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NEGATIVE BENDING OF OPEN MANIFOLDS

JOACHIM LOHKAMP

1. Introduction

In this paper we will have a new look at general existence theorems for metrics with negative Ricci curvature, which is motivated from several points of view. We will mention the most significant ones:

1) A general feeling expresses that a bending of metric yields (or preserves) negative curvature iff we bend outwards. Bending is used as an intuitive collective noun for deformations which, for instance, enlarge or shrink the metric along leaves of some foliation. (Think of the growth of spheres in hyperbolic space relative to that in Euclidean space.)

The "if" part will be supported by a simple construction of complete metrics of negative Ricci curvature $\mathrm{Ric} < 0$ on each open manifold, but we will disprove the "only" part: namely we also find bendings "inwards" for $\mathrm{Ric} < 0$, which yield existence results for closed manifolds.

2) The "classical" existence proof for metrics with negative scalar curvature $S < 0$ on closed manifolds (cf. [1], [9]) starts from some metric with negative integral scalar curvature, and the integral condition suffices to find conformal deformations to get a metric with $S < 0$.

In a coarse analogue we first construct metrics of some "huge amount" of negative Ricci curvature in one small ball, and indeed a "far-reaching" conformal diffusion yields $\mathrm{Ric} < 0$ on the whole manifold.

3) In [10] we already gave a series of existence theorems for $\mathrm{Ric} < 0$ starting with "weak" local deformations of Euclidean balls, and therefore had to cover the manifold with "compatible" balls to get $\mathrm{Ric} < 0$-metrics.

The deformation described here is "strong" in the contrasting sense just mentioned (in 2)). In particular the major technical problem of making these coverings work does not appear, and we get a short and simple argument for general existence results. But let us point out that those "weak" deformations and the subsequent covering in [10] are just the key to the borderline results; for instance, the space $\mathrm{Ric}^{<\alpha}(M)$ of metrics with Ricci

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curvature $< \alpha$, $\alpha \in \mathbb{R}$, is dense in the set of all metrics with respect to the Hausdorff- and C^0-topology (cf. [11]), whereas (cf. [12]) the C^1-closure of $\mathrm{Ric}^{\leq\alpha}(M)$ is precisely $\mathrm{Ric}^{\leq\alpha}(M)$.

Thus while [10] and [11] were written before the present paper, we will now refer to it as the first, [10] the second and [11] the third step in understanding negative Ricci curvature.

The reader may wish to have a glance at [12] to get an impression of this sequence of refinements.

Now we formally state the main theorems, which are obtained by the "productive" bendings in the open manifold case and by the "preserving" ones in the closed manifold case.

Theorem 1. *Let M^n, $n \geq 2$, be an open manifold, and g_0 an arbitrary metric on M. Then we can find a smooth function f with:*

$$g = e^{2f} \cdot g_0 \text{ is a complete metric with } Ric(g) < 0.$$

Notice that this cannot be refined to give pinched Ricci (or just scalar) curvature in each conformal class according to the nonexistence results of Ni [13] (cf. also [2]).

While Theorem 1 is obtained by bending "outwards", we additionally use a bending "inwards" to get:

Theorem 2. *Each closed manifold M^n, $n \geq 3$, admits a metric with $Ric < 0$.*

This can be localized:

Theorem 3. *On $\mathbb{R}^n$, $n \geq 3$, there exists a metric g_n with $Ric(g_n) < 0$ on $B_1(0)$ and $g_n \equiv g_{\mathrm{Eucl.}}$ outside.*

Finally we give an outline of the paper: In §2 we construct conformal deformations of any prescribed metric on some open manifold leading to $\mathrm{Ric} < 0$. In principle the conformal factor can be calculated explicitly. The next two sections are devoted to performing a refined construction on the (open) complement of certain lower dimensional submanifolds of closed manifolds to obtain an additional suitable structure near the boundary.

This is used in §5 for dimension $n \geq 4$: Here we close these manifolds again and get Theorems 2 and 3. An extra argument is needed to obtain Theorem 2 for dimension 3 (§6).

2. Conformal bending

A striking differential topological (!) result of Gromov (cf. [6]) implies that each open manifold admits a metric with negative (as well as one with positive) sectional curvature. But these metrics are *not* complete.

Indeed there are known obstructions to getting complete negative sectional (resp. positive scalar) curvature metrics in dimension $n \geq 3$ (resp. $n \geq 5$) (cf. [7]).

Therefore there is no hope of finding global "outward bendings" for $\mathrm{Sec} < 0$ as are now presented for $\mathrm{Ric} < 0$ by conformal changes on some arbitrary *open* manifold M^n of dimension $n \geq 2 \mathrm{Ric}(g)$ (resp. $r(g)(\nu)$) will always denote the Ricci tensor (resp. the Ricci curvature in direction $\nu \neq 0$) of our metric g.

Proposition 2.1. *Let g_0 be any metric on M^n. Then there exists some $f \in C^\infty(M, \mathbb{R})$ such that $g = e^{2f} \cdot g_0$ is complete and $\mathrm{Ric}(g) < 0$.*

Proof. Let $M(\overset{\circ}{M}_{n+1} \supset \overline{M}_n, M_0 = \emptyset)$ be an exhaustion of M by compact manifolds with smooth boundaries, and choose for an increasing sequence c_n of real numbers a function $F \in C^\infty(M, \mathbb{R})$ with $F \equiv c_n$ near ∂M_n and $c_n \leq F \leq c_{n+1}$ on $M_{n+1} \backslash M_n$. If the c_n are chosen suitably, then $g = e^{2F} \cdot g_0$ is complete, hence we can assume g_0 to be complete.

