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ON THE REGULARITY OF SOLUTIONS TO A GENERALIZATION OF THE MINKOWSKI PROBLEM

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The Minkowski Problem concerns the existence, uniqueness, and regularity of closed convex hypersurfaces whose Gauss curvature (as a function of the outer normals) is preassigned. Major contributions to this problem were made by Minkowski [28], [29], Aleksandrov [2], [4], Lewy [23], [24], Nirenberg [30], Calabi [9], Pogorelov [34], [35], and Cheng and Yau [10]. Variants of the Minkowski Problem were presented by Gluck [16] and Singer [41]. The survey of Gluck [17] still serves as an excellent introduction to the problem. In this article we consider a generalization of the Minkowski Problem.

We first recall the analytic formulation of the classical Minkowski Problem. Suppose $u = (u^1, \cdots, u^{n-1})$ are smooth local coordinates on the standard unit sphere, S^{n-1}, in Euclidean n-space $\mathbb{R}^n$, and $e = e_{ij}\, du^i\, du^j$ is the first fundamental form of S^{n-1}. The Einstein convention on summation (over repeated lower and upper indices) is presumed everywhere (with Latin indices running from 1 to $n-1$). Let Γ^k_{ij} denote the Christoffel symbols of the second kind for the metric e. For $h \in C^2(S^{n-1})$, let

$$\nabla_{ij} h = \partial_{ij} h - \Gamma^k_{ij} \partial_k h,$$

where

$$\partial_k h = \frac{\partial h}{\partial u^k}, \qquad \partial_{ij} h = \frac{\partial^2 h}{\partial u^i \, \partial u^j},$$

and define the operator N by

$$N(h) = \frac{\det(\nabla_{ij} h + e_{ij} h)}{\det(e_{ij})}.$$

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For a smooth strictly convex hypersurface whose support function is h, the reciprocal Gauss curvature (as a function of the outer unit normals) is given by $N(h)$.

The strong solution to the Minkowski Problem is that for each positive $g \in C^m(S^{n-1})$, $m \geq 3$, such that

$$\int_{S^{n-1}} u\, g(u)\, du = 0,$$

there is a unique (up to translation) convex hypersurface with support function $h \in C^{m+1,\alpha}(S^{n-1})$, for any $\alpha \in (0, 1)$, such that

$$N(h) = g. \tag{0.1}$$

Furthermore, if g is analytic, then h is analytic as well. The solution in this form is given by Pogorelov [36] and Cheng and Yau [10].

In this paper, we consider the following generalization of this problem. Suppose $\gamma \in \mathbb{R}$. What conditions on a given $g \in C(S^{n-1})$ are required to guarantee a solution h to the partial differential equation

$$h^\gamma N(h) = g. \tag{0.2}$$

For $\gamma = 0$, this is, of course, the classical Minkowski Problem. For $\gamma = 1$, the question was posed by Firey [14].

This question will be answered under the restrictive assumptions that $\gamma \leq 0$, $\gamma \neq 1 - n$, and that g is an even function. Specifically, it will be shown that given $\gamma \in \mathbb{R}$, such that $1 - n \neq \gamma < 0$, and a positive even $g \in C^m(S^{n-1})$, $m \geq 3$, then there is a convex solution $h \in C^{m+1,\alpha}(S^{n-1})$, to equation (0.2). Moreover, the solution h is even and unique. If g is analytic, then h is analytic, as well.

1. Preliminaries

Let $\mathscr{C}^n$ denote the set of compact convex subsets of Euclidean n-space, $\mathbb{R}^n$. The subset of $\mathscr{C}^n$ consisting of the convex bodies (compact, convex sets with nonempty interiors) will be denoted by $\mathscr{K}^n$. For the set of convex bodies containing the origin in their interiors, write $\mathscr{K}_o^n$. Let $\mathscr{K}_e^n$ denote the set of centered (i.e., symmetric about the origin) convex bodies. For $K \in \mathscr{C}^n$, let $h_K = h(K, \cdot): S^{n-1} \to \mathbb{R}$ denote the support function of K; i.e., for $u \in S^{n-1}$, $h(K, u) = \max\{u \cdot x : x \in K\}$, where $u \cdot x$ denotes the standard inner product of u and x in $\mathbb{R}^n$. The set $\mathscr{C}^n$ will be viewed as equipped with the usual Hausdorff metric, d, defined by $d(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ is the sup (or max) norm on the space of continuous functions on the unit sphere, $C(S^{n-1})$. Let B denote the centered ball of unit radius in $\mathbb{R}^n$, and write ω_n for its n-dimensional volume.

For real $p \geq 1$, $K, L \in \mathscr{K}_o^n$, and $\lambda, \mu \geq 0$ (not both zero), the Firey linear combination $\lambda \cdot K +_p \mu \cdot L \in \mathscr{K}_o^n$, is defined by

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p.$$

Note that " $\cdot$ " rather than " $\cdot_p$ " is written for Firey scalar multiplication. Firey combinations were introduced in [12] (see also [5, §24.6]). For $p = 1$, these linear combinations are the classical Minkowski combinations of convex bodies.

Let $V(Q)$ denote the volume of $Q \in \mathscr{K}^n$. For $p \geq 1$, the p-mixed volume, $V_p(K, L)$, of $K, L \in \mathscr{K}_o^n$, was defined in [26] by:

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{1.1}$$

The existence of this limit was demonstrated in [26]. For $p = 1$, this mixed volume reduces to a classical mixed volume of Minkowski.

It was shown in [26] that there is an extension of the Minkowski inequality: If $K, L \in \mathscr{K}_o^n$, and $p > 1$, then

$$V_p(K, L)^n \geq V(K)^{n-p} V(L)^p, \tag{1.2}$$

with equality if and only if K and L are dilates. For $p = 1$, this is the Minkowski mixed volume inequality. For this classical inequality, there is equality precisely when K and L are homothetic.

