# CONVEX DECOMPOSITIONS OF REAL PROJECTIVE SURFACES II: ADMISSIBLE DECOMPOSITIONS

SUHYOUNG CHOI

*Dedicated to the memory of Professor D. S. Rim*

**Abstract**

A real projective surface is a differentiable surface with an atlas of charts to real projective plane $\mathbf{RP}^2$ such that transition functions are restrictions of projective automorphisms of $\mathbf{RP}^2$. Let $\Sigma$ be an orientable compact real projective surface with convex boundary and negative Euler characteristic. Then $\Sigma$ uniquely decomposes along mutually disjoint imbedded closed projective geodesics into compact subsurfaces that are maximal annuli, trivial annuli, or maximal purely convex real projective surfaces. This is a positive answer to a question by Thurston and Goldman raised around 1977.

We assume that surfaces in this paper are orientable always. Let $S$ be a real projective surface with convex boundary. We say that $S$ is the *sum* of subsurfaces $S_1, \cdots, S_n$ if $S$ is the union of $S_1, \cdots, S_n$, and if $S_i \cap S_j$ is the union of imbedded closed geodesics disjoint from one another or the empty set whenever $i$ and $j$ are integers satisfying $1 \leq i < j \leq n$ (compare with §3.1 of [14]). If $S$ is the sum of $S_1, \cdots, S_n$, then we say that $S$ *decomposes* into $S_1, \cdots, S_n$ (*along closed geodesics*) and that $\{S_1, \cdots, S_n\}$ is a *decomposition collection* of $S$. (See Appendix B of [16], and [26] for examples of this summation process.) This definition is slightly different from the one by Goldman [14] since we do not have the principal boundary conditions.

Let $D$ be an arbitrary compact simply convex domain in a 2-dimensional sphere $\mathbf{S}^2$ such that there is a segment $\alpha$ and a compact smooth arc $\beta$ with two common endpoints $p$ and $q$ such that the boundary of $D$ $\delta D$ is $\alpha \cup \beta$. The quotient projective surface of $D - \{p, q\}$ by a properly discontinuous and free action of $\langle \vartheta \rangle$ for a hyperbolic or quasi-hyperbolic projective automorphism $\vartheta$ is called a *primitive trivial annulus*. (See [5]

for the definitions of simple convexity and quotient projective surfaces.) It is a compact annulus with convex boundary. One of its boundary components is not geodesic, and the other is geodesic. A *trivial annulus* in $S$ is a primitive trivial annulus $A$ included in $S$ such that the nongeodesic component of $\delta A$ is a component of $\delta S$. (Clearly, a trivial annulus does not include two components of $\delta S$ if the Euler characteristic of $S$ is negative.) For example, given a hyperbolic projective surface $R$ with convex boundary and an imbedded closed geodesic $\alpha$ in the interior $R^0$ of $R$ freely homotopic to a component $\beta$ of $\delta R$, the annulus with boundary $\alpha \cup \beta$ is a trivial annulus. In general, a compact convex projective surface with geodesic boundary can be extended to a compact projective surface with convex boundary. In this case, the difference between the extended surface and the original surface is often given by trivial annuli.

A *maximal annulus* in $S$ is a compact annulus with geodesic boundary that is not a proper subset of a compact annulus with geodesic boundary in $S$.

A *purely convex surface* is a convex compact surface $A$ with negative Euler characteristic that does not include a compact annulus with geodesic boundary freely homotopic to a component of $\delta A$ or include a trivial annulus. A *maximal purely convex surface* in $S$ is a purely convex surface in $S$ that is not a proper subset of a purely convex surface in $S$.

We call trivial annuli, maximal annuli, and maximal purely convex surfaces in $S$ *admissible subsurfaces* in $S$. We put the admissible subsurfaces of $S$ into three different categories: (1) trivial annuli, (2) maximal annuli, (3) maximal purely convex surfaces. If $S$ decomposes into admissible subsurfaces of $S$, then the decomposition collection is said to be *admissible*.

In this paper, we prove the admissible decomposition theorem using the main theorem of our previous paper [5]. We will use the notation and results of the paper [5].

**Admissible Decomposition Theorem.** *Let $\Sigma$ be an orientable compact real projective surface with convex boundary and negative Euler characteristic. Then $\Sigma$ admits a unique admissible decomposition collection.*

As we said in [5], this answers a question of Thurston and Goldman raised in 1977. A similar theorem is true when $\Sigma$ is not orientable [7]. It is also claimed that if $\Sigma$ is closed, then $\Sigma$ decomposes into convex compact surfaces [6] (for some consequences, see [8]).

Let us outline the contents of this paper. In §1 we classify projective automorphisms of $S^2$ according to action and identify hyperbolic, quasi-

hyperbolic, and planar automorphisms. We classify elementary annuli, building blocks of annuli.

We next discuss three important geometric objects: tight curves, $\pi$-annuli, and convex surfaces. A tight curve is a closed geodesic in $S$, which lifts to a convex open line in the universal covering $\tilde{S}$. In §2, we show that tight curves have similar properties to those of closed geodesics in hyperbolic surfaces.

In §3, we discuss $\pi$-annuli, important objects in the article [5]. We classify $\pi$-annuli in terms of elementary annuli, and show that given a $\pi$-annulus with a projective map to $S$, the map is a finite covering map onto an imbedded $\pi$-annulus. Finally, we discuss the intersection properties of maximal annuli that include imbedded $\pi$-annuli.

In §4, we discuss properties of convex compact surfaces generalizing results of Kuiper [20]. First we show that the holonomy of each essential closed curve in $S$ is hyperbolic or quasi-hyperbolic. Next, given an essential simple closed curve, we show that $S$ includes an imbedded tight curve freely homotopic to it and that this is the unique one unless it is freely homotopic to a component of $\delta S$. We show that a convex compact surface decomposes into subsurfaces that are elementary annuli, trivial annuli, or purely convex surfaces. We end with discussing the intersection property of purely convex surfaces, annuli with geodesic boundary, and trivial annuli.

In §5, we give the proof of the admissible decomposition theorem. The idea of the proof is to collect all imbedded $\pi$-annuli in $\Sigma$ and find maximal annuli that include them. We subtract these maximal annuli and trivial annuli from $\Sigma$. The main theorem of [5] implies by §3 that the closure of each component of the complement in $\Sigma$ is a purely convex surface.

In Appendix A, we present various standard facts about curves in surfaces. Since we prove these by using hyperbolic structures, we separate this material out from the main text (see Casson and Bleiler [3]). We will follow the standard terminology of hyperbolic geometry in this section (see Maskit [21]).

In Appendix B, for our purposes in this paper, we present a proof of a version of the annulus decomposition theorem of Goldman [14].

in this paper. I thank Professors E. Calabi, H. Gluck, C. T. Yang, and W. Ziller for introducing me to geometry and topology and their relationships. It is my pleasure to thank the members of the Departments of Mathematics of both the University of Illinois at Urbana and Kyungpook National University for their warm hospitality during the time when this paper was written.

## 1. Projective automorphisms and elementary annuli

**1.1.** The one-dimensional sphere $S^1$ has a one-dimensional real projective structure induced from its double covering map to the one-dimensional real projective space $\mathbf{RP}^1$. (An *antipodal pair* in $S^1$ or $S^2$ is a pair of points antipodal to each other.) A projective automorphism of $S^1$ is induced by a unique element of the group $SL_\pm(2, \mathbf{R})$ of linear automorphisms of $\mathbf{R}^2$ of determinant $\pm 1$. Hence, the action of $\langle \vartheta \rangle$ where $\vartheta$ is a projective automorphism of $S^1$ preserving orientation and corresponding to a matrix with nonnegative eigenvalues can be described as one of the following:

(1) $\vartheta$ has four fixed points composing two antipodal pairs. One pair consists of attractors, and the other pair consists of repellers. The action is said to be *hyperbolic*.

(2) $\vartheta$ has an antipodal pair of fixed points. $\vartheta$ restricted to each component of the complement is a nontrivial affine translation if the component is given the natural affine structure. The action is said to be *parabolic*.

(3) $\vartheta$ has no fixed points, is an isometry of $S^1$ equipped with the standard metric, and is not the identity or the antipode map. The action is said to be *elliptic*.

(4) $\vartheta$ is the identity.

**1.2.** A projective automorphism of $S^2$ is induced by a unique element of the group $SL_\pm(3, \mathbf{R})$ of linear automorphisms of $\mathbf{R}^3$ of determinant $\pm 1$. Hence, the projective automorphism group $\mathrm{Aut}(S^2)$ is isomorphic to $SL_\pm(3, \mathbf{R})$. Note that a projective automorphism is orientation preserving if and only if it corresponds to a matrix in $SL(3, \mathbf{R})$ (see [5]).

Let us classify projective automorphisms of $S^2$. An element of $SL(3, \mathbf{R})$ is conjugate to exactly one of the following matrices where $\lambda_1$, $\lambda_2$, and $\lambda_3$ are mutually distinct real numbers in $\mathbf{R} - \{0\}$:

$$(1) \qquad \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad \lambda_1 \lambda_2 \lambda_3 = 1;$$

(2) $\qquad \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \qquad \lambda_1^2 \lambda_2 = 1, \ \lambda_1, \lambda_2 \in \mathbf{R} - \{1, 0, -1\};$

(3) $\qquad \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \qquad \lambda_1^2 \lambda_2 = 1, \ \lambda_1, \lambda_2 \in \mathbf{R} - \{1, 0, -1\};$

(4) $\qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix};$

(5) $\qquad \begin{bmatrix} u & 1 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad u = \pm 1;$

(6) $\qquad \begin{bmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad u = \pm 1;$

(7) $\qquad \begin{bmatrix} r\cos(\theta) & -r\sin(\theta) & 0 \\ r\sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & r^{-2} \end{bmatrix}, \qquad r > 0, \ 0 < \theta < 2\pi, \ \theta \neq \pi.$

We say that a matrix in $SL(3, \mathbf{R})$ is of *type* (i) if it is conjugate to a matrix (i) above, $i = (1), (2), \cdots, (7)$. An orientation-preserving projective automorphism corresponding to a matrix of type (1) with positive eigenvalues is said to be *hyperbolic* (or *positive hyperbolic*). A projective automorphism corresponding to a matrix of type (2) with positive eigenvalues is said to be *quasi-hyperbolic*. A projective automorphism corresponding to a matrix of type (3) with positive eigenvalues is said to be *planar*. (Compare to Goldman [14], [16].)

1.3. Let us describe the fixed points and invariant great circles of orientation-preserving projective automorphisms corresponding to the above matrices when eigenvalues are nonnegative. (For each case, there are no other fixed points or invariant great circles other than what are described.)

(1) There are six fixed points, which compose three antipodal pairs. There is an antipodal pair of attractors, an antipodal pair of repellers, and an antipodal pair of points that are not attractors or repellers. Each great circle containing two fixed points not antipodal to each other is an

invariant great circle. There are three invariant great circles. The action on each of the great circles is hyperbolic.

(2) There are four fixed points which compose two antipodal pairs. There is an invariant great circle including only one of the pairs, the action on which is parabolic. The antipodal pair not in the great circle consists of attractors unless it consists of repellers. The only other invariant great circle passes through all fixed points. The action on it is hyperbolic.

(3) There is a great circle $S^1$ consisting of fixed points and an antipodal pair of fixed points not on $S^1$. The antipodal pair consists of attractors unless it consists of repellers. Except for $S^1$, a great circle is invariant if and only if it passes through the antipodal pair and an antipodal pair in $S^1$. The action on each of the great circles is hyperbolic.

(4) There is an antipodal pair of fixed points and a unique invariant great circle, which includes the pair. The action on the great circle is parabolic.

(5) The set of fixed points composes a great circle $S^1$. It includes an antipodal pair of points such that a great circle passes through them if and only if it is invariant. The action on each invariant great circle is parabolic whenever the great circle does not equal $S^1$.

(6) The action is that of the identity map.

(7) The action has an invariant great circle and an antipodal pair of fixed points not on the great circle. The action on the great circle is elliptic. The pair consists of attractors if $r < 1$ and consists of repellers if $r > 1$. If $r = 1$, then each point of the pair is neither an attractor nor a repeller.

Let us state a convenient criterion to determine the type of a projective automorphism.

**Lemma.** *Let $\vartheta$ be an orientation-preserving projective automorphism of $S^2$. Suppose that $\vartheta$ has two fixed points not antipodal to each other and that the action of $\langle \vartheta \rangle$ on the segment connecting the two points is not trivial. Then $\vartheta$ is hyperbolic, quasi-hyperbolic, or planar.*

*Proof.* Since there are two fixed points, at least two eigenvalues are positive; all eigenvalues are thus positive. $\vartheta$ does not correspond to a matrix of type (4) or (7) since the two fixed points of $\vartheta$ are not antipodal. If $\vartheta$ corresponds to a matrix of type (5) or (6), then the great circle containing the two fixed points is the set of fixed points. Hence, $\vartheta$ corresponds to a matrix of type (1), (2), or (3), and the lemma is proved.