Now using paracompactness of M we find a locally finite covering of balls B_i, $i = 1, 2, \ldots$, together with diffeomorphisms $f_i \colon B_6(0) \to B_i$, $B_6(0) \subset \mathbb{R}^n$, with $\bigcup_i f_i(B_4(0)) = M$, and $f_i(B_2(0)) \subset f_{i+1}(B_4(0) \backslash B_3(0))$. Hence $\bigcup_i f_i(B_4(0) \backslash B_2(0)) = M$, and we define for $d_i, s_i > 0$, $i \geq 1$:

$$g(0) := g_0, \quad g(n) := \prod_{i \leq n} \exp(2 \cdot F_i) \cdot g_0, \quad g(\infty) := \prod_i \exp(2 \cdot F_i) \cdot g_0$$

with $F_i = s_i \cdot \exp(-d/5 - \|f_i^{-1}(z)\| \cdot h(\|f_i^{-1}(z)\|)$ for z with $\|f_i^{-1}(z)\| < 5$ and $F_i \equiv 0$ otherwise. $\| \cdot \|$ denotes the Euclidean norm on $B_6(0)$, $h \in C^\infty(\mathbb{R}, [0, 1])$ with $h \equiv 0$ on $\mathbb{R}^{\leq 1}$, $h \equiv 1$ on $\mathbb{R}^{\geq 2}$.

Using Lemma 2.2 below we can find a $d_n > 0$ for each $n \geq 1$ such that (for fixed $d_i, s_i > 0$, $i < n$, if $n > 1$) and *each*(!) $s_n > 0$:

$$\exp(2 \cdot F_n) \cdot r(g(n))(\nu) - r(g(n-1))(\nu) < \begin{cases} 0 \text{ on } f_n(B_5(0) \backslash B_4(0)), \\ -s_n \cdot e^{-d_n} \text{ on } f_n(B_4(0) \backslash B_2(0)), \end{cases}$$

where $g(n-1)(\nu, \nu) = 1$. Thus we get by induction $d_i, s_i > 0$ for each $i \geq 1$ such that $r(g(n)) < 0$ on $(\bigcup_{i \leq n} f_i(B_4(0) \backslash B_2(0))) \backslash f_n(B_2(0))$. This yields $r(g(\infty)) < 0$, since $B_m \cap K = \emptyset$ for each compact $K \subset M$ and $m \geq n$, $n = n(K)$ large enough. Furthermore $g(\infty)$ is conformal to g_0 with a pointwise conformal factor ≥ 1. Hence it is complete.

Lemma 2.2. *Let g_0 be any metric on $N \times \mathbb{R}$ for some closed manifold N. Then there exists a $d_0 > 0$ such that for all $d \geq d_0$, $s > 0$:*

$$\exp(2se^{-d/t}) \cdot r(\exp(2se^{-d/t}g_0)(\nu) - r(g_0)(\nu) < \begin{cases} 0 \text{ on } N \times]0, 1], \\ -se^{-d} \text{ on } N \times [1, 10] \end{cases}$$

for $\nu \in T(N \times \mathbb{R})$, $\|\nu\|_{g_0} = 1$ and $(x, t) \in N \times \mathbb{R}$.

Lemma 2.2 already appeared in [10, (2.5)] to make those local deformations mentioned in §1 compatible. To get an impression of how the (elementary) proof works, and in particular where d_0 comes in, we will include a brief outline:

Recall from [3, (1.J.)] the transformation law for $g_f = e^{2f} \cdot g$, $f \in C^\infty(M^m, \mathbb{R})$:

$$\text{(1)} \quad \begin{aligned} e^{2f} \cdot r(g_f)(\nu) - r(g)(\nu) &= (m-2)(|df|^2(\nu) - \|\nabla^g f\|_g^2) \\ &\quad - ((m-2)\operatorname{Hess}_g f(\nu, \nu) + \Delta_g f) \end{aligned}$$

for $\|\nu\|_g = 1$, and the index g means that $\|\cdot\|$ etc. is with respect to g. Note that the right-hand side is $\leq -(m-2) \cdot \operatorname{Hess} f(\nu, \nu) - \Delta f$.

Now if we "bend" along a foliation $M \equiv N \times \mathbb{R}$, i.e., if we take $f(x, t) \equiv F(t)$ for $(x, t) \in N \times \mathbb{R}$ and some smooth function F with $F', F'' \geq 0$, then we get an upper estimate for the last expression:

$$\text{(2)} \quad -c(g) \cdot F'(t) - k(g) \cdot F''(t)$$

for some constants c, k with $k > 0$, which are easily seen to depend only on the geometry (i.e., g) but *not* on F (resp. f).

Assuming this we take $F = s \cdot \exp(-d/t)$ and calculate (2):

$$-s \cdot (c(g) \cdot d/t^2 + k(g) \cdot (-2d/t^3 + d^2/t^4)) \cdot \exp(-d/t) =: -s \cdot \Phi_d(t) \cdot \exp(-d/t).$$

Using $k(g) > 0$ we easily get a d_0 such that $\Phi_d > 0$ on $]0, 10]$ and $\Phi_d > +1$ on $[1, 10]$ for any $d \geq d_0$.