It was shown in [26] that, corresponding to each $K \in \mathscr{K}_o^n$, there is a positive Borel measure, $S_p(K, \cdot)$ on S^{n-1} such that

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h(Q, u)^p \, dS_p(K, u), \tag{1.3}$$

for all $Q \in \mathscr{K}_o^n$. The measure $S_1(K, \cdot)$ is the surface area measure of K introduced by Aleksandrov [1] and Fenchel and Jessen [11]. When $p = 1$, the subscript will frequently be suppressed.

It turns out (see [26]) that the measure $S_p(K, \cdot)$ is absolutely continuous with respect to $S(K, \cdot)$, and that

$$h(K, \cdot)^p dS_p(K, \cdot) = h(K, \cdot) dS(K, \cdot). \tag{1.4}$$

A convex body $K \in \mathscr{K}_o^n$ will be said (see [27]) to have a p-curvature function

$$f_p(K, \cdot) : S^{n-1} \to \mathbb{R},$$

if $S_p(K, \cdot)$ is absolutely continuous with respect to Lebesgue measure on S^{n-1} and the Radon–Nikodym derivative of $S_p(K, \cdot)$ with respect to spherical Lebesgue measure is $f_p(K, \cdot)$; i.e., if

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h(Q, u)^p f_p(K, u)\, du, \tag{1.5}$$

for all $Q \in \mathcal{K}_o^n$. For $p = 1$, the p-curvature function is the ordinary curvature function of a convex hypersurface, and the subscript will often be suppressed.

It is easily seen that a body in $\mathcal{K}_o^n$ has a positive continuous p-curvature function (for $p > 1$) if and only if the body has a positive continuous curvature function. Let $\mathcal{F}^n$, $\mathcal{F}_o^n$, $\mathcal{F}_e^n$ denote the set of all bodies in $\mathcal{K}^n$, $\mathcal{K}_o^n$, $\mathcal{K}_e^n$, respectively, that have a positive continuous curvature function. Obviously, for $K \in \mathcal{F}_o^n$,

$$f_p(K, \cdot) = h(K, \cdot)^{1-p} f(K, \cdot). \tag{1.6}$$

It was shown in [26] that for $K, L \in \mathcal{K}_o^n$ and $n \neq p > 1$,

$$S_p(K, \cdot) = S_p(L, \cdot) \quad \text{if and only if } K = L. \tag{1.7}$$

For the case $p = n$, one has only the weaker conclusion that $S_p(K, \cdot) = S_p(L, \cdot)$ if and only if K and L are dilates. In slightly less general form, this result is due to Simon [39] (see also [40]).

In [27], the p-affine surface area, $\Omega_p(K)$, of $K \in \mathcal{F}_o^n$ was defined by:

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K, u)^{n/(n+p)} du. \tag{1.8}$$

For $p = 1$, the p-affine surface area reduces to the classical affine surface area of affine differential geometry as extended by Petty (see [22]).

It was shown in [27] that the p-affine surface area of a body is invariant under $SL(n)$-transformations of the body. Unlike its classical counterpart, the p-affine surface area, for $p > 1$, is not invariant under translations of the body.

A special case of Theorem 4.8 in [27] is that if $K \in \mathcal{F}_e^n$, then

$$\Omega_p(K)^{n+p} \leq n^{n+p} \omega_n^{2p} V(K)^{n-p}, \tag{1.9}$$

with equality if and only if K is an ellipsoid. For $p = 1$, inequality (1.9) is the classical affine isoperimetric inequality of affine differential geometry (as extended by Petty [33]).

Define the p-surface area, $S_p(K)$, of $K \in \mathcal{K}_o^n$ by

$$S_p(K) = nV_p(K, B). \tag{1.10}$$

For $p = 1$, the p-surface area is the ordinary surface area, and the subscript is suppressed. From (1.5) and the definition of p-surface area, it follows that for $K \in \mathcal{F}_o^n$,

$$S_p(K) = \int_{S^{n-1}} f_p(K, u)\, du. \tag{1.11}$$

It was shown in [27, Proposition 5.2] that using the Hölder inequality we obtain that for $K \in \mathcal{K}_o^n$ and $p > 1$,

$$S_p(K) \geq n^{1-p} V(K)^{1-p} S(K)^p. \tag{1.12}$$

The weak solution of the classical Minkowski Problem (see e.g., [2], [36], and [37]) is that corresponding to each Borel measure, μ, on S^{n-1}, whose support does not lie in a great sphere of S^{n-1}, and that satisfies the orthogonality condition

$$\int_{S^{n-1}} u\, d\mu(u) = 0,$$

there exists a convex body $K \in \mathcal{K}^n$, such that

$$S(K, \cdot) = \mu.$$

Furthermore, the body K is unique up to translation.

A Borel measure on S^{n-1} is said to be even if it assumes the same values at antipodal Borel sets. Note that an even measure trivially satisfies the above orthogonality condition.

The following existence theorem was proved in [26].

Theorem 1.13. *If $n \neq p > 1$ and μ is an even Borel measure on S^{n-1}, whose support does not lie in a great sphere of S^{n-1}, then there exists a $K \in \mathcal{K}_o^n$, such that*

$$S_p(K, \cdot) = \mu,$$

or equivalently,

$$h_K^{1-p}\, dS(K, \cdot) = d\mu.$$

Moreover, the body K is unique and centered.

The statement of Theorem 1.13 is an open question for all $p < 1$.

2. Bounds for the circumradii and inradii

For $K \in \mathcal{C}^n$, let $b(K, u)$ denote the width of K in the direction $u \in S^{n-1}$; i.e., $b(K, u) = h(K, u) + h(K, -u)$. Let

$$W(K) = \frac{1}{n\omega_n} \int_{S^{n-1}} b(K, u)\, du$$

denote the mean width of K, and let $d(K)$ and $D(K)$ denote the minimal width and diameter of K; i.e.,

$$d(K)=\min\{b(K, u): u \in S^{n-1}\}, D(K) = \max\{b(K, u): u \in S^{n-1}\}.$$

For $u \in S^{n-1}$, let $\bar{u}$ denote the centered closed line segment of unit length in the direction of u. For $K \in \mathscr{K}^n$ and $u \in S^{n-1}$, let $K|u^\perp$ denote the image of the orthogonal projection of K onto $u^\perp$, the codimension 1 subspace of $\mathbb{R}^n$ that is orthogonal to u.

