J. DIFFERENTIAL GEOMETRY
40 (1994) 563-594

ON AFFINE CRYSTALLOGRAPHIC GROUPS

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Dedicated to Professor J. L. Mennicke, on the occasion of his 60th birthday

1. Introduction

Setting the scene. An *affine crystallographic group* (ACG) is a properly discontinuous group Γ of affine transformations on some (finite-dimensional) real vector space V, such that the quotient space $\Gamma\backslash V$ is compact. If Γ is also torsion free, then $\Gamma\backslash V$ is a compact *affine space form*, with fundamental group isomorphic to Γ; every flat, complete, compact connected differentiable manifold arises this way [24, Corollary 1.9.6]. If $\Gamma \leq A$, where A is a given subgroup of $\mathrm{Aff}(V)$, the group of all affine transformations of V, we shall call Γ an ACG *of type* A (for example, when A is the group of all Euclidean motions, an ACG of type A is a Bieberbach group); and if Γ is torsion free, we call $\Gamma\backslash V$ a *space form of type* A (every flat, complete, compact connected pseudo-Riemannian manifold is one of these, A being a suitable group of isometries; see [24, Theorem 2.4.9]).

Bieberbach proved that every Bieberbach group is a finite extension of its (free abelian) translation subgroup, that in each dimension there are only the finitely many isomorphism types of the Bieberbach group, and that isomorphic Bieberbach groups are conjugate in the affine group (see [24, §3.2]). None of these results is true of ACGs of more general type, but there are weaker analogues which do generalize, at least conjecturally. We shall explore some of these.

A long-standing conjecture [16] asserts that *every ACG is virtually polycyclic* (i.e., has a polycyclic subgroup of finite index). It has been proved for ACGs in dimension ≤ 3 [10], for ACGs of type A whenever A is an extension of the translation group by (the real points of) a reductive algebraic group of real rank at most 1 (for example, the group of affine Lorentz transformations) [13], and in some other cases (see [11], [20], [21].) In this paper we deal exclusively with virtually polycyclic ACGs. Throughout,

Received September 4, 1992 and, in revised form, December 16, 1993.

V will denote a fixed finite-dimensional real vector space, and A a fixed Zariski-closed subgroup of $\mathrm{Aff}(V)$.

Suppose Θ is an ACG of type A, where A is the Euclidean group, and let Δ be the translation subgroup of Θ. The essentially geometric part of the Bieberbach theory is that Δ is a full lattice in the group of all translations, and that Δ is the maximal abelian subgroup of Θ. The main finiteness result, which is essentially algebraic, then states that given such a full lattice Δ, there are only finitely many possibilities, up to conjugacy, for the crystallographic group Θ. We discuss generalizations of the "geometric" theory later; our main finiteness result is the analogue of the second part of Bieberbach's theory.

Recall that each virtually polycyclic group Θ has a unique maximal nilpotent normal subgroup, its *Fitting subgroup*. We denote this by $\mathrm{Fitt}(\Theta)$.

Definition. Let $\Delta \le \Theta$ be virtually polycyclic groups. Then Θ is a *strict extension* of Δ if the index $|\Theta:\Delta|$ is finite and $\mathrm{Fitt}(\Theta) \le \Delta$.

We say that Θ is a *normal* extension of Δ if $\Delta \triangleleft \Theta$, and call $|\Theta:\Delta|$ the *index* of the extension. The normalizer of Δ in A is denoted $N_A(\Delta)$.

Theorem A. *Let Δ be a virtually polycyclic ACG of type A. Then*:

(i) *the strict normal extensions of Δ in A lie in finitely many conjugacy classes in $N_A(\Delta)$*;

(ii) *the strict extensions of Δ in A have bounded index, and they lie in finitely many conjugacy classes in A.*

The proof of Theorem A depends on finiteness properties of *arithmetic groups* (analogously to the role of the Jordan-Zassenhaus Theorem in the Bieberbach theory). Our second main result provides the means whereby these can be exploited; here, we shall say that a Lie group K is *of type* $\mathcal{NP}$ if its identity component K_0 is nilpotent and K/K_0 is polycyclic.

Theorem B. *Let Δ be a virtually polycyclic ACG of type A. Then $N_A(\Delta)$ contains a closed normal subgroup K of type $\mathcal{NP}$ such that $N_A(\Delta)/K$ is isomorphic to an arithmetic group.*

These results can be phrased in geometric terms, when applied to a torsion-free ACG Δ. Suppose $M = \Theta\backslash V$ and $N = \Phi\backslash V$ are space forms of type A. Let us (for convenience) define an *isometry* of M onto N to be a homeomorphism $\tilde{\alpha}: M \to N$ which lifts to an automorphism α of V with $\alpha \in A$. Then $\alpha\Theta\alpha^{-1} = \Phi$, so M and N are isometric if and only if Θ and Φ are conjugate in A. When $A = \mathrm{Aff}(V)$, "isometric" simply means "affinely isomorphic"; for the pseudo-Riemannian case, see [24, Lemma 2.5.6], which shows that our "isometries" are precisely the isometries in the usual sense.

The group of all self-isometries of M is denoted $\mathrm{Aut}_A(M)$; this is exactly the image of $N_A(\Theta)$ in the group of all self-maps of M. We shall show (in §6) that $\mathrm{Aut}_A(M) \cong N_A(\Theta)/\Theta$, and hence infer from Theorem B:

Corollary B. *If M is a space form of type A with virtually polycyclic fundamental group, then $\mathrm{Aut}_A(M)$ is an extension of a Lie group of type $\mathcal{NP}$ by an arithmetic group.*

Now suppose $p\colon L = \Delta\backslash V \to M = \Theta\backslash V$ is a covering map, induced by the inclusion $p^*\colon \Delta \hookrightarrow \Theta$ of torsion-free ACGs of type A. Call p a *strict* (*normal*) covering if $p^*(\Delta) \geq \mathrm{Fitt}(\Theta)$ (respectively, $p^*(\Delta) \lhd \Theta$; see [24, p. 35]). We say that p is *equivalent* to a covering $q\colon L \to N = \Phi\backslash V$ if there exist an isometry $\beta\colon M \to N$ and an element $\alpha \in \mathrm{Aut}_A(L)$ such that the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\alpha} & L \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{\beta} & N \end{array}$$

Theorem A can then be formulated as follows:

Corollary A. *Let L be a space form of type A, with virtually polycyclic fundamental group. Then*

(i) *the strict normal covering $L \to M$ of type A lie in finitely many equivalence classes;*

(ii) *the strict coverings $L \to M$ of type A have bounded multiplicities, and the corresponding space forms M lie in finitely many isometry classes.*

Note that although the covering maps we consider are supposed to preserve the affine structure (or "A-structure"), the concept of *strictness* is purely topological, depending as it does only on the induced mapping of the fundamental groups.

A version of Theorem A for nilpotent groups was stated in [15] and recently proved in [9].

Standard groups. Of course, every Bieberbach group is a strict normal extension of its translation subgroup, so Theorem A is an honest generalization of what we called the "second part" of the Bieberbach theory. For this to be of any use, however, we have to know that every ACG is a strict extension of some well-understood kind of group, the analogue of a full lattice in the group of all translations. Our candidate for this rôle is the *standard* ACG.

First, some notation. Putting $W = V \oplus \mathbb{R}$, $\mathbb{R}$ being the set of real numbers, we identify $\mathrm{Aff}(V)$ with the subgroup of $\mathrm{GL}(W)$ consisting of matrices $\left(\begin{smallmatrix} g & v \\ 0 & 1 \end{smallmatrix}\right)$, where $g \in \mathrm{GL}(V)$ and $v \in V$ (think of V as consisting of column vectors). For a subgroup H of $\mathrm{GL}(W)$, write

$$\begin{aligned} \overline{H} &= \text{Zariski closure of } H \text{ in } \mathrm{GL}(W) \\ H^0 &= \text{identity component of } H \text{ in the Zariski topology} \\ H_0 &= \text{identity component of } H \text{ in the Lie group topology} \\ \mathrm{u}(H) &= \text{maximal normal unipotent subgroup of } H \end{aligned}$$

Note that

$$(\overline{H})_0 \leq_f (\overline{H})^0 \leq_f \overline{H},$$

where $A \leq_f B$ stands for "*A is a subgroup of finite index in B*". Note also that if H is unipotent, then H is Zariski-closed in $\mathrm{GL}(W)$ if and only if H is closed and connected (see [17, Chapter II]). Note finally that if H is virtually soluble, then $(\overline{H})^0$ is soluble, $\mathrm{u}(H)$ consists of all the unipotent element sc f H, and $\mathrm{u}(H) \leq \mathrm{u}((\overline{H})_0) = \mathrm{u}(\overline{H})$, $(\overline{H})^0/\mathrm{u}(\overline{H})$ is abelian; these facts, which follow easily from the Lie-Kolchin Theorem and the preceding sentence, will be used without special mention.

We shall say that H is *u-connected* if $\mathrm{u}(H)$ is connected (topological terms will always refer to the Lie group topology induced from $\mathrm{GL}(W)$, unless prefixed by "Zariski").

An important observation is

Lemma C. *If Δ is a virtually polycyclic ACG, then* $\mathrm{Fitt}(\Delta) = \mathrm{u}(\Delta)$.

With these preliminaries out of the way, we can make the

Definition. A subgroup Γ of $\mathrm{GL}(W)$ is *standard* if Γ is discrete and polycyclic, $\Gamma \leq (\overline{\Gamma})_0$, and $\Gamma/\mathrm{u}(\Gamma)$ is torsion-free.

In a Bieberbach group, the translation subgroup is the unique maximal standard subgroup. A virtually polycyclic ACG need not, in general, have a unique maximal standard subgroup. However, for the purposes of classification, it would suffice to have a canonical way of assigning to each ACG Θ a unique standard subgroup θ^* so that Θ is a strict normal extension of Θ^*; then Theorem A reduces the classification of virtually polycyclic ACGs of type A to (a) the classification of standard ACGs of type A, and (b) producing for each such standard ACG a finite list of representatives for its strict normal extensions.

We may proceed as follows. Let Θ be a virtually polycyclic ACG. Start by defining $\Theta^\dagger = \Theta \cap (\overline{\Theta})_0$. Then $\mathrm{u}(\Theta^\dagger) = \mathrm{u}(\theta)$ and $\Theta^\dagger/\mathrm{u}(\Theta)$ is finitely generated abelian group. Denote by $m(\Theta)$ the exponent of the torsion subgroup of this abelian group, and put $\Theta^* = (\Theta^\dagger)^{m(\Theta)}\mathrm{u}(\theta)$. It is easy to see that Θ^* is standard, and Lemma C shows that Θ is a strict normal extension of Θ^*. All our requirements are thus met by this definition.

Now we must justify the claim that the standard ACGs form a well-understood class of groups, analogous to the full lattices in the translation group. The place of the translation group in the Bieberbach theory will be taken by a *simply transitive* group of affine transformations; we say that a subgroup G of $\mathrm{Aff}(V)$ is *simply transitive* if G is closed in $\mathrm{Aff}(V)$ and the operation of G on V is simply transitive, i.e., if the orbit map $g \mapsto g \cdot 0$ of G not V is a bijection. This map is then a homeomorphism, so G is connected and simply connected; and it is known that G must be soluble [1], [16]. (It has recently been shown by Benoist [2] that these properties are *not* sufficient to imply that a Lie group can be realized as a simply transitive group of affine transformations; his counterexamples are nilpotent of dimension 11. See also [12] for more examples.)

We shall prove

Theorem D. *A group is a standard ACG if and only if it is a Zariski-dense uniform lattice in a u-connected simply transitive group of affine transformations.*

Here, a *uniform lattice* means a discrete cocompact subgroup. Theorem D amplifies a result due to Fried and Goldman [10], as does the following corollary, which is immediate from Theorem D and the preceding discussion:

Corollary D. *Every compact affine space form with virtually polycyclic fundamental group has a strict normal covering by an affine solvmanifold.*

Crystallographic hulls. What Fried and Goldman show in [10] is that if Δ is a virtually polycyclic ACG, then there exists a simply transitive subgroup G of $\mathrm{Aff}(V)$ such that $G \cap \Delta$ is a Zariski-dense uniform lattice in G and $G \cap \Delta \leq_f \Delta$. This suggests the following question: *given a simply transitive group G and a Zariski-dense uniform lattice Γ in G, how much restriction is there on the subgroups Δ of* $\mathrm{Aff}(V)$ *such that $\Gamma \leq_f \Delta$ and* $G \cap \Delta = \Gamma$? On attempting to answer this, one finds that here the strict analogy with the Euclidean crystallographic case breaks down: we show by examples in §9 that *these groups Δ may lie in infinitely many conjugacy classes in* $\mathrm{Aff}(V)$, *and the indices $|\Delta : \Gamma|$ may be unbounded.*

However, a weaker analogy survives.

