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THE MAXIMUM PRINCIPLE FOR HYPERSURFACES WITH VANISHING CURVATURE FUNCTIONS

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Abstract

We extend the maximum principle, known for hypersurfaces of a Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$ with positive constant curvature function $\sigma_k = c > 0$, to a generic class of hypersurfaces with vanishing curvature $\sigma_k = 0$, $1 \leq k < n$. Using the Alexandrov reflection method, this result can be extended to hypersurfaces with vanishing curvature function having certain symmetry and uniqueness properties that were known for minimal surfaces.

0. Introduction

Consider a hypersurface S of $\mathbb{R}^{n+1}$ with principal curvature vector $\vec{\kappa} = (\kappa_1, \cdots, \kappa_n)$ satisfying the relation

$$\sigma_k(\kappa_1, \cdots, \kappa_n) = 0, \tag{0.1}$$

where σ_k, $1 \leq k \leq n$, is the kth elementary symmetric function. The function σ_k is positive on the positive cone $\Gamma \subset \mathbb{R}^n$ defined by $\kappa_i > 0$, $i = 1, \cdots, n$. The connected component of the set $\sigma_k > 0$ that contains Γ is an open convex cone C. A well-known maximum principle states that if two hypersurfaces S and S' with curvatures $\vec{\kappa}$ and $\vec{\kappa}'$ satisfying $\sigma_k(\vec{\kappa}) = f(x)$, $f > 0$, are tangent at a point p with normal vectors pointing in the same direction and both $\vec{\kappa}(p)$ and $\vec{\kappa}'(p)$ belong to C, then one hypersurface cannot remain above the other in a neighborhood of p unless they coincide in a full neighborhood of p. This principle is used, for instance, in Alexandrov's reflection method [1], [4]. The main fact there is that the hypothesis $f > 0$ causes the local relevant PDE to be elliptic. When dealing with the first curvature function σ_1, i.e., the mean curvature function, the differential equation is quasilinear and elliptic for arbitrary f, and this allows the application of Alexandrov's method to minimal surfaces [6].

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In contrast with the case of flat hypersurfaces that satisfy (0.1) for all $k > 1$ and provide plenty of examples of distinct pairs of tangent surfaces, one above the other, the second author showed that for hypersurfaces of $\mathbb{R}^4$ with zero scalar curvature (that is, satisfying (0.1) for $k = 2$ and $n = 3$), equation (0.1) is elliptic precisely at the points where the Gauss map has rank > 1, and derived a maximum principle [5]. Here we extend this result to arbitrary k and n showing that a maximum principle holds for equation (0.1) provided that the rank of the Gauss map of one of the surfaces at the point p is $> k - 1$. This condition on the rank cannot be removed. We now describe the result more precisely.

The subset of $x \in \mathbb{R}^n$ where $\sigma_k(x) = 0$ can be decomposed as the union of k continuous leaves $Z_1, \cdots, Z_k$. Any straight line parallel to the vector $a = (1, \cdots, 1)$ intersects $\sigma_k(x) = 0$ precisely at k points (counted with multiplicity). If Π is the plane $x_1 + \cdots + x_n = 0$ orthogonal to a and $p \in Z_j$, p can be written uniquely as $p = x - \lambda_j(x)a$, $x \in \Pi$, and Z_j may be identified with the graph of a continuous function $-\lambda_j(x)$, $x \in \Pi$, if we label the numbers $\lambda_1(x) \leq \cdots \leq \lambda_k(x)$. Then, Z_1 and Z_r are, respectively, the boundary of the cones C and $-C$.

Theorem 0.1. (a) *Let S and S' be hypersurfaces of $\mathbb{R}^{n+1}$ satisfying equation* (0.1) *for some $1 \leq r < n$. Assume that they are tangent at p, with normal vectors pointing in the same direction, and that $\vec{\kappa}(p)$ and $\vec{\kappa}'(p)$ belong to the same leaf. Then, if S remains on one side of S' and the rank of the Gauss map of either S or S' is $\geq r$, S and S' must coincide in a neighborhood of p. In particular, a generic S cannot remain on one side of its tangent plane.*

