J. DIFFERENTIAL GEOMETRY
41 (1995) 185-214

SYMMETRIES OF FIBERED KNOTS

MONICA NICOLAU & SUSAN SZCZEPANSKI

Let $\mathscr{Z} \subset \mathbf{C}^{n+1}$ be an algebraic (analytic) hypersurface with an isolated singularity at the origin, which is given as the zero set of $f : \mathbf{C}^{n+1} \to \mathbf{C}$. Recall that the link of such a singularity (S^{2n+1}, K^{2n-1}) consists of a highly connected manifold K, embedded in the sphere S, as a codimension-two submanifold. Moreover, the complement $S - K$ of this embedding fibers over the circle, with the projection map given by the Milnor fibration $f(\mathbf{z})/|f(\mathbf{z})$. Thus these knots belong to a larger class of knots known as simple fibered knots. From one point of view, simple fibered knots are more general than the objects of study in spherical knot theory, since the submanifold K need not be a sphere; yet they are also more refined, since they are fibered knots.

Here we begin our investigation of finite cyclic actions on simple fibered knots (S^{2n+1}, K^{2n-1}) of dimension $n \geq 3$. Recall that a high dimensional knot is *simple* if its complement has the homotopy type of S^1 up to but not including its middle dimension. In particular, we consider simple fibered knots for which the submanifold K is a rational homology sphere. The more general situation, which requires modification of the proofs and techniques given here, as well as the introduction of some further invariants will be discussed in a separate paper [15]. We consider both the free and the semifree cases. We obtain a classification of both types of actions, as well as a determination of the number theoretic conditions which guarantee their existence.

We say that (S, K) admits a free $\mathbf{Z}_m$ action if $\mathbf{Z}_m$ acts freely on S leaving K invariant; we say that (S, K) admits a semifree action if the action on S is semifree with fixed set precisely K. Our results mirror those concerning spherical knots, found in [8], [13], [14], and [17], reflecting the fact that the objects of study are a generalization of these; the methods of proof necessarily address the nonvanishing of the homology of K and exploit the existence of the fibration of the complement.

Received November 12, 1993. The first author was partially supported by NSF grant DMS84-21371; the second author was partially supported by NSF grant DMS 89-11955.

In §1 we establish a connection between the two types of actions (compare [14].) We identify a large class of free actions on (S, K) which can be related to and classified in terms of semifree actions on (S, K). We observe that every semifree action on (S, K) can be used to construct free actions. Each free action so constructed has the following two properties: 1) the restriction to the invariant set K is a homologically trivial action, and 2) the action on $K \times S^1 = \partial(N(K))$, the boundary of the equivariant tubular neighborhood of K, projects to a free action on S^1. We say that an action with the first property is *homologically trivial* and one with the second property is *normally free.* We show that to every free $\mathbf{Z}_m$ action on (S, K) which is homologically trivial and normally free, we may associate a semifree $\mathbf{Z}_m$ action on a manifold pair (M, K), which we call the *derived semifree action.* We will show that whenever K is a $\mathbf{Z}_m$ homology sphere, the manifold M is a sphere.

Theorem 1–7. *A simple fibered knot (S, K) with K a $\mathbf{Q}$ homology sphere admits a semifree $\mathbf{Z}_m$ action if and only if it admits a free $\mathbf{Z}_m$ action which is homologically trivial and normally free. Moreover, the restrictions of these actions to the knot complement may be assumed to be the same.*

Of course, by Smith theory we know that the existence of a semifree action implies that K is a $\mathbf{Z}_m$ homology sphere. Our techniques establish that a homologically trivial free $\mathbf{Z}_m$ action which leaves invariant a codimension two $\mathbf{Z}_m$ homology sphere is necessarily normally free. Hence, we obtain:

Theorem 1–7b. *A simple fibered knot (S, K) with K a $\mathbf{Z}_m$ homology sphere admits a semifree $\mathbf{Z}_m$ action if and only if it admits a homologically trivial free $\mathbf{Z}_m$ action.*

This correspondence between semifree actions and free actions which are homologically trivial and normally free leads (via Smith theory) to nonexistence results (Theorem 1–2). For example, consider the link of the A_k singularity

$$f(z, w_1, \cdots, w_{2n}) = z^k + w_1^2 + \cdots + w_{2l}^2.$$

The link K is a rational homology sphere and so (S, K) does not admit a free homologically trivial normally free $\mathbf{Z}_m$ action unless k and m are relatively prime. (Here $\dim K \equiv 1 \pmod 4$.) For the other stabilization of A_k,

$$f(z, w_1, \cdots, w_{2l+1}) = z^k + w_1^2 + \cdots + w_{2l+1}^2$$

$\dim K \equiv 3 \pmod 4$ and K need not be a rational homology sphere. For certain m, there may indeed exist free $\mathbf{Z}_m$ actions which are homologically trivial normally free, but the derived semifree action is defined on a *nonspherical* manifold M. As one further example, consider the link of the singularity of

$$f(z_0, z_1, \cdots, z_n) = z_0^k + z_1^k + \cdots + z_n^k, \qquad k \geq 3.$$

Note that for this singularity, the link K is never a rational homology sphere. The familiar S^1 action on $\mathbf{C}^{n+1}$ certainly restricts to a homologically trivial free $\mathbf{Z}_m$ action on (S, K) for each m. However, these will be normally free if and only if m and k are relatively prime. Further, if the free action is normally free, the manifold M on which the derived semifree action is defined is necessarily nonspherical [15].

The main tool used to obtain the existence and classification results here is the *derived knot*, as defined in §1 (compare [13] and [14]). This is identified as the quotient space of the derived semifree action associated to the free, homologically trivial normally free $\mathbf{Z}_m$ action on (S, K). We show that, if (as is true for algebraic knots) (S, K) is a fibered knot, then so too is the derived knot. Consequently, the equivariant classification, as well as the realization result for $\mathbf{Z}_m$ actions, can be stated in terms of the invariants associated to the derived knot. Specifically, we make use of the monodromy operator h, the intersection form Q, and the Seifert form B, all of which are defined on the homology of the fiber. It is often convenient to use (integral) $\mu \times \mu$ matrices to represent these invariants; here μ is the Milnor number, i.e. the middle Betti number of the fiber. We will also make use of the fact that the Seifert linking form is unimodular in the case of a highly connected fibered knot (see [4] and [10].)

Theorem 3–1(A). *Let (S, K) be a simple fibered knot with intersection form and monodromy operator given by matrices Q and h_0 respectively. Then (S, K) admits a semifree $\mathbf{Z}_m$ action if and only if there exists $h_1 \in GL_\mu(\mathbf{Z})$ such that*:

1) $h_1^m = h_0$,
2) $h_1^t Q h_1 = Q$,
3) $\det(I + h_1 + \cdots + h_1^{m-1}) = \pm 1$.

In light of Theorems 1–7 and 1–7b, the three conditions above are necessary and sufficient to determine:

1) the existence of homologically trivial free $\mathbf{Z}_m$ actions when K is known to be a $\mathbf{Z}_m$ homology sphere and

2) the existence of free homologically trivial normally free $\mathbf{Z}_m$ actions when K is known to be a $\mathbf{Q}$ homology sphere.

We also obtain a statement about the existence of free $\mathbf{Z}_m$ actions for a more general class of knots.

Theorem 3–1(B). *Let* (S, K) *be a simple fibered knot with intersection form and monodromy operator given by matrices* Q *and* h_0, *respectively. Then* (S, K) *admits a homologically trivial normally free* $\mathbf{Z}_m$ *action if and only if there exists* $h_1 \in GL_\mu(\mathbf{Z})$ *such that*:

1) $h_1^m = h_0$,
2) $h_1^t Q h_1 = Q$,
3) $\operatorname{coker}(I + h_1 + \cdots + h_1^{m-1}) = \oplus_k \mathbf{Z}_m$, *where* $k = \operatorname{rank} H_{n+1}(S - K)$.

There is a strong relationship between our results (Theorem 1–7 and Theorem 3–1) and the Durfee-Kauffman [5] description of the link of $g(z, w) = f(z) + w^m$ as the m-fold branched cyclic cover of S^{2n+1} with branching set the link of $f(z)$. We mention this in the following:

Theorem. *Let* $f(\mathbf{z})$ *be algebraic* (*analytic*) *with an isolated singularity.*

A) *If the link of* $f(\mathbf{z}) + w^m$ *is spherical, then the link of* $f(\mathbf{z})$ *is the derived knot for a homologically trivial free* $\mathbf{Z}_m$ *action which is normally free.*

B) *Suppose that the link of* $f(\mathbf{z})$ *is the derived knot for a homologically trivial free* $\mathbf{Z}_m$ *action which is normally free. Then, the link of* $f(\mathbf{z})$ *is a* $\mathbf{Z}_m$ *homology sphere if and only if the link of* $f(\mathbf{z}) + w^m$ *is spherical.*

Our classification of equivariant simple fibered knots is stated in terms of the notions of equivariant homeomorphism, action equivalence, and the Seifert pairing of the associated derived knot. Two knots (S, K_0) and (S, K_1) with $\mathbf{Z}_m$ actions denoted by T_0 and T_1 are said to be *equivariantly homeomorphic* if there exists an orientation-preserving (PL or smooth) self-homeomorphism θ of the sphere, such that $\theta \circ T_0 = T_1 \circ \theta$, and such that $\theta(K_0) = K_1$. We say that the *free* actions T_0 and T_1 are *action-equivalent* if the normal bundles $\nu(K_i, S)$ are equivariantly isomorphic.

Theorem 2–1a. *Two simple fibered knots admitting homologically trivial free* $\mathbf{Z}_m$ *actions which are normally free are equivariantly homeomorphic if and only if* a) *they are action-equivalent and* b) *the* (*unimodular*) *Seifert forms of their associated derived knots are equivalent. Two simple fibered knots admitting semifree* $\mathbf{Z}_m$ *actions are equivariantly homeomorphic if and only if the* (*unimodular*) *Seifert forms of their associated derived knots are equivalent.*

In §4 we consider several types of singularities and the $\mathbf{Z}_m$ actions which they admit. The main invariant we investigate is the Alexander polynomial, which for simple fibered knots is the characteristic polynomial of the monodromy operator h. By the Monodromy Theorem, in the case of algebraic knots this polynomial is a product of cyclotomic factors; thus, its rich structure enables us to unravel the number theoretic conditions in Theorem 3–1 (existence), and identify all possible actions for a large class of algebraic knots.