**1.4.** We introduce the notion of elementary annuli (compare with §3 of [14] and [4]).

Let $\vartheta$ be an arbitrary hyperbolic projective automorphism. Then there are three noncollinear fixed points $s$, $m$, and $w$. The respective antipo-
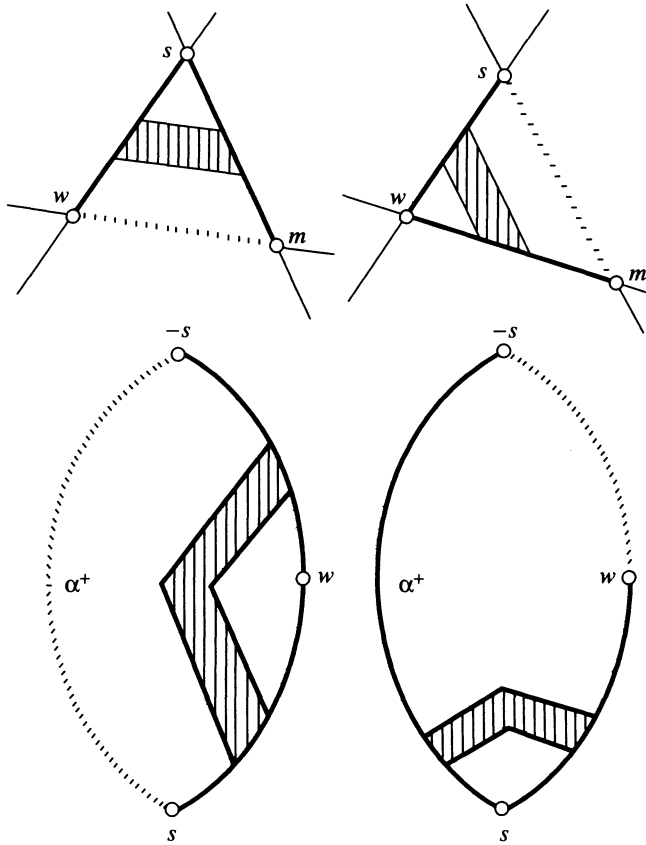
FIGURE 1. ELEMENTARY ANNULI.

dal points $-s$, $-m$, and $-w$ are also fixed. We assume that the respective eigenvalues of $s$, $m$, and $w$ are strictly decreasing. Then there is an invariant closed triangle $\Delta(smw)$. The quotient projective surface of $\Delta(smw)^o$ under $\langle \vartheta \rangle$ is called an *open elementary annulus of type I*.

We may make this into compact annuli with geodesic boundary. The quotient surface of $\Delta(smw)^o \cup \overline{sw}^o \cup \overline{sm}^o$ under the action of $\langle \vartheta \rangle$ and the quotient surface of $\Delta(smw)^o \cup \overline{sw}^o \cup \overline{mw}^o$ under the action of $\langle \vartheta \rangle$ are called *elementary annuli of type I*. Projective annuli projectively homeomorphic to these are also called *elementary annuli of type I*. A boundary component of an elementary annulus of type I is said to be *strong* if it corresponds to $\overline{sw}^o$, and *weak* otherwise. See Figure 1.

As an aside, let us note that the respective interiors of the above two annuli are projectively homeomorphic but that they are not projectively

homeomorphic by an orientation preserving map. In general, they are not projectively homeomorphic by any map unless one of the eigenvalues of $\vartheta$ equals 1.

Let $\vartheta$ be a quasi-hyperbolic projective automorphism. $\vartheta$ has two fixed points $s$ and $w$, where their respective antipodal points $-s$ and $-w$ are also fixed. Assume that the respective eigenvalues of $s$ and $w$ are strictly decreasing. Let $\mathbf{S}^1$ be the great circle on which the action of $\vartheta$ is parabolic. Then either $w$ or $s$ belongs to $\mathbf{S}^1$. Suppose that $s \in \mathbf{S}^1$. The great circle $\mathbf{S}^1$ includes an invariant segment $\alpha^+$ with endpoints $s$ and $-s$ such that the sequence $\{\vartheta^n(x)\}$, $n = 1, 2, \cdots$, converges to $s$ for each point $x$ of the open line $\alpha^{+,o}$. Let $B$ be the lune bounded by $\alpha^+$, $\overline{sw}$, and $\overline{-sw}$. The quotient of $B^o$ by the action of $\langle \vartheta \rangle$ is called an *open elementary annulus of type II* (Figure 1).

Let us consider the quotient surface of $B^o \cup \overline{sw}^o \cup \overline{-sw}^o$ under the action of $\langle \vartheta \rangle$. This is a compact projective annulus with geodesic boundary (a $\pi$-annulus in this case). We call it and a projective annulus projectively homeomorphic to it *elementary annuli of type IIa*.

The next is the quotient surface of $B^0 \cup \alpha^{+,o} \cup \overline{sw}^o$ under the action of $\langle \vartheta \rangle$. This is again a compact projective annulus with geodesic boundary. We call it and a projective annulus projectively homeomorphic to it *elementary annuli of type IIb*. A boundary component of an elementary annulus of type IIa or IIb is said to be *strong* if the component corresponds to $\overline{sw}^o$ or $\overline{-sw}^o$, and *weak* otherwise.

We remark that there is no projective homeomorphism between an elementary annulus of type I and an elementary annulus of type IIa or IIb. An elementary annulus of type IIa and an elementary annulus of type IIb are not projectively homeomorphic although their respective manifold interiors are.

## 2. Tight curves

Let $S$ be a real projective surface with convex boundary. Let $(\mathbf{dev}, h)$ be the development pair of $S$, and let $\tilde{S}$ be its universal cover with the covering map pr. A closed geodesic $\alpha: \mathbf{S}^1 \to S$, where $\mathbf{S}^1$ is a circle, is called a *tight curve* in $S$ if its lift to $\tilde{S}$ is a geodesic imbedding onto a convex open line. For example, geodesics realizing boundary component curves of elementary annuli are tight curves. A closed curve in a convex projective surface of negative Euler characteristic is a geodesic if and only if it is a tight curve. This may be verified by Lemma 1.5 of [5].

Given a closed curve $\alpha$ in $S$, it corresponds to an element $[\alpha]$ of the fundamental group $\pi_1(S)$ of $S$. If $h([\alpha])$ is hyperbolic (resp. quasi-hyperbolic, planar), then $\alpha$ is said to be *hyperbolic* (resp. *quasi-hyperbolic, planar*). This is well-defined, independently of the choices of the homotopy classes and development pairs of $S$. A principal closed geodesic in [16] is an example of a hyperbolic closed curve. See §4 for further examples of these curves. (For topological aspects of closed curves, refer to Appendix A.)

The main aim of this section is to show that tight curves in $S$ behave almost like closed geodesics in hyperbolic surfaces (see [10]). We show that a tight curve freely homotopic to a simple closed curve is an imbedding onto an imbedded closed curve, that two imbedded tight curves intersect minimally, and that given a hyperbolic or quasi-hyperbolic closed curve, at most finitely many tight curves are freely homotopic to it. Next, we show that tight curves are not planar. Lastly, we discuss characterizing properties of elementary annuli and trivial annuli and show that $S$ includes only finitely many trivial annuli.

**2.1.** Let us list a few basic properties of tight curves. Let $\alpha: \mathbf{S}^1 \to S$ be a tight curve. Then its lift $\tilde{\alpha}: \mathbf{R} \to \widetilde{S}$ is injective. Thus $\alpha$ is essential. Next, Lemma 3.4 of [5] implies that if $\alpha$ passes through a point of $\delta S$, then $\alpha$ is a covering map onto an imbedded tight curve that is a component of $\delta S$. In other words, either $\alpha$ maps into $S^o$ or $\alpha$ is a covering map onto a component of $\delta S$. If $\beta$ is an imbedded tight curve in $S$, then it follows that $\beta$ is either a subset of $S^o$ or a component of $\delta S$.

**2.2.** We assume that $S$ is homeomorphic to a compact annulus or a cover of a compact surface with negative Euler characteristic. We will be using a generic hyperbolic metric $d$ on $S$ obtained as in Appendix A. The induced metric on an arbitrary cover of $S$ will also be denoted by $d$.

**Proposition.** *Let* $\alpha$ *and* $\beta$ *be tight curves in* $S$ *freely homotopic to simple closed curves.*

(1) *Suppose that* $\gamma: \mathbf{S}^1 \to S$ *is a tight curve freely homotopic to a finite covering map of an imbedded closed curve. Then* $\gamma$ *is a finite covering map of an imbedded tight curve.*

(2) $\alpha$ *is an imbedding onto a one-dimensional submanifold of* $S$; *so is* $\beta$.

(3) *The respective image submanifolds* $\alpha_1$ *and* $\beta_1$ *of* $\alpha$ *and* $\beta$ *either are identical or have minimal intersection.*

(4) *If* $\alpha$ *and* $\beta$ *intersect at a point and intersect trivially in homotopy, then* $\alpha_1$ *and* $\beta_1$ *are identical.*

(5) *Suppose that $\alpha$ and $\beta$ intersect trivially in homotopy. Then $\alpha_1$ and $\beta_1$ are disjoint if $\alpha_1$ is not freely homotopic to $\beta_1$, and are disjoint or equal otherwise.*

(6) *Let $\gamma: \mathbf{S}^1 \to S$ be a tight curve. Then there is a finite cover $S'$ of $S$ such that $\gamma$ lifts to a simple tight curve $\gamma': \mathbf{S}^1 \to S'$.*

(7) *Assume that $S$ is compact. Let $\gamma: \mathbf{S}^1 \to S$ be a hyperbolic or quasi-hyperbolic closed curve. Then there are only finitely many tight curves in $S$ freely homotopic to $\gamma$, whose images are distinct from one another.*

*Proof.* (1) Suppose that two simply convex segments in $S$ share two endpoints, and are respectively realized by maps homotopy equivalent with endpoints fixed. Then it is clear that they are identical.

Let $\tilde{\gamma}: \mathbf{R} \to \tilde{S}$ be a lift of $\gamma$, which is a geodesic in $\tilde{S}$ and whose image is a convex maximal line, say $l$. A deck transformation $\vartheta$ corresponding to $\gamma$ and $\tilde{\gamma}$ satisfies $\vartheta(l) = l$. Let $\varphi$ be an arbitrary deck transformation. Suppose that $\varphi(l) \neq l$. Then $\varphi(l)$ and $l$ are transversal. The first paragraph implies that $l$ and $\varphi(l)$ may not meet at more than two points. Lemma 3 of Appendix A shows that $l \cap \varphi(l) = \varnothing$. Therefore, we have either $l = \varphi(l)$ or $l \cap \varphi(l) = \varnothing$ for each deck transformation $\varphi$. It follows that $\gamma$ is a finite covering map onto an imbedded tight curve.

(2) By (1), $\alpha$ is a finite covering map onto an imbedded closed curve. Since $S$ is orientable, $\alpha$ is an imbedding onto an imbedded closed curve.

(3) Suppose that $\alpha_1 \neq \beta_1$. Then $\alpha_1$ and $\beta_1$ intersect transversally. The first paragraph of the proof of (1) and the definition of minimal intersection (see Casson and Bleiler [3]) imply that $\alpha_1$ and $\beta_1$ intersect minimally. Hence, $\alpha_1$ and $\beta_1$ either are identical or intersect minimally.

(4) By (3), $\alpha_1$ and $\beta_1$ either are identical or have minimal intersection. Since the latter is not true, it follows that $\alpha_1 = \beta_1$.

(5) This follows from (4).

(6) There is a finite cover $S'$ of $S$ such that $\gamma$ lifts to a closed curve $\gamma': \mathbf{S}^1 \to S'$ such that $\gamma'$ is freely homotopic to a simple closed curve (see Scott [23], [24]). (2) implies the conclusion.

(7) Suppose not. We may assume without loss of generality that $\gamma$ is a simple closed curve by lifting to a finite cover of $S$. Let $\{\gamma_i | i \in \mathbf{N}\}$ be a countable collection of tight curves freely homotopic to $\gamma$ such that $\gamma_j$ and $\gamma_k$ are maps with distinct images whenever $j \neq k$. Parts (2) and (5) imply that $\gamma_i$ for each $i$ is a simple tight curve and that the images of $\gamma_i$ are imbedded tight curves disjoint from one another.

Let $\vartheta$ be the deck transformation of $\tilde{S}$ corresponding to $\gamma$. Let $S'$ be the cover of $S$ *corresponding to* $\vartheta$; that is, it is the quotient space of $\tilde{S}$

by the action of the group of deck transformations generated by $\vartheta$. Then $\gamma$ lifts to a simple closed curve $\gamma'$ in $S'$. Hence, each $\gamma_i$ lifts to a tight curve $\gamma_i'$ in $S'$ freely homotopic to $\gamma'$. By (2), $\gamma_i'$ is simple. For each $i$, let $\gamma_i''$ denote the image submanifold of $\gamma_i'$. Since every $\gamma_i'$ is freely homotopic to $\gamma'$ in $S'$, (5) implies that $\gamma_i''$ and $\gamma_j''$ are disjoint whenever $i \neq j$.

Let $\mathrm{pr}'': S' \to S$ be the covering map induced from the universal covering map $\mathrm{pr}: \widetilde{S} \to S$. $\mathrm{pr}''$ is a local isometry. Since $S'^o$ is homeomorphic to an open annulus, $\gamma_1''$ and $\gamma_i''$ for each $i$, $i > 1$, bound a unique annulus in $S'$. Let us denote it by $A_i'$. The above two paragraphs and a result of §6 of Appendix A show that $\mathrm{pr}'' | A_i'$ is an imbedding and, hence, that $\{d\text{-area}(A_i')\}$, $i > 1$, is a bounded sequence.

If the number of elements of $\{\gamma_i''\}$ intersecting each compact subset of $S'$ is finite, then
$$\{d\text{-area}(A_i')\} \to \infty \quad \text{as } i \to \infty.$$
Hence, $\{\gamma_i''\}$ is not locally finite.

This implies that $S'$ contains a point $p$ that is an accumulation point of $\bigcup_i \gamma_i''$ and is not a point of it. Let $l_i$ denote the inverse image of $\gamma_i''$ under the covering map from $\widetilde{S}$ to $S'$ for each $i$. Let $\tilde{p}$ denote a point of $\widetilde{S}$ corresponding to $p$, and let $B(\tilde{p})$ be a tiny disk of $\tilde{p}$ (see §1.11 in [5]). Let us list obvious properties of $l_i$:

(i) Each $l_i$ is an open $\vartheta$-invariant convex line.

(ii) Infinitely many elements of $\{l_i\}$ intersect the interior of $B(\tilde{p})$.

(iii) $l_i$ and $l_j$ are disjoint whenever $i \neq j$.

The above properties of $l_i$ yield the following properties of $\mathbf{dev}(l_i)$.