Let $W_0, W_1, \cdots, W_n$ denote the Quermassintegrals in $\mathbb{R}^n$. One definition of the Quermassintegrals is inductive: For $n = 2$ and $K \in \mathscr{C}^2$, define $W_0(K)$ to be the area of K, $2W_1(K)$ to be the perimeter of K, and $W_2(K) = \omega_2 = \pi$. Let $w_0, w_1, \cdots, w_{n-1}$ denote the Quermassintegrals in $\mathbb{R}^{n-1}$. For $K \in \mathscr{C}^n$, define $W_0(K), W_1(K), \cdots, W_n(K)$ by letting $W_0(K) = V(K)$, and for $i > 0$,

$$W_i(K) = \frac{1}{n\omega_{n-1}} \int_{S^{n-1}} w_{i-1}(K|u^\perp)\, du.$$

By definition, $W_0(K) = V(K)$, the volume of K. By the Cauchy surface area formula, $nW_1(K) = S(K)$, the surface area of K. Obviously, $W_n(K) = V(B) = \omega_n$, while $(2/\omega_n)W_{n-1}(K) = W(K)$, the mean width of K.

The mixed Quermassintegrals $W_0(K, L), W_1(K, L), \cdots, W_{n-1}(K, L)$ of $K, L \in \mathscr{C}^n$ are defined by

$$(n-i)W_i(K, L) = \lim_{\varepsilon\to 0^+} \frac{W_i(K+\varepsilon L) - W_i(K)}{\varepsilon}.$$

From $W_i(\lambda K) = \lambda^{n-i}W_i(K)$, it follows that $W_i(K, K) = W_i(K)$, for all i. Since the Quermassintegral W_{n-1} is Minkowski linear, $W_{n-1}(K, L) = W_{n-1}(L)$ for all K. Note that the mixed Quermassintegral $W_0(K, L)$ is just $V_1(K, L)$.

Lemma 2.1. *If $K \in \mathscr{K}^n$ and i is an integer such that $0 \le i < n-1$, then*

$$D(K) \min_{u\in S^{n-1}} w_i(K|u^\perp) \le n\, W_i(K).$$

Proof. Let $u_o \in S^{n-1}$ be any direction for which

$$b(K, u_o) = \max_{u\in S^{n-1}} b(K, u) = D(K).$$

Since, up to translation, $D(K)\bar{u}_o \subset K$, from the translation invariance and monotonicity of the mixed Quermassintegrals it follows that

$$W_i(K, D(K)\bar{u}_o) \le W_i(K, K).$$

Hence,

$$D(K)\, W_i(K, \bar{u}_o) \le W_i(K).$$

To complete the proof, we need the easily established fact (see, e.g., Schneider [37]) that $n\, W_i(K, \bar{u}) = w_i(K|u^{\perp})$.

For $i = 0$, the inequality of Lemma 2.1 was obtained by Firey [13] using other methods. Since $w_{n-1}(\cdot) = \omega_{n-1}$, the inequality of Lemma 2.1, for $i = n-1$, becomes: If $K \in \mathscr{K}^n$, then

$$D(K) < \frac{n\omega_n}{2\omega_{n-1}} W(K). \tag{2.2}$$

A right circular cylinder with a base whose radius is small relative to its height shows that the constant, $n\omega_n/2\omega_{n-1}$, in (2.2) is best possible.

The projection body ΠK of $K \in \mathscr{K}^n$ is the body whose support function, for $u \in S^{n-1}$, is given by

$$h(\Pi K, u) = w_0(K|u^{\perp}).$$

Recall that $w_0(K|u^{\perp})$ is the $(n-1)$-dimensional volume of the image of the orthogonal projection of K onto the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ that is orthogonal to u.

Lemma 2.3. *If $K \in \mathscr{K}^n$ and $p \ge 1$, then*

$$d(K) S_p(K)^{1/p} > 2n^{(1-p)/p}\, V(K)^{1/p}.$$

Proof. Let $u_o \in S^{n-1}$ be any direction for which

$$b(K, u_o) = \min_{u \in S^{n-1}} b(K, u) = d(K).$$

Since a translate of K is obviously contained in the right cylinder with base $K|u_o^{\perp}$ and altitude $d(K)\bar{u}_o$, we have

$$d(K) \max_{u \in S^{n-1}} h(\Pi K, u) \ge V(K).$$

But (2.2) shows that

$$\begin{aligned} D(\Pi K) &< \frac{n\omega_n}{2\omega_{n-1}} W(\Pi K) = \frac{1}{2\omega_{n-1}} \int_{S^{n-1}} b(\Pi K, u)\, du \\ &= \frac{1}{\omega_{n-1}} \int_{S^{n-1}} w_0(K|u^{\perp})\, du = n W_1(K). \end{aligned}$$

The observations that $nW_1(K) = S(K)$ and

$$D(\Pi K) = 2 \max_{u \in S^{n-1}} h(\Pi K, u),$$

together with (1.12), complete the proof.

A right circular cylinder with a base whose radius is large relative to its height shows that, for $p = 1$, the constant in Lemma 2.3 is best possible.

Let $C^+(S^{n-1})$ denote the set of nonnegative functions in $C(S^{n-1})$. For $p \geq 1$, define the p-cosine transform, $\mathrm{C}_p g$, of $g \in C^+(S^{n-1})$, by

$$(\mathrm{C}_p g)(u) = \left\{ \frac{1}{n} \int_{S^{n-1}} |u \cdot v|^p g(v)\, dv \right\}^{1/p}, \tag{2.4}$$

for $u \in S^{n-1}$. For $p = 1$, the p-cosine transform is the classical cosine transform. (See Goodey and Weil [18] for some useful connections between the cosine transform and the spherical Radon transform.) The cosine transform of the curvature function of a convex body is proportional to the support function of the projection body of the convex body. Specifically, if $K \in \mathscr{F}^n$, then

$$2h(\Pi K, \cdot) = n\,\mathrm{C}f(K, \cdot).$$

(See Schneider and Weil [38] and Goodey and Weil [19] as references regarding projection bodies.)