(b) *Let S and S' be hypersurfaces of $\mathbb{R}^{n+1}$ with boundaries ∂S and $\partial S'$ satisfying equation* (0.1) *for some $1 \leq r < n$. Assume that $p \in \partial S \cap \partial S'$, that both S and S' as well as ∂S and $\partial S'$ are tangent at p, with normal vectors point in the same direction, and that $\vec{\kappa}(p)$ and $\vec{\kappa}(p)$ belong to the same leaf. If S remains on one side of S' and the rank of the Gauss map of either S or S' is $\geq r$, then S and S' must coincide in a neighborhood of p.*

Theorem 0.2. (a) *Let S and S' be hypersurfaces of $\mathbb{R}^{n+1}$ and let $1 \leq r < n$. Assume that S satisfies equation* (0.1) *and that the curvature function σ_r' of the surface S' satisfies $\sigma_r' \geq 0$. Assume that S and S' are tangent at p, that $\kappa'(p)$ belongs to $Z_1 \cup C = \overline{C}$ and the rank of the Gauss map of either S or S' is $\geq r$. If S remains above S' (in the sense of the normal of S'), then S and S' must coincide in a neighborhood of p.*

(b) *Let S and S' be hypersurfaces of $\mathbb{R}^{n+1}$ with boundaries ∂S and*

$\partial S'$ and let $1 \leq r < n$. Assume that S satisfies equation (0.1) and that the curvature function σ_k' of the surface S' satisfies $\sigma_r' \geq 0$. Assume that S and S' as well as their boundaries are tangent at $p \in \partial S \cap \partial S'$, that $\kappa'(p)$ belongs to $Z_1 \cup C = \overline{C}$, and that the rank of the Gauss map of either S or S' is $\geq r$. If S remains above S' (in the sense of the normal of S'), then S and S' must coincide in a neighborhood of p.

The main step in the proof of the theorems is showing that if S is expressed as the graph of a function u in a neighborhood of p, then the nonlinear equation $G_r(D^2u, Du) = 0$ representing (0.1) is elliptic for u at p if and only if the rank of the Gauss map of S at p is $\geq r$. This follows from the properties of the symmetric functions which are shown in §1 and §2. The theorems are proved in §3. Notice that in the second theorem it is not required that the normal vectors point in the same direction.

Theorem 0.1 is useful for deriving symmetries by comparing different portions of the same hypersurface satisfying (0.1) (obtained, for instance, by reflection as in Alexandrov's method). Theorem 0.2 can be used, for instance, to compare the hypersurface with a cylinder having nonnegative principal curvatures. These are the basic operations in the applications of Alexandrov's principle made by Schoen to derive symmetries and embeddedness of minimal surfaces. In §4 we illustrate how Schoen's ideas can be applied to derive similar symmetry results for embedded hypersurfaces of zero higher-order curvatures. Unlike Schoen, we must require embeddedness, since hypersurfaces satisfying $\sigma_r = 0$, $1 < r < n$, do have distinguished sides locally. Among the symmetry theorems presented in §4, an easy one to state is

Corollary 4.3. *Let B be the union of two $(n-1)$-spheres lying in parallel hyperplanes of $\mathbb{R}^{n+1}$, with the line l joining their centers being orthogonal to these hyperplanes. Then any $\sigma_r = 0$ embedded hypersurface M spanning B, whose Gauss map has rank at least r everywhere, is of revolution with axis l.*

1. Hyperbolic polynomials

Here we summarize some of the results of Gårding [3] concerning hyperbolic polynomials that will be useful for us. A homogeneous polynomial P of degree $m > 0$ on $\mathbb{R}^n$ is said to be hyperbolic with respect to $a \in \mathbb{R}^n$ if the equation $P(sa + x) = 0$ has m real zeros for every $x \in \mathbb{R}^n$. This implies that $P(a) \neq 0$ and we have the factorization

$$P(sa+x) = P(a)\prod_{1}^{m}(s+\lambda_j(a,x)),$$

where we may order the roots decreasingly, $\lambda_j(a,x) \le \lambda_{j+1}(a,x)$, $j = 1, \cdots, m-1$. It turns out that $P(x)/P(a)$ has real coefficients and we may restrict ourselves to real polynomials. The functions $\lambda_j(a,x)$ are continuous functions of x, homogeneous of degree one. The set $C = C(P,a) = \{x\colon \lambda_1(a,x) > 0\}$ is an open convex cone and coincides with the connected component of $P \neq 0$, that contains a. It is convenient to assume that $P(a) > 0$; in this case, C is the component of $P > 0$, that contains a. An important property is that P is also hyperbolic with respect to b for every $b \in C$ and $C(P,a) = C(P,b)$. If P is hyperbolic with respect to a so is

$$Q(x) = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} P(x), \tag{1.1}$$

and $C(P,a) \subset C(Q,a)$. Starting from the polynomial

$$P(x) = \sigma_n(x) = x_1 \cdots x_n$$

—obviously hyperbolic with respect to $a = (1, \cdots, 1)$—and applying repeatedly (1.1) we see that the elementary symmetric functions σ_r, $r = 1, \cdots, n$ are hyperbolic with respect to a.