Theorem 4–1. *Let (S, K) be an algebraic knot with Alexander polynomial $\Delta(t) = \prod c_{d_i}(t)$ such that $\{d_i\}$ are distinct and not* 1. *Then (S, K) admits a homologically trivial free $\mathbf{Z}_m$ action which is normally free if and only if $(m, d_i) = 1$ for each i. Moreover, if m is odd then $\Delta(t) = \Delta_D(t)$. If m is even, there always exists an action for which $\Delta(t) = \Delta_D(t)$, and in general $\Delta_D(t) = \prod c_{\varepsilon_i d_i}(t)$, where*

$$\varepsilon_i = \begin{cases} 1 & \text{if } d_i \text{ is a prime power,} \\ 1 \text{ or } 2 & \text{otherwise.} \end{cases}$$

Here $\Delta_D(t)$ is the Alexander polynomial of the derived knot, and $c_d(t)$ is the cyclotomic polynomial for d. Notice that the condition in Theorem 4–1 is generic, in that the set of Alexander polynomials with distinct roots is a Zariski open set.

We mention two examples (for a much more extensive analysis see §4):

1) Characterization 2–A. The link of singularity for the polynomial

$$f(z_0, \cdots, z_n, w_1, \cdots, w_k) = z_0^{a_0} + \cdots + z_n^{a_n} + w_1^2 + \cdots + w_k^2,$$

a_i relatively prime, $a_i > 2$, admits a (free, homologically trivial and normally free, or semifree) $\mathbf{Z}_m$ action in precisely the following situations:

a) k *even*: if and only if m is prime to each a_i;

b) k *odd*, a_i *odd*: if and only if m is odd and prime to each a_i;

c) k *odd*, a_0 *even*, $n > 0$: if and only if m is prime to each a_i.

Moreover, when m is odd, the action is uniquely determined on the knot complement.

Thus, except possibly when k is even and every a_i is odd, each $\mathbf{Z}_m$ action on the complement is uniquely determined by the integer m.

As an example, consider the polynomial

$$z_0^6 + z_1^5 + w_1^2 + w_2^2.$$

This admits $\mathbf{Z}_m$ actions if and only if m and 30 are relatively prime. Moreover, m uniquely determines the action on the knot complement.

In the one remaining case [15],

c′) k **odd**, a_0 **even**, $n = 0$:

$$\Delta(t) = (t-1)c_d(t)c_{2d}(t), \quad \text{where } a_0 = 2d.$$

Hence K is not a $\mathbf{Q}$ homology sphere, and semifree $\mathbf{Z}_m$ actions never exist; homologically trivial free $\mathbf{Z}_m$ actions can exist, but only if m is prime to a_0. The derived semifree action is defined on a nonspherical manifold, M.

2) Example 5–a. The unimodal singularity

$$x^3 z + az^3 + w_1^2 + \cdots + w_k^2, \qquad a \neq 0.$$

i) When the number of squares k is odd, $\Delta(t) = c_2(t)c_{18}(t)$, and so actions exist if and only if m and 6 are relatively prime; m uniquely determines the action on the knot complement.

ii) When the number of squares k is even, $\Delta(t) = c_1(t)c_9(t)$, and so the above theorem does not apply.

Finally we remark that our methods in the first three sections use only the fibered structure of the knot (S, K) together with the topological properties of the manifold K. In the last section, the only extra condition used is the factorization of the Alexander polynomial into cyclotomic factors. This property is shared by algebraic knots and many fibered knots. Hence our results hold for this more general class of knots.

1. The work which follows is motivated by our interest in algebraic knots, that is, pairs (S^{2n+1}, K^{2n-1}) which are realized as the link of an isolated singularity of a function $f : \mathbf{C}^{n+1} \to \mathbf{C}$. Such knots are always simple, and the knot complement always fibers over the circle. These two features and their consequences will be utilized below. Hence, our results pertain to simple, fibered knots.

Let (S^{2n+1}, K^{2n-1}), $n \geq 3$, be a knot admitting a free or a semifree $\mathbf{Z}_m$ action T. If T is semifree, it follows from Smith theory that K is a mod p homology sphere for every prime p dividing m (see for example [1]). Here we will establish conditions under which the existence of a free $\mathbf{Z}_m$ action has a similar consequence for the case of a simple fibered knot (S, K). In particular, we establish that for K a rational homology sphere, the existence of a semifree $\mathbf{Z}_m$ action is equivalent to that of a free $\mathbf{Z}_m$ action of a certain type.

We begin by noting some of the properties of a simple fibered knot. The tubular neighborhood ν of K in S^{2n+1} is a trivial 2-disk bundle. Denote by X the closed knot complement of ν; then X fibers over the circle with highly connected fiber F (that is, $\pi_i(F)$ is trivial for $i < \frac{1}{2}\dim(F)$). The boundary of F is diffeomorphic to K, which is highly connected as well. The restriction of the fibration to the boundary ∂X is the trivial fibration. In particular, if the monodromy on F is denoted by h, then the restriction $h|_{\partial F} = id$.

We introduce the following two notions. A $\mathbf{Z}_m$ action on (S, K) is *homologically trivial* if the induced action on the homology of the invariant submanifold K is trivial. A $\mathbf{Z}_m$ action is *normally free* if the action on $\partial X = K \times S^1$ projects to a free action on S^1. Both of these properties are possessed by every semifree $\mathbf{Z}_m$ action.

Let T be a homologically trivial normally free, free or semifree $\mathbf{Z}_m$ action on (S, K) with equivariant exterior pair $(X, \partial X)$, and quotient pair $(X/T, \partial X/T) = (X^*, \partial X^*)$. We begin by establishing that the quotient pair is the exterior pair of a simple fibered knot. We start with the following:

Proposition 1-1. *The quotient X^* fibers over S^1, with fiber F.*

Proof. By the tubular neighborhood theorem, we have the identification $\partial X^* = S^1 \times_T K$. By assumption, the action is normally free, and so there is a fibration $K \times_T S^1 \to S^1$ with fiber K. Since K is highly connected, in particular simply connected, $\pi_1(\partial X^*) \cong \mathbf{Z}$. (Actually, by Remark 1–3.3, in the case of a free action, the action is normally free if and only if $\pi_1(\partial X^*) \cong \mathbf{Z}$.) As T acts freely on X, and since $H_i(X, \partial X) = 0$ for $i \leq 2$, it follows from [1, Chapter III, Theorem 5.5] that $H_i(X^*, \partial X^*) = 0$ for $i \leq 2$ as well. Hence, $H_1(X^*) \cong \mathbf{Z}$ and the map $H_1(X) \to H_1(X^*)$ is injective. Consideration of the diagram:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X) \arrow[r, "p_1"] \arrow[d, "\cong"', "h_1"] & \pi_1(X^*) \arrow[r] \arrow[d] & \mathbf{Z}_m \arrow[r] & 1 \\
1 \arrow[r] & H_1(X) \arrow[r, "p_2"] & H_1(X^*) & &
\end{tikzcd}$$

shows that $\pi_1(X^*)$ is abelian, hence $\mathbf{Z}$. For if $x \in [\pi_1(X^*), \pi_1(X^*)]$, then $x = p_1(y)$ and $p_2 h_1(y) = 1$. Finally, since the inclusion $\partial X \hookrightarrow X$ induces an isomorphism on π_1, so does the inclusion $\partial X^* \hookrightarrow X^*$. It now follows from [2] that the fibration $S^1 \times_T K$ extends to a fibration of X^*. q.e.d.

Note that since the following diagram commutes up to homotopy,

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "\pi"'] & X^* \arrow[d, "\pi'"] \\
S^1 \arrow[r, "\times m"] & S^1
\end{tikzcd}$$

we may assume (after perhaps replacing π by a homotopic fibration) that the diagram is a fibered square. Notice too that the generator of a $\mathbf{Z}_m$ action can be identified with h_1, the monodromy map of the fibration $\pi' : X^* \to S^1$. And so we have that the monodromies of the two fibrations in the above diagram satisfy $h_1^m = h$.

We now concentrate on the case of free actions. Notice that if T is a *semifree* action then, by the equivariant tubular neighborhood theorem, the quotient boundary $\partial X^* = S^1 \times_T K = S^1 \times K$. If X^* arises as a knot complement, then it is necessarily the case that ∂X^* is homeomorphic to the product $K \times S^1$. We now set about showing that, under our assumptions, the same is true of a free action T which is homologically trivial and normally free.

Theorem 1–2. *Let (S, K) be a simple fibered knot which admits a free $\mathbf{Z}_m$ action T. Then the following hold:*

1) *If $K \times_T S^1 \approx K \times S^1$, then the action is normally free and homologically trivial.*

2) *If the action is normally free and homologically trivial, with K a $\mathbf{Q}$ homology sphere, then $K \times_T S^1 \approx K \times S^1$. In this case,*

i) *K is a* mod p *homology sphere for each prime p dividing m, and*

ii) *$h_1|_K$ is homotopic to the identity map.*

3) *If the action is homologically trivial with K a $\mathbf{Z}_p$ homology sphere for each prime p dividing m, then $K \times_T S^1 \approx K \times S^1$. In this case,*

i) *the action is normally free, and*

ii) *$h_1|_K$ is homotopic to the identity map.*

Proof. We will show subsequently (Remark 1–3.3) that the normally free condition is equivalent to $\pi_1(\partial X^*) \cong \mathbf{Z}$. Assume this for the moment. The monodromy h_1 of the fibration $X^* \to S^1$, identified as the generator of the $\mathbf{Z}_m$ action, satisfies $h_1^m = h$, and $(h_1|_K)^m = h|_K = id$. (Recall that h is the monodromy of the fibration $X \to S^1$). To obtain the necessity of the homological triviality of $h_1|_K$ (statement 1)) we argue as follows. If $\partial X^* \approx K \times S^1$, then $\pi_1(\partial X^*) \cong \mathbf{Z}$, and hence the action is normally free. Calculating the homology of $S^1 \times_T K = \partial X^*$ via the Wang sequence of the fibration $K \times_T S^1 \to S^1$ produces the homology of $K \times S^1$ precisely in the case that $(h_1 - id)_*|K$ is trivial. Statements 2) and 3) are deduced from the following lemmas.

Lemma 1–3. *If the action is homologically trivial and normally free, then* $\mathrm{Tor}(H_{n-1}(K), \mathbf{Z}_p) = 0$ *for all p dividing m. In particular, if K is a $\mathbf{Q}$ homology sphere, then K is a* mod p *homology sphere for p dividing m.*

Lemma 1–4. *If the action is homologically trivial and normally free, and K is a $\mathbf{Q}$ homology sphere, then $h_1|_K$ is homotopic to the identity. Also, if the action is homologically trivial and K is a* mod p *homology* *sphere for primes p dividing m, then the action is normally free and $h_1|_K$ is homotopic to the identity.*

Lemma 1-5. *If the action is homologically trivial, K is a $\mathbf{Z}_m$ homology sphere, and $h_1|_K$ is homotopic to the identity, then $K \times_T S^1 \approx K \times S^1$.*

We choose to segment the proof of Theorem 1-2 in this manner, as the proof of each of these lemmas requires some effort, and the techniques in each proof are different. This also enables us to address the redundancy of the homologically trivial and normally free conditions in certain cases, and to make note of some more general cases in which partial results hold. The assumption common to all of these is that (S, K) is a simple fibered knot admitting a free $\mathbf{Z}_m$ action T, with invariant submanifold K.