The p-cosine transform, $\mathrm{C}_p g$, of $g \in C^+(S^{n-1})$, is not only continuous, but also, in fact, the support function of a centered convex body. To see this, extend the definition of $\mathrm{C}_p g$ to $\mathbb{R}^n$ by homogeneity of degree 1; i.e., for $x \in \mathbb{R}^n$, define

$$(\mathrm{C}_p g)(x) = \left\{ \frac{1}{n} \int_{S^{n-1}} |x \cdot v|^p g(v)\, dv \right\}^{1/p}.$$

From the Minkowski integral inequality (see, e.g., [20]), it follows that $\mathrm{C}_p g$ is subadditive; i.e., for all $x, y \in \mathbb{R}^n$,

$$(\mathrm{C}_p g)(x + y) \leq (\mathrm{C}_p g)(x) + (\mathrm{C}_p g)(y).$$

Hence $\mathrm{C}_p g$ is the support function of a convex body. Obviously, $\mathrm{C}_p g$ is strictly positive, unless g is identically 0. The body whose support function is $\mathrm{C}_p f_p(K, \cdot)$ will be denoted by $\mathrm{P}_p K$; i.e.,

$$h(\mathrm{P}_p K, \cdot) = \mathrm{C}_p f_p(K, \cdot).$$

For $p = 1$, the subscript will be suppressed. Clearly, $n\,\mathrm{P} = 2\Pi$.

Associated with each $K \in \mathscr{K}_o^n$ is its radial function $\rho(K, \cdot)\colon S^{n-1} \longrightarrow (0, \infty)$, defined for $u \in S^{n-1}$, by

$$\rho(K, u) = \max\{\lambda \geq 0 \colon \lambda u \in K\}.$$

For $K \in \mathscr{K}_o^n$, let $r(K)$ and $R(K)$ denote the inradius and circumradius of K, relative to the origin; i.e.,

$$r(K) = \max\{\lambda > 0 \colon \lambda B \subset K\} = \min\{\rho(K, u) \colon u \in S^{n-1}\},$$

and

$$R(K) = \min\{\lambda > 0 : K \subset \lambda B\} = \max\{\rho(K, u) : u \in S^{n-1}\},$$

where B is the centered unit ball. Note that for $K \in \mathscr{K}_e^n$ (i.e., when K is centered),

$$r(K) = d(K)/2 = \min\{h(K, u) : u \in S^{n-1}\}, \tag{2.5a}$$

and

$$R(K) = D(K)/2 = \max\{h(K, u) : u \in S^{n-1}\}. \tag{2.5b}$$

The following crude estimate will be needed.

Lemma 2.6. *If $K \in \mathscr{F}_e^n$ and $p \geq 1$, then*

$$\rho(K, u)^{-1} V(K)^{1/p} > [\mathrm{C}_p f_p(K, \cdot)](u),$$

for all $u \in S^{n-1}$.

Proof. From the definitions of the support and radial function, it follows immediately that for all $u, v \in S^{n-1}$,

$$\rho(K, u)^{-1} \geq |u \cdot v| / h(K, v).$$

But by (1.6),

$$\begin{aligned}[\mathrm{C}_p f_p(K, \cdot)](u) &= \left\{\frac{1}{n}\int_{S^{n-1}} \left(\frac{|u\cdot v|}{h(K, v)}\right)^p h(K, v) f(K, v)\, dv\right\}^{1/p} \\ &\leq \left\{\frac{1}{n}\int_{S^{n-1}} h(K, v) f(K, v)\, dv\right\}^{1/p} \rho(K, u)^{-1},\end{aligned}$$

and the last quantity is just $V_1(K, K)^{1/p}\rho(K, u)^{-1} = V(K)^{1/p}\rho(K, u)^{-1}$.

3. A priori estimates involving the p-curvature function

For $K \in \mathscr{F}_o^n$ and $n \neq p \geq 1$, define $\delta_p^-(K)$ and $\delta_p^+(K)$ by

$$\delta_p^-(K) = \begin{cases} (\Omega_p(K)/n\omega_n)^{(n+p)/n(n-p)} & \text{if } p < n, \\ (S_p(K)/n\omega_n)^{1/(n-p)} & \text{if } p > n, \end{cases}$$

and

$$\delta_p^+(K) = \begin{cases} (S_p(K)/n\omega_n)^{1/(n-p)} & \text{if } p < n, \\ (\Omega_p(K)/n\omega_n)^{(n+p)/n(n-p)} & \text{if } p > n. \end{cases}$$

Note that $\delta_p^-(K)$ and $\delta_p^+(K)$ involve only $f_p(K, \cdot)$, and also that since $f_p(K, \cdot) \geq 0$ is not identically 0, we have $\delta_p^-(K), \delta_p^+(K) > 0$.

Inequalities (1.9) and (1.12) immediately yield:

Lemma 3.1. *If* $K \in \mathcal{F}_e^n$ *and* $n \neq p \geq 1$, *then*

$$\delta_p^+(K) \geq [V(K)/\omega_n]^{1/n} \geq \delta_p^-(K) > 0.$$

Combine Lemma 2.3, (2.5a), and Lemma 3.1, to get:

Lemma 3.2. *If* $K \in \mathcal{F}_e^n$ *and* $n \neq p \geq 1$, *then*

$$r(K) > \omega_n^{1/p} n^{(1-p)/p} \delta_p^-(K)^{n/p} S_p(K)^{-1/p}.$$

Note that the lower bound given in Lemma 3.2 for $r(K)$ depends only on $f_p(K, \cdot)$.