Returning to a general hyperbolic polynomial of degree m, we see that the variety $P = 0$ can be written as the union of the m sets

$$Z_j = \{x - \lambda_j(a,x)a\colon x \in \Pi\}, \qquad j = 1, \cdots, m,$$

where Π is the hyperplane through the origin orthogonal to a. The leaves Z_j are independent of the choice of $a \in C$.

The functions λ_j satisfy the important inequality

$$\lambda_j(a,x) < \lambda_j(a,x+b), \qquad x \in \mathbb{R}^n, \ b \in C. \tag{1.2}$$

Lemma 1.1. *Let P be a real hyperbolic homogeneous polynomial of degree $m > 0$ in $\mathbb{R}^n$ and let y be a root of P. There exists $\varepsilon > 0$ such that the function*

$$C \ni b \mapsto P(y+b), \qquad b \in C, \ |b| < \varepsilon,$$

does not change sign. Hence, $\nabla P(y) \cdot b$ does not change sign for $b \in \overline{C}$.

Proof. Write $y = x + sa$, $x \in \Pi$, $s = -\lambda_j(a,x)$. Then,

$$P(y+b) = P(sa+x+b) = P(a)\prod_{i=1}^{m}(s+\lambda_i(a, x+b))$$
$$= P(a)\prod_{i=1}^{m}(\lambda_i(a, x+b) - \lambda_i(a, x) + \lambda_i(a, x) - \lambda_j(a, x)).$$

For those values of i such that $\lambda_i(a, x) = \lambda_j(a, x)$, the corresponding factor does not change sign for $b \in C$ in view of (1.2). If $\lambda_i(a, x) - \lambda_j(a, x) \neq 0$, we have that $|\lambda_i(a, x+b) - \lambda_i(a, x)| < |\lambda_i(a, x) - \lambda_j(a, x)|$ for small b so those terms do not change sign either.

Lemma 1.2. *Let $y \neq w$ belong to a leaf Z_j of $P = 0$, and assume that $w - y \in \overline{C}$. Then, $w - y \in \partial C$ and P vanishes on the line determined by y and w.*

Proof. Fix $a \in C$ and write $\lambda_j(x) = \lambda_j(a, x)$. Then, $y \in Z_j$ if and only if $\lambda_j(y) = 0$. We know from (1.2) that $c = w - y$ cannot belong to the open cone C so it must lie in its point set boundary ∂C. Let c_k be a sequence in C, converging to c. For $0 < t < 1$, (1.2) implies that $\lambda_j(y) < \lambda_j(y + tc_k) < \lambda_j(y + c_k)$. Letting $k \to \infty$ we get that $\lambda_j(y + t(w-y)) = 0$, so P vanishes on the segment $[y, w]$ and therefore on the whole line.

Consider now a homogeneous polynomial P, hyperbolic with respect to $a = (1, \cdots, 1)$ and assume that the positive cone $\Gamma = \{x : x_i > 0\}$ is contained in $C = C(P, a)$. This happens, for instance, for the elementary symmetric functions σ_r. If $P(x) = 0$ it follows from Lemma 1.1 that $\nabla P(x) \cdot b$ does not change sign for $b \in \overline{\Gamma}$; in particular, either

$$(1.3)\qquad \begin{aligned} &\frac{\partial P(x)}{\partial x_i} \geq 0, \quad i = 1, \cdots, n, \qquad \text{or} \\ &\frac{\partial P(x)}{\partial x_i} \leq 0, \quad i = 1, \cdots, n. \end{aligned}$$

We say that x is an elliptic root if $P(x) = 0$, and either $\partial_j P(x) > 0$, $j = 1, \cdots, n$ or $\partial_j P(x) < 0$, $j = 1, \cdots, n$; i.e., either $\nabla P(x)$ or $-\nabla P(x)$ belongs to Γ.