(i) For each $i$, $\mathbf{dev}(l_i)$ is an $h(\vartheta)$-invariant convex line, where $h$ is the holonomy homomorphism.

(ii) Since $\mathbf{dev}|B(\tilde{p})$ is an imbedding, $\mathbf{dev}(l_i)$ is distinct from $\mathbf{dev}(l_j)$ whenever $l_i$ and $l_j$ intersect $B(\tilde{p})$ with $i \neq j$.

These contradict the fact that $\mathbf{S}^2$ includes only finitely many $h(\vartheta)$-invariant convex lines and proves (7).

**2.3.** We will now prove the following important property.

**Proposition.** *Suppose that $S$ is compact and that $\chi(S) < 0$. If $\alpha: \mathbf{S}^1 \to S$ is a tight curve in $S$, then $\alpha$ is not planar.*

*Proof.* Suppose that $\alpha$ is planar. We may assume without loss of generality that $\alpha$ is simple by lifting to a finite cover of $S$ by Proposition 2.2(6). Let $S'$ be the cover of $S$ *corresponding* to $\alpha$; that is, $S'$ is the quotient of $\widetilde{S}$ by the action of the group generated by a deck transformation $\vartheta$ corresponding to $\alpha$ and its lift $\tilde{\alpha}: \mathbf{R} \to \widetilde{S}$. Let $\mathrm{pr}': \widetilde{S} \to S'$

denote the covering map, and $\alpha' : S^1 \to S'$ a lift of $\alpha$. By Proposition 2.2(2), $\alpha'$ imbeds onto an imbedded tight curve $\alpha''$ in $S'$.

Let $A$ be the subset of points of $S'$ through which a tight curve freely homotopic to $\alpha'$ passes. We claim that $A$ is an open subset of $S'$. Let $x \in A$. Then $x$ belongs to an imbedded tight curve $\alpha_x$ freely homotopic to $\alpha'$. Let $\tilde{\alpha}_x$ be the image of a lift of $\alpha_x$ to $\tilde{S}$ corresponding to $\vartheta$. We have either $\tilde{\alpha}_x \subset \tilde{S}^o$ or $\tilde{\alpha}_x \subset \delta\tilde{S}$ by §2.1. In the first case, an open neighborhood $\mathscr{U}$ of $\tilde{\alpha}_x$ satisfies the following conditions:

• $\mathrm{pr}' | \mathscr{U}$ is a covering map onto an annulus, an open neighborhood of $\alpha_x$.

• $\mathbf{dev} | \mathscr{U}$ is injective (thus, it is an imbedding).

The image $\mathbf{dev}(\mathscr{U})$ includes an $h(\vartheta)$-invariant open triangle $T$ including $\mathbf{dev}(\tilde{\alpha}_x)$ (see §1.3). Let $T' = (\mathbf{dev}|\mathscr{U})^{-1}(T)$. The set $T'$ is a $\vartheta$-invariant open neighborhood of $\tilde{\alpha}_x$. Since $h(\vartheta)$ is planar, $T'$ is foliated by $\vartheta$-invariant convex lines. Thus, $\mathrm{pr}' | T'$ is a covering map onto an open annulus that is an open neighborhood of $\alpha_x$ and is foliated by imbedded tight curves freely homotopic to $\alpha_x$. If we have $\tilde{\alpha}_x \subset \delta\tilde{S}$, then a similar argument shows that an open neighborhood of $\alpha_x$ is foliated by imbedded tight curves freely homotopic to $\alpha_x$. Thus, $A$ is open.

Let $\mathrm{pr}'' : S' \to S$ be the covering map induced from the universal covering map $\mathrm{pr} : \tilde{S} \to S$. We claim that $\mathrm{pr}'' | A$ is injective. Let $x$ and $y$ be two distinct points of $A$. Let $\alpha_x$ and $\alpha_y$ be the imbedded tight curves freely homotopic to $\alpha''$ containing $x$ and $y$ respectively. Proposition 2.2(2) implies that $\mathrm{pr}'' | \alpha_x$ and $\mathrm{pr}'' | \alpha_y$ are injective maps onto imbedded tight curves. Thus, if $\alpha_x = \alpha_y$, then $\mathrm{pr}''(x) \neq \mathrm{pr}''(y)$. Suppose that $\alpha_x \neq \alpha_y$. By Lemma 5 of Appendix A, the respective images are distinct from each other. From Proposition 2.2(5) it follows that the respective images are disjoint, so that $\mathrm{pr}''(x) \neq \mathrm{pr}''(y)$. Thus, $\mathrm{pr}'' | A$ is injective.

The injectivity of $\mathrm{pr}'' | A$ yields that the $d$-area of $A$ is finite. Let $A'$ be the component of $A$ including $\alpha''$. Clearly, $A'$ is an annulus foliated by imbedded tight curves freely homotopic to $\alpha''$. We may assume without loss of generality that $\alpha''$ is geodesic with respect to $d$ (see Appendix A). Let us consider for each point $x$ of $\alpha''$ the maximal $d$-geodesic $\lambda_x$ in $A'$ perpendicular to $\alpha''$ at $x$. Since the $d$-area of $A'$ is finite, $\alpha''$ contains a point $x$ such that the $d$-length of the component $\mu_x$ of $\lambda_x \cap A'$ containing $x$ is finite. There is a monotone sequence of points $\{p_i\}$ on a component of $\mu_x - \{x\}$ converging to the endpoint $p$ of $\mu_x$ that is not $x$ and satisfies $p \notin A$. Then for each $i$, $p_i$ is an element of an imbedded tight curve

$\alpha_i$ freely homotopic to $\alpha''$. Let us denote by $A_i$ the annulus bounded by $\alpha_i$ and $\alpha''$ for each $i$. Clearly, $A_i \subset A'$. Let $\mu_{x,i}$ be the closure of the component of $\mu_x \cap A_i^o$ containing $x$ for each $i$. Let $q_i$ for each $i$ be the endpoint of the $d$-geodesic segment $\mu_{x,i}$ that is not $x$. (See Figure 2.) Note that $\alpha'' \cap \mu_{x,i} = \{x\}$, $\alpha_i \cap \mu_{x,i} = \{q_i\}$, and $\mu_{x,i}^o \subset A_i^o$ hold for each $i$.

We have the following possibilities:
  (i) $\{d\text{-length}\,(\alpha_i)\}$ is a bounded sequence.
  (ii) $\{d\text{-length}\,(\alpha_i)\}$ is an unbounded sequence.
We will show below that each of these possibilities are contradictions. This will complete the proof.

  (i) Assume without loss of generality that each $\alpha_i$ is parametrized by $d$-length. Then each $\alpha_i$ lies in a compact subset $K$ of $S'$, $\alpha_i$ is a distance decreasing map, and, hence, $\{\alpha_i\}$ is bounded and uniformly continuous. Thus by the Ascoli-Arzelà theorem, a subsequence of $\{\alpha_i\}$ converges to a continuous closed curve $\alpha_\infty : \mathbf{S}^1 \to S'$ passing through $p$. It is straightforward to show by a local argument that $\alpha_\infty$ is a tight curve and that $\alpha_\infty$ is freely homotopic to $\alpha''$. Hence $p \in A$, and this is a contradiction.

  (ii) First, we cut open the annulus $A_i$ along $\mu_{x,i}$ and lift it to $\widetilde{S}$: To begin, we lift $\alpha$. Let $\tilde{\alpha}^*$ be the image of the lift $\tilde{\alpha}$ of $\alpha$ to $\widetilde{S}$. Recall that $\vartheta$ corresponds to $\tilde{\alpha}^*$ and $\alpha$. Let $x^*$ be a point of $\tilde{\alpha}^*$ corresponding to $x$. Then $\tilde{\alpha}^*$ includes a compact arc $\alpha^*$ with endpoints $x^*$ and $\vartheta(x^*)$. Here $\mathrm{pr}'\,|\alpha^*$ maps onto $\alpha''$ and is injective in the complement of the set of the endpoints of $\alpha^*$. Now, we lift $\lambda_x$ to a maximal $d$-geodesic $\lambda_x^*$ that is $d$-perpendicular to $\alpha^*$ at $x^*$. A maximal $d$-geodesic $\vartheta(\lambda_x^*)$ is $d$-perpendicular to $\alpha^*$ at $\vartheta(x^*)$. The $d$-geodesic $\lambda_x^*$ includes a compact arc $\mu_{x,i}^*$ with endpoints $x^*$ and $q_i^*$, the point on $\lambda_x^*$ corresponding to $q_i$. Note that $\mathrm{pr}'\,|\mu_{x,i}^*$ is an embedding onto $\mu_{x,i}$ and that so is $\mathrm{pr}'\,|\vartheta(\mu_{x,i}^*)$. There is also an arc $\alpha_i^*$ with endpoints $q_i^*$ and $\vartheta(q_i^*)$ such that $\mathrm{pr}'\,|\alpha_i^*$ is onto $\alpha_i$ and is injective in the complement of the set of the endpoints. Since we have

$$\alpha^* \cap \mu_{x,i}^* = \{x^*\} \qquad \mu_{x,i}^* \cap \alpha_i^* = \{q_i^*\},$$
$$\alpha_i^* \cap \vartheta(\mu_{x,i}^*) = \{\vartheta(q_i^*)\}, \qquad \vartheta(\mu_{x,i}^*) \cap \alpha^* = \{\vartheta(x^*)\},$$

the surface $\widetilde{S}$ includes a compact disk $A_i^*$ bounded by four arcs $\alpha^*$, $\mu_{x,i}^*$, $\alpha_i^*$, and $\vartheta(\mu_{x,i}^*)$. It is clear that $\mathrm{pr}'\,|A_i^*$ maps onto $A_i$ and is injective in the complement of $\mu_{x,i}^* \cup \vartheta(\mu_{x,i}^*)$ (see Figure 2, next page).
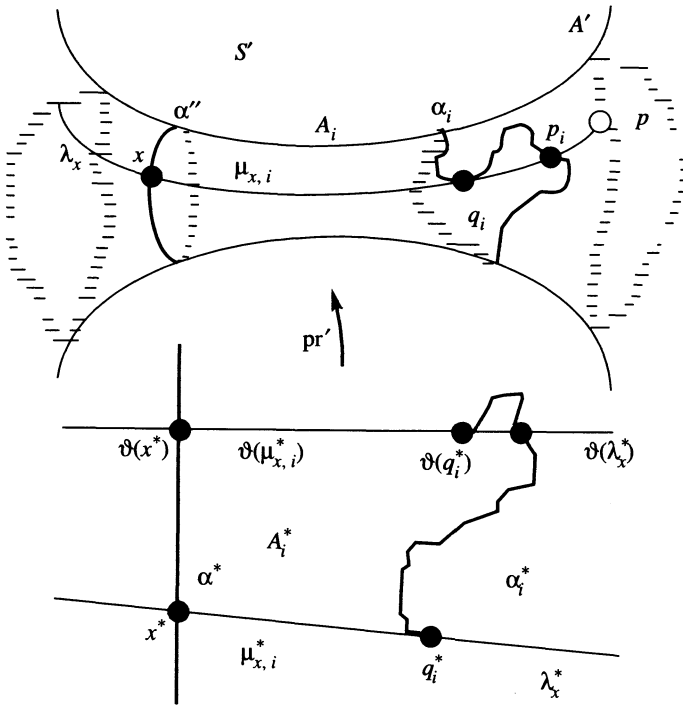
FIGURE 2

Let $B_r$ denote the compact $d$-ball of radius $r$, $r > 0$, with center $x^*$. There are two possibilities:

($\alpha$) There exists a positive constant $r$ such that $\alpha_i^* \subset B_r$ for every $i$.

($\beta$) $\{\alpha_i^*\}$ is unbounded.

We will derive contradictions for these two cases. This will show that (ii) is a contradiction also.

($\alpha$) Cover $B_r$ by a finite collection of open $d$-balls of radius $c_S/2$ where $c_S$ is the constant of §2.4. Let $N$ be the number of $d$-balls in the collection. For each $i$, let us choose $N+1$ points $r_{i,j}$, $j = 1, \cdots, N+1$, on $\alpha_i^*$ so that the following properties hold:

• $r_{i,1} = q_i^*$, and $r_{i,N+1} = \vartheta(q_i^*)$.

• Let $\beta_{i,j}$ for each $j$, $j = 1, \cdots, N$, be the arc in $\alpha_i^*$ with endpoints $r_{i,j}$ and $r_{i,j+1}$. Then $\beta_{i,j}^0$ does not contain $r_{i,k}$ for any $k$.

• The $d$-length of $\beta_{i,j}$ is independent of $j$.

We will now state two consequences that contradict each other: First, since $\{d\text{-length}(\tilde{\alpha}_i)\}$ is unbounded, it follows that $\{d\text{-length}(\beta_{i,j})\}$ for

each fixed $j$ is unbounded. Second, let us fix $i$. The so-called pigeon-hole principle implies that there are at least two elements of $\{r_{i,j}|j = 1, \cdots, N+1\}$ in one of the covering $d$-balls. Let us say that they are $r_{i,k}$ and $r_{i,l}$ where $k < l$. Then by Lemma 2.4, the $d$-length of the arc on $\alpha_i^*$ connecting $r_{i,k}$ and $r_{i,l}$ is less than $c_S'$. Hence, $d$-length$(\beta_{i,k}) < c_S'$. This means that $d$-length$(\beta_{i,j}) < c_S'$ for every $j$. Since this inequality holds for every $i$, we have a contradiction.

($\beta$) Since $\mu_{x,i} \subset \mu_x$ for every $i$, it follows that the $d$-distance from every point of $\mu_{x,i}^*$ or $\vartheta(\mu_{x,i}^*)$ to $x^*$ is bounded above by a constant $C$ independent of $i$. Let $R$ be an arbitrary fixed real number greater than $C$; let us choose an arbitrary fixed integer $i$ for $R$ such that $\alpha_i^*$ is not a subset of $B_{3R}$.

