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SOME BLOWUP FORMULAS FOR SU(2) DONALDSON POLYNOMIALS

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1. Introduction

Donaldson's vanishing theorem [4] settles the question of the behavior of his polynomial invariants under the connected sum operation, provided that both summands have $b_+^2 > 0$. This leaves open the question of what happens when one adds a negative-definite manifold to a manifold which admits Donaldson invariants. Indeed, this question arises naturally in the differential topology of algebraic surfaces, since the process of "blowing up" a surface X can be realized topologically as the operation of forming the connected sum of X with $\overline{CP}^2$. (Here the bar indicates that the projective plane is given the orientation making its intersection form negative-definite.)

Let X be an oriented, simply-connected 4-manifold with $b_+^2 > 1$ and odd, and let β be some "homology orientation" for X as in [3]. These data determine a sequence of multilinear functions on $H_2(X)$, the SU(2) Donaldson polynomials, which are indexed by $k \in \mathbf{Z}$ and are denoted by $\gamma_k: S^{d(k)}(H_2(X)) \to \mathbf{Z}$. In the above expression, $S^{d(k)}$ denotes the $d(k)^{\text{th}}$ symmetric power of the vector space $H_2(X)$, and $d(k)$ is given by the relation

$$d(k) = 4k - \tfrac{3}{2}(1 + b_+^2(X)).$$

Identifying $H_2(X)$ and $H_2(\overline{CP}^2)$ with their respective images in $H_2(X\#\overline{CP}^2)$ under the natural inclusions, one can expand

$$\gamma_k(X\#\overline{CP}^2) = \sum_i \binom{d(k)}{i} \lambda_{k,i}(e^*)^i,$$

where the $\lambda_{k,i}$ are multilinear functions in $H_2(X)$, and e^* is the linear function dual to the generator e of $H_2(\overline{CP}^2)$. We have omitted the homology orientation in the above notation with the understanding that

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$X\#\overline{CP}^2$ is given the homology orientation which it inherits from the one for X. The combinatorial factors are an artifact of the accepted conventions for multiplying symmetric forms; another way to express the above expansion is to define the $\lambda_{k,i}$ to be the symmetric function on $H_2(X)$ given by

$$\lambda_{k,i}(x_1, \cdots, x_{d(k)-i}) = \gamma_k(X\sharp\overline{\mathbb{CP}}^2)(x_1, \cdots, x_{d(k)-i}, \overbrace{e, \cdots, e}^{i}).$$

Since $X\#\overline{CP}^2$ admits an automorphism sending e to $-e$, the naturality of the invariants ensures that all the $\lambda_{k,2i+1}$ must vanish, hence all the $\lambda_{k,i}$ in the above expansion are independent of the choice of (a sign for) e. It is conjectured that all the $\lambda_{k,i}$ are linear combinations of Donaldson polynomials of X; indeed it is known that $\lambda_{k,0} = \gamma_k(X)$, $\lambda_{k,2} = 0$, $\lambda_{k,4} = -2\gamma_{k-1}(X)$, and $\lambda_{k,6} = -2\gamma_{k-1}(X)(\mathscr{P})$ (see [4], [5]).

The purpose of this paper is to compute the next two coefficients of this expansion. We obtain the relations

$$\begin{aligned}\lambda_{k+2,8}(x_1, \cdots, x_{d(k)}) = &-2\gamma_{k+1}(X)(x_1, \cdots, x_{d(k)}, \wp, \wp) \\ &-4\gamma_k(X)(x_1, \cdots, x_{d(k)})\end{aligned} \tag{1}$$

and

$$\begin{aligned}\lambda_{k+2,10}(x_1, \cdots, x_{d(k)-2}) = &-2\gamma_{k+1}(X)(x_1, \cdots, x_{d(k)-2}, \wp, \wp, \wp) \\ &-24\gamma_k(X)(x_1, \cdots, x_{d(k)-2}, \wp).\end{aligned} \tag{2}$$

Here, $\wp$ is (a suitable extension of) the Pontryagin class of the $SO(3)$ "framing bundle" over the top stratum of the Yang-Mills moduli space $\mathscr{M}_k^{\text{top}}(X)$ described by $\widetilde{\mathscr{M}}_k^{\text{top}}(X) \times_{\mathscr{G}_k} (P_k|_x)$, where P_k denotes the principal SU(2) bundle over X with second Chern number k, $\mathscr{G}_k$ denotes the gauge group of automorphisms of P_k, and $\widetilde{\mathscr{M}}_k^{\text{top}}$ denotes the space of all anti-self-dual (ASD) connections over P_k. (There are difficulties due to the fact that the class $\wp$ does not extend over the entire compactified moduli space; these difficulties can be avoided as in [5].)

One can profitably interpret the Donaldson invariant as a signed intersection number. That is to say, if $\Sigma \hookrightarrow X$ is an embedded surface, there is an associated "divisor" (codimension-2 stratified subspace of $\mathscr{M}_k(X)$ with oriented normal bundle) D_Σ which represents $\mu([\Sigma])$, in the sense that the Thom class of its normal bundle is $\mu([\Sigma])$. Thus, the Donaldson polynomials (when evaluated on classes coming from $H_2(X; \mathbf{Z})$) count the points in the intersection of sufficiently many such divisors. We will often blur the distinction bewtween the embedded curve and the homology class it represents, using D_x with $x \in H_2(X; \mathbf{Z})$ to mean D_Σ, where Σ is some embedded surface representing x.

The computations in this paper rely on a local description due to Taubes [11] (see also [5]) of the moduli space for a connected sum. To state this result we must introduce a bit more notation. The SO(3) framing bundle referred to earlier admits a natural extension as an SO(3) space over all of $\mathscr{M}_k(X)\backslash\mathscr{I}(x)$, where $\mathscr{I}(x) \subset \mathscr{M}_k(X)$ denotes the subset of classes of generalized connections which concentrate at x. This extended SO(3) space, the space of "framed connections," will be denoted $(\mathscr{M}(X)\backslash\mathscr{I}(X))^x$. Taubes' result parameterizes open subsets of moduli spaces of connected sums, endowed with metrics with sufficiently long connected sum tubes, in terms of framed moduli spaces of the connected summands. These open subsets, the so-called Taubes neighborhoods, consist of those classes whose curvature is sufficiently small on the tube. In the form we will need it, Taubes' result states that for suitable metrics on $X\#\overline{CP}^2$, the Taubes neighborhoods whose background curvature is bounded away from zero on the X side of the connect sum are (orientation-preserving) diffeomorphic to subsets of

$$\coprod_{r+s=k\,;\,r>0} \{\mathscr{M}_r(X)^x \times \mathscr{M}_s(\overline{CP}^2)^y\}/\,\mathrm{SO}(3)$$

consisting of classes whose curvature is small near the connected sum points $x \in X$ and $y \in \overline{CP}^2$. Moreover, given $x_i \in H_2(X)$, $\mu(x_i)$ restricted to a component of such a Taubes neighborhood parametrized by $\{\mathscr{M}_r(X)^x \times \mathscr{M}_s(\overline{CP}^2)^y\}/\,\mathrm{SO}(3)$ can be represented by $\pi_X^*(D_{x_i})$, where π_X is the natural projection map

$$\pi_X\colon \{\mathscr{M}_r(X)^x \times \mathscr{M}_s(\overline{CP}^2)^y\}/\,\mathrm{SO}(3) \to \mathscr{M}_r(X),$$

and the $D_{x_i} \subset \mathscr{M}_r(X)$. Slightly more care must be taken in describing cohomology classes associated to e, taking into account the fact that the moduli space for $\overline{CP}^2$ generically contains nontrivial reducible connections. The classes $\mu(e)$ fail to extend as honest cohomology classes over these "reducible singularities" in the $\overline{CP}^2$ moduli spaces, so one must pass to an equivariant setting (see [5] and [7]). The class $\mu(e)$ extends as an equivariant cohomology class $\mu(e)^y \in H^2_{SO(3)}((\mathscr{M}_s(\overline{\mathbf{CP}}^2)\backslash\mathscr{I}(y))^y)$, and the $\mu(e)$ restrict over the Taubes neighborhoods as pullbacks of these classes under the natural equivariant projection map. (Of course, Taubes' results hold in greater generality than we have stated. The results are slightly more complicated to write down when the connections on the X side are

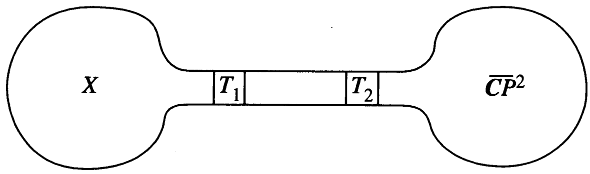

FIGURE 1. PARTITIONING THE CONNECTED SUM

allowed to have trivial background, due to the fact that the trivial connection is not a "smooth point" of the moduli space of a manifold with $b_2^+ > 0$. We will not encounter such connections; and the same complications do not arise on the $\overline{CP}^2$ side, since in the latter case, the trivial connection is a smooth point.)

The terms $\lambda_{k,i}$ with $k < 8$ have the property that the intersection of all the divisors defining the relevant Donaldson invariant lie in a single Taubes neighborhood, as can be seen from the dimension counting argument presented in [5] (arguments of this kind will be encountered in §3). This property fails for the $\lambda_{k,8}$ and $\lambda_{k,10}$, providing the main challenge in performing the calculation (and accounting for the presence of two terms rather than one in (1) and (2)). It will be notationally convenient to restrict attention for the time being to the $\lambda_{k,8}$ calculation, although most of the comments in this section apply with minor modifications in both cases. We will return to a discussion of $\lambda_{k,10}$ in a later section.

Dimension counts ensure that for metrics on the connected sum with sufficiently long tube length, the ASD connections in the defining intersection for $\lambda_{k,8}$

$$D_{x_1} \cap \cdots \cap D_{x_{d(k)}} \cap \overbrace{D_e \cap \cdots \cap D_e}^{8}$$

fall into two categories: those with nearly one unit of charge over the $\overline{CP}^2$ part of the connected sum, and those with nearly two units. We separate the two possibilities by taking two annular regions along the connected sum neck T_1 and T_2 (closer to X and $\overline{CP}^2$ respectively, as specified in Figure 1), and letting $\mathcal{U}_1^{k+1}$ be the subset of connections in $D_{x_1} \cap \cdots \cap D_{x_{d(k)}} \cap D_e \cap D_e \cap D_e$ whose energy distributes as nearly 1 on $\overline{CP}^2$ and small on T_2 (and hence nearly $k+1$ on the rest of the connected sum); $\mathcal{U}_2^k$ be the subset of connections whose energy distributes as nearly k on X and small on T_1 (and hence nearly 2 on the rest of the connected sum). In other words, in $\mathcal{U}_1^{k+1}$, the "bubble" (if it has formed) is trapped to sit to the left of T_2, and in $\mathscr{U}_2^k$ it is trapped to the right of T_1. These two open sets cover the entire $D_{x_1} \cap \cdots \cap D_{x_{d(k)}} \cap D_e \cap D_e \cap D_e$, and each sits in a Taubes neighborhood. However, each open set is noncompact, so one cannot simply evaluate the $\mu(e)^5$ on a fundamental class; and moreover these sets intersect, so one must be careful not to count points on this intersection twice.

Via the Taubes description, $\mathscr{U}_1^{k+1}$ can be thought of as the sphere bundle associated to the base bundle over a family of connections of charge $k+1$ over X with at most one charge concentrating. More precisely, let $\mathscr{X}^8$ denote the eight-dimensional "cut down" moduli space

$$\mathscr{X}^8 = D_{x_1} \cap \cdots \cap D_{x_{d(h)}} \subset \mathscr{M}_{d(k+1)}(X).$$

This space has a subset $\mathscr{Y}^8$ consisting of connections not too concentrated near x, hence framable at that point. Moreover, as we shall later see, the equivariant intersection $D_e^y \cap D_e^y \cap D_e^y \subset \mathscr{M}_1(\overline{CP}^2)^y$ picks out the single reducible orbit with S^1 stabilizer; in other words, this triple intersection is $SO(3)$-equivariantly diffeomorphic to S^2 endowed with the standard $SO(3)$ action. The Taubes descriptions then give an isomorphism

$$\mathscr{U}_1^{k+1} \cong (\mathscr{Y}^8)^x \times S^2/SO(3).$$

We will see that $\mathscr{Y}^8$, hence $\mathscr{U}_1^{k+1}$, is noncompact in general; one expects $\mathscr{X}^8$ to contain families of connections whose charge concentrates arbitrarily close to x.

A similar description exists for $\mathscr{U}_2^k$. Cutting down in $\mathscr{M}_k(X)$, we have a zero-dimensional, compact space

$$\mathscr{X}^0 = D_{x_1} \cap \cdots \cap D_{x_{d(k)}} \subset \mathscr{M}_k(X).$$

This zero-dimensional space lives entirely in the top stratum of $\mathscr{M}_k(X)$, hence it can be framed at x. It is worthwhile noting that (by definition)

$$\#\mathscr{X}^0 = \gamma_k(X)(x_1, \cdots, x_{d(k)}).$$

Moreover, the 16-dimensional (noncompact) $\mathscr{M}_2(\overline{CP}^2)^y$ admits $SO(3)$-equivariant divisors D_e, three of which intersect in a 10-dimensional SO(3)-space

$$\mathscr{N}_7^y = D_e \cap D_e \cap D_e$$

(the subscript denotes the dimension of the $SO(3)$-quotient). In this notation, we have

$$\mathscr{U}_2^k \cong \{(\mathscr{X}^0)^x \times \mathscr{N}_7^y\}/SO(3).$$

The noncompactness of $\mathcal{N}_7$ (once again due to charge bubbling off towards y) is reflected in the noncompactness of $\mathscr{U}_2^k$.