Proof of Lemma 1-3. Recall that K is a highly connected, closed $(2n-1)$-dimensional submanifold of S^{2n+1}. Using duality and the exact cohomology sequence of the pair (S, K) we can identify the only two interesting homology (cohomology) groups of K with the homology (cohomology) groups of X; that is:

$$H_{j-1}(K) \cong H_j(S-K) \cong H_j(X) \quad \text{for } j = n, n+1,$$

and

$$H^{j-1}(K) \cong H^j(S-K) \cong H^j(X) \quad \text{for } j = n, n+1.$$

We note that $\operatorname{rank} H^{n+1}(X) = \operatorname{rank} H^n(X)$; we let $\mathrm{TOR}(K)$ denote $\mathrm{Ext}(H_n(X), \mathbf{Z})$, which is identified, by the Universal Coefficient Theorem, with the torsion subgroup of $H^{n+1}(X)$ and also with the torsion subgroup of $H_{n-1}(K)$. The result will follow once we have shown that $\mathrm{Tor}(H^{n+1}(X), \mathbf{Z}_m) \cong \mathrm{Tor}(\mathrm{TOR}(K), \mathbf{Z}_m) = 0$.

We now consider the cohomology spectral sequence of the homotopy fibration

$$X \xrightarrow{p} X^* \longrightarrow B\mathbf{Z}_m.$$

Since $(h_1|_K)_* = id$ can be identified with the generator of $\pi_1(B\mathbf{Z}_m)$, the coefficient system is simple, hence we have:

$$E_2^{i,j} \cong H^i(\mathbf{Z}_m; H^j X) \cong \begin{cases} H^j(X) & \text{if } i = 0 \\ \mathbf{Z}_m & \text{if } i \neq 0, \text{ even}, j = 0, 1 \\ H^j(X) \otimes \mathbf{Z}_m & \text{if } i \neq 0, \text{ even}, j = n, n+1 \\ \mathrm{Tor}(\mathbf{Z}_m, H^{n+1} X) & \text{if } i \text{ odd}, j = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Consideration of the cohomology spectral sequence leads to the following:

i) By virtue of the cup product, the map

$$E_2^{i,1} \cong \mathbf{Z}_m \to E_2^{i+2,0} \cong \mathbf{Z}_m \quad \text{for } i > 0$$

is determined by the map, for $i = 0$,

$$\delta : E_2^{0,1} \cong \mathbf{Z} \to E_2^{2,0} \cong \mathbf{Z}_m.$$

Moreover, the cokernels of these maps for $i \geq 0$, i even are isomorphic. (For i odd, the groups are zero.) Therefore the following hold:

a) $E_3^{2,0} \cong E_\infty^{2,0} \cong \text{cokernel } \delta \cong H^2(X^*)$;

b) $p_1^* : \ker \delta \cong H^1(X^*) \hookrightarrow H^1(X)$ is the inclusion $d\mathbf{Z} \hookrightarrow \mathbf{Z}$ and so defines the integer d;

and

c) $H^2(X^*) \cong \text{Ext}(H_1(X^*), \mathbf{Z}) \cong \text{Ext}(\mathbf{Z} \oplus \mathbf{Z}_{m/d}, \mathbf{Z}) \cong \mathbf{Z}_{m/d}$. In fact, $H^i(X^*) \cong \mathbf{Z}_{m/d}$ for $2 \leq i \leq n-1$.

The integer d can be understood as follows: for the projection of the action onto the circle, the subgroup $\mathbf{Z}_{m/d} \hookrightarrow \mathbf{Z}_m$ fixes S^1 and $\mathbf{Z}_d$ acts freely on the S^1. Thus, the action is normally free if and only if $d = m$.

ii) We now develop some consequences of the fact that the cohomology of X^* must be finitely generated.

a) For i *odd*, we have that the differential

$$E_2^{i,n+1} \cong \text{Tor}(\mathbf{Z}_m, H^{n+1}(X)) \to E_2^{i+2,n} \cong 0.$$

So, the group on the left is identified as $E_3^{i,n+1}$. By part i), it must be finite cyclic of order $t \leq m/d$, or else the cohomology of X^* would fail to be finitely generated. Further, if the action is normally free, this group must be trivial, since in this case, $E_k^{i,j} \cong 0$ for $k \geq 3$, $j < n$, and $i \geq 1$.

b) For i even, $i > 2$, we deduce properties of the kernel and the cokernel of the maps

$$E_2^{i,n+1} \cong \mathbf{Z}_m \otimes H^{n+1}(X) \to E_2^{i+2,n} \cong \mathbf{Z}_m \otimes H^n(X).$$

We note that the target group in this case is a direct sum of a finite number (at most μ, the Milnor number) of copies of $\mathbf{Z}_m$ and that the source group is isomorphic to the direct sum of this with $\text{Tor}(\mathbf{Z}_m, H^{n+1}(X))$. Again, by the finiteness of the cohomology and part i), the cokernel of this map must be a finite cyclic group of order $q \leq m/d$, and the kernel must be a finite cyclic group of order $tq = m/d$. These observations are consequences of the fact that the kernel and cokernel of these maps are identified as $E_3^{i,n+1}$ and $E_3^{i+2,n}$, respectively. (The last statement is clear for $n > 3$, but also follows for $n = 3$.) We conclude that the action is normally free if and only if this differential is an isomorphism. Thus, if the action is normally free, then $\mathrm{Tor}(H_{n-1}(K), \mathbf{Z}_m) \cong \mathrm{Tor}(\mathbf{Z}_m, H^{n+1}(X)) \cong 0$.

iii) Finally, if a homologically trivial free action leaves invariant a $\mathbf{Q}$ homology sphere K, then $H^n(X) \cong 0$ and $\mathrm{Tor}(H_{n-1}(K), \mathbf{Z}_m)$ is a subgroup of $\mathbf{Z}_m$, in fact, a subgroup isomorphic to $\mathbf{Z}_{m/d}$. Hence, if the action is normally free, then K must be a $\mathbf{Z}_m$ homology sphere. q.e.d.

We mention, for further reference, the following:

Corollary 1–3A. *If (S, K) admits a homologically trivial free $\mathbf{Z}_m$ action, then* $\mathrm{Tor}(\mathrm{TOR}(K), \mathbf{Z}_m)$ *is isomorphic to a subgroup of $\mathbf{Z}_m$.*

Proof. This follows from step ii) of the proof of Lemma 1–3.

Corollary to 1–3B. *Let T be a free, homologically trivial normally free $\mathbf{Z}_m$ action on a simple fibered knot (S, K). Then X^*, the quotient of the knot complement, fibers over the circle with h_1, the monodromy of the fibration, satisfying $h_1^m = h$. Further,*

i) *the map*

$$\alpha = (I + h_1 + h_1^2 + \cdots + h_1^{m-1})_* : H_n(F) \to H_n(F)$$

is an injection with $\mathrm{cokernel} \cong H_{n+1}(X) \otimes \mathbf{Z}_m$.

ii) $\det(\alpha) = \det(I + h_1 + h_1^2 + \cdots + h_1^{m-1})_* = \pm 1$ *if and only if K is a* mod p *homology sphere for all primes p dividing m.*

Proof of Corollary to 1–3B. The statements of the corollary follow from Theorem 1–1, Lemma 1–3 and their implications for the following commutative diagram relating the homology of X, and X^* and the fiber F:

$$\begin{array}{ccccccccc}
0 \longrightarrow & H_{n+1}(X) & \longrightarrow & H_n(F) & \xrightarrow{(I-h)_*} & H_n(F) & \xrightarrow{\iota} & H_n(X) & \longrightarrow 0 \\
& \downarrow{\scriptstyle p-\{*,n+1\}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle p_{*,n}} & \\
0 \longrightarrow & H_{n+1}(X^*) & \longrightarrow & H_n(F) & \xrightarrow{(I-h_1)_*} & H_n(F) & \xrightarrow{\iota} & H_n(X^*) & \longrightarrow 0
\end{array}$$

Note that the horizontal rows are given by the appropriate Wang sequence, and that

$$\alpha = (I + h_1 + \cdots + h_1^{m-1})_*,$$

since $h_1^m = h$. The spectral sequence argument in the proof of Lemma 1–3 establishes that $p_{*,n}$ and $p - \{*, n+1\}$ are injective. Hence, $p_{*,n}$ is an isomorphism and α is an isomorphism if and only if $p - \{*, n+1\}$ is an isomorphism. q.e.d.

We will return to this result in §§3 and 4, where we discuss the existence of actions on algebraic knots.

Remark 1–3C. The cohomology spectral sequence for the homotopy fibration

$$X \xrightarrow{p} \partial X^* \longrightarrow B\mathbf{Z}_m$$

can be analyzed in an identical manner. We readily deduce the equivalence of the normally free condition and the fact that $H_1(\partial X^*) \cong \mathbf{Z}$. As K is simply connected, we conclude that, in the case of a free action, $\pi_1(\partial X^*) \cong \mathbf{Z}$ if and only if the action is normally free.

Proof of Lemma 1–4. By assumption, we have that $h_1|_K$ is a periodic degree-one map which induces the identity homomorphism on homology. We consider the extension problem:

$$\begin{tikzcd} K \times \{0\} \arrow[d, phantom, "\cap"] \arrow[dr, "h_1|_K"] & \\ K \times I \arrow[r, dashed, "\exists ?"] & K \times I \\ K \times \{1\} \arrow[u, phantom, "\cup"] \arrow[ur, "id"'] & \end{tikzcd}$$

Once defined, the obstructions σ_i will lie in

$$H^{i+1}(K \times I, K \times \{0\} \cup K \times \{1\}; \pi_i(K \times I)) \cong H^i(K; \pi_i(K)).$$

By assumption, K is a $\mathbf{Q}$ homology sphere, and hence, by Lemma 1–3, a mod p homology sphere for p dividing m. The nonzero vanishing obstruction groups are

$$H^{n-1}(K; \pi_{n-1}(K)) \cong H_n(K; \pi_{n-1}(K)) \cong \mathrm{Tor}(H_{n-1}(K), \pi_{n-1}(K)),$$
$$H^n(K; \pi_n(K)) \cong H_{n-1}(K; \pi_n(K)) \cong H_{n-1}(K) \otimes \pi_n(K),$$

and

$$H^{2n-1}(K; \pi_{2n-1}(K)) \cong \pi_{2n-1}(K).$$

All of these are m-torsion free. The first is m-torsion free as a consequence of Lemma 1–3; the second and third are consequences of the generalized Hurewicz Isomorphism Theorem which implies that $\pi_n(K)$ and $\pi_{2n-1}(K)$ have no m-torsion.

