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SESHADRI CONSTANTS, GONALITY OF SPACE CURVES, AND RESTRICTION OF STABLE BUNDLES

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1. Introduction

There exist many situations in algebraic geometry where the extrinsic geometry of a variety is reflected in clear restrictions in the way that it can map to projective spaces. For example, it is well-known that the gonality of a smooth plane curve C of degree d is $d-1$, and that every minimal pencil has the form $\mathscr{O}_C(H-P)$, where H denotes the hyperplane class and $P \in C$.

In fact, there are to date various statements of this kind concerning the existence of morphisms from a divisor to $\mathbf{P}^1$. The first general results in this direction are due to Sommese [37] and Serrano [55]. Reider [34] then showed that at least part of Serrano's results for surfaces can be obtained by use of vector bundle methods based on the Bogomolov-Gieseker inequality for semistable vector bundles on a surface.

In [3], a generalization of these methods to higher dimensional varieties is used to obtain the following statement:

Theorem 1.1. *Let X be a smooth projective n-fold, and let $Y \subset X$ be a reduced irreducible divisor. If $n \geq 3$ assume that Y is ample, and if $n = 2$ assume that $Y^2 > 0$ (so that in particular it is at least nef). Let $\phi: Y \to \mathbf{P}^1$ be a morphism, and let F denote the numerical class of a fiber.*

(i) *If $F \cdot Y^{n-2} < \sqrt{Y^n} - 1$, then there exists a morphism $\psi: X \to \mathbf{P}^1$ extending ϕ. Furthermore, the restriction $H^0(X, \psi^*\mathscr{O}_{\mathbf{P}^1}(1)) \to H^0(Y, \phi^*\mathscr{O}_{\mathbf{P}^1}(1))$ is injective. In particular, ψ is linearly normal if ϕ is.*

(ii) *If $F \cdot Y^{n-2} = \sqrt{Y^n} - 1$ and $Y^n \neq 4$, then either there exists an extension $\psi: X \to \mathbf{P}^1$ of ϕ, or else we can find an effective divisor D on X such that $(D \cdot Y^{n-1})^2 = (D^2 \cdot Y^{n-2})Y^n$ and $D \cdot Y^{n-1} = \sqrt{Y^n}$, and an inclusion $\phi^*\mathscr{O}_{\mathbf{P}^1}(1) \subset \mathscr{O}_Y(D)$.*

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However, a much less understood range of situations is the one where $\operatorname{codim}(Y) \geq 2$. In some particular cases there are rather precise statements. In curve theory, in particular, one has a clear picture of the gonality of Castelnuovo extremal curves [1]. In even degree, for example, $C \subset \mathbf{P}^3$ is a smooth complete intersection of a smooth quadric and a hypersurface of degree $a \geq 2$, and the gonality is attained by restricting to C the rulings on the quadric. More generally, unpublished work of Lazarsfeld shows that if $C \subset \mathbf{P}^3$ is a smooth complete intersection of type (a, b), with $a \geq b$, then $\operatorname{gon}(C) \geq a(b-1)$. Lazarsfeld's argument is also based on Bogomolov's instability theorem. In a somewhat more general direction, Ciliberto and Lazarsfeld have studied linear series of low degree on various classes of space curves. Their method is based on the number of conditions imposed by a linear series on another.

Naturally enough, one is led to investigate more general situations. We shall focus on the gonality of space curves, and then show how the methods developed apply to other circumstances as well. In the codimension-1 case we have seen that the self-intersection of the divisor governs the numerical constraint on a free pencil on Y. Loosely speaking, in the higher codimension case a similar role is played by the Seshadri constant of the curve, which is defined as follows. Consider a smooth curve $C \subset \mathbf{P}^3$, and denote the blowup of $\mathbf{P}^3$ along C by $f\colon X_C \to \mathbf{P}^3$, and the exceptional divisor by $E = f^{-1}C$. The *Seshadri constant* of C is

$$\varepsilon(C) = \sup\{\eta \in \mathbf{Q} | f^*H - \eta E \text{ is ample}\}.$$

This is a very delicate invariant, and it gathers classical information such as what secants the curve has and the minimal degree in which powers of $\mathcal{I}_C$ are globally generated. For example, if $C \subset \mathbf{P}^3$ is a complete intersection of type (a, b), with $a \geq b$, then $\varepsilon(C) = 1/a$. More generally, if $C \subset \mathbf{P}^3$ is defined as the zero locus of a regular section of a rank-two vector bundle $\mathcal{E}$, then we have an estimate $\varepsilon(C) \geq \gamma(\mathcal{E})$, where $\gamma(\mathcal{E})$ is the Seshadri constant of $\mathcal{E}$, defined as

$$\gamma(\mathcal{E}) = \sup\{n/m | S^n\mathcal{E}(m) \text{ is globally generated}\}.$$

It is always true that $\varepsilon(C) \geq 1/d$. However, the problem of finding general optimal estimates $\varepsilon(C)$ for an arbitrary curve seems to be a hard one. Something can be said, for example, as soon as C can be expressed as an irreducible component of a complete intersection of smooth surfaces.

Interest in Seshadri constants, of course, is not new. In fact, if Y is a subvariety of any projective variety X, one can define in an obvious way the Seshadri constant of Y with respect to any polarization H on X. Seshadri constants of points, in particular, have received increasing attention recently, partly in relation to the quest for Fujita-type results. A differential geometric interpretation has been given by DeMailly [9]. Seshadri constants of points on a surface have been investigated by Ein and Lazarsfeld [11], who have proved the surprising fact that they can be bounded away from zero at all but countably many points of S. However, Seshadri constants of higher dimensional subvarieties have apparently never been put to use.

What a bound on the gonality of a space curve might look like is suggested by Lazarsfeld's result. In fact, we may write $a(b-1) = \deg(C) - 1/\varepsilon(C)$, so that for a complete intersection we have the optimal bound

$$\operatorname{gon}(C) \geq d - 1/\varepsilon(C).$$

Keeping the above notation, let us define

$$H_\eta = f^*H - \eta E \quad \text{and} \quad \delta_\eta(C) = \eta \cdot \deg(N) - d,$$

where N is the normal bundle of C. For example, for a complete intersection of type (a, b) with $a \geq b$ we have $\delta_{1/a}(C) = b^2$, and $\delta_\eta(C)$ has a simple geometric meaning, that we explain at the end of §3. Our result is

Theorem 1.2. *Let $C \subset \mathbf{P}^2$ be a smooth curve of degree d and Seshadri constant $\varepsilon(C)$. Set $\alpha = \min\{1, \sqrt{d}(1 - \varepsilon(C)\sqrt{d})\}$. Then*

$$\operatorname{gon}(C) \geq \min\left\{\frac{\delta_{\varepsilon(C)}(C)}{4\varepsilon(C)}, \alpha\left(d - \frac{\alpha}{\varepsilon(C)}\right)\right\}.$$

This reproduces Lazarsfeld's result if $a \geq b + 3$. As another example, it says that if $a \gg b$ and C is residual to a line in a complete intersection of type (a, b), then $\operatorname{gon}(C) = ab - (a + b - 2)$ (consider the pencil of planes through the line). In view of the above, one would expect the above bound to hold with $\alpha = 1$ always, but I have been unable to prove it.

The idea of the proof is as follows. If A is a minimal pencil on C, and $\pi: E \to C$ is the induced projection, one can define a rank-2 vector bundle on X_C by the exactness of the sequence

$$0 \to \mathcal{F} \to H^0(C, A) \otimes \mathcal{O}_{X_C} \to \pi^* A \to 0.$$

The numerical assumptions then force $\mathcal{F}$ to be Bogomolov unstable with respect to $H_{\varepsilon(C)}$ (see §2), and therefore a maximal destabilizing line bundle $\mathcal{O}_{X_C}(-D) \subset \mathcal{F}$ comes into the picture. D and A are related by the inequalities coming from the instability of $\mathcal{F}$, and from this one can show that $\deg(A)$ is forced to satisfy the above bound.

By its general nature, this argument can be applied to the study of linear series on arbitrary smooth subvarieties of $\mathbf{P}^r$. We will not detail this generalization here.

In another direction, similar methods have been used by Bogomolov [4], [6] to study the behavior of a stable bundle on a surface under restriction to a curve C that is linearly equivalent to a multiple of the polarization at hand. For example, it follows from Bogomolov's theorem that if S is a smooth surface with $\mathrm{Pic}(S) \simeq \mathbf{Z}$, and $\mathcal{E}$ is a stable rank 2 vector bundle on S, then $\mathcal{E}|_C$ is also stable, for every irreducible curve $C \subset S$ such that $C^2 > 4c_2(\mathcal{E})^2$. A more complicated statement holds for arbitrary surfaces. One can see, in fact, that this result implies a similar one for surfaces in $\mathbf{P}^3$.

In the spirit of the above discussion, one is then led to consider the problem of the behavior under restriction to subvarieties of higher codimension. The inspiring idea, suggested by the divisor case, should be that when some suitable invariants, describing some form of "positivity" of the subvariety, become large with respect to the invariants of the vector bundle, then stability is preserved under restriction. Furthermore, if in the divisor case one needs the hypothesis that $\mathcal{E}$ be $\mathcal{O}_S(C)$-stable, in the higher codimension case one should still expect some measure of the relation between the geometry of subvariety and the stability of the vector bundle to play a role in the solution of the problem.

In fact, in the case of space curves the same kind of argument that proves the theorem about gonality can be applied to this question. Before explaining the result, we need the following definition. Recall that if X is a smooth projective threefold, $\mathcal{F}$ is a vector bundle on X, and L and H are two nef line bundles on X, $\mathcal{F}$ is said to be (H, L)-stable if for every nontrivial subsheaf $\mathcal{G} \subset \mathcal{F}$ we have $(fc_1(\mathcal{G}) - gc_1(\mathcal{F}))\cdot H \cdot L < 0$, where $f = \mathrm{rank}(\mathcal{F})$ and $g = \mathrm{rank}(\mathcal{G})$. Let then $\mathcal{E}$ be a rank-two vector bundle on $\mathbf{P}^3$, and consider a curve $C \subset \mathbf{P}^3$. Let us define the *stability constant of* $\mathcal{E}$ *with respect to* C as

$$\gamma(C, \varepsilon) = \sup\{\eta \in [0, \varepsilon(C)] | f^*\mathcal{E} \text{ is } (H, H_\eta)\text{-stable}\}.$$

For example, if C is a complete intersection of type (a, b) and the restriction of $\mathcal{E}$ to one of the two surfaces defining C is stable (with respect to the hyperplane bundle, then $\gamma(C, \mathcal{E}) = \varepsilon(C)$.