Lemma 1.3. *Let P be a real homogeneous polynomial hyperbolic with respect to $a = (1, \cdots, 1)$, and assume that $\Gamma \subset C(P, a)$. If y, w belong to a leaf Z_j, $w - y \in \overline{\Gamma}$ and either y or w is an elliptic root, then $y = w$.*

Proof. Say y is an elliptic root. Since $\overline{\Gamma} \subset \overline{C}$, it follows from Lemma 1.2 that P vanishes on the segment $[y, w]$. We also have that $y_j \leq w_j$, $j = 1, \cdots, n$ with strict inequality for some j if $y \neq w$. If $\phi(s) = P(y + s(w-y))$, then $\phi'(0) = \nabla P(y) \cdot (w - y) \neq 0$, contradicting the fact that P vanishes on $[y, w]$.

2. Elementary symmetric functions

As remarked in §1, the elementary symmetric function

$$\sigma_r(x) = \sum_{i_1<\cdots<i_r} x_{i_1}\cdots x_{i_r}$$

is a hyperbolic polynomial with respect to $a = (1, \cdots, 1)$, positive on the positive cone $\Gamma = \{x : x_i > 0\}$. We say that a point $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ has rank r if exactly r components of x do not vanish, for instance, points in Γ have rank n and if a point has rank 0 it is the origin.

Lemma 2.1. *Let $x \in \mathbb{R}^n$, $2 \leq r \leq n$, and assume that $\sigma_r(x) = \sigma_{r-1}(x) = 0$. Then the rank of x is $\leq r-2$.*

Proof. We use induction on $r \geq 2$. If $r = 2$ we must show that $x = 0$ when $\sigma_2(x) = \sigma_1(x) = 0$. Since $|x|^2 = \sigma_1^2 - 2\sigma_2(x)$, the result follows. Now let $r \geq 3$ and assume that the result has been proved for $r-1$ and all $n \geq r-1$. With a slight abuse of notation we may write

$$\begin{aligned} \sigma_r(x) &= \sigma_r(x_1, \cdots, \hat{x}_j, \cdots, x_n) + x_j\sigma_{r-1}(x_1, \cdots, \hat{x}_j, \cdots, x_n) \\ &= 0, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \sigma_{r-1}(x) &= \sigma_{r-1}(x_1, \cdots, \hat{x}_j, \cdots, x_n) + x_j\sigma_{r-2}(x_1, \cdots, \hat{x}_j, \cdots, x_n) \\ &= 0, \end{aligned} \tag{2.2}$$

where $\hat{x}_j$ indicates that the jth component x_j has been omitted. Substitution of (2.2) in (2.1) yields

$$\begin{aligned} \sigma_r(x_1, \cdots, \hat{x}_j, \cdots, x_n) &= x_j^2\sigma_{r-2}(x_1, \cdots, \hat{x}_j, \cdots, x_n) \\ &= x_j^2\frac{\partial\sigma_{r-1}}{\partial x_j}(x). \end{aligned} \tag{2.3}$$

Adding the identities (2.3) for $j = 1, \cdots, n$ we obtain

$$0 = (n-r)\sigma_r(x) = \sum_{j=1}^n \sigma_r(x_1, \cdots, \hat{x}_j, \cdots, x_n) = \sum_{j=1}^n x_j^2\frac{\partial\sigma_{r-1}}{\partial x_j}(x). \tag{2.4}$$

From (1.3) applied to $P = \sigma_{r-1}$, all partial derivatives in (2.4) have the same sign so each term must vanish. Assume without loss of generality that $x_n \neq 0$. Then, $x_n^2\sigma_{r-2}(x_1, \cdots, x_{n-1}) = 0$ implies that the second factor $\sigma_{r-2}(x_1, \cdots, x_{n-1})$ vanishes. Again, (2.2) for $j = n$ shows that $\sigma_{r-1}(x_1, \cdots, x_{n-1}) = 0$. By the inductive hypothesis the rank of $(x_1, \cdots, x_{n-1})$ is $\leq r-3$ and this yields that the rank of x is $\leq r-2$.

Corollary 2.2. *Let $x \in \mathbb{R}^n$, $2 \le r \le n$, and assume that for some $1 \le j \le n$*

$$\sigma_r(x) = \frac{\partial \sigma_r}{\partial x_j}(x) = 0. \tag{2.5}$$

Then the rank of x is $\le r-1$.