Let $f$ be a function on $\widetilde{S}$, that measures $d$-distances from $x^*$. For each $r$, $r > C$, we can divide $A_i^*$ into three sets: $A^*(r)_{i,+}$, $A^*(r)_{i,0}$ and $A^*(r)_{i,-}$ consisting of points whose values under $f$ are greater than, equal to, and less than $r$ respectively. For a regular value $r$, $r > C$, of $f|\alpha_i^{*,o}$, $A^*(r)_{i,0}$ is a one-dimensional submanifold in $A_i^*$, each component of which is a compact arc intersecting $\alpha_i^{*,o}$ precisely at the set of endpoints. Thus, we have constructed our first object: $A^*(r)_{i,0}$ on the level set $f^{-1}(r)$.

Let

$$\delta_R = \inf\{d\text{-length}(A^*(r)_{i,0})|R < r < 2R,$$
$$r \text{ is a regular value of } f|\alpha_i^{*,o}\},$$

and let

$$\varepsilon_R = \max\{\delta_R, d\text{-area}(A)/R\}.$$

Clearly, $\varepsilon_R > 0$. Let $r'$ be a regular value of $f|\alpha_i^{*,o}$ with $R < r' < 2R$ such that the $d$-length of $A^*(r')_{i,0}$ is less than $2\varepsilon_R$. Let $x_i^*$ denote a point of $\alpha_i^* - B_{3R}$. Clearly, $x_i^* \in A^*(r')_{i,+}$, and $x^* \in A^*(r')_{i,-}$. There is a component arc $\zeta_i$ of $A^*(r')_{i,0}$ that separates $x^*$ from $x_i^*$; i.e., every path from $x^*$ to $x_i^*$ in $A_i^*$ intersects $\zeta_i$. Let $n_i$ and $m_i$ denote the endpoints of $\zeta_i$ in $\alpha_i^{*,o}$; let $\gamma_i$ be the arc on $\alpha_i^*$ sharing endpoints $n_i$ and $m_i$ with $\zeta_i$. Then $\gamma_i \ni x_i^*$. Thus, we have constructed our second and last objects for $R$: $\zeta_i$ and $\gamma_i$.

Since $\zeta_i$ is a subset of $A^*(r')_{0,i}$, the $d$-length of $\zeta_i$ is less than $2\varepsilon_R$. Since $n_i, m_i \in f^{-1}(r')$, and $x_i^* \notin B_{3R}$, the $d$-length of $\gamma_i$ is greater than $2R$.

Hyperbolic geometry and the Sard theorem imply that the $d$-area of $A_i^*$ is greater than or equal to $\delta_R R$. From the facts that $d$-area$(A)$ is finite, $A_i \subset A$, and $\text{pr}'$ is a local isometry, it follows that $\delta_R \leq d$-area$(A)/R$. This means that $\varepsilon_R \to 0$ as $R \to \infty$, so that $d(n_i, m_i) \to 0$ and $d$-length$(\gamma_i) \to \infty$ as $R \to \infty$, which contradict Lemma 2.4, since $\gamma_i$ is a convex segment.

**2.4.** In a real projective surface, a *tiny disk* is a compact disk simply convex with respect to the real projective structure (see §1.11 in [5]). Choose a covering of $S$ by interiors of tiny disks and choose a finite subcovering. The finite subcovering corresponds to a locally finite covering $\mathscr{O}$ of $\widetilde{S}$ by tiny disks. It follows that there is a positive constant $c_S$ such that for every point $p$ of $\widetilde{S}$, the covering $\mathscr{O}$ contains a tiny disk $B$ of $p$ such that $d(p, \text{bd} B) > c_S$.

**Lemma.** *Suppose that $p$ and $q$ are points of $\widetilde{S}$ connected by a convex segment $\alpha$. Then there is a positive constant $c_S'$ independent of $p$ and $q$ such that if $d(p, q) < c_S$, then $d$-length$(\alpha) < c_S'$.*

*Proof.* Let $B$ be a tiny disk belonging to $\mathscr{O}$. Given the Hausdorff metric associated with $d$, the collection of compact subsets of $B$ forms a compact metric space. All segments and point sets in $B$ form a closed subset of this metric space. (A subset is called a *point set* if it is the set consisting of a single point.) Let us call the subspace $\mathscr{B}$. If we assign the $d$-length of every point subset to be zero, then the real-valued function defined on $\mathscr{B}$ assigning each element to its $d$-length is a continuous function. Hence, there is the supremum of the set of values of this function. The facts that for tiny disks mapping onto each other under deck transformations, the respective supremum are equal to each other and the covering $\mathscr{O}$ is locally finite imply that there exists an upper bound of the collection of these supremums for tiny disks in $\mathscr{O}$. Let $c_S'$ be this upper bound multiplied by two. This satisfies the conclusion of the lemma.

**2.5.** For the remainder of this section, let us discuss about facts on trivial annuli and elementary annuli. We call a compact annulus with convex boundary a *hyperbolic* (resp. *quasi-hyperbolic*) annulus if the boundary components are hyperbolic (resp. quasi-hyperbolic). Let us state a criterion for an annulus to be an elementary or trivial annulus.

**Lemma.** *Let $A$ be a hyperbolic or quasi-hyperbolic annulus in $S$.*

(1) *Suppose that $\delta A$ is a geodesic. Then $A$ includes exactly two imbedded tight curves if and only if $A$ is an elementary annulus.*

(2) *Suppose that the nongeodesic boundary components of $\delta A$ are components of $\delta S$. Then a boundary component of $\delta A$ is a unique imbedded*

*tight curve in $A$ if and only if $A$ is a trivial annulus.*

(3) *An elementary or trivial annulus in $S$ is not a proper subset of another elementary or trivial annulus in $S$.*

*Proof.* (1), (2) These are straightforward consequences of §1 and the annulus decomposition theorem of Appendix B.

(3) This follows from (1) and (2).

**2.6.** Let us discuss trivial annuli in $S$. Let $T_1$ and $T_2$ be trivial annuli in $S$. Then their respective geodesic components $\alpha_1$ and $\alpha_2$ of $\delta T_1$ and $\delta T_2$ are imbedded tight curves. Since $\alpha_1$ and $\alpha_2$ are freely homotopic to components of $\delta S$, by Proposition 2.2, either $\alpha_1$ and $\alpha_2$ are disjoint or $\alpha_1 = \alpha_2$. The facts that a trivial annulus is the closure of a component of the complement of an imbedded tight curve, its geodesic boundary component, in $S$ and $\chi(S) < 0$ and the above lemma imply that $T_1 = T_2$ or $T_1 \cap T_2 = \varnothing$. As a consequence, for each nongeodesic component of $\delta S$, there is at most one trivial annulus including the component. Therefore, $S$ includes finitely many trivial annuli, which are mutually disjoint.

## 3. $\pi$-Annuli

**3.1.** Let $M$ be a projective surface, and let $\widetilde{M}$ be its universal cover. Let $D\colon \widetilde{M} \to \mathbf{S}^2$ be a developing map for $M$, and let $H$ be the holonomy homomorphism associated with $D$. A *strong tight curve* in $M$ is a hyperbolic or quasi-hyperbolic tight curve $\alpha\colon \mathbf{S}^1 \to M$ such that for its lift $\tilde{\alpha}\colon \mathbf{R} \to \widetilde{M}$, the map $D \circ \tilde{\alpha}$ is an imbedding onto a convex line connecting a fixed point of the largest eigenvalue with a fixed point of the smallest eigenvalue of $H(\vartheta)$ where $\vartheta$ is the deck transformation of $\widetilde{M}$ corresponding to $\tilde{\alpha}$ and $\alpha$. A *weak tight curve* is a tight curve in $M$ that is not strong. For boundary components of elementary annuli, the definitions given in §1.4 agree with those given here. The geodesic boundary component of a trivial annulus is a strong tight curve. A component of $\delta M$ is a principal closed geodesic if and only if it is a strong hyperbolic tight curve (see [16]).

These notions are invariant under projective maps and covering maps.

**Lemma.** *Let $\alpha\colon \mathbf{S}^1 \to M$ be a closed curve, and let $f\colon \mathscr{U} \to M'$ be a projective map defined on a neighborhood $\mathscr{U}$ of the image of $\alpha$ mapping to a projective surface $M'$. Let $\beta$ be a closed curve in $M'$, and let $c_1$ and $c_2$ be finite covering maps from $\mathbf{S}^1$ to $\mathbf{S}^1$. Suppose that $f \circ \alpha \circ c_1$ is a reparametrization of $\beta \circ c_2$. Then $\alpha$ is a strong tight curve if and only if $\beta$ is a strong tight curve.*

*Proof.* Straightforward.

**3.2.** Let $B$ be a great disk or a lune in $\mathbf{S}^2$. Let $\eta_1$ and $\eta_2$ be two great segments with common endpoints forming an antipodal pair and with $\eta_1 \cup \eta_2 = \delta B$. The quotient projective surface $A$ of $(B^o \cup \eta_1^o) - \{x\}$ for a point $x$ of $\eta_1^o$ under the action of $\langle \vartheta \rangle$ for a projective automorphism $\vartheta$ in $\mathrm{Aut}(\mathbf{S}^2)$ is a $\pi$-annulus (see [5]).

The projective automorphism $\vartheta$ has two invariant segments $\eta_1$ and $\eta_2$ and a fixed point $x$. Since $\vartheta$ is orientation-preserving, the endpoints of $\eta_1$ and $\eta_2$ are also fixed points. Lemma 1.3 implies that $\vartheta$ is hyperbolic, quasi-hyperbolic, or planar.

Suppose that $S$ is compact, and that there is a projective map $f \colon A \to S$. Then $f|\alpha$ for a component $\alpha$ of $\delta A$ is a tight curve in $S$. Thus $A$ must be hyperbolic or quasi-hyperbolic by Proposition 2.3.

Suppose that $A$ is hyperbolic. It is clear that $B$ is the union of two invariant triangles. It follows that $A$ is the sum of two elementary annuli of type I. Suppose that $A$ is quasi-hyperbolic. Then $B$ is either an invariant lune or the union of two invariant lunes. In the first case, $A$ is an elementary annulus of type IIa; in the second case, $A$ is the sum of two elementary annuli of type IIb.

Let us observe that if $A$ is a hyperbolic $\pi$-annulus, then the components of $\delta A$ are either both strong or both weak. Moreover, $A^o$ includes a unique imbedded tight curve, which is weak if the components of $\delta A$ are strong and is strong otherwise. If $A$ is a $\pi$-annulus that is an elementary annulus of type IIa, then the components of $\delta A$ are strong. In this case, $A^o$ includes no imbedded tight curve. If $A$ is a $\pi$-annulus that is the sum of two elementary annuli of type IIb, then the components of $\delta A$ are strong. Moreover, $A^o$ includes a unique tight curve, which is weak.

**3.3.** The aim of this section is to prove the following proposition:

**Proposition.** *Let $f \colon A \to S$ be a projective map. Then $S$ includes an imbedded $\pi$-annulus $F$ such that $f = i_F \circ c_A$ for an inclusion map $i_F \colon F \to S$ and a finite covering map $c_A \colon A \to F$; that is, the following diagram is commutative:*

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & S \\
{\scriptstyle c_A}\big\downarrow & & \big\| \\
F & \xrightarrow{\ i_F\ } & S.
\end{array}
$$

**3.4.** A consequence of Proposition 3.3 and the main theorem in [5] is as follows (this was claimed in the introduction in [5]).

**Theorem.** *S is convex if and only if S does not include an imbedded π-annulus.*

*Proof.* Suppose that $S$ is not convex. Then there is a π-annulus with a projective map to $S$ by the main theorem [5]. Thus the above proposition shows that $S$ includes an imbedded π-annulus. The converse portion is a consequence of the following lemma.

**3.5. Lemma.** *Let M be a real projective surface. Suppose that M includes a π-annulus E. Then M is not convex.*

*Proof.* Suppose that $M$ is convex. Let $(D, H)$ be a development pair of $M$. Since $\widetilde{M}$ is tame, $D$ is an imbedding onto a convex domain $\Omega$ in $\mathbf{S}^2$ by §1.4 in [5]. Let $p: \widetilde{M} \to M$ be the universal cover, and let $\widetilde{E}$ be a component of $p^{-1}(E)$. There exists a deck transformation $\varphi$ of $\widetilde{M}$ acting on $\widetilde{E}$, and $D|\widetilde{E}$ is an imbedding onto $(K^o \cup \zeta^o) - \{y\}$ for a lune or great disk $K$, a convex segment $\zeta$ in $\delta K$, and a point $y$ of $\zeta^o$. Since $\Omega$ is convex, and $(K^o \cup \zeta^o) - \{y\} \subset \Omega$, it follows that $y \in \Omega$. However, $y$ is a fixed point of $H(\varphi)$. Since the action of the holonomy group on $\Omega$ is free, this is a contradiction. Thus, $M$ is not convex.

**3.6.** We need some preliminary material before proving Proposition 3.3. Let $M$ be a compact real projective surface with convex boundary. A *geodesic complex* $K$ in $M$ is a compact subset with the following property: for each point $p$ of $K$, the surface $M$ includes an open neighborhood $\mathscr{U}$ of $p$ such that

$$\mathscr{U} \cap K = \bigcup_{i=1}^{n} l_i$$

holds where each $l_i$, $i = 1, \cdots, n$, is a maximal line in $\mathscr{U}$ passing through $p$. For example, the union of finitely many imbedded closed geodesics and maximal geodesic segments in $M$ is a geodesic complex. Also, the image of every closed geodesic in $M$ is a geodesic complex.

Let $K$ be a geodesic complex. A *regular point* of $K$ is a point of $K$ with a neighborhood in $K$ that is a line, a *vertex* of $K$ is a point of $K$ that is not regular, and a *regular arc* of $K$ is a component arc of $K$ removed the set of vertices. Regular arcs of $K$ are imbedded geodesics.