Then we have

Theorem 1.3. *Let $\mathscr{E}$ be a stable rank-2 vector bundle on $\mathbf{P}^3$ with $c_1(\mathscr{E}) = 0$. Let $C \subset \mathbf{P}^3$ be a smooth curve of degree d and Seshadri constant $\varepsilon(C)$, and let $\gamma = \gamma(C, \mathscr{E})$ be the stability constant of $\mathscr{E}$ with respect to C. Suppose that $\mathscr{E}|_C$ is not stable. Then*

$$c_2(\mathscr{E}) \geq \min\left\{\frac{\delta_\gamma(C)}{4}, \alpha\gamma\left(d - \frac{\alpha}{\gamma}\right)\right\},$$

where $\alpha =: \min\{1, \sqrt{d}(\sqrt{3/4} - \gamma\sqrt{d})\}$.

The problem of the behavior of stable bundles on $\mathbf{P}^r$ under restriction to curves has been studied by many researchers. In particular, a well-known fundamental theorem of Mehta and Ramanathan [25] shows that $\mathscr{E}|_C$ is stable if C is a *general* complete intersection curve of type $(a_1, a_2, \dots)$, and all the $a_i \gg 0$. Flenner [12] has then given an explicit bound on the $a_i s$ in term of the invariants of $\mathscr{E}$, which makes the conclusion of Mehta and Ramanathan's Theorem true. On the other hand, here we give numerical conditions that imply stability for $\mathscr{E}|_C$, with no generality assumption and without restricting C to be a complete intersection.

We have the following applications.

Corollary 1.1. *Let $\mathscr{E}$ be a stable rank-2 vector bundle on $\mathbf{P}^3$ with $c_1(\mathscr{E}) = 0$ and $c_2(\mathscr{E}) = c_2$. Suppose that $b \geq c_2 + 2$. If $V \subset \mathbf{P}^3$ is a smooth surface of degree b, then $\mathscr{E}|_V$ is $\mathscr{O}_V(H)$-stable.*

Corollary 1.2. *Let $\mathscr{E}$ be a stable bundle on $\mathbf{P}^3$ with $c_1(\mathscr{E}) = 0$. Suppose that $C = V_a \cap V_b \subset \mathbf{P}^3$ is an irreducible smooth complete intersection curve and that V_a is smooth. Assume furthermore that $a \geq 4b/3 + 10/3$ that and that $b \geq c_2(\mathscr{E}) + 2$. Then $\mathscr{E}|_C$ is stable.*

Corollary 1.3. *Let $c_2 \geq 0$ be an integer and let $\mathscr{M}(0, c_2)$ denote the moduli space of stable rank-two vector bundles on $\mathbf{P}^3$. If $a \gg b \gg c_2$ and $C \subset \mathbf{P}^3$ is an irreducible smooth complete intersection of type (a, b), then $\mathscr{M}(0, c_2)$ embeds in the moduli space of stable vector bundles of degree* 0 *on C.*

This paper covers part of the content of my Ph.D. thesis at UCLA. I want to thank Robert Lazarsfeld, my advisor, for introducing me to algebraic geometry and taking continuous interest in my progress. This work was inspired by his approach to complete intersections, and it owes a lot to his geometric point of view and to his suggestions.

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2. Preliminaries

In this section we state some results that will be used in the sequel. The following fact is well known.

Lemma 2.1. *Let X be a smooth projective variety and let $Y \subset X$ be a divisor. Suppose that we have an exact sequence $0 \to \mathcal{F} \to \mathcal{E} \to A \to 0$, where A is a line bundle on Y, and $\mathcal{E}$ is a rank-2 vector bundle on X. Let $[Y] \in A^1(X)$ be the divisor class of Y, and let $[A] \in A^2(X)$ be the image of the divisor class of A under the push forward $A^1(Y) \to A^2(X)$. Then $\mathcal{F}$ is a rank-2 vector bundle on X, having Chern classes $c_1(\mathcal{F}) = c_1(\mathcal{E}) - [Y]$ and $c_2(\mathcal{F}) = c_2(\mathcal{E}) + [A] - Y \cdot c_1(\mathcal{E})$.*

Lemma 2.2. *Let X be a smooth projective threefold, and let $C \subset X$ be a smooth curve in X. Denote by $f: X_C \to X$ the blowup of X along C, and let E be the exceptional divisor. Then $E^3 = -\deg(N)$, where N is the normal bundle of C in X. Furthermore, let A be any line bundle on X, and by abuse of language let A also denote its pullback to X_C. Then $E^2 \cdot A = -C \cdot A$.*

Proof. Both statements follow from a simple Segre class computation (see, for example, [14]). q.e.d.

We now recall some known results about instability of rank-2 vector bundles on projective manifolds, which are one of the main tools in the following analysis. Recall the following notation.

Definition 2.1. If S is a smooth projective surface, $N(S)$ is the vector space of the numerical equivalence classes of divisors in S; $K^+(S) \subset N(S)$ is the (positive) cone spanned by those divisors D such that $D^2 > 0$ and $D \cdot H > 0$ for some polarization on S. In general, if X is a smooth projective n-fold and H is a polarization on it, we shall denote by $K^+(X, H)$ the cone of all numerical classes D in $N(X)$ such that $D^2 \cdot H^{n-2} > 0$ and $D \cdot H^{n-1} > 0$ (or, equivalently, $D \cdot R \cdot H^{n-2} > 0$ for any other polarization R on X).

Definition 2.2. Let X be a smooth projective n-fold, and let $\mathcal{E}$ be a rank-2 vector bundle on X, with Chern classes $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$. The *discriminant* $\Delta(\mathcal{E}) \in A^2(X)$ is

$$\Delta(\mathcal{E}) = c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}).$$

Lemma 2.3. *Let X be a smooth projective n-fold, and let $\mathcal{E}$ be a rank-2 vector bundle on X. Fix a polarization H on X. Suppose that $\mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{E}$ are two line bundles in $\mathcal{E}$. Let us denote by l_1 and l_2 their H-degrees, respectively (i.e., $l_i = \mathcal{L}_i \cdot H^{n-1}$) and let $e = \deg_H(\mathcal{E}) = \bigwedge^2 \mathcal{E} \cdot H^{n-1}$ be the H-degree of $\mathcal{E}$. Suppose that $2l_i > e$ for $i = 1$ and $i = 2$ (in other words, $\mathcal{L}_1$ and $\mathcal{L}_2$ make $\mathcal{E}$ H-unstable). If $\mathcal{L}_2$ is saturated in $\mathcal{E}$, then $\mathcal{L}_1 \subseteq \mathcal{L}_2$.*

Proof. Set $l = \min\{l_1, l_2\}$. By assumption, we have $2l - e > 0$.

Claim 2.1. If the statement is false, the morphism of vector bundles $\phi: \mathcal{L}_1 \oplus \mathcal{L}_2 \to \mathcal{E}$ is generically surjective.

Proof. Set $\mathcal{Q} = \mathcal{E}/\mathcal{L}_\in$. Then $\mathcal{Q}$ is a rank-1 torsion free sheaf. The morphism $\mathcal{L}_1 \to \mathcal{Q}$ is therefore either identically zero or generically nonzero. If $\mathcal{L}_1 \not\subset \mathcal{L}_2$ the morphism $\mathcal{L}_1 \to \mathcal{Q}$ is then generically nonzero. But this implies that ϕ is generically surjective. q.e.d.

Therefore, $\bigwedge^2 \mathcal{E} \otimes \mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1}$ is an effective line bundle; it follows that $0 \le e - (l_1 + l_2) \le e - 2l$, a contradiction.

Corollary 2.1. *Let X and $\mathcal{E}$ be as above, and let $\mathcal{A} \subset \mathcal{E}$ be a saturated H-destabilizing line bundle. Then $\mathcal{A}$ is the maximal H-destabilizing line bundle.*

Corollary 2.2. *Let X be a smooth projective n-fold, and fix a very ample linear series $|V|$ on X, with $V \subset H^0(X, H)$. Suppose that $\mathcal{E}$ is a rank-2 vector bundle on X which is H-unstable. Let $C \subset X$ be a general complete intersection of $n - 1$ divisors in $|V|$. Then the maximal destabilizing line bundle of $\mathcal{E}|_C$ is the restriction to C of the maximal destabilizing line bundle of $\mathcal{E}$.*

Proof. Let $\mathcal{A}$ be the maximal destabilizing line bundle of $\mathcal{E}$. Then the inclusion $\psi: \mathcal{A} \to \mathcal{E}$ drops rank in codimension 2, because $\mathcal{A}$ is saturated in $\mathcal{E}$. Let Z be the locus where ψ drops rank. For a general complete intersection curve, we have $C \cap Z = \emptyset$. Hence $\mathcal{A}|_C$ is the maximal destabilizing line bundle of $\mathcal{E}|_C$. q.e.d.

The basic result is the following.

Theorem 2.1 (Bogomolov). *Let S be a smooth projective surface, and let $\mathcal{E}$ be a rank-2 vector bundle on S. Let $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$ be its Chern classes, and suppose that $c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) > 0$. Then there exists an exact sequence $0 \to A \to \mathcal{E} \to \mathcal{I}_Z \otimes B \to 0$, where A and B are line bundles on S, and Z is a codimension-2 (possibly empty) local complete intersection subscheme, with the property that $A - B \in K^+(S)$.*

For a proof, see [5], [27], [33], [15], or [22].

Corollary 2.3. *Let S and $\mathcal{E}$ be a smooth projective surface and a rank-2 vector bundle on it such that the hypothesis of the theorem are satisfied.*

Let $\mathscr{A}$ and $\mathscr{B}$ be the line bundles in the above exact sequence. Then the following inequalities hold: $(\mathscr{A} - \mathscr{B}) \cdot H > 0$ for all polarizations H on S, and $(\mathscr{A} - \mathscr{B})^2 \geq c_1(\mathscr{E})^2 - 4c_2(\mathscr{E})$.

Proof. The first inequality follows from the condition $A - B \in K^+(S)$. To obtain the second, just use the above exact sequence to compute $c_1(\mathscr{E})$ and $c_2(\mathscr{E})$: we obtain

$$c_1(\mathscr{E})^2 - 4c_2(\mathscr{E}) = (A+B)^2 - 4A \cdot B - 4\deg[Z] \leq (A-B)^2.$$

Corollary 2.4. *Let S and $\mathscr{E}$ satisfy the hypothesis of Bogomolov's theorem, and let H be any polarization on S. Then $\mathscr{E}$ is H-unstable, and $\mathscr{A}$ is the maximal H-destabilizing subsheaf of $\mathscr{E}$.*

Recall the fundamental theorem of Mumford-Mehta-Ramanatan (cf. [27]).

Theorem 2.2. *Let X be a smooth projective n-fold, and let H be a polarization on X. Consider a vector bundle $\mathscr{E}$ on X. If $m \gg 0$, and $V \in |mH|$ is general, then the maximal $H|_V$-destabilizing subsheaf of $\mathscr{E}|_V$ is the restriction of the maximal H-destabilizing subsheaf of $\mathscr{E}$.*

This theorem is very powerful, because it detects global instability from instability on the general complete intersection curve.

