supported by non-directed Research Fund, Korea Research Foundation and the Basic Science Research Institute Program, Ministry of Education, Project No. 94-1416.

Such Kähler metrics are of interest because not only are they absolute minima of the conformal energy functional

$$\int_M \|\mathscr{W}\|_h^2 \cdot d\mu_h$$

as any other half-conformally-flat metric, but they also minimize the full curvature energy functional

$$\int_M \|\mathscr{R}\|_h^2 \cdot d\mu_h$$

over all smooth Riemannian metrics on M [14]. They also are obvious examples of extremal Kähler metrics in the sense of Calabi, and it is hoped that their understanding might be useful in this more general context.

Excluding Ricci-flat Kähler metrics for which the existence problem has been solved by Yau [27], compact scalar-flat Kähler surfaces have negative Kodaira dimension by a vanishing theorem of Yau [26] and therefore are either ruled or biholomorphic to $\mathbb{CP}_2$; the latter possibility is however excluded because with the complex orientation, $\mathbb{CP}_2$ has positive signature and therefore cannot admit anti-self-dual metrics by Chern-Weil theory.

As a generalized Calabi problem we are concerned with existence of (non-Ricci-flat) scalar-flat Kähler metrics on compact ruled surfaces of genus g. In [14] LeBrun-Singer have classified such Kähler surfaces with holomorphic vector fields and studied their small deformations. The genus is $g \geq 2$ in this case while no examples are known for $g = 0, 1$.

In this paper we will address the following question: *given a compact scalar-flat Kähler surface M, does the blow-up of M at one point admit metrics of the same kind?*

Although the answer is known to be negative for the simplest case $\mathbb{CP}_1 \times \Sigma_g$, $g \geq 2$, of the product of two Riemann surfaces with constant curvature of opposite sign, LeBrun [9] has shown that the blow-up of this scalar-flat Kähler surface at two well-chosen points admits scalar-flat Kähler metrics.

Blowing up points on a complex surface M is topologically equivalent to performing the connected sum of M with $\overline{\mathbb{CP}}_2$. Therefore our results may also be viewed in connection with a powerful theorem of Taubes asserting that the connected sum of *any* oriented compact 4-manifold with sufficiently many $\overline{\mathbb{CP}}_2$'s admits anti-self-dual metrics [25]. One of the main results in the work of LeBrun-Singer is a complex analogue of the above Taubes' theorem; our techniques also yield a more direct proof of their result; see Corollary 5.11.

Our main source of inspiration is the work of Donaldson-Friedman [6] where they study the existence of anti-self-dual metrics on the connected sum of two given anti-self-dual 4-manifolds. Their idea is to relate the problem to the existence of smooth complex deformations of a singular 3-fold arising from the given twistor spaces. As anti-self-dual hermitian metrics correspond to twistor spaces equipped with a certain divisor, our techniques involve a relative deformation theory of singular 3-folds with a singular divisor.

Before giving a brief outline of our paper we wish to explain these last points. The twistor space Z of $(M^4, h, \textit{orientation})$ is the S^2-bundle of almost complex structures on M, which are compatible with both h and the orientation; Z admits a natural almost complex structure which makes it into a complex 3-manifold exactly when h is anti-self-dual [1]. Twistor theory will play a fundamental role in this work; we will denote by $t : Z \to M$ the bundle projection and notice that a hermitian structure J on M is exactly a section of t. Furthermore if D is the image of this section then the important point is that the pair (Z, D) completely determines the conformal class $[h]$ of the metric as well as the complex structure J.

In §2, following suggestions of LeBrun, we adapt the geometric construction of Donaldson-Friedman to deal with the hermitian case and we construct a complex 3-fold $\mathscr{Z}$ with a divisor $\mathscr{D}$ both having normal crossing singularities. Theorem 2.6 gives us a very precise understanding of the complex structure on the smooth deformations of $\mathscr{D}$. In §3 we apply the theory of Ran [23], [24] on deformations of holomorphic maps to our geometric situation of the pair $(\mathscr{Z}, \mathscr{D})$. In §4 we prove general results about existence of hermitian anti-self-dual metrics on the blow-up of a hermitian anti-self-dual surface. We then consider the Kähler case and the main result of §5 is the following theorem which, in most cases, gives an affirmative answer to our question:

Theorem 5.2. *Let M be a compact scalar-flat Kähler surface with $c_1^{\mathbb{R}}(M) \neq 0$. Then the blow-up of M at any collection of points (distinct or not) and any of its small deformations admit scalar-flat Kähler metrics unless M is the projectivization of a split rank-2 holomorphic vector bundle of zero degree over a Riemann surface of genus $g \geq 2$.*

In fact, thanks to the work of LeBrun-Singer [14], the obstructions that we found in §4 are very often trivial for Kähler surfaces. Among other applications we show that the smooth manifold $(\mathbb{CP}_1 \times \Sigma_g)\#\overline{\mathbb{CP}}_2$ admits scalar-flat Kähler metrics and more generally

Theorem 5.8. *In any versal deformation of nonminimal ruled surfaces of genus $g \geq 2$ there exists an open dense subset of scalar-flat Kähler surfaces.*

2. Geometric construction

Let us start this section by briefly recalling the Donaldson-Friedman construction of what we may call "a singular twistor space $\mathscr{Z}$ over the connected sum" of two anti-self-dual manifolds M_1 and M_2 with twistor spaces Z_1 and Z_2, respectively. After blowing up arbitrary twistor lines l_1 in Z_1 and l_2 in Z_2 we can glue the resulting manifolds $\tilde{Z}_1$ and $\tilde{Z}_2$ along their exceptional divisors Q_1 and Q_2, respectively. This is because the normal bundle $\nu_{l_i/Z_i} \cong \mathscr{O}(1) \oplus \mathscr{O}(1)$ for $i = 1, 2$, and therefore its projectivization Q_i is a smooth quadric hypersurface with normal bundle $\nu_{Q_i/Z_i} \cong \mathscr{O}_{\mathbb{CP}_1 \times \mathbb{CP}_1}(1, -1)$, $i = 1, 2$. $\mathscr{Z}$ is then formed from $\tilde{Z}_1$ and $\tilde{Z}_2$ by identifying Q_1 and Q_2 in such a way that the two $\mathbb{CP}_1$-factors, the base and the fiber, of each quadric are reversed. The end result is a complex 3-dimensional space $\mathscr{Z} = \tilde{Z}_1 \cup_Q \tilde{Z}_2$ with an antiholomorphic involution σ_0 and only mild singularities. More precisely $\mathscr{Z}$ has normal crossing singularities along Q satisfying the d-semistable condition; that is $\nu_{Q/Z} \cong \mathscr{O}_{Q_1}(1, -1) \otimes \mathscr{O}_{Q_2}(-1, 1) = \mathscr{O}_Q$. This is a necessary condition for the complex space $\mathscr{Z}$ to admit "smoothing", and one of the main results of Donaldson-Friedman is the following sufficient condition which they call the unobstructed case:

Theorem 2.1 [6, Theorem 4.1, Corollary 5.1]. *If $H^2(Z_i, \Theta_{Z_i}) = 0$ for $i = 1, 2$, then $\mathscr{Z}$ admits a standard deformation that is a complex analytic family $\varpi : \mathfrak{Z} \to S$ where $\mathfrak{Z}$ is smooth, ϖ is proper, and S is a small open ball in $\mathbb{C}^n$ centered at the origin such that:*

(1) *$\mathfrak{Z}$ has an antiholomorphic involution σ compatible with complex conjugation in S, and let $S \cap \mathrm{Fix}(\sigma) = \{(r_1, r_2, \cdots, r_n) \mid r_i \in \mathbb{R},\ i = 1, \cdots, n\}$.*
(2) *The central fiber $\varpi^{-1}(0)$ is the singular complex space $\mathscr{Z}$ whose antiholomorphic involution σ_0 coincides with the induced one from σ.*
(3) *For any $\vec{t} = (t_1, t_2, \cdots, t_n) \in S \backslash \{t_1 = 0\}$, $\varpi^{-1}(\vec{t})$ is a smooth complex manifold, and for any real vector $\vec{r} = (r_1, r_2, \cdots, r_n)$ with $r_1 \neq 0$ the fiber $\varpi^{-1}(\vec{r})$ is a twistor space of a self-dual metric on the connected sum $M_1 \# M_2$.*

Let us now introduce some notation: throughout the paper Θ_X will denote the sheaf of holomorphic vector fields on the smooth complex manifold X, while $\Theta_{X,Y}$ denotes the sheaf of holomorphic vector fields on X which are tangent to Y, the smooth submanifold, along Y. From the work of Kodaira-Spencer we know that the deformation theory of the complex structure of a compact complex manifold X is governed by the cohomology of the sheaf Θ_X. Infinitesimal deformations are $H^1(\Theta_X)$, and obstructions lie in $H^2(\Theta_X)$. The deformation of the pair (X, Y) where Y is a smooth complex submanifold is described in a similar way by the cohomology of $\Theta_{X,Y}$.

When $\mathscr{Z}$ is a compact singular reduced complex space as above, there is also a theory for the deformations of the complex structure of $\mathscr{Z}$ in terms of the global Ext groups $\mathrm{Ext}^i(\Omega_{\mathscr{Z}}, \mathscr{O}_{\mathscr{Z}}) := T^i_{\mathscr{Z}}$. Similarly to the smooth case, $T^1_{\mathscr{Z}}$ describes infinitesimal deformations of the complex space $\mathscr{Z}$ and obstructions lie in $T^2_{\mathscr{Z}}$. These are usually computed from the Ext sheaves $\mathscr{E}xt^i(\Omega_{\mathscr{Z}}, \mathscr{O}_{\mathscr{Z}}) := \tau^i_{\mathscr{Z}}$ by means of the "local to global" spectral sequence $E_2^{p+q} = H^p(\tau^q_{\mathscr{Z}}) \Rightarrow T^{p,q}_{\mathscr{Z}}$; see [6] for an overview of these facts.

Our main concern will be to use a deformation theory of holomorphic maps on singular complex spaces to prove results about existence of special anti-self-dual metrics on 4-manifolds in the same spirit of [6]. Before starting with our geometric construction let us briefly recall some ideas about the proof of 2.1. For simplicity we will also assume that S can be taken to be an open neighborhood of the origin in $T^1_{\mathscr{Z}}$, and by abuse we will just write $\mathfrak{Z} \to T^1_{\mathscr{Z}}$ for the above family and notice that $\mathfrak{Z}$ is actually a versal deformation of $\mathscr{Z}$ in this case. Then from the local to global spectral sequence above we have an exact sequence which in the case of normal crossings is [6, (5.3)]

$$(2.2) \qquad 0 \longrightarrow H^1(\tau^0_{\mathscr{Z}}) \longrightarrow T^1_{\mathscr{Z}} \longrightarrow H^0(\tau^1_{\mathscr{Z}}) \longrightarrow H^2(\tau^0_{\mathscr{Z}}) \longrightarrow T^2_{\mathscr{Z}} \longrightarrow 0.$$

The hypothesis $H^2(\Theta_{\mathscr{Z}_i}) = 0$ for $i = 1, 2$ then implies that $H^2(\tau^0_{\mathscr{Z}}) = 0$ so that $T^2_{\mathscr{Z}} = 0$, and the deformation theory is unobstructed and modeled on $T^1_{\mathscr{Z}}$. Furthermore $\tau^1_{\mathscr{Z}} \cong \nu_Q \cong \mathscr{O}_Q$, and an infinitesimal deformation gives rise to a smoothing of $\mathscr{Z}$ if and only if its image under the projection $T^1_{\mathscr{Z}} \to H^0(\tau^1_{\mathscr{Z}}) \cong \mathbb{C}$ is nonzero. The "locally trivial" deformations then lie in $H^1(\tau^0_{\mathscr{Z}})$ which was the hyperplane $t_1 = 0$ in the statement of 2.1. Finally a basic feature of twistor spaces is that they have a fixed-point-free antiholomorphic involution, also called "real structure"; then the natural real structure of $\mathscr{Z}$ induced by Z_1 and Z_2 gives rise to antiholomorphic involutions on global and sheaf Ext, and by naturality these are compatible under the maps of (2.2). If furthermore $T^0_{\mathscr{Z}} = 0$, then the deformation over $T^1_{\mathscr{Z}}$ is universal, and the total space is equipped with an antiholomorphic involution; this deformation is then standard and gives rise to twistor spaces over the connected sum of M_1 and M_2.

If $T^0_{\mathscr{Z}} \neq 0$, then instead Donaldson-Friedman construct a standard deformation $\mathfrak{Z} \to S$ and a submersion $e: S \to T^1_{\mathscr{Z}}$ compatible with real structures.

We want to generalize this construction and result to a relative situation where we have a singular pair $(\mathscr{Z}, \mathscr{D})$ consisting of a singular complex hypersurface $\mathscr{D} \subset \mathscr{Z}$. The main ideas here are due to LeBrun and already appeared in [17].