Fortunately, the intersection $\mathscr{U}_1^{k+1} \cap \mathscr{U}_2^k$ sits in another Taubes neighborhood. This region can be thought of as composed of connections of charge k on X glued to connections of charge 1 on a tubular region (between T_1 and T_2) and charge 1 on $\overline{CP}^2$. From the point of view of the ASD equation, the tube behaves like S^4 (the flat tube is conformally equivalent to $S^4 \backslash \{n, s\}$, where n and s are two distinct points in S^4), so we have in fact that

$$\mathscr{U}_1^{k+1} \cap \mathscr{U}_2^k \subset \mathscr{M}_k(X)^x \times \mathscr{M}_1(S^4)^{n,s} \times \mathscr{M}_1(\overline{CP}^2)^y / SO(3) \times SO(3),$$

where $\mathscr{M}_1(S^4)^{n,s}$ indicates that the moduli space over S^4 must be based at two separate points. Indeed, letting $\mathscr{W}^{n,s} \subset \mathscr{M}_1(S^4)^{n,s}$ denote those connections which are not concentrated near either point n or s (i.e., the subset where it makes sense to frame), we have that

$$\mathscr{U}_1^{k+1} \cap \mathscr{U}_2^k \cong (\chi^0)^x \times \mathscr{W}^{n,s} \times S^2 / SO(3) \times SO(3).$$

The key to dealing with the noncompactness issue is to find "caps" $\mathscr{A}$ and $\mathscr{B}$ to attach to the ends of $\mathscr{U}_1^{k+1}$ and $\mathscr{U}_2^k$ to compactify them (as stratified spaces) in such a way that the classes $\mu(e)$ extend over the compactifications and so that the fundamental classes extend. Then by an excision argument,

$$\begin{aligned}\mu(e)^5[\mathscr{U}_1^{k+1} \cup \mathscr{U}_2^k] = \mu(e)^5[\mathscr{U}_1^{k+1} \cup \mathscr{B}] + \mu(e)^5[\mathscr{A} \cup \mathscr{U}_2^k] \\ - \mu(e)^5[\mathscr{A} \cup (\mathscr{U}_1^{k+1} \cap \mathscr{U}_2^k) \cup \mathscr{B}].\end{aligned} \tag{3}$$

The first term on the right-hand side will be referred to as the "contribution from the X side", as it is records connections where the "extra" unit of charge sits over X; similarly, the second term will be referred to as the "contribution from the $\overline{CP}^2$ side", and the final term the "correction term".

The caps for $\mathscr{U}_1^{k+1}$ and $\mathscr{U}_2^k$ will come from caps for the moduli space for S^4. More precisely, consider the $SO(3)_n \times SO(3)_s$-space $\mathscr{W}^{n,s}$. This is a stratified space with two ends, one for each pole. The subscripts on the $SO(3)$'s are intended to call attention to the fact that the first factor acts on frames over n while the second acts on frames over s. Two sets A and B will be found which satisfy the following properties.

A is an $SO(3) \times SO(3)$-space whose group action is compatible with the action on $\mathscr{W}^{n,s}$, in the sense that there is an $SO(3) \times SO(3)$-equivariant identification ϕ_A of a subspace of A with a subspace of $\mathscr{W}^{n,s}$. Moreover, restricting to the second factor, the action on A is free, giving the identification space $A \cup_{\phi_A} \mathscr{W}^{n,s}$ the structure of an $SO(3)_s$-bundle.

The base of the above $SO(3)_s$-bundle projection, $(A \cup_{\phi_A} \mathscr{W}^{n,s})/SO(3)_s$, has one end which corresponds to instantons concentrated near s.

B is a rational S^2-fibration over instantons over the southern hemisphere based at n.

The rational S^2-fibration structure extends the S^2-bundle structure on $\mathscr{W}^{n,s}/S^1_s$ associated to the $SO(3)_s$ quotient, in the sense that there is an identification ϕ_B of a subset of $\mathscr{W}^{n,s}/S^1_s$ with a subset of B which, when composed with the fibration map for B, commutes with the natural $SO(3)_s$ quotient map. Moreover, the identification ϕ_B is a rational homotopy equivalence.

Note that since B is merely a rational S^2-bundle, the extended space $\{(A \cup_{\phi_A} \mathscr{W}^{n,s})/S^1_s\} \cup_{\phi_B} B$ cannot be a compact space. However, it will have a rational fundamental class, since the space has the rational homotopy type of a rational sphere fibration over a compact stratified space. This is sufficient for calculations. Moreover, the above spaces support classes naturally extending $\mu(e)$. Over $(A \cup_{\phi_A} \mathscr{W}^{n,s})/S^1_s$, the class comes from the free S^1_s action on $A \cup_{\phi_A} \mathscr{W}^{n,s}$ induced from $S^1_s \hookrightarrow SO(3)_s$; and over B it comes from a natural extension (which will be described in detail later) of a generator of $H^2(S^2_{(0)}; \mathbf{Q})$. Note that no $SO(3)_n$-equivariance is required of B. Although this makes the construction of B simpler, it also introduces somewhat of an asymmetry in the compactification.

The caps $\mathscr{A}$ and $\mathscr{B}$ can be readily constructed from the A and B described above. Since a neighborhood of the end of $\mathscr{U}_1^{k+1}$ is modeled on

$$\{(\chi^0)^x \times (\mathscr{W}^{n,s}/S^1_s)\}/SO(3)_s = \coprod_{\gamma_k} (\mathscr{W}^{n,s}/SO(e)_s),$$

this space will be sealed off by attaching $\coprod_{\gamma_k} B$, where γ_k is shorthand for $\gamma_k(X)(x_1, \cdots, x_{d(k)})$. Similarly, $\mathscr{U}_2^k$ will be capped off by the same number of copies of A.

Thus, before the calculation can proceed, one must have a more explicit understanding of these caps over S^4. This is undertaken in §2. After this, the next three sections are devoted to computing the terms appearing on the right-hand side in (3). Section 3 computes the first term, the contribution from the X side. This calculation is similar in spirit to the calculations of $\lambda_{k,4}$ and $\lambda_{k,6}$, the latter of which is also included in this section. Section

4 employs some of the lemmas developed in the previous two sections together with the concrete description of the compactification from §2 to calculate the correction term. In §5, the contribution from the $\overline{CP}^2$ side is reduced to characteristic class computation which is then performed. The issues arising in this section are similar to those arising in Yang's work [12] on computing wall-crossing formulas. In §6, we apply the techniques developed in the earlier sections to compute the formula for $\lambda_{k,10}$ stated in (2). Section 7 contains some remarks on possible future computations.

2. The caps

Let $\mathscr{W}$ denote the subset of the moduli space of generalized Chern class-one instantons over S^4 consisting of those classes which are not too concentrated near either pole. This space can be based twice, once over n and once over s, to obtain a right $SU(2)\times SU(2)$ space. Dividing out by the natural $\{\pm 1\}\times\{\pm 1\}$, one gets a space which will be denoted $\mathscr{W}^{n,s}$, and which we will refer to as a "doubly-based" moduli space (although this name is slightly misleading; the space is really the $\{\pm 1\}$ quotient of a space of doubly-based connections, a technical point which can be safely ignored). We wish to identify this space explicitly. (The definition of $\mathscr{W}$ is somewhat ambiguous. The condition "not too concentrated near either pole" could be made more precise, say, by giving an upper bound less than $8\pi^2$ to the total energy of the connections over two precise, disjoint neighborhoods of n and s. There is, however, no benefit to doing so at this time, and although we will give an exact formulation later, these choices are by nature arbitrary and not really relevant to the topology of the spaces involved.)

The challenge here is to understand the structure of the space, along with the given group action, at "infinity"; i.e., at the completely concentrated connections. As a preliminary remark, we point out that although either of the $SO(3)$'s act freely on $\mathscr{W}^{n,s}$, they do not act freely at once. This is due to these concentrated connections with trivial background, which have three-dimensional stabilizers and three-dimensional orbits in the doubly-based space.

Before proceeding, we introduce some notation.

Identify $\mathbf{R}^4\cong S^4\backslash s$ under stereographic projection from s. Given a positive real number λ and a quaternion $a\in\mathbf{R}^4=\mathbf{H}$, μ_λ^a will denote the unique conformal automorphism of S^4 which restricts to the $\mathbf{R}^4\subset S^4$ as the affine map $q\to\lambda q+a$. Since the (left) action of $\mathrm{Diff}(S^4)$ on S^4 lifts naturally to the total space of P_1, under the identification of $P_1 \cong \Lambda_2^-(S^4)$, it induces an action (on the right) on the set of connections over P_1 which descends to the set of connections modulo gauge transformations. The group of conformal diffeomorphisms preserves the set of ASD connections, and it is a fact that any nonconcentrated $[\nabla] \in \mathscr{M}_1(S^4)$ can be uniquely written as

$$[\nabla] = [\mu_\lambda^a(\Gamma)],$$

where Γ is the "standard" instanton (see, for example, [1]). To simplify notation we will write ∇_λ^a for $\mu_\lambda^a(\Gamma)$ if $\lambda \neq 0$. We can extend this notation to the case of $\lambda = 0$ by writing ∇_0^a for the connection completely concentrated at a. We make the definition of $\mathscr{W}$ precise now by declaring it to consist of connections of the form ∇_λ^a where $\lambda < 3/2$ and $|a| < 1$ and $\lambda > 7/8$ if $|a| < 1/4$. (These constants are included for definiteness only).

We will need to decompose $\mathscr{W} = \mathscr{W}_1 \cup \mathscr{W}_2$, with $\mathscr{W}_1$ consisting of somewhat concentrated connections whose center is near the equator and $\mathscr{W}_2$ consisting of rather diffuse connections. For concreteness, let

$$\mathscr{W}_1 = \{\nabla_\lambda^a \in \mathscr{W} | \lambda < 3/4\} \quad \text{and} \quad \mathscr{W}_2 = \{\nabla_\lambda^a \in \mathscr{W} | \lambda > 1/4\}.$$

These open sets, along with those which will appear later in this section, are indicated in Figure 2. (The topology of the situation is captured in this picture; the precise geometry is not important.)

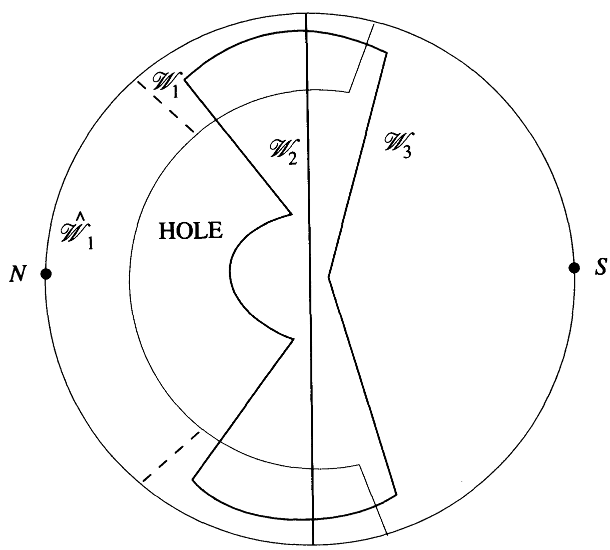

FIGURE 2. PARTITIONING THE MODULI SPACE OF THE SPHERE

If $\nabla^a_\lambda \in \mathscr{W}_1^{n,s}$, there is a unique associated great circle γ_a which passes from n to s through the point antipodal to a. We will also need to fix some point p in the S^3 equator of S^4 throughout the discussion. The expression $T^\nabla_\gamma(F)$ will denote parallel transport with respect to ∇ of the frame F along the curve γ (from $\gamma(0)$ to $\gamma(1)$).

Let F_s denote some initial frame over s; or equivalently, let $[\Gamma, F_s]$ be a framing for the standard instanton. This can be viewed as a simultaneous framing for all classes of connections in $\mathscr{W}_2$ as follows. The group of conformal diffeomorphisms which fix s acts naturally on the top stratum of $\mathscr{M}_1(S^4)^s$ under $\psi \times [\nabla, F_s] \to [\psi(\nabla), \psi_s^{-1}(F_s)]$, where $\psi(\nabla)$ denotes the action of the conformal diffeomorphism ψ on the ASD connection ∇, and ψ_s denotes the restriction of ψ on the frame bundle of $\Lambda^-_2(S^4)$ to the fiber over s. (The crucial point to note here is that for $u \in \mathscr{G}_1$,

$$[\psi(u(\nabla)), \psi_s^{-1}F_s] = [(\psi^{-1}u\psi)(\psi(\nabla)), \psi_s^{-1}F_s] \sim [\psi(\nabla), \psi_s^{-1}u(F_s)].)$$

Moreover, the subgroup whose differential at s is simply a rescaling (i.e., not a rotation) actually fixes the frames over s; so it makes sense to frame any $\nabla^a_\lambda \in \mathscr{W}_2$ by $\mu^a_\lambda[\Gamma, F_s] = [\nabla^a_\lambda, F_s]$. This construction will be implicitly used throughout the following discussion.

Parallel transport will be used to identify frames at n with frames at s. A language will then be needed for comparing two frames over the same point. Suppose F is a free, homogeneous G-space (i.e., F is equivalent as a G-space to G itself with the action given by right translations). The example to keep in mind is the fiber over a point of a principal G-bundle, such as $P_1/\{\pm 1\}$. Let $p_1, p_2 \in F$ be two points; then $\Delta(p_1, p_2) \in G$ is defined to be the element such that $p_2 = p_1\Delta(p_1, p_2)$.