Theorem 2.3. *Let X be a smooth projective n-fold, and let H be a polarization on X. Consider a rank-2 vector bundle $\mathscr{E}$ on X, and suppose that $(c_1(\mathscr{E})^2 - 4c_2(\mathscr{E})) \cdot H^{n-2} > 0$. Then there exists an exact sequence $0 \to \mathscr{A} \to \mathscr{E} \to \mathscr{B} \otimes \mathscr{I}_Z \to 0$, where $\mathscr{A}$ and $\mathscr{B}$ are invertible sheaves, and Z is a locally complete intersection of codimension two (possibly empty) such that $\mathscr{A} - \mathscr{B} \in K^+(X, H)$ and*

$$(\mathscr{A} - \mathscr{B})^2 \cdot H^{n-2} \geq (c_1(\mathscr{E})^2 - 4c_2(\mathscr{E})) \cdot H^{n-2}.$$

Furthermore, $\mathscr{A}$ is the maximal $(H, \cdots, H, L)$-destabilizing subsheaf of $\mathscr{E}$, for every ample line bundle L on X.

Proof. The case $n = 2$ is just the content of Theorem 2.1; for $n \geq 3$, the statement follows by induction using Theorem 2.2. q.e.d.

Definition 2.3. If $\mathscr{E}$ satisfies the hypothesis of the theorem, we shall say that $\mathscr{E}$ is *Bogomolov-unstable with respect to H*.

Lemma 2.4. *Let $f: X \to Y$ be a morphism of projective varieties. Let $\mathscr{F}$ and A be, respectively, a vector bundle and an ample line bundle on X. For $y \in Y$, let $X_y = f^{-1}y$ and denote by $\mathscr{I}_{X_y}$ the ideal sheaf of X_y. Then there exists $k > 0$ such that $H^i(X, \mathscr{F} \otimes A^n \otimes \mathscr{I}_{X_y}) = 0$ for all $i > 0$, $n \geq k$, and for all $y \in Y$.*

Proof. For all $y \in Y$, there is an exact sequence

$$0 \to \mathcal{F} \otimes A^n \otimes \mathcal{I}_{X_y} \to \mathcal{F} \otimes A^n \to \mathcal{F} \otimes A^n|_{X_y} \to 0.$$

Furthermore, there exists a flattening stratification of Y with respect to f, $Y = \coprod_{l=1}^r Y_l$, with the following property [28]. The Y_l are locally closed subschemes of Y, and if $X_l := f^{-1}Y_l$, $l = 1, \cdots, r$, and $f_l\colon X_l \to Y_l$ is the restriction of f, then f_l is a flat morphism. Let us then start by finding k_1 such that for all $n \geq k_1$ we have

$$H^i(X, \mathcal{F} \otimes A^n) = 0 \quad \text{and} \quad H^i(X, \mathcal{F} \otimes A^n \otimes \mathcal{I}_{X_l}) = 0$$

for all $i > 0$ and all $l = 1, \cdots, r$. Then it is easy to see that the statement is equivalent to saying that there is $k \geq k_1$ such that for all $n \geq k$ the restriction maps $\phi_y\colon H^0(X, \mathcal{F} \otimes A^n) \to H^0(X_y, \mathcal{F} \otimes A^n|_{X_y})$ are all surjective, and that $H^i(X_y, \mathcal{F} \otimes A^n|_{X_y}) = 0$, for all $y \in Y$ and all $i > 0$. If $y \in Y_l$, and $\mathcal{I}_{X_y}^{X_l}$ denotes the ideal sheaf of X_y in X_l, then we have an exact sequence $0 \to \mathcal{I}_{X_l} \to \mathcal{I}_{X_y} \to \mathcal{I}_{X_y}^{X_l} \to 0$.

Claim 2.2. The lemma is true if there exists k such that for all $n \geq k$, for $l = 1, \cdots, r$ and for all $y \in Y_l$ we have that $H^i(X_l, \mathcal{F} \otimes A^n|_{X_l} \otimes \mathcal{I}_{X_y}^{X_l}) = 0$ for $i > 0$.

Proof. It follows from an obvious exact sequence. q.e.d.

This means that we can reduce to the case where f is flat. For $y_0 \in Y$, we can find k_0 such that for $n \geq k_0$ and, $i > 0$ we have $H^i(X, \mathcal{F} \otimes A^n \otimes \mathcal{I}_{X_{y_0}}) = 0$. Therefore, the morphism

$$\beta_{y_0}\colon \lim_{y_0 \in U} H^0(f^{-1}U, \mathcal{F} \otimes A^n) \to H^0(X_y, \mathcal{F} \otimes A^n)$$

is onto, and then so is

$$\psi_{y_0} =: \beta_{y_0} \otimes k(y_0)\colon f_*(\mathcal{F} \otimes A^n)(y) \to H^0(X_{y_0}, \mathcal{F} \otimes A^n|_{X_{y_0}}).$$

By Grauert's theorem [21], ψ_{y_0} is an isomorphism, and that the same holds for ψ_y, for y in a suitable open neighbourhood U_0 of y_0. Therefore the restriction morphism $H^0(X, \mathcal{F} \otimes A^n) \to H^0(X_y, \mathcal{F} \otimes A^n|_{X_y})$ comes from a morphism of sheaves, and hence they are onto for all $y \in V_0$, for a suitable open set $V_0 \subset U_0$. We can then invoke the quasi-compatness of Y to conclude that there exists k such that $H^1(X, \mathcal{F} \otimes A^n \otimes \mathcal{I}_{X_y}) = 0$ for all $y \in Y$. As to $i \geq 2$, we have isomorphisms

$$H^i(X_y, \mathcal{F} \otimes A^n|_{X_y}) \simeq H^{i+1}(X, \mathcal{F} \otimes A^n \otimes \mathcal{I}_{X_y}) = 0$$

for all $i > 0$, and so we need to show that $H^i(X_y, \mathscr{F} \otimes A^n) = 0$ for $n \gg 0$, $i > 0$ and all $y \in Y$. But for $n \gg 0$ we have $R^i f_*(\mathscr{F} \otimes A^n) = 0$ if $i > 0$ and then this implies $h^i(X_y, \mathscr{F} \otimes A^n|_{X_y}) = 0$ for all $y \in Y$ [28]. q.e.d.

We record here a trivial numerical lemma that will be handy in the sequel.

Lemma 2.5. *If $s \geq \alpha$, $a \geq 2s$ and $b \geq as - s^2$, then $b \geq a\alpha - \alpha^2$.*

Proof. $as - s^2$ is increasing in s if $a \geq 2s$. The statement follows.

3. Seshadri constants of curves

Let $C \subset \mathbf{P}^3$ be a smooth curve and let H denote the hyperplane bundle on $\mathbf{P}^3$. We shall let $f: X_C \to \mathbf{P}^3$ be the blowup of $\mathbf{P}^3$ along C, and $E = f^{-1}C$ be the exceptional divisor.

Definition 3.1. The *Seshadri constant of C* is

$$\varepsilon(C) := \sup\{\eta \in \mathbf{Q} | f^*H - \eta E \text{ is ample}\}.$$

In other terms, $\varepsilon(C)$ is the supremum of the ratio n/m, where n and m are such that $mH - nE$ is ample (or, equivalently, very ample). In the sequel we shall use the shorthand $H_\eta := H - \eta E$ for $\eta \in \mathbf{Q}$; furthermore, we shall generally write H for f^*H (as we just did).

Lemma 3.1. *H_η is ample if and only if $0 < \eta < \varepsilon(C)$. It is nef if and only if $\eta \in [0, \varepsilon(C)]$.*

Proof. Since the ample cone of a projective variety is convex, the line $H - tE \subset N^1(X)$ intersects $K^+(X)$ in a segment $(H - t_1 E, H - t_2 E)$. Let F denote the numerical class of a fiber of $\pi: E \to C$. Then $H_\eta \cdot F = \eta$, and therefore if H_η is ample we must have $\eta > 0$. Hence $t_1 \geq 0$. On the other hand, it is well known that $H - tE$ is ample for $t > 0$ sufficiently small, and therefore $t_1 = 0$. By definition, $t_2 = \varepsilon_2(\mathscr{E})$. The remaining part of the statement is clear.

Corollary 3.1. *We have*

$$\varepsilon(C) = \sup\{\eta | H_\eta \cdot D \geq 0 \textit{ for all curves } D \subset X_C\}.$$

Lemma 3.2. *Let $C \subset \mathbf{P}^3$ be a smooth curve, and let $\mathscr{I}_C$ be its ideal sheaf. Let M and n be nonnegative integers. Then $\mathscr{O}_{X_C}(mH - nE)$ is globally generated if $\mathscr{I}_C^n(m)$ is.*

Proof. Suppose that $\mathscr{I}_C^n(m)$ is globally generated, and let $F_1, \cdots, F_k \in H^0(\mathbf{P}^3, \mathscr{I}_C^n(m))$ be a basis. Let $P \in C$ and let U be some open neighbourhood of P. By assumption, $F_1, \cdots, F_k$ generate $\mathcal{I}_C$ in U. By abuse of language, let us write F_i for the pullbacks f^*F_i. If e is a local equation for E in a Zariski open set $V \subset f^{-1}U$, then the ideal generated by the $F_i s$ is $\langle\{F_i\}\rangle = (e^n)$. Hence we can write

$$\sum_{i=1}^{k} P_i F_i = e^n$$

for some P_i regular on V. However, by construction we can write $F_i = \widetilde{F}_i e^n$, and therefore we have

$$\sum_{i=1}^{k} \widetilde{F}_i P_i = 1$$

in V. Hence the $\widetilde{F}_i$ are base point free, and they can be extended to global sections of $\mathcal{O}_{X_C}(mH - nE)$, which is therefore globally spanned.

Corollary 3.2. *Let $C \subset \mathbf{P}^3$ be a smooth curve. Then*

$$\varepsilon(C) \geq \sup\{n/m | \mathcal{I}_Y^n(m) \textit{ is globally generated}\}.$$

Let us look at some examples.

Example 3.1. If $L \subset \mathbf{P}^3$ is a line, then $\mathcal{I}_L(1)$ is globally generated. Therefore $\mathcal{E}(L) \geq 1$. On the other hand, let $\Lambda \subset \Pi^3$ be a hyperplane containing L, and let $D \subset \Lambda$ be any irreducible curve distinct from L. Then $H_1 \cdot \widetilde{D} = \deg(D) - L \cdot_\Lambda D = 0$, where $\widetilde{D} \subset Bl_L(\mathbf{P}^3)$ is the proper transform of L. Hence $\varepsilon(L) = 1$. As we shall see shortly, this generalizes to the statement that if $C \subset \mathbf{P}^3$ is a smooth complete intersection of type (a, b) and $a \geq b$, then $\varepsilon(C) = 1/a$.

Example 3.2. If $C \subset \mathbf{P}^3$ has an l-secant line, then $\varepsilon(C) \leq 1/l$. To see this, let L be the l-secant; denoting by $\widetilde{L} \subset X_C$ the proper transform of L in $Bl_C(\mathbf{P}^3)$ we have $H \cdot \widetilde{L} = 1$ and $E \cdot \widetilde{L} = l$. Hence $0 \leq H_\varepsilon \cdot \widetilde{L}$ implies $\varepsilon \leq 1/l$.