Proof. Assuming without loss of generality that $j = n$ and writing $x' = (x_1, \cdots, x_{n-1})$, from the identity

$$\sigma_r(x) = \sigma_r(x') + x_n \frac{\partial \sigma_r}{\partial x_n}(x)$$

and (2.5) we obtain that $\sigma_r(x') = 0$, and we are also assuming that $\partial \sigma_r / \partial x_n(x) = \sigma_{r-1}(x') = 0$. Now, it follows from Lemma 2.1 that the rank of x' is $\le r-2$, so the rank of x is $\le r-1$.

Remark. If $x \in \mathbb{R}^n$ has rank equal to r, then $\sigma_r(x) \ne 0$. Thus, if $\sigma_r(x) = 0$ the rank of x is either $< r$ or $> r$.

Corollary 2.3. *Let $x \in \mathbb{R}^n$, $1 \le r \le n$, and assume that $\sigma_r(x) = 0$ and the rank of x is $\ge r$ (this can only happen if $r < n$). Then, the rank of $\nabla \sigma_r(x)$ is n; i.e., no partial derivative of σ_r of order one vanishes at x and x is an elliptic root. Conversely, if the rank of x is $< r$, x is not an elliptic root.*

Proof. If $r \ge 2$, this follows from Corollary 2.2 and the remarks made before Lemma 1.3. The case $r = 1$ is immediate. Finally, the last statement is readily verified: if the rank of x is $j < r$, at lest $n-r+1$ components of x vanish, say, $x = (x_1, \cdots, x_{r-1}, 0, \cdots, 0)$ so all monomials in $\sigma_r(x)$ and $\partial_1 \sigma_r(x)$ vanish.

3. Proofs of the theorems

The proofs follow from the standard arguments that we now recall. We first prove (a) of the first theorem. We choose local coordinates centered at the point of tangency of S and S' so that they are expressed as the graphs of functions u and u' with $\nabla u(0) = \nabla u'(0) = 0$. Condition (0.1) is expressed by a nonlinear equation

$$G_r(D^2 u, Du) = 0, \tag{3.1}$$

that both u and u' satisfy. We want to show that if $u \ge u'$ (or $u \le u'$), u and u' cannot be distinct. At the origin, the components of $w = \vec{\kappa}(0)$ and $y = \vec{\kappa}'(0)$ are the ordered eigenvalues of the Hessian matrix of u and u' respectively. Notice that the rank of the Gauss map of S (S')

is precisely the rank of w (y) as a point of $\mathbb{R}^n$. If $u \geq u'$ we have that $D^2u(0) \geq D^2u'(0)$ as matrices, so the minimax characterization of eigenvalues (or an application of (1.2) for a suitable P; cf. [3]) shows that $w - y = \vec{\kappa}(0) - \vec{\kappa}'(0) \in \overline{\Gamma}$. Since either y or w is an elliptic root of σ_r and both belong to the same leaf of $\sigma_r = 0$, Lemma 1.3 implies that $\vec{\kappa}(0) = \vec{\kappa}'(0)$. Thus, $D^2u(0) = D^2u'(0)$ and the linearizations of G_r at u and u' coincide at 0. It is known [2] that (3.1) is elliptic for u at 0 precisely when $\vec{\kappa}(0)$ is an elliptic root of σ_r (in particular, (3.1) is elliptic at the origin for both u and u'). Therefore, the difference $v = u' - u$ satisfies a linear elliptic equation

$$\sum a_{ij}\partial_i\partial_j v + b_i\partial_i v = 0$$

in a neighborhood of the origin and, assuming a maximum at an interior point, has to be constant and thus it has to vanish.

The proof of (b) is essentially the same except that this time the difference $v = u' - u$ will assume a maximum at a boundary point instead of an interior point. Since the boundary is smooth the conclusion now follows from the boundary point lemma.

Finally, we prove (a) of Theorem 0.2 and leave part (b) to the reader. First observe that if we change the orientation of a normal of S, the new surface will still satisfy the hypotheses of the theorem, so we may assume that the normals of S and S' are equally oriented. If S and S' are locally represented by the graphs of u and u', with horizontal common tangent plane at the origin and normal pointing upwards, we obtain as before from $u \geq u'$ that $\vec{\kappa}(0) - \vec{\kappa}'(0) \in \overline{\Gamma} \subset Z_1 \cup C$. Hence, it follows that $\sigma_r\vec{\kappa}(0)) \geq \sigma_r\vec{\kappa}'(0))$ which implies that $\sigma_r(\vec{\kappa}'(0)) = 0$. Thus, $\vec{\kappa}'(0) \in \partial C = Z_1$ and so does $\vec{\kappa}(0)$ because Z_1 is above all the other leaves Z_j in the order given by C. Since both curvature vectors belong to the same leaf, we may apply part (a) of the previous theorem and the proof is finished.