Definition. Let Γ be a polycyclic subgroup of $\mathrm{GL}(W)$, and G a subgroup of $\mathrm{GL}(W)$. Then G is a *syndetic hull* for Γ if G is closed and connected, Γ is a Zariski-dense uniform lattice in G, and $\dim G = h(\Gamma)$.

Here, $h(\Gamma)$ denotes the Hirsch length ("rank" in [10]) of the polycyclic group Γ. This definition is a slight modification of one from [10]. It is shown in [10] and §4 that if Γ is an ACG, then $h(\Gamma) = \dim V$; so if Γ and G are as in the previous paragraph, then G is a syndetic hull for Γ.

Definition. Let Δ be a subgroup of $\mathrm{GL}(W)$, and Γ a polycyclic subgroup of Δ. Then Δ is a *geometrically strict extension* of Γ if $|\Delta : \Gamma|$ is finite and there exists a syndetic hull G for Γ such that $G \cap \Delta = \Gamma$.

We shall prove

Theorem E. *If Γ is a polycyclic ACG, then the geometrically strict extensions of Γ lie in finitely many isomorphism classes.*

The geometric meaning of Theorem E is that given the space form $\Gamma\backslash V$, the space forms $\Delta\backslash V$ corresponding to geometrically strict extensions Δ of Γ lie in finitely many homeomorphism classes; this follows from Theorem E in view of [10, Theorem 1.20].

In spite of the counterexamples mentioned above, we can retrieve the full strength of Theorem A, in the present framework, by using a more stringent concept of hull. Let us call a syndetic hull G for Γ a *good hull* if $u(G)$ is connected, i.e., if G is *u-connected.* Define a *good geometrically strict extension* as we defined "geometrically strict extension", replacing "syndetic hull" with "good hull". We shall prove

Lemma F. *If Γ is a polycyclic ACG, then every good geometry strict extension of Γ is a strict extension of Γ.*

Thus as a special case of Theorem A we can state

Theorem G. *Let Γ be a virtually polycyclic ACG of type A. Then:*

(i) *the good geometrically strict normal extensions of Γ in A lie in finitely many conjugacy classes in $N_A(\Gamma)$;*

(ii) *the good geometrically strict extensions of Γ in A have bounded index, and they lie in finitely many conjugacy classes in A.*

Arrangement of the paper. Section 2 is technical. Section 3 gives some general conditions under which the normalizer of a polycyclic linear group can be represented as an arithmetic group; this is a major ingredient in the proof of Theorem B. Section 4 examines syndetic hulls in some detail, and contains the proof of Theorem D. Section 5 gives a characterisation of simply transitive groups in terms of unipotent simply transitive groups; it also contains the proofs of Lemmas C and F.

Theorem B and Corollary B are proved in §6. Theorem A is proved in §7 which also contains the proof of Theorem E, apart from a technical step which is made in §8. Section 9 gives examples of nonconjugate geometrically strict extensions.

Notation.

$g^x = x^{-1}gx$

$[a, b] = a^{-1}b^{-1}ab$

$\langle X\rangle$: group generated by the set X

$[A, B] = \langle [a, b] | a \in A,\ b \in B\rangle$

$\Gamma' = [\Gamma, \Gamma]$: derived group of Γ
$N_A(\Gamma)$: normalizer of Γ in A
$C_A(\Gamma)$: centralizer of Γ in A
$\mathrm{Der}(\Gamma, M)$: additive group of all derivations of Γ in the Γ-module M
$A \triangleleft B$: A is a normal subgroup of B
$A \leq_f B$: A is a subgroup of finite index in B
$\mathrm{Fitt}(\Gamma)$: maximal nilpotent normal subgroup of Γ
$u(\Gamma)$: maximal unipotent normal subgroup of Γ
$x = x_u x_s = x_s x_u$ denotes the multiplicative Jordan decomposition of a matrix x.

2. The "unipotent shadow" of a polycyclic group

The main results become quite easy to prove when restricted to ACGs which are unipotent. Our strategy is to associate to each polycyclic ACG a certain unipotent group (which will turn out also to be an ACG, though we do not make this explicit), and then to build our arguments around this unipotent group. The construction is explained in this preliminary section.

In this section and the next, we work over an arbitrary field k of characteristic zero; the proofs are the same as in the special case where $k = \mathbb{R}$, and there is a genuine gain in generality.

We fix a positive integer n and a polycyclic subgroup Γ of $GL_n(k)$. We write

$$\begin{aligned} \overline{\Gamma} &= \text{Zariski closure of } \Gamma \text{ in } GL_n(k), \\ U &= u(\overline{\Gamma}), \\ F &= u(\Gamma) = U \cap \Gamma, \end{aligned}$$

and assume that

$$\Gamma' \leq F, \tag{1}$$

which implies also that

$$[\overline{\Gamma}, \overline{\Gamma}] \leq \overline{F} \leq U. \tag{2}$$

We write

$$F^{\mathbb{Q}} = \langle x \in U \mid x^m \in F \text{ for some } m \neq 0 \rangle,$$

and for $m \in \mathbb{N}$ put

$$F^{1/m} = \langle x \in F^{\mathbb{Q}} \mid x^m \in F \rangle.$$

Every element of $F^{\mathbb{Q}}$ has some power in F; every finitely generated subgroup of $F^{\mathbb{Q}}$ is contained in $F^{1/m}$ for some m; and for each m, the group $F^{1/m}$ is finitely generated and satisfies $|F^{1/m}: F| < \infty$.

If X is a nilpotent subgroup of $\overline{\Gamma}$, then $X_u = \{x_u | x \in X\}$ and $X_s = \{x_s | x \in X\}$ are subgroups of $\overline{\Gamma}$, the map $X \to X_u \times X_s$, $x \mapsto (x_u, x_s)$ is a homomorphism, and X_s is abelian because $[\overline{\Gamma}, \overline{\Gamma}] \leq U$; see [18, Chapter 7, Proposition 3]. It follows that $X \leq \mathrm{C}_{\overline{\Gamma}}(X_s)$.

Put $Y = \mathrm{N}_{\mathrm{GL}_n(k)}(\Gamma)$.

Proposition 2.1. *There exist a natural number m and a nilpotent subgroup D of $F^{1/m}\Gamma$ such that:*

(i) $F^{1/m}D = F^{1/m}\Gamma$,
(ii) $Q := \langle F^{1/m}, D_u \rangle$ *is a finitely generated subgroup of* U,
(iii) Y *normalizes* Q,
(iv) Q *is Zariski dense in* U,
(v) $|Y: F\mathrm{N}_Y(D)| \leq |Y: F\mathrm{C}_Y(D_S)| < \infty$,
(vi) $\mathrm{h}(Q) \leq \mathrm{h}(\Gamma)$.

Proof. Put $G = F^{\mathbb{Q}}\Gamma$. By [18, Chapter 7, Exercise 9], G contains a nilpotent subgroup X such that $G = F^{\mathbb{Q}}X$ (the exercise assumes that $G/F^{\mathbb{Q}}$ is free abelian, but this is not needed for the proof). Put $Z = \mathrm{C}_G(X_s)$. Then $X \leq Z \leq \mathrm{C}_U(X_s) \times X_s$, so Z is nilpotent and $X_s = Z_s$. It follows that Z is a maximal nilpotent supplement for $F^{\mathbb{Q}}$ in G (for if $Z \leq T \leq G$ and T is nilpotent, then $T_s \leq Z_s$ implies $T \leq \mathrm{C}_G(T_s) \leq \mathrm{C}_G(Z_s) = Z$).

For $m \in \mathbb{N}$, put $D(M) = Z \cap F^{1/m}\Gamma$. Since Γ is finitely generated and $F^{\mathbb{Q}}\Gamma = G = F^{\mathbb{Q}}Z$, there exists e such that $F^{1/e}\Gamma = F^{1/e}D(e)$; then $F^{1/m}\Gamma = F^{1/m}D(m)$ and $F^{\mathbb{Q}}\Gamma = F^{\mathbb{Q}}D(m)$ for every multiple m of e. Given any such m, we see as above that $D(m)_s = Z_s$, and (similarly to the above) that $D(m)$ is a maximal nilpotent supplement for $F^{1/m}$ in $F^{1/m}\Gamma$. Moreover,

$$\mathrm{N}_Y(D(m)) \leq \mathrm{N}_Y(D(m)_s) = \mathrm{N}_Y(Z_s) \leq \mathrm{N}_Y(\mathrm{C}_{F^{1/m}\Gamma}(Z_s)) = \mathrm{N}_Y(D(m));$$

consequently

$$\mathrm{N}_Y(D(m)) = \mathrm{N}_Y(Z_s) = N,$$

say, independently of m.

According to [18, Chapter 3, Theorem 4], the maximal nilpotent supplements for $F^{1/e}$ in $F^{1/e}$ lie in finitely many conjugacy classes. As $D(e)^y$ is such a supplement for each $y \in Y$, and $|F^{1/e}: F|$ is finite, it follows that $|Y: \mathrm{N}_Y(D(e))F|$ is finite. In other words, $|Y: NF|$ is finite.

Let $y_1, \cdots, y_r$ represent the right cosets of NF in Y. By [18, Chapter 7, Exercise 9] (quoted above), the maximal nilpotent supplements for $F^{\mathbb{Q}}$ in G are all conjugate. Hence for $i = 1, \cdots, r$ there exists $x_i \in F^{\mathbb{Q}}$ such that $Z^{y_i} = Z^{x_i}$. We now choose a multiple m of e so that $\{x_1, \cdots, x_r\} \subseteq F^{1/m}$.

Put $D = D(m)$. Then (i) holds, as clearly does (ii). Suppose $y \in Y$. Then $y \in NFy_i = Ny_iF$ for some i, so there exists $x \in F$ with

$$Z_x^y = Z_s^{y_i x} = Z_s^{x_i x}.$$

Since $D = C_{F^{1/m}\Gamma}(Z_s)$ this implies $D^y = D^{x,x}$, and hence $D_u^y = D_u^{x_i x}$. As $x_i x \in F^{1/m}$ it follows that Y normalizes $Q = \langle F^{1/m}, D_u \rangle$; so we have (iii).

Let $\overline{Q}$ be the Zariski closure of Q in $\overline{\Gamma}$. Then $\overline{Q} \triangleleft \Gamma$, since $Q \geq F \geq [\Gamma, \Gamma]$. We may assume without loss of generality that k is algebraically closed. Then $\overline{\Gamma}/\overline{Q}$ is an algebraic group containing the Zariski-dense subgroup $\overline{Q}\Gamma/\overline{Q} = \overline{Q}D_s/\overline{Q}$. Since D_s is a diagonalizable group, so is $\overline{\Gamma}/\overline{Q}$. It follows that $U \leq \overline{Q}$; thus (iv) holds.

As for (v), we have seen that $N_Y(D) = N_Y(D_s) = N$ and that $|Y : NF|$ is finite. Since D_s is a diagonalizable group, $|N_Y(D_s) : C_Y(D_s)|$ is also finite.