Lemma 3.3. *Let $C \subset \mathbf{P}^3$ be a smooth curve of degree d. Then* $1/\sqrt{d} \geq \varepsilon(C) \geq 1/d$.

Proof. It is well-known that a smooth subvariety of degree d of projective space is cut out by hypersurfaces of degree d. Hence $\mathcal{I}_C(d)$ is globally generated, and this proves the second inequality. As to the first, we must have $0 \leq H \cdot H_\varepsilon^2 = 1 - \varepsilon^2 d$, by a simple Segre class computation. q.e.d.

The right inequality is sharp if the curve is degenerate; the left one is sharp for a complete intersection curve of type (a, a). If the curve is nondegenerate, however, one can say something more.

Definition 3.2. Let $C \subset \mathbf{P}^3$ be a smooth curve, and let $\mathcal{I}_C$ be its ideal sheaf. C is said to be l-regular if $H^i(\mathbf{P}^3, \mathcal{I}_C(l-i)) = 0$ for all $i > 0$. The regularity of C, denoted by $m(C)$, is the smallest l such that C is l-regular [7], [29], [16].

Remark 3.1. By a celebrated theorem of Castelnuovo, we have $m(C) \leq d-1$ [7], [20].

Proposition 3.1. *Let $C \subset \mathbf{P}^3$ be a smooth space curve, and let $m = m(C)$ be its regularity. Then $2/(m-1) \geq \varepsilon(C) \geq 1/m$.*

Proof. By a classical theorem of Castelnuovo-Mumford, the homogeneous ideal of C is saturated in degree $m(C)$ and therefore $\varepsilon(C) \geq 1/m(C)$. By definition, to prove the first inequality it is enough to show that $H^i(\mathbf{P}^3, \mathcal{I}_C(k)) = 0$ for $k \geq \lceil 2/\varepsilon(C) \rceil - 3$ because this implies $m(C) \leq 2/\varepsilon(C)+1$ and then the statement. To prove the above vanishing, observe that $\{2/(\lceil 2/\varepsilon(C)\rceil+1\} < \varepsilon(C)$ and therefore $(\lceil 2/\varepsilon(C)\rceil+1)H-2E$ is an ample integral divisor X_C. Since $\omega_{X_C} = \mathcal{O}_{X_C}(-4H+E)$, the Kodaira vanishing theorem gives

$$H^i(X_C, \mathcal{O}_{X_C}((\lceil 2/\varepsilon(C)\rceil - 3)H - E)) = 0$$

for $i > 0$, as desired.

Remark 3.2. Using vanishing theorems on the blowup to obtain bounds on the regularity is a well-known technique: see [3] for various results in this direction.

Remark 3.3. It is not possible, in the above vanishing, to replace the condition on k by $k \geq \lceil 1/\varepsilon \rceil$. To see this, suppose that C is a complete intersection of type (a, b) so that we have a Koszul resolution

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-b) \to \mathcal{O}_{\mathbf{P}^3} \oplus \mathcal{O}_{\mathbf{P}^3}(a-b) \to \mathcal{I}_C(a) \to 0.$$

It follows that $H^2(\mathbf{P}^3, \mathcal{I}_C(a)) \simeq H^3(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(-b)) \neq 0$, for $b \geq 4$.

Corollary 3.3. *Let $C \subset \mathbf{P}^3$ be a nondegenerate smooth curve. Then $\varepsilon(C) \geq 1/(d-1)$.*

Equality is attained in the previous corollary in the case of a twisted cubic.

It is convenient to introduce the following definition.

Definition 3.3. Let $C \subset \mathbf{P}^3$ be a smooth curve. For an irreducible curve $D \subset \mathbf{P}^3$ different from C let $\widetilde{D}$ be its proper transform in the blowup of $\mathbf{P}^3$ along C. Define

$$\varepsilon_1(C) := \sup\{\eta \in \mathbf{Q} | (H - \eta E)|_E \text{ is ample}\}$$

and

$$\varepsilon_2(C) := \sup\{\eta \in \mathbf{Q} | H_\eta \cdot \widetilde{D} \geq 0 \ \forall \text{ irreducible curves } D \neq C\}.$$

Remark 3.4. $\varepsilon(c) = \min\{\varepsilon_1(C), \varepsilon_2(C)\}$.

We are interested in estimating the Seshadri constant of a space curve C. It is convenient to examine $\varepsilon_1(C)$ and $\varepsilon_2(C)$ separately. We shall see that $\varepsilon_1(C)$ is determined by the structure of the normal bundle, while $\varepsilon_2(C)$ depends on the "linkage" of C, and is generally much harder to estimate. We start with an analysis of $\varepsilon_1(C)$.

Definition 3.4. Let C be a smooth projective curve, and let $\mathscr{E}$ be a rank 2 vector bundle on it. For all finite morphisms $f: \widetilde{C} \to C$ and all exact sequences of locally free shaves on $\widetilde{C}$ of the form $0 \to L \to f^*\mathscr{E} \to M \to 0$, consider the ratio $\deg(L)/\deg(f)$. Let $\Sigma_{\mathscr{E}}$ denote the set of all the numbers obtained in this way. Define $s(\mathscr{E}) =: \Sigma_{\mathscr{E}}$.

Remark 3.5. As in [39], $s(\mathscr{E})$ can be interpreted as a measure of the instability of $\mathscr{E}$. More precisely, we have $s(\mathscr{E}) = \frac{1}{2}\deg(\mathscr{E})$ if $\mathscr{E}$ is semistable, and $s(\mathscr{E}) = \deg(L)$ if $\mathscr{E}$ is unstable, and $L \subset \mathscr{E}$ is the maximal destabilizing line subbundle of $\mathscr{E}$. In order words, $s(\mathscr{E}) - \frac{1}{2}\deg(\mathscr{E}) \geq 0$ always, and equality holds if and only if $\mathscr{E}$ is semistable.

We then have

Proposition 3.2. *Let $C \subset \mathbf{P}^3$ be a smooth curve. Denote by N the normal bundle of C in $\mathbf{P}^3$, and let $\varepsilon_1(C)$ be as above. Then*

$$\varepsilon_1(C) = \deg(C)/s(N).$$

Proof. Let $f: P_C \to \mathbf{P}^3$ be the blowup of $\mathbf{P}^3$ along C, and let E be the expectional divisor; recall that E can be identified with the relative projective space of lines in the vector bundle N. Set $\pi = f|_E$, and denote by F a fiber of π. Let $D \subset E$ be any reduced irreducible curve. If D is fiber of π, then $\eta > 0$ ensures that $H_\eta \cdot D > 0$. Hence we may assume that $D \to C$ is a finite map, whose degree is given by $a = D \cdot F$. Let $\psi: \widetilde{D} \to D \subset X_C$ be the normalization of D, and let $p: \widetilde{D} \to C$ be the induced morphism. Then ψ is equivalent to the assignment of a sub-line bundle $L \subset p^*N$, given by $L = \psi^*\mathscr{O}_{\mathbf{P}N}(-1)$. Since $\mathscr{O}_{\mathbf{P}N}(-1) \simeq \mathscr{O}_E(E)$, we have $\deg(L) = D \cdot E$. Hence $H_\eta \cdot D = aH \cdot C - \eta \cdot \deg(L)$; the condition $\eta \leq \varepsilon_1(C)$ translates therefore in the condition $\eta \leq \inf\{(H \cdot C)/\deg(L)/a\}$. In other words, then, it is equivalent to $\eta \leq (H \cdot C)/s(N)$.

Example 3.3. Let $C \subset \mathbf{P}^3$ be a smooth complete intersection curve of type (a, b), with $a \geq b$. Then we have a Koszul resolution of the ideal sheaf of C, from which it is easy to conclude that $\varepsilon(C) \geq 1/a$. On the other hand, $s(N) = a^2 b$ and therefore by Proposition 3.2 $\varepsilon_1(C) = 1/a$. Hence we have $\varepsilon(C) = 1/a$.

Example 3.4. Let $C \subset \mathbf{P}^3$ be given as the zero locus of a regular section of a rank-2 vector bundle $\mathscr{E}$ on $\mathbf{P}^3$. It is well known that this is always the case provided that the determinant of the normal bundle N extends. The Koszul resolution then is

$$0 \to \det(\mathscr{E})^{-1} \to \mathscr{E}^* \to \mathscr{I}_C \to 0.$$

By Corollary 3.2 and Proposition 3.2, we then conclude that

$$(H \cdot C)/s(\mathscr{E}|_C) \geq \varepsilon(C) \geq \varepsilon(\mathscr{E}),$$

where

$$\varepsilon(\mathscr{E}) = \sup\{n/m | S^n\mathscr{E}^*(m) \text{ is spanned}\}.$$

We shall call $\varepsilon(\mathscr{E})$ the Seshadri constant of the vector bundle $\mathscr{E}$. It has the following geometric interpretation. Let $\mathbf{P}\mathscr{E}$ be the relative projective space of lines in $\mathscr{E}$. $\text{Pic}(\mathbf{P}\mathscr{E})$ is generated by two line bundles H and $\mathscr{O}(1)$, where H is the pullback of the hyperplane bundle on $\mathbf{P}^3$. Let R be some divisor associated to the line bundle $\mathscr{O}(1)$. It is well known that the rational divisor $H + \eta R$ is ample, for sufficiently small $\eta \in \mathbf{Q}^+$ [21].

Proposition 3.3. $\varepsilon(\mathscr{E}) = \sup\{\eta \in \mathbf{Q} | H + \eta R \in \text{Div}_{\mathbf{Q}}(\mathbf{P}\mathscr{E})$ *is ample*$\}$.

Proof. Provisionally denoted by $\gamma(\mathscr{E})$ the right-hand side of the statement. Also, for brevity let us set $X = \mathbf{P}\mathscr{E}$, and let X_z stand for the fiber over a point $z \in \mathbf{P}^3$. Let us first prove that $\varepsilon(\mathscr{E}) \leq \gamma(\mathscr{E})$. Suppose then that $\eta = n/m < \varepsilon(C)$, where n and m are such that $S^n\mathscr{E}^*(m)$ is globally generated. Since

$$S^n\mathscr{E}^*(m) = f_*\mathscr{O}_X(mH + nR),$$

we have the identifications

$$H^0(X, \mathscr{O}_X(mH + nR)) \simeq H^0(\mathbf{P}^3, S^n\mathscr{E}^*(m))$$

and

$$H^0(X_z, \mathscr{O}_{X_z}(mH + nR)) \simeq S^n E^*(m)(z).$$

With this in mind, we then have a surjection

$$H^0(X, \mathscr{O}_X(mH + nR)) \to H^0(X_z, \mathscr{O}_{X_z}(mH + nR))$$

for all $z \in \mathbf{P}^3$, and since $\mathscr{O}_X(mH + nR)$ is generated along the fibers, it is also globally generated.