3. Opening of closed manifolds

To prove the existence of metrics with Ric < 0 on *closed* manifolds of dimension $n \geq 4$ we first notice

Lemma 3.1. $\mathbb{R}^n$, $n \geq 4$, *contains a closed manifold N^{n-2} with trivial normal bundle and admitting a metric with Ric < 0.*

Proof. $n = 4$: Each closed orientable surface F admits an embedding into $\mathbb{R}^3$ and hence into $\mathbb{R}^4$. In this situation the normal bundle is trivial since $F \subset \mathbb{R}^3$ is a hypersurface, which always fulfills this condition. Thus taking a surface of genus 2, this admits a hyperbolic metric.

$n = 5$: Again each closed orientable three-manifold N^3 admits an embedding in $\mathbb{R}^5$ with trivial normal bundle (cf. [8], Corollary 4). Thus take some orientable hyperbolic three-manifold, or alternatively take the standard 3-sphere $S^3 \supset \mathbb{R}^5$ and use the existence of a $\mathrm{Ric} < 0$-metric on S^3 (cf. §6).

$n \geq 6$: We can use induction: In §5 we will prove that each N^{n-2} admits a metric with $\mathrm{Ric} < 0$, thus we take $S^{n-2} \subset \mathbb{R}^n$. q.e.d.

Now let us briefly indicate how to proceed in the proof of Theorem 2 (and 3): we will choose a ball $B \subset M^n$, $n \geq 4$ and use $N^{n-2} \subset B$ as in the previous lemma and consider $M \backslash N$. This is an open manifold and admits a metric with $\mathrm{Ric} < 0$ as in Proposition 2.1. Next (in §§3–5) we will use the conditions on N to bend $M \backslash N$ to get a Riemannian structure with $\mathrm{Ric} < 0$, which has M as a natural completion.

Thus let M^n, $n \geq 4$, be an arbitrary manifold, $B \subset M^n$ a ball, and $N^{n-2} \subset B$ as in Lemma 3.1, and denote by V, W open tubular neighborhoods of N with $\overline{V} \subset W \subset \overline{W} \subset B$. We will introduce a second bending (additionally to Proposition 2.1), this time for standardization of the boundary structure.

Proposition 3.2. *Let g_0 be a metric on $M \backslash N$ with $Ric(g_0) \leq 0$. Then there is a metric g on $M \backslash N$ with $g \equiv g_0$ on $M \backslash W$, $Ric(g) < 0$ on $W \backslash N$ and such that $(V \backslash N, g)$ is isometric to*

$$(]0, 1[\times S^1 \times N, g_{\mathbb{R}} + \sinh^2 m(r+\rho)/m^2 \cdot g_{S^1} + \kappa^2 \cdot g_N),$$

where g_N is a metric with $Ric(g_N) < 0$, $m \in \mathbb{Z}^{>0}$, $\rho > 0$, $\kappa > 0$.

Proof. Using the triviality of the normal bundle of N we can find a diffeomorphism Φ from $]-2, 12[\times S^1 \times N$ into $B \backslash N$ with $\Phi(]0, 12[\times S^1 \times N) = W \backslash N$ and $\Phi(]3, 12[\times S^1 \times N) \to V \backslash N$ and admitting a continuous extension with $\overline{\Phi}(\{12\} \times S^1 \times N) = N$.

Thus take the following metric on $\mathbb{R}^{>-2} \times S^1 \times N$:

$$g_1 = h \cdot \Phi^*(g_0) + (1-h) \cdot (g_{\mathbb{R}} + g_{S^1} + g_N)$$

for some $h \in C^\infty(\mathbb{R}_1[0, 1])$ with $h \equiv 0$ on $\mathbb{R}^{\geq 3}$, $h \equiv 1$ on $\mathbb{R}^{\leq 2}$; as in Lemma 2.2 we get a d_0 such that for $s > 0$, $\nu \in T(\mathbb{R} \times S^1 \times N)$, $\|\nu\|_{g_1} = 1$, $(t, z, x) \in \mathbb{R} \times S^1 \times N$,

$$\exp(2 \cdot s \cdot e^{-d_0/t}) \cdot r(\exp(2 \cdot s \cdot e^{-d_0/t}) \cdot g_1)(\nu) - r(g_1)(\nu) \\ < \begin{cases} 0 & \text{on }]0, 1] \times S^1 \times N, \\ -s \cdot e^{-d_0} & \text{on } [1, 10[\times S^1 \times N. \end{cases}$$

Hence for s large enough we get $r(\exp(2 \cdot s \cdot e^{-d_0/t}) \cdot g_1)(\nu) < 0$ on $]0, 10[\times S^1 \times N$, and $(]5, 10[\times S^1 \times N, \exp(2 \cdot s \cdot e^{-d_0/t}) \cdot g_1)$ is isometric to $(]\alpha, \beta[\times S^1 \times N, g_{\mathbb{R}} + F^2 \cdot (g_{S^1} + g_N))$ (via some isometry $\varphi = (\varphi_{\mathbb{R}}, id_{S^1 \times N})$ $\varphi_{\mathbb{R}}$ is uniquely determined), where for some $\alpha < \alpha + 5 < \beta$ and $F \in C^\infty(]\alpha, \beta[, \mathbb{R}^{>0})$ with $F', F'' > 0$ (obtained by rescaling $\mathbb{R}$): namely $F' > 0$ is independent of scaling $\mathbb{R}$ and $F'' > 0$ is clear from the warped product formula (cf. [3]):

$0 > \mathrm{Ric}(g_{\mathbb{R}} + F^2 \cdot (g_{S^1} + g_N))(\nu, \nu) = -(n-1)(F''/F) g_{\mathbb{R}}(\nu, \nu)$, for ν tangent (i.e., horizontal) to $\mathbb{R}$.