In order to give the geometric construction in this case, we need to introduce the notion of a degree-1 divisor D in a twistor space Z. The (twistor) degree of a divisor D in Z is defined to be the intersection number of D with a generic twistor line l in Z. For a degree-1 divisor there are only two possibilities:

(a) D meets every twistor line at exactly one point.
(b) There exists a unique real twistor line $l \subset D$, and D is $\mathbb{CP}_2$ blown up at n points.

After showing that D must be smooth [12] and that l is a rational curve with normal bundle $\nu_{l/D} \cong \mathscr{O}_{\mathbb{CP}_1}(1)$, this follows immediately from [2, V.4.3]. In particular any two $(+1)$-curves in D must intersect and therefore cannot both be real twistor lines.

Remark 2.3. The existence of a degree-1 divisor gives some very powerful informations on the anti-self-dual manifold (M, h). In fact D exactly represents a complex structure compatible with the metric and orientation, in case (a). In case (b) the twistor map $t: Z \to M$ shows that after reversing the orientation, M is diffeomorphic to the connected sum of n-copies of $\mathbb{CP}_2$, which we indicate by $n\mathbb{CP}_2$ if $n \geq 1$, while M is the sphere S^4 if $n = 0$. We will only consider the case $n \geq 1$. All the known examples of twistor spaces with degree-1 divisors of type (b) are as follows: for $n = 1, 2$ they must be the twistor space of the symmetric metric on $\mathbb{CP}_2$ or of Poon's metrics on $\mathbb{CP}_2 \# \mathbb{CP}_2$ [21], respectively. While for $n \geq 3$ we have explicit examples due to LeBrun [8] and their small deformations. See also [11] and [22] for many of their interesting properties. Examples of twistor spaces with degree-1 divisors of type (a) can be found in [4], [18], [19], [9], [10].

Given this motivation we can now proceed to construct a singular twistor space $\mathscr{Z} = \tilde{Z}_1 \cup_Q \tilde{Z}_2$ together with a singular divisor $\mathscr{D}$, over the connected sum $M_1 \# M_2$.

For this purpose we need to be in the following situation: there is a degree-1 divisor of type (b) D_1 in Z_1 and a degree-1 divisor D_2 in Z_2 of any type. Assuming that this holds we will let $\tilde{Z}_1$ be the blow-up of

Z_1 along the unique real twistor line $l_1 \subset D_1$, and $\tilde{Z}_2$ be the blow-up of Z_2 along any (generic) real twistor line l_2 meeting D_2 transversely (at exactly one point); we can carry out Donaldson-Friedman identification of $\tilde{Z}_1$ and $\tilde{Z}_2$ along the exceptional divisor in such a way, explained below, that the resulting space is the singular twistor space $\mathscr{Z}$ described above, together with a connected singular divisor $\mathscr{D} = \tilde{D}_1 \cup_{\mathbb{CP}_1} \tilde{D}_2$.

Here $\tilde{D}_1$ and $\tilde{D}_2$ denote the proper transform in $\tilde{Z}_1$ and $\tilde{Z}_2$, of D_1 and D_2 respectively. Notice then that $\tilde{D}_1$ is biholomorphic to D_1, while $\tilde{D}_2$ is biholomorphic to D_2 with one point blown up. Let us now focus our attention on the singular locus Q in $\mathscr{Z}$. Each $Q_i \subset \tilde{Z}_i$ is the total space of the projectivized normal bundle $\mathbb{P}(\mathscr{O}(1) \oplus \mathscr{O}(1)) \cong \mathbb{CP}_1 \times \mathbb{CP}_1$ of $l_i \subset Z_i$, for $i = 1, 2$; and we will think of the first $\mathbb{CP}_1$-factor as the base and of the second as the fibers, so that we have the following situation for the two rational curves to be identified: $C_1 := \tilde{D}_1 \cap Q_1$ is the base with normal bundle $\mathscr{O}(1)$ in $\tilde{D}_1$, while $C_2 := \tilde{D}_2 \cap Q_2$ is a fiber whose normal bundle in $\tilde{D}_2$ is $\mathscr{O}(-1)$.

When we carry out the Donaldson-Friedman construction, we will identify Q_1 and Q_2 by switching the factors in such a way that $\tilde{D}_1$ and $\tilde{D}_2$ are glued together along C_1 and C_2 to form a singular hypersurface $\mathscr{D} = \tilde{D}_1 \cup_{\mathbb{CP}_1} \tilde{D}_2$ inside $\mathscr{Z}$. Again $\mathscr{D}$ is a singular complex space of dimension 2 with only normal crossing singularities satisfying the d-semistable condition $\nu_{\mathbb{CP}_1/\mathscr{D}} = \mathscr{O}(1) \otimes \mathscr{O}(-1) = \mathscr{O}$. Then as in the previous theorem of Donaldson-Friedman one can check that $H^2(\Theta_{D_i}) = 0$ for $i = 1, 2$ is a sufficient condition for the existence of smooth deformations of $\mathscr{D}$. Our goal is to find conditions under which the pair $(\mathscr{Z}, \mathscr{D})$ admits simultaneous smoothings.

After some preliminaries we will specialize to study anti-self-dual hermitian structures on the blow-up at one point of a compact anti-self-dual complex surface M with twistor space Z. Topologically this is the 4-manifold $M \# \overline{\mathbb{CP}}_2$. Therefore we will take $M_1 = M$ and $Z_1 = Z$, while $M_2 = \overline{\mathbb{CP}}_2$ is the complex projective space with reversed orientation and Fubini-Study metric so that its twistor space $Z_2 = \mathbb{F}$ is the complete flag of $\mathbb{C}_3$ which can be realized as the hypersurface $\sum_{i=0}^{2} z_i w_i = 0$ in $\mathbb{CP}_2 \times \mathbb{CP}_2^*$, using obvious notation. We will have $\mathscr{Z} = \tilde{Z} \cup_Q \tilde{\mathbb{F}}$ with divisor $\mathscr{D} = \tilde{M} \cup_l \widetilde{\mathbb{CP}}_2$, and any smoothing (Z_t, D_t) of the pair with a real structure will be shown to be a twistor space of an anti-self-dual hermitian metric on the smooth 4-manifold $M \# \overline{\mathbb{CP}}_2$ which is diffeomorphic to D_t and acquires a complex structure from it.

Remark 2.4. The following theorem gives precise informations on the complex structure of D_t by looking at the versal deformation of the singular complex surface $\mathscr{D}$ and will be used several times in §4. By abuse, throughout this paper, we will identify the base space of a deformation with its tangent space; the notation are as in the previous theorem, we just replace $\mathscr{Z}$ by $\mathscr{D}$ and consider the following analogue of (2.2):

$$0 \longrightarrow H^1(\tau^0_{\mathscr{D}}) \longrightarrow T^1_{\mathscr{D}} \longrightarrow H^0(\tau^1_{\mathscr{D}}) \longrightarrow 0. \tag{2.5}$$

The proof of the theorem shows that any versal deformation of $\mathscr{D}$ contains both a versal deformation of $\tilde{M}$ and a 1-parameter trivial deformation of $\tilde{M}$. As pointed out by LeBrun the latter can be seen explicitly by blowing up the total space of the trivial deformation $\tilde{M} \times \mathbb{C}$ along the exceptional rational curve in $\tilde{M} \times \{0\}$ which has normal bundle $\mathscr{O}(-1) \oplus \mathscr{O}$. The result is a smooth 3-fold equipped with a projection to $\mathbb{C}$. The central fiber is $\mathscr{D} = \tilde{M} \cup_{\mathbb{CP}_1} \widetilde{\mathbb{CP}}_2$ and any other fiber is $\tilde{M}$ itself. This deformation must be contained in the deformation over $T^1_{\mathscr{D}}$ by versality, and it also shows that any smooth fiber of the versal family is a small deformation of $\tilde{M}$.

Theorem 2.6. *If $H^2(M, \Theta_M) = 0$, $\mathscr{D}$ admits a versal deformation $\varpi : \mathfrak{M} \longrightarrow T^1_{\mathscr{D}}$ with smooth total space such that if $\mathscr{D}_t$ denotes the fiber $\varpi^{-1}(t)$ then the following hold*:

(1) *$\mathscr{D}_t$ is singular if and only if $t \in H^1(\tau^0_{\mathscr{D}})$.*
(2) *There is a natural identification $H^1(\tau^0_{\mathscr{D}}) \cong H^1(\Theta_{\tilde{M}})$ given by restriction.*
(3) *The induced Kodaira-Spencer map produces a splitting of the sequence* (2.5) *so that $T^1_{\mathscr{D}} \cong H^1(\Theta_{\tilde{M}}) \oplus H^0(\tau^1_{\mathscr{D}})$. Furthermore under this splitting we have*:
(4) *$\mathscr{D}_t$ is biholomorphic to $\tilde{M}$ for any $t = (0, t_2)$, $0 \neq t_2 \in H^0(\tau^1_{\mathscr{D}})$.*
(5) *Any small deformation of $\tilde{M}$ is biholomorphic to $\mathscr{D}_t$ for some $t = (t_1, t_2)$ with $0 \neq t_2 \in H^0(\tau^1_{\mathscr{D}})$.*

Proof. The existence of the deformation and property (1) are just as in Theorem 2.1.

To show (2) consider the normalization $q: \mathscr{D}' = \tilde{D}_1 \amalg \tilde{D}_2 \rightarrow \mathscr{D}$ and inclusion $i: l \rightarrow \mathscr{D}$. Then we have a "normalization sequence" $0 \longrightarrow \tau^0_{\mathscr{D}} \longrightarrow q_*\Theta_{\mathscr{D}', l_1 \cup l_2} \longrightarrow i_*\Theta_l \longrightarrow 0$, and since the restriction map $H^0(\Theta_{\widetilde{\mathbb{CP}}_2, l_2}) \longrightarrow$

$H^0(\Theta_{l_2})$ is surjective, we have a natural identification

$$H^1(\tau^0_{\mathscr{D}}) \cong H^1(\Theta_{\tilde{M}}), \tag{2.7}$$

which means that the infinitesimal deformations of $\mathscr{D} = \tilde{M} \cup_{\mathbb{CP}_1} \widetilde{\mathbb{CP}}_2$ which preserve the form of the singularities all come from the infinitesimal deformations of $\tilde{M}$. This proves part (2), and it may now be useful to give a brief outline of the rest of the proof of the theorem: to produce the required splitting of (2.5) we will blow down the total space of the versal deformation $\varpi : \mathfrak{M} \longrightarrow T^1_{\mathscr{D}}$ to a smooth manifold $\mathfrak{N}$ and get a complex analytic family $\nu : \mathfrak{N} \longrightarrow T^1_{\mathscr{D}}$ in the sense of Kodaira, containing a versal deformation of $\nu^{-1}(0) = \tilde{M}$ (corresponding to the $H^1(\Theta_{\tilde{M}})$-factor) as well as a trivial deformation of $\tilde{M}$ (corresponding to the $H^0(\tau^1_{\mathscr{D}})$-factor).

Let us start by taking any smoothing direction $t \in T^1_{\mathscr{D}} \setminus H^1(\tau^0_{\mathscr{D}})$, and consider the 1-parameter family obtained by restricting the family $\mathfrak{M}$ to this direction, by abuse we will just write it as $\varpi : \mathfrak{M} \longrightarrow \mathbb{C}$. We then have that $\mathfrak{M}$ is a smooth 3-fold, $\varpi^{-1}(0) = \mathscr{D}$ and that $\varpi^{-1}(t) = D_t$ is a smooth complex surface for each $0 \neq t \in \mathbb{C}$.