Let us prove that $\gamma(\mathscr{E}) \leq \varepsilon(\mathscr{E})$. Let $\eta = n/m < \gamma(\mathscr{E})$, where n and m have been chosen so that $mH + nR$ is ample. After perhaps multiplying m and n by some large positive integer we may suppose that $mH + nR$

is very ample and that $H^i(X, \mathcal{I}_{X_z}(mH+nR)) = 0$ for all $i > 0$ and all $z \in \mathbf{P}^3$ (see Lemma 2.4). But then we have surjective restriction maps

$$H^0(X, \mathcal{O}_X(mH+nR)) \to H^0(X_z, \mathcal{O}_{X_z}(mH+nR))$$

for all $z \in \mathbf{P}^3$, and the lemma thus follows from the above identification.

Remark 3.6. The equation $\varepsilon(C) \geq \varepsilon(\mathcal{E})$ from Example 3.4 can then be explained as follows. For each $n \geq 0$ we have the surjective morphism $S^n\mathcal{E}^* \to \mathcal{I}_C^n$ and therefore a surjection of sheaves of graded algebras $\bigoplus_{n\geq 0} S^n\mathcal{E}^* \to \bigoplus_{n\geq 0}\mathcal{I}_C^n$, which yields a closed embedding $i: X_C \hookrightarrow \mathbf{P}\mathcal{E}$. On the other hand, $i^*\mathcal{O}_{\mathbf{P}\mathcal{E}}(R) = \mathcal{O}_{X_C}(-E)$ and the above inequality is just saying that if $H + \eta R$ is ample, it restricts to an ample divisor on X_C.

We now consider ways to estimate $\varepsilon_2(C)$. $\varepsilon_2(C)$ gathers more global information than $\varepsilon_1(C)$, because it relates to how C is "linked" to the curves in $\mathbf{P}^3$. Recall that our definition was:

$$\varepsilon_2(C) := \sup\{\eta \in \mathbf{Q} | H_\eta \cdot \widetilde{D} \geq 0 \ \forall \text{ irreducible curves } D \subset \mathbf{P}^3,\ D \neq C\}.$$

As usual, $\widetilde{D}$ denotes the proper transform of D in the blowup of C.

In order to estimate $\varepsilon_2(C)$, we assume given two distinct irreducible hypersurfaces V_a and V_b through C and divide the problem in two parts: (a) to estimate the intersection numbers $\widetilde{D}\cdot E$ for $D \not\subset V_a \cap V_b$, and (b) to control the same numbers for D, a component of the residual curve to C in $V_a \cap V_b$.

Let us start with the following simple observation. Let C and D be reduced curves in $\mathbf{P}^3$, and let $t: D_n \to D \subset \mathbf{P}^3$ be the normalization of D. If $f: X_C \to \mathbf{P}^3$ is the blowup of C, and E_C is the expectional divisor, then clearly t factors through f, i.e., there exists $u: D_n \to X_C$ such that $t = f \circ u$. On the other hand, $t^{-1}C = u^{-1}f^{-1}C = u^{-1}E_C$ and therefore

$$\widetilde{D}\cdot E_C = D_n \cdot_u E_C = \deg\{t^{-1}C\}. \tag{1}$$

We can now attack part (a), which is the easiest. We have the following.

Lemma 3.4. *Let $C \subset \mathbf{P}^3$, V_a and V_b be two distinct reduced irreducible surfaces through C. Suppose that $a \geq b$, and let $\eta \leq 1/a$. Then for every irreducible curve $D \not\subset V_a \cap V_b$ we have $\widetilde{D}\cdot H_\eta \geq 0$.*

Proof of the lemma. Let D be reduced and have degree s, and set $G := V_a \cap V_b$. G is a complete intersection curve, and then we know from the Koszul resolution of its ideal sheaf that its Seshadri constant satisfies $\varepsilon(G) \geq 1/a$. Let $X_G \to \mathbf{P}^3$ be the blowup of $\mathbf{P}^3$ along G, and let E_G be the exceptional divisor. For $\eta \in \mathbf{Q}$, let $H'_\eta = g^*H - \eta E_G$. By what we have just said, $H'_{1/a}$ is a nef divisor on X_G. Therefore, if we let $D' \subset X_G$ denote the proper transform of D in X_G, we have $D' \cdot H'_\eta \geq 0$, and this can be written as $D' \cdot E_G \leq as$. Now let as above $t\colon D_n \to D \subset \mathbf{P}^3$ be the normalization of D, and let $\widetilde{D} \subset X_C$ denote the proper transform of D in the blowup of C. Then by (1) we have

$$\widetilde{D} \cdot E_C = \deg\{t^{-1}C\} \leq \deg\{t^{-1}G\} = D' \cdot E_G,$$

since $G \supset C$ as schemes. Therefore,

$$H_{1/a} \cdot \widetilde{D} \geq H'_{1/a} \cdot D' \geq 0, \tag{2}$$

and the statement follows. q.e.d.

There does not seem to be much that one can say about part (b) in general; if we throw in some extra geometric information, however, there is a possible estimate.

Proposition 3.4. *Let $C \subset \mathbf{P}^3$ be a smooth curve. Suppose that C is contained in the intersection of two distinct reduced and irreducible hypersurfaces V_a and V_b of degree a and b, respectively. Suppose that all the residual curves to C in the complete intersection $V_a \cap V_b$ are reduced and that at least one of the two hypersurfaces is smooth. Then*

$$\varepsilon_2(C) \geq 1/(a+b-2),$$

where the equality holds if and only if the residual curve to C is the union of disjoint lines.

Example 3.5. It is well-known that a curve which is linked to a line L in a complete intersection of type (a, b) is cut out by the hypersurfaces V_a and V_b and by a third equation of degree $a+b-2$. Therefore its ideal sheaf is generated in degree $a+b-2$, so that $\varepsilon(C) \geq 1/(a+b-2)$. On the other hand, it is easy to check that $\widetilde{L} \cdot E = a+b-2$. Thus in this case we find directly that $\varepsilon(C) = 1/(a+b-2)$. More generally, the same argument works whenever C is linked to a union of (reduced) disjoint lines.

Example 3.6. The assumption that the residual curves be all reduced is necessary. To see this, let $L \subset \mathbf{P}^3$ be a line, and let V be a smooth surface of degree v through L. We have $L \cdot_V L = 2 - v$. Let H be the hyperplane bundle restricted to V. Then for $s \gg 0$ the linear series $|sH - 2L|$ is very ample. Choose a smooth curve $C \in |sH - 2L|$. Then C is linked to a double line supported on L in the complete intersection $V \cap W$, where W is a suitable hypersurface of degree s in $\mathbf{P}^3$. Thus we have

$$\widetilde{L} \cdot E_C = (sH - 2L) \cdot_V L = s - 2L^2 = s + 2v - 4,$$

and so $\varepsilon_2(C) \leq 1/(s + 2v - 4)$.

Proof. The following simple argument was suggested by the referee. Suppose that V_a is smooth. Let $X \subset V_a$ be the residual curve to C in $V_a \cap V_b$, and let $D \subset X$ be one of its components; write $G = D + D'$. We then have

$$\begin{aligned} 2p_D - 2 &= D \cdot (D + K_{V_a}) \\ &= D \cdot [bH - C - D' + (a-4)H] \\ &= (a+b-4)H - D \cdot D' - \widetilde{D} \cdot E. \end{aligned}$$

From this one then gets

$$\widetilde{D} \cdot E = (a+b-4)\deg(D) - D \cdot D' + 2 - 2p_D,$$

and this is always $\leq (a+b-2)\deg(D)$, with equality holding if and only if $\deg(D) = 1$, $p_D = 0$ and $D \cdot D' = 0$. q.e.d.

We now define two auxiliary invariants related to the Seshadri constant that will be useful shortly.

Definition 3.5. Let $C \subset \mathbf{P}^3$ be a smooth curve of degree d, and let $\varepsilon(C)$ be its Seshadri constant. Let N be the normal bundle of C in $\mathbf{P}^3$. For $0 \leq \eta \leq \varepsilon(C)$ a rational number, define

$$\delta_\eta(C) := \eta \cdot \deg(N) - d$$

and

$$\lambda_\eta(C) := \eta^2 d^2 - \delta_\eta(C).$$

It is easy to check that

$$\delta_\eta(C) := E^2 \cdot H_\eta. \tag{3}$$

More explicitly, suppose that $0 < \eta < \varepsilon(C)$, and let m and n be large positive integers such that $\eta = n/m$, and $mH - nE$ is very ample. Then for a general $S \in |mH - nE|$ the intersection $C' = E \cap S$ is an irreducible smooth curve, and the induced morphism $C' \to C$ has degree n. Thus

$$\delta_\eta(C) = \frac{C' \cdot_S C'}{H \cdot_S H}. \tag{4}$$

Similarly,

$$\lambda_\eta(C) = \frac{(H \cdot C')^2}{(H \cdot_S H)^2} - \frac{C' \cdot_S C'}{H_S H}. \tag{5}$$

Remark 3.7. If we let $x = \eta d$, we have $\lambda_\eta(C) = f(x)$, where

$$f(x) = x^2 - (4 + (2g-2)/d)x + d,$$

g being the genus of C. For C sucanonical, f is the polynomial introduced by Halphen in his celebrated speciality theorem [18], [19], [32] given by

$$g(X) = x^2 - (4+e)x + d,$$

where $e = \max\{k | H^1(C, \mathscr{O}_C(k)) \neq 0\}$. Observe that $e \leq (2g-2)/d$ always.

Corollary 3.4. *Suppose that there exists an irreducible surface of degree m through C, having multiplicity n along C. If $\eta = n/m$, then $\lambda_\eta(C) \geq 0$. In particular, $\lambda_\eta(C) \geq 0$ for all $0 \leq \eta \leq \varepsilon(C)$. Equality holds if and only if $\mathscr{O}_S(C')$ is numerically equivalent to a multiple of $\mathscr{O}_S(H)$. In particular, $\lambda_{\varepsilon(C)}(C) \geq 0$ and equality holds if C is a complete intersection. If C is subcanonical, and ηd is an integer, then $\lambda_\eta(C) = 0$ forces C to be a complete intersection.*

Proof. A straightforward application of the Hodge index theorem. The last part follows from the corresponding statement of the speciality theorem (see [18]).

Corollary 3.5. *Let $C \subset \mathbf{P}^3$ have genus g, degree d and Seshadri constant $\varepsilon(C)$. Then*

$$g \leq \tfrac{1}{2}d^2\varepsilon(C) + d(1/(2\varepsilon(C)) - 2) + 1.$$

The right-hand side of the above inequality is a decreasing function of ε in the interval $(1/d, 1/\sqrt{d})$. In other words, higher Seshadri constants impose tighter conditions on the genus. For a Castelnuovo extremal curve of even degree we have $\varepsilon = 2/d$ and the right-hand side, as a function of d, is asymptotic to $d^2/4$.