We can assume $\max r(g_N) = -1$. Then (using $F', F'' < 0$) there are $f, g \in C^\infty(]\alpha, \beta[, \mathbb{R}^{>0})$ with $\mathrm{Ric}(g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g^2 \cdot g_N) < 0$ and $f \equiv g \equiv F$ near α, $f(t) = (\sinh m(t-c))/m$, $g = \kappa > 0$ on $]\beta - 1, \beta[$ for some $m \in \mathbb{Z}^{>0}$, $\kappa > 0$, $c \in]\alpha, \beta - 1[$. (This will be proved in a moment; cf. Proposition 4.1, (i) below.) Note that this is just the desired boundary structure. Thus we are left to install it on $\beta \backslash N$: define a diffeomorphism $\phi :]-2, 12[\to]-2, 10[$, with $\phi = id$ on $]-2, 1[$, $\phi(]2, 12[) =]5, 10[$, and $\varphi_{\mathbb{R}} \circ \phi$ is linear on $]3, 12[$ with $\varphi_{\mathbb{R}} \circ \phi(]3, 12[) =]\beta - 1, \beta[$. Then we are ready to define

$$g_2 := \begin{cases} (\varphi \circ (\phi, id_{S^1 \times N}))^*(g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g^2 \cdot g_N) & \text{on } [2, 12[\times S^1 \times N, \\ (\phi, id_{S^1 \times N})^*(\exp(2 \cdot s \cdot e^{-d_0/t}) \cdot g_1) & \text{on }]-2, 2[\times S^1 \times N. \end{cases}$$

In particular $g_2 \equiv g_1$ on $]-2, 1] \times S^1 \times N$ and $\mathrm{Ric}(g_2) < 0$ on $]0, 12[\times S^1 \times N$. Thus define the push-forward metric $g := \Phi_*(g_2)$. It is easily checked that g fulfills the claim.

4. Smoothings and warpings

This section is devoted to describing deformations used in §3 as well as §5 to smooth singularities by means of certain warped product arguments. This also generalizes results in [5] and [4]; cf. the remarks on Proposition (6.1) below.

Thus we consider $(]a, b[\times S^1 \times N^{n-2}, g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g^2 \cdot g_N)$ for some $f, g \in C^\infty(]a, b[, \mathbb{R}^{>0})$. If $\max r(g_N) = -1$, then $r(g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g^2 \cdot g_N) < 0$ is easily seen to be equivalent to the following three inequalities:

$$(1) \qquad (n-3) \cdot \frac{1 + (g')^2}{g^2} + \frac{g''}{g} + \frac{f'}{f} \cdot \frac{g'}{g} > 0,$$

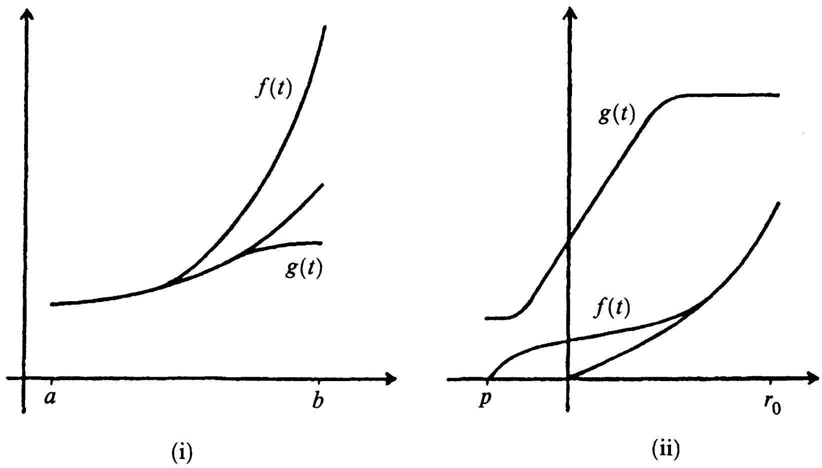


FIGURE A

$$(2) \qquad (n-2)\cdot\frac{g''}{g}+\frac{f''}{f}>0,$$

$$(3) \qquad (n-2)\cdot\frac{g'}{g}\cdot\frac{f'}{f}+\frac{f''}{f}>0.$$

Proposition 4.1.
(i) *Let* $F \equiv G \in C^\infty(]a, b[, \mathbb{R}^{>0})$ *with* $F', F'' > 0$. *Then there are* $f, g \in C^\infty(]a, b[, \mathbb{R}^{>0})$ *with* $Ric(g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g^2 \cdot g_N) < 0$ *such that for some* $m \in \mathbb{Z}^{>0}$, $\kappa > 0$,

$$f(t) = \begin{cases} F(t) \\ (\sinh m(t-c))/m \end{cases}, \quad g(t) = \begin{cases} G(t) & \textit{near } a \\ \kappa & \textit{near } b, \textit{ for some } c \in]a, b[. \end{cases}$$

(ii) *Let* $F \equiv \sinh/\alpha$, $\alpha > 1$, $G \equiv m > 0$ *be defined on* $]0, r_0[$ *for some* $r_0 > 0$. *Then there are* $f, g \in C^\infty(]p, r_0[, \mathbb{R}^{>0})$ *for some* $p < 0$ *and some* $\kappa > 0$ *with* $Ric(g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g^2 \cdot g_N) < 0$ *such that*

$$f(t) = \begin{cases} (\sinh t)/\alpha, \\ \sinh(t-p), \end{cases} \qquad g(t) = \begin{cases} m & \textit{near } r_0, \\ \kappa & \textit{near } p. \end{cases}$$

Proof. We construct f, g which fulfill the "boundary conditions" and (1)–(3) by glueing together functions defined on disjoint intervals.