Let us state properties of $K$. (Let $d'$ denote a complete metric on $M$). First, it is easy to see that there are finitely many vertices and regular arcs in $K$. Second, let $\gamma: [a, b] \to K$, $a < b$, be a $d'$-length parametrized geodesic where $\gamma(a)$ or $\gamma(b)$ does not belong to $\delta M$. Then the local condition of $K$ implies that there is a $d'$-length parametrized geodesic $\gamma': I' \to K$, where $I'$ is an interval including $[a, b]$ properly, and $\gamma'|[a, b] = \gamma$; that is, a $d'$-length parametrized geodesic $\gamma'$ extends $\gamma$.

Lastly, we claim that one of the following holds for each point $p$ of $K \cap M^o$:

- There is a closed geodesic in $K \cap M^o$ passing through $p$.
- There is a geodesic path $\gamma: I \to K$ for a compact interval $I$ such that $\gamma(\delta I) \subset \delta M$, and $\gamma$ passes through $p$.

Let us prove this. Choose a $d'$-length parametrized geodesic $\eta: J \to K$ for an interval $J$ passing through $p$. Let $\eta': J' \to K$ for an interval $J'$ be a maximal geodesic extending $\eta$; that is, we assume that $\eta'$ is a $d'$-length parametrized geodesic, that $J$ is a subset of $J'$, that $\eta'|J = \eta$, and that there is no $d'$-length parametrized geodesic extending $\eta'$. The interval $J'$ either is unbounded or is compact. If $J'$ is unbounded, then the compactness of $K$ implies that there exists a point of a regular arc that $\eta'$ passes and then passes again in the direction of the previous visit. This implies that $\eta'$ passes $p$ and then passes $p$ again in the direction of the previous visit. Hence, $\eta'$ corresponds to a closed geodesic $\eta''$ passing through $p$. Since $\eta''$ passes $p$, it follows that the image of $\eta''$ is a subset of $M^o$ by Lemma 34 in [5]. If $J'$ is compact, then $\eta'(\delta J') \subset \delta M$. Otherwise, $\eta'$ is not maximal by the above paragraph. Thus the claim is proved.

**3.7.** We now begin the proof of Proposition 3.3. Let $\alpha$ and $\beta$ be components of $\delta A$. As a first step, we will prove that $f|\alpha$ is a covering map onto an imbedded tight curve in $S$. If $f(\alpha)$ intersects $\delta S$, then $f(\alpha)$ is a component of $\delta S$ by §2.1. Hence, the claim is true in this case.

Suppose that $f(\alpha) \subset S^o$. Then the image $f(\alpha)$ is a geodesic complex. Since for each point of $A$, there is a neighborhood such that $f$ restricted to it is a diffeomorphism onto a simply convex open disk or a disk projectively homeomorphic to a simply convex open disk in $\mathbf{S}^2$ intersected with a closed hemisphere, $f^{-1}(f(\alpha))$ is a geodesic complex in $A$.

Let $L = f^{-1}(f(\alpha))$. By §3.6, there are the following possibilities:

(1) There is a closed geodesic in $L \cap A^o$.

(2) There is a geodesic $\gamma: I \to A$ for an interval $I$ passing through a point of $A^o$ such that $\gamma(\delta I) \subset \delta A$.

(3) $L \subset \delta A$.

In case (1), Proposition 2.2(1) implies that the closed geodesic is a covering map onto an imbedded tight curve $\eta$ in $A^o$. Let $\eta': \mathbf{S}^1 \to A^o$ be a simple tight curve realizing $\eta$, and let $\alpha': \mathbf{S}^1 \to A$ be a tight curve realizing $\alpha$. Since $\eta'$ is a closed geodesic, the image of $f \circ \eta'$ is $f(\alpha)$. This implies that $f \circ \eta' \circ c$ is a reparametrization of $f \circ \alpha' \circ c'$ for finite covering maps $c$ and $c': \mathbf{S}^1 \to \mathbf{S}^1$. Thus, if $\alpha$ is strong, then $\eta$ is also strong by

Lemma 3.1. However, this contradicts §3.2. If $\alpha$ is weak, then $\eta$ is also weak; this is also a contradiction.

In case (2), the geodesic corresponds to a path in $(B^o \cup \eta^o) - \{x\}$ connecting two points of $\eta^o - \{x\}$ and passing through a point of $B^o$ (see §3.2). Such a path cannot be geodesic, and this is a contradiction.

Since cases (1) and (2) do not occur, $L \subset \delta A$. In this case, $L$ is either a component of $\delta A$ or $\delta A$ itself, and every point of $L$ is regular. Thus $f(\alpha)$ is an imbedded tight curve, and $f|\alpha$ is a finite covering map onto $f(\alpha)$, which we claimed. Similarly, it follows that $f|\beta$ is a finite covering map onto an imbedded tight curve $f(\beta)$.

Let $S'$ be the cover of $S$ corresponding to $f(\alpha)$. The map $f$ lifts to a projective map $f': A \to S'$. Proposition 2.2 implies that $f'|\alpha$ and $f'|\beta$ are finite covering maps onto the imbedded tight curves $f'(\alpha)$ and $f'(\beta)$ respectively.

Let $H$ be the continuous function defined on $S'$ that measures $d$-distances from $f'(\alpha)$. Since $f'$ is an immersion, $H \circ f'$ achieves the maximum value at points of $\delta A$. This implies that $f'(\alpha) \neq f'(\beta)$. Since $f'(\alpha) \cap f'(\beta) = \varnothing$ by Proposition 2.2, $f'(\alpha)$ and $f'(\beta)$ bound a unique compact annulus, say $A'$, in $S'$. Using $H$, we can show similarly that the image of $f'$ lies in $A'$. In fact, $f'(A^o) \subset A'^o$.

Since $f'(\alpha) \neq f'(\beta)$, Lemma 5 of Appendix A implies that $f(\alpha) \neq f(\beta)$. Proposition 2.2 implies that $f(\alpha) \cap f(\beta) = \varnothing$. Hence, $f(\alpha)$ and $f(\beta)$ bound a unique compact annulus $F$. Section 6 of Appendix A shows that the covering map from $S'$ to $S$ restricted to $A'$ is an imbedding onto $F$. Hence, the image of $f$ lies in $F$. Let $c_A: A \to F$ be the immersion such that $f = i_F \circ c_A$ for the inclusion map $i_F: F \to S$. Each of $c_A|\alpha$ or $c_A|\beta$ is a covering map onto an appropriate boundary component of $F$. Since $A$ is compact, the inverse image of each point of $F$ under $c_A$ is a finite set. It is clear that $c_A(x) \in \delta F$ if and only if $x \in \delta A$. These imply that $c_A$ is a covering map. By the definition of $\pi$-annuli, an orientable quotient projective surface of a $\pi$-annulus is a $\pi$-annulus. Thus, $F$ is a $\pi$-annulus. This completes the proof of Proposition 3.3.

**3.8.** Let us now discuss imbedded $\pi$-annuli and maximal annuli including them. We need the following lemma.

**Lemma.** *Let $A$ be an imbedded $\pi$-annulus in $S$. Let $\alpha: \mathbf{S}^1 \to S$ be a tight curve. Then $\alpha$ intersects trivially in homotopy with every closed curve realizing a component of $\delta A$. Moreover, if $\alpha$ is simple, then we have either $\alpha_1 = \beta$ or $\alpha_1 \cap \beta = \varnothing$ for an arbitrary component $\beta$ of $\delta A$ and the image $\alpha_1$ of $\alpha$.*

*Proof.* Suppose that $\alpha$ intersects essentially with a component $\beta$ of $\delta A$. By §3.6, $\alpha_1$ is a geodesic complex. Since $\alpha_1 \cap A^o \neq \varnothing$ and $\delta A$ is geodesic, $\alpha_1 \cap A$ is a geodesic complex in $A$. Since no regular arc of $\alpha_1$ is tangent to $\beta$ and $\alpha_1 \cap \beta \neq \varnothing$, a regular arc of $\alpha_1 \cap A$ ends at a point of $\beta$ transversally. Let $p$ be a point of $A^o$ on the regular arc. Section 3.6 implies that there is a geodesic $\gamma \colon I \to A$ passing through $p$ such that $\gamma(\delta I) \subset \delta A$. This is a contradiction as in the case (2) of §3.7. The rest of the conclusion follows from Proposition 2.2.

**3.9.** We claim that $S$ can include only finitely many $\pi$-annuli. Proposition 2.2(7) and the fact that an imbedded $\pi$-annulus is the closure of a component of the complement of two imbedded tight curves imply that there are only finitely many imbedded $\pi$-annuli freely homotopic to a given simple closed curve. If two $\pi$-annuli $A$ and $B$ are not freely homotopic to each other, then each component of $\delta A$ is disjoint from the components of $\delta B$ by the above lemma, and it follows that $A$ and $B$ are disjoint. Since given a compact surface, every collection of imbedded essential closed curves disjoint from one another and not freely homotopic to one another has finitely many elements, $S$ includes only finitely many imbedded $\pi$-annuli.

**3.10.** We now discuss maximal annuli. Given an arbitrary hyperbolic or quasi-hyperbolic annulus $A$ with geodesic boundary, it is a subset of a unique maximal annulus by Proposition 2.2(7) and the annulus decomposition theorem of Appendix B. We restrict our study to a particular class of maximal annuli. We denote by $M(S)$ the collection of maximal annuli that are either freely homotopic to imbedded $\pi$-annuli or freely homotopic to the components of $\delta S$ that have hyperbolic or quasi-hyperbolic holonomy. Note that if the above $A$ is freely homotopic to a component of $\delta S$, then $A$ is a subset of an element of $M(S)$. By Proposition 2.2 and Lemma 3.8, we obtain the next lemma.

**Lemma.** *Each imbedded $\pi$-annulus is a subset of an element of $M(S)$, for every distinct elements $a$ and $a'$ of $M(S)$, the annuli $a$ and $a'$ are not freely homotopic and satisfy $a \cap a' = \varnothing$, and $M(S)$ is finite.*

## 4. Convex surfaces

**4.1.** Suppose that $S$ is convex and compact. Then $S$ is projectively homeomorphic to the quotient projective surface $\Omega/\Gamma$ for a convex domain $\Omega$ of $S^2$ and a projective automorphism group $\Gamma$ acting properly discontinuously and freely on $\Omega$ by Lemma 1.5 of [5]. Moreover, $Cl(\Omega)$

is a simply convex disk in $S^2$, including no antipodal pair. Let us identify $\Omega$ with $\widetilde{S}$ in this section; this identifies $\Gamma$ with the deck transformation group of $\widetilde{S}$. (For examples of convex surfaces, see Goldman [16].)

It is well known that $Cl(\Omega)$ is not a triangle, a simply convex disk bounded by the union of three segments (see [1], [15], [17], [25]). Suppose not. Then there is a homomorphism from $\Gamma$ to the permutation group of the three vertices of the triangle, and the kernel of this homomorphism is abelian since the vertices are fixed under the action of the kernel. Since $\pi_1(S)$ is not virtually abelian, this is a contradiction.

**4.2.** Suppose that $\vartheta$ is a hyperbolic automorphism in $\Gamma$. Then $Cl(\Omega)$ contains a unique attractor and a unique repeller. Let us denote the points by $s$ and $w$ respectively. $Cl(\Omega)$ contains at most one fixed point that is not an attractor or a repeller. Let us denote by $m$ the third fixed point if it exists. Hence, $\overline{sw}$ is a unique invariant segment in $Cl(\Omega)$ if $m$ does not exist. $\overline{sw}$, $\overline{sm}$, and $\overline{mw}$ are all the invariant segments in $Cl(\Omega)$ if $m$ exists. They bound a unique invariant triangle $\Delta(smw)$ in $Cl(\Omega)$. Suppose that $\varphi$ is a quasi-hyperbolic element of $\Gamma$. Then, similarly, $Cl(\Omega)$ contains a unique fixed point of the largest eigenvalue and a unique fixed point of the smallest eigenvalue. Let us denote them by $h$ and $l$ respectively. Then the unique $\varphi$-invariant segment in $Cl(\Omega)$ is $\overline{hl}$.

We will need the following preliminary lemma, on the properties of invariant segments. Note that the proof of (3) differs from that of Kuiper [20]. However, the lemma itself may be proved along Kuiper's argument.

**Lemma.** (1) *If $m$ exists, then $\overline{sm}$, $\overline{mw} \subset \mathrm{bd}\,\Omega$.*

(2) $\overline{hl} \subset \mathrm{bd}\,\Omega$.

(3) *Each nontrivial element $\gamma$ of $\Gamma$ does not correspond to a matrix of type* (4).

(4) $\overline{sw}^o \subset \Omega$.

(5) *If $m$ exists, then one of $\overline{sm}^o$ and $\overline{mw}^o$ is a subset of $\Omega$, and the other is disjoint from $\Omega$.*

(6) $\overline{hl}^o \subset \Omega$.

*Proof.* (1) Suppose that $\overline{sm}^o \cap \Omega^o \neq \varnothing$. Let $p$ be a point of this intersection. Let $S^1$ be the $\vartheta$-invariant great circle including $\overline{sm}$. Let $U$ be an open disk in $\Omega$ containing $p$ such that $U - S^1$ has two components. Choose a point $x$ from a component of $U - S^1$, and choose a point $y$ from the other component. Then $\{\vartheta^{-n}(x)\}$ converges to a repeller of $\vartheta$, and $\{\vartheta^{-n}(y)\}$ converges to the antipodal point of the repeller. These points are in $Cl(\Omega)$. This contradicts the fact that $\Omega$ is simply convex.
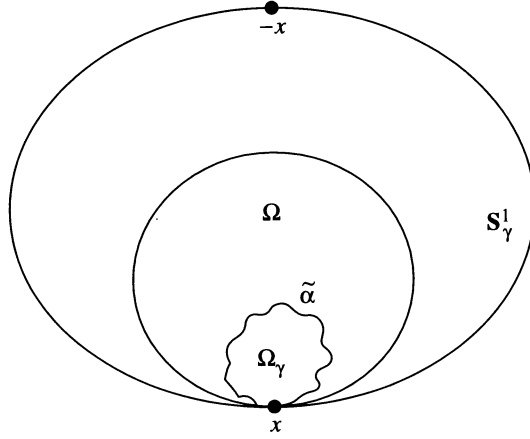
FIGURE 3. THE ACTION ON $\Omega$ OF A PROJECTIVE AUTO-
MORPHISM CORRESPONDING TO A MATRIX OF TYPE (4).