Now that all the relevant language is in place, we wish to trivialize $\mathscr{W}^{n,s}$ as an $SO(3)$-bundle over $\mathscr{W}^s$. This is done over $\mathscr{W}_1^{n,s}$ by transporting the frame from n to s using the canonically associated great circles γ_a and the specified connections. That is,

$$\tau_1 : \mathscr{W}_1^{n,s} \to SO(3) \times \mathscr{W}_1^s$$

is given by

$$\tau_1[F_n, \nabla^a, F_s] = \Delta(F_s, T^{\nabla^a}_{\gamma a} F_n) \times [\nabla^a, F_s]$$

with inverse

$$\Theta_1 : SO(3) \times \mathscr{W}_1^s \to \mathscr{W}_1^{n,s}$$

given by

$$\Theta_1(g, [\nabla^a, F_s]) = [(T^{\nabla^a}_{\gamma a})^{-1} F_s g, \nabla^a, F_s].$$

These maps extend to the completely concentrated connections, since the γ_a are guaranteed to avoid passing through the centers of concentration. They also do not extend through all of $\mathscr{W}$ essentially because there is no way to continuously extend the map $\nabla_\lambda^a \to \gamma_a$ through the interior of $\mathscr{M}_1(S^4)$ (a manifestation of the fact that the canonical map from $S^3 \to \Omega(S^4)$ is homotopically nontrivial).

Thus, $\mathscr{W}^{n,s}$ must be trivialized differently over $\mathscr{W}_2$. This is done by comparing the frame over s with the frame obtained by translating the frame over n to s along the path antipodal to the point of concentration (if it exists), restoring the background connection to the "standard" instanton, transporting the frame back to n, and then coming back along the fixed great circle γ_p. That is, define

$$\tau_2 \colon \mathscr{W}_2^{n,s} \to SO(3) \times \mathscr{W}_2^s$$

by

$$\tau_2[F_n, \nabla_\lambda^a, F_s] = \Delta(F_s, T_{\gamma_p}^\Gamma T_{\gamma a}^{\Gamma-1} T_{\gamma a}^{\nabla_\lambda^a} F_n) \times [\nabla_\lambda^a, F_s].$$

This map is a priori not defined when ∇_λ^a is nonconcentrated or concentrated over n, i.e., when $a = n$. However, it admits a natural extension to such connections, since for any a, $T_{\gamma a}^{\nabla_\lambda^n{}^{-1}} T_{\gamma a}^{\nabla_\lambda^a} = \mathrm{id}$. Thus, one can safely define

$$\tau_2[F_n, \nabla_\lambda^n, F_s] = \Delta(F_s, T_{\gamma_p}^\Gamma F_n) \times [\nabla_\lambda^n, F_s].$$

To make the notation a bit more concise, define

$$\Psi^a(F_s) = T_{\gamma_p}^\Gamma T_{\gamma a}^{\Gamma-1}(F_s).$$

In this notation the inverse,

$$\Theta_2 \colon SO(3) \times \mathscr{W}_2^s \to \mathscr{W}_2^{n,s},$$

is given by

$$\Theta_2(g, [\nabla^a, F_s]) = [(\Psi^a T_{\gamma a}^{\nabla^a})^{-1} F_s g, \nabla^a, F_s].$$

It should be pointed out that τ_1 provides more than merely a trivialization of $\mathscr{W}_1^{n,s}$ as an $SO(3)$-space. Because of the properties of Δ, it gives an identification of $\mathscr{W}_1^{n,s}$ as an $SO(3)_n \times SO(3)_s$-space with $SO(3) \times \mathscr{W}_1^s$ given the right $SO(3)_n \times SO(3)_s$-action

$$(g, [\nabla^a, F_s]) \times (a_n, b_s) = (b_s^{-1} g a_n, [\nabla^a, F_s b_s]).$$

Indeed, this description allows one to construct at least a piece of the first cap. Letting $\widehat{\mathscr{W}_1^s}$ denote the set of all sufficiently concentrated connections whose concentration point sits in the northern hemisphere (say, all connections ∇^a_λ with $\lambda < 3/4$), one can endow $SO(3) \times \widehat{\mathscr{W}_1^s}$ with the natural extension of the above $SO(3)_n \times SO(3)_s$ action. In this way, one obtains an enlarged $SO(3)_n \times SO(3)_s$ space

$$\widehat{\mathscr{W}_2^{n,s}} = \widehat{\mathscr{W}_1^{n,s}} \coprod \mathscr{W}^{n,s}/\Theta_1.$$

Note now that this space is only "formally framed" near n, in the sense that points concentrated over n in $\widehat{\mathscr{W}_1^{n,s}}$ do not correspond to geometrically twice framed instantons. Moreover, the space has two ends. One end corresponds to connections somewhat concentrated near s ($a > 1$ or $\lambda > 3/2$ or, of course, the connection which is totally concentrated over s), the other corresponds to connections almost concentrated near n ($a < 1/4$ but $1/4 < \lambda < 7/8$).

Similarly, for Θ_2, the space $SO(3) \times \mathscr{W}_2^s$ can be endowed with an $SO(3)_n \times SO(3)_s$ action making Θ_2 an equivariant map.

Dividing out by the $SO(3)_s$ action, it becomes clear how to fill in the remaining holes.

Lemma 1. *Under the free $SO(3)_s$ action, $\widehat{\mathscr{W}_2^{n,s}}$ is an $SO(3)_s$-bundle over a space which is homeomorphic to a punctured $P_{-1}(S^4) \times_{SU(2)} c(SO(3))$. More concretely, trivialize $P_{-1}(S^4)$ near s, so that*

$$D^4 \times c(SO(3)) \hookrightarrow P_{-1}(S^4) \times_{SU(2)} c(SO(3)).$$

If $\frac{1}{2}c(SO(3)) \hookrightarrow c(SO(3))$ denotes a proper subcone, then the "puncture" consists of the image of $D^4 \times \frac{1}{2}c(SO(3))$ under the above inclusion.

Proof. This proof involves writing down natural quotients by the $SO(3)_s$ action (i.e., $SO(3)_s$-invariant maps) over the two pieces which compose $\widehat{\mathscr{W}_2^{n,s}}$ and seeing how they glue together. A quotient on $\widehat{\mathscr{W}_1^s}$,

$$\Pi_1 : SO(3) \times \widehat{\mathscr{W}_1^s} \to \widehat{\mathscr{W}_1^s},$$

can be defined by

$$\Pi_1(g, [\nabla^a, F_s]) = [\nabla^a, F_s g];$$

similarly on $SO(3) \times \mathscr{W}_2^s$, a quotient

$$\Pi_2 : SO(3) \times \mathscr{W}_2^s \to \mathscr{W}_2^s$$

can be defined by

$$\Pi_2(g, [\nabla^a, F_s]) = [\nabla^a, F_s g].$$

Both maps are clearly $SO(3)_s$ invariant; however they do not match up; i.e., $\Pi_1 \neq \Pi_2 \tau_2 \Theta_1$. Instead, since

$$\tau_2 \Theta_1(g \times [\nabla^a, F_s]) \simeq \Delta(F_s, \Psi^a F_s g) \times [\nabla^a, F_s],$$

we see that

$$\Pi_1(g \times [\nabla^a, F_s]) = [\nabla^a, F_s g],$$

whereas

$$\Pi_2 \tau_2 \Theta_1(g \times [\nabla^a, F_s]) \simeq [\nabla^a, \Psi^a F_s g].$$

Thus, the Π_i glue together properly to give a map to

$$\widehat{\mathscr{W}_1^s} \coprod \mathscr{W}_2^s / [\nabla^a, F_s] \sim [\nabla^a, \Psi^a F_s].$$

The Ψ are transition functions for $P_1(S^4)$ in the sense that they describe how to get from the trivialization about s to the one about n, but this is the opposite of the identification taking place in the image of the Π_i; so we have effectively described $\widehat{\mathscr{W}_2^{n,s}}$ as an $SO(3)$ bundle over a punctured version (at s) of $P_{-1}(S^4) \times_{SU(2)} c(SO(3))$. q.e.d.

From the above description, one can see that one of the holes in the base space $\widehat{\mathscr{W}_2^{n,s}}/SO(3)_s$ is bounded by $P_{-1}(S^4) \times_{SU(2)} SO(3)$, which is homeomorphic to $\mathbf{R}P^7$. By attaching a $c(\mathbf{R}P^7)$, one can identify the base space as a punctured $\mathbf{H}P^2$ modulo the involution given by $[q, r, s] \to [-q, -r, s]$. It is important to note that the orientation induced from this description on our punctured $\mathbf{H}P^2/\{\pm 1\}$ agrees with the one induced from the positive-definition orientation on $\mathbf{H}P^2$; in other words, the self-intersection number of the cone section is positive. This is true because the identification of $c(SU(2))$, which we think of as being the quaternion algebra $\mathbf{H}$, with $\mathbf{C}^2$ taking left quaternionic multiplication to the left action by $SU(2)$ matrices reverses orientation. Explicitly, the map $\mathbf{C}^2 \to \mathbf{H}$ defined by $[w, z] \to w + \mathbf{j}z$ takes the oriented basis (induced from the S^1 action) of $\mathbf{C}^2$ given by $\langle (1, 0), (i, 0), (0, 1), (0, i) \rangle$ to the negatively oriented basis (of $\mathbf{H}$) $\langle 1, \mathbf{i}, \mathbf{j}, \mathbf{ji} = -\mathbf{k} \rangle$.

The total space of the bundle can also be extended over the hole bounded by $\mathbf{R}P^7$. This is a consequence of the following lemma.

Lemma 2. *The $SO(3)_s$ bundle $\widehat{\mathscr{W}_2^{n,s}}$ restricts to a product bundle over the $\mathbf{R}P^7$ boundary corresponding to the connections nearly concentrated at n.*

Proof. Under the natural map $\widehat{\mathscr{W}_2^{n,s}} \to \mathscr{M}_1(S^4)$, this boundary component maps to the interior of the moduli space. Moreover, the above map lifts in a canonical $SO(3)_s$-equivariant way to $\widehat{\mathscr{W}_2^{n,s}} \to \mathscr{M}_1(S^4)^s$, and the latter space is a trivial $SO(3)_s$ bundle over the interior of the moduli space. q.e.d.

The above lemma suggests that one should fill in the total space by attaching $c(\mathbf{R}P^7) \times SO(3)$, since the boundary is identified as $\mathbf{R}P^7 \times SO(3)$.

Indeed, it is not difficult to see that one can fill in the entire $SO(3)_n \times SO(3)_s$ action over this region by letting $SO(3)_n$ act on the first factor only (acting as bundle transformations on the bundle over S^4) and letting $SO(3)_s$ act on the second factor only as right translations.

This extended $SO(3)_n \times SO(3)_s$ space constitutes what was referred to earlier as the cap A, with the identification $\phi_A \colon \mathscr{W}^{n,s} \to A$ induced from the natural inclusion. Before we move on to the other cap, we identify the $SO(3)_s$-bundle more explicitly. (This will be needed for later calculations.)

Lemma 3. *Under the embedding*

$$\widehat{\mathscr{W}_2^{n,s}}/SO(3)_s \hookrightarrow P_{-1}(S^4) \times_{SU(2)} c(SO(3)),$$

the bundle $\widehat{\mathscr{W}_2^{n,s}}$ *is identified with* $\pi^* P_{-1}(S^4) \times_{SU(2)} SO(3)$, *where* $\pi \colon P_{-1}(S^4) \times_{SU(2)} c(SO(3)) \to S^4$ *denotes the projection to cone point.*

Proof. To see this, we construct two maps $\phi_1, \phi_2 \colon D^4 \to \widehat{\mathscr{W}_2^{n,s}}$ which differ along the bounding S^3 by the $SO(3)_s$ action. This difference gives a map $S^3 \to SO(3)$ which is then shown to be given by Ψ. Specifically, $\phi_1 \colon D^4 \to \widehat{\mathscr{W}_1^{n,s}}$ is defined by

$$\phi_1(x) = \begin{cases} e \times [\nabla_0^x, *] & \text{if } |x| \le \frac{1}{2}, \\ e \times [\nabla_{|x|-1/2}^{x/2|x|}, (\Psi^{x/|x|})^{-1} F_s] & \text{if } |x| \ge \frac{1}{2}, \end{cases}$$

where F_s is once more taken to be a fixed framing of the standard instanton (hence, as discussed earlier, a framing of all instantons in $\widehat{\mathscr{W}_2}$). The " $*$ " is used above to draw attention to the fact that the framing is irrelevant for completely concentrated connections. Furthermore, $\phi_2 \colon D^4 \to \mathscr{W}_2^{n,s}$ is defined by

$$\phi_2(x) = e \times [\nabla_{1/2}^{x/2}, F_s].$$

If $|x| = 1$, then

$$\Theta_1 \circ \phi_1(x) = [(\Psi^x T_{\gamma_s}^{\nabla^s})^{-1} F_s, \nabla^x, \Psi^x F_s],$$

whereas

$$\Theta_2 \circ \phi_2(x) = [(\Psi^x T_{\gamma_s}^{\nabla^s})^{-1} F_s, \nabla^x, F_s],$$

so the two maps must agree along ∂D^4 in the $SO(3)_s$ quotient hence can be glued together to form a map of $S^4 \to \widehat{\mathscr{W}_2^{n,s}}/SO(3)_s$. Indeed, it is easy to see that this map describes a section of the cone bundle over S^4 from Lemma 3. Since a section of a cone bundle is a homotopy equivalence with the total space of the cone bundle, the bundle $\widehat{\mathscr{W}_2^{n,s}}$ over $\widehat{\mathscr{W}_2^{n,s}}/SO(3)_s$

is determined by its restriction to this sphere. But a bundle over a sphere is determined by its equatorial transition function, which in this case is visibly given by Ψ. q.e.d.