The total space of the new deformation $\nu : \mathfrak{N} \longrightarrow T^1_{\mathscr{D}}$ will simply be the blow-down of the 3-manifold $\mathfrak{M}$ along the hypersurface $\widetilde{\mathbb{CP}}_2 \subset \mathscr{D} = \varpi^{-1}(0)$; this blowing-down process will contract $\widetilde{\mathbb{CP}}_2$ to $\mathbb{CP}^1$ so that the (singular) central fiber $\varpi^{-1}(0) = \tilde{M} \cup \widetilde{\mathbb{CP}}_2$ is contracted to $\tilde{M} = \nu^{-1}(0)$. To prove that this can be done we have to show that the necessary and sufficient condition of Fujiki-Nakano [7] applies. Namely, the normal bundle $\nu_{\widetilde{\mathbb{CP}}_2/\mathfrak{M}}$ should be the tautological line bundle over the projectivization $\widetilde{\mathbb{CP}}_2 = \mathbb{P}(E)$ of a rank-2 vector bundle $E \longrightarrow \mathbb{CP}_1$. Then $\mathfrak{M}$ can be blown down along $\widetilde{\mathbb{CP}}_2$ to yield a smooth 3-manifold $\mathfrak{N}$ containing a copy of $\mathbb{CP}_1$ with normal bundle E. Of course we know a priori what the normal bundle $E \longrightarrow \mathbb{CP}_1$ should be, namely $E = \mathscr{O} \oplus \mathscr{O}(-1)$ because when $\tilde{M} \cup \widetilde{\mathbb{CP}}_2$ is contracted to $\tilde{M}$ then the image of $\widetilde{\mathbb{CP}}_2$ is the exceptional divisor l in $\tilde{M}$ so that $\nu_{l/\mathfrak{N}} = \mathscr{O} \oplus \mathscr{O}(-1)$ because $\nu_{\tilde{M}/\mathfrak{N}}$ is trivial. Also notice that in fact $\mathbb{P}(\mathscr{O} \oplus \mathscr{O}(-1)) \cong \widetilde{\mathbb{CP}}_2$. To compute the conormal bundle of $\widetilde{\mathbb{CP}}_2 \subset \mathfrak{M}$, we have the sequence

$$0 \longrightarrow \nu^*_{\mathscr{D}/\mathfrak{M}|\widetilde{\mathbb{CP}}_2} \longrightarrow \nu^*_{\widetilde{\mathbb{CP}}_2/\mathfrak{M}} \longrightarrow \nu^*_{\widetilde{\mathbb{CP}}_2/\mathscr{D}} \longrightarrow 0, \tag{2.8}$$

whose exactness can easily be checked from the local model 3.3. On one hand, $\nu^*_{\mathscr{D}/\mathfrak{M}}$ is trivial, and, on the other hand, $\nu^*_{\widetilde{\mathbb{CP}}_2/\mathscr{D}}$ is supported on the intersection $l = \tilde{M} \cup \widetilde{\mathbb{CP}}_2$ and is just the conormal bundle of l in $\tilde{M}$, i.e., $\nu^*_{\widetilde{\mathbb{CP}}_2/\mathscr{D}} \cong \nu^*_{l/\tilde{M}} \cong \mathscr{O}_{\mathbb{CP}_1}(1)$. Therefore the sequence

$$0 \longrightarrow \mathscr{O}_{\widetilde{\mathbb{CP}}_2} \longrightarrow \nu^*_{\widetilde{\mathbb{CP}}_2/\mathfrak{M}} \longrightarrow \mathscr{O}_l(1) \longrightarrow 0$$

is exact showing that $\nu^*_{\widetilde{\mathbb{CP}}_2/\mathfrak{M}}$ is the line bundle associated to the divisor l. Now recall that l has self-intersection $+1$ in $\widetilde{\mathbb{CP}}_2$, and it follows that $\nu_{\widetilde{\mathbb{CP}}_2/\mathfrak{M}}$ restricts to $\mathscr{O}_{\mathbb{CP}_1}(-1)$ on any fiber of the ruling $\widetilde{\mathbb{CP}}_2 \longrightarrow \mathbb{CP}_1$. We have shown that $\mathfrak{M}$ can be blown down to $\mathfrak{N}$ while $\widetilde{\mathbb{CP}}_2$ is contracted to $\mathbb{CP}_1$. Then an easy computation of the tautological line bundle T of a Hirzebruch surface $\mathbb{P}(\mathscr{O}(a) \oplus \mathscr{O}(b))$ shows that if $T = (l)$ and $l^2 = +1$, then $a = -1$ and $b = 0$. Finally by writing the blowing-down process in local coordinates we see that the naturally defined map ν has maximal rank everywhere; so that $\nu\colon \mathfrak{N} \longrightarrow \mathbb{C}$ is a complex analytic family in the sense of Kodaira, i.e., a 1-dimensional deformation of the central fiber $\tilde{M}$. But in fact the same argument applies to the total space of the versal deformation $\varpi : \mathfrak{M} \longrightarrow T^1_{\mathscr{D}}$ and yields a complex analytic family $\nu : \mathfrak{N} \longrightarrow T^1_{\mathscr{D}}$ which is a deformation of $\tilde{M}$ over $T^1_{\mathscr{D}}$.

Now the hypothesis $H^2(\Theta_{\tilde{M}}) = 0$ implies the existence of a versal deformation of $\tilde{M}$ over $H^1(\Theta_{\tilde{M}})$; that is, every other deformation of $\tilde{M}$ is induced from this by means of a linear map called the Kodaira-Spencer map. In our case $ks : T^1_{\mathscr{D}} \to H^1(\Theta_{\tilde{M}})$ produces a splitting of (2.5) because the isomorphism $H^0(\tau^1_{\mathscr{D}}) \cong H^1(\Theta_{\tilde{M}})$ was induced by restriction, and our deformation $\nu\colon \mathfrak{N} \longrightarrow T^1_{\mathscr{D}}$ also comes from restricting the deformation $\varpi : \mathfrak{M} \longrightarrow T^1_{\mathscr{D}}$ to $\tilde{M} \subset \mathscr{D}$, by means of the above blowing-down process. This concludes the proof of (3).

To finish the proof of the theorem just notice that the two deformations $\varpi : \mathfrak{M} \longrightarrow T^1_{\mathscr{D}}$ and $\nu : \mathfrak{N} \longrightarrow T^1_{\mathscr{D}}$ coincide outside of $H^1(\tau^0_{\mathscr{D}})$, so that it is enough to look at $\nu : \mathfrak{N} \longrightarrow T^1_{\mathscr{D}}$ and notice that under the above splitting the kernel of $ks : T^1_{\mathscr{D}} \to H^1(\Theta_{\tilde{M}}) \cong H^1(\tau^0_{\mathscr{D}})$ is identified with $H^0(\tau^1_{\mathscr{D}})$. As a result the complex structure of $\nu^{-1}(t)$ coincides with that of $\tilde{M}$ for any $t = (0, t_2)$, while the restriction of ν to any hyperplane of the form $H^1(\Theta_{\tilde{M}}) \times \{t_2\}$ is a versal deformation of $\tilde{M}$. q.e.d.

Before we begin the discussion on relative deformations we have to make an important observation. In this section we discuss some geometric aspects of relative deformations of a singular twistor space $\mathscr{Z}$, and for this purpose it is more convenient for the reader to describe the construction of a single degree-1 divisor $\mathcal{D} \subset \mathcal{Z}$. In the next section we will consider the existence of deformations of the singular pair $(\mathcal{Z}, \mathcal{D})$ into a smooth twistor space Z with a degree-1 divisor in it. However, since twistor spaces need to possess real structures, we now have to modify the previous setting and consider instead the pair $(\mathcal{Z}, \mathcal{D}\bar{\mathcal{D}})$ where $\bar{\mathcal{D}}$ is the image of $\mathcal{D}$ under the real structure σ_0 of $\mathcal{Z}$. The point is that now the pair $(\mathcal{Z}, \mathcal{D}\bar{\mathcal{D}})$ does have a real structure so that we can consider its real deformations.

3. Relative singular deformations

In this section we are going to apply a deformation theory of holomorphic maps developed by Ran in [23] to the geometric situation of §2. We consider the deformations of a holomorphic imbedding $f : \mathcal{D}\bar{\mathcal{D}} \hookrightarrow \mathcal{Z}$, where $\mathcal{Z}$ is the 3-fold with normal crossing singularities described before and with reducible divisor $\mathcal{D}\bar{\mathcal{D}}$, again with normal crossings. Following the notation of Ran [24], we have an exact sequence of groups:

$$
\begin{aligned}
&0 \longrightarrow T_f^0 \longrightarrow T^0_{\mathcal{D}\bar{\mathcal{D}}} \oplus T^0_{\mathcal{Z}} \longrightarrow \operatorname{Ext}^0_f(\Omega^1_{\mathcal{Z}}, \mathcal{O}_{\mathcal{D}\bar{\mathcal{D}}}) \longrightarrow \\
(3.1) \qquad &\longrightarrow T_f^1 \xrightarrow{\phi\prime\oplus\phi} T^1_{\mathcal{D}\bar{\mathcal{D}}} \oplus T^1_{\mathcal{Z}} \xrightarrow{\psi\prime-\psi} \operatorname{Ext}^1_f(\Omega^1_{\mathcal{Z}}, \mathcal{O}_{\mathcal{D}\bar{\mathcal{D}}}) \longrightarrow \\
&\longrightarrow T_f^2 \longrightarrow T^2_{\mathcal{D}\bar{\mathcal{D}}} \oplus T^2_{\mathcal{Z}} \longrightarrow \cdots
\end{aligned}
$$

where $T^p_{\mathcal{Z}} = \operatorname{Ext}^p(\Omega^1_{\mathcal{Z}}, \mathcal{O}_{\mathcal{Z}})$ and similarly for $\mathcal{D}\bar{\mathcal{D}}$; while in general, for a morphism of ringed spaces $f : X \longrightarrow Y$, an $\mathcal{O}_X$-module A and an $\mathcal{O}_Y$-module B, $\operatorname{Ext}^p_f(B, A)$ will denote the derived functor of $\operatorname{Hom}_f(B, A) :=$ $\operatorname{Hom}_{\mathcal{O}_X}(f^*B, A) = \operatorname{Hom}_{\mathcal{O}_Y}(B, f_*A)$ in either variable. T^i_f is also defined as a derived functor in a natural way [23], and finally $\psi\prime$ is induced by the imbedding map $f : \mathcal{D}\bar{\mathcal{D}} \hookrightarrow \mathcal{Z}$ and ψ by restriction from $\mathcal{Z}$ to $\mathcal{D}\bar{\mathcal{D}}$. We start with the following result of Ran:

Proposition 3.2 [23, Proposition 3.1]. *Infinitesimal deformations of the singular pair $f : \mathcal{D}\bar{\mathcal{D}} \hookrightarrow \mathcal{Z}$ are given by T^1_f with obstructions lying in T^2_f.*

Before starting with a few computations, we give the local description of the pair $(\mathcal{Z}, \mathcal{D}\bar{\mathcal{D}})$ near the singular locus.

Remark 3.3. We can take coordinates w_1, w_2, w_3, w_4 in $\mathbb{C}^4$ so that the local model of the normal crossing singularity of $\mathcal{Z} = \tilde{Z}_1 \cup_Q \tilde{Z}_2$ is given by $\mathcal{Z} = \{w_1 w_2 = 0\}$ with the smooth threefold $\tilde{Z}_i$ corresponding to $w_i =$

0, $i = 1, 2$. They intersect along the smooth surface $Q = \{w_1 = w_2 = 0\}$, and the divisor $\mathscr{D}$ is given by the equation $w_4 = 0$, and it has again a normal crossing singularity $\mathscr{D} = \tilde{D}_1 \cup_l \tilde{D}_2$ where $\tilde{D}_i = \{w_4 = w_i = 0\}$, $i = 1, 2$. A similar description holds for $\bar{\mathscr{D}}$.

When the divisors $\mathscr{D}$ and $\bar{\mathscr{D}}$ are disjoint, the singular locus of the pair $(\mathscr{Z}, \mathscr{D}\bar{\mathscr{D}})$ is contained in Q, and "smoothings" of the pair correspond to anti-self-dual metrics.

On the other hand, if $\mathscr{D} \cap \bar{\mathscr{D}} \neq \varnothing$, then their intersection is a $\mathbb{CP}_1$ lying in the smooth part of $\mathscr{Z}$ and the singularity (of $\mathscr{D}\bar{\mathscr{D}}$) there can be described locally by $\mathscr{D}\bar{\mathscr{D}} = \{z_1 z_2 = 0\}$ in $\mathbb{C}^3$.

Remark 3.4. As a useful exercise we can now give an explicit description of $\Omega^1_{\mathscr{Z}}$ and its dual $\tau^0_{\mathscr{Z}} := \mathscr{H}\mathrm{om}(\Omega^1_{\mathscr{Z}}, \mathscr{O}_{\mathscr{Z}})$ in terms of local sections in a neighborhood of the singularity locus. Recall that for a complex space $\mathscr{Z}$ the sheaf of Kähler differentials $\Omega^1_{\mathscr{Z}}$ is defined by imbedding $\mathscr{Z}$ in a smooth ambient space $\tilde{\mathscr{Z}}$ with ideal sheaf $\mathscr{I}_{\mathscr{Z}}$ and then taking the cokernel of the natural map $d : \mathscr{I}_{\mathscr{Z}}/\mathscr{I}^2_{\mathscr{Z}} \to \Omega^1_{\tilde{\mathscr{Z}}_{|\mathscr{Z}}}$. If we do this for our local model $\mathscr{Z} = \{w_1 w_2 = 0\}$ in $\mathbb{C}^4$, we have that local sections of $\Omega^1_{\mathscr{Z}}$ are equivalence classes $\sum_{i=1}^4 f_i\, dw_i$ with $f_i \in \mathscr{O}_{\mathscr{Z}} = \mathscr{O}_{\mathbb{C}^4}/\mathscr{I}_{\mathscr{Z}}$ modulo the relation $0 = d(w_1 w_2) = \omega_2\, dw_1 + w_1\, dw_2$. Let us now consider the sheaf of derivations $\tau^0_{\mathscr{Z}} := \mathscr{H}om(\Omega^1_{\mathscr{Z}}, \mathscr{O}_{\mathscr{Z}})$. First notice that a holomorphic function on $\mathscr{Z}$ is just a pair of holomorphic functions $(f, g) \in \mathscr{O}_{\tilde{Z}_1} \oplus \mathscr{O}_{\tilde{Z}_2}$ with $f = f(w_2, w_3, w_4)$ and $g = g(w_1, w_3, w_4)$ which agree on the intersection Q: $f(0, w_3, w_4) = g(0, w_3, w_4)$. This can be seen by using power series for example. It is now easy to see that the local sections of $\tau^0_{\mathscr{Z}}$ are vector fields on $\mathbb{C}^4$, $\sum_{i=1}^2 f_i \frac{\partial}{\partial w_i}$ with $f_i \in \mathscr{O}_{\mathscr{Z}}$ satisfying the condition that $f_1 = 0$ on $w_1 = 0$ and $f_2 = 0$ on $w_2 = 0$ because these are the homomorphisms from $\Omega^1_{\mathscr{Z}}/\langle d(w_1 w_2)\rangle$ to $\mathscr{O}_{\mathbb{C}^4}/\langle w_1 w_2\rangle$. Notice also that since $\mathscr{Z}$ is the union of the two hyperplanes $w_1 = 0$ and $w_2 = 0$, a function f_1 which vanishes on $w_1 = 0$ is just a function on the hyperplane $w_2 = 0$ and so can be written as $f_1 = w_1 \cdot g_1(w_1, w_3, w_4)$. Similarly $f_2 = w_2 \cdot g_2(w_2, w_3, w_4)$. With this description at hand it is now straightforward to check exactness of the "normalization" sequence in [6, (5.4)].