Next note that if the ith obstruction to a homotopy from the identity to $h_1|K$ is σ_i, then the ith obstruction to a homotopy from $h_1|_K$ to $(h_1|_K)^2$ is $(h_1|_K)^*(\sigma_i) = \sigma_i$. By the general principle of additivity of obstructions, the obstruction to a homotopy from the identity to $(h_1|_K)^j$ is $j\sigma_i$. Finally, since $h_1|_K$ has period m, $m\sigma_i = 0$, and so $\sigma_i = 0$. q.e.d.

Remark. The above ideas can be applied to the case of $H_n(K; \mathbf{Q}) \not\cong 0$. Again there are three nonvanishing obstruction groups. As above, in dimension n, the cohomology group has no m-torsion. In dimension $n+1$, the homotopy excision theorem implies that $\pi_n(K)$ can have only 2-torsion, and the generalized relative Hurewicz theorem establishes that the torsion of $\pi_{2n-1}(K)$ is "bounded" by the torsion of $\pi_{2n-1}(S^{2n-1})$ and $\pi_{2n-1}(S^{n-1})$. Hence, for certain m, it can be argued in this case as well, that $h_1|K$ is homotopic to the identity. For details, see [15].

Proof of Lemma 1–5. Let $H : K \times I \to K \times I$ be a homotopy from the identity to $h_1|_K$. If H is in fact a diffeomorphism, we can deduce our claim by way of the following diagram:

$$\begin{array}{ccc} K \times I & \xrightarrow{H} & K \times I \\ \downarrow & & \downarrow \\ K \times S^1 & \xrightarrow{\overline{H}} & K \times_T S^1 = K \times I/(x, 0) \sim (h_1(x), 1) \end{array}$$

Here $\overline{H}$ is the map induced on the quotient. In other words, it will suffice to show that under the stated assumptions, $h_1|_K$ is pseudoisotopic to the identity, and hence, $K \times_T S^1 \approx K \times S^1$. To begin, we note that each homotopy H represents an element in the structure set $\mathcal{S}(K \times I, \partial(K \times I))$, since $H|K \times (j)$, $j = 0, 1$ is a diffeomorphism, where the structure set, $\mathcal{S}(K \times I, \partial(K \times I))$, can be given a group structure by "stacking" two maps $H_i : X_i \to K \times I$, $i = 1, 2$, and glueing the domains of these maps along the appropriate pieces of the boundary via the diffeomorphism $(\delta_2^0)^{-1} \circ \delta_1^1$. Our notation is understood as $\delta_i^j = H_i^{-1}|K \times (j)$. Lastly, recall that $[H]$ is the class of homotopies between the identity on K and $h_1|_K$ which are homotopic to H, relative to the boundary, $K \cup K$.

Consider the surgery exact sequence for this structure group:

$$L_{2n+1}(\pi_1) \longrightarrow \mathcal{S}(K \times I, \partial(K \times I)) \xrightarrow{\eta} [K \times I/\partial(K \times I), G/PL]$$
$$\xrightarrow{\sigma} L_{2n}(\pi_1).$$

Since $L_{2n+1}(\pi_1) = 0$, the map η is an injective homomorphism; in particular, $\mathcal{S}(K \times I, \partial(K \times I)) \cong \ker \sigma$. The equivalence class, $[H]$, is represented by a diffeomorphism if and only if $\eta([H]) = 0$. If $\eta([H])$ is realized as the normal invariant of a homotopy from the identity to itself, J, then H and $-J$ can be "stacked" as described above, and, by virtue of the additivity of obstructions, $\eta([H \cup -J]) = 0$. The resulting map, $H \cup -J$, is another homotopy between the identity and $h_1|_K$, where $-J$

denotes

$$(id_K \times (1-t)) \circ J : X \to K \times [0, 1] \to K \times [0, 1].$$

It suffices to show that $[H]$ is realized as an element of $\widetilde{\mathscr{S}}(K \times I, \partial(K \times I))$, the subgroup of the structure group which is generated by the homotopies from the identity to itself.

First we notice that, since $(h_1|_K)^m = id|_K$, we can "stack" the homotopies $(h_1^j|_{K \times id_I} \circ H,\ j = 1, \cdots, m)$ to obtain H'. Clearly, $[H'] \in \widetilde{\mathscr{S}}$, and $\eta[H'] = m\eta[H]$.

Next we show that the group $\mathscr{S}(K \times I, \partial(K \times I))$, and hence its subgroup, $\widetilde{\mathscr{S}}(K \times I, \partial(K \times I))$, are finite and m-torsion free. This follows from the obstruction theory. An element of $[K \times I/\partial(K \times I), G/PL]$ can be viewed in terms of the obstructions to a representative of the class being homotopic to the trivial map. In particular, $\mathscr{S}(K \times I, \partial(K \times I)) \cong \ker \sigma$ is related to the cohomology groups of K by the exact sequence

$$H^n(K; \pi_{n+1} G/PL) \to \ker \sigma \to H^{n-1}(K; \pi_k G/PL),$$

since there are in fact only two dimensions in which these cohomology groups are nonzero. (Note that the top dimensional obstruction is zero as it is detected by the surgery obstruction.) As a consequence of our assumptions, it follows from the previous lemmas that these cohomology groups have no m-torsion. The same is therefore true of $\ker \sigma$, and of $\widetilde{\mathscr{S}}(K \times I, \partial(K \times I))$.

This implies that the map $(\times m)$ on $\widetilde{\mathscr{S}}$ is injective. The finiteness of $\mathscr{S}(K \times I, \partial(K \times I))$ follows from the fact that $G/PL \otimes \mathbf{Q}$ can be identified with the product of Eilenberg-MacLane space, $\prod_i K(\mathbf{Q}, 4i)$, and hence, $[K \times I/\partial(K \times I), G/PL] \otimes \mathbf{Q} \cong \sum_i H^{4i}(K \times I/\partial(K \times I); \mathbf{Q})$. For K a rational homology sphere, these will all be zero and the set of normal maps into G/PL is finite. It follows that the map $(\times m)$ on $\widetilde{\mathscr{S}}(K \times I, \partial(K \times I))$ is an isomorphism, and so $[H]$ is realized as a homotopy from the identity on K to itself. Hence, $H_1|_K$ is pseudoisotopic to the identity, and $K \times_T S^1 \cong K \times S^1$. This finishes the proof of Lemma 1–5, and hence of Theorem 1–2.

Remark. In the case where K is not a $\mathbf{Q}$ homology sphere (that is, $H_n(K)$ is nontrivial), the above arguments establish that the map $(\times m)$ is injective. In some cases, it is possible to show directly that the groups $\mathscr{S}(K \times I, \partial(K \times I))$ and $\widetilde{\mathscr{S}}(K \times I, \partial(K \times I))$ are in fact finite. For example, for the singularity, $f(\mathbf{z}) = z_0^2 + z_1^2 + \cdots + z_n^2$, the groups in question are finite for $n \neq 3(4)$, and hence, in these cases, $h_1|_K$ is necessarily pseudoisotopic to the identity for m odd. For n odd, the link of the singularity, K, is not a rational homology sphere. These examples illustrate the possibility that although the corresponding spaces X^* fiber over the circle, and have boundary diffeomorphic to $K \times S^1$, the derived semifree action is defined on a nonspherical manifold, M. Refer to [15] for details.

We are now ready to construct the derived knot for a $\mathbf{Z}_m$ action T. Let T be either free or semifree, acting as before on (S, K) with equivariant exterior pair $(X, \partial X)$ and quotient pair $(X^*, \partial X^*)$. Our aim is to construct a manifold M, with a semifree $\mathbf{Z}_m$ action, T_1, which fixes $K \hookrightarrow M$ and which is related to the original action by the condition that the closed complement of K in $M/_{T_1}$ is diffeomorphic to X^*. Hence we consider those actions, T, for which $\partial X^* \approx K \times S^1$. For example, T can be any semifree action, or any free, homologically trivial, normally free action with invariant K a $\mathbf{Q}$ homology sphere. By Theorem 1–2, a free action here is necessarily homologically trivial and normally free. Form $\sum = X^* \cup_{\partial X^*} (D^2 \times K)$. We examine $(X^*, \partial X^*)$ and show that in the cases under study, it is a knot exterior pair. That is, we show $\sum$ is a sphere. Note that if T is semifree, then $\sum \approx S^{2n+1}/_T$.

Proposition 1–6. *The quotient of an equivariant knot exterior pair is a knot exterior pair so long as the action satisfies one of the following*:

i) *T is a semifree $\mathbf{Z}_m$ action*;

or,

ii) *T is a homologically trivial $\mathbf{Z}_m$ action which is normally free with invariant submanifold a $\mathbf{Q}$ homology sphere.*

Proof. In each of the cases listed in the theorem, $\partial X^* \approx K \times S^1$. We will show that $\sum$ is diffeomorphic to a sphere. In each case, it follows that K is a $\mathbf{Z}_m$ homology sphere. It is readily deduced from the Mayer-Vietoris sequence for the decomposition $S^{2n+1} = X \cup K \times S^1$ and the exact sequence for the pair that $H_i(X, \partial X; \mathbf{Z}_m) = 0$ for $i \leq 2n-1$. Once again, by virtue of the result in [1, Chapter III, Theorem 5.5], the above implies that the homology of the quotient pair $(X^*, \partial X^*)$ is likewise trivial for $i \leq 2n-1$. Thus, $H_i(\partial X^*; \mathbf{Z}_m) = 0$ except for $i = 0, 2n-1$, and $2n$. In particular, if $n \geq 2$, then $H_{2n-2}(\partial X^*; \mathbf{Z}_m) = 0$. Further, the inclusion map induces an isomorphism

$$H_{2n-1}(\partial X^*; \mathbf{Z}_m) \to H_{2n-1}(K \times D^2; \mathbf{Z}_m);$$

the boundary map

$$H_{2n}(X^*, \partial X^*; \mathbf{Z}_m) \to H_{2n-1}(\partial X^*; \mathbf{Z}_m)$$

is an isomorphism and the connecting homomorphism in the Mayer-

Vietoris sequence

$$H_{2n+1}(\sum; \mathbf{Z}_m) \to H_{2n}(K \times S^1; \mathbf{Z}_m)$$

is an isomorphism as well. So the exact sequence of the pair gives:

$$H_i(X^*; \mathbf{Z}_m) = \begin{cases} \mathbf{Z}_m & i = 0, 1 \\ 0 & 2 \leq i \leq 2n. \end{cases}$$

Hence $H_i(\sum; \mathbf{Z}_m) \cong H_i(S^{2n+1}; \mathbf{Z}_m)$. Also, using the spectral sequence of the m-fold cover $(X, \partial X) \to (X^*, \partial X^*)$, we obtain that $\sum$ must be a $\mathbf{Z}[1/m]$ homology $(2n+1)$ sphere. Hence it remains to show that $\sum$ is simply connected. But this follows immediately from our discussion of the proof of Proposition 1-1 in which we establish that $\pi_1(X^*) \cong \mathbf{Z}$. We conclude that $\sum$ is a PL sphere, and so it can be made into a smooth sphere after changing its smoothness structure (if necessary) at a point outside X^*. q.e.d.