For $K \in \mathcal{F}_o^n$, define $\sigma_p(K)$ by

$$\sigma_p(K) = 1/\min_{u \in S^{n-1}} [\mathrm{C}_p f_p(K, \cdot)](u).$$

Recall that the p-cosine operator, C_p, maps (not identically 0) functions in $C^+(S^{n-1})$ into strictly positive even functions which are the support functions of centered convex bodies. Specifically, the function $\mathrm{C}_p f_p(K, \cdot) > 0$ is the support function of the centered body $\mathrm{P}_p K \in \mathcal{K}_e^n$. Thus, by (2.5a), we have

$$\sigma_p(K) = 1/r(\mathrm{P}_p K) = 2/d(\mathrm{P}_p K).$$

From Lemma 2.6 and Lemma 3.1 follows:

Lemma 3.3. *If* $K \in \mathcal{F}_e^n$ *and* $n \neq p \geq 1$, *then*

$$\omega_n^{1/p} \delta_p^+(K)^{n/p} \sigma_p(K) > R(K).$$

Again note that the upper bound on $R(K)$ depends only on $f_p(K, \cdot)$.

4. Regularity for the p-Minkowski Problem

An immediate consequence of Theorem 1.13 is that corresponding to each $p \in \mathbb{R}$, such that $n \neq p > 1$, and to each continuous even function, $g: S^{n-1} \longrightarrow (0, \infty)$, there exists a centered $K \in \mathcal{F}_e^n$, such that

$$f_p(K, \cdot) = g$$

or equivalently,

$$h(K, \cdot)^{1-p} f(K, \cdot) = g.$$

In addition, the body K is unique. The uniqueness claimed is not in the class $\mathcal{K}_e^n$, but rather in the larger class $\mathcal{K}_o^n$, and is an immediate consequence of (1.7).

It will now be shown that if g is sufficiently smooth, then the support function of the body $K \in \mathcal{F}_e^n$ is also smooth.

Theorem 4.1. *Suppose* $g\colon S^{n-1} \longrightarrow (0, \infty)$ *is an even function of class* $C^m(S^{n-1})$, $m \geq 3$. *Then there exists a unique convex body* K *with support function* $h = h(K, \cdot) \in C^{m+1,\alpha}(S^{n-1})$, *for any* $\alpha \in (0, 1)$, *such that*

$$h(K, \cdot)^{1-p} f(K, \cdot) = g. \tag{4.2}$$

Furthermore, if g *is analytic, then* h *is analytic as well.*

The proof of this theorem is based on analytic arguments applied to the differential equation corresponding to (4.2).

First we reformulate (4.2) analytically. Hartman and Wintner [21] have shown that if ∂K is of class C^2 and the Gauss curvature of ∂K does not vanish, then ∂K can be parameterized by points on S^{n-1} by using the inverse of the Gauss map of ∂K onto S^{n-1}. In this parameterization, $h = h(K, \cdot) \in C^2(S^{n-1})$, and

$$f(K, \cdot) = \frac{\det(\nabla_{ij} h + e_{ij} h)}{\det(e_{ij})},$$

where the notation is as in the introduction.

Thus (4.2) can be rewritten as

$$h^{1-p} \frac{\det(\nabla_{ij} h + e_{ij} h)}{\det(e_{ij})} = g \quad \text{on } S^{n-1},$$

where K and u have been suppressed. For $h \in C^2(S^{n-1})$, define

$$N_p(h) = h^{1-p} \frac{\det(\nabla_{ij} h + e_{ij} h)}{\det(e_{ij})}.$$

Our objective is to show the solvability for h of

$$N_p(h) = g, \tag{4.3}$$

for a given even g. For that we use a continuity scheme as in [30].

For $t \in [0, 1]$, let

$$g_t(u) = (1 - t) + t g(u),$$

and consider the family of equations

$$N_p(h_t) = g_t, \qquad t \in [0, 1]. \tag{4.4}$$

We begin with the case $g \in C^\infty(S^{n-1})$. The case where $g \in C^m(S^{n-1})$, $m \geq 3$, will be treated at the end, by approximation.

Let Λ denote the set of all $t \in [0, 1]$ for which (4.4) admits a positive solution, $h_t \in C^\infty(S^{n-1})$, on which the operator N_p is elliptic and such that $h_t = h_t(K_t, \cdot)$, for some $K_t \in \mathscr{K}_e^n$. Ellipticity here is equivalent to the requirement that the quadratic form

$$(\nabla_{ij} h_t + e_{ij} h_t)\xi^i \xi^j, \qquad \xi \in \mathbb{R}^{n-1}, \tag{4.5}$$

be positive definite at every $u \in S^{n-1}$.

Obviously, if $t = 0$, then (4.4) admits a solution $h = 1$. Hence, $\Lambda \neq \varnothing$. The desired conclusion that $\Lambda = [0, 1]$ will follow after it is shown that Λ is both open and closed on $[0, 1]$. From $\Lambda = [0, 1]$, it will follow that (4.4) is solvable at $t = 1$.

The proof that Λ is closed will be given in several steps. We begin by establishing uniform estimates, in t, of $\max_{S^{n-1}} |h_t|$ for solutions of (4.4). It will also be shown that $h_t > 0$, and in fact, that the h_t are uniformly bounded away from 0.

Define $c_{np} > 0$ by letting

$$c_{np} = \left\{ \frac{1}{n} \int_{S^{n-1}} |u \cdot v|^p \, dv \right\}^{1/p},$$

for some $u \in S^{n-1}$. Obviously, the integral is independent of the particular choice of $u \in S^{n-1}$.

From Lemmas 3.2 and 3.3, it follows that for $K_t \in \mathscr{K}_e^n$ and $p < n$,

$$\left(\frac{\omega_n^{1/p}}{c_{np}} \right) \frac{\left[\max_{u \in S^{n-1}} g_t(u) \right]^{n/p(n-p)}}{\left[\min_{u \in S^{n-1}} g_t(u) \right]^{1/p}} > R(K_t),$$

and

$$r(K_t) > \frac{1}{n} \frac{\left[\min_{u \in S^{n-1}} g_t(u) \right]^{n/p(n-p)}}{\left[\max_{u \in S^{n-1}} g_t(u) \right]^{1/p}},$$

while for $K_t \in \mathscr{K}_e^n$ and $p > n$,

$$\left(\frac{\omega_n^{1/p}}{c_{np}} \right) \left[\min_{u \in S^{n-1}} g_t(u) \right]^{1/(n-p)} > R(K_t),$$

and

$$r(K_t) > \frac{1}{n} \left[\max_{u \in S^{n-1}} g_t(u) \right]^{1/(n-p)}.$$

Since $g_t(u) = (1-t) + tg(u)$, we have

$$\max_{u\in S^{n-1}} g_t(u) = (1-t) + t \max_{u\in S^{n-1}} g(u)$$

and

$$\min_{u\in S^{n-1}} g_t(u) = (1-t) + t \min_{u\in S^{n-1}} g(u).$$

Hence, there exist $a_1, a_2, b_1, b_2 \in (0, \infty)$, depending only on g, such that

$$a_1 \le \max_{u\in S^{n-1}} g_t(u) \le a_2, \; b_1 \le \min_{u\in S^{n-1}} g_t(u) \le b_2, \quad \text{for all } t \in [0, 1]. \tag{4.6}$$