Finally, we consider $h(Q)$. Now D has a subgroup E of finite index such that E_u normalizes $F^{1/m}$ [18, Chapter 7, Lemma 7]. Then $E_u \leq_f D_u$ and $D_u \cong D/D_uD_s$, so

$$h(E_u) = h(D) - h(D \cap D_s).$$

Also

$$F^{1/m} \cap E \leq F^{1/m} \cap E_u \leq F^{1/m} \cap D_u \leq C_{F^{1/m}}(D_s) = F^{1/m} \cap D,$$

so $h(F^{1/m} \cap E_u) = h(F^{1/m} \cap D)$. Therefore

$$\begin{aligned} h(F^{1/m}E_u) &= h(F^{1/m}) + h(E_u) - h(F^{1/m} \cap E_u) \\ &= h(F^{1/m}) + h(D) - h(F^{/1m} \cap D) - h(D \cap D_s) \\ &= h(F^{1/m}D) - h(D \cap D_s). \end{aligned}$$

But $F^{1/m}E_u \leq_f Q$, since Q is a finitely generated nilpotent group, generated by the elements each of which has some positive power lying in $F^{1/m}E_u$; so $h(Q) = h(F^{1/m}E_u)$. Also $\Gamma \leq_f F^{1/m}\Gamma = F^{1/m}D$. Thus $h(Q) \leq h(\Gamma)$.

3. Normalizers of polycyclic subgroups

We keep the notation of §2; in addition, we fix a Zariski-closed subgroup A of $\mathrm{GL}_n(k)$, so that $A = \mathscr{A}(k)$ is the group of k-rational points of a linear algebraic k-group $\mathscr{A}$. For a subgroup H of $\mathscr{A}$, the Zariski-closure of H in $\mathscr{A}$ is denoted $\widehat{H}$; thus if $H \leq A$ we have $\overline{H} = \widehat{H}(k) = \widehat{H} \cap \mathrm{GL}_n(k)$.

We assume that the polycyclic group Γ is contained in A, and put

$$Y_A = \mathrm{N}_A(\Gamma) = A \cap Y, \qquad \mathscr{U} = u(\widehat{\Gamma}).$$

Note that then $U = \mathscr{U}(k)$. We assume further that the following hold:

$$\widehat{\Gamma} \text{ is Zariski connected}, \tag{3}$$

$$\dim \mathscr{U} = \mathrm{h}(\Gamma), \tag{4}$$

and

$$\mathrm{C}_{\widehat{Y}_A}(\Gamma) \text{ is unipotent}. \tag{5}$$

Since $\widehat{\Gamma} \lhd \widehat{Y}_A$, we also have $\mathscr{U} \lhd \widehat{Y}_A$, so the adjoint representation of $\widehat{Y}_A$ restricts to a k-rational morphism $\rho\colon \widehat{Y}_A \to \mathrm{GL}(\mathscr{L})$, where $\mathscr{L}$ is the Lie algebra of $\mathscr{U}$.

A subgroup $\Delta \leq \mathrm{GL}(\mathscr{L})$ is *arithmetic* if, for some $\mathbb{Q}$-structure on $\mathscr{L}$, there exists an algebraic $\mathbb{Q}$-subgroup $\mathscr{H}$ of $\mathrm{GL}(\mathscr{L})$ such that Δ is commensurable with $\mathscr{H}(\mathbb{Z})$. This is equivalent to the following, which may be taken as an alternative definition: there exists a full $\mathbb{Z}$-lattice Λ in $\mathscr{L}$ such that Δ is a subgroup of finite index in a Zariski-closed subgroup of $\mathrm{GL}(\Lambda)$.

The aim of this section is to establish

Proposition 3.1. *$\rho(Y_A)$ is an arithmetic group.*

This is the main step in the proof of Theorem B, which will be completed in §6.

Once we have defined Y_A, the algebraic group $\mathscr{A}$ plays no further role. So to simplify notation, we may as well assume that $A = \widehat{Y}_A$. Thus $\widehat{\Gamma} \lhd \mathscr{A}$, and ρ is defined on $\mathscr{A}$. Note that now $\ker \rho = \mathrm{C}_{\mathscr{A}}(\mathscr{U})$.

In the following, we write $\mathrm{stab}_{\mathscr{B}(k)}(\Lambda)$ for the set of all $x \in \mathscr{B}(k)$ such that $\rho(x)$ fixes Λ.

Lemma 3.2. *There exist a full $\mathbb{Z}$-lattice Λ in $\mathscr{L}$ and a Zariski k-closed subgroup $\mathscr{B}$ of $\mathscr{A}$ such that:*

(i) $C_{\mathscr{B}}(\mathscr{U}) = C_{\mathscr{A}}(\Gamma)$,
(ii) $\rho(Y_A)$ *stabilizes* Λ, *and putting* $\Pi = \mathrm{stab}_{\mathscr{B}(k)}(\Lambda)$,
(iii) $\Pi \cap Y_A \leq_f \Pi$,
(iv) $(\Pi \cap Y_A)F \leq_f Y_A$.

Recall that $F = U \cap \Gamma$.

Before proving this, we complete the

Proof of Proposition 3.1. Fix a $\mathbb{Z}$-basis for Λ and thereby identify $\mathrm{GL}(\mathscr{L})$ with the algebraic matrix group GL_h, where $h = \dim L$ $(= \mathrm{h}(\Gamma)$, by (4)). Then

$$\langle \rho(Y_A), \rho(\Pi) \rangle \leq \mathrm{GL}_h(\mathbb{Z}). \tag{6}$$

Now hypothesis (5), with Lemma 3.2(i), implies that $\rho(\mathscr{B}(k)) = \rho(\mathscr{B})(k)$ [4, Corollary 15.7]. Also $\rho(\mathscr{B}(k)) \cap \mathrm{GL}_h(\mathbb{Z}) = \rho(\Pi)$, by the definition of Π. It follows that

$$\rho(\Pi) = \rho(\mathscr{B})(k) \cap \mathrm{GL}_k(\mathbb{Z}) = \rho(\mathscr{B}) \cap \mathrm{GL}_h(\mathbb{Z}).$$

Thus $\rho(\Pi)$ is Zariski closed in $\mathrm{GL}_h(\mathbb{Z})$, and so $\rho(\Pi)$ is an arithmetic group. With Lemma 3.2(iii) this shows that $\rho(\Pi \cap Y_A)$ is arithmetic.

The group $\rho(F)$ is a unipotent subgroup of $\mathrm{Gl}_h(\mathbb{Z})$, so $\rho(F)$ is arithmetic. Since $F \lhd Y_A$, $\rho(\Pi \cap Y_A)$ normalizes $\rho(F)$. It follows that $\rho((\Pi \cap Y_A)F)$ is arithmetic also (see Lemma 3.3). Then Lemma 3.2(iv), with (6), shows that $\rho(Y_A)$ is an arithmetic group. This establishes Proposition 3.1, modulo Lemma 3.2 and the following

Lemma 3.3. *If* $\Theta, \Psi \leq \mathrm{GL}_h(\mathbb{Z})$ *are arithmetic groups, and* Θ *normalizes* Ψ, *then* $\Psi\Theta$ *is an arithmetic group.*

Proof. Let R, N be the Zariski closures in Gl_h of Θ and Ψ respectively. Then R normalizes N. Let G be the Zariski closure of RN. Then $N \lhd G$, so there exists a $\mathbb{Q}$-rational epimorphism $\theta: G \to H$, for some algebraic Q-group H, with $\ker\theta = N$. Thus $\theta(R)$ is Zariski closed in H, and $RN = \theta^{-1}\theta(R)$ is also so in G, whence $G = RN$, and $H = \theta(R)$. Since Θ is an arithmetic subgroup of R, Borel's theorem [6, Theorem 6] shows that $\theta(\Theta)$ is an arithmetic subgroup of H. Similarly, $\theta(G(\mathbb{Z}))$ is an arithmetic subgroup of H; as $\Theta \leq G(\mathbb{Z})$, it follows that $\theta(\Theta) \leq_f \theta(G(\mathbb{Z}))$. Therefore $N\Theta \leq_f NG(\mathbb{Z})$, and by taking the intersection of $NG(\mathbb{Z})$ with $\mathrm{GL}_h(\mathbb{Z})$ we obtain $N(\mathbb{Z})\Theta \leq_f N(\mathbb{Z})G(\mathbb{Z}) = G(\mathbb{Z})$. Since $|N(\mathbb{Z}): \Psi|$ is finite, it follows that $\Psi\Theta \leq_f G(\mathbb{Z})$, showing that $\Psi\Theta$ is an arithmetic subgroup of G.

Remark. Readers familiar with the Galois cohomology of algebraic groups will see that the hypothesis (5) could be weakened to that

$H^1(k, C_{\widehat{Y}_A}(\Gamma))$ is finite. The only change in the proof is that now $\rho(\Pi)$ has finite index in $\rho(\mathscr{B}) \cap \mathrm{GL}_h(\mathbb{Z})$, instead of being equal to it. Note that this weaker condition is satisfied whenever k is a local field (including the case $k = \mathbb{R}$; see [7]).

Proof of Lemma 3.2. Let $Q = \langle F^{1/m}, D_u \rangle$ be the subgroup of U given in Proposition 2.1. We take $\Lambda = \mathbb{Z} \log Q \subseteq \mathscr{L}$, where log denotes the inverse of the exponential map $\mathscr{L} \to \mathscr{U}$. Since Y normalizes Q, certainly $\rho(Y_A)$ stabilizes Λ, and (ii) holds.

Since Q is Zariski dense in $\mathscr{U}$, the set $\log Q$ is Zariski dense in $\mathscr{L}$, and consequently Λ spans $\mathscr{L}$. Now $Q^{\mathbb{Q}} = \exp(\mathbb{Q} \log Q)$ is the Mal′cev completion of Q, and it follows from the theory of the Mal′cev correspondence that $\dim_{\mathbb{Q}}(\mathbb{Q} \log Q) = \mathrm{h}(Q)$. Therefore Λ is a free $\mathbb{Z}$-module of rank $\mathrm{h}(Q)$. Since $\mathrm{h}(Q) \le \mathrm{h}(\Gamma) = \dim \mathscr{L}$ and Λ spans $\mathscr{L}$, this implies that $\mathrm{h}(Q) = \mathrm{h}(\Gamma)$ and that Λ is a full $\mathbb{Z}$-lattice in $\mathscr{L}$.

Next, we define $\mathscr{B}$ by

$$\mathscr{B} = C_{\mathscr{A}}(D_s),$$

where D is the subgroup of $F^{1/m}\Gamma$ given in Proposition 2.1. Since $\Gamma \le UD_s \le \widehat{\Gamma}$, it is clear that $C_{\mathscr{B}}(\mathscr{U}) = C_{\mathscr{A}}(\widehat{\Gamma})$, so we have (i). Since Y_A stabilizes Λ, it follows from the definition of $\Pi = \mathrm{stab}_{\mathscr{B}}(\Lambda)$ that $\Pi \cap Y_A = C_{Y_A}(D_s)$. Hence, by Proposition 2.1(v), $(\Pi \cap Y_A)F \le_f Y_A$, giving (iv).

Finally we prove (iii). First of all, we claim that

$$[\widehat{\Gamma}, \mathscr{A}^0] \le \widehat{F}. \tag{7}$$

To see this, note that $\widehat{\Gamma}/\mathscr{U}$ is an algebraic torus, normal in $\mathscr{A}/\mathscr{U}$. This implies that $\widehat{\Gamma}/\mathscr{U}$ is central in $\mathscr{A}^0/\mathscr{U}$ [4, §8.10], so $[\widehat{\Gamma}, \mathscr{A}^0] \le \mathscr{U}$. Then $[\Gamma, Y_A \cap \mathscr{A}^0] \le \Gamma \cap \mathscr{U} = F$. But $Y_A \cap \mathscr{A}^0$ is Zariski dense in $\mathscr{A}^0$ (since we have assumed that $\widehat{Y}_A = \mathscr{A}$), so (7) follows. Next, put

$$Q_1 = \langle \exp \Lambda \rangle, \qquad M = Q_1 \cap \widehat{F}.$$

Then Q_1 is a finitely generated subgroup of $Q^{\mathbb{Q}}$ (this can be deduced, for example, from [18, Chapter 6, Lemma 1]); and M is the isolator of F in Q_1, because the vector space $\mathbb{Q} \log F$ is Zariski closed in the space $\mathbb{Q} \log Q_1$. Thus $F \le_f M$, and it follows that $\Gamma \le_f MD$; note that D normalizes M, since $D \le F^{1/m}\Gamma \le QY$ implies that D normalizes Q_1.