It follows from the above proof and the fact that $X \to X^*$ is a covering, that $\pi_i(X) \cong \pi_i(X^*)$ for all i. Therefore the exterior knot pair $(X^*, \partial X^*)$ is fibered and highly connected, and so by [4], the knot $(\sum, K)$ is determined by its knot exterior pair. We now define the *derived knot* of an equivariant simple fibered knot $(S, K; T)$ (with free action T homologically trivial normally free) to be this highly connected fibered knot with exterior pair $(X^*, \partial X^*)$.

If we consider the m-fold branched cover of $\sum$, branching over K, we obtain a manifold M, with, T_1, a semifree $\mathbf{Z}_m$ action fixing K. We define the *derived semifree action* of a homologically trivial free $\mathbf{Z}_m$ action which is normally free to be $(M, K; T)$. In particular, we may identify $M = X \cup_\psi (K \times D^2)$, where $\psi : K \times S^1 \to K \times S^1$ is a diffeomorphism. Note that if the $\mathbf{Z}_m$ action, T, is semifree, then $(S, K; T) \approx (M, K; T_1)$. If the action T is a free, homologically trivial normally free $\mathbf{Z}_m$ action with K a $\mathbf{Q}$ homology sphere, then M is also a sphere. In this case as well, the complement of the knot pair (M, K) fibers over the circle and so determines the knot (M, K). Hence, (M, K) is equivalent to the knot (S, K).

Remark. We have already noted the possibility of free, homologically trivial normally free $\mathbf{Z}_m$ actions in the case of $H_n(K) \neq 0$. Here too we can deduce that $(X^*, \partial X^*)$ is a knot exterior pair provided that we know that $\partial X^* \approx K \times S^1$. It will still hold that $H_{n-1}(K)$ is m-torsion free. But, since $H_n(K) \neq 0$, we have that $H_i(K \times S^1; \mathbf{Z}_m) \cong H_i(\partial X^*; \mathbf{Z}_m)$ is nontrivial in dimension $i = n-1$, $n, n+1$ as well as in dimensions $i = 0, 2n-1$ and $2n$, as above. In general, whether or not a union such as $\sum = X^* \cup_\phi K \times D^2$ is a homology sphere depends upon the choice of the diffeomorphism ϕ. We claim that since the action is necessarily homologically trivial normally free, there is always a diffeomorphism, ϕ, which is obtained from a pseudoisotopy of $h_1|_K$ to the identity as in the proof of Lemma 1–5, and the resulting $\sum$ is a sphere. In this case, it is also possible to define the derived semifree action. However, a careful review of the discussion, in particular, the "weave diagrams" for the homology of the decompositions

$$M = X \cup_\psi (K \times D^2),$$

and

$$\sum = X^* \cup_\phi (K \times D^2)$$

establishes that M is definitely not a homology sphere. (This is, of course, consistent with the results of Smith theory.) As this is not needed for the main results here, we do not include a proof. Details of the argument can be found in [15]. This phenomenon is illustrated by

Example. For the link of the singularity, $f(\mathbf{z}) = z_0^2 + z_1^2 + \cdots + z_n^2$, there is a free S^1 action on the knot (S, K), with K invariant, and which is given by multiplication of the coordinates z_i by $\lambda \in S^1$, that is,

$$\lambda \circ (z_0, z_1, \cdots, z_n) = (\lambda z_0, \lambda z_1, \cdots, \lambda z_n).$$

Any restriction of the S^1 action to a finite cyclic group $\mathbf{Z}_m \in S^1$ gives a homologically trivial, normally free $\mathbf{Z}_m$ action on (S, K), provided m is odd. We consider the two cases, n odd and n even. First, for n odd, recall that the link, K, is not a $\mathbf{Q}$ homology sphere. In fact, $H_n(K) \cong H_{n-1}(K)$ is free of rank one. The derived semifree action is defined on (M, K). The manifold M is nonspherical and depends upon the restriction of the free $\mathbf{Z}_m$ action to the submanifold K. For n even, the link, K, is a $\mathbf{Q}$ homology sphere with $H_{n-1}(K) \cong \mathbf{Z}_2$. In this case, the branched cover of the derived knot is a sphere, and so the derived semifree action is an action on (S, K). Before we prove classification results, we make the following remark:

Remark. If $k \mid m$ and (S, K) admits a $\mathbf{Z}_m$ action, then it trivially admits a $\mathbf{Z}_k$ action. The properties of homological triviality and normally free are inherited by the $\mathbf{Z}_k$ action. The existence of the derived knot allows us to break down a $\mathbf{Z}_m$ action into pieces, each of which is a $\mathbf{Z}_p$ action for a prime p. Note that this is a consequence of the special nature of the actions considered (free and semifree), and because of the existence of the derived knot at each stage. In particular, since $K \times S^1 \approx \partial X_p^*$, for each $p \mid m$, for any other prime factor of m, say q, the $\mathbf{Z}_q$ action on the quotient of X by the $\mathbf{Z}_p$ action, X_p^*, extends to an action on $K \times D^2$. This observation is useful in §4, when we calculate the Alexander polynomial.

The existence of a derived knot for a free or a semifree $\mathbf{Z}_m$ action also gives, as in [14], the following:

Theorem 1–7. *A simple fibered knot* (S, K) *with* K *a* $\mathbf{Q}$ *homology sphere admits a semifree* $\mathbf{Z}_m$ *action* T^s *if and only if it admits a free, homologically trivial normally free* $\mathbf{Z}_m$ *action* T^f. *Moreover,* T^s *and* T^f *may be assumed to agree on an equivariant knot complement.*

The proof is almost identical to that in [14]. To construct a free action on (S, K) from a semifree action, the existence of a free $\mathbf{Z}_m$ action on the submanifold K is required. By Smith theory in the semifree case and Theorem 1–2 in the free case, K is a $\mathbf{Z}_p$ (hence rational) homology sphere, and is simply connected. By [11] and [19], homologically trivial free $\mathbf{Z}_m$ actions exist on all simply connected $\mathbf{Q}$ homology spheres.

2. We reduce classification and existence questions to the nonequivariant case. In the previous section, we discussed the existence of the derived knot for semifree $\mathbf{Z}_m$ actions and homologically trivial free $\mathbf{Z}_m$ actions which are normally free; we also discussed the existence of the derived semifree action. Here, for K a $\mathbf{Q}$ homology sphere, we use the derived knot to reduce classification and existence questions to the nonequivariant case. In the more general situation (see [15]), the derived semifree action is the appropriate generalization of the derived knot. For the classification we need to consider, in the free case, the derived knot together with the following normal bundle information: we define two invariant knots $(S, K_i; T_i)$ $i = 1, 2$ to be *action-equivalent* if (compare [13]) there exists an equivariant normal bundle isomorphism $\Theta : \nu(K_1, S) \cong \nu(K_2, S)$.

Theorem 2–1. *Two simple fibered knots admitting free, homologically trivial normally free* $\mathbf{Z}_m$ *actions are equivariantly homeomorphic if and only if they* a) *are action-equivalent and* b) *have isotopic derived knots. Two simple fibered knots admitting semifree* $\mathbf{Z}_m$ *actions are equivariantly homeomorphic if and only if they have isotopic derived knots.*

Before giving the proof of 2–1, we mention as a corollary the classification in terms of Seifert linking forms. We define a *derived* Seifert form for $(S, K; T)$ to be a Seifert form for the derived knot [13]. It was shown, by J. Levine [10] for simple spherical knots and by A. Durfee [4] for fibered knots, that the isotopy classification is equivalent to the s-equivalence of the Seifert matrix. Moreover, in the fibered case, it follows from Alexander duality that the Seifert form corresponding to the fiber is unimodular, and so the relations of s-equivalence and equivalence coincide [18], [4]. Using this *unimodular* Seifert form we have:

Theorem 2–1a. *Two simple fibered knots admitting free, homologically trivial normally free* $\mathbf{Z}_m$ *actions are equivariantly homeomorphic if and only if* a) *they are action-equivalent and* b) *the unimodular derived Seifert forms are equivalent. Two simple fibered knots admitting semifree* $\mathbf{Z}_m$ *actions are equivariantly homeomorphic if and only if the unimodular derived Seifert forms are equivalent.*

Proof of 2–1. Clearly if $(S, K_1; T_1)$ and $(S, K_2; T_2)$ are equivariantly homeomorphic, conditions a) and b) are satisfied, since the derived knots are simple and fibered, hence [4] determined by their exterior pair $(X_j^*, \partial X_j^*)$. To show that an isotopy of the derived knots lifts to an equivariant homeomorphism we use the fibration to S^1. Let $(\sum, K_j)$, $j = 1, 2$, be the derived knots. By [4] we may assume these to be isotopic through fibered knots, in particular that the final equivalence $\Psi^* : (X_1^*, \partial X_1^*) \to (X_2^*, \partial X_2^*)$ preserves the fibration to S^1. Then Ψ^* lifts to $\Psi : (X_1, \partial X_1) \to (X_2, \partial X_2)$ such that Ψ preserves the equivariant fibration to S^1. In particular, $\Psi|_{\partial X_1} : S^1 \times K_1 \to S^1 \times K_2$ preserves the equivariant product structure. Hence by the equivariant tubular neighborhood theorem, Ψ extends equivariantly to the normal bundles $\nu(K_i)$ if and only if these are equivariantly isomorphic. In the case of a free $\mathbf{Z}_m$ action this happens exactly when the knots are action-equivalent. If the action is semifree, an equivariant isomorphism of the normal bundles of the fixed knots is equivalent to an isomorphism of the normal bundles in the derived knots. But the latter is ensured by the isotopy of these derived knots.