To summarize these results, we have augmented a region of (a $\{\pm 1\}$ quotient of) the doubly-based moduli space $\mathscr{W}^{n,s}$ by attaching the cap A. The $SO(3)_n \times SO(3)_s$ action on the doubly-based space can be extended over the composite space so as to induce three different orbit types: along the first stratum of the moduli space (the points which project to completely concentrated connections) the orbits have three-dimensional stabilizers; in the "central points" of A, which correspond to the cone point factor in $c(\mathbf{R}P^7) \times SO(3)$, the $SO(3)_n$ acts trivially while the $SO(3)_s$ acts freely; everywhere else, the orbits are free $SO(3)_n \times SO(3)_s$ orbits. Moreover, the $SO(3)_n$ quotient of the augmented space is naturally identified with the portion $\mathscr{M}_1(S^4)^s$ lying over the instantons not too concentrated near s.

The above construction necessarily had to be delicate in order to extend the $SO(3)_s$-action *freely* to the partially compactified space. It is precisely this feature which complicates the construction of A, a complication which is reflected in the topology of the $SO(3)_s$ quotient of the extended space (which is homotopy equivalent to a punctured $\mathbf{H}P^2/\{\pm 1\}$). The freeness will be used to extend the $\mu(e)$ class. More precisely, $\mathscr{W}^{n,s}/S^1_s$ is the space which is to be compactified, and, as we shall later see, the class to be extended is a class which naturally arises from the free S^1_s action on $\mathscr{W}^{n,s}$; it is the first Chern class of the complex line bundle $\mathscr{W}^{n,s} \times_{S^1_s} \overline{\mathbf{C}} \to \mathscr{W}^{n,s}/S^1_s$. But $\mathscr{W}^{n,s}/S^1_s$ is augmented by A/S^1_s, over which the complex line bundle naturally extends as $A \times_{S^1_s} \overline{\mathbf{C}} \to A/S^1_s$.

Constructing the other cap B need not be nearly so complicated, since rather less is expected of this cap. To see what is involved, consider the S^2 bundle projection

$$(4) \qquad (A \cup_{\phi_A} \mathscr{W}^{n,s})/S^1_s \to (\widehat{\mathscr{W}_2^{n,s}}/SO(3)_s)\backslash \mathscr{W}_3^n,$$

where $\mathscr{W}_3$ consists of the instantons sufficiently concentrated anywhere near the southern hemisphere; more precisely, we could say that $\mathscr{W}_3$ consists of all ∇^a_λ for which $|a| > 7/8$ or $\lambda > 5/4$. This bundle is identified in Lemma 1 as being an S^2 bundle over $(\mathbf{H}P^2/\{\pm 1\})\backslash D^4 \times c(SO(3))$. The cohomology class to be extended comes naturally from the sphere bundle structure; it is the "cotangents along the fiber" of the sphere bundle.

Suppose now that we could extend the S^2 bundle over the missing $\mathscr{W}_3^n \cong D^4 \times c(SO(3))$. Then, the total space of this extended bundle restricted to $D^4 \times c(SO(3))$ would form the cap B. This sphere bundle would admit a fundamental class, as it would be an S^2 bundle over a space with fundamental class. Moreover, the cotangents along the fiber would extend automatically, as it extends naturally with the sphere bundle structure.

Unfortunately, the actual situation is slightly more complicated than this ideal case. Although there are no rational obstructions to extending the above S^2-bundle (as we will shortly see), there are certain finite obstructions to doing so. More precisely, one can show that the explicit trivializations of the bundle $\mathscr{W}^{n,s}/\mathscr{W}^s$ given in this section, when restricted to a central slice, express it as being a quotient of the $SU(2)$ bundle over S^7 whose transition functions over $S^3 \times S^3$ are given by the commutator map $SU(2) \times SU(2) \to SU(2)$. The fact that this bundle is nontrivial, despite the fact that its second Chern class vanishes, is a classical result of homotopy theory [10]. This subtlety can be circumvented by appealing to rational homotopy theory (see [6]), a theory which was designed to address precisely these sorts of problems. Concretely, given a simply-connected CW complex X, the theory ensures the existence of a new CW complex denoted $X_{(0)}$ with

$$\pi_i(X_{(0)}) \cong \pi_i(X) \otimes \mathbf{Q}$$

and a "localization map" $X \to X_{(0)}$ which is uniquely characterized (up to homotopy) by the property that the induced map

$$\pi_i(X) \to \pi_i(X_{(0)}) \cong \pi_i(X) \otimes \mathbf{Q}$$

is the canonical (algebraic) localization map.

As the following lemma shows, any sphere bundle maps naturally to an associated $S^2_{(0)}$-fibration under a map that induces a rational homotopy equivalence. If the Pontryagin class of the sphere bundle extends over some larger set, the associated $S^2_{(0)}$-fibration does, too, along with the (rational) cohomology class determined by the cotangents along the fiber. Now, it is evident that the Pontryagin class of the bundle considered in expression (4) extends over the excluded $\mathscr{W}_3^n$ (as $H^4(D^4 \times c(SO(3)); \mathbf{Q}) = H^4(\partial\{D^4 \times c(SO(3)); \mathbf{Q}) = H^4(\partial\{D^4 \times c(SO(3))\}; \mathbf{Q}) = 0)$. Thus, by analogy with the simplistic case considered above, we let B be the restriction to $D^4 \times c(SO(3))$ of the associated $S^2_{(0)}$-fibration, and ϕ_B be the natural map from the sphere bundle $(\mathscr{W} \cap \mathscr{W}_3)^{n,s}/S^1_s \to (\mathscr{W} \cap \mathscr{W}_3)^n$ to its associated $S^2_{(0)}$-fibration.

We now state and prove the lemma alluded to above.

Lemma 4. *If $\pi\colon P \to X$ is an $SO(3)$-bundle with first Pontryagin class $\wp \in H^4(X;\mathbf{Q})$ and associated sphere bundle $S = P/S^1$ and $i\colon X \hookrightarrow \overline{X}$ has the property that $\wp$ extends; i.e., there is a $\overline{\wp} \in H^4(\overline{X};\mathbf{Q})$ such that $i^*(\overline{\wp}) = \wp$, then there is a naturally associated $S^2_{(0)}$ fibration $\overline{\pi}\colon \overline{S} \to \overline{X}$ and an inclusion $j\colon S \to \overline{S}$ covering i. Moreover, the class in $H^2(S;\mathbf{Q})$ given by $c_1(P \times_{S^1} \overline{\mathbf{C}}) \to S$ extends naturally to a class i $H^2(\overline{S};\mathbf{Q})$.*

Proof. Throughout the proof, we will use standard notation from algebraic topology [6], where E_G/B_G denotes the universal bundle over the classifying space for the group G, $K(\Pi, n)$ denotes the Eilenberg-Mac Lane space representing the functor $S \to H^n(S, \Pi)$, and $X_{(0)}$ denotes the localization of the space X at 0.

Naturality allows us to restrict attention to the universal case, that is, $X = B_{SO(3)}$, $P = E_{SO(3)}$, and $\overline{X} = \{B_{SO(3)}\}_{(0)}$. In fact, $\{B_{SO(3)}\}_{(0)} = K(\mathbf{Q}, 4)$ where the Pontryagin class $\wp$ is precisely what gives the localization map. We wish to extend the universal sphere bundle $E_{SO(3)}/S^1 \to B_{SO(3)}$ to an $S^2_{(0)}$ bundle over all of $\{B_{SO(3)}\}_{(0)}$. This is done by localizing the above map to obtain $K(\mathbf{Q}, 2) \to K(\mathbf{Q}, 4)$, noting that

$$E_{SO(3)}/S^1 \cong B_{S^1} \cong K(\mathbf{Z}, 2).$$

Moreover, the class to be extended, $c_1(P \times_{S^1} \overline{\mathbf{C}})$, can be easily seen to be the (rational image of the) tautological class of $K(\mathbf{Z}, 2)$ times -2, and this class is canonically extended over $K(\mathbf{Q}, 2)$. (The factor of -2 is due to the fact that $SO(3) \times_{s^1} \overline{\mathbf{C}}$ is the cotangent bundle of S^2.) q.e.d.

Remark. We will use the term "cotangents along the fiber" to denote both the class in $H^2(S, \mathbf{Z})$ given by $c_1(P \times_{s^1} \overline{\mathbf{C}})$ and also its natural extension to $H^2(\overline{S};\mathbf{Q})$ constructed in the proof of the lemma above.

This terminology is originates from the case where $\pi\colon P \to X$ is an $SO(3)$ bundle in the category of $\mathscr{C}^\infty$ manifolds. In this case, the bundle $P \times_{s^1} \overline{\mathbf{C}}$ is indeed the cokernel $T^*S/(\pi/S^1)^*(T^*X)$; it restricts over each S^2 fiber as the cotangent bundle of S^2. q.e.d.

In summary, whereas A is a stratified space extending the $SO(3)_n \times SO(3)_s$ action, B is a rather more formal space. Little effort has been made to extend the $SO(3)_n \times SO(3)_s$ action to all of B. Indeed, the only remnant of the $SO(3)_s$ action on $\widehat{\mathscr{W}}_2^{n,s}$ which extends over B is the rational $S^2_{(0)}$ fibration structure; and the only remnant of the $SO(3)_n$ action is the natural action on the base of the fibration.

3. Contribution from the X side

Computing the contribution from the X side is analogous to the computation of the $\lambda_{k,6}$. We present this latter computation to elucidate the computation of the contribution from the X side.

Indeed, we show that

$$\lambda_{k+1,6}(x_1, \cdots, x_{d(k)-2}) = -2\gamma_k(X)(x_1, \cdots, x_{d(k)-2}, \wp)$$

(see [5]). First, we show

$$D_{x_1} \cap \cdots \cap D_{x_{d(k)-2}} \cap D_e \cap D_e \cap D_e \subset \mathcal{M}_{k+1}(X \# \overline{\mathbf{C}P}^2)$$

must lie in Taubes neighborhoods of the form $\mathcal{M}_k(X)^x \times \mathcal{M}_1(\overline{\mathbf{C}P}^2)^y / SO(3)$. More specifically, connections in this set are obtained by gluing generalized connections on the X side to the reducible connection on the $\overline{\mathbf{C}P}^2$ side.

Following the notation of §1 we let

$$\mathscr{X}^4 = D_{x_1} \cap \cdots \cap D_{x_{d(k)-2}} \subset \mathcal{M}_k(X).$$

This compact, four-dimensional space generically (i.e., for suitably perturbed divisors D_{x_i}) hits the codimension-four first stratum $X \times \mathcal{M}_{k-1}(X)$ in a compact, zero-dimensional set, which we can assume does not contain the connected sum point x. (This is true because of the identification of D_{x_i} on the first stratum

$$D_{\Sigma_i | X \times \mathcal{M}_{k-1}(X)} = \pi_1^*(\Sigma_i) + \pi_2^*(D_{\Sigma_i}),$$

where Σ_i is some geometric representative of x_i which we can assume to be supported away from the base point x. It follows from this description that any point of $\mathscr{X}^4 \cap \{X \times \mathcal{M}_{k-1}(X)\}$ which does not concentrate near some Σ_i must pull back from $D_{x_1} \cap \cdots \cap D_{x_{d(k)-2}} \subset \mathcal{M}_{k-1}(X)$, an intersection which we can arrange to be empty.) Hence, the connections in this cut down moduli space can be safely framed over the connected sum point. As in the $\lambda_{k,4}$ computation in [5], the intersection of based divisors $D_e^y \cap D_e^y \cap D_e^y$ in $\mathcal{M}_1(\overline{\mathbf{C}P}^2)^y$ must cluster around the sphere obtained by basing the reducible connection. The reason for this is that $\mathcal{M}_1(\overline{\mathbf{C}P}^2)^y$ is an eight-dimensional $SO(3)$-space, with the intersection of three divisors contained (generically) in the top stratum. If this top stratum admitted a free $SO(3)$ action, we could compute the intersection in the quotient space, which is five-dimensional; and the intersection of three codimension-2 subspaces of a five-dimensional space is generically empty, so the intersection number would be zero. But the action on the top stratum is not free. There is one orbit with S^1 stabilizer which is responsible for a nonmanifold point in the quotient space. So, if the triple intersection is to be nonempty, it must be a multiple of this orbit. From a local description of this orbit, one can show that this multiplicity is in fact one, and that the additional $\mu(e)$ restrict to this sphere as the Euler class of the cotangent bundle.

Reinterpreted in terms of the connected sum, this is saying that in fact all of $D_{x_1} \cap \cdots \cap D_{x_{d(k)-2}} \cap D_e \cap D_e \cap D_e$ lies in a subset of a single Taubes neighborhood which can be modeled on the sphere bundle $(\mathscr{X}^4)^x/S^1$ over $\mathscr{X}^4$ associated to the basing bundle at x. Moreover, the normal bundle to D_e restricts to the sphere bundle as the bundle of cotangents along the fiber, and the orientation on the above intersection coming from the homology orientation of the moduli space is the natural one $(\mathscr{X}^4)^x/S^1$ inherits from the S^2 bundle structure [5]. (In general, we orient a sphere bundle over an oriented space $\pi: S \to X$ by declaring the volume form to be the class

$$\nu_s = \pi^*(\nu_X) \cup \nu_{\text{fib}},$$

where ν_{fib} is some class which restricts along an S^2 fiber to be the volume form of that fiber.)