We will use the following simple observation (see Figure A):

(*) If $f_1, f_2 \in C^\infty(]\alpha, \gamma[, \mathbb{R}^{>0})$ fulfill $f_1(\beta) = f_2(\beta)$ for some $\beta \in]\alpha, \gamma[$, $0 < f_1' < f_2'$, and $f_1'', f_2'' \geq 0$ (resp. > 0), then there is a function $h \in C^\infty(]\alpha, \gamma[, \mathbb{R}^{>0})$ with $h \equiv f_1$ near α, $h \equiv f_2$ near γ, and $h' > 0, h'' \geq 0$ (resp. $h'' > 0$).

(i) We choose $\kappa := G(b) + 1$. Then we can find a g with $g \equiv G$ on $]a, b - 3\varepsilon[$, $g \equiv \kappa$ on $[b - \epsilon, b[$, $g' > 0$ on $]a, b - \epsilon[$, $g'' > 0$ on $]a, b - 2\epsilon[$ for some small $\epsilon > 0$, $10 \cdot \epsilon < |b - a|$. For m large enough (say $\geq m_0$) we can find a $c_m \in]b - 4\epsilon, b - 3\epsilon[$ such that there is exactly one $t_m \in]c_m, b - 3\epsilon[$ such that $(\sinh m(t_m - c_m))/m = F(t_m)$ and near t_m: $0 < F^{(\kappa)} < ((\sinh m(t - c_m))/m)^{(\kappa)}$, $\kappa = 1, 2$.

Now we use (*) to get a function $f_m \in C^\infty(]a, b[, \mathbb{R}^{>0})$ with $f_m', f_m'' > 0$ and $f_m \equiv F$ near a, $f_m \equiv (\sinh m(t - c_m))/m$ on $]b - 3\epsilon, b[$. Thus for each $m \geq m_0$ (3) is fulfilled on $]a, b[$ for f_m and g, (2) is fulfilled on $]a, b - 3\epsilon]$. Furthermore $|g'/g|, |g''/g| < K$ and $f_m''/f_m = m^2$, $f_m'/f_m \geq m$ on $[b - 3\epsilon, b[$. Hence (2) is fulfilled on $]a, b[$ for $m^2 > (n - 2) \cdot K$.

Finally to get (1) we notice $|g''/g| < |(n-3)/g^2|$ on $]b - \epsilon - 2\delta, b[$ for some small $\delta > 0$ and $|g'/g| > K' > 0$ on $]b - 3\epsilon, b - \epsilon - \delta[$. Thus (1) is fulfilled on $]b - 3\epsilon, b[$ for each $m \geq m_0$ with $m \cdot K' > K$. Hence define $f = f_m$, $c = c_m$ for some $m \geq \max\{m_0, K, K', ((n-2) \cdot K)^{1/2}\} + 1$.

(ii) We start from $F \equiv \sinh/\alpha, \alpha < 1, G \equiv m > 0$ on $]0, r_0[$ which obviously fulfill (1)–(3). For $\gamma \in]0, \frac{m}{r_0}[$, define a function $G_\gamma \in C^\infty(]t_\gamma, r_0[, \mathbb{R}^{>0})$ with $t_\gamma = -m/\gamma + r_0/2 < 0$ and $G_\gamma = \gamma \cdot (t - r_0/2) + m$ on $]t_\gamma, r_0/4[$, $G_\gamma = m$ on $]r_0/2, r_0[$ and $G_\gamma' > 0$ on $]r_0/4, r_0/2[$. If $\gamma > 0$ is small enough, G_γ can be defined such that F and G_γ again fulfill (1)–(3) on $]0, r_0[$.

Fix such a $\gamma > 0$ and G_γ. Then define a function $f \in C^\infty(]t_\gamma - 3, r_0[, \mathbb{R}^{>0})$ with $f' > 0$ on $]t_\gamma - 3, r_0[$, $f'' > 0$ on $]t_\gamma - 1, r_0[$ and

$$f(t) = \begin{cases} (\sinh t)/\alpha & \text{on }]r_0/4, r_0[, \\ \sinh(t - (t_\gamma - 3)) & \text{on }]t_\gamma - 3, t_\gamma - 3 + 4\delta[\text{ for small } \delta \in]0, 1/10[. \end{cases}$$

Next we consider $g(\kappa, m) := \max\{G_\gamma, \sinh m(t - (t_\gamma - 3))/\kappa\}$.

For each $m \in \mathbb{Z}^{>0}$ we can find a $\kappa = \kappa(m) \in \mathbb{Z}^{>0}$ such that there is a unique $t_m \in]t_\gamma, 0[$ with $G_\gamma(t_m) = \sinh m(t_m - (t_\gamma - 3))/\kappa(m)$ and $G_\gamma' > (\sinh m(t - (t_\gamma - 3))/\kappa(m))'$ in t_m. Since $(\cdot)''$ of both functions is nonnegative we can find, using (*) in two points a function $g_m \in C^\infty(]t_\gamma - 3, r_0[, \mathbb{R}^{>0})$ with $g_m', g_m'' \geq 0$ and $g_m' > 0$ on $]t_\gamma - 3 + \delta, r_0[$ and

$$g_m(t) = \begin{cases} G_\gamma & \text{on }]0, r_0[\\ (\sinh m(t - (t_\gamma - 3)))/K(m) & \text{on }]t_\gamma - 3 + 3\delta, t_\gamma[\\ \kappa_m & \text{on }]t_\gamma - 3, t_\gamma - 3 + \delta[\\ & \text{for some suitable } \kappa_m > 0. \end{cases}$$