$$(3.5) \qquad 0 \longrightarrow \tau^0_{\mathscr{Z}} \longrightarrow q_* \Theta_{\mathscr{Z}', Q_1 \cup Q_2} \longrightarrow i_* \Theta_Q \longrightarrow 0,$$

where $q : \mathscr{Z}' = \tilde{Z}_1 \amalg \tilde{Z}_2 \to \mathscr{Z}$ is the normalization map, and $i : Q \to \mathscr{Z}$ the inclusion. The details are as follows: the first map sends the section

$\sum_{i=1}^{2} f_i \frac{\partial}{\partial w_i}$ to the pair

$$\left(w_2 \cdot g_2(w_2, w_3, w_4)\frac{\partial}{\partial w_2} + f_3(w_1, 0, w_3, w_4)\frac{\partial}{\partial w_3} + f_4(w_1, 0, w_3, w_4)\frac{\partial}{\partial w_4}, \omega_1 \cdot g_1(w_2, w_3, w_4)\frac{\partial}{\partial w_1} + f_3(0, w_2, w_3, w_4)\frac{\partial}{\partial w_3} + f_4(0, w_2, w_3, w_4)\frac{\partial}{\partial w_4} \right),$$

while the second map is given by

$$\left(w_2 \cdot h_2(w_2, w_3, w_4)\frac{\partial}{\partial w_2} + h_3(w_2, w_3, w_4)\frac{\partial}{\partial w_3} + h_4(w_2, w_3, w_4)\frac{\partial}{\partial w_4}, w_1 \cdot k_1(w_1, w_3, w_4)\frac{\partial}{\partial w_2} + k_3(w_1, w_3, w_4)\frac{\partial}{\partial w_3} + k_4(w_1, w_3, w_4)\frac{\partial}{\partial w_4} \right)$$
$$\mapsto (h_3(0, w_3, w_4) - k_3(0, w_3, w_4))\frac{\partial}{\partial w_3} + (h_4(0, w_3, w_4) - k_4(0, w_3, w_4))\frac{\partial}{\partial w_4}.$$

Now we want to understand the terms $\mathrm{Ext}^p_f(\Omega^1_{\mathcal{Z}}, \mathcal{O}_{\mathcal{D}\bar{\mathcal{D}}})$ in the exact sequence (3.1). For this purpose we notice that in general there exists a spectral sequence with $E_2^{r,q} = \mathrm{Ext}^r(L^q f^* \Omega^1_{\mathcal{Z}}, \mathcal{O}_{\mathcal{D}\bar{\mathcal{D}}})$ abutting to $\mathrm{Ext}^p_f(\Omega^1_{\mathcal{Z}}, \mathcal{O}_{\mathcal{D}\bar{\mathcal{D}}})$, but in our local model the qth left derived functor $L^q f^* \Omega^1_{\mathcal{Z}} = 0$, for any $q > 0$ and therefore

(3.6) $$\mathrm{Ext}^p_f(\Omega^1_{\mathcal{Z}}, \mathcal{O}_{\mathcal{D}\bar{\mathcal{D}}}) = \mathrm{Ext}^p(\Omega^1_{\mathcal{Z}_{|\mathcal{D}\bar{\mathcal{D}}}}, \mathcal{O}_{\mathcal{D}\bar{\mathcal{D}}}).$$

This means that our groups Ext^p_f are actually isomorphic to the usual Ext groups of some sheaf; therefore we have the following local to global spectral sequence:

(3.7) $$E_2^{q,r} = H^q(\mathcal{E}xt^r(\Omega^1_{\mathcal{Z}_{|\mathcal{D}\bar{\mathcal{D}}}}, \mathcal{O}_{\mathcal{D}\bar{\mathcal{D}}})) \Rightarrow \mathrm{Ext}^p_f(\Omega^1_{\mathcal{Z}}, \mathcal{O}_{\mathcal{D}\bar{\mathcal{D}}}).$$

We will now forget about $\bar{\mathcal{D}}$ and just compute the cohomology of $\mathcal{E}xt^p(\Omega^1_{\mathcal{Z}_{|\mathcal{D}}}, \mathcal{O}_{\mathcal{D}})$ on the complex space $\mathcal{D}$. We start with a description of these sheaves.

Lemma 3.8. *If $\mathcal{J}_{\mathscr{D}}$ is the ideal sheaf of $\mathscr{D}$ in $\mathscr{Z}$, then the conormal sheaf $\nu^*_{\mathscr{D}/\mathscr{Z}} := \mathcal{J}_{\mathscr{D}}/\mathcal{J}^2_{\mathscr{D}}$ is a locally free $\mathscr{O}_{\mathscr{D}}$-module of rank* 1, *and the conormal sheaf sequence*

$$0 \longrightarrow \nu^*_{\mathscr{D}/\mathscr{Z}} \longrightarrow \Omega^1_{\mathscr{Z}_{|\mathscr{D}}} \longrightarrow \Omega^1_{\mathscr{D}} \longrightarrow 0$$

is exact.

Proof. This is a local statement which only needs to be verified near the singularity. But then our local model 3.3 tells us that the local sections of $\mathcal{J}_{\mathscr{D}}/\mathcal{J}^2_{\mathscr{D}}$ can be identified with $\{f\cdot w_4 | f \in \mathscr{O}_{\mathscr{D}}\}$ so that $\nu^*_{\mathscr{D}/\mathscr{Z}}$ is locally free as well as its dual: the normal bundle.

Of course $\mathscr{O}_{\mathscr{D}}$ and $\Omega^1_{\mathscr{D}}$ have the same description as $\mathscr{O}_{\mathscr{Z}}$ and $\Omega^1_{\mathscr{Z}}$ but with $\omega_4 = 0$, because $\mathscr{Z}$ and $\mathscr{D}$ have the same type of singularity. Therefore the map $\Omega^1_{\mathscr{Z}_{|\mathscr{D}}} \longrightarrow \Omega^1_{\mathscr{D}}$ is onto with kernel $\mathcal{J}_{\mathscr{D}}/\mathcal{J}^2_{\mathscr{D}}$. q.e.d.

A more concrete description of the normal sheaf $\nu_{\mathscr{D}}$, in the spirit of (3.5), can be given in terms of the normalization $q : \mathscr{D}' = \tilde{D}_1 \amalg \tilde{D}_2 \to \mathscr{D}$ and inclusion $i : l \to \mathscr{D}$. The following can easily be checked by the techniques of 3.4.

Lemma 3.9. *The normal bundle $\nu_{\mathscr{D}/\mathscr{Z}}$ fits into the exact sequence*

$$0 \longrightarrow \nu_{\mathscr{D}/\mathscr{Z}} \longrightarrow q_*\nu_{\mathscr{D}'/\mathscr{Z}'} \longrightarrow i_*\nu_{l/Q} \longrightarrow 0.$$

We are now ready to compute the local Ext sheaves which appear in the spectral sequence of (3.7). The following lemma shows that in our case $\mathscr{E}\mathrm{xt}^p(\Omega^1_{\mathscr{Z}_{|\mathscr{D}}}, \mathscr{O}_{\mathscr{D}})$ is isomorphic to the restriction of $\tau^p_{\mathscr{Z}}$ to $\mathscr{D}$.

Lemma 3.10.

$$\mathscr{E}\mathrm{xt}^p(\Omega^1_{\mathscr{Z}_{|\mathscr{D}}}, \mathscr{O}_{\mathscr{D}}) \cong \begin{cases} \tau^0_{\mathscr{Z}_{|\mathscr{D}}}, & \text{for } p = 0, \\ \tau^1_{\mathscr{D}} \cong \mathscr{O}_l, & \text{for } p = 1, \\ 0, & \text{for } p \geq 2. \end{cases}$$

Furthermore we have the following exact sequence of $\mathscr{O}_{\mathscr{D}}$-modules:

$$0 \longrightarrow \tau^0_{\mathscr{D}} \longrightarrow \tau^0_{\mathscr{Z}_{|\mathscr{D}}} \longrightarrow \nu_{\mathscr{D}/\mathscr{Z}} \longrightarrow 0.$$

Proof. From the conormal exact sequence of 3.8 we have a long exact sequence of local Ext sheaves:

$$0 \longrightarrow \tau^0_{\mathscr{D}} \longrightarrow \mathscr{E}\mathrm{xt}^0(\Omega^1_{\mathscr{Z}_{|\mathscr{D}}}, \mathscr{O}_{\mathscr{D}}) \xrightarrow{\rho} \mathscr{E}\mathrm{xt}^0(\nu^*_{\mathscr{D}/\mathscr{Z}}, \mathscr{O}_{\mathscr{D}}) \longrightarrow$$
$$\longrightarrow \tau^1_{\mathscr{D}} \longrightarrow \mathscr{E}\mathrm{xt}^1(\Omega^1_{\mathscr{Z}_{|\mathscr{D}}}, \mathscr{O}_{\mathscr{D}}) \longrightarrow \mathscr{E}\mathrm{xt}^1(\nu^*_{\mathscr{D}/\mathscr{Z}}, \mathscr{O}_{\mathscr{D}}) \longrightarrow \cdots$$

We know from §2 that $\tau^p_{\mathscr{D}} = 0$ for $p \geq 2$, $\tau^1_{\mathscr{D}} \cong \mathscr{O}_l$ and $\tau^0_{\mathscr{D}}$ is the sheaf of derivations of $\mathscr{O}_{\mathscr{D}}$. Next, from the first part of Lemma 3.8

it follows that $\mathscr{E}\mathrm{xt}^0(\nu^*_{\mathscr{D}}, \mathscr{O}_{\mathscr{D}}) = \nu_{\mathscr{D}}$ and that $\mathscr{E}\mathrm{xt}^p(\nu^*_{\mathscr{D}}, \mathscr{O}_{\mathscr{D}}) = 0$ for all $p \geq 1$. Therefore it is enough to show that the map ρ is onto and that $\mathscr{E}\mathrm{xt}^0(\Omega^1_{\mathscr{Z}_{|\mathscr{D}}}, \mathscr{O}_{\mathscr{D}}) \cong \tau^0_{\mathscr{Z}_{|\mathscr{D}}}$. That is, we will show the exactness of the sequence

$$0 \longrightarrow \tau^0_{\mathscr{D}} \longrightarrow \tau^0_{\mathscr{Z}_{|\mathscr{D}}} \longrightarrow \nu_{\mathscr{D}/\mathscr{Z}} \longrightarrow 0.$$

Again it is enough to do this using the local model near the singularity locus. The argument for $\mathscr{H}\mathrm{om}(\Omega^1_{\mathscr{Z}_{|\mathscr{D}}}, \mathscr{O}_{\mathscr{D}}) \cong \mathscr{E}\mathrm{xt}^0(\Omega^1_{\mathscr{Z}_{|\mathscr{D}}}, \mathscr{O}_{\mathscr{D}}) \cong \tau^0_{\mathscr{Z}_{|\mathscr{D}}}$ is the same we gave in 3.4; just set $w_4 = 0$. But then obviously $\rho\colon \sum_{i=1}^4 f_i \frac{\partial}{\partial w_i} \mapsto f_4 \frac{\partial}{\partial w_4}$ is onto for $f_i \in \mathscr{O}_{\mathscr{D}}$. q.e.d.

We can now compute the cohomology group we are interested in:

Lemma 3.11. *If* $H^2(\Theta_{Z_i}) = 0$ *for* $i = 1, 2$, *then* $H^2(\tau^0_{\mathscr{Z}_{|\mathscr{D}}}) = 0$.

Proof. We start by considering the exact sequence (3.5) and noticing that when we restrict it to $\mathscr{D}$, the sequence is still exact:

$$0 \longrightarrow \tau^0_{\mathscr{Z}_{|\mathscr{D}}} \longrightarrow q_*\Theta_{(Z', Q')_{|\mathscr{D}}} \longrightarrow i_*\Theta_{Q_{|l}} \longrightarrow 0.$$

This can be checked directly as before, and it just amounts to set the last coordinate $w_4 = 0$, also recall that $\mathscr{D}$ and Q intersect transversely.