Corollary 3.6. *Let D be a divisor on X_C, and set $s = D \cdot H_\eta \cdot H$. Then for $0 \leq \eta \leq \varepsilon(C)$ we have*

$$D^2 \cdot H_\eta - D \cdot H_\eta \cdot E \leq s^2 - s\eta d.$$

Proof. Write

$$D = xH + yE.$$

Then

$$D^2 \cdot H_\eta = x^2 + y^2\delta_\eta(C) + 2xyd$$

and

$$D \cdot H_\eta \cdot E = x\eta d + y\delta_\eta(C).$$

From this we obtain

$$D^2 \cdot H_\eta - D \cdot H_\eta \cdot E = s^2 - s\eta d - \lambda_\eta(C)(y^2 - y).$$

Since y is an integer, the statement then follows from Corollary 3.4.

Remark 3.8. From the inequality (see Remark 3.5) $s(N) \geq \frac{1}{2}\deg(N)$ and the definition of $\delta_\eta(C)$, it is easy to see that $d \geq \delta_\eta(C)$.

4. Gonality of space curves and free pencils on projective varieties

We have seen that if $C \subset S$ is a smooth curve with $C^2 > 0$, then one can give lower bounds on the gonality of C [35]. We deal here with the next natural question: if $C \subset \mathbf{P}^3$, what can be said about $\operatorname{gon}(C)$ in terms of the invariants of this embedding, and exactly which invariants should one expect to play a direct role? A hint to this is given by Lazarsfeld's result, to the effect that if C is nondegenerate complete intersection of type (a, b) with $a \geq b$ then $\operatorname{gon}(C) \geq a(b-1)$.

For $C \subset \mathbf{P}^r$ a smooth curve, we let

$$\delta_\eta(C) = E^2 \cdot H_\eta^{r-2}.$$

We then have $\delta_\eta(C) = \eta^{r-3}(\eta \deg(N) - \deg(C))$.

Theorem 4.1. *Let $C \subset \mathbf{P}^r$ be a smooth curve of degree d, $r \geq 3$. Let $\varepsilon(C)$ be the Seshadri constant of C, and set*

$$\alpha = \min\{1, \sqrt{\varepsilon(C)^{r-3}d}(1 - \varepsilon(C)\sqrt{\varepsilon(C)^{r-3}d})\}.$$

Then

$$\operatorname{gon}(C) \geq \min\left\{\frac{\delta_{\varepsilon(C)}(C)}{4\varepsilon(C)^{r-2}}, \alpha\left(\deg(C) - \frac{\alpha}{\varepsilon(C)^{r-2}}\right)\right\}.$$

Although we state the result for curves in $\mathbf{P}^r$ for the sake of simplicity, it is easy to see that the same considerations apply when $\mathbf{P}^r$ is replaced by a general smooth projective manifold X with $\operatorname{Pic}(X) \simeq \mathbf{Z}$. Later in this section we shall indicate how these results generalize to higher dimensional varieties in $\mathbf{P}^r$.

Proof. To avoid cumbersome notation, we shall assume that $r = 3$. The proof applies to higher value of r, with no significant change. We want then to show that

$$\operatorname{gon}(C) \geq \min\left\{\frac{\delta_{\varepsilon(C)}(C)}{4\varepsilon(C)}, \alpha\left(d - \frac{\alpha}{\varepsilon(C)}\right)\right\}, \tag{6}$$

where $\alpha = \min\{1, \sqrt{d}(1 - \varepsilon(C)\sqrt{d})\}$.

Suppose, to the contrary, that the statement is false: if $k = \operatorname{gon}(C)$, then k is strictly smaller than both terms within the braces in the last inequality. For $\eta < \varepsilon(C)$ sufficiently close to $\varepsilon(C)$ the same holds. More precisely, if we let $\alpha_\eta = \min\{1, \sqrt{d}(1-\eta\sqrt{d})\}$, we have

$$k < \delta_\eta(C)/4\eta \tag{7}$$

and

$$k < \alpha_\eta(d - \alpha_\eta/\eta). \tag{8}$$

Pick a minimal pencil $A \in \mathrm{Pic}^k(C)$, and set $V =: H^0(C, A)$. Then V is a two-dimensional vector space. On C we have an exact sequence of locally free sheaves $0 \to -A \to V \otimes \mathscr{O}_C \to A \to 0$. Consider the blowup diagram. Define

$$\mathscr{F} := \mathrm{Ker}(\psi\colon V \otimes \mathscr{O}_{X_C} \to \pi^*A), \tag{9}$$

where π^*A is a line bundle on E, and ψ is surjective. Since E is a Cartier divisor in X_C, $\mathscr{F}$ is a rank-2 vector bundle on X_C. As usual we set $H_\eta = H - \eta E$, where η is a rational number.

Claim 4.1. Let η be a rational number in the interval $(0, \varepsilon(C))$. If $k < \frac{1}{4}\delta_\eta(C)/\eta$, then $\mathscr{F}$ is Bogomolov-unstable with respect to H_η.

Proof. By Lemma 2.1, the Chern classes of $\mathscr{F}$ are $c_1(\mathscr{F}) = -E$ and $c_2(\mathscr{F}) = \pi^*[A]$, where $[A]$ denotes the divisor class in $A^1(C)$ of any element in $|V|$, and we implicitly map $A^1(E)$ to $A^2(X_C)$. Then the discriminant of $\mathscr{F}$ (Definition 2.2) is given by

$$\Delta(\mathscr{F}) = E^2 - 4[A].$$

Therefore by the assumption we have

$$\Delta(\mathscr{F}) \cdot H_\eta = \delta_\eta(C) - 4\eta k > 0, \tag{10}$$

which implies that $\mathscr{F}$ is Bogomolov-unstable with respect to H_η. q.e.d.

Hence, by Theorem 2.3, there exists a unique saturated invertible subsheaf $\mathscr{L} \subset \mathscr{F}$ having the following properties:

(i) $\mathscr{L}$ is the maximal destablizing subsheaf of $\mathscr{F}$ with respect to any pair (H_η, R), with R an arbitrary ample divisor on X_C. In particular, for any such pair we have $(2c_1(\mathscr{L}) - c_1(\mathscr{F})) \cdot H_\eta \cdot R > 0$. Incidentally, this implies that $\mathscr{L}$ is the same for all the values of $0 < \eta < \varepsilon(C)$ which make the hypothesis of the claim true.

(ii) $(2c_1(\mathscr{L}) - c_1(\mathscr{F}))^2 \cdot H_\eta \geq \Delta(\mathscr{F}) \cdot H_\eta$.

Given the inclusions $\mathscr{L} \subset \mathscr{F} \subset \mathscr{O}^2_{X_C}$, we have

$$\mathscr{L} = \mathscr{O}_{X_C}(-D) \tag{11}$$

for some effective divisor D on X_C. We can write

$$D = xH + yE,$$

with x and y integers and $x \geq 0$. Set

$$s =: D \cdot H_\eta \cdot H = x + y\eta d.$$

Since $\mathscr{F}$ has no sections, $D \neq 0$. The same applies for the restriction to any ample surface. Hence $s \geq 0$ for $0 < \eta < \varepsilon(C)$.

Lemma 4.1. *Assume that $s \geq \alpha$. Then $k \geq \alpha(d - \alpha/\eta)$.*

Proof. Given (11), from (ii) and (10) we get

$$(E - 2D)^2 \cdot H_\eta \geq \delta_\eta(C) - 4\eta k. \tag{12}$$

Since $E^2 \cdot H_\eta = \delta_\eta(C)$, this can be rewritten

$$D^2 \cdot H_\eta - D \cdot H_\eta \cdot E \geq -\eta k.$$

By Corollary 3.6, we then have

$$s^2 - s\eta d \geq -\eta k. \tag{13}$$

On the other hand, we have the destabilizing condition (i)

$$(E - 2D) \cdot H_\eta \cdot H \geq 0. \tag{14}$$

Now

$$E \cdot H_\eta \cdot H = \eta d,$$

and therefore (14) can be written

$$\eta d \geq 2s. \tag{15}$$

Therefore we can apply Lemma 2.5 with $a = \eta d$ and $b = \eta k$ to obtain $\eta k \geq \eta d\alpha - \alpha^2$. This proves the lemma. q.e.d.

The proof of the theorem is then reduced to the following lemma.

Lemma 4.2. $s \geq \alpha$.

Proof. We shall argue that $s \geq \alpha_\eta$ for all rational $\eta < \varepsilon(C)$ such that the inequalities (7) and (8) hold. For all such η we are then in the situation of Claim 4.1.

Claim 4.2 $\mathscr{L}$ is saturated in $V \otimes \mathscr{O}_X$.

Proof. By construction, $\mathscr{L} = \mathscr{O}_X(-D)$ is saturated in $\mathscr{F}$. Therefore, if the claim is false, then the inclusion $\mathscr{L} \subset V \otimes \mathscr{O}_X$ drops rank along E. Hence there exists an inclusion $\mathscr{O}_X(E - D) \subset \mathscr{O}_X^2$. This implies that $D - E$ is effective, and in particular $(D - E) \cdot H_\eta^2 \geq 0$. Together with the instability condition $(E-2D)\cdot H_\eta^2 > 0$, this would yield $D \cdot H_\eta^2 < 0$, against the fact that D is effective. q.e.d.

By Claim 4.2, there is an exact sequence

$$0 \to \mathscr{O}_X(-D) \to V \otimes \mathscr{O}_X \to \mathscr{O}_X(D) \otimes \mathscr{I}_Y \to 0,$$

where Y is a closed subscheme of X of codimension two or empty. Computing $c_2(\mathscr{O}_X^2) = 0$ from this sequence, we obtain $D^2 = [Y]$, and therefore $D^2 \cdot H \geq 0$. On the other hand, $D^2 \cdot H = x^2 - y^2 d$, and so $x \geq |y|\sqrt{d}$. Now,

$$s = x + y\eta d \geq x - |y|\eta d \geq |y|\sqrt{d}(1 - \eta\sqrt{d}).$$

By construction, $H^0(X, \mathscr{F}) = 0$, and therefore $D \neq 0$. Hence, if $y = 0$, then $s = x \geq 1$. If $y \neq 0$, then the above inequality shows that $s \geq \sqrt{d}(1 - \eta\sqrt{d})$. q.e.d.

This completes the proof of the theorem.

Example 4.1. Let $C \subset \mathbf{P}^3$ be a smooth complete intersection curve of type (a, b), with $a \geq b + 3$, $b \geq 2$. Then $\operatorname{gon}(C) \geq a(b-1)$.

Remark 4.1. This shows that the result is generally optimal. However, the theorem is void for a complete intersection of type (a, a). But for complete intersections one knows more than just the Seshadri constant: not only $\varepsilon(C) = 1/a$, but in fact the linear series $|aH - E|$ is base point free, and the general element is smooth. An ad hoc argument proves that $\operatorname{gon}(C) \geq a(b-1)$ [23].

Example 4.2. Let C be a nondegenerate smooth complete curve in $\mathbf{P}^3$ that is linked to a line in a complete intersection of type (a, b). Then for $a \gg b \gg 0$ we obtain $\operatorname{gon}(C) \geq \deg(C) = (a + b - 2)$. This is clearly optimal, because a base point free linear series of that degree is obtained by considering the pencil of planes through the line. The same considerations as in Remark 4.1 apply.