Now we will choose a large m such that f and g_m fulfill (1)–(3):

(1) is always fulfilled. (2) and (3) are fulfilled on $]t_\gamma - 3, t_\gamma - 3 + 3\delta[$ and on $]t_\gamma - 1, r_0[$. Since $g_m'/g_m \geq m$, $g_m''/g_m = m^2$ on $[t_\gamma - 3 + 2\delta, t_\gamma - 1/2]$ and $f' > 0$ on $]t_\gamma - 3, r_0[$ we find (3) fulfilled on $]t_\gamma - 3 + 2\delta, t_\gamma - 1/2[$ for large m. Finally $f''/f > -c$ for some $c > 0$, hence $(n-2) \cdot g_m''/g_m + f''/f > 0$ for large m, which yields (2).

Hence we choose these f and $g = g_m$, $\kappa = \kappa_m$ for some large m.

5. Closing of manifolds

We reformulate Propositions 2.1 and 3.2 in our context for closed M^n, $n \geq 4$, $N^{n-2} \subset B \subset M^n$, and a metric g_N on N with $\mathrm{Ric}(g_N) < 0$. We obtain metrics on $M \backslash N$ with Ric < 0 and some nice behavior near the boundary:

Corollary 5.1.

(i) $M^n \backslash N^{n-2}$ *admits a metric* g *with Ric* < 0, *and there is a tube* V *of* N *such that* $V \backslash N$ *is isometric to*

$$(]0, 1[\times S^1 \times N, g_{\mathbb{R}} + (\sinh^2 m(r + \rho))/m^2 \cdot g_{S^1} + c^2 \cdot g_N),$$

for some $c, \rho, m > 0$.

(ii) $\mathbb{R}^n \backslash N^{n-2}$, $n \geq 4$ *admits a metric* g *with Ric*$(g) < 0$ *on* $W \backslash N$, $(\overline{W} \subset B_1(0))$, $g = g_{\text{Eucl.}}$ *on* $\mathbb{R}^n \backslash W$, *and there is a tube* V *of* N *such that* $V \backslash N$ *is isometric to*

$$(]0, 1[\times S^1 \times N, g_{\mathbb{R}} + (\sinh^2 m(r + \rho))/m^2 \cdot g_{S^1} + c^2 \cdot g_N)$$

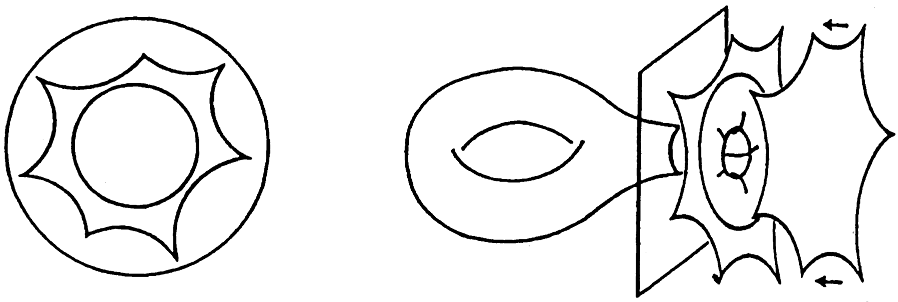
for some $c, \rho, m > 0$.

Now we will "bend inward" = "close" $M^n \backslash N$ (resp. $\mathbb{R}^n \backslash N$) to get a metric with Ric < 0 on M (resp. on $B_1(0) \subset \mathbb{R}^n$) using the following lemma.

Closing Lemma 5.2. *For each pair* $m, R > 0$ *there is a metric* $g(m, R)$ *on* $S^2 \times N$ *with Ric* < 0 *and a subset* $D_R \times N$ *canonically isometric to*

$$(B_R(0) \times N^{n-2}, g_{\text{hyp.}} + c^2 m^2 \cdot g_N) \subset (\mathbb{H}^2, g_{\text{hyp.}}) \times (N^{n-2}, c^2 m^2 \cdot g_N).$$

We shall prove this lemma in a moment, but we first derive the



$B_{2R}(0) \subset P \subset \mathbb{H}^2$ $\quad M^n\backslash V$ $\quad$ glue P and P' $\times N$

FIGURE B

Proof of Theorems 2 *and* 3. Scale the metric of g of Corollary 5.1(i) and (ii) by m^2. Now the tubular neighborhood $V\backslash N$ is isometric to

$$\begin{aligned}(&]m\rho, m+m\rho[\times S^1 \times N, g_{\mathbb{R}} + \sinh^2 r \cdot g_{S^1} + c^2 m^2 g_N) \\ &= (B_{m+m\cdot\rho}(0)\backslash B_{m\cdot\rho}(0) \times N, g_{\text{hyp.}} + c^2 m^2 g_N).\end{aligned}$$

Thus in Closing Lemma 5.2 choose $R = m+m\cdot\rho$ and glue $(S^2\times N\backslash D_R \times N, g(m, R))$ with $M^n\backslash N$ and $\mathbb{R}^n\backslash N$ along their (isometric) boundaries. This yields M^n equipped with a smooth metric with $\text{Ric} < 0$.