Next notice that the normal bundle of l in Q is trivial so that $\Theta_{Q_{|l}} \cong \mathscr{O}_{\mathbb{CP}_1} \oplus \mathscr{O}_{\mathbb{CP}_1}(2)$ and therefore $H^2(\tau^0_{\mathscr{Z}_{|\mathscr{D}}}) \cong \bigoplus_{i=1}^2 H^2(\Theta_{(\tilde{Z}_i, Q_i)_{|\tilde{D}_i}})$, which reduces the computation of our cohomology group to the vanishing of the cohomology of some natural sheaves on the smooth surfaces $\tilde{D}_i$, $i = 1, 2$. Since our argument is almost independent of $i = 1, 2$ we now drop the subscript.

We consider the usual exact sequence of the relative tangent sheaf of the smooth pair $(\tilde{Z}, Q)$, and notice again that by transversality it remains exact when restricted to $\tilde{D}$:

$$0 \longrightarrow \Theta_{(\tilde{Z}, Q)_{|\tilde{D}}} \longrightarrow \Theta_{\tilde{Z}_{|\tilde{\mathscr{D}}}} \longrightarrow \nu_{(Q/\tilde{Z})_{|l}} \longrightarrow 0.$$

Now we claim that the induced long exact sequence gives isomorphisms of the following cohomology groups on $\tilde{D}$: $H^2(\Theta_{(\tilde{Z}, Q)_{|\tilde{D}}}) \cong H^2(\Theta_{\tilde{Z}_{|\tilde{D}}})$, because the restriction $\nu_{Q/\tilde{Z}_{|l}} \cong \mathscr{O}_{\mathbb{CP}_1}(\pm 1)$, so that its first cohomology is always zero. The reason is that $l = \tilde{D} \cap Q$ and l has trivial normal bundle on Q while its self-intersection in $\tilde{D}$ is ± 1 depending on whether the original twistor line to be blown up belongs to the divisor D or not (see the construction in §2).

Finally, it remains to show $H^2(\Theta_{\tilde{Z}_{|\tilde{D}}}) = 0$ under our hypothesis. This follows from the restriction exact sequence

$$0 \longrightarrow \Theta_{\tilde{Z}} \otimes \mathcal{I}_{\tilde{D}} \longrightarrow \Theta_{\tilde{Z}} \longrightarrow \Theta_{\tilde{Z}_{|\tilde{D}}} \longrightarrow 0,$$

because our hypothesis easily implies that $H^2(\Theta_{\tilde{Z}}) = 0$ [6, p. 218], while by Serre duality $H^3(\Theta_{\tilde{Z}} \otimes \mathcal{I}_{\tilde{D}}) \cong H^0(K_{\tilde{Z}} \otimes \Omega^1_{\tilde{Z}} \otimes [\tilde{D}])$. Now, this group vanishes by restriction to the lines of normal bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$ which fill up the open set $\tilde{Z}\setminus Q \cong Z\setminus\{\text{a twistor line}\}$ because they have intersection number 1 with the divisor $\tilde{D}$; the proof is now complete.

4. Anti-self-dual hermitian metrics

In this section we will assume that $\mathcal{D} \cap \bar{\mathcal{D}} = \varnothing$. As a consequence all the results proven for $\mathcal{D}$ in the last section hold for $\mathcal{D}\bar{\mathcal{D}}$ just as well, because we can identify any natural sheaf on $\mathcal{D}\bar{\mathcal{D}}$ with the direct sum of the corresponding sheaves on $\mathcal{D}$ and $\bar{\mathcal{D}}$. Let us recall once more that this assumption corresponds to the geometric condition that in the construction of the singular space $\mathcal{Z} = \tilde{Z}_1 \cup \tilde{Z}_2$ we take Z_1 to be the twistor space of a hermitian anti-self-dual metric on a compact complex surface M and this in turn is equivalent to the case where D_1 is biholomorphic to M with the complex structure J, its conjugate $\bar{D}_1$ is biholomorphic to $\bar{M}$ with $-J$, and they are disjoint degree-1 divisors in Z_1.

Let us start with one definition.

Definition 4.1. If Y is a complex subspace of a complex space X, we define the sheaf of relative derivations of X with respect to Y to be the subsheaf of the derivations of X preserving the ideal $\mathcal{I}_Y$, and we will use the notation $\tau^0_{X,Y} := \{v | v \in \mathrm{Der}(\mathcal{O}_X, \mathcal{O}_X),\ v : \mathcal{I}_Y \longrightarrow \mathcal{I}_Y\}$.

When X and Y are both smooth, then $\tau^0_{X,Y}$ is nothing but the sheaf $\Theta_{X,Y}$ of smooth vector fields on X which are tangent to Y along Y. Now we consider the homomorphism of $\mathcal{O}_{\mathcal{Z}}$-modules $\tau^0_{\mathcal{Z}} \oplus \tau^0_{\mathcal{D}\bar{\mathcal{D}}} \rightarrow \tau^0_{\mathcal{Z}_{|\mathcal{D}\bar{\mathcal{D}}}}$ defined by restriction on the first factor minus inclusion of the second factor; this map is surjective because $\tau^0_{\mathcal{Z}} \rightarrow \tau^0_{\mathcal{Z}_{|\mathcal{D}\bar{\mathcal{D}}}}$ is also so. Then we can check as in Remark 3.4 that its kernel is $\tau^0_{\mathcal{Z},\mathcal{D}\bar{\mathcal{D}}}$. So we have the following exact sequence of sheaves of $\mathcal{O}_{\mathcal{Z}}$-modules:

$$0 \longrightarrow \tau^0_{\mathcal{Z},\mathcal{D}\bar{\mathcal{D}}} \longrightarrow \tau^0_{\mathcal{D}\bar{\mathcal{D}}} \oplus \tau^0_{\mathcal{Z}} \longrightarrow \tau^0_{\mathcal{Z}_{|\mathcal{D}\bar{\mathcal{D}}}} \longrightarrow 0. \tag{4.2}$$

In the spirit of [6], the group $H^1(\tau^0_{\mathcal{Z},\mathcal{D}\bar{\mathcal{D}}})$ corresponds to locally trivial infinitesimal deformations of the pair, i.e., the singularities remain locally a product. For the second cohomology group instead, we have the following vanishing result.

Lemma 4.3. *If* $H^2(\Theta_{Z_i, D_i\bar{D}_i}) = 0$ *for* $i = 1, 2$ *then* $H^2(\tau^0_{\mathscr{Z}, \mathscr{D}\bar{\mathscr{D}}}) = 0$.

Proof. To start with we notice that just as for the exact sequence (3.1.5), there is a "normalization" exact sequence of $\mathscr{O}_{\mathscr{Z}}$-modules:

$$0 \longrightarrow \tau^0_{\mathscr{Z}, \mathscr{D}\bar{\mathscr{D}}} \longrightarrow q_*\Theta_{\mathscr{Z}', (\mathscr{D}\bar{\mathscr{D}})', Q'} \longrightarrow i_*\Theta_{Q, l\bar{l}} \longrightarrow 0$$

here $(\mathscr{D}\bar{\mathscr{D}})' = (\tilde{D}_1 \amalg \tilde{\bar{D}}_1) \amalg (\tilde{D}_2 \amalg \tilde{\bar{D}}_2)$ and $Q' = Q_1 \amalg Q_2$ while $l = Q_1 \cap \tilde{D}_1 = Q_2 \cap \tilde{\bar{D}}_2$ and $\bar{l} = Q_1 \cap \tilde{\bar{D}}_1 = Q_2 \cap \tilde{D}_2$.

Of course the cohomology of $i_*\Theta_{Q, l\bar{l}}$ coincides with that of $\Theta_{Q, l\bar{l}}$ so that

$$H^2(\tau^0_{\mathscr{Z}, \mathscr{D}\bar{\mathscr{D}}}) = H^2(q_*\Theta_{\mathscr{Z}', (\mathscr{D}\bar{\mathscr{D}})', Q'})$$

because $0 \to \Theta_{Q, l\bar{l}} \to \Theta_Q \to \nu_l \oplus \nu_{\bar{l}} \to 0$ is exact and $H^0(\Theta_Q) \to H^0(\nu_l \oplus \nu_{\bar{l}})$ is surjective. But the cohomology of $\Theta_{\mathscr{Z}', (\mathscr{D}\bar{\mathscr{D}})', Q'}$ coincides with that of its direct image so that we are left to show that $H^2(\Theta_{\tilde{Z}_i, \widetilde{D_i\bar{D}_i}, Q_i}) = 0$ for $i = 1, 2$. Now if $b : \tilde{Z}_i \to Z_i$ is the blowing-down map, the cohomology of the above sheaf coincides with the cohomology of its direct image, again by Leray spectral sequence, namely it will be enough to show $H^2(\Theta_{Z_i, D_i\bar{D}_i, l_i}) = 0$ in the twistor space Z_i for $i = 1, 2$.

Next we have to consider the two cases separately; when $i = 2$, $l_2 = D_2 \cap \bar{D}_2$ so that $\Theta_{Z_2, D_2\bar{D}_2, l_2} = \Theta_{Z_2, D_2\bar{D}_2}$ whose second cohomology vanishes by hypothesis; on the other hand, when $i = 1$, l_1 intersects D_1 and $\bar{D}_1$ transversely, and one can check easily that the following sequence is exact:

$$0 \longrightarrow \Theta_{Z_1, D_1\bar{D}_1, l_1} \longrightarrow \Theta_{Z_1, D_1\bar{D}_1} \longrightarrow \nu_{l_1/Z_1} \longrightarrow 0.$$

Finally the lemma follows since $H^p(\nu_{l_1/Z_1}) = H^p(\mathscr{O}_{\mathbb{CP}_1}(1)^{\oplus 2}) = 0$ for $p = 1, 2$. q.e.d.

We are now ready for the following theorem.

Theorem 4.4. *If* $H^2(\Theta_{D_i}) = H^2(\Theta_{Z_i}) = H^2(\Theta_{Z_i, D_i\bar{D}_i}) = 0$ *for* $i = 1, 2$, *then* T^2_f *vanishes, and moreover any element in a small open ball centered at the origin of* T^1_f *and away from the hyperplane* $H^1(\tau^0_{\mathscr{Z}, \mathscr{D}\bar{\mathscr{D}}})$ *corresponds to a simultaneous smoothing of* $\mathscr{Z}$ *and* $\mathscr{D}\bar{\mathscr{D}}$.

Proof. To show this we consider the exact sequence (3.1) and insert the relevant part, modified by (3.6), as the middle row of the following diagram:

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\Delta & \xrightarrow{d' \oplus d} & H^0(\tau^1_{\mathscr{D}\mathscr{D}}) \oplus H^0(\tau^1_{\mathscr{Z}}) & \xrightarrow{\hat{\psi}' - \hat{\psi}} & H^0(\tau^1_{\mathscr{Z}|\mathscr{D}\mathscr{D}}) & \longrightarrow & 0 \\
\uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle r \oplus \rho} & & \uparrow{\scriptstyle \varrho} & & \\
T^1_f & \xrightarrow{\phi' \oplus \phi} & T^1_{\mathscr{D}\mathscr{D}} \oplus T^1_{\mathscr{Z}} & \xrightarrow{\psi' - \psi} & \operatorname{Ext}^1(\Omega^1_{\mathscr{Z}|\mathscr{D}\mathscr{D}}, \mathscr{O}_{\mathscr{D}\mathscr{D}}) & \longrightarrow & T^2_f \\
\uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle i \oplus j} & & \uparrow{\scriptstyle \iota} & & \\
H^1(\tau^0_{\mathscr{Z},\mathscr{D}\mathscr{D}}) & \xrightarrow{\phi'_- \oplus \phi_-} & H^1(\tau^0_{\mathscr{D}\mathscr{D}}) \oplus H^1(\tau^0_{\mathscr{Z}}) & \xrightarrow{\psi'_- - \psi_-} & H^1(\tau^0_{\mathscr{Z}_{|\mathscr{D}\mathscr{D}}}) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

This diagram is commutative with exact rows and columns, and the result will follow from some straightforward diagram chasing; let us explain why. Notice that, putting $d' \oplus d$ aside for a moment, the horizontal arrows are all induced from either restrictions or inclusions of sheaves of $\mathscr{O}_{\mathscr{Z}}$-modules. They are induced homomorphisms in either cohomology or Ext groups. The map α in the first column can be defined naturally and in fact is a monomorphism; any element in $H^1(\tau^0_{\mathscr{Z},\mathscr{D}\mathscr{D}})$ corresponds to a locally trivial infinitesimal deformation of the pair and can therefore be thought as sitting inside T^1_f. The left-lower square of the diagram is commutative by naturality, and β is the quotient map. The map $d' \oplus d$ is also defined to be the quotient of $\phi' \oplus \phi$. The last two columns come from local-to-global spectral sequences; commutativity of the right-hand side of the diagram then follows by naturality of these algebraic constructions.