3. Let (S^{2n+1}, K) be an algebraic knot, hence a simple fibered knot, with (unimodular) Seifert form B for a Seifert manifold F^{2n}, and associated intersection form Q on $H_n(F)$, $Q = B + (-1)^n B^t$. Let $h_0 \in \operatorname{Aut} H_n(F)$ be the monodromy operator. (Recall that in §1, h was used to denote the monodromy map $h : F \to F$ while the monodromy operator was h_*. For simplicity of notation, we now use h for the monodromy *operator.*) First, we discuss the situation when K is a $\mathbf{Q}$ homology sphere, and then state a result for more general simple fibered knots.

If (S, K) admits a $\mathbf{Z}_m$ action such that the derived knot is defined (see §1, Proposition 1–6), then F can be taken to be a lift from the derived knot, since the fibration $(S^{2n+1} - K) \to S^1$ may be assumed to be equivariant. Suppose the corresponding derived knot invariants are: h_1 the derived monodromy operator, B^* the derived Seifert form, and Q^* the derived intersection form. Then $Q = Q^*$, and $h_1^m = h_0$. (Compare [17], [13]). Note too that $Q = B(I - h_0) = B^*(I - h_1)$, and that both h_0 and h_1 are Q-isometries; that is:

$$Q(h_i x, h_i y) = Q(x, y) \qquad i = 0, 1,$$

or in matrix notation

$$h_i^t Q h_i = Q.$$

If K is a mod p homology sphere for each $p \mid m$ (in particular a *rational* homology sphere), we conclude from the Wang sequence of the fibrations that $\det(I - h_j) \neq 0$ for $j = 0, 1$. Thus, since

$$\begin{aligned} Q &= B^*(I - h_1) = B(I - h_0) \\ &= B(I - h_1^m) = B(I - h_1)(I + h_1 + \cdots + h_1^{m-1}) \end{aligned}$$

and since both B and B^* are unimodular,

$$\det(I + h_1 + \cdots + h_1^{m-1}) = \pm 1.$$

The above also follows from the Corollary 1–3B which establishes that for simple fibered knots admitting homologically trivial normally free actions,

$$\operatorname{coker}(I + h_1 + \cdots + h_1^{m-1}) = \oplus_k \mathbf{Z}_m,$$

where $k = \operatorname{rank} H_{n+1}(S - K)$.

Finally, note that, since both (S, K) and its derived knot are fibered, the monodromy operators $h_0, h_1 \in \operatorname{Aut}(H_n(F))$. In matrix notation, $h_0, h_1 \in GL_\mu(\mathbf{Z})$, where μ, the Milnor number, equals the rank of $H_n(F)$. We now show that the above conditions are also sufficient for the existence of an action. Notice that, if $\det(I - h_i) \neq 0$, the *isometry* (Q, h_i) determines the knot. We state the result in matrix notation:

Theorem 3–1. *Let (S, K) be a simple fibered knot with K a $\mathbf{Q}$ homology sphere K, with isometry (Q, h_0). Then (S, K) admits a (semifree or free, homologically trivial normally free) $\mathbf{Z}_m$ action if and only if there exists $h_1 \in GL_\mu(\mathbf{Z})$ such that*

1) $h_1^m = h_0$,
2) $h_1^t Q h_1 = Q$,
3) $\det(I + h_1 + \cdots + h_1^{m-1}) = \pm 1$.

Proof. We need only show the sufficiency. In the process, we will also establish that the three conditions imply that K is a mod p homology sphere for $p \mid m$. Note that $(I-h_0) = (I-h_1)(I+h_1+\cdots+h_1^{m-1})$. Since K is a $\mathbf{Q}$ homology sphere, $\det(I-h_0) \neq 0$. Hence it follows from 3) that $\det(I-h_1) = \pm\det(I-h_0) \neq 0$, and $(I-h_1)^{-1} \in GL_\mu(\mathbf{Q})$. Define $B^* = Q(I-h_1)^{-1}$, and recall that $B = Q(I-h_0)^{-1}$ where B is the Seifert form for (S, K). Then B^* is a unimodular integer matrix, for

$$B^* = Q(I-h_0)^{-1}(I+h_1+\cdots+h_1^{m-1}) = B(I+h_1+\cdots+h_1^{m-1}),$$

and both B and $(I+h_1+\cdots+h_1^{m-1})$ are unimodular integer matrices. Hence the Kervaire construction produces a (derived) knot $(\sum, K)$ with Seifert form B^*, intersection form Q, and monodromy operator h_1.

Let $(X^*, \partial X^*)$ (respectively $(X, \partial X)$) be the exterior pair for $(\sum, K)$ (respectively (S, K)). By lifting to the universal abelian cover we see that, since $h_1^m = h_0$, (S, K) is an m-fold branched cover of $(\sum, K)$ with branching set K. Hence a semifree $\mathbf{Z}_m$ action T exists. (By Smith theory, K must be a mod p homology sphere for $p \mid m$.) Moreover $\partial X \to \partial X^*$ is the m-fold cover $S^1 \times K \to S^1/_{\mathbf{Z}_m} \times K$. Define the following natural $\mathbf{Z}_m$ action τ on $D^2 \times K$: τ acts on D^1 via a rotation by some (any) primitive mth root of unity, and on K via some (any) homologically trivial free $\mathbf{Z}_m$ action. (The latter always exist [11], [19] since K is a rational homology sphere.) Then $\tau|_{\partial(D^2\times K)}$ is equivalent to the standard $\mathbf{Z}_m$ action on $S^1 \times K$; the quotient $S^1 \times_{\mathbf{Z}_m} K$ is equivalent to the quotient space of $T\_\partial X$. This follows from the homologically trivial condition as in the proofs of the lemmas in §1. Glueing the free actions T restricted to $(X, \partial X)$, and τ on $(D^2 \times K, \partial(D^2 \times K))$ gives the desired free action.

Remark. For homologically trivial normally free $\mathbf{Z}_m$ actions on simple fibered knots (S, K) we have established the necessity of the certain conditions. Namely, if a simple fibered knot (S, K) with isometry (Q, h_0) admits a homologically trivial free $\mathbf{Z}_m$ action which is normally free, then the following hold:

A. $\mathrm{Tor}(H_{n-1}(K), \mathbf{Z}_m) \cong 0$.

B. There exists $h_1 \in GL_\mu(\mathbf{Z})$ such that

1) $h_1^m = h_0$,

2) $h_1^t Q h_1 = Q$,

3) the map $\alpha = (I+h_1+\cdots+h_1^{m-1})$ induces an injection with $\operatorname{coker}\alpha_* = \oplus_k \mathbf{Z}_m$, where $k = \operatorname{rank} H_n(K)$.

We have considered the sufficiency of these conditions. Of course, if $k = 0$, this is just a restatement of the theorem. For $k \neq 0$, we must have some condition which replaces the result of [18], [10], and guarantees the existence of a homologically trivial action on K and that the natural action on $K \times D^2$ restricts to an action on $K \times S^1$ with $(K \times S^1)/_\tau \cong K \times S^1$. In certain cases, the latter will follow from the conditions listed above by arguments similar to those in the proof of Lemma 1–5. In any case, once it is known that K admits a homologically trivial free $\mathbf{Z}_m$ action with generating diffeomorphism isotopic to the identity, the sufficiency of the conditions listed above follows.

4. Let (S, K) be an algebraic knot admitting a $\mathbf{Z}_m$ action. Recall that Proposition 1–6 and subsequent discussion establish conditions which ensure the existence of the derived knot. Here, we restrict our attention to those actions which satisfy one of the following:

i) T is a semifree $\mathbf{Z}_m$ action;

ii) T is a homologically trivial normally free $\mathbf{Z}_m$ action with a $\mathbf{Q}$ homology sphere as its invariant submanifold;

or

iii) T is a free $\mathbf{Z}_m$ action with $\partial X^* \approx K \times S^1$.

Let h_0, and h_1 denote the monodromy operator of the knot (S, K) and the derived knot $(\sum, K)$, respectively. We now consider the two Alexander polynomials: $\Delta(t) = \det(t - h_0)$ for (S,K) and $\Delta_D(t) = \det(t - h_1)$ for the derived knot. When (S, K) is algebraic, $\Delta(t)$ is a product of cyclotomic polynomials [12], and the rich number theoretic properties of these allow for concrete computations covering a multitude of examples. In what follows, all examples are stabilized so that the dimension of K is at least 5, and K is simply connected. The stabilization is accomplished by considering polynomials of the form

$$f(z_0, \cdots, z_k) + w_1^2 + w_2^2 + \cdots + w_l^2.$$

The relationship between the algebraic invariants of

$$f(z_0, \cdots, z_k)$$

and

$$f(z_0, \cdots, z_k) + w_1^2 + w_2^2 + \cdots + w_l^2$$

is well understood [7]. In particular, for $l = 0(4)$, the algebraic invariants (namely, the intersection form, the Seifert form and the monodromy operator) of f and its stabilization coincide.

We start with the following:

Observation. If (S, K) is an algebraic knot with Alexander polynomial $\Delta(t)$ such that $\Delta(1) = 0$, then

i) (S, K) admits no semifree $\mathbf{Z}_m$ actions;

ii) (S, K) admits no homologically trivial free $\mathbf{Z}_m$ actions if

$$\operatorname{Tor}(H_{n-1}(K), \mathbf{Z}_m)$$

is not cyclic;

iii) (S, K) admits no free, homologically trivial normally free $\mathbf{Z}_m$ actions if

$$\operatorname{Tor}(H_{n-1}(K), \mathbf{Z}_m) \neq 0.$$

Proof. Statement i) is a consequence of Smith theory. Statements ii) and iii) follow from our computations in the proof of Lemma 1–3.

Example 1. *Brieskorn polynomials.* Recall the Alexander polynomial for the singularity

$$f(z_0, \cdots, z_l) = z_0^{a_0} + \cdots + z_l^{a_l}$$

is

$$\Delta(t) = \prod (t - \rho_0 \cdots \rho_l)$$

where ρ_i runs through all a_ith roots of unity other than 1 [11]. We already have considered the singularity with $a_i = 2$ for all i. In this case, $\Delta(t) = t - (-1)^{l+1}$ and $H_{n-1}(K) \cong \mathbf{Z}$ or $\mathbf{Z}_2$, depending upon the parity of l. In particular, we saw that the homologically trivial actions given by restricting the S^1 action, Λ,

$$\lambda \circ (z_0, z_1, \cdots, z_l) = (\lambda z_0, \lambda z_1, \cdots, \lambda z_l)$$

are normally free precisely when m is odd.