On $\mathbb{R}^n$ we obtain a metric $\overline{g}_n$ which fulfills $\text{Ric}(\overline{g}_n) < 0$ on W and $\overline{g}_n \equiv g_{\text{Eucl.}}$ on $\mathbb{R}^n\backslash W$. Now as in Proposition 2.1 we consider a diffeomorphism $f\colon B_6(0) \to B_2(0)$ with $f(B_5(0)) = B_1(0)$ and $f(B_3(0)) \subset W$, and define $g(s, d) = \exp(2\cdot s\cdot \exp(-d/5 - \|f^{-1}(z)\|))\cdot h(\|f^{-1}(z)\|)\cdot \overline{g}_n$ on $B_1(0)$, $(= \overline{g}_n$ otherwise) such that for suitable $d > 0$ and *small* $s > 0$ we have $r(g(s, d)) < 0$ on $B_1(0)$, $g(s, d) \equiv g_{\text{Eucl.}}$ outside. Then we take $g_n \equiv g(s, d)$.

Proof of Closing Lemma 5.2. Consider $B_{2R}(0) \in \mathbb{H}^2$, a compact, convex geodesic polygon P with $B_{2R}(0) \subset P$, and a second copy P'. Now we glue P and P' along their common boundary and obtain S^2 with a singular hyperbolic metric g^-; there are only finitely many singularities (corresponding to the vertices $p_1, \cdots, p_k$ of P). Near the singular points the metric can be written with respect to the polar coordinates:

$([0, r[\times S^1, dr^2 + \sinh^2 r d\Theta^2/\alpha^2)$ for some $r > 0$, $\alpha > 1$. Take such an $r_0 < R/2$ and start with $(S^2 \times N, g^- + c^2 m^2 \cdot g_N)$. We may assume that $B_{r_0}(p_i) \cap B_{r_0}(p_j) = \emptyset$ for vertices $p_i \neq p_j \in S^2$. Then we will smooth the metric on $B_{r_0}(p_i) \times N$ as follows (see Figure B):

As in Proposition 4.1(ii) we can find a $\rho<0$ and $f, g\in C^{\infty}(]\rho, r_0[, \mathbb{R}^{>0})$ with:

$\mathrm{Ric}\,(g_{\mathbb{R}}+f^2\cdot g_{S^1}+g^2\cdot g_N)<0$ on $]\rho, r_0[\times S^1\times N$ and:

$$f(r)\equiv\begin{cases}\sinh r/\alpha,\\ \sinh(r-\rho),\end{cases}\qquad g\equiv\begin{cases}cm>0 & \text{near } r_0,\\ \kappa>0 & \text{near } \rho.\end{cases}$$

This is a smooth metric identical to $g(m, R)$ near $\partial B_r(p_i)\times N$ and we substitute $([0, r_0[\times S^1\times N, dr^2+\sinh^2 r\, d\theta^2/\alpha^2)$ by $([\rho, r_0[\times S^1\times N, g_{\mathbb{R}}+f^2\cdot g_{S^1}+g^2\cdot g_N)$ and get a smooth metric with $\mathrm{Ric}<0$ on $S^2\times N$, which contains a set canonically isometric to $(B_R(0)\times N, g_{\mathrm{hyp.}}+c^2m^2\cdot g_N)$.

6. Closed three manifolds

The only closed codimension-2 submanifold of a three-manifold is S^1, hence we cannot argue as in the higher dimensional case where we were able to use that the corresponding submanifold admits a metric with $\mathrm{Ric}<0$. On the other hand closed three-manifolds are subject to Thurston's hyperbolic Dehn surgery, and this was already used to derive

Proposition 6.1. *Each closed three-manifold admits a metric with* $Ric<0$.

Namely Gao and Yau [5] and Brooks [4] pointed out that each of these manifolds admits a hyperbolic metric which is regular outside some closed curves, and they managed to smooth these particular singularities to get metrics with $\mathrm{Ric}<0$. Moreover the author gave an elementary proof in [10].

For these reasons we skip some minor technical arguments in the following short proof of Proposition 6.1, but the reader will easily fill in the details.

The proof starts with some outward bending as before. The problem occurs if we try to close the manifold:

The proof of Proposition 3.2 also includes the following result for a three-manifold M with $S^1\times B_r(0)\subset B\subset M$, B a ball, and $S^1\times B_r(0)$ a (trivial) solid torus:

Lemma 6.2. $M\backslash S^1\times B_r(0)$ *and* $S^1\times B_r(0)$ *admit metrics* g_1, g_2 *with*:

(i) $g_i=g_{\mathbb{R}}+\gamma^2\cdot r^2\cdot(g_{S^1}+g_{S^1})$ *on* $]1, 2[\times S^1\times S^1\equiv$ *neck of the boundary, for some possibly large* $\gamma>0$;

(ii) $Ric(g_i)<0$ *elsewhere.*

We would like to glue $M\backslash S^1 \times B_r(0)$ and $S^1 \times B_r(0)$ along their boundaries, but these metrics do not fit together smoothly.

Therefore we have to straighten them before glueing:

Proposition 6.3. *There is a metric* g *on* $]2, b[\times S^1 \times S^1$ *for some* $b > 3$ *and some* $c > 0$, *with* $Ric(g) \leq 0$ *such that*

$$g = \begin{cases} g_{\mathbb{R}} + \gamma^2 \cdot r^2 \cdot (g_{S^1} + g_{S^1}) & \textit{near } \{2\} \times S^1 \times S^1, \\ g_{\mathbb{R}} + c^2 \cdot (g_{S^1} + g_{S^1}) & \textit{near } \{b\} \times S^1 \times S^1. \end{cases}$$

Proof. We first construct a metric g_0 on $]-3, 3[\times S^1 \times S^1$ with:

$$g_0 \equiv g_{\mathbb{R}} + g_{S^1} + g_{S^1} \text{ on }]-3, -2] \cup [2, 3[\times S^1 \times S^1,$$

and $\mathrm{Ric}(g_0) < 0$ on $]-2, 2[\times S^1]-2, 2[\times S^1 \times S^1$.