Thus, there exist $a = a(g)$, $b = b(g) \in (0, \infty)$, such that

$$a \le r(K_t) \quad \text{and} \quad R(K_t) \le b \qquad \text{for all } t \in [0, 1],$$

or equivalently,

$$a \le h_t = h(K_t, \cdot) \le b \quad \text{for all } t \in [0, 1]. \tag{4.7}$$

Next, we show that the h_t admit a uniform a priori C^1 estimate. The arguments here are similar to those in [31]. Set

$$v_t = \tfrac{1}{2}(h_t^2 + |\nabla h_t|^2),$$

where

$$|\nabla h_t|^2 = e^{ij}\partial_i h_t \partial_j h_t,$$

and $(e^{ij}) = (e_{ij})^{-1}$. At any critical point of v_t on S^{n-1} we have

$$0 = \nabla_i v_t = (\nabla_{is} h_t + h_t e_{is}) e^{sk} \partial_k h_t, \qquad i = 1, \cdots, n-1.$$

From (4.4) and (4.7) it follows that the determinant of this system does not vanish on S^{n-1}. Hence $\partial_k h_t = 0$ at critical points of v_t. Thus,

$$\tfrac{1}{2}|\nabla h_t|^2 \le \max_{u\in S^{n-1}} v_t(u) \le \tfrac{1}{2} \max_{u\in S^{n-1}} h_t(u)^2 \le \tfrac{1}{2} b^2. \tag{4.8}$$

This establishes a uniform estimate of C^1-norm of h_t.

The estimates (4.7) and (4.8) show that the family $\{h_t : t \in \Lambda\}$ is compact in $C^{0,\beta}(S^{n-1})$ for any $\beta \in (0, 1)$. Suppose $t_k \in \Lambda$ is a sequence such that $\lim_{k\to\infty} t_k = t_0$. The Blaschke Selection Theorem (see, e.g., Schneider [37]) will yield a subsequence of the t_k also denoted by t_k and a body in $\mathscr{K}_e^n$ denoted by K_{t_0}, such that

$$\lim_{k\to\infty} h_{t_k} = h(K_{t_0}, \cdot),$$

uniformly on S^{n-1}. Abbreviate $h(K_{t_0}, \cdot)$ by h_{t_0}.

We now show that $h_{t_0} \in C^{2,\alpha}(S^{n-1})$ for any $\alpha \in (0, 1)$. The arguments here are similar to those in [31, §5.2].

For $u_0 \in S^{n-1}$, let S^{u_0} denote the open hemisphere for which u_0 is the pole. We will show that $h_{t_0} \in C^{2,\alpha}$ in some neighborhood of u_0. Let H denote the hyperplane tangent to S^{u_0} at u_0, and let H' denote the hyperplane parallel to H that passes through the center of S^{n-1}. Choose cartesian coordinates so that $x^1, \cdots, x^{n-1}$ are coordinates for H' (with origin at the center of S^{n-1}) and x^n is directed toward u_0. Project S^{u_0} radially onto H and then orthogonally onto H'. Thus we introduce on S^{u_0} the coordinates $x = (x^1, \cdots, x^{n-1})$, and in these coordinates S^{u_0} has the parametric representation

$$u(x) = (x, 1)/Q(x) \quad \text{where } Q(x) = \sqrt{1 + |x|^2}.$$

With each function h_{t_k}, associate the function

$$\tilde{h}_{t_k} = Q h_{t_k}.$$

Clearly, the sequence $\tilde{h}_{t_k}$ converges uniformly to $\tilde{h}_{t_0}$ on any compact subset of the hyperplane H'. In addition, $\tilde{h}_{t_k} \geq a$, where a is as in (4.7), and $\tilde{h}_{t_k} \longrightarrow \infty$, when $|x| \to \infty$.

Equations (4.4) will now be rewritten in terms of $\tilde{h}_{t_k}$. A few preliminary computations are needed.

Since $Q(x)u(x) = (x, 1)$, we have

$$\partial_i Q\, u + Q\, \partial_i u = (\underbrace{0, \cdots, 0}_{i-1}, 1, 0, \cdots, 0). \tag{4.9}$$

Differentiating (4.9) yields

$$\partial_{ij} Q\, u + \partial_i Q\, \partial_j u + \partial_j Q\, \partial_i u + Q\, \partial_{ij} u = 0. \tag{4.10}$$

By the Gauss derivation formulas,

$$\partial_{ij} u = \Gamma^k_{ij}\, \partial_k u - e_{ij} u.$$

Taking the inner product of (4.10) with $\partial_s u$, and noting that $(\partial_s u, u) = 0$ and $(\partial_s u, \partial_k u) = e_{sk}$, we get

$$\partial_i Q\, e_{sj} + \partial_j Q\, e_{si} + Q\, \Gamma^k_{ij}\, e_{ks} = 0.$$