Now let $x \in \Pi \cap \mathscr{A}^0$, $h \in M$, and $g \in D$. Then $g_u \in Q_1$ and $g_s^x = g_s$. Clearly $Q_1^x = Q_1$ which together with (7) shows that $h^x \in M$

and $g_u^x g_u^{-1} \in M$, giving

$$(hg)^x = h^x \cdot g_u^x g_u^{-1} \cdot g_u g_s \in Mg.$$

Thus $\Pi \cap \mathscr{A}^0$ normalizes MD. As MD has only finitely many subgroups of index equal to $|MD\colon \Gamma|$, some subgroup of finite index in $\Pi \cap \mathscr{A}^0$ normalizes Γ, and since $|\Pi\colon \Pi \cap \mathscr{A}^0|$ is finite, we conclude that $|\Pi\colon \Pi \cap Y_A|$ is finite, as required.

4. Hulls

Here we collect a number of results concerning syndetic hulls, and give the proof of Theorem D.

We keep the notation of §1, so W is a real vector space, $W = V \oplus \mathbb{R}$, and $\mathrm{Aff}(V) \leq \mathrm{GL}(W)$. By an ACG we shall mean a discrete subgroup of $\mathrm{Aff}(V)$ which is acting properly discontinuously and cocompactly on V.

Fix a standard subgroup Γ of $\mathrm{GL}(W)$, and write $F = \mathrm{u}(\Gamma)$, $U = \mathrm{u}(\overline{\Gamma})$. Then $\Gamma \leq (\overline{\Gamma})_0$, so $\Gamma' \leq F = U \cap \Gamma$ and $\overline{\Gamma'} = \overline{\Gamma}' \leq \overline{F} \leq U \leq (\overline{\Gamma})_0$; to see the first equality, note that $\overline{\Gamma'}$ is Zariski closed since it is closed, connected and unipotent.

Proposition 4.1. *There exists a syndetic hull G for Γ satisfying $\overline{F} \leq G$.*

Proof. Let $\pi\colon E \to (\overline{\Gamma})_0/\overline{F}$ be a universal cover of the connected abelian Lie group $(\overline{\Gamma})_0/\overline{F}$ and put $K = \ker \pi$. Then E is a vector group, and K is a discrete subgroup of E.

Now $F = \Gamma \cap \overline{F}$ is discrete and cocompact in $\overline{F}$, and $\dim \overline{F} = \mathrm{h}(F)$ (see [17, Chapter II]). As Γ is discrete in $(\overline{\Gamma})_0$, this implies that $\Gamma\overline{F}/\overline{F}$ is discrete in $(\overline{\Gamma})_0/\overline{F}$. It follows that $\pi^{-1}(\Gamma\overline{F}/\overline{F}) = S$, say, is a discrete subgroup of E. Since Γ is standard, Γ/F is a free abelian group. Therefore so is S/K, and also so $S = K \oplus P$ for some subgroup P. Note that $\pi/_P$ maps P isomorphically onto $\Gamma\overline{F}/\overline{F}$.

Let $\overline{P}$ be the vector subspace of E spanned by P, and define $G \leq (\overline{\Gamma})_0$ by $G/\overline{F} = \pi(\overline{P})$. We claim that G is a syndetic hull for Γ.

Certainly G is closed and connected, and $\Gamma \leq G \leq \overline{\Gamma}$. Also $\dim G = \mathrm{h}(\Gamma)$, since $\dim(G/\overline{F}) = \dim(\overline{P}) = \mathrm{h}(P) = \mathrm{h}(\Gamma/F)$, as P is a discrete subgroup of E, and $\dim(\overline{F}) = \mathrm{h}(F)$ as observed above. We have seen that $\overline{F}/(\Gamma \cap \overline{F})$ is compact; and $G/(\gamma\overline{F})$ is compact because it is homeomorphic to $\overline{P}/P$. It follows that G/Γ is compact. This completes the proof.

Remark. The above construction was inspired by the work of Fried and Goldman [10]. The reader is warned, however, that the construction given in §1.10 of [10] does not work: the group it produces has too large a dimension, in general. The fault is easily repaired by using our argument instead. For an explicit example, see §9.

Lemma 4.2. *Let* Γ *be as above and assume that* $\mathrm{Fitt}(\Gamma) = \mathrm{u}(\Gamma)$. *Let* G *be a syndetic hull for* Γ. *Then the following are equivalent*:

(a) $\mathrm{u}(G)$ *is connected*;
(b) $\mathrm{u}(G) = \overline{F}$;
(c) $\overline{F} \leq G$.

Proof. Since $\mathrm{u}(G)$ is connected if and only if it is Zariski closed, and since $F \leq \mathrm{u}(G)$, we see that (b) $\Rightarrow$ (a) $\Rightarrow$ (c). Assume now that $\overline{F} \leq G$. Let L be the Lie algebra of $\overline{F}$ and put $\Lambda = \mathbb{Z}\log F \subset L$; then Λ is a full $\mathbb{Z}$-lattice in L, and we identify Λ with $\mathbb{Z}^d$ and L with $\mathbb{R}^d$ by choosing a basis for Λ. Write $\rho\colon \overline{\Gamma} \to \mathrm{GL}|(L) = \mathrm{GL}_d(\mathbb{R})$ for the adjoint representation of $\overline{\Gamma}$ on L. Then $\rho(\Gamma) \leq \mathrm{GL}_d(\mathbb{Z})$, so the Zariski closure H of $\rho(\Gamma)$ is defined over $\mathbb{Q}$, as is the group $\mathrm{u}(H)$. Note that $\rho(G) \leq H$ since $G \leq \overline{\Gamma}$. Put $L_0 = 0$, and for $i \geq 1$ let L_i/L_{i-1} be the fixed-point space of $\mathrm{u}(H)$ in L/L_{i-1}. Then $0 = L_0 < L_1 < \cdots < L_r = L$ for some r, and each L_i is an H-invariant subspace of L defined over $\mathbb{Q}$. The action of H on L induces a $\mathbb{Q}$-rational representation

$$\psi\colon H \to \prod_{i=1}^{r} \mathrm{GL}(L_i/L_{i-1}),$$

with $\ker\psi = u(H)$; and $\psi(H(\mathbb{Z}))$ is contained in an arithmetic subgroup of $\psi(H)$, by an elementary property of $\mathbb{Q}$-rational representations. Therefore, $\psi(H(\mathbb{Z}))$ is discrete in $\psi(H)$; as $\rho(\Gamma) \leq H(\mathbb{Z})$ it follows that $\mathrm{u}(H)\rho(\Gamma) = \psi^{-1}\psi\rho(\Gamma)$ is closed in H.

Put $X = G \cap \rho^{-1}(u(H))$. Then $\rho(X) = \rho(G) \cap u(H)$, so

$$\rho(X\Gamma) = \rho(G) \cap \mathrm{u}(H)\rho(\Gamma)$$

by the modular law. Therefore $\rho(X\Gamma)$ is closed in $\rho(G)$, and so $X\Gamma$ is closed in G. Since G/Γ is compact, so is $X\Gamma/\Gamma$. Therefore $X/(X \cap \Gamma)$ is compact. But X is nilpotent, since $G/\overline{F}$ is abelian, so

$$X \cap \Gamma \leq \mathrm{Fitt}(\Gamma) = F \leq X \cap \Gamma,$$

showing that X/F is compact. Now clearly

$$F \leq \overline{F} \leq \mathrm{u}(G) \leq X,$$

and $u(G)$ is closed in X. Therefore $\mathrm{u}(G)/F$ is compact, and as $\mathrm{u}(G)$ is unipotent it follows that $\mathrm{u}(G) \leq \overline{F}$. Thus (c) $\Rightarrow$ (b) and the proof is complete.

Lemma 4.3. *If* $\Delta \leq \mathrm{Aff}(V)$ *is a virtually polycyclic ACG, then* $\mathrm{h}(\Delta) = \dim(V)$.

Proof. By going to a subgroup of finite index, we may assume that Δ is torsion free and polycyclic. Then the cohomological dimension $\mathrm{cd}(\Delta)$ of Δ is equal to $\mathrm{h}(\Delta)$ [3]. On the other hand, the compact manifold $\Delta\backslash V$ is a $K(\Delta, 1)$ (see, e.g., [8, Chapter I, §4]); this implies that $\mathrm{cd}(\Delta) = \dim(\Delta\backslash V) = \dim(V)$ (see [8, Chapter VIII, Proposition 8.1]).

Proposition 4.4. *Let* Γ *as above be an ACG, and let* G *be a syndetic hull for* Γ. *Then* $G \leq \mathrm{Aff}(V)$, *and the affine action of* G *on* V *is simply transitive.*

Proof. Let K be a maximal compact subgroup of G. According to [8, Chapter VIII, §9, Example 4],

$$\mathrm{cd}(\Gamma) = \dim(G) - \dim(K).$$

As $\dim(G) = \mathrm{h}(\Gamma)$, by definition, it follows that $\dim(K) = 0$, and since G is connected this implies that $K = 1$ (see for example [22, Chapter 3, Exercises 42 and 36]). Thus G has no nontrivial compact subgroups.

Since $G \leq \overline{\Gamma}$ by definition, $G \leq \mathrm{Aff}(V)$. As Γ acts freely on V and G/Γ is compact, the stabilizer in G of each point of V is compact. Hence, by the previous paragraph, G acts freely on V. Lemma 4.3 shows that $\dim(G) = \mathrm{h}(\Gamma) = \dim(V)$; therefore the orbits of G in V are open, and as V is connected it follows that the action of G on V is transitive.

Remark. This argument is similar to one used in [10]. Fried and Goldman do not require that $\dim(G) = \mathrm{h}(\Gamma)$, merely that $\dim(G) \leq \mathrm{h}(\Gamma)$; this will clearly suffice for our proof also, which then *implies* $\dim(G) = \mathrm{h}(\Gamma)$.

Lemma 4.5. *Let* S *be an abelian subgroup of* $\mathrm{Aff}(V)$ *which is diagonalizable over* $\mathbb{C}$. *Then* S *has a fixed point in* V.

Proof. $\mathbb{A}(V)$ acts on $\mathbb{C}^{n+1} = \mathbb{C}V \times \mathbb{C}$ and fixes the complex hyperplane $\mathbb{C}V \times 1$. Now S has an eigenvector $(w, \lambda) \in \mathbb{C}V \times \mathbb{C}$ with $\lambda \neq 0$. Since $S \leq \mathrm{Aff}(V)$, the point $(\lambda^{-1}w, 1)$ is fixed by S. Then $v = \frac{1}{2}(\lambda^{-1}w + \overline{(\lambda^{-1}w)})$ is a fixed point of S in V, where the bar denotes complex conjugate.

Lemma 4.6. *Let* Γ *as above be an ACG and let* G *be a syndetic hull for* Γ. *Let* N *be the maximal nilpotent closed normal subgroup of* G. *Then* $N = \mathrm{u}(G)$ *and* $N \cap \Gamma = \mathrm{Fitt}(\Gamma)$. *Consequently,* $\mathrm{Fitt}(\Gamma) = \mathrm{u}(\Gamma)$.

Proof. The group $N_s = \{u_s | u \in N\}$ is a diagonalizable (over $\mathbb{C}$) abelian subgroup of $\mathrm{Aff}(V)$. By Lemma 4.5, the set T of fixed points of n_s in V is nonempty. But G fixes T and acts transitively on V, so $T = V$. It follows that $N_s = 1$, so N is unipotent. As $\mathrm{U}(G) \leq N$ we get $N = \mathrm{u}(G)$.

As $\overline{\mathrm{Fitt}(\Gamma)} \triangleleft \overline{\Gamma}$ and $G \leq \overline{\Gamma}$ we see that $\overline{\mathrm{Fitt}(\Gamma)} \cap G \leq N$. From this it is clear that $\Gamma \cap N = \mathrm{Fitt}(\Gamma)$.

Proof of Theorem D. We have to show that a standard ACG is the same thing as a Zariski-dense uniform lattice in a simply transitive u-connected subgroup of $\mathrm{Aff}(V)$.