Define a metric g_s on $]-3, 3[\times S^1$ which has exactly one hyperbolic orbifold singularity in $(1.5, e^0)$ and (Gaussian) curvature $K \leq 0$ elsewhere. Furthermore, making use of the negative curvature obtained by introducing a singularity we can assume

$$g_s = \begin{cases} g_{\mathbb{R}} + g_{S^1} & \text{on }]-3, -1] \cup [2, 3[\times S^1, \\ g_{\mathbb{R}} + (1 + s \cdot \exp(-d/t + 1))^2 \cdot g_{S^1} & \text{on }]-1, 1[\times S^1 \end{cases}$$

for suitable large d and small $s > 0$.

Moreover define the "mirror image" g_s' as the pullback of g_s by the canonical reflection along $\{0\} \times S^1$.

Now fit these metrics together to get a metric $g(s)$ on $]-3, 3[\times S^1 \times S^1$:

$$g(s) = \begin{cases} g_{S^1} + g_s' \text{ and } g_s + g_{S^1} \quad \text{on }]-3, -1] \cup [1, 3[\times S^1 \times S^1, \\ g_{\mathbb{R}} + (1 + s \cdot \exp(-d/t + 1))^2 \cdot g_{S^1} \\ \qquad + (1 + s \cdot \exp(-d/1 - t))^2 g_{S^1} \text{ on }]-1, 1[\times S^1 \times S^1. \end{cases}$$

A brief look at the calculations after Lemma 2.2 and the three inequalities (1)–(3) before Proposition 4.1 should convince that for suitable small $s > 0$ and large $d > 0$: $\mathrm{Ric}(g(s)) < 0$ on $]-1, 1[\times S^1 \times S^1$.

Using Lemma 2.2 we can deform $g(s)$ to some metric $\overline{g}(s)$, which has Ric < 0 on $]-2, 2[\times S^1 \times S^1$, is $g_{\mathbb{R}} + g_{S^1} + g_{S^1}$ outside and has two singular curves (coming from the orbifold singularities) such that the metric is again a hyperbolic orbifold singularity in a neighborhood of these curves. Hence they can be smoothed and we get the desired metric g_0 similar to that in §4.

Now we can slightly, e.g., using Lemma 2.2, deform this metric to g_ε with:

$$g_\varepsilon = \begin{cases} g_{\mathbb{R}} + (1+\varepsilon\cdot t)^2\cdot(g_{S^1}+g_{S^1}) & \text{on }]-3,-2[\times S^1\times S^1, \\ g_{\mathbb{R}} + g_{S^1} + g_{S^1} & \text{on }]2,3[\times S^1\times S^1, \end{cases}$$

$\mathrm{Ric}(g_\varepsilon) \le 0$, for some small $\varepsilon > 0$.

Let p (we may assume $p < -10$) fulfill $1+\varepsilon\cdot p = 0$ and prolongate g_ε onto $]p,-3]\times S^1\times S^1$ by $g_{\mathbb{R}} + (1+\varepsilon t)^2\cdot(g_{S^1}+g_{S^1})$. Next take the covering $\pi_n :]p,3[\times S^1\times S^1 \to]p,3[\times S^1\times S^1$ defined by $\pi_n(t,e^{ix},e^{iy}) = (t,e^{nix},e^{niy})$, for $n\in\mathbb{Z}^{>0}$, and consider $g_n := \pi_n^*(g_\varepsilon)$ which has some nice properties:

(i) $$g_n = \begin{cases} g_{\mathbb{R}} + n^2(g_{S^1}+g_{S^1}) & \text{on }]2,3[\times S^1\times S^1, \\ g_{\mathbb{R}} + n^2\cdot(1+\varepsilon\cdot t)^2\cdot(g_{S^1}+g_{S^1}) & \text{on }]p,p+5[\times S^1\times S^1, \end{cases}$$

(ii) $\mathrm{Ric}(g_n) \le 0$.

Thus take some large n to ensure $n\cdot(1+\varepsilon t) > \gamma\cdot(t-p)$ on $\mathbb{R}^{>p}$. This metric can be easily deformed to g_γ with $\mathrm{Ric}(g_\gamma)\le 0$, $g_\gamma \equiv g_n$ on $]p+6,3[\times S^1\times S^1$, and $g_\gamma \equiv g_{\mathbb{R}} + \gamma^2\cdot(t-p_0)^2\cdot(g_{S^1}+g_{S^1})$ on $]p_0,p+5[\times S^1\times S^1$ for some $p_o \le p$.

This is just our claim. q.e.d.

Now we combine Lemma 6.2 and Proposition 6.3 as follows: take g_1 and g_2 as in Lemma 6.2, deform Proposition 6.3, and glue the resulting Riemannian manifolds. This yields a smooth metric on M^3 which is easily deformed into some metric with $\mathrm{Ric}(g) < 0$, using Lemma 2.2.

This argument also implies as in the higher dimensional case:

Corollary 6.4. *On $\mathbb{R}^3$ there is a metric g_3 with $Ric(g_3) < 0$ on $B_1(0)$ and $g_3 \equiv g_{\mathrm{Eucl.}}$ outside.*

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