Remark 4.2. An analysis of "small" linear series on special classes of space curves is carried out by Ciliberto and Lazarsfeld in [8]. It would be interesting to know whether the present method can be adapted to give a generalization of their results.

From the theorem, we immediately get

Corollary 4.1. *Let $X \subset \mathbf{P}^r$ be a smooth projective variety. Let d be the degree of X, and $\varepsilon(X)$ be its Seshadri constant. Suppose that A is a line bundle on X with a pencil of sections $V \subset H^0(X, A)$ whose base locus has codimension at least 2. Let F be any divisor in the linear series*

$|A|$. *Then*

$$\deg(F) \geq \min\left\{\frac{1}{4}\left(c_1(N)\cdot_X H^{n-1} - \frac{d}{\varepsilon(X)}\right), \alpha\left(d - \frac{\alpha}{\varepsilon(X)^{c-1}}\right)\right\},$$

where

$$c = \operatorname{codim}(X, \mathbf{P}^r)$$

and

$$\alpha = \min\left\{1, \sqrt{\varepsilon(X)^{c-2}d}(1 - \varepsilon(X)\sqrt{\varepsilon(X)^{c-2}d})\right\}.$$

Proof. Let $C \subset X$ be a curve of the form $X \cap \Lambda$, where $\Lambda \subset \mathbf{P}^3$ is a linear subspace of dimension $c+1$, with c the codimension of X. Then V restricts to a base point free pencil on C, and the result follows by applying the theorem.

5. Stability of restricted bundles

We deal with the following problem.

Problem 5.1. Let $\mathscr{E}$ be a rank-2 vector bundle on $\mathbf{P}^3$, and let $C \subset \mathbf{P}^3$ be a smooth curve. If $\mathscr{E}$ is stable, what conditions on C ensure that $\mathscr{E}|_C$ is also stable?

Remark 5.1. This question has been considered by Bogomolov [4], [6] in the case of vector bundle on surfaces. In particular, he shows that if S is a smooth projective surface with $\operatorname{Pic}(S) \simeq \mathbf{Z}$, $\mathscr{E}$ is a stable rank-2 vector bundle on S with $c_1(\mathscr{E}) = 0$, and $C \subset S$ is a smooth curve with $C^2 > 4c_2(\mathscr{E})^2$, then $\mathscr{E}|_C$ is stable.

After a suitable twisting, we may also assume that $\mathscr{E}$ is *normalized*, i.e., $c_1(\mathscr{E}) = 0$ or -1. We shall suppose here that $c_1(\mathscr{E}) = 0$, the other case being similar.

As usual we adopt the following notation: $f: X_C \to \mathbf{P}^3$ is the blowup of $\mathbf{P}^3$ along C, $E = f^{-1}C$ is the exceptional divisor, and $\pi: E \to C$ is the induced projection. Recall that for $\eta \in \mathbf{Q}$ we set $H_\eta := H - \eta E$, where we write H for f^*H. If $0 < \eta < \varepsilon(C)$, H_η is a polarization on X_C.

Definition 5.1. We define the *stability constant of $\mathscr{E}$ with respect to C* as

$$\gamma(C, \mathscr{E}) = \sup\{\eta \in [0, \varepsilon(C)] | f^*\mathscr{E} \text{ is } (H_\eta, H)\text{-stable}\}.$$

Remark 5.2. Recall that $f^*\mathscr{E}$ is (H, H_η)-stable if for all line bundles $\mathscr{L} \subset f^*\mathscr{E}$ we have $\mathscr{L}\cdot H \cdot H_\eta < 0$.

Lemma 5.1. *Suppose* $0 \leq \eta < \varepsilon(C)$. *Then* $f^*\mathscr{E}$ *is* (H, H_η)*-semistable if and only if* $\eta \leq \gamma(C, \mathscr{E})$. *If* $\eta < \gamma(C, \mathscr{E})$, $f^*\mathscr{E}$ *is* (H, H_η)*-stable.*

Proof. The collection of the numerical classes of nef divisors D with respect to which $f^*\mathscr{E}$ is (H, D)-semistable (or stable) is convex, hence it contains the segment $[H, H_{\gamma(C, \mathscr{E})}]$. Since $f^*\mathscr{E}$ is (H, H)-stable, the second statement follows.

Lemma 5.2. *Suppose that* $V \subset \mathbf{P}^3$ *is a smooth surface of degree* a *containing* C, *and that* $\mathscr{E}|_V$ *is* $\mathscr{O}_V(H)$*-stable. Then*

$$\gamma(C, \mathscr{E}) \geq \min\{1/a, \varepsilon(C)\}.$$

Proof. Let $\widetilde{V}$ be the proper transform of V in X_C. Then $\widetilde{V} \simeq V$ and $\widetilde{V} \in |aH_{1/a}|$. The hypothesis implies that for every line bundle $\mathscr{L} \subset f^*\mathscr{E}$ we have $\mathscr{L} \cdot H_{1/a} \cdot H < 0$. Hence the same holds for every η with $0 \leq \eta \leq 1/a$.

Remark 5.3. Note that the same argument actually proves the following stronger statement: Let $V \supset C$ be a reduced irreducible surface through C having degree m and multiplicity n along C, and such that $f^*\mathscr{E}|_{\widetilde{V}}$ is $\mathscr{O}_{\widetilde{V}}(H)$-stable. Then $\gamma(C, \mathscr{E}) \geq \min\{n/m, \varepsilon(C)\}$.

Lemma 5.3. *Fix* $c_2 \geq 0$ *an integer. Then there exists an integer* k *with the following property. If* $\mathscr{E}$ *is a stable rank-*2 *vector bundle on* $\mathbf{P}^3$ *with* $c_1(\mathscr{E}) = 0$ *and* $c_2(\mathscr{E}) = c_2$, *and if* $V \subset \mathbf{P}^3$ *is a smooth surface of degree* $a \geq k$, *then* $\mathscr{E}|_V$ *is* $\mathscr{O}_V(H)$*-stable.*

Proof. We start by finding s such that for a general surface S of degree s we have $\mathrm{Pic}(S) \simeq \mathbf{ZH}$ ($s \geq 4$ will do) and furthermore the restriction $\mathscr{E}|_S$ is $\mathscr{O}_S(H)$-stable. Bogomolov's theorem (Remark 5.1) then implies that for any curve $C \subset S$ such that $C^2 > 4c_2(\mathscr{E})^2 s^2$ the restriction $\mathscr{E}|_C$ is also stable. Let now $a > 0$ be such that $a^2 > 4c_2(\mathscr{E})^2 s$. Suppose that V is a smooth surface of degree a and that $\mathscr{E}|_V$ is not stable. Then the same is true for $C = V \cap S$. For a general choice of S, C is a smooth curve, and since $C \cdot_S C = a^2 s > 4c_2(\mathscr{E})^2 s^2$, we have a contradiction. q.e.d.

We can in fact restate the previous lemma as follows:

Let s *be the smallest positive integer such that for a general surface of degree* s *we have* $\mathrm{Pic}(S) \simeq \mathbf{Z}$ *and* $\mathscr{E}|_S$ *stable. If* $a > 2c_2(\mathscr{E})\sqrt{s}$, *and* $V \subset \mathbf{P}^3$ *is any smooth surface of degree* a, *then* $\mathscr{E}|_V$ *is* $\mathscr{O}_V(H)$*-stable.*

Corollary 5.1. *Let* $\mathscr{E}$ *be a rank-*2 *stable bundle on* $\mathbf{P}^3$ *with* $c_1(\mathscr{E}) = 0$ *but* $c_2(\mathscr{E}) \neq 1$ (*i.e.,* $\mathscr{E}$ *is not a null correlation bundle* [30]). *If* $a > 2c_2(\mathscr{E})$, *and* $V \subset \mathbf{P}^3$ *is a smooth hypersurface of degree* a, *then* $\mathscr{E}|_V$ *is* $\mathscr{O}_V(H)$*-stable.*

Proof. In fact, a theorem of Barth implies that in this case we can take $s = 1$ [2].

Remark 5.4. In light of Barth's restriction theorem, by induction these statements generalize to $\mathbf{P}^r$ for any $r \geq 2$ (for $r = 2$ this is just Bogomolov's theorem, and the hypothesis $c_2 \neq 1$ is not needed).

Remark 5.5. In the proof of Corollary 5.1, we use stability on the hyperplane section to deduce stability on the whole surface. What makes this work is Bogomolov's theorem (cf. Remark 5.1), which gives us a control of the behaviour of stability under restriction to plane curves. On the other hand, if we are given an arbitrary smooth surface V, it may well be that $\mathscr{E}|_V$ is H-stable while $\mathscr{E}|_C$ is not, where C is an hyperplane section of V. In that case, however, $\mathscr{E}|_{V\cap W}$ will be stable, if W is a smooth surface of very large degree such that $V \cap W$ is a smooth curve. To improve the above result, therefore, one would need to control the behavior of stability under restriction to curves not necessarily lying in a plane. After proving the restricted Theorem 5.1 we shall strengthen the above corollary (cf. Corollary 5.4).

Definition 5.2. If X is a smooth variety, and $c_i \in A^i(X)$ for $i = 1$ and 2, let $\mathscr{M}_X(c_1, c_2)$ denote the moduli space of stable rank-2 vector bundles with Chern classes c_1 and c_2.

Corollary 5.2. *Fix integers $r \geq 3$ and $c_2 \geq 0$. Then for any sufficiently large positive integer a the following holds: if $V \subset \mathbf{P}^r$ is a smooth hypersurface of degree a, then $\mathscr{M}_{\mathbf{P}^r}(0, c_2)$ embeds as an open subset of $\mathscr{M}_V(0, c_2 a)$.*

Proof. $\mathscr{M}_{\mathbf{P}^r}(0, c_2)$ forms a bounded family of vector bundles, and therefore so does the collection of the vector bundles $\operatorname{End}(\mathscr{E}, \mathscr{F})$, with $\mathscr{E}, \mathscr{F} \in \mathscr{M}_{\mathbf{P}^r}(0, c_2)$. Therefore, if $k \gg 0$, we have $H^i(\mathbf{P}^r, \operatorname{End}(\mathscr{E}, \mathscr{F}) \times (-a)) = 0$ for all $i \leq 2$, $a \geq k$ and for all $\mathscr{E}, \mathscr{F} \in \mathscr{M}_{\mathbf{P}^r}(0, c_2)$. Furthermore, by the above lemma we can assume that $E|_V$ is $\mathscr{O}_V(H)$-stable for all $\mathscr{E} \in \mathscr{M}_{\mathbf{P}^r}(0, c_2)$. From the long exact sequence in cohomology associated to the exact sequence of sheaves

$$0 \to \operatorname{End}(\mathscr{E}, \mathscr{F})(-a) \to \operatorname{End}(\mathscr{E}, \mathscr{F}) \to \operatorname{End}(\mathscr{E}|_V, \mathscr{F}|_V) \to 0,$$

we then obtain isomorphisms

$$H^0(\mathbf{P}^r, \operatorname{End}(\mathscr{E}, \mathscr{F})) \simeq H^0(V, \operatorname{End}(\mathscr{E}|_V, \mathscr{F}|_V))$$

and

$$H^1(\mathbf{P}^r, \operatorname{End}(\mathscr{E}, \mathscr{F})) \simeq H^1(V, \operatorname{End}(\mathscr{E}|_V, \mathscr{F}|_V)).$$

Since there cannot be any nontrivial homomorphism between two nonisomorphic stable bundles of the same slope, the first one implies that

$\mathscr{E} \to \mathscr{E}|_V$ is a one-to-one morphism of $\mathscr{M}_{\mathbf{P}^r}(0, c_2)$ into $\mathscr{M}_V(0, c_2 a)$, and the second that the derivative of this morphism is an isomorphism [24].