Suppose $G \leq \mathrm{Aff}(V)$ is simply transitive and u-connected, and let Δ be a Zariski-dense uniform lattice in G. As Δ is discrete in G, its action on V is properly discontinuous; and $\Delta\backslash V$ is compact because it is homeomorphic to $\Delta\backslash G$. So Δ is an ACG. Since $\Delta \leq G \leq \overline{\Delta}$ and G is connected, $\Delta \leq (\overline{\Delta})_0$. Finally,

$$\Delta/\mathrm{u}(\Delta) \cong \Delta\mathrm{u}(G)/\mathrm{u}(G) \leq G/\mathrm{u}(G);$$

since G is soluble, connected and simply connected, and $\mathrm{u}(G)$ is connected, $G/\mathrm{u}(G)$ is a vector group. Hence $\Delta/\mathrm{u}(\Delta)$ is torsionfree. Thus Δ is standard.

For the converse, we may take Γ, as above, to be an ACG. By Proposition 4.1, Γ has a syndetic hull G with $\overline{F} \leq G$. Lemma 4.6 shows that $\mathrm{Fitt}(\Gamma) = \mathrm{u}(\Gamma) = F$, and it follows from Lemma 4.2 that $\mathrm{u}(G)$ is connected. Proposition 4.4 implies that G is a simply transitive subgroup of $\mathrm{Aff}(V)$. As Γ is a Zariski-dense uniform lattice in G by definition, this completes the proof.

In the course of the argument, we have more or less proved the following:

Proposition 4.7. *Let Δ be a polycyclic ACG.*

(i) Δ is standard if and only if Δ possesses a good hull.

(ii) Suppose $\Delta \leq G \leq \overline{\Delta}$; then G is a syndetic hull for Δ if and only if G is a simply transitive subgroup of $\mathrm{Aff}(V)$.

We leave it to the reader to fill in the details; note that if Δ has a syndetic hull G, then G is also a syndetic hull for Δ^*, so G is a simply transitive subgroup of $\mathrm{Aff}(V)$ by Proposition 4.4.

5. Simply transitive groups

As in the introduction, A will denote a Zariski-closed subgroup of $\mathrm{Aff}(V)$; though as far as this section is concerned, we may as well take $A = \mathrm{Aff}(V)$.

Lemma 5.1. *If $H \leq \mathrm{Aff}(V)$ and H acts transitively on V, then $\mathrm{C}_A(H)$ is unipotent.*

Proof. Suppose $x \in \mathrm{C}_A(H)$ is semisimple. By Lemma 4.5, x has a fixed point w, say, in V. Then $h \cdot w$ is fixed by x for every $h \in H$, and as H is transitive on V it follows that $x = 1$. Thus $\mathrm{C}_A(H)$ contains no nonidentity semisimple elements, and as $\mathrm{C}_A(H)$ is Zariski closed it must be unipotent.

Proposition 5.2. *If G is a simply transitive subgroup of* $\mathrm{Aff}(V)$, *then* $\mathrm{u}(\overline{G})$ *is also simply transitive.*

This result is due to Auslander [1]; it is included in Theorem 5.5, which we shall prove below.

Lemma 5.3. *Let $G \leq \mathrm{Aff}(V)$ be simply transitive, and let Δ be a virtually polycyclic ACG. Then the following groups are all unipotent:* $\mathrm{C}_A(\Delta)$, $\mathrm{C}_A(\mathrm{u}(\overline{\Delta}))$, *and* $\mathrm{C}_A(\mathrm{u}(\overline{G}))$.

Proof. The result for $\mathrm{C}_A(\mathrm{u}(\overline{G}))$ follows from Proposition 5.2 and Lemma 5.1. Now put $\Gamma = \Delta^*$. Then Γ is a standard ACG, so by Theorem D we may suppose that $\Gamma \leq G \leq \overline{\Gamma}$. Thus

$$\mathrm{C}_A(\Delta) \leq \mathrm{C}_A(\Gamma) = \mathrm{C}_A(\overline{\Gamma}) \leq \mathrm{C}_A(G);$$

and as $(\overline{\Delta})^0 = (\overline{\Gamma})^0$ we have

$$\mathrm{u}(\overline{\Delta}) = \mathrm{u}(\overline{\Gamma}) \geq \mathrm{u}(G),$$

so $\mathrm{C}_A(\mathrm{u}(\overline{\Delta})) \leq \mathrm{C}_A(\mathrm{u}(G))$. Hence the remaining claims follow from Lemma 5.1 and the first part.

Lemma C. *If Δ is a virtually polycyclic ACG, then* $\mathrm{Fitt}(\Delta) = \mathrm{u}(\Delta)$.

Proof. Put $R = \mathrm{Fitt}(\Delta)$ and let $\Gamma = \Delta^*$. Then $R \cap \Gamma = \mathrm{Fitt}(\Gamma)$, which is unipotent by Lemma 4.6. Since $R \cap \Gamma$ has finite index in R, the group $R_s = \{x_s | x \in R\}$ is finite. Since Γ clearly normalizes R_s, the subgroup $\mathrm{C}_\Gamma(R_s) = C$, say, has a finite index in Γ. As $\overline{\Gamma}$ is Zariski connected, this implies that $\overline{C} = \overline{\Gamma}$, so that $R_s \leq \mathrm{C}_{\mathrm{Aff}(V)}(\overline{\Gamma})$, which is unipotent by Lemma 5.3. Hence $R_s = 1$. Therefore $R \leq \mathrm{u}(\Delta)$, and the result follows.

Lemma F. *Let Γ be a polycyclic ACG. Then every good geometrically strict extension of Γ is a strict extension of Γ.*

Proof. Suppose Δ is a good geometrically strict extension of Γ. Then Γ has a good hull G such that $G \cap \Delta = \Gamma$. Now $\mathrm{u}(G)$ is connected, so Zariski closed. Therefore $\mathrm{u}(\Gamma) = \mathrm{u}(G) \cap \mathrm{u}(\Delta)$ is Zariski closed in $\mathrm{u}(\Delta)$. But $\mathrm{u}(\Gamma)$ has finite index in $\mathrm{u}(\Delta)$, so $\mathrm{u}(\Gamma) = \mathrm{u}(\Delta)$. Lemma C shows that $\mathrm{Fitt}(\Delta) = \mathrm{u}(\Delta)$. Thus $\mathrm{Fitt}(\Delta) \leq \Gamma$, and the result follows.

For technical reasons, we shall need the following stronger version of Lemma 5.3; here, $\widehat{A}$ denotes the Zariski closure of A in the algebraic group $\mathrm{GL}(\mathbb{C}W)$.

Lemma 5.4. *If Δ is a virtually polycyclic ACG, then $\mathrm{C}_{\widehat{A}}(\Delta)$ is unipotent.*

Proof. We may assume, without loss of generality, that $A = \mathrm{Aff}(V)$. Put $C = \mathrm{C}_A(\Delta)$ and $C = \mathrm{C}_{\widehat{A}}(\Delta)$. Lemma 5.3 implies that C is unipotent, so it will suffice to show that C is Zariski dense in $\widetilde{C}$.

Of course, $C = \widetilde{C}(\mathbb{R})$, so we only require the $\mathbb{R}$-rational points to be Zariski dense in the algebraic group $\widetilde{C}$. That this is so follows from the fact that $\widetilde{C}$ is defined by linear equations over $\mathbb{R}$; namely $gh = hg$ and $(g-1)w \in V$, where $g \in \widetilde{C}$, $h \in \Delta$, $w \in W$.

We conclude this section with a digression. The following theorem shows that in order to classify simply transitive groups of affine transformations, it is sufficient, in principle, to classify the unipotent ones (this idea is used implicitly in [13], for example). This may be seen as a contribution to the problem of classifying standard ACGs.

Theorem 5.3. (i) *A closed subgroup G of A is simply transitive if and only if the following holds: there exist a simply transitive unipotent subgroup U of A and a diagonalizable subgroup S of $\mathrm{N}_A(U)$ such that*

$$(8) \qquad [U, S] \le G, \quad GS = US, \quad \text{and} \quad G \cap S = 1.$$

In this case, $U = \mathrm{u}(\overline{G})$.

(ii) *Given U and S as in* (i), *the closed subgroups G of A satisfying* (8) *are precisely the sets of the form*

$$(9) \qquad G = \{x \cdot x\theta \mid x \in U\},$$

where $\theta: U \to S$ is a continuous homomorphism with $[U, S] \le \ker\theta$.

(iii) *The group G in* (9) *is u-connected if and only if* $\ker\theta$ *is connected.*

Proof. (i) Suppose G is simply transitive. Then G is soluble, connected, and simply connected. Put $U = \mathrm{u}(\overline{G})$, $N = (U \cap G)_0$, and let H be a Cartan subgroup of G. Then H is nilpotent and $G = NH$. Put $S = H_s$. Then S is diagonalizable, since H is connected, and S normalizes U because $S \le \widetilde{G}$.

Now NH_u is connected and unipotent, so it is Zariski closed. Since $G \le NH_u \cdot S$ it follows that $\overline{G}/NH_u$ is contained in a diagonalizable algebraic group, whence $\mathrm{u}(\overline{G}/NH_u) = 1$. This implies that $U = NH_u$. Thus

$$GS = NHH_s = NH_uH_s = US.$$

Since $S \le \overline{H}$ and H normalizes N, which is Zariski closed, S normalizes N.. So

$$[U, S] = [NH_u, S] = [N, S] \le N \le G.$$

Also $G \cap S = 1$ because every semisimple affine transformation has a fixed point, by Lemma 4.5.

Conversely, suppose G satisfies (8). By Lemma 4.5 the group S has a fixed point w on V. Then

$$Gw = GSw = USw = Uw,$$

so G acts transitively on V if and only if U does. If G is transitive, and $g \in G$ fixes a point of V, then some conjugate h of g fixes w. Now $hs \in U$ for some $s \in S$, and hs fixes w; so if U acts freely we have $hs = 1$. Therefore $h \in G \cap S = 1$ imply that $g = 1$. Thus G acts freely on V. The same reasoning with the roles of U and G reversed shows that if G acts freely on V, then so does U.

Since U is connected, $U = \overline{U}$, and so $U \lhd \overline{G}$. We see as above that $\mathrm{u}(\overline{G}/U) = 1$. This shows that $\mathrm{u}(\overline{G}) = U$.

(ii) An elementary calculation yields that the set G in (9) is indeed a group satisfying (8). Using the fact that the projection mapping $\pi \colon U\overline{S} \to \overline{S}$ is continuous, it is also easy to verify that G is closed in A provided θ is continuous.

Conversely, suppose we have a group G satisfying (8). Then for each $x \in U$ there is a unique $s \in S$ with $xs \in G$, and we define $\theta \colon U \to S$ by $x\theta = s$. Thus $\ker\theta = G \cap U \geq [U, S]$, and it follows that θ is a homomorphism. Since θ can be written as a composition $U \to V \to G \xrightarrow{\pi} S$, where $U \to V$ and $V \to G$ are homeomorphisms, we see that θ is continuous.

(iii) is clear, since $(G) = G \cap U = \ker\theta$.

For an example of a group as in (9) and some further comments on the classification of simply transitive affine groups, see §9.

6. The normalizer of an ACG

Now we are going to prove Theorem B. We shall say that a group G is of type $\mathcal{NPFA}$ if G has a chain of normal subgroups

$$(10) \qquad G_1 \leq G_2 \leq G_3 \leq G,$$

such that G_1 is a connected nilpotent Lie group, G_2/G_1 is polycyclic, G_3/G_2 is finite, and G/G_3 is isomorphic to an arithmetic group. G is of type $\mathcal{NPF}$ (respectively, $\mathcal{NP}$, $\mathcal{NPA}$) if $G = G_3$ (respectively, $G = G_2$, $G_2 = G_3$).

Lemma 6.1. *Let G be a group of type $\mathcal{NPFA}$, and let m be a natural number. Then the subgroups of order dividing m in G lie in finitely many conjugacy classes.*

Proof. Let G_i $(i = 1, 2, 3)$ be as in (10). A theorem of Borel and Harish-Chandra (see [5]) shows that the finite subgroups of G/G_3 lie in finitely many conjugacy classes. So it will suffice to consider the finite subgroups of G_4, where G_4/G_3 is some fixed finite subgroup of G/G_3. Then G_4/G_1 is virtually polycyclic. A theorem of Mal′cev (see [18, Chapter 8, Theorem 5]) implies that the finite subgroups of G_4/G_1 lie in finitely many conjugacy classes. Thus we are reduced to considering the finite subgroups of H, where H/G_1 is a finite subgroup of G_4/G_1.