Evaluating the Donaldson polynomial now amounts to computing $\mu(e)^3[(\mathscr{X}^4)^x/S^1]$. This computation can be performed after making two observations about sphere bundles.

Lemma 5. *Suppose $\pi: S \to X$ is a 2-sphere bundle associated to some $SO(3)$ bundle P over X with first Pontryagin class $\wp$. Let $c \in H^2(S; \mathbf{Z})$ be the Euler class of the cotangents along the fiber, i.e., $c = c_1(P \times_{S^1} \overline{\mathbf{C}})$. Then, $c^2 = \pi^*(\wp)$.*

Lemma 6. *Suppose $\pi: S \to X^n$ is a 2-sphere bundle over a space with volume form ν_X, c is as above. Then $-2\nu_s = \pi^*(\nu_x) \cup c$, where ν_S is chosen according to the conventions outlined above. In particular, for any $\alpha \in H^n(X)$,*

$$\pi^*(\alpha) \cup c[S] = -2\alpha[X].$$

Putting these facts together with the definition of γ_{k-1}, we see that

$$\begin{aligned}\mu(e)^3[(\mathscr{X}^4)^x/S^1] &= \mu(e) \cup \pi^*(\wp)[(\mathscr{X}^4)^x/S^1] = -2\wp[\mathscr{X}^4] \\ &= -2\gamma_k(X)(x_1, \cdots, x_{d(k)-2}, \wp).\end{aligned}$$

So we turn attention to the proof of Lemma 5. (The second lemma is purely formal.)

We can assume (by universality) that the S^2-bundle is the bundle associated to an $SU(2)$-bundle E; i.e., $S = \mathbf{P}E$, which supports a "tautological bundle" associated to $\tau \in H^2(\mathbf{P}E; \mathbf{Z})$. Under such circumstances, $c = 2\tau$ and $-4\wp(\overline{E}) = c_2(E)$. To see the latter relation, assume that E reduces to S^1, i.e., that there is some line bundle $\mathscr{L}$ with the property that $E \times_{SU(2)} \mathbf{C}^2 = \mathscr{L} \oplus \mathscr{L}^{-1}$. Then, the associated real 3-plane bundle decomposes as

$$\overline{E} \times_{SO(3)} \mathbf{R}^3 = \mathbf{R} \oplus \mathscr{L}^{\otimes 2}.$$

Employing the definition of the Pontryagin class, we see

$$\wp(\overline{E}) = -c_2(\overline{E} \otimes \mathbf{C}) = -c_2(\mathscr{L}^{\otimes 2} \oplus \mathscr{L}^{-\otimes 2}) = 4c_1(\mathscr{L})^2 = -4c_2(E).$$

So, $c^2 = (-2\tau)^2 = 4\tau^2 = -4c_2(E) = \wp$. q.e.d.

Next, we proceed to the problem at hand. Of course, the dimension counts for $\lambda_{k+2,8}$ no longer ensure that concentration takes place away from the connected sum point; on the contrary, the 10-dimensional $\mathscr{U}_1^{k+1}$ is expected to lie in $\mathscr{M}_{k+1}(X)^x \times S^2/SO(3)$, forming an S^2 bundle $(\mathscr{V}^8)^x \times S^2/SO(3)$ over an eight-dimensional subspace $\mathscr{V}^8 \subset \mathscr{M}_{k+1}(X)$ (this notation is explained in §1). This eight-dimensional set is expected to hit the codimension-four first stratum of $\mathscr{M}_{k+1}(X)$ in a set of dimension 4. This is saying precisely that one expects concentration to occur arbitrarily close to $x \in X$; and this is the reason for adding the cap B.

Recall that $\mathscr{U}_1^{k+1}$ is augmented by the addition of caps to localize to an $S^2_{(0)}$-fibration over the compact space $\mathscr{V}^8 \cup \{(\chi^0)^x \times \mathscr{W}_3^n/SO(3)\}$, where the second set is glued onto the first using the natural identification of the end of $\mathscr{V}^8$ with $(\chi^0)^x \times \mathscr{W}^n/SO(3)$. But this base space admits a natural identification with

$$\mathscr{X}^8 = D_{x_1} \cap \cdots \cap D_{x_{d(k)}} \subset \mathscr{M}_{k+1}(X).$$

Thus, applying (the rational analogues of) Lemmas 5 and 6, we see that

$$\mu(e)^5[\mathscr{U}_1^{k+1} \cup \mathscr{B}] = -2\wp \cup \wp[\chi^8],$$

where $\wp$ is the Pontryagin class determining the $S^2_{(0)}$-fibration extending the S^2-bundle $(\mathscr{V}^8)^x/S^1$ over $\mathscr{V}^8$. Indeed, since the inclusion map $\mathscr{V}^8 \hookrightarrow \mathscr{X}^8$ induces an isomorphism on H^4 and $\wp$ restricts to $\mathscr{V}^8$ as the Pontryagin class of the base bundle, we can interpret the above equation as expressing the relation

$$\mu(e)^5[\mathscr{U}_1^{k+1} \cup \mathscr{B}] = -2\gamma_{k-1}(X)(x_1, \cdots, x_{d(k-1)}, \wp, \wp). \tag{5}$$

It should be stressed that the class $\wp$ appearing above is not the restriction of the Pontryagin class of an $SO(3)$ bundle over $\mathscr{M}_{k=1}(X)$. Over $\mathscr{V}^8$, it is the restriction of the Pontryagin class of the $SO(3)$-space $(\mathscr{M}_{k+1}(X)\backslash\mathscr{I}(x))^x$, and homotopy theoretic considerations (see for example [5]) allow one to extend the Pontryagin class of this $SO(3)$-space uniquely over all strata of $\mathscr{M}_{k+1}(X)$ with nontrivial background. Thus, the class $\wp \in H^4(\chi^8; \mathbf{Q})$ referred to in the above formula must be the restriction of this class.

Two other technical points raised in the discussion above should be addressed for completeness. The first is the appeal to "rational analogues" of Lemmas 5 and 6. They hold for tautological reasons. Lemma 5 states that the fibration

$$\pi\colon K(\mathbf{Z}, 2) \cong B_{S^1} \to B_{SO(3)}$$

satisfies

$$(-2\iota_{K(\mathbf{Z},2)})^2 = \pi^*(\wp),$$

where $\iota_{K(\mathbf{Z},2)} \in K(\mathbf{Z}, 2)$ is the tautological two-dimensional integral class, and $\wp$ is the universal Pontryagin class. Hence, when the fibration is localized, we should still have

$$(-2\iota_{K(\mathbf{Q},2)})^2 = \pi^*(\wp).$$

The general case of Lemma 5 follows from universality.

The second technical point is the fact that $\mathscr{U}_1^{k+1} \cup B$ is not strictly speaking an $S^2_{(0)}$-fibration. From its construction, it is the union of a geometric S^2 bundle with an $S^2_{(0)}$ fibration. However, due to the compatibility of these two fibration structures on the overlap, the space is rational homotopy equivalent to an $S^2_{(0)}$-fibration.

4. The spherical correction term

With the lemmas on sphere bundles from the previous section and the explicit compactification from the section before, we can readily compute the contribution coming from the neck region, i.e., "spherical correction term" in (3).

In the notation of §1,

$$\begin{aligned}&\mu(e)^5[\mathscr{A} \cup (\mathscr{U}_1^{k+1} \cap \mathscr{U}_2^k) \cup \mathscr{B}] \\ &\quad = \gamma_k(x_1, \cdots, x_{d(k)})\mu(e)^5[\{(A \cup_{\phi_A} \mathscr{W}^{n,s})/S^1_s\} \cup_{\phi_B} B].\end{aligned}$$

But the results of §2 exhibit $\{(A \cup_{\phi_A} \mathscr{W}^{n,s})/S^1_s\} \cup_{\phi_B} B$ as a rational S^2-fibration over $\mathbf{H}P^2/\{\pm 1\}$, a space which will be referred to in this section as Z. Moreover, according to Lemma 4, $\mu(e)$ is the "cotangents along the fiber" of this $S^2_{(0)}$-fibration (see the remark following Lemma 4 for a definition of this terminology), so we can apply (the rational analogues of) Lemmas 5 and 6 to obtain that

$$\mu(e)^5[\{(A \cup_{\phi_A} \mathscr{W}^{n,s})/S^1_s\} \cup_{\phi_B} B] = -2\wp^2[Z],$$

where $\wp$ is the Pontryagin class defining the fibration over Z.

Letting $S^4 \hookrightarrow \mathbf{H}P^2$ denote the fixed point set of the involution, we can deduce from Lemma 3 that $\wp$ restricted to this S^4 is the Pontryagin class of the $SO(3)$-reduction of an $SU(2)$-bundle with c_2 generating $H^4(S^4; \mathbf{Z})$. Thus, letting $q: \mathbf{H}P^2 \to Z$ denote the quotient map, we see that

$$-2\wp^2[Z] = -\wp^2(q_*([\mathbf{H}P^2])) = -16.$$

Thus, we see that

$$\mu(e)^5[\mathscr{A} \cup (\mathscr{U}_1^{k+1} \cap \mathscr{U}_2^k) \cup \mathscr{B}] = -16\gamma_k(x_1, \cdots, x_{d(k)}). \tag{6}$$

5. Contribution from the $\overline{\mathbf{C}P}^2$ side

Calculating the $\overline{\mathbf{C}P}^2$-side contribution amounts to evaluating $\mu(e)^5$ on the fundamental class of the 10-dimensional space formed by capping off

$$\mathscr{N}_7^y = \{D_e \cap D_e \cap D_e \cap \mathscr{M}_2(\overline{\mathbf{C}P}^2)\}^y$$

and multiplying the results by $\gamma_k(X)(x_1, \cdots, x_{d(k)})$. We will refer to the capped-off triple intersection as K^0 for the purpose of this discussion. Note that $K^0 = \mathscr{N}_7^y \cup A$.

This computation is made possible by the $SO(3)$ action on K^0 and the equivariance of the $\mu(e)$ with respect to this action. In much the same way that the equivariance in §3 allows one to restrict attention to the sphere associated to the based S^1-reducible connection in $\mathscr{M}_1(\overline{\mathbf{C}P}^2)$, equivariance in this case requires that one understand the space K^0 (about which rather little is known) only near the points with nontrivial stabilizer (which we will completely describe). Formally, one has the homotopy quotient $q: K^0 \to K^0 \times E_{SO(3)}/SO(3)$, and wishes to calculate the "intersection number"

$$\langle q^*(\mu(e)^5), [K^0] \rangle = \langle \mu(e)^5, q_*([K^0]) \rangle.$$

Letting $L^0 \subset K^0$ denote the union of points with nontrivial stabilizer, the inclusion induces an isomorphism on equivariant cohomology groups

$$H_{10}^{SO(3)}(K^0) \cong H_{10}^{SO(3)}(L^0).$$

This is true because the set L^0 admits a cone bundle neighborhood, and the homotopy quotient of the compliment (which is a free $SO(3)$-space) $(K^0 \backslash L^0) \times E_{SO(3)}/SO(3)$ is homotopy equivalent to the topological quotient $(K^0 \backslash L^0)/SO(3)$, which is a seven-dimensional space. This is saying that the intersection number referred to above can be obtained by restricting to a neighborhood of L^0.

These are three components of L^0. One of these, the critical one, is due to the S^1-reducible connection in $\mathscr{M}_1(\overline{\mathbf{C}P}^2)$; these are the points in L^0 with S^1 stabilizer. We will denote this component by R^0. Another component comes from the strata with trivial background connection. It is not difficult to see that both points of concentration for any class of connections in this set are bounded away from y (they must cluster about a neighborhood of the exceptional curve). Moreover, the equivariant class associated to $\mu(e)$ in this case pulls up from a class on the geometric quotient (just as in the definition for the usual μ classes when the moduli space contains no reducible). The third component is an artifact of the construction of the cap used to compactify the moduli space. It corresponds to the cone point factor in the $c(\mathbf{R}P^7) \times SO(3) \subset A$ introduced in the discussion following Lemma 1. Since it is A/S^1 which is added to K^0, the component of L^0 considered is the central S^2 in $c(\mathbf{R}P^7) \times S^2$. (Recall that $SO(3)_n$ acts on the first factor only, acting freely on the concentric $\mathbf{R}P^7$'s away from the cone point and trivially on the central points.) It is true once again that this equivariant class pulls up from the geometric quotient; indeed it pulls up from the projection to the second factor (and this projection factors through the $SO(3)$ quotient).

It is immediate that the local contributions due to these latter two components are zero. This is true because over each component, the equivariant class determined by $\mu(e)$, hence $\mu(e)^5$, pulls up from the geometric quotient. But H^{10} of the geometric quotient is zero (the space is seven dimensional). Consequently, the $\overline{\mathbf{C}P}^2$-side contribution is entirely due to the local contribution about the S^1-reducible connections in the compactified $\mathscr{M}_2(\overline{\mathbf{C}P}^2)^y$, all of which come from the first stratum. So, we turn our attention to such connections.

Let ∇ be a reducible connection on P_1. ∇ decomposes P_1 as $Q \times_{S^1} SU(2)$, where Q is a principal S^1 bundle whose first Chern class is a square root (in $H^*(\overline{\mathbf{C}P}^2; \mathbf{Z})$) of $-\nu_{\overline{\mathbf{C}P}^2}$. The Q is not uniquely determined by this property;

$$Q \times_{S^1} SU(2) \cong Q^{-1} \times_{S^1} SU(2),$$

or equivalently,

$$c_1(Q)^2 = c_1(Q^{-1})^2.$$

A choice of such a reduction is equivalent to a choice of isomorphism

$$S^1 \cong \mathrm{Stab}(\nabla).$$

We know that $\mathscr{H}^1(\overline{\mathbf{C}P}^2; \mathrm{ad}(\nabla))$, which will be abbreviated $\mathscr{H}^1_\nabla$, is a six-dimensional real vector space with a linear action of $\mathrm{Stab}(\nabla)$. An explicit identification of $\mathrm{Stab}(\nabla)$ with S^1 as above identifies this cohomology vector space with $\mathbf{C}^3$ given the square of the standard right S^1 action.