On the other hand,

$$\partial_i \tilde{h}_{t_k} = \partial_i Q\, h_{t_k} + Q\, \partial_i h_{t_k},$$

which leads to

$$\begin{aligned}\partial_{ij}\tilde{h}_{t_k} &= \partial_{ij}Q\,h_{t_k} + \partial_i Q\,\partial_j h_{t_k} + \partial_j Q\,\partial_i h_{t_k} + Q\,\partial_{ij}h_{t_k}\\ &= \partial_{ij}Q\,h_{t_k} + (\partial_i Q\,e_{js} + \partial_j Q\,e_{is})\,e^{sl}\partial_l h_{t_k} + Q\,\partial_{ij}h_{t_k}\\ &= \partial_{ij}Q\,h_{t_k} - Q\Gamma^l_{ij}\partial_l h_{t_k} + Q\,\partial_{ij}h_{t_k}\\ &= \partial_{ij}Q\,h_{t_k} + Q\nabla_{ij}h_{t_k}.\end{aligned}$$

Now, taking the inner product of (4.10) with u gives

$$\partial_{ij}Q = Qe_{ij}.$$

Hence,[1]

$$\partial_{ij}\tilde{h}_{t_k} = Q(\nabla_{ij}h_{t_k} + h_{t_k}e_{ij}). \tag{4.11}$$

Equations (4.4) now become

$$Q^{p-n}\tilde{h}_{t_k}^{1-p}\frac{\det(\partial_{ij}\tilde{h}_{t_k})}{\det(e_{ij})} = g_{t_k}. \tag{4.12}$$

On the other hand, it follows from (4.9) that

$$e_{ij} = \frac{1}{Q^2}\left[\delta_{ij} - \frac{\partial_i Q\,\partial_j Q}{Q^2}\right],$$

and

$$\det(e_{ij}) = Q^{-2n}.$$

Substituting this into (4.12), we obtain the equations satisfied by $\tilde{h}_{t_k}$,

$$\det(\partial_{ij}\tilde{h}_{t_k}) = g_{t_k}\tilde{h}_{t_k}^{p-1}Q^{-n-p}. \tag{4.13}$$

It follows from (4.11) that the matrix $(\partial_{ij}\tilde{h}_{t_k})$ is positive definite whenever the matrix $(\nabla_{ij}h_{t_k} + h_{t_k}e_{ij})$ is positive definite. Since the h_{t_k} are elliptic solutions of (4.4), and (4.5) is satisfied, we conclude that the graphs $x^n = \tilde{h}_{t_k}(x)$, for $x \in H'$, are strictly concave. Consequently, $\tilde{h}_{t_0}$ is a concave function on H'.

Since $\tilde{h}_{t_0} \to \infty$ when $|x| \to \infty$, we can choose λ, sufficiently large, so that the set $A = \{x \in H' : \tilde{h}_{t_0}(x) \le \lambda\}$ is a nonempty compact convex set

[1] The counterpart of the next equation in [31] contains a misprint. Specifically, equation (5.3) in [31] should read

$$\nabla_{ij}h_t + h_t e_{ij} = \frac{\partial^2 \tilde{h}_t}{\partial x_i \partial x_j}(1+q^2)^{-1/2}.$$

in H'. Since the $\tilde{h}_{t_k}$ converge to $\tilde{h}_{t_0}$, uniformly on each compact subset of H', we may choose λ so that some compact set Θ is contained strictly inside all the sets $A_{t_k} = \{x \in H' : \tilde{h}_{t_k}(x) \le \lambda\}$ as well as the set A.

Now consider the functions

$$\tilde{h}'_{t_k} = \tilde{h}_{t_k} - \lambda.$$

From (4.13) it follows that, on H', these functions satisfy

$$\det(\partial_{ij}\tilde{h}'_{t_k}) = g_{t_k}\left(\tilde{h}'_{t_k} + \lambda\right)^{p-1} Q^{-n-p}. \tag{4.14}$$

Obviously $\tilde{h}'_{t_k}$ vanishes on the boundary of A_{t_k}. Furthermore, by (4.6) and (4.7), the right-hand side of (4.14) is bounded away from 0, uniformly in t. We now apply Theorem 4.2 of [31], according to which, for any $i, j = 1, \cdots, n-1$, the following bound exists:

$$|\partial_{ij}\tilde{h}'_{t_k}(x)| \le c, \tag{4.15}$$

where c depends only on the C^1-norm of $\tilde{h}'_{t_k}$ in A_{t_k}, the C^2-norm of $g_{t_k}Q^{-n-p}$ in A_{t_k}, and the distance of the point x to the boundary of A_{t_k}. From our preceding C^0 and C^1 estimates of h_{t_k}, we conclude that on the set Θ, the C^1-norm of $\tilde{h}'_{t_k}$ can be estimated uniformly in t. Clearly, the same is true for the C^2-norm of $g_{t_k}Q^{-n-p}$. Consequently, on the set Θ, the estimate (4.15) holds uniformly in t (possibly with a suitably adjusted c).

It follows from an estimate of Calabi (see Theorem 4.5 in [31], or [8, §3]) that at every interior point $x \in A_{t_k}$, the third derivatives of $\tilde{h}'_{t_k}$ admit an estimate depending on the C^2-norm of $\tilde{h}'_{t_k}$ in A_{t_k}, the C^3-norm of $g_{t_k}Q^{-n-p}$ in A_{t_k}, and the distance of the point x to the boundary of A_{t_k}. Clearly, this together with (4.15), and the uniform boundedness of the C^3-norm of $g_{t_k}Q^{-n-p}$, in Θ, implies that the third derivatives of $\tilde{h}'_{t_k}$ are uniformly bounded in Θ.

The Arzela–Ascoli Theorem will now yield a subsequence of the $\tilde{h}'_{t_k}$ that converges uniformly to $\tilde{h}'_{t_0}$ in $C^{2,\alpha}$, for any $\alpha \in (0, 1)$. Thus, $\tilde{h}'_{t_0}$ is an elliptic solution of (4.14) in Θ. Under such circumstances, standard arguments and results from the theory of elliptic PDE's (see, e.g., [15, §17.5]) can be used to deduce that $\tilde{h}'_{t_0}$ is, in fact, in $C^\infty(\Theta)$. This implies that $\tilde{h}_{t_0} \in C^\infty(\Theta)$, and finally that $h_{t_0} \in C^\infty(S^{n-1})$. Hence we have proved that Λ is closed.