Now we explain exactness: the bottom row is the long exact sequence induced from (4.2), and its last entry is zero by Lemma 4.3. When $H^2(\Theta_{Z_i}) = 0$, the unobstructed case of Donaldson-Friedman [6, p. 218], we have that $H^2(\tau^0_{\mathscr{Z}}) = 0$ which implies that $T^2_{\mathscr{Z}} = 0$. Similarly, $H^2(\Theta_{D_i}) = 0$ yields $H^2(\tau^0_{\mathscr{D}\mathscr{D}}) = T^2_{\mathscr{D}\mathscr{D}} = 0$, which means that the middle column is exact, and that the term following T^2_f in the middle row is zero. Similarly, the exactness of the last column comes from the exact sequence induced by the first few terms of the spectral sequence (3.7) together with the vanishing result $H^2(\tau^0_{\mathscr{Z}_{|\mathscr{D}\mathscr{D}}}) = 0$ of 3.11.

Therefore the vanishing of T^2_f is equivalent to the surjectivity of $\psi' - \psi$.

The top row is particularly simple to understand because the sheaves τ^1 are all trivial; in fact $\tau^1_{\mathscr{Z}} = \nu_{Q_1/\tilde{Z}_1} \otimes \nu_{Q_2/\tilde{Z}_2} \cong \mathscr{O}_Q(1, -1) \otimes \mathscr{O}_Q(-1, 1) = \mathscr{O}_Q \cong \mathscr{O}_{\mathbf{CP}_1 \times \mathbf{CP}_1}$ and $\tau^1_{\mathscr{D}} = \nu_{l_1/\tilde{D}_1} \otimes \nu_{l_2/\tilde{D}_2} \cong \mathscr{O}_l(1) \otimes \mathscr{O}_l(-1) = \mathscr{O}_l \cong \mathscr{O}_{\mathbf{CP}_1}$; similarly $\tau^1_{\bar{\mathscr{D}}} \cong \mathscr{O}_{\bar{l}} \cong \mathscr{O}_{\mathbf{CP}_1}$ and $\tau^1_{\mathscr{Z}_{|\mathscr{D}\bar{\mathscr{D}}}} = \mathscr{O}_{Q|l\bar{l}} = \mathscr{O}_l \oplus \mathscr{O}_{\bar{l}}$. Therefore once we have fixed a trivialization of $\tau^1_{\mathscr{Z}} \cong \mathscr{O}_Q$, we have trivializations, on $\tau^1_D \cong \mathscr{O}_l$ and $\tau^1_{\bar{D}} \cong \mathscr{O}_{\bar{l}}$ induced from inclusion, which coincide with the trivialization induced by restriction on $\tau^1_{\mathscr{Z}_{|\mathscr{D}\bar{\mathscr{D}}}} \cong \mathscr{O}_l \oplus \mathscr{O}_{\bar{l}}$.

This simple discussion just amounts to saying that, after a canonical identification of $H^0(\tau^1_{\mathscr{Z}})$ with $H^0(\tau^1_{\mathscr{D}})$ and $H^0(\tau^1_{\bar{\mathscr{D}}})$ and of $H^0(\tau^1_{\mathscr{Z}_{|\mathscr{D}\bar{\mathscr{D}}}})$ with their direct sum, the map $\hat{\psi}'$ is the identity while $\hat{\psi}$ is the diagonal imbedding.

Also notice that from the surjectivity of $\hat{\psi}' - \hat{\psi}$ and that of $\psi'_- - \psi_-$ we can conclude after some diagram chasing (five lemma) that $\psi' - \psi$ is onto and so $T^2_f = 0$. One can then compute the dimension of T^1_f to be $t^1_f = h^1(\tau^0_{\mathscr{Z}, \mathscr{D}\bar{\mathscr{D}}}) + 1$.

Now observe that Δ is isomorphic to the kernel of $\hat{\psi}' - \hat{\psi}$, which consists of the elements (c, c, c) in $H^0(\tau^1_{\mathscr{D}}) \oplus H^0(\tau^1_{\bar{\mathscr{D}}}) \oplus H^0(\tau^1_{\mathscr{Z}})$, so $d' \oplus d$ is a monomorphism. Therefore any element in the complement of the hyperplane $H^1(\tau^0_{\mathscr{Z}, \mathscr{D}\bar{\mathscr{D}}})$ of T^1_f gives smoothings of both $\mathscr{Z}$ and $\mathscr{D}\bar{\mathscr{D}}$, because its image under $(r \oplus \rho) \circ (\phi' \oplus \phi)$ is $(c, c, c) \neq 0$ in the space of smoothings $H^0(\tau^1_{\mathscr{D}\bar{\mathscr{D}}}) \oplus H^0(\tau^1_{\mathscr{Z}})$. q.e.d.

Since we have some understanding of the smoothings of the pair $(\mathscr{Z}, \mathscr{D}\bar{\mathscr{D}})$, we want to apply it to prove the existence of twistor spaces with degree-1 divisors. This means that we have to start to discuss the real structures. For this purpose let us look at the second column in the above diagram. We will work under the assumption that $T^2_{\mathscr{Z}} = T^2_{\mathscr{D}} = 0$, as we recalled in §2 the real structure of $\mathscr{Z}$ induces a real structure on the base of the versal deformation $T^1_{\mathscr{Z}}$, and this is compatible with real structures on $H^1(\tau^0_{\mathscr{Z}})$ and $H^0(\tau^1_{\mathscr{Z}})$; similarly we have that $T^1_{\mathscr{D}}$ is the base of a versal deformation $\mathfrak{D} \to T^1_{\mathscr{D}}$ of $\mathscr{D}$. Now $T^1_{\bar{\mathscr{D}}}$ is naturally isomorphic to the conjugate of $T^1_{\mathscr{D}}$, and by taking the conjugate complex space of $\mathfrak{D}$ we obtain a versal deformation $\bar{\mathfrak{D}} \to T^1_{\bar{\mathscr{D}}}$ of $\bar{\mathscr{D}}$. Then $\mathfrak{D} \times \bar{\mathfrak{D}} \to T^1_{\mathscr{D}} \oplus T^1_{\bar{\mathscr{D}}} = T^1_{\mathscr{D}\bar{\mathscr{D}}}$ is a versal deformation of $\mathscr{D}\bar{\mathscr{D}} = \mathscr{D} \amalg \bar{\mathscr{D}}$, and the point is that this particular versal deformation has an obvious real structure of the type $(w_1, w_2) \mapsto (\bar{w}_2, \bar{w}_1)$. This holds for both the total space and the base. Then a real element of $T^1_{\mathscr{D}\bar{\mathscr{D}}}$ is of the type $t = (t_1, \bar{t}_1)$ with $t_1 \in T^1_{\mathscr{D}}$, and the fiber over it is a pair $(D_t, \bar{D}_t)$ where $\bar{D}_t = (D_t, \bar{\mathscr{O}}_t)$ is the conjugate complex space so that the fiber over a real element $t \in T^1_{\mathscr{D}\bar{\mathscr{D}}}$ has an obvious real structure. This real structure on $T^1_{\mathscr{D}\bar{\mathscr{D}}} = T^1_{\mathscr{D}} \oplus T^1_{\bar{\mathscr{D}}}$ is compatible with similar ones on $H^1(\tau^0_{\mathscr{D}\bar{\mathscr{D}}}) = H^1(\tau^0_{\mathscr{D}}) \oplus H^1(\tau^0_{\bar{\mathscr{D}}})$ and on $H^0(\tau^1_{\mathscr{D}\bar{\mathscr{D}}}) = H^0(\tau^1_{\mathscr{D}}) \oplus H^0(\tau^1_{\bar{\mathscr{D}}})$. For example, the real structure on the "space of smoothings" $H^0(\tau^1_{\mathscr{D}\bar{\mathscr{D}}}) \oplus H^0(\tau^1_{\mathscr{Z}}) \cong H^0(\mathscr{O}_l) \oplus H^0(\mathscr{O}_{\bar{l}}) \oplus H^0(\mathscr{O}_Q) \cong \mathbb{C}^3$ induced by restricting the involution σ_0 of $\mathscr{Z}$ is exactly given by restricting σ_0 to the invariant quadric Q. Now, $\sigma_0(l) = \bar{l}$, and therefore the real structure is $(w_1, w_2, z) \mapsto (\bar{w}_2, \bar{w}_1, \bar{z})$ where $(w_1, w_2, z) \in \mathbb{C}^3$ are the coordinates for the space of smoothings as above; its fixed points are $(w_1, \bar{w}_1, r)$ with $r \in \mathbb{R}$.

We are now ready to apply our results to the case of the blow-up of a surface M at one point. We keep the notation that $M_1 = M$ and $M_2 = \overline{\mathbb{CP}}_2$. Let us recall here that some standard computations show that for the twistor space $\mathbb{F}$ of $\overline{\mathbb{CP}}_2$,

$$H^2(\Theta_{\tilde{\mathbb{F}}} \otimes \mathscr{I}_{\widetilde{D_2\bar{D}_2}}) = H^2(\Theta_{\tilde{\mathbb{F}}, \widetilde{D_2\bar{D}_2}}) = H^2(\Theta_{\tilde{\mathbb{F}}}) = 0.$$

Theorem 4.5. *Let M be a compact anti-self-dual hermitian surface with twistor space Z, and let $D \subset Z$ be the divisor defined by the complex structure of M. If $H^2(\Theta_M) = H^2(\Theta_Z) = H^2(\Theta_{Z, D\bar{D}}) = 0$, then there exist anti-self-dual hermitian metrics on the blow-up at one point of some small deformations of M.*

Proof. We show first that there exists $t \in T^1_f$ such that Z_t is a twistor space with real structure σ_t and disjoint degree-1 divisors D_t and $\bar{D}_t$ such that $\sigma_t(D_t) = \bar{D}_t$. Using the notation of the previous diagram, one can show by simple diagram chasing that there exists $t \in T^1_f$ which satisfies the following conditions.

(1) $\beta(t) = (r, r, r) \in \Delta$ with $0 \neq r \in \mathbb{R}$.
(2) $\phi(t) \in T^1_{\mathscr{Z}}$ is a real element.
(3) $\phi'(t) \in T^1_{\mathscr{D}\bar{\mathscr{D}}}$ is a real element.

One can see by a standard argument that $D_t\bar{D}_t$ imbeds into Z_t for small t. By Theorem 4.4, the condition $\beta(t) \neq 0$ means that we have deformations of smooth complex manifolds pairs $(Z_t, D_t\bar{D}_t)$ with $D_t\bar{D}_t \subset Z_t$. D_t and $\bar{D}_t$ remain disjoint degree-1 divisors by topological stability of the intersection degree under deformation. Next we use that $\phi(t)$ is a real element in $T^1_{\mathscr{Z}}$ such that $\rho \circ \phi(t) \neq 0 \in H^1(\tau^0_{\mathscr{Z}})$: if the versal deformation over $T^1_{\mathscr{Z}}$ is standard in the sense of Donaldson-Friedman, i.e., has a real structure, then Z_t is a smooth twistor space with real structure σ_t, and let (M_t, h_t) denote the corresponding anti-self-dual conformal structure on $M\#\overline{\mathbb{CP}}_2$. We are left to show that $\sigma_t(D_t) = \bar{D}_t$.

This follows from the following reasoning: the fact that $\phi'(t) \in T^1_{\mathscr{D}\bar{\mathscr{D}}} = T^1_{\mathscr{D}} \oplus T^1_{\bar{\mathscr{D}}}$ is a real element just amounts to saying, by the construction of our versal deformation of $\mathscr{D}\bar{\mathscr{D}}$, that $\bar{D}_t$ is the complex conjugate of D_t. On the other hand as degree-1 divisors each of them defines a hermitian structure on (M_t, h_t), denote it by J_t and $\tilde{J}_t$ respectively, with the property that (M_t, J_t) is biholomorphic to D_t while $(M_t, \tilde{J}_t)$ is biholomorphic to $\bar{D}_t$. Then $\tilde{J}_t = -J_t$ so that $\sigma_t(D_t) = \bar{D}_t$ as wanted, because Z_t can be thought of as the bundle of h_t-hermitian structures J with σ_t sending J to $-J$. The proof is now complete in the case where the versal deformation is standard.

Otherwise there always exist a standard deformation $\mathfrak{S} \to S$ of $\mathscr{Z}$ and a submersion $e : S \to T^1_Z$ compatible with real structures [6, p. 225]. Therefore we can choose a real element $s \in S$ such that $e(s) = t$, and then the fiber $Z_s \subset \mathfrak{S}$ is biholomorphic to Z_t (by versality) which is a smooth complex manifold, and also carries a real structure σ_s so that the pair (Z_s, σ_s) is a smooth twistor space of an anti-self-dual structure (M_s, h_s) with M_s diffeomorphic to $M\#\overline{\mathbb{CP}}_2$. If we now think of D_t and $\bar{D}_t$ as divisors in $Z_s \cong Z_t$ we have that they are σ_s-invariant by the same argument as before.

Finally, by Theorem 2.6 the complex structure of D_t is determined by the element $\phi'(t) = (t_1, t_2) \in T^1_{\mathscr{D}\bar{\mathscr{D}}}$. Of course $t_2 = r \neq 0$ so that D_t is a small deformation of the blow-up of M at the point corresponding to the twistor line $l_1 \subset Z_1 = Z$. q.e.d.