In general (following the discussion in [5]), a neighborhood in $\mathscr{M}_{k+1}(X)$ of the points in the first stratum consisting of generalized connections with background ∇ is modeled on

$$\mathscr{H}^1_\nabla \times_{\mathrm{Stab}(\nabla)} \{P_k \times [\Lambda^2_+ \times_{SO(3)} c(SO(3))]/SU(2)\}.$$

Here, the $\mathrm{Stab}(\nabla)$ acts on $\mathscr{H}^1$ on the right by pullback, on P_k on the left as gauge transformations; the $SU(2)$ acts diagonally on the right, acting on P_k as bundle transformations and on the $c(SO(3))$ from the standard right action of $SU(2)$ on $SO(3)$. In our situation, when the underlying four-manifold is $\overline{\mathbf{C}P}^2$ and the background is given with reduction to Q, this turns into $\mathbf{C}^3 \times_{S^1} \{Q \times [\Lambda^2_+ \times_{SO(3)} c(SO(3))]/U(1)\}$ (with the right action of $U(1)$ on $c(SO(3))$ given by the square of the standard action, as it is viewed as a one-parameter subgroup of the double-cover, $SU(2)$). Moreover, the "standard orientation" for the above agrees with the canonical orientation the above space inherits from the specified actions. It will be convenient to reexpress this space as a right quotient; i.e., to write it as

$$\mathbf{C}^3 \times \{Q \times [\Lambda^2_+ \times_{SO(3)} c(SO(3))]/U(1)\}/S^1 \tag{7}$$

where now the S^1 acts on the right on $Q \times [\Lambda^2_+ \times_{SO(3)} c(SO(3))]/U(1)$ by the inverse of the left action on Q.

We actually want a model for the based moduli space near the singular stratum; that is, we are interested in looking at the preimage of the above neighborhood in $\mathscr{M}_2(\overline{\mathbf{C}P}^2)^y$. Strictly speaking, this makes sense only when the connection concentrates away from y, in which case the data can be seen to be given by

$$\mathbf{C}^3 \times \{Q \times [\Lambda_+^2 \times_{SO(3)} c(SO(3))]/U(1)\} \times_{S^1} SU(2),$$

a space which will be referred to as the "framed gluing bundle away from y", $\mathscr{GB}^0|_{\overline{\mathbf{C}P}^2}\backslash\{y\}$. Here, the final $SU(2)$ should be thought of as $(P_1)_y$, with the left action of S^1 coming from $S^1 \cong \mathrm{Stab}(\nabla) \hookrightarrow \mathscr{G}_1$.

It is a rather straightforward matter to extend the bundle $\mathscr{GB}^0|_{\overline{\mathbf{C}P}^2\backslash\{y\}}$ over the omitted point y. To see how, first consider a neighborhood of the reducible stratum of $\mathscr{M}_2(\overline{\mathbf{C}P}^2)$ as described in expression (7). This bundle can be trivialized over a neighborhood of y via a Taubes parameterization, $(\mathbf{C}^3 \times_{S^1} SO(3)) \times (\widehat{\mathscr{W}_1^s})/SO(3)$ (with the $SO(3)$ acting on $\widehat{\mathscr{W}_1^s}$ as $SO(3)_s$), using notation from §2. The above trivialization lifts to an $SO(3)$-equivalent trivialization

$$(\mathbf{C}^3 \times_{s^1} SO(3)) \times (\mathscr{W}_1^{n,s})/SO(3) \hookrightarrow \mathscr{GB}^0|_{\overline{\mathbf{C}P}^2\backslash\{y\}}$$

of the latter bundle over a punctured neighborhood of y, with the $SO(3)$ action on the first space coming from the $SO(3)_n$ action on the second factor. By the definition of $\widehat{\mathscr{W}_1^{n,s}}$, this parametrization extends to a trivialization

(8)

$$(\mathbf{C}^3 \times_{s^1} SO(3)) \times (\widehat{\mathscr{W}_1^{n,s}})/SO(3) \hookrightarrow \mathbf{C}^3 \times \{Q \times [\Lambda_+^2 \times_{SO(3)} c(SO(3))]/U(1)\} \times_{s^1} SU(2)$$

of the latter bundle over y. Thus, the "framed gluing bundle" $\mathscr{GB}^0$ defined by

$$\mathbf{C}^3 \times \{Q \times [\lambda_+^2 \times_{SO(3)} c(SO(3))]/U(1)\} \times_{s^1} SU(2),$$

provides a natural extension of the neighborhood of the reducible stratum in $\mathscr{M}_2(\overline{\mathbf{C}P}^2)^y$.

To see how this extension fits in with the constructions from §2, we must turn our attention to the divisors used to cut down the space. Following the discussion in [5], we know that $\mu(e)$ restricted to $\mathscr{GB}^0 \cap \mathscr{M}_2(\overline{\mathbf{C}P}^2)^y$ is represented by the divisor $-\Delta^0 \cup +\mathscr{GB}^0|_\Sigma$, where

$$\Delta^0 = \mathbf{C}^2 \times \{Q \times [\Lambda_+^2 \times_{SO(3)} c(SO(3))]/U(1)\} \times_{S^1} SU(2)$$

is the subbundle of the gluing bundle corresponding to the vanishing set of the Dirac operator coupled to ∇, and Σ denotes the embedded sphere whose homology class is e (i.e., it is the exceptional curve in the blowup). Intersecting three of these divisors, we can restrict the parameterization of (8) to get a parameterization

$$\widehat{\mathscr{W}_1^{n,s}}/S^1_s \hookrightarrow \bigcap^3(-\Delta^0 \cup +\mathscr{GB}^0|_\Sigma),$$

with the divisor restricted to this subset representing the cohomology class coming from this S^1 action.

In summary, we have the following proposition.

Proposition 1. *The space $\mathscr{GB}^0$ endowed with the $SO(3)$-equivariant divisor $-\Delta^0 \cup +\mathscr{GB}^0|_\Sigma$, has the property that intersecting three copies of this divisor models a neighborhood of R^0 in the capped off space, K^0, and the divisor restricted to this triple intersection represents $\mu(e)$.*

Thus, dropping the coefficient of $\gamma_k(X)(x_1, \cdots, x_{d(k)})$, the contribution from the $\overline{CP}^2$-side is given by the signed intersection number

$$\bigcap^8(-\Delta^0 + \mathscr{GB}^0|_\Sigma) = \bigcap^8\Delta^0 - 8\bigcap^7\Delta^0|_\Sigma + 28\bigcap^6\Delta^0|_{\Sigma\cdot\Sigma}. \tag{9}$$

Assume for the moment that the principal $SO(3)$-bundle Λ^2_+ can be lifted to a principal $SU(2)$-bundle $\widetilde{\Lambda^2_+}$ (i.e., its w_2 vanishes). Then, the bundle $Q\times[\Lambda^2_+ \times_{SO(3)} c(SO(e))]/U(1)$ admits an S^1-equivariant fiberwise double-cover by the bundle $Q\times[\widetilde{\Lambda^2_+} \times_{SU(2)} c(SU(2))]/U(1)$. This space in turn is orientation-*reversing* S^1-equivariantly identifiable with the associated complex vector bundle $\mathscr{L}_{Q^{-1}} \otimes \Lambda^2_{+,C}$ (see the discussion following Lemma 1), where $\Lambda^2_{+,\mathbf{C}} = \widetilde{\Lambda^2_+} \times_{SU(s)} \mathbf{C}^2$. Here the right S^1 action is obtained by identifying S^1 with the unit norm complex numbers, and letting them act canonically on the vector bundle. Moreover, the right S^1-space $\mathbf{C}$ endowed with the square of the standard action is also equivariantly covered twice by $\mathbf{C}$ given the standard S^1 action. Thus, the entire (unframed) gluing bundle $\mathbf{C}^3\times\{Q\times[\Lambda^2_+\times_{SO(3)} c(SO(3))]/U(1)\}/S^1$ is the image of a projective cone bundle $-\mathbf{C}^3 \oplus (\mathscr{L}_{Q^{-1}} \otimes \Lambda^2_{+,\mathbf{C}})S^1$ under a map of degree 8 (before dividing out by the S^1 action, the map is of degree 16).

Interpreting this discussion from the framed point of view, we obtain the following proposition.

Proposition 2. *Suppose N is a four-manifold given with a lift of the principal $SO(3)$ bundle Λ^2_+ to an $SU(2)$ bundle $\widetilde{\Lambda^2_+}$ and some principal*

S^1 bundle Q. Then the bundle

$$\mathbf{C}^3 \times \{Q \times [\Lambda^2_+ \times_{SO(3)} c(SO(3))]/U(1)\} \times_{S^1} SU(2)$$

is covered by a fiberwise degree 16 map from the bundle

$$-\mathbf{C}^3 \oplus (\mathscr{L}_{Q^{-1}} \otimes \Lambda^2_{+,\mathbf{C}}) \times_{s^1} SU(2),$$

a bundle which will be referred to as $\widetilde{\mathscr{GB}}^0$. Moreover, the divisor defined by the subbundle

$$\mathbf{C}^2 \times \{Q \times [\Lambda^2_+ \times_{SO(3)} c(SO(3))]/U(1)\} \times_{s^1} SU(2)$$

lifts back to $-2\{\mathbf{C}^2 \oplus (\mathscr{L}_{Q^{-1}} \otimes \Lambda^2_{+,\mathbf{C}}) \times_{S^1} SU(2)\}$.

Remark. The multiplicity is due to the fact that the degree of the covering restricted to

$$\mathbf{C}^2 \times \{Q \times [\Lambda^2_+ \times_{SO(3)} c(SO(3))]/U(1)\} \times_{S^1} SU(2)$$

is 8. q.e.d.

If $\overline{\mathbf{C}P}^2$ satisfied the hypotheses of Proposition 2, calculating the contribution from the $\overline{\mathbf{C}P}^2$-side would amount to calculating intersection numbers in $\widetilde{\mathscr{GB}}^0$ which, as we shall soon see, is quite tractable using the technology of Chern classes. However, $\overline{\mathbf{C}P}^2$ fails to satisfy these hypotheses since $\widetilde{\Lambda^2_+}$ (a *Spin* structure) exists precisely when the intersection form of the relevant manifold is even. More generally, the unavoidability of the reducible connection (the condition $b^2_+ = 0$) is inherently incompatible with the existence of a *Spin* structure (by Donaldson's diagonalization theorem [2]). This turns out not to be a serious problem, though. Computing as if $\mathscr{GB}^0$ did indeed lift to $\widetilde{\mathscr{GB}}^0$ as above, one derives a formula for the intersection of divisors involving characteristic numbers of Q and Λ^2_+, and then employing a method used in Yang's thesis, one can show that this characteristic class formula remains valid in the non-*Spin* situation.

So, for the time being we will simply feign the existence of $\widetilde{\mathscr{GB}}^0$. Then, combining (9) with Proposition 2, we see that the contribution sought is

$$\begin{aligned}
&\bigcap^8 \Delta^0 - 8\bigcap^7 \Delta^0|_\Sigma + 28\bigcap^6 \Delta^0|_{\Sigma\cdot\Sigma} \\
(10)\qquad &= \frac{1}{16}\left(2^8\bigcap^8 \tilde{\Delta}^0 - 8\cdot 2^7\bigcap^7 \tilde{\Delta}^0|_\Sigma + 28\cdot 2^6\bigcap^6 \tilde{\Delta}^0|_{\Sigma\cdot\Sigma}\right) \\
&= 16\bigcap^8 \tilde{\Delta}^0 - 64\bigcap^7 \tilde{\Delta}^0|\Sigma + 112\bigcap^6 \tilde{\Delta}^0|_{\Sigma\cdot\Sigma}.
\end{aligned}$$

The computations now will rely on the following lemma on projective bundles.

Lemma 7. *Let $\mathbf{C}\oplus V$ be a vector bundle which admits a trivial line bundle summand as indicated. The submanifold $\mathbf{P}(V)\hookrightarrow\mathbf{P}(\mathbf{C}\oplus V)$ induces a homology class whose Poincaré dual $[\mathbf{P}(V)]^*\in H^2(\mathbf{P}(\mathscr{L}\oplus V))$ is the inverse of the tautological class, τ^*. In particular, in the $(n+1)$-plane bundle $\mathbf{C}\oplus V$,*

$$(\mathbf{P}(V)^*)^{n+1}+\sum_{i=1}^{n}c_i(V)([\mathbf{P}(V)]^*)^{n-i}=0.$$

Moreover, a lemma is needed to handle spaces of the form of $\widetilde{\mathscr{GB}}^0$.