Let Z be the center of G_1. Arguing by induction on the nilpotency class of G_1, we may suppose that the subgroups of H/Z of order dividing m lie in finitely many conjugacy classes. Thus we may fix $K/Z \leq H/Z$, with $|K/Z|$ dividing m, and consider subgroups Δ of K such that $|\Delta|\,|m$ and $Z\Delta = K$. Now Z is a connected abelian Lie group, so the elements of order dividing m in Z form a finite subgroup. Hence there are only finitely many possibilities for the group $\Delta \cap Z$. Fixing a finite subgroup F of Z, with $F \lhd K$, we are left with showing that the complements for Z/F in K/F lie in finitely many conjugacy classes. Let $\zeta\colon Z/F \to Z/F$ be the map $x \mapsto x^m$. The exact sequence $1 \to \ker\zeta \to Z/F \xrightarrow{\zeta} Z/F \to 1$ yields an exact sequence

$$H^1(K/Z\,,\ker\zeta) \to H^1(K/Z\,, Z/F) \xrightarrow{\zeta^*} H^1(K/Z\,, Z/F).$$

But $|K/Z|\,|m$ implies that ζ^* is the zero map, so $H^1(K/Z\,, Z/F)$ is finite since $\ker\zeta$ is finite. Hence the result follows.

Lemma 6.2. *Let A be an arithmetic group, G a polycyclic normal subgroup of A, and F a finite normal subgroup of A. Let $\overline{G}$ be the Zariski closure of G in A. Then $\overline{G}$ is polycyclic, and both $A/\overline{G}$ and A/F are isomorphic to arithmetic groups.*

Proof. F is Zariski closed in A, and $\overline{G} \lhd A$. That $A/\overline{G}$ and A/F are arithmetic follows from Borel's theorem on homomorphic images of arithmetic groups, [6, Theorem 6]. $\overline{G}$ is polycyclic because it is soluble and linear over $\mathbb{Z}$ [18, Chapter 2, Corollary 1].

Now we recall the conventions of §1: W denotes a finite-dimensional vector space, $W = V \oplus \mathbb{R}$, and A is a Zariski-closed subgroup of $\mathrm{Aff}(V) \leq \mathrm{GL}(W)$.

Proposition 6.3. *Let J be a closed subgroup of $\mathrm{GL}(W)$, Γ a polycyclic closed normal subgroup of J, and Δ/Γ a finite subgroup of J/Γ. Put $H = \mathrm{N}_J(\Delta)$. Suppose there exists an $\mathbb{R}$-rational homomorphism of J onto an arithmetic group, with unipotent kernel. Then*:

(i) *H is of type $\mathcal{NPA}$, and contains a closed normal subgroup K of type $\mathcal{NP}$ such that H/K is isomorphic to an arithmetic group*;

(ii) *$K\Delta/\Delta$ is of type $\mathcal{NP}$, $H/K\Delta$ is isomorphic to an arithmetic group, and H/Δ is of type $\mathcal{NPA}$*;

(iii) *for each positive integer* m, *the finite subgroups of* H/Δ *with order dividing* m *lie in finitely many conjugacy classes.*

Proof. Let ρ be the given homomorphism of J. Put $C = \ker \rho$, and $M = \rho^{-1}(\overline{\rho(\Gamma)})$ where $\overline{\rho(\Gamma)}$ denotes the Zariski closure of $\rho(\Gamma)$ in the arithmetic group $\rho(J)$. Then M is closed and normal in J. Lemma 6.2 shows that J/M is isomorphic to an arithmetic group and that M/C is polycyclic. Also C/C_0 is polycyclic (see [17, Chapter II]), so $M/C_0\Gamma$ is polycyclic. Since $C_0\Gamma/\Gamma \cong C_0/(C_0 \cap \Gamma)$ is a connected nilpotent Lie group, $M\Delta/\Gamma$ is a group of type $\mathcal{NPF}$. Lemma 6.1 now implies that the J-conjugates of Δ/Γ lie in finitely many conjugacy classes of $M\Delta/\Gamma$. Consequently HM has finite index in J. Thus $HM/M \leq_f J/M$ is isomorphic to an arithmetic group.

Since Γ is closed in J, so also are Δ in H. Therefore $H \cap C$ is a closed unipotent subgroup of $\mathrm{GL}(W)$; as above, we conclude that $(H \cap M)/(H \cap C)_0$ is polycyclic. Now put $K = H \cap M$. Then K is of type $\mathcal{NP}$ and $H/K \cong HM/M$, so (i) follows since the connected nilpotent Lie group $(H \cap C)_0$ is normal in H.

For (ii), note that $K\Delta/K$ is a finite normal subgroup of H/K, since $\Gamma \leq H \cap M$, so $H/K\Delta$ is arithmetic by Lemma 6.2. The rest is clear.

Part (iii) follows from (ii) and Lemma 6.1.

Proof of Theorem B. Δ is virtually polycyclic ACG, contained in A. Putting $H = \mathrm{N}_A(\Delta)$, we shall show that H satisfies the conclusions of Proposition 6.3.

Put $\Gamma = \Delta^*$, as defined in §1, and put $J = \mathrm{N}_A(\Gamma)$. Since, clearly, $\Gamma \triangleleft H$, we have $H = \mathrm{N}_J(\Delta)$, so the notation is consistent with that of Proposition 6.3. Thus the result will follow if we can exhibit an $\mathbb{R}$-rational homomorphism ρ of J onto an arithmetic group, such that $\ker \rho$ is unipotent.

To do so, we invoke Proposition 3.1. Put $U = \mathrm{u}(\overline{\Gamma})$ and $C = \mathrm{C}_J(U)$. We must verify the statements (3), (4), and (5) of §3 which are hypotheses for Proposition 3.1.

(3) The Zariski closure of Γ in $\mathrm{Aff}(\mathbb{C}V)$ is Zariski connected. This is clear since $\Gamma \in (\overline{\Gamma})_0$.

(4) This is equivalent to $\dim(U) = \mathrm{h}(\Gamma)$, which follows from Lemma 4.3 and Proposition 5.2.

(5) $C_{\hat{J}}(\Gamma)$ is unipotent, where $\hat{J}$ is the Zariski closure of J in $\mathrm{Aff}(\mathbb{C}V)$. This follows from Lemma 5.4.

Proposition 3.1 therefore shows that the adjoint representation ρ of J on the Lie algebra of U maps J onto an arithmetic group. Clearly

$\ker \rho = C$; and Lemma 5.3 implies that C is unipotent, completing the proof.

Now suppose that Δ, as above, is torsion free.

Lemma 6.4. *The kernel of the natural epimorphism* $\pi : \mathrm{N}_A(\Delta) \to \mathrm{Aut}_A(\Delta \backslash V)$ *is exactly* Δ.

Proof. Certainly $\Delta \leq \ker \pi$. Now suppose $x \in \ker \pi$. Then for each $v \in V$ there exists $\gamma_v \in \Delta$ such that $x \cdot v = \gamma_v \cdot v$. Thus V is the union of its affine subspaces

$$V_{x^{-1}\gamma} = \{v \in V | x \cdot v = \gamma \cdot v\},$$

as γ runs over the countable group Δ. But a countable collection of proper affine subspaces cannot cover V. Therefore $V_{x^{-1}\gamma} = V$ for some $\gamma \in \Delta$, and then $x = \gamma \in \Delta$. Thus $\ker \pi \leq \Delta$.

Proof of Corollary B. By Lemma 6.4 we may identify $\mathrm{Aut}_A(\Delta \backslash V)$ with H/Δ. We have shown that the conclusions of Proposition 6.3 hold. Thus $K\Delta$ is a closed normal subgroup of H. So $K\Delta$ is a Lie group, and $K\Delta/\Delta$ is a Lie group of type $\mathcal{NP}$. Hence $H/K\Delta$ is isomorphic to an arithmetic group.

7. Strict extensions

Let $\mathrm{Aut}(\Delta)$, $\mathrm{Out}(\Delta)$ denote the automorphism group and outer automorphism group of a group Δ. It is known that if Δ is virtually polycyclic, then the finite subgroups of $\mathrm{Aut}(\Delta)$ lie in finitely many conjugacy classes [18, Chapter 8, Theorem 5]). Using the technique of Lemma 6.1, one can show that the same is true for $\mathrm{Out}(\Delta)$. However, the following weaker result suffices for present purposes:

Lemma 7.1. *If* Δ *is a virtually polycyclic group, then the finite subgroups of* $\mathrm{Out}(\Delta)$ *have bounded order.*

Proof. A recent theorem of Wehrfritz [23] shows that $\mathrm{Out}(\Delta)$ is isomorphic to a linear group over $\mathbb{Z}$, and therefore virtually torsion free. Hence the lemma follows.

We shall denote by $g(\Delta)$ the l.c.m. of the orders of all finite subgroups of $\mathrm{Out}(\Delta)$.

Lemma 7.2. *Let* Δ *be a virtually polycyclic ACG. If* Θ *is a strict normal extension of* Δ, *then* $|\Theta : \Delta| \,|\, g(\Delta)$.

Proof. By definition, $\mathrm{u}(\Theta) \leq \Delta$. Lemma 5.3 shows that $C_\Theta(\Delta) \leq \mathrm{u}(\Theta)$. The result follows since $\Theta/\Delta C_\theta(\Delta)$ is isomorphic to a subgroup of $\mathrm{Out}(\Delta)$.

Proof of Theorem A (i). Now Δ is a virtually polycyclic ACG of type A; the claim is that the strict normal extensions of Δ in A lie in finitely many conjugacy classes in A.

Proposition 6.3(iii), together with the proof of Theorem B in §6, show that the subgroups of $N_A(\Delta)/\Delta$ of order dividing $g(\Delta)$ lie in finitely many conjugacy classes. Thus the result follows from Lemma 7.2.

For the second part of Theorem A, and for Theorem E, we have to consider extensions which are not normal. We fix a standard ACG Γ, and use the notation of §4, so $U = \mathrm{u}(\overline{\Gamma})$. Put $D = \overline{\Gamma}'$, and recall that $D = \overline{\Gamma'} \leq U$. Let

$$H = N_{GL(W)}(\overline{\Gamma}).$$

Then H/U is the group of real points of a linear algebraic group [4, Corollary 15.7], and $\overline{\Gamma}/U$ is (the group of real points of) a normal algebraic torus in H/U. Therefore $|H: C_H(\overline{\Gamma}/U)|$ is finite [4, §8.10]. Put $s = |H: C_H(\overline{\Gamma}/U)|$, and $r = \dim(U/D)$, and denote by $g(r)$ the l.c.m. of the orders of finite subgroups of $GL_r(\mathbb{Z})$, so $g(r) = g(\mathbb{Z}^r)$ in our previous notation.

Lemma 7.3. *If* $\Gamma \leq_f \Theta \leq Gl(W)$ *then*

(i) $|\Theta: C_\Theta(\overline{\Gamma}/U)| \,|\, s$,

(ii) $|\Theta: C_\Theta(\overline{\Gamma}/D)| \,|\, sg(r)$.

Proof. Since $\overline{\Gamma}$ is Zariski connected, we have $\overline{\Gamma} = \overline{\Gamma}_1$ whenever $\Gamma_1 \leq_f \Gamma$. Hence for the purposes of this proof, we may replace Γ by a suitable Γ_1 and assume that $\Gamma \lhd \Theta$. Therefore $\theta \leq H$ and $|\Theta: C_\Theta(\overline{\Gamma}/U)| \,|\, s$. This establishes (i). For (ii), let us write $\Delta = C_\Theta(\overline{\Gamma}/U)$, $\Phi = C_\Theta(\overline{\Gamma}/D)$, and note that $\Gamma \leq \Phi \leq \Delta$.

Now Proposition 2.1 shows that U contains a Zariski-dense finitely generated subgroup Q such that $\mathrm{u}(\Gamma) \leq Q$, Q is normalized by Δ, and $\mathrm{h}(Q) \leq \mathrm{h}(\Gamma)$. As $\mathrm{h}(\Gamma) = \dim(V) = \dim(U)$, by Lemma 4.3 and Proposition 5.2, Q is a uniform lattice in U. Since $\Gamma' \leq \mathrm{u}(\Gamma) \leq Q$, $Q \cap D$ is Zariski dense in D and $D/(Q \cap D)$ is compact. It follows that QD/D is a uniform lattice in $U/D \cong \mathbb{R}^r$, so that $QD/D \cong \mathbb{Z}^r$.