More precise informations on the complex structure of the resulting non-minimal complex surface are given by the following:

Theorem 4.6. *Under the same hypothesis of* 4.5, *if in addition* $H^2(\Theta_Z \otimes \mathscr{I}_{D\bar{D}}) = 0$, *where* $\mathscr{I}_{D\bar{D}}$ *is the ideal sheaf of* $D\bar{D}$, *then there exist anti-self-dual hermitian metrics on the blow-up of* M *at any point, as well as on any of its small deformations.*

Proof. We are going to study the image of ϕ' in the diagram of Theorem 4.4.

We write $T^1_{\mathscr{D}\bar{\mathscr{D}}} = T^1_{\mathscr{D}} \oplus T^1_{\bar{\mathscr{D}}}$ and recall the splitting given by Theorem 2.6 (3). Namely $T^1_{\mathscr{D}} \cong H^1(\tau^0_{\mathscr{D}}) \oplus H^0(\tau^1_{\mathscr{D}})$, the first factor corresponds to the infinitesimal deformations of $\tilde{M}$ and the second factor corresponds to the smoothings of D. We know from 4.5 that there are real elements in T^1_f giving rise to the smoothings of $\mathscr{D}$. Therefore we only have to show that ϕ'_- sends the real part of $H^1(\tau^0_{\mathscr{Z},\mathscr{D}\bar{\mathscr{D}}})$ onto the real part of $H^1(\tau^0_{\mathscr{D}\bar{\mathscr{D}}})$, which of course is isomorphic to $H^1(\tau^0_{\mathscr{D}})$ under projection on the first factor.

Moreover, since ϕ'_- is compatible with real structures, it will be enough to show that $\phi'_- : H^1(\tau^0_{\mathscr{Z},\mathscr{D}\bar{\mathscr{D}}}) \to H^1(\tau^0_{\mathscr{D}}) \oplus H^1(\tau^0_{\bar{\mathscr{D}}})$ is onto.

From the "normalization" exact sequence of Lemma 4.3, we have

$$\xrightarrow{\epsilon_1} H^1(\tau^0_{\mathscr{Z},\mathscr{D}\bar{\mathscr{D}}}) \longrightarrow H^1(\Theta_{\tilde{Z}_1,\widetilde{D_1\bar{D}_1},Q_1}) \oplus H^1(\Theta_{\tilde{Z}_2,\widetilde{D_2\bar{D}_2},Q_2}) \longrightarrow 0$$

and using Leray spectral sequences one can show that

$$H^1(\Theta_{\tilde{Z}_1,\widetilde{D_1\bar{D}_1},Q_1}) \cong H^1(\Theta_{Z_1,D_1\bar{D}_1,l_1})$$

while

$$H^1(\Theta_{\tilde{Z}_2,\widetilde{D_2\bar{D}_2},Q_2}) \cong H^1(\Theta_{Z_2,D_2\bar{D}_2});$$

furthermore ϵ_1 is the zero map because $H^0(\Theta_{\tilde{Z}_1,\widetilde{D_1\bar{D}_1},Q_1}) \to H^0(\Theta_{Q,l\bar{l}})$ is surjective.

As a result,

$$H^1(\tau^0_{\mathscr{Z},\mathscr{D}\bar{\mathscr{D}}}) \cong H^1(\Theta_{Z_1,D_1\bar{D}_1,l_1}) \oplus H^1(\Theta_{Z_2,D_2\bar{D}_2}).$$

Next we use the exact sequence

$$0 \longrightarrow \tau^0_{\mathscr{D}} \longrightarrow \Theta_{\tilde{D}_1,E_1} \oplus \Theta_{\tilde{D}_2,E_2} \longrightarrow \Theta_l \longrightarrow 0$$

with $E_i = Q_i \cap \tilde{D}_i$ for $i = 1, 2$. From the induced long exact sequence one easily infers an isomorphism

$$H^1(\tau^0_{\mathscr{D}}) \cong H^1(\Theta_{\tilde{D}_1,E_1}) \oplus H^1(\Theta_{\tilde{D}_2,E_2}).$$

However from the Leray spectral sequence of the blow-up it follows that $H^1(\Theta_{\tilde{D}_2,E_2}) \cong H^1(\Theta_{D_2,l_2}) = 0$ while $H^1(\Theta_{\tilde{D}_1,E_1}) \cong H^1(\Theta_{D_1,p})$.

Of course the same holds for $\tau^0_{\bar{\mathscr{D}}}$, and we have shown that

$$H^1(\tau^0_{\mathscr{D}}) \oplus H^1(\tau^0_{\bar{\mathscr{D}}}) \cong H^1(\Theta_{D_1,p}) \oplus H^1(\Theta_{\bar{D}_1,\bar{p}}).$$

These cohomology groups fit into the long exact sequence induced by

$$0 \longrightarrow \Theta_{Z_1,l_1} \otimes \mathscr{I}_{D_1\bar{D}_1} \longrightarrow \Theta_{Z_1,D_1\bar{D}_1,l_1} \longrightarrow \Theta_{D_1,p} \oplus \Theta_{\bar{D}_1,\bar{p}} \longrightarrow 0,$$

defined on the smooth twistor space Z_1.

From this we see that the map $H^1(\Theta_{Z_1,D_1\bar{D}_1,l_1}) \longrightarrow H^1(\Theta_{D_1,p}) \oplus H^1(\Theta_{\bar{D}_1,\bar{p}})$ is surjective because $H^2(\Theta_{Z_1,l_1} \otimes \mathscr{I}_{D_1\bar{D}_1})$ vanishes since it fits into the long exact sequence induced by

$$0 \longrightarrow \Theta_{Z_1,l_1} \otimes \mathscr{I}_{D_1\bar{D}_1} \longrightarrow \Theta_{Z_1} \otimes \mathscr{I}_{D_1\bar{D}_1} \longrightarrow \nu_{l_1/Z_1} \otimes \mathscr{I}_{D_1\bar{D}_1} \longrightarrow 0,$$

in which $\nu_{l_1/Z_1} \otimes \mathcal{J}_{D_1\bar{D}_1} \cong \mathcal{O}_{\mathbb{CP}_1}(-1) \oplus \mathcal{O}_{\mathbb{CP}_1}(-1)$ has vanishing cohomology while $H^2(\Theta_{Z_1} \otimes \mathcal{J}_{D_1\bar{D}_1}) = 0$ by hypothesis.

Our claim now follows from the naturality of all isomorphisms involved in this proof. q.e.d.

Note that the condition $H^2(\Theta_M) = H^2(\Theta_Z \otimes \mathcal{J}_{\mathcal{D}\bar{\mathcal{D}}}) = 0$ implies $H^2(\Theta_{Z,D\bar{D}}) = 0$.

We now strengthen Theorem 4.6 by applying simultaneous smoothings; see §6.2 in [6].

Theorem 4.7. *If* $H^2(\Theta_M) = H^2(\Theta_{Z,D\bar{D}}) = 0$ *and either* $H^2(\Theta_Z) = 0$ *or* $H^2(\nu_{D/Z}) = 0$, *then there exist anti-self-dual hermitian metrics on the blow-up of* M *at any collection of distinct points. The same conclusion holds for some small deformation of* M, *and it holds for any small deformation when* $H^2(\Theta_Z \otimes \mathcal{J}_{\mathcal{D}\bar{\mathcal{D}}}) = 0$.

Proof. Let $p_1, p_2, \cdots, p_k \in M$, and blow-up the twistor lines $l_1, \cdots, l_k$ above them to obtain $\tilde{Z}_1$. The 3-dimensional complex manifold $\tilde{Z}_1$ contains a divisor $\tilde{D}_1$ which is biholomorphic to the blow-up of M at $p_1, \cdots, p_k$. Let $Q_1, \cdots, Q_k$ be the resulting quadrics in $\tilde{Z}_1$, and attach k copies of $\tilde{\mathbb{F}}$ to obtain a singular space with normal crossing $\mathcal{Z} = \tilde{Z} \cup_{Q_1} \tilde{\mathbb{F}}_1 \cdots \cup_{Q_k} \tilde{\mathbb{F}}_k$ and a singular divisor $\mathcal{D}\bar{\mathcal{D}}$ just as in the case $k = 1$. Then the singularities of $(\mathcal{Z}, \mathcal{D}\bar{\mathcal{D}})$ are on the disjoint union of the quadrics, and therefore the same arguments as before apply, there are no obstructions to smoothings because

$$H^2(\Theta_{\tilde{\mathcal{Z}}, \widetilde{\mathcal{D}\bar{\mathcal{D}}}, Q_1 \amalg \ldots \amalg Q_k}) \cong H^2(\Theta_{Z, D\bar{D}, l_1 \amalg \ldots \amalg l_k}) = 0,$$

as it follows from the short exact sequence

$$0 \longrightarrow \Theta_{Z, D\bar{D}, l_1 \amalg \ldots \amalg l_k} \longrightarrow \Theta_{Z, D\bar{D}} \longrightarrow \nu_{l_1/Z} \oplus \cdots \oplus \nu_{l_k/Z} \longrightarrow 0.$$

The argument for real structures is also just like in the previous situation where we have only one point to blow up. q.e.d.

5. Scalar-flat Kähler surfaces

If h is an anti-self-dual hermitian metric on the complex surface (M, J), the same is true for any other metric in the conformal class $[h]$; as seen in the introduction if $[h]$ admits a Kähler representative h, then the scalar curvature of h vanishes identically. Furthermore when M is compact, it was shown by Boyer [5, Theorem 1] that this only depends on the topology: $[h]$ admits a scalar-flat Kähler representative if and only if $b_1(M)$ is even.

In this section we will apply the results of §4 to the situation of a compact scalar-flat Kähler surface M with $c_1^{\mathbb{R}}(M) \neq 0$. The assumption on the first Chern class is not restrictive because the blow-up of M cannot admit such metrics when $c_1(M) = 0$ [5, Theorem 5].

Indeed when M is scalar-flat Kähler rather than just hermitian and anti-self-dual, the obstruction space $H^2(\Theta_Z \otimes \mathcal{I}_{D\bar{D}})$ that we found in the previous section has been described very precisely by LeBrun-Singer [14] and, perhaps surprisingly, turns out to vanish in most cases.

The point is that in the Kähler case the ideal sheaf $\mathcal{I}_{D\bar{D}}$ is isomorphic to the $1/2$ power of the canonical bundle of Z [19, 2.1] but then the Penrose transform identifies the obstruction space $H^2(\Theta_Z \otimes K_Z^{1/2})$ with the space of holomorphic vector fields Ξ on M such that the derivative of the restricted Futaki invariant of the Kähler class in the direction of Ξ vanishes [14, proof of Theorem 2.7]; in particular there are no obstructions when M has no nontrivial holomorphic vector fields.

Using the notation $\mathcal{I}_{D\bar{D}} \cong K_Z^{1/2}$ we can rephrase their results in

Proposition 5.1 [14]. *Let M be a compact scalar-flat Kähler surface with $c_1^{\mathbb{R}}(M) \neq 0$ and twistor space Z. The following statements are equivalent*:

(1) $H^2(\Theta_Z \otimes K_Z^{1/2}) \neq 0$.
(2) *M is a minimal ruled surface of genus $g \geq 2$ with nontrivial holomorphic vector fields.*
(3) *M is biholomorphic to the projectivization $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to \Sigma_g$ of a split rank-2 holomorphic vector bundle over a Riemann surface of genus $g \geq 2$ and $\deg \mathcal{L} = 0$.*

Proof. The Matsushima-Lichnerowicz and Futaki obstructions must vanish; therefore, the first two statements are equivalent by [14, 2.7, 3.5, and 3.1], while the equivalence between (2) and (3) follows from [14, 3.1 and 3.4]. q.e.d.

We can now state our main result.

Theorem 5.2. *Let M be a compact scalar-flat Kähler surface with $c_1^{\mathbb{R}}(M) \neq 0$ which does not satisfy the equivalent conditions of* 5.1. *Then the blow-up of M at any collection of points (distinct or not) and any of its small deformations admit scalar-flat Kähler metrics.*

Proof. By induction it is enough to show that $\tilde{M}$, the blow-up of M at one point, admits scalar-flat Kähler metrics because $c_1^{\mathbb{R}}(\tilde{M}) \neq 0$ and $\tilde{M} \not\cong \mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to \Sigma_g$.

To do this we show that M satisfies the hypothesis of Theorem 4.6. This is an easy task and the following arguments can also be found in [14].

Of course $H^2(\Theta_Z \otimes K_Z^{1/2}) \neq 0$ holds by hypothesis, while $H^2(\Theta_M) = 0$ for any ruled surface by Serre duality and restriction to the rational curves of trivial normal bundle.