Lemma 8. *Let*

$$\Delta=V\times_{S^1}SU(2)\hookrightarrow(\mathbf{C}\oplus V)\times_{S^1}SU(2)$$

be an $SU(2)$-equivariant divisor of an N-dimensional space (which is the total space of a $\mathbf{C}^{n+1}\times_{S^1}SU(2)$-bundle over Z). Then Δ admits $SU(2)$-equivariant perturbations $\Delta_1,\cdots,\Delta_N$ which intersect transversally such that

$$\#(\Delta_1\cap\cdots\cap\Delta_N)=-(\mathbf{P}(V)^*)^{N-2}[\mathbf{P}(\mathbf{C}\oplus V)].$$

Proof. By generically perturbing the V's, we get $N-2$ vector subbundles $\{V_i\}$ of $\mathbf{C}\oplus V$ whose intersection (in this vector bundle) consists of $(\mathbf{P}(V)^*)^{N-2}[\mathbf{P}(\mathbf{C}\oplus V)]$ many complex lines which are linear subspaces of certain fibers of $\mathbf{C}\oplus V$. Thus, the intersection of the associated divisors $\bigcap_{i=1}^{N-2}(V_i\times_{S^1}SU(2))$ picks out that many spaces of the form $\mathbf{C}\times_{S^1}SU(2)$. But this latter space is the $c_1=-1$ line bundle over S^2 (see the $\lambda_{k,4}$ calculation from [5]). One more divisor picks out the S^2 "zero sections", and the final divisor contributes the sign -1. q.e.d.

Due to the above two lemmas, the intersection number constituting the first term in (10) can be expressed in terms of the Chern classes of (a complex 5-plane bundle which determines) the projective bundle $-\widetilde{\mathscr{GB}}^0/SU(2)$. These classes can be calculated by the Whitney sum formula [8] to be

$$c_1(\mathbf{C}^3\oplus(\mathscr{L}_Q^{-1}\otimes\Lambda^2_{+,\mathbf{C}}))=-2e^*,$$
$$c_2(\mathbf{C}^2\oplus(\mathscr{L}_Q^{-1}\otimes\Lambda^2_{+,\mathbf{C}}))=c_2(\Lambda^2_{+,\mathbf{C}})+(e^*)^2.$$

Since the base space is $\overline{\mathbf{C}P}^2$, which is four-dimensional, one need not worry about any other Chern classes. Hence,

$$(11)\quad \begin{aligned} 16\bigcap^{8}(\tilde{\Delta}^0) &= 16(\mathbf{P}(\mathbf{C}^2 \oplus (\mathscr{L}_Q^{-1} \otimes \Lambda^2_{+,\mathbf{C}}))^*)^6[\mathbf{P}(\mathbf{C}^3 \oplus (\mathscr{L}_Q^{-1} \otimes \Lambda^2_{+,\mathbf{C}}))] \\ &= 16\tau^{*6}[\mathbf{P}(\mathbf{C}^3 \oplus (\mathbf{L}_Q^{-1} \otimes \Lambda^2_{+,\mathbf{C}}))]; \end{aligned}$$

but using

$$\tau^{*5} + c_1\tau^{*4} + c_2\tau^{*3} = 0,$$

we see

$$\tau^{*6} = ((c_1)^2 - (c_2))\tau^{*4} = (3(e^*)^2 - c_2(\Lambda^2_{+,\mathbf{C}}))\tau^{*4}.$$

Substituting

$$(e^*)^2 = -\nu_{\overline{\mathbf{C}P}^2}$$

and

$$c_2(\Lambda^2_{+,\mathbf{C}}(\overline{\mathbf{C}P}^2)) = -\tfrac{1}{4}(3\sigma(\overline{\mathbf{C}P}^2) + 2\chi(\overline{\mathbf{C}P}^2))\nu_{\overline{\mathbf{C}P}^2} = -\tfrac{3}{4}\nu_{\overline{\mathbf{C}P}^2}$$

into (11), we obtain

$$(12)\quad 16\bigcap^{8}(\tilde{\Delta}^0) = 16(-3 + 3/4) = -36.$$

The next term in (10), $-64\bigcap^{7}\tilde{\Delta}^0|_\Sigma$, is easier to calculate, as the base is only two-dimensional, so one needs to keep track of only one characteristic class, c_1. By the same arguments as in the previous case, using the relations

$$\tau^{*5} = -c_1\tau^{*4} = -2e^*\tau^{*4},$$

and $e^*[\Sigma] = +1$, we see

$$(13)\quad -64\bigcap^{6}\tilde{\Delta}^0|_\Sigma = -64\tau^{*5}[\mathbf{P}(\mathbf{C}^3 \oplus (\mathscr{L}_Q^{-1} \otimes \Lambda^2_{+,\mathbf{C}})|_\Sigma)] = 128e^*[\Sigma] = 128.$$

The last term $112\bigcap^{6}\Delta^0|_{\Sigma\cdot\Sigma}$ is in fact the easiest to compute; $\Sigma\cdot\Sigma = -1$, so the contribution is -112.

Adding these three terms together (i.e., substituting into (10)) and multiplying by $\gamma_k(X)(x_1, \cdots, x_{d(k)})$, we see that the total contribution from the $\overline{\mathbf{C}P}^2$-side is

$$(14)\quad \mu(e)^5[\mathscr{A} \cup \mathscr{U}_2^k] = -20\gamma_k(X)(x_1, \cdots, x_{d(k)}).$$

We now turn our attention to justifying the use of $\mathscr{GB}^0$. Following Yang [12], one observes that there is always another four-manifold M and a map of nonzero degree $f\colon M \to \overline{\mathbf{C}P}^2$ such that $f^*(\Lambda^2_+)$ indeed does admit a lift to an $SU(2)$ bundle, $\widetilde{f^*(\Lambda^2_+)}$. Hence, $f^*(\mathscr{GB}^0)$ admits a lift, so turning our attention to the first term in (9), we get

$$\begin{aligned}
\bigcap^8(\Delta^0) &= \frac{1}{\deg f}\bigcap^8 f^*(\Delta^0)\\
&= \frac{1}{\deg f}16(\mathbf{P}(\mathbf{C}^2 \oplus (\mathscr{L}_{f^*Q^{-1}} \otimes \widetilde{f^*(\Lambda^2_+)}_{+,\mathbf{C}}))^*)^6\\
&\qquad\qquad \cdot [\mathbf{P}(\mathbf{C}^3 \oplus (\mathscr{L}_{f^*Q^{-1}} \otimes \widetilde{f^*(\Lambda^2_+)}_{+,\mathbf{C}}))]\\
&= \frac{1}{\deg f}16\{3c_1(f^*(\mathscr{L}_Q^{-1}))^2 - c_1(\widetilde{f^*(\Lambda^2_+)}_{+,\mathbf{C}})\}[M]\\
&= 16\{3c_1(\mathscr{L}_Q^{-1})^2[\overline{\mathbf{C}P}^2] + (1/4)\wp(f^*(\Lambda_2^+))[M]/(\deg f)\}\\
&= 16\{3c_1(\mathscr{L}_Q^{-1})^2[\overline{\mathbf{C}P}^2] + (1/4)\wp(\Lambda_2^+)[\overline{\mathbf{C}P}^2]\}.
\end{aligned}$$

Thus, (12) remains valid. (Note that this argument is necessary for handling only the first term, as Λ_2^+ lifts when restricted to exceptional curve or the point.)

Obtaining the result announced in (1) is now a matter of plugging the results from (5), (14), and (6) into (3).

6. Computing $\lambda_{k,10}$

We now apply the same techniques to calculate $\lambda_{k,10}$, obtaining the result stated in (2).

Formally, the setting is very similar to the one of the $\lambda_{k,8}$ computation. We cut down by the all the divisors coming from the X side and three coming from the $\overline{\mathbf{C}P}^2$ side. The same dimension counts as earlier indicate that this set is covered by two open sets, $\mathscr{U}_1^{k+1}$, the subset of connections whose energy distributions as nearly 1 on $\overline{\mathbf{C}P}^2$ and small on T_2 (and hence nearly $k+1$ on the rest of the connected sum), and $\mathscr{U}_2^k$, the subset of connections whose energy distributes as nearly k on X and small on T_1 (and hence nearly 2 on the rest of the connected sum). Similarly, the overlap is parameterized as a double gluing; just as earlier,

$$\mathscr{U}_1^{k+1} \cap \mathscr{U}_2^k \subset \mathscr{M}_k(X)^x \times \mathscr{M}_1(S^4)^{n,s} \times \mathscr{M}_1(\overline{\mathbf{C}P}^2)^y/SO(3) \times SO(3).$$

Two caps (which will also be referred to as $\mathscr{A}$ and $\mathscr{B}$) should be added to perform the calculations in a compact space. Formally, we have

$$\begin{aligned}
\mu(e)^7[\mathscr{U}_1^{k+1} \cup \mathscr{U}_2^k] &= \mu(e)^7[\mathscr{U}_1^{k+1} \cup \mathscr{B}] + \mu(e)^7[\mathscr{A} \cup \mathscr{U}_2^k]\\
&\quad - \mu(e)^7[\mathscr{A} \cup (\mathscr{U}_1^{k+1} \cap \mathscr{U}_2^k) \cup \mathscr{B}]
\end{aligned} \tag{15}$$

(compare with (3)). These caps will also be constructed from the caps to the doubly-based moduli space of the sphere constructed in §2, in a way we will outline presently.

Letting

$$\mathscr{X}^4 = D_{x_1} \cap \cdots \cap D_{d(k)-2} \subset \mathscr{M}_k(X),$$

we can model the intersection of $\mathscr{U}_2^k$ with $\mathscr{U}_1^{k+1}$ as follows:

$$\begin{aligned} \mathscr{U}_1^{k+1} \cap \mathscr{U}_2^k &\cong (\mathscr{X}^4)^x \times \mathscr{W}^{n,s} \times S^2/SO(3)_n \times SO(3)_s \\ &= (\mathscr{X}^4)^x \times (\mathscr{W}^{n,s}/S_3^1)/SO(3)_n. \end{aligned} \tag{16}$$

Note that the four-dimensional space $\mathscr{X}^4$ can be expected to hit the first stratum of $\mathscr{M}_{k-1}(X)$ in finitely many points, generalized connections which we can arrange to concentrate away from x (as in the $\lambda_{k,6}$ calculation in §3). Thus, it makes sense to frame $\mathscr{X}^4$, obtaining an $SO(3)$-space $(\mathscr{X}^4)^x$ whose associated Pontryagin class is (by definition) the restriction to $\mathscr{X}^4$ of the class $\wp \in H^4(\mathscr{M}_k(X); \mathbf{Q})$. The analogue of $\mathscr{X}^4$ for the $\lambda_{k,8}$ computation was the zero-dimensional $\mathscr{X}^0$. Thus, whereas in the $\lambda_{k,8}$ computation the space to be compactified was a finite disjoint union of S^4 moduli space data, now the space to be compactified is a bundle of S^4 moduli space data over another space. This is the primary conceptual difference between the two calculations.

Since the cap A used to compactify $\mathscr{W}^{n,s}$ is $SO(3)_n \times SO(3)_s$ equivariant, it makes sense to define $\mathscr{A}$ by the expression $(\mathscr{X}^4)^x \times (A/S_s^1)/SO(3)_n$, attaching it to $(\mathscr{U}_1^{k+1} \cap \mathscr{U}_2^k)$ via the natural

$$\mathscr{A} \cup (\mathscr{U}_1^{k+1} \cap \mathscr{U}_2^k) \cong (\mathscr{X}^4)^x \times (\widehat{\mathscr{W}_2^{n,s}}/S_s^1)/SO(3)_n$$

(compare with equation (16)). This space is an S^2-bundle over $(\mathscr{X}^4)^x \times (\widehat{\mathscr{W}_2^{n,s}}/SO(3)_s)/SO(3)_n$, which in turn is a fiber bundle over $\mathscr{X}^4$ with fiber $\widehat{\mathscr{W}_2^{n,s}}/SO(3)_s$ and structure group $SO(3)_n$. Lemma 1 identifies this fiber as a punctured $Z = \mathbf{H}P^2/\{\pm 1\}$.

The discussion following the proof of the aforementioned lemma also identifies the $SO(3)_n$ action on Z in a concrete way. Consider the right action of $\mathbf{H}^*$ on H^3 given by $q \times (a, b, c) \to (a, b, q^{-1}c)$. This linear action descends to an action on $\mathbf{H}P^2$. Indeed, it induces an $SO(3)$ action on $\mathbf{Z}$, which models the $SO(3)_n$ action on $\widehat{\mathscr{W}_2^{n,s}}/SO(3)_s \subset Z$.

Now, the construction of $\mathscr{B}$ is analogous to our earlier construction of B. We argue that the Pontryagin class associated to the S^2-bundle

$$(\mathscr{X}^4)^x \times (\mathscr{W}^{n,s}/S_s^1)/SO(3)_n \to (\mathscr{X}^4)^x \times \mathscr{W}^n/SO(3)_n$$

extends over all of $(\mathscr{X}^4)^x \times (\mathscr{W} \cup \mathscr{W}_3)^n/SO(3)$. This is true because the inclusion map

$$(\mathscr{X}^4)^x \times \mathscr{W}^n/SO(3)_n \subset (\mathscr{X}^4)^x \times (\mathscr{W} \cup \mathscr{W}_3)^n/SO(3)$$

can be viewed as an inclusion of fiber bundles over $\mathscr{X}^4$ restricting along the fibers (up to homotopy) to the inclusion $\partial D^4 \times c(SO(3)) \subset D^4 \times c(SO(3))$. But this inclusion of fibers induces an isomorphism up through H^4, so it follows that the inclusion of fiber bundles must induce an isomorphism through this range, too (by a Leray spectral sequence, for example). In particular, the Pontryagin class considered above must extend as claimed. Hence, we can define B as the associated $S^2_{(0)}$ fibration restricted to $(\mathscr{X}^4)^x \times \mathscr{W}_3^n/SO(3)$.