As an aside, we note that instead of using the a priori estimates of Calabi for the third derivatives of $\tilde{h}'_{t_k}$, one can instead use the interior Hölder estimates of Evans and Krylov for the second derivatives (see [15, §17.4] and the references given there).

To see that Λ is open we proceed as follows. Suppose $\bar{t} \in [0, 1)$, and $\bar{h} = h_{\bar{t}}$ is a C^∞ solution of (4.4) for $t = \bar{t}$. Consider the map

$$P(t, h) = N_p(h) - g_t \colon C^\infty(S^{n-1}) \longrightarrow C^\infty(S^{n-1}).$$

We require the kernel of the derivative

$$P'(\bar{t}, \bar{h})(\xi) = \frac{d}{ds} P(\bar{t}, \bar{h} + s\xi)\bigg|_{s=0}, \qquad \xi \in C^\infty(S^{n-1}),$$

or equivalently, the solutions of the equation

$$(4.16)\quad \begin{aligned} &P'(\bar{t}, \bar{h})(\xi) \\ &= (1-p)\bar{h}^{-p} \frac{\det(\nabla_{ij}\bar{h} + e_{ij}\bar{h})}{\det(e_{ij})}\xi \; + \; \bar{h}^{1-p} \frac{E^{ij}}{\det(e_{ij})}(\nabla_{ij}\xi + e_{ij}\xi) = 0, \end{aligned}$$

where $E^{ij} = \operatorname{cof}(\nabla_{ij}\bar{h} + e_{ij}\bar{h})$. Recall that

$$N(h) = \frac{\det(\nabla_{ij}h + e_{ij}h)}{\det(e_{ij})},$$

and rewrite (4.16) as

$$(1-p)N(\bar{h})\xi \; + \; \bar{h}\frac{E^{ij}}{\det(e_{ij})}(\nabla_{ij}\xi + e_{ij}\xi) = 0.$$

After integration over S^{n-1}, we have

$$(4.17)\quad (1-p)\int_{S^{n-1}} N(\bar{h})\xi \, du \; + \int_{S^{n-1}} \bar{h}\frac{E^{ij}}{\det(e_{ij})}(\nabla_{ij}\xi + e_{ij}\xi)\, du \; = \; 0.$$

As is well known (see [32, Lemma 2.1]), the operator $[E^{ij}/\det(e_{ij})]\nabla_{ij}$ is selfadjoint, in the sense that

$$\int_{S^{n-1}} \bar{h}\frac{E^{ij}}{\det(e_{ij})}\nabla_{ij}\xi \, du = \int_{S^{n-1}} \xi \frac{E^{ij}}{\det(e_{ij})}\nabla_{ij}\bar{h}\, du.$$

Since

$$E^{ij}(\nabla_{ij}\bar{h} + e_{ij}\bar{h}) = (n-1)\det(\nabla_{ij}\bar{h} + e_{ij}\bar{h}),$$

and $p \neq n$, it follows from (4.17) that

$$\int_{S^{n-1}} N(\bar{h})\xi \, du \; = \; 0,$$

which we rewrite as

$$(4.18)\qquad \int_{S^{n-1}} \xi \frac{E^{ij}}{\det(e_{ij})}(\nabla_{ij}\bar{h} + e_{ij}\bar{h})\, du = 0.$$

A theorem of Hilbert and Aleksandrov (see [3, §6]) shows that (4.18) implies

$$(4.19)\qquad \int_{S^{n-1}} \xi \frac{E^{ij}}{\det(e_{ij})}(\nabla_{ij}\xi + e_{ij}\xi)\, du \le 0.$$

However, multiplying (4.16) by $\bar{h}^{p-1}\xi$ and integrating over S^{n-1} give:

$$(1-p)\int_{S^{n-1}} \bar{h}^{-1}N(\bar{h})\xi^2\, du + \int_{S^{n-1}} \xi \frac{E^{ij}}{\det(e_{ij})}(\nabla_{ij}\xi + e_{ij}\xi)\, du = 0.$$

Since $p > 1$, this and (4.19) yield that

$$\int_{S^{n-1}} \bar{h}^{-1}N(\bar{h})\xi^2\, du \le 0,$$

and hence ξ vanishes identically.

Thus, $\operatorname{Ker} P'(\bar{t}, \bar{h}) = \{0\}$. The implicit function theorem can now be used to conclude that the equation $P(t, h) = 0$ is solvable in $C^\infty(S^{n-1})$ for all t sufficiently close to $\bar{t}$. The solution h_t is close to $\bar{h} = h_{\bar{t}}$ in the C^k-norm, for any $k \ge 0$, and therefore it is also elliptic and the corresponding body K_t belongs to $\mathcal{F}_e^n$. Hence the set Λ is open.

To complete the proof of the theorem, the assumption that $g \in C^\infty(S^{n-1})$ will be removed. Suppose g is a positive, even function of class $C^m(S^{n-1})$, where $m \ge 3$. Approximate g in $C^m(S^{n-1})$ by positive analytic functions g_i. Since g is even, the sequence of even functions $\tilde{g}_i$, defined by $2\tilde{g}_i(u) = g_i(u) + g_i(-u)$, also approximates g in $C^m(S^{n-1})$. Let h_i be the elliptic solutions of

$$h_i^{1-p}N(h_i) = \tilde{g}_i \quad \text{in } S^{n-1}, \qquad i = 1, 2, \cdots.$$

The a priori estimates described in the proof showing that Λ is closed, can now be used again to see that there is a subsequence of the h_i that converges to h in $C^{2,\alpha}(S^{n-1})$, for any $\alpha \in (0, 1)$, and that h satisfies the equation

$$h^{1-p}N(h) = g \quad \text{in } S^{n-1}.$$

Again, by standard results from elliptic PDE theory we can conclude that $h \in C^{m+1,\alpha}(S^{n-1})$, for any $\alpha \in (0, 1)$, and that h is analytic provided that g is analytic.

If in Theorem 4.1 we only assume that the function g is continuous and positive, then it follows from a result of Caffarelli [7, Theorem 1] that the solution (in the weak sense of Aleksandrov) of (4.2) is in the class $W^{2,k}$ for any $k < \infty$.

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