4. Symmetry of a class of surfaces

We will follow Schoen's steps in [6], pointing out the necessary adjustments in order to apply Theorems 0.1 and 0.2 to the geometric situation $\sigma_r = 0$, $1 < r < n$. It is important to observe that the symmetry theorems, valid for $\sigma_1 = 0$, will be proved here: (i) for $\sigma_r = 0$ generically, since the maximum principles presented here hold under a generic assumption, namely, that the rank of the normal map is $> r - 1$; (ii) for embedded hypersurfaces, since some of the arguments used by Schoen do not necessarily work when $r > 1$, as we remark after the proof of Theorem 4.1.

Following the notation in [6, §1], $B = B^{n-1} \subset \mathbb{R}^{n+1}$ denotes a compact embedded C^2 boundary of dimension $n-1$, and M^n is a smooth embedded generic (in the above sense) $\sigma_r = 0$ hypersurface in $\mathbb{R}^{n+1}$ with $\partial M = B$; $\Omega \subset \mathbb{R}^n$ denotes an open bounded connected set in $\mathbb{R}^n$, whose boundary $\partial\Omega$ is connected and has curvature vector in $\overline{C}$, in particular $\sigma_r \geq 0$, with respect to the inward pointing unitary normal. Observe that $\sigma_r \geq 0$, r odd, implies that $\partial\Omega$ is connected. The meanings of Π_t, Σ_t, Σ_{t+}, Σ_{t-}, and Σ^* remain the same, as well as the definition of graph with locally bounded slope and the notion of a set being above another one, which is denoted by $A \geq B$.

Finally, we keep the numbering of theorems and corollaries in [6], so Corollary 4.3 stated in the Introduction here corresponds to Corollary 3 there.

Theorem 4.1. *Let B and Ω as above, such that* (i) $B \subset \partial\Omega \times \mathbb{R}$, (ii) B_{0+} *is a graph with locally bounded slope, and* (iii) $B^*_{0+} \geq B_{0-}$. *If M is any embedded surface with $\partial M = B$ having all interior points inside $\Omega \times \mathbb{R}$, and satisfies $\sigma_r = 0$ with rank of the Gauss map at least r everywhere, then* (i) M_{0+} *is a graph with locally bounded slope, and* (ii) $M^*_{0+} \geq M_{0-}$.

Remark. The hypothesis that $\operatorname{int}(M) \subset \Omega \times \mathbb{R}$ is not restrictive. Indeed, let S be a connected component of M having an interior point outside $\Omega \times \mathbb{R}$, so S will be tangent to an outer cylinder homothetic to $\partial\Omega \times \mathbb{R}$. As $\vec{k}'(\partial\Omega) \in \overline{C}$ in $\mathbb{R}^{n-1}$, one has that the curvature vector in $\mathbb{R}^n$ of the outer cylinder also belongs to $\overline{C}$. Now Theorem 0.2(a) yields that they coincide locally, hence $S \subset \partial\Omega \times \mathbb{R}$. Removing such components, the theorem may be applied as stated.

Proof. Following [6], let $\bar{t}$ be the maximum height of B. The case where $\bar{t} \leq 0$ is easy. Either M_{0+} has no interior points, so $M_{0+} = B_{0+}$, or M_{0+} has an interior point of maximal height, which forces M to remain below its tangent space, so Theorem 0.1(a) implies it is contained in Π_0 locally. By connectedness, M_{0+} is a region in Π_0. Anyway, the theorem holds.

Otherwise, let us consider the subset T of $[0, \bar{t}]$ satisfying: (i) M_{t+} is a graph with locally bounded slope, and (ii) $M^*_{t+} \geq M_{t-}$. By the same argument used in the case $\bar{t} \leq 0$, one has that $\bar{t} \in T$. If we show that T is closed and open in $[0, \bar{t}]$, it will follow that $0 \in T$, so the theorem will be proved.