Now, we can proceed to the X side computation. Let

$$\mathscr{X}^{12} = D_{x_1} \cap \cdots \cap D_{d(k)-2} \subset \mathscr{M}_{k+1}(X).$$

$\mathscr{U}_1^{k+1}$ is an S^2 bundle over the open subset $\mathscr{V}^{12} \subset \mathscr{X}^{12}$ consisting of connections not concentrated too badly near x. A neighborhood of infinity of $\mathscr{V}^{12}$ is modelled on $(\mathscr{X}^4)^x \times \mathscr{W}^n/SO(3)$. Indeed, $(\mathscr{U}_1^{k+1}/SO(3)) \cup ((\mathscr{X}^4)^x \times \mathscr{W}_3^n/SO(3))$ gives all of $\mathscr{X}^{12}$. A direct application of the sphere bundle lemmas shows that

$$\begin{aligned} \mu(e)^7[\mathscr{U}_1^{k+1} \cup \mathscr{B}] &= -2\wp \cup \wp \cup \wp[\mathscr{X}^{12}] \\ &= -2\gamma_{k+1}(X)(x_1, \cdots, x_{d(k)-2}, \wp, \wp, \wp). \end{aligned} \tag{17}$$

We need a bit more information to perform the correction term calculation for $\lambda_{k,10}$. The correction term space is an $S^2_{(0)}$-bundle over the space $(\mathscr{X}^4)^x \times Z/SO(3)_n$, which (up to the quotient by an involution) looks like the projectivization of a quaternionic three-plane bundle. The next goal is to find out which element of $H^4((\mathscr{X}^4)^x \times Z/SO(3); \mathbf{Q})$ determines this fibration.

We begin with a lemma about $\mathbf{H}P^2$ bundles.

Lemma 9. *Let $V = \mathbf{H} \oplus \mathbf{H} \oplus \mathscr{L}$ be a quaternionic three-plane bundle over some space X admitting two trivial summands as shown, so that the only nonvanishing integral characteristic class is $c_2(V) = c_2(\mathscr{L})$, which we will denote simply by c. Let $\mathbf{P}(V)$ denote the quaternionic projectivization. Then there is a "quaternionic hyperplane class" $h \in H^4(\mathbf{P}(V))$ and an isomorphism*

$$H^*(\mathbf{P}(V)) \cong H^*(X)[h]/(h^3 + ch^2).$$

This lemma follows from the Leray-Hirsch theorem and the quaternionic analogues of the Grothendieck definition of characteristic classes.

This lemma states in particular that any element $x \in H^4(\mathbf{P}(V))$ can be uniquely written as $x = \pi^*(y) + ah$, where $y \in H^4(X)$ and $a \in \mathbf{Z}$. The integer a is determined by restricting x to a fiber, and using an isomorphism $H^4(\mathbf{H}P^2; \mathbf{Z}) \cong \mathbf{Z}$ which takes the restriction of k to 1.

One determines y with the help of a natural section of π, which we call $\sigma: X \to \mathbf{P}(V)$, given by taking $\sigma(x)$ to the quaternionic subline of V_x determined by $\mathscr{L}_x$. More precisely, we have the following lemma.

Lemma 10. *Let V be a quaternionic three-plane as in the previous lemma, and $x \in H^4(\mathbf{P}(V))$ be some cohomology class. Define $a \in \mathbf{Z}$ by the relation*

$$x|_{\mathbf{H}P^2} = ah|_{\mathbf{H}P^2}.$$

Then

$$x = \pi^*(\sigma^* x + ac) + ah.$$

Proof. The key fact to note here is that $\sigma^*(h) = -c$. This is true because the tautological bundle (which is $-h$) restricts to the section $\sigma(X)$ as $\mathscr{L}$. q.e.d.

We now have all the necessary ingredients to determine the correction term. The term is given by

$$\mu(e)^7[\mathscr{A} \cup \{(\mathscr{X}^4)^x \times (\mathscr{W}^{n,s}/S^1_s)/SO(3)\} \cup \mathscr{B}],$$

where $\mu(e)$ is the seventh power of the cotangents along the fiber of a rational S^2 fibration. Letting Π denote the Pontryagin class of this fibration, we have that this must be given by $-2\Pi^3[(\mathscr{X}^4)^x \times Z/SO(3)]$, using the sphere bundle lemmas and our explicit description of the base of the $S^2_{(0)}$ fibration.

The bundle $(\mathscr{X}^4)^x \times Z/SO(3)$ behaves like the projective three-plane bundles studied above. In particular, it admits a rational hyperplane class H satisfying

$$(-4H)^3 + \wp(-4H)^2 = 0.$$

Since Π agrees with $-4H$ along the Z fiber (this is essentially the content of Lemma 3), we have that

$$\Pi = \pi^*(\sigma^*(\Pi) + \wp) - 4H.$$

But $\sigma^*(\Pi) = 0$, since the $SO(3)_n$ action is trivial on the "central points" in the cap A (see the discussion following the proof of Lemma 2).

Consequently,

$$
\begin{aligned}
-2\Pi^3[(\mathscr{X}^4)^x \times Z/SO(3)] &= -2(\pi^*(\wp) - 4H)^3[(\mathscr{X}^4)^x \times Z/SO(3)] \\
&= -2\{(-4H)^3 + 3(-4H)^2\pi^*(\wp)\} \\
&\quad \cdot [(\mathscr{X}^4)^x \times Z/SO(3)] \\
&= -2\{-(-4H)^2\pi^*(\wp) + 3(-4H)^2\pi^*(\wp)\} \\
&\quad \cdot [(\mathscr{X}^4)^x \times Z/SO(3)] \\
&= -32\wp[\mathscr{X}^4].
\end{aligned}
\tag{18}
$$

Here we use the fact that $(-4H)^2$ restricts along the fiber to $8\nu_Z$. Thus, we have the correction term.

We turn our attention now to the $\overline{\mathbf{C}P}^2$-side contribution. In analogy with the correction term space, our space to be capped off is contained in $(\mathscr{X}^4)^x \times \mathscr{M}_2(\mathbf{C}P^2)^y/SO(3)$; it is the subset consisting of the triple intersection of divisors D_e. Once more, it is the nonfreeness of the $SO(3)$ action on $\mathscr{M}_2(\mathbf{C}P^2)^y$ which allows the answer to be nonzero. By the arguments presented in §5, we restrict our attention to a neighborhood of the S^1-reducible stratum. Indeed, we have the following analogue of Proposition 1.

Proposition 3. *The space $(\mathscr{X}^4)^x \times \mathscr{GB}^0/SO(3)$, endowed with the divisor $(\mathscr{X}^4)^x \times (-\Delta^0 \cup \mathscr{GB}^0|_\Sigma)/SO(3))$ models the capped off neighborhood of our reducible stratum in the sense that the triple intersection of the divisor gives a neighborhood of $\mathscr{A} \cup \mathscr{U}_2^k$, with the divisor restricting over this set representing $\mu(e)$.*

Thus, calculating the $\overline{\mathbf{C}P}^2$-side amounts to computing

$$
\begin{aligned}
&\bigcap^{10}(\mathscr{X}^4)^x \times (-\Delta^0 + \mathscr{GB}^0|_\Sigma)/SO(3) \\
&\quad = \bigcap^{10}(\mathscr{X}^4)^x \times \Delta^0/SO(3) - 10\bigcap^{9}(\mathscr{X}^4)^x \times \Delta^0|_\Sigma/SO(3) \\
&\qquad + \binom{10}{2}\bigcap^{8}(\mathscr{X}^4)^x \times \Delta^0|_{\Sigma\cdot\Sigma}/SO(3).
\end{aligned}
$$

The propositions of §5 suggest that $\bigcap^7\Delta^0$ picks out a certain number of spheres S^2, with the remaining Δ^0 restricting along these spheres as the cotangent bundle. This number is computed using the characteristic class data for the gluing bundle. More precisely, (12) translates into the statement that $\bigcap^7\Delta^0 = 18S^2$. Similarly, (13) states that $\bigcap^6\Delta^0|_\Sigma = 8S^2$, and the computation for the third term suggests that $\bigcap^5\Delta^0|_\Sigma = 2S^2$. Substituting these relations into (19), we get that

$$\bigcap^{10}(\mathscr{X}^4)^x \times (-\Delta^0 + \mathscr{G}\mathscr{B}^0|_\Sigma)/SO(3) = 28\mu(e)^3[(\mathscr{X}^4)^x \times S^2/SO(3)].$$

We are again in a familiar situation; $\mu(e)$ restricts to the S^2 bundle $(\mathscr{X}^4)^x \times S^2/SO(3)$ as the cotangents along the fiber, so

$$\bigcap^{10}(\mathscr{X}^4)^x \times (-\Delta^0 + \mathscr{G}\mathscr{B}^0|_\Sigma)/SO(3) = -56\wp[\mathscr{X}^4], \tag{19}$$

where $\wp$ is the Pontryagin class of $(\mathscr{X}^4)^x/\mathscr{X}^4$, i.e., the restriction of the class from $\mathscr{M}_k(X)$.

Putting together the results of (17), (18), and (19) in (15), we get the result stated in (2).

7. Conclusion

It should be pointed out first that our arguments depend on only the homotopy type of $\overline{\mathbf{C}P}^2$. Although the computations did use the moduli spaces for $\overline{\mathbf{C}P}^2$, which *a priori* might contain some delicate diffeomorphism information, our computations localized to the part of the moduli space of $\overline{\mathbf{C}P}^2$ near the reducible connections, which, as explained in §5, depends only on the intersection form of $\overline{\mathbf{C}P}^2$. Thus, (1) and (2) hold for any manifold homotopy equivalent to $\overline{\mathbf{C}P}^2$.

Indeed, if N is any negative-definite, simply-connected differentiable four-manifold, and e is a generator of $H_2(N;\mathbf{Z})$, then we will show that

$$\begin{aligned}\gamma_{k+2}(X\#N)(x_1, \cdots, x_{d(k)}, \overbrace{e, \cdots, e}^{8}) = &- 2\gamma_{k+1}(X)(x_1, \cdots, x_{d(k)}, \wp, \wp) \\ &- 4\gamma_k(X)(x_1, \cdots, x_{d(k)}),\end{aligned}$$

and

$$\begin{aligned}\gamma_{k+2}(x_1, \cdots, x_{d(k)-2}, \overbrace{e, \cdots, e}^{10}) = &- 2\gamma_{k+1}(X)(x_1, \cdots, x_{d(k)-2}, \wp, \wp, \wp) \\ &- 24\gamma_K(X)(X_1, \cdots, x_{d(k)-2}, \wp).\end{aligned}$$

If N splits smoothly into a connected sum of projective planes, the result follows immediately from the formulas for $\lambda_{k,0}$, $\lambda_{k,8}$, and $\lambda_{k,10}$. It is, however, not known if any such N must split in this way; the only theorem in this direction is Donaldson's result [2] which states that the intersection form of N is diagonalizable. This fact, together with a slight modification of our earlier arguments, is sufficient to prove the formula as stated.

Let $e, f_1, \cdots, f_m$ be a standard basis for the intersection form. Proceeding as before, we must look at the reducible singularities in the moduli spaces for N. Although $\mathcal{M}_1(N)$ now contains $m+1$ reducible singularities, those associated to e and those associated to $f_1, \cdots, f_m$, the latter reductions do not contribute; $\mu(e)^y$ is zero at all reductions orthogonal to e (see for example [5]). Thus, the only local contribution coming from $\mathcal{M}_1(N)$ is the one corresponding to the reduction e, so that the X side contribution and the correction term are the same as before. However, the N side computation is different for two reasons. On the one hand, the value of the local contribution about the first stratum reducible corresponding to e changes. The part of the contribution computed in (12) actually involves $c_2(\Lambda^2_{+,\mathbf{C}}(N))$ which is given by $(m-3)/4$ when $b_2^-(N) = m+1$. This is the only term which depends on $b_2(N)$, so one can see that the first stratum answer is increased by $4m$. On the other hand, we have new top stratum reducible singularities in $\mathcal{M}_2(N)$. These singularities correspond to the $e \pm f_i$ and $f_i \pm f_j$ (where $i \neq j$). The latter singularities once more do not contribute, as the corresponding cohomology classes are orthogonal to e, but each of the $2m$ singularities $e \pm f_i$ contribute -2 by the usual arguments. (This time, the neighborhoods are modelled on $\mathbf{C}^7 \times_{S^1} SO(3)$, with D_e represented by $-\mathbf{C}^6 \times_{S^1} SO(3)$.) These two effects, the change in the first stratum contribution and the addition of top stratum singularities, cancel fortuitiously, and we obtain the results promised above.

The results in this paper agree with the blowup formulas of Kronheimer and Mrowka [8] which hold when X has "simple type" (and N is $\overline{\mathbf{C}P}^2$). This simple type condition on a four-manifold X posits that

$$\gamma_{k+1}(X)(x_1, \cdots, x_{d(k)}, \wp, \wp) = (-8)^2 \gamma_k/(X)(x_1, \cdots, x_{d(k)}).$$

In this case, the two Donaldson polynomials appearing on the right-hand sides of (1) and (2) are linearly dependent. Thus, the blowup formulas can be written as constants of proportionality between $\lambda_{k,l}$ and $\gamma_{k-l/4}(X)$ or $\gamma_{k-l/4}(X)$ contracted with $\wp$, depending on the parity of $l/2$. These constants are -132 and 1216 for $l = 8$ and 10 respectively.

In calculating $\lambda_{k,i}$ for $i > 10$, one expects to encounter more terms in the answer, corresponding in part to energy crawling across the tube in greater quantities and in part to encountering more reductions in the higher Chern class moduli spaces for $\overline{\mathbf{C}P}^2$. Dealing with these problems requires a good understanding of spaces of higher instantons on S^4 (much as the compactification in §2 relied rather heavily on the explicit understanding of the space of charge one instantons), although pieces of the formula can be calculated without a complete understanding.

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