Next we use two standard exact sequences on the twistor space Z of M:

$$0 \longrightarrow \Theta_Z \otimes \mathcal{I}_{D\bar{D}} \longrightarrow \Theta_{Z,D\bar{D}} \longrightarrow \Theta_{D\bar{D}} \longrightarrow 0$$

$$0 \longrightarrow \Theta_{Z,D\bar{D}} \longrightarrow \Theta_Z \longrightarrow \nu_{D\bar{D}/Z} \longrightarrow 0.$$

From the first one we infer that $H^2(\Theta_{Z,D\bar{D}}) = 0$ because $H^2(\Theta_{D\bar{D}}) \cong H^2(\Theta_M) \oplus H^2(\Theta_{\bar{M}})$ which vanishes. But then $H^2(\Theta_Z) = 0$ from the second sequence since by [19, 3.1] $\nu_{D\bar{D}/Z}$ is isomorphic to $K_M^{-1} \oplus K_{\bar{M}}^{-1}$ whose second cohomology vanishes by Serre duality and Yau's vanishing theorem [26, Corollary 2].

Remarks 5.3. (a) The order in which the two operations of deforming and blowing up are carried out is not important because of the stability of (-1)-lines and also the exact sequence $H^1(\Theta_{\tilde{M}}) \to H^1(\Theta_M) \to 0$.

(b) Our result includes Theorem 3.10 in [14] by just taking the collection of points to be empty.

Sometimes the obstruction that we found cannot be overcome, and the following observation means that the hypothesis of the previous theorem cannot be relaxed:

Remark 5.4. On the contrary to 5.2 suppose $M \cong \mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to \Sigma_g$ is scalar-flat Kähler with deg $\mathcal{L} = 0$. Let $\tilde{M}$ be the blow-up of M at $m \geq 1$ points lying on the zero section of $\mathcal{L}$ so that $H^0(\Theta_{\tilde{M}}) \neq 0$ with Euler field Ξ. Then $\tilde{M}$ cannot admit scalar-flat Kähler metrics because the Futaki invariant $\mathcal{F}(\Xi, [\omega]) \neq 0$ for any admissible Kähler class $[\omega]$ [14, Corollary 3.4 (b)].

In the next application we give new examples of complex surfaces which admit scalar-flat Kähler metrics. We start by recalling the construction of their minimal models.

If Σ_g is a Riemann surface of genus $g \geq 2$ with hyperbolic metric of curvature -1, and $\mathbb{CP}_1$ is given the metric of curvature $+1$, then the product metric on the complex surface $S_0 = \Sigma_g \times \mathbb{CP}_1$ is scalar-flat Kähler. For its twistor space Z_0 one has $H^2(\Theta_{Z_0} \otimes K_{Z_0}^{1/2}) \cong \mathbb{C}^3$ by [14, p. 296], and in fact the blow-up of S_0 at one point cannot admit Kähler metrics of constant scalar curvature because the group of biholomorphisms is not reductive (Matsushima-Lichnerowicz obstruction [14, p. 282]). On the other hand, it should be noticed here, especially in view of 5.2, that one can blow up two or more points on S_0 to obtain a scalar-flat Kähler surface [9].

As a result one may wonder whether the *smooth* 4-manifold $(\Sigma_g \times \mathbb{CP}_1)\#\overline{\mathbb{CP}}_2$ admits scalar-flat Kähler metrics. We give a positive answer below.

The construction is as follows: let S be a unitary, flat $\mathbb{CP}_1$-bundle over Σ_g, that is, we have a representation of $\pi_1(\Sigma_g)$ into $SU(2)$. If E is the associated flat unitary bundle, then we have that $S \cong \mathbb{P}(E)$ holomorphically, and furthermore $H^0(\Theta_S) = 0$ if E does not split as the sum of two line bundles; also notice that E is stable exactly when the representation is irreducible, by a celebrated theorem of Narasimhan and Seshadri [16]. Now the local product metric is scalar-flat Kähler just as in the product case, and S is homeomorphic to $\Sigma_g \times \mathbb{CP}_1$ (however the Kähler classes are usually different as pointed out in [3]); this leads to the following observation:

Proposition 5.5. *Let $M_0 = \mathbb{P}(E_0)$ be the projectivization of a rank-2 vector bundle of degree zero over a compact Riemann surface Σ_g^0 of genus $g \geq 2$. Let $\{M_t\}_{t\in T}$ be a versal family of deformations of M_0. Then there exists an open dense subset $T' \subset T$ such that $M_{t'}$ admits scalar-flat Kähler metrics, and $H^0(M_{t'}, \Theta) = 0$ for each $t' \in T'$.*

Proof. Given a vector bundle E over a compact Riemann surface Σ_g of genus $g \geq 2$, a result of Narasimhan-Ramanan [15, Proposition 2.6] states that stable bundles fill up an open dense set in the versal deformation of E over Σ_g. In our case these stable bundles will correspond to scalar-flat Kähler surfaces without holomorphic vector fields in the versal deformation of M_0, by the above-mentioned theorem of Narasimhan-Seshadri.

The result then follows from the relations among the deformations of E_0, Σ_g^0 and M_0; see [6, (2.9)] for example. q.e.d.

By contrast, when $M = \mathbb{P}(E)$ with $\deg E = 1$, there are no scalar-flat Kähler surfaces in the versal deformation of M, for example, because of the classification given in [20]. For minimal surfaces this is the fairly complete picture that has probably been known for some time, and we are now ready to apply 5.2 to gain informations on the nonminimal case.

Corollary 5.6. *Let $M_{t'}$ be as above. Then there exist scalar-flat Kähler metrics on the blow-up of $M_{t'}$ at any collection of points.*

In particular, since the surfaces M_t are all topologically trivial, one has

Corollary 5.7. *For any $n \geq 0$, there exist scalar-flat Kähler metrics on the smooth 4-manifold $(\mathbb{CP}_1 \times \Sigma_g)\#n\overline{\mathbb{CP}}_2$.*

In fact the previous examples are generic in the following sense.

Theorem 5.8. *In any versal deformation of nonminimal ruled surfaces of genus $g \geq 2$ there exists an open dense subset of scalar-flat Kähler surfaces.*

Proof. First notice that the minimal model of a nonminimal ruled surface is not unique and can always be assumed to be the projectivization of a degree-0 vector bundle. If $\mathfrak{M} = \{M_t\}_{t \in T}$ is such a versal family, then we can use the techniques of 2.6 to simultaneously blow down the exceptional curves of each M_t and obtain a family $\check{\mathfrak{M}} = \{\check{M}_t\}_{t \in T}$ where each $\check{M}_t$ is the projectivization of a rank-2 vector bundle of degree-0. The statement thus follows from 5.5 because of the surjection $H^1(\Theta_{\check{M}_t}) \to H^1(\Theta_{M_t})$. q.e.d.

A final application in the case that M has a trivial Lie algebra of biholomorphisms comes from the work of LeBrun-Simanca [13, Corollary 1] where they prove that scalar-flat Kähler metrics can sometimes be perturbed to produce Kähler metrics of arbitrary constant scalar curvature.

Corollary 5.9. *When $H^0(\Theta_M) = 0$, there exist Kähler metrics with constant scalar curvature of any sign on the complex surfaces of Theorem 5.2.*

Next we point out a restatement of 5.2 in the case where $H^0(\Theta_M) \neq 0$ but M is nonminimal; it is related to the notion of parabolic stability introduced by Seshadri; see, for example, [14, p. 306 and Corollary 3.9].

Corollary 5.10. *Suppose that M is obtained by blowing up $m \geq 1$ points on the zero section of $\mathscr{L}$ in $\mathbb{P}(\mathscr{L} \oplus \mathscr{O})$ and that the corresponding parabolic bundle is quasi-stable. Then any blow-up of M and any of its small deformations admit scalar-flat Kähler metrics.*

The following result appears in [14, main theorem], whose proof involves a deformation argument after constructing explicit examples. Here we present a more direct argument using 5.2.

Corollary 5.11 [14]. *Let M be any ruled surface of genus $g \geq 2$. Then the blow-up of M at sufficiently many points admits scalar-flat Kähler metrics.*

Proof. Any two ruled surfaces of the same genus are bimeromorphically equivalent [5]; therefore, some blow-up of M can also be obtained by blowing up points on the projectivization of a stable rank-2 vector bundles of degree 0. This surface admits scalar-flat Kähler metrics with no holomorphic vector fields and we can now apply Theorem 5.2. q.e.d.

We conclude the paper with some open questions:

Remark 5.12. The main open problem about scalar-flat Kähler surfaces is to determine whether a ruled surface S of genus $g \leq 1$ can admit such metrics. It is known however that these surfaces cannot be minimal [4, Theorem 4]; therefore, by Theorem 5.2, if S is a ruled surface of genus $g = 0, 1$ admitting a scalar-flat Kähler metric, the same will be true for the blow-up of S at any number of points. Furthermore provided that one has a *single* example of a scalar-flat Kähler surface of genus 0 and 1, Corollary 5.11 would hold without any genus assumption, and Conjecture 1 in [14] would therefore be proven.

Remark 5.13. It is plausible that our construction can work in the obstructed case too, i.e., when $H^2(\Theta_Z \otimes \mathscr{I}_{D\bar{D}}) \neq 0$; see [6] and 5.4. The interested reader may be able to pursue this to reconstruct all the scalar-flat Kähler metrics with holomorphic vector fields.

Acknowledgment

We wish to express our gratitude to M. S. Narasimhan and Claude LeBrun for their help and interest in this work, and to Alberto Verjovsky and D. S. Nagaraj for discussions and encouragement; we also thank I.C.T.P. (International Center for Theoretical Physics) and S.I.S.S.A. (Scuola Internazionale di Studi Superiori Avanzati) for providing us with an ideal environment.

References

[1] M.F. Atiyah, N.J. Hitchin & I.M. Singer, *Self-duality in four dimensional riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978) 425-461.

[2] W. Barth, C. Peters & A. Van de Ven, *Compact complex surfaces*, Springer, Berlin, 1984.

[3] A. Besse, *Einstein manifolds*, Springer, Berlin, 1987.

[4] D. Burns & P. de Bartolomeis, *Stability of vector bundles and extremal metrics*, Invent. Math. **92** (1988) 403-407.

[5] C.P. Boyer, *Conformal duality and compact complex surfaces*, Math. Ann. **274** (1986) 517-526.

[6] S.K. Donaldson & R. Friedman, *Connected sums of self-dual manifolds and deformations of singular spaces*, Nonlinearity **2** (1989) 197-239.

[7] A. Fujiki & S. Nakano, *Supplement to "On the inverse of monoidal transformations"*, Publ. Res. Inst. Math. Sci. 7 (1971/72) 637-644.

[8] C. LeBrun, *Explicit self-dual metrics on* $\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2$, J. Differential Geometry **34** (1991) 223-253.

[9] ——, *Scalar-flat Kähler metrics on blown-up ruled surfaces*, J. Reine Angew. Math. **420** (1991) 161-177.

[10] ——, *Anti-self-dual hermitian metrics on the blow-up Hopf surface*, Math. Ann. **289** (1991) 383-392.

[11] C. LeBrun & Y. S. Poon, *Twistors, Kähler manifolds and bimeromorphic geometry*. I, J. Amer. Math. Soc. **5** (1992) 289-316.

[12] ——, *Self-dual manifolds with symmetry*, preprint, 1990.

[13] C. LeBrun & S. Simanca, *On Kähler surfaces of constant positive scalar curvature*, J. Geom. Anal. (to appear).

[14] C. LeBrun & M. Singer, *Existence and deformation theory for scalar-flat Kähler metrics on compact complex surfaces*, Invent. Math. **112** (1993) 273-313.

[15] M.S. Narasimhan & S. Ramanan, *Deformations of the moduli space of vector bundles over an algebraic curve*, Ann. of Math. (2) **101** (1975) 391-417.

[16] M.S. Narasimhan & C.S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2) **82** (1965) 540-567.

[17] Pontecorvo, M., Ph.D. thesis, Stony Brook, 1989.

[18] ——, *Hermitian surfaces and a twistor space of algebraic dimension* 2, Proc. Amer. Math. Soc. **113** (1991) 177-186.

[19] ——, *On twistor spaces of anti-self-dual hermitian surfaces*, Trans. Amer. Math. Soc. **331** (1992) 653-661.

[20] ——, *Uniformization of conformally flat hermitian surfaces*, Differential Geom. Appl. **2** (1992) 295-305.

[21] Y.S. Poon, *Compact self-dual manifolds with positive scalar curvature*, J. Differential Geometry **24** (1986) 97-132.

[22] ——, *On the algebraic structure of twistor spaces*, J. Differential Geometry **36** (1992) 451-491.

[23] Z. Ran, *Deformation of maps*, Algebraic Geometry Proceedings (Trento, 1988). Lecture Notes in Math. (E. Ballico & C. Ciliberto, eds.), vol. 1389, Springer, Berlin, 247–253.

[24] ——, *Stability of certain holomorphic maps*, J. Differential Geometry **34** (1991) 37-47.

[25] C. Taubes, *The existence of anti-self-dual metrics*, J. Differential Geometry **36** (1992) 163–253.

[26] S.T. Yau, *On the curvature of compact Hermitian manifolds*, Invent. Math. **25** (1974) 213-239.

[27] ——, *On the Ricci curvature of a complex Kähler manifold and complex Monge-Ampere equations*, Comm. Pure Appl. Math. **31** (1978) 339-411.

I.C.T.P.
S.I.S.S.A. and Universita' di Napoli