Clearly T is an interval, so assume $(t, \bar{t}] \subset T$. We must show that

$t \in T$. To see that M_{t+} is a graph, we proceed like Schoen, taking two points in M_{t+}, over the same $\vec{x}$ in Π_0, say $(\vec{x}, t)$ and $(\vec{x}, x_{n+1})$, with $x_{n+1} > t$. As $B_{t+} \subset B_{0+}$ is a graph, it follows that $\vec{x} \in \Omega$, so $(\vec{x}, t)$ is an interior point of M. The slope of M at $(\vec{x}, x_{n+1})$ is finite, since $x_{n+1} \in (t, \bar{t}]$, so a neighborhood of this point in M can be represented as a graph G over a neighborhood V of $(\vec{x}, t)$ in Π_t, with $G \geq V$ strictly. Given $(\vec{y}, t) \in V$, one has that $p^{-1}\{y\}$ intersects G at a level $s > t$, thus in a unique point of M_{s+}. Since G is disjoint from V, it follows that a neighborhood of $(\vec{x}, t)$ in M lies below Π_t, and by Theorem 0.1(a) this neighborhood is contained in Π_t.

A continuation argument gives us that the component of $(\vec{x}, t)$ has boundary $\subset \Pi_t$, hence is equal to $\partial\Omega \times \{t\}$, with $\bar{t} > t \geq 0$, so one of its points is strictly below the highest, contradicting the fact that B_{0+} is a graph. Therefore M_{t+} is a graph and surely has locally bounded slope. Clearly $M^*_{t+} \geq M_{t-}$, for otherwise a point $(\vec{x}, x_{n+1}) \in M_{t+}$ would be reflected through Π_t into a point below $(\vec{x}, y_{n+1}) \in M_{t-}$, and the same would happen at a level t' slightly greater than t. That finishes the proof that T is closed.

Now we prove that T is open. Given $t > 0$, $t \in T$, one first proves that the vertical vector $\mathbf{e}_{n+1}$ is not tangent to M at a point $p \in \Pi_t$. If $p \in B \cap \Pi_t$, as $\mathbf{e}_{n+1}$ is not tangent to $\partial\Omega$ at $t > 0$, one has that M is tangent to the cylinder $\partial\Omega \times \mathbb{R}$. Theorem 0.2(b) applies here, since $\vec{k}(\partial\Omega \times \mathbb{R}) \in \overline{C}$, yielding that a neighborhood of p in M would be contained in $\partial\Omega \times \mathbb{R}$, contrary to the assumption. Now let p be an interior point of M, with $p \in \Pi_t$. One has that M^*_{t+} and M_{t-} meet at the boundary point p, with M^*_{t+} above M_{t-}, since $t \in T$. If $\mathbf{e}_{n+1} \in T_pM$, then M^*_{t+} and M_{t-} meet tangentially at p with curvature vectors in the same leaf, since the principal curvatures at p do not change with a reflection through the plane Π_t which contains the normal. Theorem 0.1(b) implies that M^*_{t+} and M_{t-} will coincide in a neighborhood of p, so the connected component of p, up to the boundary, will be symmetric with respect to Π_t, with $t > 0$, contradicting $B^*_{0+} \geq B_{0-}$.

Since M has finite slope at all points in Π_t, we can find $\varepsilon > 0$ such that $\forall \varepsilon' \in (0, \varepsilon]$ the set $M \cap S_{\varepsilon'}$ is a graph with bounded slope over a subset of Π_0, where $S_{\varepsilon'}$ denotes the strip $|x_{n+1} - t| < \varepsilon'$.

If we take $s \in (0, \bar{t}]$ with $|s - t| < \varepsilon/4$, and denote by ρ_s the reflection in Π_s, then $\rho_s(S_{\varepsilon/2}) \subset S_\varepsilon$, hence $\rho_s(M_{s+} \cap S_{\varepsilon/2}) \geq M_{s-}$. As a matter of fact, the intersection of the latest two sets inside the strip S_ε occurs in $M \cap \Pi_s$, as $M \cap S_\varepsilon$ is a graph. On the other hand, the complement

$M_{s+}\backslash S_{\varepsilon/2}$ is a compact subset of M_{t+}, thus $\rho_t(M_{s+}\backslash S_{\varepsilon/2}) \geq M_{t-}$. We observe that $M^*_{t+} \cap M_{t-} = M \cap \Pi_t$, since the equality holds at the boundary and M is embedded. Otherwise, they would be tangent at an interior point and the reflected normal of M_{t+} would coincide with the normal of M_{t-} at the tangency point, by the embeddedness of M, so they would coincide locally, contradicting the equality at the boundary. As $M_{s+}\backslash S_{\varepsilon/2}$ is strictly above Π_t, one has that $\rho_t(M_{s+}\backslash S_{\varepsilon/2}) \geq M_{t-}$ disjointly, so the same is true for s near t, thus finishing the proof that $\rho_s(M_{s+}) \geq M_{s^-}$. The fact that M_{s^+} is a graph with locally bounded slope follows from $M \cap S_\varepsilon$ being a graph with bounded slope. The proof of Theorem 4.1 is finally complete.