Let $\mu: \Delta \to GL_r(\mathbb{Z})$ denote the conjugation representation of Δ on QD/D. Then $\mu(\Delta)$ is finite, since $\Gamma \leq \ker \mu$. Therefore $|\mu(\Delta)| \,|\, g(r)$. It follows that $\Sigma = \ker \mu$ satisfies $|\Theta: \Sigma| \,|\, sg(r)$, $[U, \Sigma] \leq D$, and $[\overline{\Gamma}, \Sigma] \leq U$. Now Σ/Φ embeds in the vector group $\mathrm{Der}(\overline{\Gamma}/U, U/D)$, via the map $\sigma\Phi \mapsto (Ux \mapsto [{}'\sigma, x]D)$ $(\sigma \in \Sigma,\ x \in \overline{\Gamma})$. But Σ/Φ is finite since $\Gamma \leq \Phi$; so $\Phi = \Sigma$ and (ii) follows.

Proof of Theorem A (ii). Given a virtually polycyclic ACG Δ of type A, we have to show that the strict extensions of Δ in A have bounded index, and lie in finitely many conjugacy classes in A.

We take $\Gamma = \Delta^*$ in the above discussion, and for each strict extension Θ of Δ write $\Theta_1 = C_\Theta(\overline{\Gamma}/U)$. Then Lemma 7.3(i) shows that $|\Theta: \Theta_1| \mid s$. On the other hand,

$$[\Gamma, \Theta_1] \leq \Theta_1 \cap U \leq u(\Theta) \leq u(\Delta) = u(\Gamma),$$

so $\Gamma \triangleleft \Theta_1$. Hence by Lemma 7.2 we have $|\Theta_1 : \Gamma| \mid g(\Gamma)$. Thus each strict extension Θ of Δ satisfies

$$|\Theta: \Delta| \mid |\Theta: \Gamma| \mid sg(\Gamma) = sg(\Delta^*),$$

which establishes the first claim.

Now put $m = sg(\Gamma)$. For each Θ as above we have

$$\Delta^m u(\Delta) \leq \Theta^m u(\Delta) \leq \Delta,$$

and θ is a strict normal extension of $\Theta^m u(\Delta)$ since $u(\Delta) = u(\Theta)$. As $\Delta/\Delta^m u(\Delta)$ is finite, there are only finitely many possibilities for the group $\Theta^m u(\Delta)$; call them $\Delta_1, \cdots, \Delta_h$, and for each i let $\mathscr{X}_i$ be the set of strict extensions θ of Δ such that $\Theta \leq A$ and $\Theta^m u(\Delta) = \Delta_i$. Theorem A(i) shows that $\mathscr{X}_i$ consists of finitely many conjugacy classes of subgroups in $N_A(\Delta_i)$; the second claim thus follows.

The remainder of this section is devoted to the proof of Theorem E. We keep the notation introduced above, and fix an ACG Δ such that $\Gamma = \Delta^*$.

Lemma 7.4. *If Θ is a geometrically strict extension of Δ then* $|\Theta: N_\Theta(\Delta)| \mid sg(r)$.

Proof. By definition, there exists a syndetic hull G for Δ such that $\Theta \cap G = \Delta$. Then $G \leq (\overline{\Delta})_0 = (\overline{\Gamma})_0$ and $G' \geq \Delta' \geq \Gamma'$. As G' is a closed connected subgroup of the unipotent group U, it follows that G' is Zariski closed and hence that $G' = \overline{\Gamma'} = D$. So if $\Sigma = C_\Theta(\overline{\Gamma}/D)$ we have

$$[\Delta, \Sigma] \leq \Theta \cap [G, \Sigma] \leq \Theta \cap D \leq \Delta.$$

Thus $\Sigma \leq N_\Theta(\Delta)$ and the result follows from Lemma 7.3(ii).

Let Z denote the center of Δ, put $K = Z \cap \Delta'$, and let T/K be the torsion subgroup of Z/K.

Lemma 7.5. *Let Θ be a geometrically strict extension of Δ, and let $\Psi = \Delta C_\Theta(\Delta)$. Let X be the center of Ψ. Then the following hold:*

(i) $\Psi = \Delta X$.

(ii) *$Z \leq X$ and X/T is torsion free.*

Proof. Put $X_1 = C_\theta(\Delta)$. Then X_1 is unipotent, by Lemma 5.3, so $X_1 \leq U$ since $|\overline{\Theta}: \overline{\Gamma}|$ is finite. As $U \leq \overline{\Gamma} \leq \overline{\Delta}$ it follows that $\Psi = \Delta X_1 \leq \overline{\Delta}$. Therefore X_1 centralizes Ψ, so $X_1 = X$ and (i) holds.

Now let G be a syndetic hull for Δ such that $\Theta \cap G = \Delta$. As we have seen before, G' is Zariski closed; since $K = Z \cap \Delta' \leq G'$ it follows that $\overline{K} \leq G'$. Hence

$$X \cap \overline{K} \leq X \cap G = X \cap \Delta \leq Z.$$

It is clear that $Z \leq X$; thus $X \cap \overline{K} = Z \cap \overline{K}$. As X is unipotent, this implies that $X/(Z \cap \overline{K})$ is torsion free. As Z is unipotent, $Z \cap \overline{K} = T$. Therefore we have (ii).

For the final step we make a purely group theoretic observation: *given the polycyclic group* Δ, *there are only finitely many isomorphism types of groups* Ψ *which contain* Δ *as a subgroup of finite index and satisfy* (i) *and* (ii) *of Lemma* 7.5 (with X the center of Ψ). Postponing the proof to the next section, we now complete the

Proof of Theorem E. The claim is that the geometrically strict extensions of Δ lie in finitely many isomorphism classes. By Lemma 7.5 and the remark above, the groups $\Delta C_\Theta(\Delta)$ lie in finitely many isomorphism classes, when Θ ranges over all geometrically strict extensions of Δ. It will suffice, therefore, to consider such groups Θ for which $\Delta C_\Theta(\Delta)$ is isomorphic to a fixed group Ψ, say.

Let Θ be one of these groups. Then $N_\Theta(\Delta)/\Delta C_\Theta(\Delta)$ is isomorphic to a subgroup of $\mathrm{Out}(\Delta)$, so $|N_\Theta(\Delta) \colon \Delta C_\Theta(\Delta)| \mid g(\Delta)$. Lemma 7.4 shows that $|\Theta \colon N_\Theta(\Delta)| \mid sg(r)$. Thus Θ contains a subgroup isomorphic to Ψ and of index dividing $sg(r)g(\Delta)$. By [18, Chapter 8, Theorem 6], the groups Θ satisfying this condition lie in finitely many isomorphism classes. This completes the proof, modulo the result of §8.

8. Finite extensions of polycyclic groups

Let $Z(\Psi)$ denote the center of a group Ψ. Fix a polycyclic group Δ, put $Z = Z(\Delta)$, $K = Z \cap \Delta'$, and let T/K be the torsion subgroup of Z/K.

Proposition 8.1. *There are only finitely many isomorphism types of groups* Ψ *such that*

(11) $$\Delta \leq_f \Psi = \Delta Z(\Psi), \qquad Z(\Psi)/T \text{ is torsion free.}$$

Before embarking on the proof, we make some definitions. For any virtually polycyclic group Γ, define

$$K(\Gamma) = Z(\Gamma) \cap \Gamma', \qquad T(\Gamma)/K(\Gamma) = \tau(Z(\Gamma)/K(\Gamma)),$$
$$P(\Gamma)/\Gamma' Z(\Gamma) = \tau(\Gamma/\Gamma' Z(\Gamma)),$$

where $\tau(A)$ denotes the torsion subgroup of an abelian group A. Put

$$e(\Gamma) = |P(\Gamma)\colon \Gamma'\mathrm{Z}(\Gamma)| \cdot |T(\Gamma)\colon K(\Gamma)|,$$
$$f(\Gamma) = \mathrm{h}(\mathrm{Z}(\Gamma)/K(\Gamma)).$$

Call a subgroup Q of Γ a *core* of Γ if $\Gamma' \leq Q$ and

$$\Gamma/\Gamma' = P(\Gamma)/\Gamma' \times Q/\Gamma'.$$

Lemma 8.2. *Let Γ be a virtually polycyclic group. Then the following hold*:

(i) $\mathrm{Z}(\Gamma) = T(\Gamma) \times A$ *for some subgroup A, and Γ possesses a core.*
(ii) *Let A be as in* (i) *and let Q be a core of* Γ. *Then $AQ = A \times Q$, $|\Gamma\colon AQ| = e(\Gamma)$, and $A \cong \mathbb{Z}^{f(\Gamma)}$.*

Proof. For (i), note that both $\mathrm{Z}(\Gamma)/T(\Gamma)$ and $\Gamma/P(\Gamma)$ are free abelian groups. (ii) is best verified by drawing a lattice diagram.

Lemma 8.3. *Let Q be a virtually polycyclic group, and let $e \geq 1$ and $f \geq 0$. Then the virtually polycyclic groups Γ such that $e(\Gamma) = e$, $f(\Gamma) = f$, and Q is a core of Γ lie in finitely many isomorphism classes.*

Proof. Lemma 8.2(ii) shows that each such group Γ has a normal subgroup of index e isomorphic to $\mathbb{Z}^f \times Q$. The result now follows from [18, Chapter 8, Theorem 6].

Remark. Although we have quoted [18, Chapter 8, Theorem 6] twice (see the proof of Theorem E, above), it is easy to see that it is only really needed once.

Proof of Proposition 8.1. Let Q be a core of Δ. In view of Lemma 8.3, it will suffice to show that if Ψ satisfies (11), then $e(\Psi) = e(\Delta)$, $f(\Psi) = f(\Delta)$, and Q is a core of Ψ.

Note that $\Psi' = \Delta'$ and $\mathrm{Z}(\Psi) \cap \delta = Z$. Put $P = P(\Delta)$. Then $\Psi/PZ(\Psi) \cong \Delta/P$ and $PZ(\Psi)/\Psi'\mathrm{Z}(\Psi) \cong P/\Delta'Z$. It follows that $PZ(\Psi) = P(\Psi)$ and that $|P(\Psi)\colon \Psi'\mathbb{Z}(\Psi)| = |P\colon \Delta'Z|$. We also see that

$$P(\Psi) \cap Q = P\mathrm{Z}(\Psi) \cap Q = P \cap Q = \Delta' = \Psi',$$

and

$$P(\Psi)Q = P\mathrm{Z}(\Psi)Q = \Delta\mathrm{Z}(\Psi) = \Psi;$$

thus Q is a core of Ψ.

Since $|\mathrm{Z}(\Psi)\colon Z|$ is finite, so also is $K(\Psi)/K$, and as $\mathrm{Z}(\Psi)/T$ is torsion free this implies that $K(\Psi) \leq T \leq Z$. With $\Psi' = \Delta'$ this shows that $K(\Psi) = K$, and hence that $T(\Psi) = T$. Thus $|T(\Psi)\colon K(\Psi)| = |T\colon K|$; together with what we showed above this yields $e(\Psi) = e(\Delta)$.

Finally, we see that $f(\Psi) = f(\Delta)$, because $Z(\Psi)/K(\Psi)$ is a finite extension of Z/K. This completes the proof.

9. An example

Here we exhibit a simply transitive subgroup G of $\mathrm{Aff}(V)$, where $V = \mathbb{R}^6$, a Zariski-dense uniform lattice Γ in G, and a family of pairwise nonconjugate subgroups $\Gamma_q \le \mathrm{Aff}(V)$, $q \in \mathbb{N}$, such that for each q, $\Gamma \triangleleft \Gamma_q$, $|\Gamma_q : \Gamma| = q$, $\Gamma_q \cap G = \Gamma$. It follows from the theory (and will anyway be visible) that $\mathrm{h}(\Gamma) = 6 = \dim(G)$, so G is a syndetic hull for Γ and hence each Γ_q is a geometrically strict extension of Γ.

We shall omit a number of routine calculations (all quite simple).