Hence, $\overline{sm} \subset \operatorname{bd}\Omega$. It follows similarly that $\overline{mw} \subset \operatorname{bd}\Omega$.

(2) A similar argument also shows that $\overline{hl} \subset \operatorname{bd}\Omega$.

(3) By lifting to a finite cover of $S$, if necessary, we may assume without loss of generality that $\gamma$ corresponds to an imbedded closed curve $\alpha$ in $S^o$. Then $\widetilde{S}$ includes a simple component arc $\tilde{\alpha}$ of the inverse image of $\alpha$ that is $\gamma$-invariant.

Suppose that $\gamma$ corresponds to a matrix of type (4). The automorphism $\gamma$ has an invariant great circle $\mathbf{S}^1_\gamma$ and two fixed points, which are on $\mathbf{S}^1_\gamma$, antipodal to each other. Since $\operatorname{Cl}(\Omega)$ is $\gamma$-invariant, it follows that $\operatorname{bd}\Omega$ contains a fixed point $x$ of $\gamma$; hence, the fixed points of $\gamma$ are $x$ and $-x$. The component of $\mathbf{S}^2 - \mathbf{S}^1_\gamma$ including $\Omega^o$ is such that for each compact subset $A$ of it, the sequence $\{\gamma^n(A)\}$ converges to $\{x\}$ as $n \to \infty$. Moreover, $\{\gamma^{-n}(A)\}$ also converges to $\{x\}$ as $n \to \infty$ (see Figure 3). Hence, $\tilde{\alpha}$ is a simple curve starting and ending at $x$ such that $\tilde{\alpha} \cup \{x\}$ is an imbedded closed curve (not necessarily differentiable).

It follows that $\Omega$ includes the unique open disk $\Omega_\gamma$ that is a component of $\Omega - \tilde{\alpha}$ and whose topological boundary set in $\mathbf{S}^2$ equals $\tilde{\alpha} \cup \{x\}$. By Lemma 4 of Appendix A, $\Gamma$ includes an element $\tau$ that sends a point $\tilde{\alpha}$ into $\Omega_\gamma$ and does not commute with $\gamma$. Since $\alpha$ is simple, $\tau(\tilde{\alpha})$ is disjoint from $\tilde{\alpha}$ and, hence, is a subset of $\Omega_\gamma$. The curve $\tau(\tilde{\alpha})$ is a simple curve starting and ending at $\tau(x)$. Since $\tau(\tilde{\alpha})$ is infinitely long under the metric $d$ of $\widetilde{S}$ (see §2.2 and Appendix A), $\tau(\tilde{\alpha})$ must end at $x$ also. Thus $\tau(x) = x$. Thus, $\tau \circ \gamma \circ \tau^{-1}$ fixes $x$ and corresponds to a matrix of

type (4). $\mathbf{S}_\gamma^1$ is the unique supporting great circle of $\mathrm{Cl}(\Omega)$ at $x$. Since $\tau(\mathbf{S}_\gamma^1)$ is also a supporting great circle of $\mathrm{Cl}(\Omega)$ at $x$, we have $\mathbf{S}_\gamma^1 = \tau(\mathbf{S}_\gamma^1)$. This means that $\gamma$ and $\tau \circ \gamma \circ \tau^{-1}$ commute since they are both of type (4) and share an invariant great circle and a fixed point. This contradicts the fact that $\gamma$ and $\tau$ do not commute (see §2 of Appendix A).

(4) The segment $\overline{sw}$ is a subset of $\mathrm{Cl}(\Omega)$. If $\overline{sw}^o \cap \Omega^o \neq \varnothing$, then $\overline{sw}^o \subset \Omega^o$. Suppose that $\overline{sw} \subset \mathrm{bd}\,\Omega$ and that $\overline{sw} \subset \mathrm{Cl}(\Omega) - \Omega$. As in (3), we assume without loss of generality that $\vartheta$ corresponds to an imbedded closed curve $\alpha$. Let $\tilde{\alpha}$ be the simple component arc of the inverse image of $\alpha$ in $\tilde{A}$, which is $\vartheta$-invariant. A description of the action of $\langle \vartheta \rangle$ implies that $\tilde{\alpha}$ is a simple arc with endpoints $s$ and $w$ and that $\tilde{\alpha} \cup \overline{sw}$ is the topological boundary in $\mathbf{S}^2$ of an open disk $\Omega_\vartheta$ that is a component of $\Omega - \tilde{\alpha}$. As before, we obtain a deck transformation $\tau$ that maps a point of $\tilde{\alpha}$ into $\Omega_\vartheta$ and that does not commute with $\vartheta$. Now an argument similar to one in (3) shows that this is a contradiction. Hence, $\overline{sw} \cap \Omega \neq \varnothing$. Let $\beta$ be a component of $\overline{sw}^o \cap \Omega$. Since each component of $\overline{sw}^o \cap \Omega$ is the image of a lift of a component curve of $\delta S$, it follows that $\beta$ is invariant under an element of $\Gamma$. Call it $\tau'$. Then the action of $\langle \tau' \rangle$ on $\overline{sw}$ cannot be nontrivial unless $\beta = \overline{sw}^o$ (see §7.1 of [5]). Hence, $\overline{sw}^o \subset \Omega$.

(5) Similarly to the proof of (4), it follows that $(\overline{sm}^o \cup \overline{mw}^o) \cap \Omega$ is not empty. Assume without loss of generality that $\overline{sm}^o \cap \Omega$ is not empty. Then similarly to the proof of (4), $\overline{sm}^o \subset \Omega$. Since the action of $\langle \vartheta \rangle$ on $\Omega$ is properly discontinuous, $\overline{mw}^o$ is disjoint from $\Omega$. Hence (5) is proved.

(6) This can be proved similarly to (4).

**4.3.** We present the proof of the following generalization of the work of Kuiper [20].

**Lemma.** *Let $\gamma$ be an essential closed curve in $S$. Then $\gamma$ is hyperbolic or quasi-hyperbolic.*

*Proof.* Let $\vartheta$ be an element of $\Gamma$ corresponding to $\gamma$. Since $\vartheta$ is orientation preserving, $\vartheta$ corresponds to a matrix $T$ belonging to $\mathrm{SL}(3, \mathbf{R})$. Suppose that $T$ is of type (1), (2), or (3) and that the eigenvalues of $T$ are positive. Then Proposition 2.3 implies that $\vartheta$ is hyperbolic or quasi-hyperbolic.

We will show that no other possibility for $T$ can happen. From the simple convexity of $\mathrm{Cl}(\Omega)$ it follows easily that $T$ is not of type (1), (2), or (3) with negative eigenvalues.

The Brouwer fixed-point theorem implies that $\mathrm{Cl}(\Omega)$ contains at least one fixed point of $\vartheta$ in $\mathrm{bd}\,\Omega$. Let us choose one, which we denote by $p$.

Let $G$ be the set of supporting great circles at $p$. It is clear that $G$ forms a topological space homeomorphic to a segment or is the set consisting of one element. The action of $\langle \vartheta \rangle$ induces an action on $G$. Hence, an element of $G$ is fixed under $\vartheta$ by the Brouwer fixed-point theorem. The corresponding supporting great circle is $\vartheta$-invariant. If $T$ is of type (7), then no great circle passing through the fixed points of $\vartheta$ is $\vartheta$-invariant. Hence, $T$ is not of type (7).

Since $\Gamma$ is isomorphic to $\pi_1(S)$, it contains no elements of order two. If $T$ is of type (6), then $T^2 = \mathrm{Id}$. Hence, $T$ is not of type (6).

Suppose that $T$ is of type (5). Let $\eta$ be an arbitrary segment with endpoints $p$ and $-p$ passing through a point $r$ of $\Omega^0$. Then $\eta \cap \mathrm{Cl}(\Omega)$ is a simply convex segment with two endpoints, one of which is $p$. The other endpoint is not $-p$. Let it be called $q$. Then $q$ is a fixed point of $\vartheta^2$ since $\eta$ is a $\vartheta^2$-invariant segment. Since $q$ is fixed, every point of $\eta$ is a fixed point of $\vartheta^2$. Hence, $r$ is fixed. This contradicts the fact that $\langle \vartheta^2 \rangle$ acts freely on $\Omega$.

By Lemma 4.2, $T$ is not of type (4). The proof is completed.

**4.4.** A *purely convex surface* is a convex compact surface $M$ with negative Euler characteristic that does not include a compact annulus with geodesic boundary freely homotopic to a component of $\delta M$ or include a trivial annulus. Since every principal closed geodesic is an imbedded strong tight curve, Proposition 4.5(8) implies that convex compact surfaces with principal geodesic boundary are examples of purely convex surfaces (see Goldman [16]).

Many convex surfaces are not purely convex. By a summation method described in §3.7 of Goldman [16], we may obtain a compact surface $M_1$ with geodesic boundary, which decomposes into a convex compact surface with principal geodesic boundary and an elementary annulus of type I. $M_1$ is convex by Lemma 5.4.

Note that one can obtain a convex surface with a boundary component that is a quasi-hyperbolic tight curve using Goldman's techniques [16]. There is a convex real projective structure on a compact pair-of-pants such that one of the boundary invariants equals $(\lambda, \tau)$ for every real numbers $\lambda$ and $\tau$ satisfying $0 < \lambda < 1$, $\tau = 2/\sqrt{\lambda}$. This can be obtained by a slight extension of Theorem 4.1 of [16]. (The proof of the remark is to be supplied in another paper.) Since the boundary invariant is as above, the boundary component is not hyperbolic but quasi-hyperbolic by Lemma 4.3.

**4.5.** A *boundary elementary annulus* in $S$ is an elementary annulus including a component of $\delta S$. The following proposition is a generaliza-

tion of the results of Kuiper [20] that every essential simple closed curve in a closed convex surface with negative Euler characteristic is freely homotopic to a closed geodesic.

**Proposition.** *Let $\alpha$ be an imbedded essential closed curve in $S$, and let $\vartheta$ be an element of $\Gamma$ corresponding to $\alpha$.*

(1) *If* bd $\Omega$ *includes a $\vartheta$-invariant segment, then $\alpha$ is freely homotopic to a tight-curve component of $\delta S$, which corresponds to a $\vartheta$-invariant open line in $\delta \Omega$.*

(2) *If $\alpha$ is hyperbolic and is freely homotopic to no component of $\delta S$, then $S^o$ includes a unique imbedded tight curve freely homotopic to $\alpha$, which is strong.*

(3) *If $\alpha$ is hyperbolic and is freely homotopic to a component of $\delta S$, then $S$ either*

(i) *includes a unique imbedded tight curve freely homotopic to $\alpha$, which is strong, or*

(ii) *includes two imbedded tight curves freely homotopic to $\alpha$, where one is strong and lies in $S^o$ (the other is weak and is a component of $\delta S$).*

(4) *If $\alpha$ is quasi-hyperbolic, then $S$ includes a unique imbedded tight curve freely homotopic to $\alpha$, which is strong and is a component of $\delta S$.*

(5) *$\alpha$ is hyperbolic if $\alpha$ is not freely homotopic to a component of $\delta S$.*

(6) *$S$ includes a unique imbedded strong tight curve freely homotopic to $\alpha$ and at most one imbedded weak tight curve freely homotopic to $\alpha$, which must be a component of $\delta S$.*

(7) *If $S$ is purely convex, then $S$ includes a unique imbedded tight curve freely homotopic to $\alpha$, which is strong.*

(8) *Every component of $\delta S$ is an imbedded strong tight curve if and only if $S$ is purely convex.*

(9) *$S$ is the sum of a purely convex surface and subsurfaces that are trivial annuli or boundary elementary annuli. Every two distinct elements of the collection of the trivial annuli and the boundary elementary annuli are disjoint. Each component $\beta$ of $\delta S$ is a strong tight curve if and only if $\beta$ is a boundary component of the purely convex surface but is not that of the trivial annuli or the boundary elementary annuli.*

*Proof.* (1) Let us denote the endpoints of the invariant segment by $p$ and $q$ which are fixed points of $\vartheta$. Suppose that $\vartheta$ is hyperbolic and that one of $p$ or $q$ is not an attractor or a repeller of $\vartheta$. We assume without loss of generality that $p$ is a repeller and that $q$ is not an attractor. Then $\mathrm{Cl}(\Omega)$ includes an attractor $r$. Lemma 4.2(1) implies that two invariant segments $\overline{pq}$ and $\overline{qr}$ are subsets of bd $\Omega$. The interior $\eta$ of one of these

segments is a subset of $\Omega$ by Lemma 4.2(5). Since $\eta$ is $\vartheta$-invariant, under the quotient map from $\Omega$, the open line $\eta$ corresponds to an imbedded tight curve that is a component of $\delta S$ and to which $\alpha$ is freely homotopic.

Suppose now that $\vartheta$ is hyperbolic and that each of $p$ and $q$ is an attractor or a repeller of $\vartheta$. Then similarly to above, Lemma 4.2(4) implies that $\overline{pq}^o$ is a subset of $\Omega$ and corresponds to a tight curve that is a component of $\delta S$ and to which $\alpha$ is freely homotopic. If $\vartheta$ is quasi-hyperbolic, then the desired conclusion follows similarly.

(2) Since $\vartheta$ is hyperbolic, bd $\Omega$ contains an attractor and a repeller of $\langle\vartheta\rangle$. Let them be denoted by $p$ and $q$ respectively. Since $\alpha$ is not freely homotopic to a boundary component, $\overline{pq}$ is not a subset of bd $\Omega$ by (1). Hence, $\overline{pq}^o \subset \Omega^o$. This line corresponds to an imbedded strong tight curve in $S^o$ freely homotopic to $\alpha$ by Proposition 2.2. Suppose that $S$ includes another tight curve freely homotopic to $\alpha$. Then it corresponds to a $\vartheta$-invariant open line in $\Omega$, which does not connect $p$ and $q$. By Lemma 4.2(1), the open line should be a subset of bd $\Omega$. Since (1) gives the contradiction that $\alpha$ is freely homotopic to a component of $\delta S$, the uniqueness follows.