Remark. The argument used by Schoen to prove that M has no points of self-intersection lying in Π_t does not work in the $\sigma_r = 0$ situation, $r > 1$. Two disks, one above the other, could have curvature vectors in distinct leaves, so the maximum principle does not apply.

Corollary 4.1. *Suppose $B \subset \partial\Omega \times \mathbb{R}$ is a graph with bounded slope, Ω as before. Then any embedded M with $\partial M = B$, satisfying $\sigma_r = 0$ with rank of the Gauss map at least r everywhere, is the graph of a smooth function defined on $\overline{\Omega}$.*

Proof. We choose coordinates so that $B \subset \{x_{n+1} > 0\}$. Theorem 4.1 applies to $B = B_{0+}$, proving the corollary.

Theorem 4.2. *Suppose the hypotheses of Theorem* 4.1 *are satisfied and in addition that $B^*_{0+} = B_{0-}$. If M is any embedded hypersurface with $\partial M = B$ having all interior points inside $\Omega \times \mathbb{R}$, and satisfies $\sigma_r = 0$ with rank of the Gauss map at least r everywhere, then* (i) *M_{0+} is a graph with locally bounded slope, and* (ii) *$M^*_{0+} = M_{0-}$.*

Proof. By applying Theorem 4.1 and changing x_{n+1} to $-x_{n+1}$ we obtain both inequalities $M^*_{0+} \geq M_{0-}$ and $M^*_{0-} \leq M_{0+}$. Let B_1 be a boundary component of M. If $B_1 \subset \Pi_0$, then $M = \overline{\Omega}$ and $B_1 = \partial\Omega$, so the theorem holds.

Otherwise, let $\mathbf{p} \in B_1 \cap \{x_{n+1} > 0\} \subset B_{0+}$, and $\mathbf{p}^*$ be its reflection. Then $\mathbf{p}^* \in B_{0-} \subset M$. Now, let $V \subset M_{0+}$ be a neighborhood of $\mathbf{p}$ in M, such that V is a smooth graph with bounded gradient, and likewise, let $V' \subset M_{0-}$ be a neighborhood of $\mathbf{p}^*$. Since $M_{0-} \leq M^*_{0+} \leq M_{0-}$, it follows that $V^* \cap V'$ coincides with a neighborhood of $\mathbf{p}^*$ in M, so $V^* \subset M$. Therefore, the set of points in M having a neighborhood V such that $V^* \subset M$ is open and closed, is hence a connected component M_1 of M, with $B_1 \subset \partial M_1$. This proves that every component of M is symmetric.

Corollary 4.2. *Suppose $B = B_1 \cup B_2$, where each B_i is connected and lies in a hyperplane P_i. Assume that P_1 and P_2 are parallel and that B*

is invariant under reflection with respect to a hyperplane Π *orthogonal to* P_1 *and* P_2. *Assume moreover that each piece of* B_i *bounded by* Π *is a graph over* Π *with locally bounded slope. Then every smooth embedded* $\sigma_r = 0$ *hypersurface bounding* B, *whose Gauss map has rank* $> r - 1$ *everywhere, is invariant under reflection with respect to* Π. *Moreover, if* M *is connected, then the piece of* M *on each side of* Π *is a graph over* Π *with locally bounded slope.*

Proof. One chooses coordinates so that $\Pi = \Pi_0$, and observes that B lies on the boundary of a suitably chosen cylinder $\Omega \times \mathbb{R}$, such that the principal curvatures of $\partial\Omega$ are all ≥ 0, with respect to the inner normal. A direct application of Theorem 4.2 gives us the result.

Proof of Corollary 4.3. Every hyperplane Π containing the line l is in a situation to where Corollary 4.2 can be applied, since the sphere is symmetric with respect to any hyperplane containing its center. Therefore, M is symmetric with respect to any Π containing the line l, so M is invariant by rotations around l.

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