Now consider the case $a_i = p$ for p prime. For all l, $\Delta(1) \neq 1$. A homologically trivial action obtained by restricting the S^1 action is normally free if and only if $(p, m) = 1$. It follows from the above that the finite subgroup of $H_{n-1}(K)$ is necessarily cyclic of order p. (This is consistent with the result of [9] which can be used to compute the homology of $K/_\Lambda$ as a subspace of $\mathbf{CP}^n$ and so the homology of K.)

There are many cases where free, homologically trivial normally free actions exist, and, as we will show, the Alexander polynomial encodes most of the information. Denote by $c_d(t)$ the cyclotomic polynomial for d.

Theorem 4–1. *Let (S, K) be a simple fibered knot with Alexander polynomial $\Delta(t) = \prod c_{d_i}(t)$ such that $\{d_i\}$ are distinct and not* 1. *Then (S, K) admits a $\mathbf{Z}_m$ action if and only if $(m, d_i) = 1$ for each i. Moreover, if m*

is odd, then $\Delta(t) = \Delta_D(t)$. *If* m *is even, there always exists an action for which* $\Delta(t) = \Delta_D(t)$, *and in general* $\Delta_D(t) = \prod c_{\varepsilon_i d_i}(t)$, *where*

$$\varepsilon_i = \begin{cases} 1 & \text{if } d_i = \text{prime power,} \\ 1 \text{ or } 2 & \text{otherwise.} \end{cases}$$

Remark. The condition that $(m, d_i) = 1$ for each i is stronger than the condition (see Theorem 3–1) that K be a mod p homology sphere for each prime p dividing m. For the latter only requires m to be prime to the collection $\{d_j\}$ for which $c_{d_j}(1) \neq \pm 1$ (equivalently when d_j is a power of a prime.)

Proof. Part 1: Suppose (S, K) admits a $\mathbf{Z}_m$ action.

First note that we may assume m to be a prime p. For if the result holds for $\mathbf{Z}_p$ actions, and (S, K) admits a $\mathbf{Z}_m$ action with m divisible by a prime p, then (S, K) admits a $\mathbf{Z}_p$ action whose derived Alexander polynomial will again be a product of *distinct* cyclotomic polynomials. Now the derived knot for the $\mathbf{Z}_p$ action (which is a highly connected, fibered knot) will admit a $\mathbf{Z}_{m/p}$ action and so on.

Let h_0 be the monodromy operator for (S, K), and for

$$\Delta(t) = \prod_{1 \leq i \leq k} c_{d_i}(t)$$

assume $d_1 > d_2 > \ldots > d_k > 1$. Let $\{\xi_{ij} \mid i = 1, \ldots, k; j = 1, \ldots, \phi(d_i)\}$ be the primitive d_ith roots of unity, hence the eigenvalues of h_0. Suppose (S, K) admits a $\mathbf{Z}_p$ action with derived monodromy operator h_1. Then $h_1^p = h_0$. If we denote the eigenvalues of h_1 by η_{ij} where $\eta_{ij}^p = \xi_{ij}$, then since p is prime, either $o(\eta_{ij}) = o(\xi_{ij}) = d_i$, or $o(\eta_{ij}) = po(\xi_{ij})$, where o denotes the order.

Claim 1. For p odd, $(p, d_i) = 1$ and $o(\eta_{ij}) = o(\xi_{ij})$.

Proof. Suppose $o(\eta_{1j}) = pd_1$. Then $c_{pd_1}(t) \mid \Delta_D(t)$, so that all the primitive (pd_1)th roots of unity occur among the $\{\eta_{ij}\}$. Since there are $\phi(pd_1) > \phi(d_1)$ of these, there exist $i > 1$ and j such that $o(\eta_{ij}) = pd_1$. But $o(\eta_{ij}) = pd_i$ or d_i, neither of which is equal to pd_1 since $d_1 > d_i$. Hence $o(\eta_{1j}) = d_1$, and since $\eta_{1j}^p = \xi_{1j}$ and η_{1j} are both primitive d_ith roots of unity, we see that $(p, d_1) = 1$. Now $c_{d_1}(t)$ divides both $\Delta_D(t)$, and $\Delta(t)$. Let $\Delta_{D2}(t) = \Delta_D(t)/c_{d_1}(t)$, and $\Delta_2(t) = \Delta(t)/c_{d_1}(t)$, apply the above argument to see that $o(\eta_{2j}) = d_2$ and that $(p, d_2) = 1$. Iteration of this procedure yields the desired result for all i.

Claim 2. If $p = 2$, then $(p, d_i) = 1$, and $\Delta_D(t) = \prod c_{\varepsilon_i d_i}(t)$, where $\varepsilon_i = 1$ or 2.

Proof. We first show that $2 \nmid d_1$. As before, $o(\eta_{1j}) = d_1$, or $o(\eta_{1j}) = 2d_1$. If $o(\eta_{1j}) = d_1$, then $2 \nmid d_1$. If $o(\eta_{1j}) = 2d_1$, then $c_{2d_1}(t) \mid \Delta_D(t)$, hence the $\phi(2d_1)$ primitive $(2d_1)$th roots of unity occur among the $\{\eta_{ij}\}$. If $2 \mid d_1$, then $\phi(2d_1) = 2\phi(d_1)$, and this is impossible as in the proof of Claim 1. Thus $2 \nmid d_1$. Again, as for Claim 1, let $\Delta_2(t) = \Delta(t)/c_{d_1}(t)$, and

$$\Delta_{D2}(t) = \begin{cases} \Delta_D(t)/c_{d_1}(t) & \text{if } o(\eta_{1j}) = d_1, \\ \Delta_D(t)/c_{2d_1}(t) & \text{if } o(\eta_{1j}) = 2d_1. \end{cases}$$

Application of the above argument to η_{2j}, $\Delta_2(t)$ and $\Delta_{D2}(t)$ produces the result for $i = 2$. Iteration of the procedure completes the proof of Claim 2. To complete the proof of Part 1 we simply note that if $\varepsilon_i = 2$, then d_i cannot be a prime power. Recall (Theorem 3–1) that $\Delta(1) = \Delta_D(1)$ and so $c_{2d_1}(1) = c_{d_1}(1)$ whenever $\varepsilon_i = 2$ and $c_d(1) \neq \pm 1$ if and only if d is a prime power.

Part 2. Suppose conversely that $(m, d_i) = 1$ for each i.

We produce an operator h_1 satisfying the conditions in Theorem 3–1. Notice first that since the Alexander polynomial is a product of distinct cyclotomic polynomials

$$\Delta(t) = \Pi c_{d_i}(t),$$

the monodromy operator h_0 is periodic with order

$$o(h_0) = d = \text{l.c.m. } \{d_i\},$$

where "l.c.m." denotes the least common multiple. Choose l such that $lm \equiv 1 \pmod d$, and let $h_1 = h_0^l$. Then h_1 satisfies conditions 1) and 2) of Theorem 3–1. Moreover, we know that K^{2n-1} is a rational homology sphere, since $\Delta(1) \neq 0$. We need only show that for each prime p dividing m, $H_{n-1}(K^{2n-1})$ has no p torsion. But this is equivalent to $(m, d_i) = 1$ for each i with $c_{d_i}(1) \neq 1$. Since by assumption $(m, d_i) = 1$ for *every* i, it follows that K is a mod p homology sphere for each prime p dividing m. Finally, condition 3) is equivalent to:

$$ch_{h_0}(1) = \pm ch_{h_1}(1).$$

Here ch denotes the characteristic polynomial, and of course $\Delta(t) = ch_{h_0}(t)$. But in fact $h_1 = h_0^l$, and

$$ch_{h_0}(t) = \prod c_{d_i}(t),$$

and $(l, d_i) = 1$. Hence by a straightforward argument, each $\eta_{ij} = \xi_{ij}^l$ is a primitive d_ith root of unity and $ch_{h_0}(t) = ch_{h_1}(t)$. Thus condition 3) is also satisfied.

This finishes the proof of the theorem.

As a consequence of the proof of Theorem 4–1 and the Classification Theorem 2–1, we have the following:

Corollary 4–2. *Let (S, K) be a simple fibered knot with Alexander polynomial $\Delta(t) = \Pi c_{d_i}(t)$ such that $\{d_i\}$ are distinct and not* 1. *If (S, K) admits a $\mathbf{Z}_m$ action and m is odd, then the restriction of this action to the knot complement is uniquely determined by m.*

Proof. Consider the monodromy operator h_0 of (S, K), and the derived monodromy operator $h_1 = h_0^l$ as in Theorem 4–1. Suppose h_2 is another derived monodromy operator satisfying the conditions in the Existence Theorem 3–1. Then $ch_{h_2}(t) = ch_{h_0}(t)$, so that $o(h_2) = o(h_0) = o(h_1)$; thus $h_2 = h_1$. Uniqueness of the action on the knot complement follows from the Classification Theorem 2–1.

Example 2. *More Brieskorn singularities.* Consider the Alexander polynomial $\Delta(t)$ of the singularity for

$$f(z_0, \cdots, z_n, w_1, \cdots, w_k) = z_0^{a_0} + \cdots + z_n^{a_n} + w_1^2 + \cdots + w_k^2,$$

where the exponents a_i are coprime and $a_i > 2$. It is easy to see that, in this case, $\Delta(t)$ is a product of distinct cyclotomic factors. In fact, from the formula for the Alexander polynomial of a Brieskorn singularity (Example 1), a direct computation shows the following:

a) If k is even

$$\Delta(t) = \prod c_{d_j}(t),$$

where $d_j = b_{0j}b_{1j}\cdots b_{nj}$, for each i, b_{ij} running through all divisors of a_i, except 1.

b) If k is odd, and each a_i is odd, then

$$\Delta(t) = \prod c_{2d_j}(t),$$

where d_j are exactly as in a) above.

c) If k is odd, and a_0 is even, then

$$\Delta(t) = \prod c_{d_j}(t),$$

where $d_j = b_{0j}b_{1j}\cdots b_{nj}$, b_{ij} as above for $i > 0$, and b_{0j} running through all divisors of a_0, *including* 1, but *excluding* 2. Notice the special case $n = 0$. This is the only case in which $d_j = 1$ occurs, hence the only case in which K fails to be a $\mathbf{Q}$ homology sphere.

Characterization 2–A. The link of singularity for the polynomial

$$f(z_0, \cdots, z_n, w_1, \cdots, w_k) = z_0^{a_0} + \cdots + z_n^{a_n} + w_1^2 + \cdots + w_k^2,$$

a_i relatively prime, $a_i > 2$, admits a (free, homologically trivial normally free or semifree) $\mathbf{Z}_m$ action in precisely the following situations:

a) k *even*: if and only if m is prime to each a_i;

b) k *odd*, a_i *odd*: if and only if m is odd and prime to each a_i;

c) k *odd*, a_0 *even*, $n > 0$: if and only if m is prime to each a_i.