Step 1. Define a map $\nu : V \to \mathrm{Aff}(V)$ by

$$\nu(r, y, \mathbf{x}, \mathbf{z}) = \begin{pmatrix} 1 & 0 & \mathbf{x} & 0 & r + \frac{1}{2}\|\mathbf{x}\| \\ & 1 & 0 & 0 & y \\ & & \mathbf{1} & 0 & \mathbf{x}' \\ & & & \mathbf{1} & \mathbf{z}' \\ & & & & 1 \end{pmatrix}$$

Here, the letters in bold type denote elements of $\mathbb{R}^2$, $\mathbf{1}$ denotes the 2×2 identity matrix, $\mathbf{x}'$ the transpose of $\mathbf{x}$, and $\|\mathbf{x}\| = \mathbf{x}\mathbf{x}'$. The missing entries are zero.

Since ν is a homomorphism, $U = \nu(V)$ is an abelian, unipotent subgroup of $\mathrm{Aff}(V)$, and clearly simply transitive. Also $\nu : V \to U$ is an isomorphism.

Step 2. For $r \in \mathbb{R}$ put

$$E(r) = \begin{pmatrix} \cos 2\pi r & \sin 2\pi r \\ -\sin 2\pi r & \cos 2\pi r \end{pmatrix},$$

and define $t : V \to \mathrm{Aff}(V)$ by

$$t(r, y, \mathbf{x}, \mathbf{z}) = \mathrm{diag}(1, 1, E(r), e^y, e^{-y}, 1).$$

Then t is a homomorphism of V onto a diagonalizable subgroup S of $\mathrm{Aff}(V)$, and so $\theta = t \circ \nu^{-1} : U \to S$ is a (continuous) homomorphism. Note that

$$\ker\theta = \nu(\ker t) = \nu(\mathbb{Z} \oplus 0 \oplus \mathbb{R}^4). \tag{12}$$

If $g = (r, y, \mathbf{x}, z_1, z_2)$ and $h = (r^*, y^*, \mathbf{x}^*, \mathbf{z}^*)$, then

$$\begin{aligned} g^h &= \nu(r, y, \mathbf{x}E(r^*), e^{-y^*} z_1, e^{y^*} z_2), \\ [g, h] &= \nu(0, 0, \mathbf{x}E(r^*) - \mathbf{x}, e^{-y^*} z_1 - z_1, e^{y^*} z_2 - z_2). \end{aligned} \tag{13}$$

Thus $[U, S] \leq \ker\theta$. It follows that the set

$$g = \{g \cdot \theta(g) | g \in U\} = \{\nu(v) \cdot t(v) | v \in V\}$$

is a simply transitive subgroup of $\mathrm{Aff}(V)$, by Theorem 5.5 (of course, this can be seen directly). Put $\psi(v) = \nu(v) \cdot t(v)$ for $v \in V$. Then ψ is a homeomorphism of V onto G.

Step 3. Suppose L is a full lattice in $\mathbb{R}^4$, and $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}\setminus\{0\}$ satisfy

$$L \begin{pmatrix} E(\alpha) & & \\ & e^{-\beta} & \\ & & e^{\beta} \end{pmatrix} = L. \tag{14}$$

Put

$$\gamma = \psi(\alpha, \beta, 0, 0, 0, 0).$$

Then (13) shows that for a subgroup of A of $\mathbb{R}^2$,

$$\nu(A \oplus L)^{\gamma} = \nu(A \oplus L)^{t(\alpha, \beta, 0, 0, 0, 0)} = \nu(A \oplus L).$$

Hence

$$\Gamma_A = \nu(A \oplus L)\langle\gamma\rangle$$

is a subgroup of $\mathrm{Aff}(V)$. For $(a, b) \in A \oplus L$ from (12) we have

$$\nu(a, b) \in G \Leftrightarrow a \in \mathbb{Z} \oplus 0 \Leftrightarrow \nu(a, b) = \psi(a, b),$$

so provided that $\mathbb{Z}\oplus 0 \leq A$, we have $\Gamma_A \cap G = \Gamma$ where we put $\Gamma = \Gamma_{\mathbb{Z}\oplus 0}$. We also have $\Gamma \triangleleft \Gamma_A$, since U is abelian and (13) implies that γ normalizes $\nu(\mathbb{Z}\oplus 0\oplus L)$. Since ν is an isomorphism, $\Gamma_A/\Gamma \cong A/(\mathbb{Z}\oplus 0)$. In particular, putting $A(q) = q^{-1}\mathbb{Z} \oplus 0$ and $\Gamma_q = \Gamma_{A(q)}$, we have $\Gamma \triangleleft \Gamma_q$, $|\Gamma_q : \Gamma| = q$, and $\Gamma_q \cap G = \Gamma$.

We claim that Γ is a uniform lattice in G. To see this, note that

$$\Gamma = \{\psi(m + n\alpha, n\beta, a) | m, n \in \mathbb{Z},\ a \in L\},$$

so $\psi^{-1}(\Gamma)$ is a full $\mathbb{Z}$-lattice in $\mathbb{R}^6 = \psi^{-1}(G)$. As Ψ is a homeomorphism, this shows that Γ is discrete and cocompact in G.

It is easy to see (if one draws the matrix of a typical element of Γ) that Γ *is Zariski dense in* G *if and only if* $\langle E(\alpha)\rangle$ *is Zariski dense in* $\mathrm{SO}_2(\mathbb{R})$, that is, *if and only if* α *is irrational.*

Step 4. Assume that α is irrational, and that L has the following properties:

$$(1 \quad 0 \quad z_1 \quad z_2) \in L \textit{ for some } z_1, z_2; \tag{15}$$

$$\textit{if } (\mathbf{x} \quad \mathbf{z}) \in L \textit{ then } \|\mathbf{x}\| \textit{ is an algebraic integer.} \tag{16}$$

Then for positive integers $p \neq q$, Γ_p and Γ_q are not conjugate in $\mathrm{Aff}(V)$.

To prove this, suppose $g \in \mathrm{Aff}(V)$ and $g^{-1}\Gamma_p g = \Gamma_p$. Since $\mathrm{u}(\Gamma_A) = \nu(A \oplus L)$, this implies

$$\nu(A_p \oplus L)g = g\nu(A_q \oplus L). \tag{17}$$

Since α is irrational, the center of Γ_A is $\nu(A \oplus 0)$; so we also have

$$\nu(A_p \oplus 0)g = g\nu(A_q \oplus 0). \tag{18}$$

Denote the (i, j)-entry of g by g_{ij}. From (18) we get

$$p^{-1}\mathbb{Z} = g_{11}q^{-1}\mathbb{Z},$$
$$0 = g_{i1}q^{-1}\mathbb{Z} \qquad (2 \leq i \leq 6),$$

so

$$g_{11} = \pm q/p, \qquad g_{21} = \cdots = g_{61} = 0. \tag{19}$$

Note that the last row of g is $(0 \cdots 01)$. Suppose $a \in A_p$, $b \in A_q$, $x, y \in L$ satisfy

$$\nu(a + x)g = g\nu(b + y).$$

From the first row of this equation we find that

$$\begin{aligned} x_1 g_3 + x_2 g_{43} &= g_{11} y_1, \\ x_1 g_{34} + x_2 g_{44} &= g_{11} y_2, \end{aligned} \tag{20}$$

$$\begin{aligned} x_1 g_{35} + x_2 g_{45} &= 0, \\ x_1 g_{36} + x_2 g_{46} &= 0. \end{aligned} \tag{21}$$

From the last column, using (19) we find

$$\begin{aligned} x_1 &= g_{33} y_1 + g_{34} y_2 + g_{35} y_3 + g_{36} y_4, \\ x_2 &= g_{43} y_1 + g_{44} y_2 + g_{45} y_3 + g_{46} y_4. \end{aligned} \tag{22}$$

In view of (17), we may allow x to vary over the four-dimensional lattice L. The vector (x_1, x_2) then takes two linearly independent values, and (21) implies that

$$g_{35} = g_{45} = g_{36} = g_{46} = 0. \tag{23}$$

From (15), we may choose $(y_1, y_2) = (1, 0)$. Then (22) and (23) give $(x_1, x_2) = (g_{33}, g_{43})$, and (20) gives $\|(x_1, x_2)\| = g_{11}$. Now (16) and (19) imply that q/p is an algebraic integer. Hence $p|q$, and $p = q$ by symmetry.

Step 5. To find L, α, and β, it will suffice to find the following: an algebraic number field k of degree 4 over $\mathbb{Q}$ having two real and two non-real complex embeddings $\sigma_1, \sigma_2 \colon k \to \mathbb{R}$, $\mu, \overline{\mu} \colon k \to \mathbb{C}$, and a unit $\varepsilon \neq \pm 1$ in the ring of integers $\mathcal{O}$ of k such that

$$\sigma_1(\varepsilon) = \sigma_2(\varepsilon)^{-1} > 0, \qquad \|\mu(\varepsilon)| = 1.$$

For L we take the image in $\mathbb{C} \oplus \mathbb{R}^2$ of $\mathcal{O}$ under the map $\tau = \mu \times \sigma_1 \times \sigma_2$, identifying $\mathbb{C}$ with $\mathbb{R}^2$ by fixing the basis $(1, i)$. Put

$$\alpha = (1/2\pi) \arg \mu(\varepsilon), \qquad \beta = -\log \sigma_1(\varepsilon).$$

It is well known that L is then a full $\mathbb{Z}$-lattice in $\mathbb{R}^4$, and one verifies directly that (14) holds. Note that $\beta \neq 0$ since $\varepsilon \neq \pm 1$, and that α is irrational because in fact ε has infinite order (since $\sigma_1(\varepsilon)$, being real, cannot be a root of unity).

Both (15) and (16) are clearly satisfied: for $(1, 0, 1, 1) = \tau(1) \in L$, and if $(\mathbf{x}, \mathbf{z}) = \tau(c)$ then $\|\mathbf{x}\| = \mu(c)\overline{\mu}(c)$.

All that remains is to exhibit a suitable ε; and then take $k = \mathbb{Q}(\varepsilon)$. A root of the following polynomial will do; we leave it as an exercise to verify the details: $X^4 - 4X^3 + 4X^2 - 4X + 1$.

Further remarks. (i) Since $\Gamma/\mathrm{u}(\Gamma)$ is infinite cyclic, Γ is a standard ACG, so Γ must have a good hull. Of course G is not one, since $\mathrm{u}(G) = \nu(\mathbb{Z} \oplus 0 \oplus \mathbb{R}^4)$ is not connected; thus we have not violated Theorem G!

If we follow the proof of Proposition 4.1, we obtain a family H_m ($m \in \mathbb{Z}$) of a good hulls for Γ; H_m is the set of all matrices

$$\begin{pmatrix} 1 & 0 & \mathbf{x} & 0 & 0 & r \\ & 1 & 0 & 0 & 0 & \beta y \\ & & E((\alpha + m)y) & 0 & 0 & \mathbf{x}' \\ & & & e^{\beta y} & 0 & z_1 \\ & & & & e^{-\beta y} & z_2 \\ & & & & & 1 \end{pmatrix}$$

with $(r, y, \mathbf{x}, z_1, z_2) \in \mathbb{R}^6$.

(ii) For each $m \in \mathbb{Z}$, H_m is a good hull for each of the groups Γ_q, which are all isomorphic to Γ. It is easy to see that H_m also contains infinitely many pairwise nonisomorphic Zariski-dense uniform lattices, which are all "abstractly commensurable", i.e., isomorphic up to finite index. In general, a simply transitive affine group may contain infinitely many pairwise abstractly noncommensurable Zariski-dense uniform lattices; for examples of this phenomenon, see [13].

(iii) A representation of a Lie group G as a simply transitive subgroup of $\mathrm{Aff}(V)$ corresponds to a *complete left-symmetric algebra* structure on the Lie algebra of G; see for example the introduction of [14]. The best way to approach the classification or construction of simply transitive affine groups is to linearise the problem and work with left-symmetric algebras. It is often quite easy in practice to endow a given nilpotent Lie algebra with a complete left-symmetric structure (although, contrary to a long-standing conjecture, it is not *always* possible [2], [12]): this gives a practical method for constructing many examples of simply transitive affine groups.

A generalization of Theorem 5.5 in the context of left-symmetric algebras is given in [19].

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