(3) Let $p$ and $q$ denote the attractor and the repeller of $\vartheta$ in $\Omega$ respectively. Suppose that no other fixed point of $\vartheta$ belongs to Cl($\Omega$). Similarly to (2), Lemma 4.2(4) and Proposition 2.2 imply that $S$ includes a unique imbedded tight curve, which is strong. Suppose that a fixed point other than $p$ or $q$ belongs to Cl($\Omega$). Let it be denoted by $r$. It follows that $\vartheta$-invariant segments in Cl($\Omega$) are $\overline{pq}$, $\overline{pr}$, and $\overline{qr}$. By Lemma 4.2(1), $\overline{pr}$ and $\overline{qr}$ are subsets of bd $\Omega$. Since $\Omega$ is not a triangle, we have $\overline{pq}^o \subset \Omega^o$. This line corresponds to an imbedded strong tight curve freely homotopic to $\alpha$ by Proposition 2.2. Exactly one of $\overline{pr}^o$ and $\overline{qr}^o$ is a subset of $\Omega$ by Lemma 4.2(5); it corresponds to a component of $\delta S$, which is a weak tight curve and is freely homotopic to $\alpha$. Since $\overline{pq}^o$, $\overline{pr}^o$, and $\overline{qr}^o$ are the only $\vartheta$-invariant open lines in $\Omega$, there is no other imbedded tight curve freely homotopic to $\alpha$.

(4) It follows similarly to (3) that $S$ includes a unique imbedded tight curve freely homotopic to $\alpha$. By Lemma 4.2(2), the tight curve is a component of $\delta S$.

(5) This follows from (4).

(6) This follows from (2), (3), and (4).

(7) This follows from (6).

(8) Suppose that one of the components of $\delta S$ is not a strong tight curve. Then $S^o$ includes a unique imbedded strong tight curve $\beta$ freely homotopic to the component by (6). Let $E$ be the compact annulus

bounded by $\beta$ and the component of $\delta S$. By Lemma 2.5 and (6), $E$ is a trivial annulus or a boundary elementary annulus. Hence, $S$ is not purely convex.

The converse result follows from (6).

(9) For each component of $\delta S$, there is a unique imbedded strong tight curve freely homotopic to it. Hence, either each component of $\delta S$ is strong or $S$ is the sum of $S_0$ and $A_1, \cdots, A_n$, where $S_0$ is a surface such that components of $\delta S_0$ are strong tight curves, and $A_i$ for each $i$ is a compact annulus with convex boundary. In the former case, $S$ is purely convex by (8), and we are done. Let us assume that we have the second case. Since $S$ includes no imbedded $\pi$-annulus by Theorem 3.4, $S_0$ includes no imbedded $\pi$-annulus. Hence, $S_0$ is a convex surface by Theorem 3.4. By (8), $S_0$ is purely convex. Lemma 2.5 and (6) imply that each $A_i$ is a boundary elementary annulus or a trivial annulus. The conclusions of (9) follow easily.

**4.6.** Let us end this section with the following corollary to Proposition 4.5. Let us assume now that $S$ is not necessarily convex. Suppose that $A$, $T$, and $P$ respectively are a compact annulus with geodesic boundary, a trivial annulus, and a purely convex surface in $S$. Since $T$ and $P$ are convex, every closed curve in $T$ and $P$ is a geodesic if and only if it is a tight curve (see §2). Notice also that if $A$ is hyperbolic or quasi-hyperbolic, then the components of $\delta A$ are imbedded tight curves by the annulus decomposition theorem in Appendix B.

**Corollary.** (1) *Each component of $\delta P$ or $\delta A$ is not a subset of $T^o$.*

(2) *Each component of $\delta A$ or $\delta T$ is not a subset of $P^o$.*

(3) *Each component of $\delta T$ or $\delta P$ is not a subset of $A^o$.*

*Proof.* (1) Suppose that a component of $\delta A$ is a subset of $T^o$. Then $A$ is hyperbolic or quasi-hyperbolic. Since components of $\delta A$ are tight curves, Lemma 2.5(2) imply contradiction. Similarly, no component of $\delta P$ is a subset of $T^o$.

(2) Let $\alpha$ and $\beta$ be components of $\delta A$. Suppose that $\alpha \subset P^o$. Then $A$ is hyperbolic or quasi-hyperbolic. We now deduce properties of $A \cap P$, which is nonempty. By Proposition 2.2, the set of boundary points of $A \cap P$ with respect to the relative topology of $A$ is the union of tight curves that are components of $\delta P$ or is the empty set. Hence, a component $F$ of $A \cap P$ including $\alpha$ can be one of the following:

(a) a tight curve that is a common component of $\delta A$ and $\delta P$;

(b) a compact annulus $B$ bounded by tight curves in $A$ where one component of $\delta B$ is a component of $\delta P$, and the other component of $\delta B$ is a component of $\delta A$; or

(c) $A$ itself.

Since $\alpha \subset P^o$, (a) is not possible. Since Proposition 4.5(7) implies that $\alpha$ is not freely homotopic in $P$ to a component of $\delta P$, (b) is not possible. Proposition 4.5(7) also implies that $A$ is not a subset of $P$. Thus, (c) is also not possible. Therefore, no component of $\delta A$ is a subset of $P^o$.

A similar argument shows that a component of $\delta T$ is not a subset of $P^o$.

(3) Suppose that $\gamma$ is a component of $\delta T$ in $A^o$. Thus, $A$ is hyperbolic or quasi-hyperbolic, and $T^o \cap A^o \neq \varnothing$. Then (1) and Proposition 2.2 imply that the components $\alpha$ and $\beta$ of $\delta A$ are disjoint from $T^o$. Hence $T^o \cap A^o = T^o \cap A$. Since $T^o \cap A$ is thus a relatively open and closed subset of $T^o$, we have $T^o \cap A = T^o$ and $T^o \subset A$. Since $T$ cannot be a subset of $A$, this is a contradiction.

Suppose that $\gamma$ is a component of $\delta P$ in $A^o$. Thus, $P^o \cap A^o \neq \varnothing$. Then (2) and Proposition 2.2 imply similarly to the above paragraph that $\alpha$ and $\beta$ are disjoint from $P^o$. Similarly to the above, a contradiction follows. Thus the desired conclusion follows.

Let us remark about a consequence of the above corollary: given arbitrary two surfaces $S_1$ and $S_2$ among $A$, $T$, and $P$, if components of $\delta S_1$ and $\delta S_2$ intersect trivially in homotopy with one another, and $A$ is hyperbolic or quasi-hyperbolic, then $S_1 \cap S_2$ is either empty or the union of common components of $\delta S_1$ and $\delta S_2$.

## 5. The proof of the admissible decomposition theorem

In this section, we prove the admissible decomposition theorem. Let $\Sigma$ be a compact real projective surface with convex boundary and negative Euler characteristic.

**5.1.** Let us denote the collection of trivial annuli in $\Sigma$ by $T(\Sigma)$, and let $\mathscr{T}(\Sigma)$ denote the union of its elements. Recall from §3.10 that $M(\Sigma)$ denotes the collection of maximal annuli freely homotopic to $\pi$-annuli or hyperbolic or quasi-hyperbolic components of $\delta\Sigma$, and let $\mathscr{M}(\Sigma)$ denote the union of its elements. Let us denote by $C(\Sigma)$ the collection of the closures in $\Sigma$ of components of $\Sigma - \mathscr{M}(\Sigma) - \mathscr{T}(\Sigma)$.

We claim the existence of an admissible decomposition collection: $\Sigma$ is the sum of all subsurfaces that are elements of $T(\Sigma)$, $M(\Sigma)$, or $C(\Sigma)$. The elements of $C(\Sigma)$ are maximal purely convex surfaces.

Let us begin the proof. Consider two arbitrary elements of $T(\Sigma)$ or $M(\Sigma)$. Then they either are disjoint or intersect precisely at the union of their common boundary components by §§2.6 and 4.6 and Lemma 3.10.

Hence, $\mathcal{T}(\Sigma) \cup \mathcal{M}(\Sigma)$ is the union of compact annuli disjoint from one another. The topological boundary of the set is the union of imbedded tight curves in $\Sigma^o$ disjoint from one another. Thus, each element of $C(\Sigma)$ is a compact surface with convex boundary.

Let $S$ be an element of $C(\Sigma)$. Suppose that $S$ is an annulus. Since $\chi(\Sigma) < 0$, a component of $\delta A$ for a maximal annulus $A$, an element of $M(\Sigma)$, equals a component of $\delta S$. The annulus decomposition theorem of Appendix B implies that $S$ is the sum of elementary annuli and trivial annuli in $S$. Since a trivial annulus in $S$ is a trivial annulus in $\Sigma$, it follows that $S$ is the sum of elementary annuli. But this gives a contradiction that $A$ is not maximal. Hence, $\chi(S) < 0$. Since every imbedded $\pi$-annulus in $\Sigma$ is a subset of an element of $M(\Sigma)$, the surface $S$ is convex by Theorem 3.4.

By construction, each component of $\delta S$ either is a component of $\delta \Sigma$ or is a boundary tight-curve component of a maximal annulus in $M(\Sigma)$ or a trivial annulus in $T(\Sigma)$ in $\Sigma^o$. Let us state two consequences of this:

(1) Each component of $\delta S$ is a strong tight curve.

(2) Corollary 4.6 implies that $S$ is not a proper subset of a purely convex surface.

Let us prove (1): Let $\alpha$ be an arbitrary component of $\delta S$.

(i) Suppose that $\alpha$ is a component of $\delta \Sigma$. If a trivial annulus or a boundary elementary annulus $A$ in $S$ includes $\alpha$, then $A$ is a trivial annulus in $\Sigma$ or a boundary elementary annulus in $\Sigma$. A boundary elementary annulus in $\Sigma$ is a subset of an element of $M(\Sigma)$ by §3.10. Since $\alpha$ is in the complement of $\mathcal{T}(\Sigma) \cup \mathcal{M}(\Sigma)$, it follows that $\alpha$ is not a subset of a trivial annulus or a boundary elementary annulus in $S$. By Proposition 4.5(9), $\alpha$ is a strong tight curve.

(ii) Suppose that $\alpha$ is a component of $\delta A$ for a maximal annulus $A$ in $M(\Sigma)$. If $\alpha$ is not strong, then $\alpha$ is a subset of a boundary elementary annulus in $S$ by Proposition 4.5(9). This yields a contradiction that $A$ is not maximal. Thus, $\alpha$ is a strong tight curve.

(iii) If $\alpha$ is a boundary tight-curve component of a trivial annulus in $T(\Sigma)$, then $\alpha$ is strong (see §3.1).

Consequently (1) and Proposition 4.5(8) imply that $S$ is purely convex. By (2), $S$ is a maximal purely convex surface. Therefore, the existence of an admissible decomposition collection follows.

**5.2.** Given two subsurfaces of a real projective surface, we say that they are *adjacent* if their intersection is the union of finitely many imbedded closed geodesics disjoint from one another.

We now claim the uniqueness: Suppose that $T'$, $M'$, and $C'$ respectively are a collection of trivial annuli, a collection of maximal annuli, and a collection of maximal purely convex surfaces in $\Sigma$. If $\Sigma$ is the sum of all the subsurfaces that are elements of $T'$, $M'$ or $C'$, then $T' = T(\Sigma)$, $M' = M(\Sigma)$, and $C' = C(\Sigma)$.

By §2.6, $T' \subset T(\Sigma)$. Because of the decomposition of $\Sigma$ into elements of $T'$, $M'$, and $C'$, the number of components of $\delta\Sigma$ that are not geodesic equals the number of elements of $T'$. Similarly, the number of nongeodesic components of $\delta\Sigma$ equals the number of the elements of $T(\Sigma)$. Thus, $T' = T(\Sigma)$.

Let $A$ be an element of $M'$. Suppose that $A$ is freely homotopic to a component of $\delta\Sigma$. Since $\chi(\Sigma) < 0$, Lemma 5.3 implies that at least one component of $\delta A$ is a boundary component of a maximal purely convex surface in $C'$. Hence, $A$ is hyperbolic or quasi-hyperbolic, and $A \in M(\Sigma)$. Suppose now that $A$ is not freely homotopic to a component of $\delta\Sigma$. Then each component of $\delta A$ is a boundary component of a maximal purely convex surface in $C'$. Hence, each component of $\delta A$ is strong by Proposition 4.5(8), and $A$ includes a $\pi$-annulus by Lemma 5.5. Thus, $A \in M(\Sigma)$, and $M' \subset M(\Sigma)$.

Suppose that $M'$ is a proper subset of $M(\Sigma)$. Then by Lemma 3.10, an element $A$ of $M(\Sigma) - M'$ is a subset of the union of some purely convex surfaces in $C'$ and trivial annuli. Since the intersection of $A$ with each trivial annulus is a geodesic boundary component of the trivial annulus or is empty by §4.6, $A$ is a subset of the union of purely convex surfaces belonging to $C'$. Since by Lemma 5.3, no two elements of $C'$ are adjacent, $A$ is a subset of a purely convex surface belonging to $C'$. This contradicts Proposition 4.5(7). Therefore, $M' = M(\Sigma)$. Since no two elements of $C(\Sigma)$ are adjacent, we also have $C' = C(\Sigma)$. Hence, the uniqueness is proved, and the proof of the admissible decomposition theorem is complete.

**5.3. Lemma.** *Suppose that two admissible subsurfaces in $\Sigma$ are adjacent. Then their types are different from each other.*

*Proof.* It is straightforward to prove this with the help of the following lemma.