Moreover, when m is odd, the action is uniquely determined on the knot complement.

Thus, except possibly when k is even and every a_i is odd, each $\mathbf{Z}_m$ action on the complement is uniquely determined by the integer m.

As an example, consider the polynomial $z_0^6 + z_1^5 + w_1^2 + w_2^2$. This admits $\mathbf{Z}_m$ actions if and only if m and 30 are relatively prime. Moreover, m uniquely determines the action on the knot complement.

In the one remaining case (see [15]),

c′) k *odd*, a_0 *even*, $n = 0$:

$$\Delta(t) = (t-1)c_d(t)c_{2d}(t), \quad \text{where } a_0 = 2d.$$

Here K is not a $\mathbf{Q}$ homology sphere, and semifree $\mathbf{Z}_m$ actions never exist; homologically trivial free $\mathbf{Z}_m$ actions can exist but only if m is prime to a_0. Then, the derived semifree action is defined on a nonspherical manifold, M.

Example 3. $\mathbf{Z}_2$ actions where $\Delta(t) \neq \Delta_D(t)$.

Consider (S, K) a spherical simple fibered knot, whose Alexander polynomial splits, as in Theorem 4–1, into distinct cyclotomic factors

$$\Delta(t) = \prod c_{d_j}(t) \qquad d_j \neq 1.$$

Suppose further that (S, K) admits a $\mathbf{Z}_2$ action, or equivalently every d_j is odd. Such is the case, for instance, when f is as in Example 2 above with: a_i odd, k even, and $n \geq 1$. (The condition $n \geq 1$ is needed to ensure sphericity.)

Let h_0 denote the monodromy operator of (S, K). Since $ch_{h_0} = \Delta(t)$, h_0 is periodic (its eigenvalues are distinct), and its order $o(h_0)$ is odd. Thus there is a power of h_0, say, $h_1 = h_0^l$, (in fact, $2l = o(h_0) + 1$) such that $h_1^2 = h_0$. Then

$$\det(I - h_0) = \det(I - h_1)\det(I + h_1),$$

and by sphericity of (S, K), all three determinants must be ± 1. Hence both h_1 and $-h_1$ satisfy the conditions of Theorem 3–1, thus exhibiting two different $\mathbf{Z}_2$ actions. It is now easy to see that

$$\Delta(t) = \Delta_{h_1}(t) = ch_{h_1}(t) = \prod c_{d_j}(t)$$

and

$$\Delta_{-h_1}(t) = ch_{-h_1}(t) = \prod c_{2d_j}(t).$$

In particular, the $\mathbf{Z}_2$ action with derived monodromy $-h_1$ has derived Alexander polynomial $\Delta_{h_1}(t) \neq \Delta(t)$.

Example 4. Let (S, K) be the link of singularity for the polynomial

$$f = z_0^q + a_1^q + w_1^2 + \cdots + w_k^2,$$

q an odd prime, k odd. Then (S, K) admits a $\mathbf{Z}_m$ action if and only if $(m, 2q) = 1$.

Proof. The Alexander polynomial is

$$\Delta(t) = (c_2(t))^{q-1}(c_{2q}(t))^{q-2}.$$

Notice that in this case Theorem 4–1 does not apply.

Denote the eigenvalues of the monodromy operator h_0 by $\xi_{q,i}$ and $\xi_{2,j}$, depending on whether ξ is a primitive $2q$th root of unity or square root of unity. Suppose $\mathbf{Z}_m$ acts with derived monodromy operator h_1, and let $\eta_{q,i}$ and $\eta_{2,j}$ be the eigenvalues of h_1, where $\eta_*^m = \xi_*$. If p is any prime dividing m, then (S, K) must admit a $\mathbf{Z}_p$ action. We show that $(p, 2q) = 1$. Notice first that $p \neq 2$, since $\Delta(1) = 2^{q-1}$. We now claim that $p \neq q$. For if $p = q$, then $\eta_{q,j}^q$ must be a primitive $2q$th root of unity, and thus η is a primitive $2q^2$ root of unity. In particular, $c_{2q^2}(t) | \Delta_D(t)$, where $\Delta_D(t)$ is the derived Alexander polynomial for the $\mathbf{Z}_p$ action. But this is impossible, since $\deg c_{2q^2}(t) > \deg \Delta(t) = \deg \Delta_D(t)$. Hence if a $\mathbf{Z}_m$ action exists, m and $2q$ must be relatively prime.

Conversely, suppose $(m, 2q) = 1$. We know [12] that h_0 must be periodic, and thus its order is $o(h_0) = 2q$. Hence for each m with $(m, 2q) = 1$, there exists an appropriate power h_1 of h_0, such that $h_1^m = h_0$. In this case $ch_{h_1}(t) = ch_{h_0}(t) = \Delta(t)$, and thus, the algebraic conditions of Theorem 3–1 (Existence) are easily seen to be satisfied. Hence a $\mathbf{Z}_m$ action exists, with derived monodromy operator h_1. q.e.d.

As a special case we mention:

Example 4–A. For $f = z_0^3 + z_1^3 + w_1^2$, a $\mathbf{Z}_m$ action exists if and only if $m \equiv \pm 1 \pmod 6$. In this case:

a) the derived knot is isotopic to the original knot, and

b) m uniquely determines the action on the knot complement.

Proof. The first part is simply a restatement of the general result above. To obtain conditions a) and b), it suffices to deduce, from the Alexander polynomial and the periodicity of h_0, that $o(h_0) = 6$. For then $h_0 = h_1^{\pm 1}$ for $m = \pm 1$ mod 6, respectively and a) and b) must hold. q.e.d.

Consider now an arbitrary algebraic knot (S, K). The Seifert matrix B and monodromy operator h_0 can in principle be calculated from a Dynkin diagram in a distinguished basis. (See for example Husein-Zade [7] for definitions of these.) Often, these diagrams can be used to compute all possible $\mathbf{Z}_m$ actions, either by using the Alexander polynomial or by investigating the monodromy operator h_0, together with its mth roots h_1. We give one more class of examples.

Example 5. $f(x, z, w_1, \dots w_l) = x^k z + az^n + \sum w_i^2$ $(k \geq 2, n \geq 2, a \neq 0.)$

The Dynkin diagrams of these singularities were calculated by Gabriélov [6]. We used these to compute, for some k and n, the Alexander polynomials which we give below. Note that there are two different Alexander polynomials, which we will denote by $\Delta_E(t)$ and $\Delta_O(t)$, corresponding to an even and odd number of squares, w_i^2.

a) $k = 3, n = 3$.

$$\Delta_E(t) = (t - 1)c_9(t).$$

No semifree actions, since $\Delta(1) = 0$; a free, homologically trivial normally free $\mathbf{Z}_m$ action only if $m \equiv \pm 1, \pm 2, \pm 4,$ (mod 9).

$$\Delta_O(t) = c_2(t)c_{18}(t).$$

$\mathbf{Z}_m$ actions (all unique) exist if and only if $m \equiv \pm 1, \pm 5, \pm 7,$ (mod 18).

b) $k = 4, n = 3$.

$$\Delta_E(t) = (t - 1)^2[c_2(t)]^2 c_3(t)[c_6(t)]^2$$

and

$$\Delta_O(t) = (t - 1)^2[c_2(t)]^2[c_3(t)]^2 c_6(t).$$

No semifree actions, since $\Delta(1) = 0$; a free, homologically trivial normally free $\mathbf{Z}_m$ action exists only if $m \equiv \pm 1$ (mod 6).

c) $k = 3, n = 8$.

$$\Delta_E(t) = (t - 1)c_3(t)c_6(t)c_{12}(t)c_{24}(t).$$

No semifree actions exist, since $\Delta(1) = 0$; a free, homologically trivial normally free $\mathbf{Z}_m$ action exists only if $m \equiv \pm 1, \pm 5, \pm 7, \pm 11$ (mod 24).

$$\Delta_0(t) = c_2(t)c_3(t)c_6(t)c_{12}(t)c_{24}(t).$$

$\mathbf{Z}_m$ actions (all unique) if and only if $m \equiv \pm 1, \pm 5, \pm 7, \pm 11$ (mod 24).

References

[1] G. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972.

[2] W. Browder & J. Levine, *Fibering manifolds over a circle*, Comment. Math. Helv. **40** (1966) 153–160.

[3] M. Davis, "The Smith Conjecture", (J. Morgan and H. Bass, eds.), Academic Press, New York, 1984.

[4] A. Durfee, *Fibered knots and algebraic singularities*, Topology **13** (1974) 47–59.

[5] A. Durfee & L. Kauffman, *Periodicity of branched cyclic covers*, Math. Ann. **218** (1975) 157–174.

[6] A. M. Gabriélov, *Dynkin diagrams for unimodal singularities*, Functional Anal. Appl. **8** (1974) 192–196.

[7] S. M. Husein-Zade, *The monodromy groups of isolated singularities of hypersurfaces*, Russian Math. Surveys **32** (1977) 23–69.

[8] C. Kearton & S. M. J. Wilson, *Cyclic group actions on odd-dimensional spheres*, Comment. Math. Helv. **56** (1981) 615–626.

[9] R. Kulkarni & J. Wood, *Nonsingular complex hypersurfaces*, Advances in Math. **35** (1978) 239–263.

[10] J. Levine, *An algebraic classification of some knots of codimension two*, Comment. Math. Helv. **45** (1970) 185–198.

[11] P. Löffler, *Die existenz freier* $\mathbf{Z}_k$ *operationen auf rationalen homologiespharen*, J. Reine Angew. Math. **326** (1981) 171–186.

[12] J. Milnor, *Singular Points of Complex Hypersurfaces*, Princeton University Press, Princeton, NJ, 1968.

[13] M. Nicolau, *A classification of invariant knots*, Duke Math. J. **58** (1989) 151–171.

[14] ——, *Semigree actions on spheres*, Pacific J. Math **156** (1992) 337–358.

[15] M. Nicolau & S. Szczepanski, *A torsion invariant of symmetric nonspherical knots*, to appear.

[16] N. Stoltzfus, *Equivariant concordance of invariant knots*, Trans. Amer. Math. Soc. **254** (1979) 1–45.

[17] P. Strickland, *Branched cyclic covers of simple knots*, Proc. Amer. Math. Soc. (3) **90** (1984) 440–444.

[18] H. Trotter, *Homology of group systems with applications to knot theory*, Ann. Math **76** (1962) 464–498.

[19] S. Weinberger, *Homologically trivial group actions*. I, Amer. J. Math **108** (1986) 1005–1022.

University of Illinois, Urbana
Lehigh University