Corollary 5.3. $\gamma(C, \mathscr{E}) > 0$.

Proof. By Lemma 5.3, for $r \gg 0$ the restriction of $\mathscr{E}$ to any smooth surface of degree r is stable with respect to the hyperplane bundle. So we just need to consider a smooth surface through C of very large degree and apply Lemma 5.2.

Example 5.1. Let $C = V_a \cap V_b \subset \mathbf{P}^3$ be a smooth complete intersection of type (a, b), with $a \geq b$. Suppose that V_a is smooth, and that $\mathscr{E}|_{V_a}$ is $\mathscr{O}_{V_a}(H)$-stable. Then

$$\gamma(C, \mathscr{E}) = 1/a = \varepsilon(C).$$

In general, $0 < \eta < \gamma(C, \mathscr{E})$ if and only if for m and n sufficiently large integers such that $\eta = n/m$, and $S \in |mH - nE|$ a smooth surface, we have that $f^*\mathscr{E}|_S$ is $\mathscr{O}_S(H)$-stable. In other words, we have a degree-m hypersurface with an ordinary singularity of multiplicity n along C, such that the pullback of $\mathscr{E}$ to the desingularization of S is H-stable.

Our result is then the following.

Theorem 5.1. *Let $\mathscr{E}$ be a stable rank-2 vector bundle on $\mathbf{P}^3$ with $c_1(\mathscr{E}) = 0$. Let $C \subset \mathbf{P}^3$ be a smooth curve of degree d and Seshadri constant $\varepsilon(C)$, and let $\gamma = \gamma(C, \mathscr{E})$ be the stability constant of $\mathscr{E}$ with respect to C. Let $\alpha = \min\{1, \sqrt{d}(\sqrt{3/4} - \gamma\sqrt{d})\}$. Suppose that $\mathscr{E}|_C$ is not stable. Then*

$$c_2(\mathscr{E}) \geq \min\{\delta_\gamma/4, \alpha\gamma(d - \alpha/\gamma)\}.$$

Proof. Suppose to the contrary that $c_2(\mathscr{E})$ is strictly smaller than both quantities on the right-hand side. We can find a rational number η with $0 < \eta < \gamma$ such that

$$c_2(\mathscr{E}) < \delta_\eta(C)/4 \tag{16}$$

and

$$c_2(\mathscr{E}) < \alpha\eta(d - \alpha/\eta). \tag{17}$$

By assumption there exists a line bundle L on C of degree $l \geq 0$ sitting in an exact sequence $0 \to L \to \mathscr{E}|_C \to L^{-1} \to 0$. Define a sheaf $\mathscr{F}$ on X_C by the exactness of the sequence

$$0 \to \mathscr{F} \to f^*\mathscr{E} \to \pi^* L^{-1} \to 0. \tag{18}$$

Then $\mathscr{F}$ is a rank-2 vector bundle on X_C having Chern classes $c_1(\mathscr{F}) = -[E]$ and $c_2(\mathscr{E}) = f^*c_2(\mathscr{E}) - \pi^*[L]$ (cf. Lemma 2.1). A straightforward computation thus gives

$$\Delta(\mathcal{F}) \cdot H_\eta = \delta_\eta(C) - 4c_2(\mathcal{E}) + 4\eta l \geq \delta_\eta(C) - 4c_2(\mathcal{E}), \tag{19}$$

which is positive by (16). Therefore $\mathcal{F}$ is Bogomolov-unstable with respect to H_η (Theorem 2.3). Let $\mathcal{L} \subset \mathcal{F}$ be the maximal destabilizing line bundle with respect to H_η. We shall write $\mathcal{L} = \mathcal{O}_{X_C}(-D)$, with $D = xH - yE$.

Claim 5.1. $x > 0$.

Proof. Pushing forward the inclusion $\mathcal{L} \subset \mathcal{F}$ we obtain an inclusion $\mathcal{O}_{\mathbf{P}^3}(-x) \subset \mathcal{E}$. Therefore the statement follows from the assumption of stability on $\mathcal{E}$ and the hypothesis $c_1(\mathcal{E}) = 0$. q.e.d.

The destabilizing condition implies $(2c_1(\mathcal{L}) - c_1(\mathcal{F})) \cdot H_\eta \cdot R \geq 0$ for all nef divisors on X_C, with strict inequality holding when R is ample. In particular, with $R = H$ we have

$$(E - 2D) \cdot H_\eta \cdot H \geq 0. \tag{20}$$

Let us set $s = D \cdot H_\eta \cdot H$. Then (20) reads

$$\eta d \geq 2s. \tag{21}$$

On the other hand, since $\mathcal{L}$ is saturated in $\mathcal{F}$, we also have $(E-2D)^2 \cdot H_\eta \geq \Delta(\mathcal{F}) \cdot H_\eta$, and with some algebra this becomes

$$c_2(\mathcal{E}) \geq D \cdot E \cdot H_\eta - D^2 \cdot H_\eta + \eta l \geq D \cdot E \cdot H_\eta - D^2 \cdot H_\eta. \tag{22}$$

Invoking Corollary 3.6 then gives

$$c_2(\mathcal{E}) \geq s\eta d - s^2. \tag{23}$$

Claim 5.2. $\mathcal{L}$ saturated in $f^*\mathcal{E}$.

Proof. Suppose not. Then there would be an inclusion

$$\mathcal{L}(E) = \mathcal{O}_{X_C}(E - D) \subset f^*\mathcal{E},$$

and therefore the (H, H_η)-stability of $f^*\mathcal{E}$ would force

$$(E - D) \cdot H_\eta \cdot H < 0.$$

On the other hand by instability we have $E \cdot H_\eta \cdot H \geq 2D \cdot H_\eta \cdot H$ and from this it follows that

$$E \cdot H_\eta \cdot H = \eta d < 0,$$

absurd. q.e.d.

Therefore there is an exact sequence

$$0 \to \mathcal{O}_{X_C}(-D) \to f^*\mathcal{E} \to \mathcal{O}_{X_C}(D) \otimes \mathcal{I}_W \to 0,$$

where W is a local complete intersection subscheme of X_C of codimension 2 or empty. Computing $c_2(f^*\mathcal{E})$ from the above sequence we then get $f^*c_2(\mathcal{E}) = W - D^2$, i.e., $D^2 \cdot H \geq -c_2(\mathcal{E})$. This can be rewritten $x^2 \geq y^2 d - c_2(\mathcal{E})$. Recall that we have (Remark 3.8) $d \geq \delta_\eta(C)$, and therefore the assumption $c_2(\mathcal{E}) < \delta_\eta(C)/4$ implies

$$c_2(\mathcal{E}) < d/4. \tag{24}$$

Lemma 5.4.

$$s \geq \min\{1, \sqrt{d}(\sqrt{3/4} - \eta\sqrt{d})\}.$$

Proof. If $y \leq 0$, then $s = x + |y|\eta d \geq 1$. If $y > 0$, we have $s = x - y\eta d \geq y\sqrt{d}(\sqrt{1 - c_2(\mathcal{E})/d} - \eta\sqrt{d})$ and therefore using (24) we obtain

$$s \geq \sqrt{d}(\sqrt{3/4} - \eta\sqrt{d}).$$

Hence we can apply Lemma 2.5 with $a = \eta d$ and $b = c_2(\mathcal{E})$ to deduce $c_2(\mathcal{E}) \geq \alpha\eta d - \alpha^2$, which contradicts (17). This completes the proof of the Theorem.

Corollary 5.4. *Let $\mathcal{E}$ be a stable rank-2 vector bundle $\mathbf{P}^3$ with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = c_2$. If $b \geq c_2 + 2$, and $V \subset \mathbf{P}^3$ is a smooth hypersurface of degree b, then $\mathcal{E}|_V$ is $\mathcal{O}_V(H)$-stable.*

Proof. Let $a \gg b$; then we may assume that if $W \subset \mathbf{P}^3$ is a surface of degree a, then $\mathcal{E}|_W$ is H-stable. If W is chosen generally, we may also assume that $C = W \cap V$ is a smooth curve. Thus use of Lemma 5.2 gives $\gamma(C, \mathcal{E}) = \varepsilon(C) = 1/a$. For a large enough, furthermore, we also have $\alpha = 1$. Hence the theorem implies that if $\mathcal{E}|_C$ is not stable, then $c_2 \geq b - 1$. From the hypothesis it therefore follows that $\mathcal{E}|_C$ is stable, and this forces $\mathcal{E}|_V$ to be stable also.

Corollary 5.5. *Let $\mathcal{E}$ be as above, let $C = V_a \cap V_b$ be a smooth complete intersection curve of type (a, b), and suppose that V_a is smooth. Assume that $a \geq 4b/3 + 10/3$ and that $b \geq c_2 + 2$. Then $\mathcal{E}|_C$ is stable.*

Proof. By Corollary 5.4, $\mathcal{E}|_{V_b}$ is H-stable. Hence by Lemma 5.2 $\gamma(C, \mathcal{E}) = 1/a$. The first hypothesis implies that $\alpha = 1$. Thus if $\mathcal{E}|_C$ is not stable, the theorem yields $c_2 \geq b - 1$, a contradiction.

Corollary 5.6. *Fix a nonnegative integer c_2. Then we can find positive integers a and b such that if $C \subset \mathbf{P}^3$ is any smooth complete intersection curve of type (a, b), then $\mathcal{M}_{\mathbf{P}^3}(0, c_2)$ embeds as a subvariety of $\mathcal{M}_C(0)$.*

Proof. The argument is similar to the one in the proof of Corollary 5.2. Here one uses the Koszul resolution

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-a-b) \to \mathcal{O}_{\mathbf{P}^3}(-a) \oplus \mathcal{O}_{\mathbf{P}^3}(-b) \to \mathcal{J}_C \to 0$$

to show that $H^i(\mathbf{P}^3, \operatorname{End}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{J}_C) = 0$ for $i \leq 1$.

Remark 5.6. Using the above corollary, we obtain a compactification of $\mathcal{M}_{\mathbf{P}^3}(c_1, c_2)$, by simply taking its closure in the moduli space of semistable bundles on the curve. It would be interesting to know whether these compactifications are intrinsic, i.e., they are independent of the choice of the curve or, if not, how they depend on the geometry of the embedding $C \subset \mathbf{P}^3$.

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