**5.4.** The following lemma generalizes Theorem 3.7 of Goldman [16]. It is also true that the lemma may be proved by an extension of the proof given by Goldman [16].

**Lemma.** *Let $S_1, \cdots, S_n$ be convex compact subsurfaces of $\Sigma$. Suppose that $\chi(S_i) \leq 0$ for each $i$, $i = 1, \cdots, n$, and that $n \geq 2$. Assume the following conditions*:

- $\{S_1, \cdots, S_n\}$ *is a decomposition collection of a connected subsurface* $S'$ *of* $\Sigma$.
- *An annulus in the decomposition collection is not adjacent to another annulus in the decomposition collection.*
- $S_i \cap S_j$ *whenever* $i \neq j$ *is either the union of strong tight curves in* $S$ *or empty.*

Then $S'$ *is a convex subsurface of* $\Sigma$. *Moreover, if* $S_1, \cdots, S_n$ *are purely convex, then* $S'$ *is purely convex.*

*Proof.* The union $S'$ is a compact subsurface of $S$ with convex boundary and with $\chi(S') < 0$. Suppose that $S'$ is not convex. Then $S'$ includes an imbedded $\pi$-annulus. Let us call it $E$. There exists an integer $k$, $1 \leq k \leq n$, such that $E^o \cap S_k \neq \varnothing$. By Lemma 3.5, $E$ is not a subset of $S_k$. Therefore, $\delta S_k \cap E^o \neq \varnothing$. Let $\alpha$ be an arbitrary component of $\delta S_k$. Each component of $\delta E$ either is disjoint from $\alpha$ or is identical with $\alpha$ by Lemma 3.8. This fact implies that $E^o$ includes a component $\beta$ of $\delta S_k$. The second condition of the premise shows that $\beta$ is a component of $\delta S_l$ for some $l$ where $\chi(S_l) < 0$. Since $\beta$ is a strong tight curve, it follows that $\beta$ is a boundary component of a purely convex surface in $S_l$ by Proposition 4.5(9). However, this contradicts Corollary 4.6. Hence, $S'$ is convex, and the desired conclusions follow.

**5.5. Lemma.** *Let* $A$ *be a compact hyperbolic or quasi-hyperbolic annulus such that each component of* $\delta A$ *is a strong tight curve. Then* $A$ *includes a* $\pi$-annulus.

*Proof.* By the annulus decomposition theorem of Appendix B, $A$ is the sum of elementary annuli $E_i$, $i = 1, \cdots, n$, where $E_i \cap E_j = \varnothing$ if $|i - j| > 1$, and $E_i \cap E_{i+1}$ is a common component of $\delta E_i$ and $\delta E_{i+1}$. Suppose that $n = 1$. Since the boundary components are strong, $E_1$ is an elementary annulus of type IIa, a $\pi$-annulus. Assume now that $n \geq 2$. Suppose that $A$ is hyperbolic. Then $E_1$ and $E_2$ are elementary annuli of type I and meet at a weak tight curve. This implies that $E_1 \cup E_2$ is a $\pi$-annulus. Suppose that $A$ is quasi-hyperbolic. If $E_1$ is an elementary annulus of type IIa, then $E_1$ is a $\pi$-annulus. If $E_1$ is an elementary annulus of type IIb, then so is $E_2$, and $E_1$ and $E_2$ meet at a weak tight curve. It follows that $E_1 \cup E_2$ is a $\pi$-annulus. In all cases, $A$ includes a $\pi$-annulus.

## Appendix A: The topology of closed curves in surfaces

**1.** Let $S$ be a surface. Let $\widetilde{S}$ be the universal cover of $S$, and let pr: $\widetilde{S} \rightarrow S$ denote the universal covering map. Let $\mathbf{S}^1$ be a circle, and

let $c: \mathbf{R} \to \mathbf{S}^1$ denote the infinite cyclic covering map from the real line $\mathbf{R}$ to $\mathbf{S}^1$. A *closed curve* in $S$ is a regular arc from $\mathbf{S}^1$ to $S$. It is said to be *simple* if it is injective. (An *imbedded closed curve* is the image submanifold of an injective closed curve.) In this paper, we consider only closed curves that map into $S^o$ or $\delta S$. (Closed geodesics in real projective surfaces always have this property; see §2.1 of the main text.) A *lift* of a closed curve $\alpha: \mathbf{S}^1 \to S$ to $\widetilde{S}$ is a map $\tilde{\alpha}$ satisfying the following commutative diagram:

$$
\begin{array}{ccc}
\mathbf{R} & \xrightarrow{\ \tilde{\alpha}\ } & \widetilde{S} \\
{\scriptstyle c}\big\downarrow & & {\scriptstyle \mathrm{pr}}\big\downarrow \\
\mathbf{S}^1 & \xrightarrow{\ \alpha\ } & S.
\end{array}
$$

A *lift* of $\alpha$ to a cover $S'$ of $S$ is the map $\alpha'$ satisfying the following commutative diagram:

$$
\begin{array}{ccc}
\mathbf{S}^1 & \xrightarrow{\ \alpha'\ } & S' \\
\big\| & & {\scriptstyle \mathrm{pr}'}\big\downarrow \\
\mathbf{S}^1 & \xrightarrow{\ \alpha\ } & S,
\end{array}
$$

where $\mathrm{pr}': S' \to S$ is the covering map. (It will be clear from the context which one we mean by a "lift.")

A closed curve is said to be *essential* if it is not null-homotopic. Two closed curves $\alpha$ and $\beta$ are said to *intersect* if their images intersect. $\alpha$ *intersects* a set $A$ if the image of $\alpha$ intersects $A$. $\alpha$ and $\beta$ are said to intersect *trivially in homotopy* if they are homotopic to disjoint closed curves and are said to intersect *essentially* otherwise. Simple closed curves $\alpha$ and $\beta$ have *minimal intersection* if their image submanifolds $\alpha_1$ and $\beta_1$ have minimal intersection (see Casson and Bleiler [3]); i.e., $\alpha_1$ and $\beta_1$ intersect transversally and there are no arcs $\mu$ and $\nu$ in $\alpha_1$ and $\beta_1$ respectively having common endpoints and such that $\mu \cup \nu$ is the boundary of a disk in $S$.

Two closed curves $\alpha: \mathbf{S}^1 \to S$ and $\beta: \mathbf{S}^1 \to S$ are said to be *freely homotopic* if $\alpha$ and $\beta$ are homotopic. We need the following definitions for convenience in the main text. A closed curve $\alpha: \mathbf{S}^1 \to S$ is said to be *freely homotopic* to an imbedded closed curve $\beta$ in $S$ if $\alpha$ is homotopic to an imbedding from $\mathbf{S}^1$ realizing $\beta$. Two imbedded closed curves $\alpha$ and $\beta$ in $S$ are said to be *freely homotopic* if an imbedding from $\mathbf{S}^1$ realizing $\alpha$ is homotopic to that realizing $\beta$. An annulus $A$ in $S$ is freely *homotopic* to $\alpha$ if a component of $\delta A$ is freely homotopic to $\alpha$. Two

annuli $A$ and $B$ in $S$ are *freely homotopic* to each other if a component of $\delta A$ is freely homotopic to a component of $\delta B$.

2.   We now prove needed facts on curves in surfaces using hyperbolic structures. Let $\mathbf{H}^2$ be the hyperbolic plane. We identify $\mathbf{H}^2$ with the interior of the upper hemisphere of the complex projective space $\mathbf{CP}^1$ using the Poincaré model. A hyperbolic structure on $S$ with convex boundary whose associated metric is complete and such that each essential closed curve has hyperbolic holonomy is called a *generic hyperbolic structure* on $S$. Suppose that $S$ is homeomorphic to a compact annulus or a cover of a compact surface of negative Euler characteristic. Since a compact surface with negative Euler characteristic admits a generic hyperbolic structure (see [27] and [3]), it follows that $S$ admits a generic hyperbolic structure. Further, given an essential simple closed curve in $S$, there is a generic hyperbolic structure on $S$ such that the closed curve is geodesic (see Casson and Bleiler [3]). We may identify $\widetilde{S}$ with a convex domain $\Omega$ in $\mathbf{H}^2$ complete under the hyperbolic metric. Then $S$ is identified with $\Omega/\Gamma$ where $\Gamma$ is a discrete subgroup of the group $\mathrm{PSL}(2, \mathbf{R})$ of fractional linear transformations with real coefficients acting properly discontinuously and freely on $\Omega$. Here, $\Gamma$ is identified with the deck transformation group of $\widetilde{S}$. (Each element of $\Gamma$ is hyperbolic.)

Let us first discuss about commuting elements of $\Gamma$. Let $\vartheta$ and $\varphi$ be two nontrivial elements of $\Gamma$. Then $\vartheta$ and $\varphi$ commute if and only if $\vartheta$ and $\varphi$ have the same fixed point set (see Proposition I.D.3 in [21]). This means that $\vartheta$ and $\varphi$ commute if and only if $\vartheta$ and $\varphi$ have a common invariant complete geodesic in $\mathbf{H}^2$.

We claim that $\vartheta$ commutes with $\varphi$ if and only if $\vartheta$ and $\varphi$ are multiples of a common element of $\Gamma$. Suppose that $\vartheta$ and $\varphi$ commute. $\mathbf{H}^2$ includes a complete geodesic $l$ that is invariant under both $\vartheta$ and $\varphi$. Since the action of the group $\{\varphi' \in \Gamma | \varphi'(l) = l\}$ on $l$ is properly discontinuous and free, the group is an infinite cyclic group. Thus, $\vartheta$ and $\varphi$ are multiples of its generator. The converse portion of the statement is trivial to show.

Lastly, suppose that $\vartheta$ commutes with $\varphi \circ \vartheta \circ \varphi^{-1}$. Then we claim that $\vartheta$ commutes with $\varphi$. The proof is as follows: $\vartheta$ has an invariant complete geodesic $l$ in $\mathbf{H}^2$. Since $\varphi \circ \vartheta \circ \varphi^{-1}$ commutes with $\vartheta$, it also leaves $l$ invariant. Since $l$ is the unique invariant complete geodesic of $\vartheta$, we have $\varphi(l) = l$. Hence, $\varphi$ commutes with $\vartheta$.

3.   We now discuss on lifts of closed curves. Let $r$ and $s$ be the endpoints of a complete geodesic $l$ corresponding to an essential simple

closed curve. Then given $\vartheta$ in $\Gamma$, we have either that $\vartheta(r) = r$ and $\vartheta(s) = s$ or that $\vartheta(r)$ and $\vartheta(s)$ lie in a common component of $\operatorname{bd} H^2 - \{r, s\}$.

**Lemma.** *Suppose that $\alpha$ is an essential closed curve freely homotopic to a finite covering of a simple closed curve, and that $\tilde{\alpha}$ is a simple arc that is the image of a lift of $\alpha$ to $\tilde{S}$, an injective map. Suppose that $\tilde{\alpha}$ and $\vartheta(\tilde{\alpha})$ for a deck transformation $\vartheta$ are transversal. Then if $\tilde{\alpha}$ and $\vartheta(\tilde{\alpha})$ intersect at a point, then they intersect at another point.*

*Proof.* Since $\alpha$ is essential, $\tilde{\alpha}$ is an arc ending at two distinct points $r$ and $s$ of $\operatorname{bd} H^2$ (see the proof of Lemma 2.3 of Casson and Bleiler [3]). Let $\varphi$ be an element of $\Gamma$ corresponding to $\alpha$ and $\tilde{\alpha}$. Then $r$ and $s$ are the endpoints of a $\varphi$-invariant complete geodesic. Suppose that we have $r = \vartheta(r)$ and $s = \vartheta(s)$. Then $\vartheta$ and $\varphi$ commute. We have thus $\vartheta^m = \varphi^n$ for two nonzero integers $m$ and $n$ by §2. This means that $\langle \varphi^n \rangle$ acts on both $\tilde{\alpha}$ and $\vartheta(\tilde{\alpha})$. Hence, the conclusion follows. Suppose now that $\vartheta(r)$ and $\vartheta(s)$ lie in the same component of $\operatorname{bd} H^2 - \{r, s\}$. The conclusion follows easily in this case by transversality.

**4.** We will need the following lemma. Let $d$ denote the complete hyperbolic metric on $S$ corresponding to the generic hyperbolic structure. We denote the induced metric on an arbitrary cover of $S$ by $d$ also.

**Lemma.** *Suppose that $S$ is compact. Let $\alpha$ be an essential simple closed curve in $S$, and let $\tilde{\alpha}$ be the simple arc that is the image of a lift of $\alpha$ to $\tilde{S}$, an injective map. Let $\vartheta$ be a deck transformation of $\tilde{S}$ corresponding to $\alpha$ and $\tilde{\alpha}$. Suppose that a component $\tilde{S}_\alpha$ of $\tilde{S} - \tilde{\alpha}$ is homeomorphic to an open disk. Then there is a deck transformation $\varphi$ not commuting with $\vartheta$, that maps a point of $\tilde{\alpha}$ to a point of $\tilde{S}_\alpha$.*

*Proof.* We may assume without loss of generality that $\alpha$ is geodesic. Since $\tilde{S}_\alpha$ is an open disk, the completeness of $\tilde{S}$ implies that $\tilde{S}_\alpha$ is a component of $H^2$ removed the complete geodesic $\tilde{\alpha}$. Hence, there is a sequence of points $\{x_i\}$ in $\tilde{S}_\alpha$ such that $\{d(x_i, \tilde{\alpha})\} \to \infty$. Let $x \in \tilde{\alpha}$. Since $S$ is compact, for each $i$, there is a deck transformation $\vartheta_i$ such that $d(\vartheta_i(x), x_i) < 2 \operatorname{diam}(S)$ for the $d$-diameter $\operatorname{diam}(S)$. Let $i$ be an integer such that $d(x_i, \tilde{\alpha}) > 2 \operatorname{diam}(S)$. Then $\vartheta_i(x) \in \tilde{S}_\alpha$. This completes the proof (compare to [1]).

**5.** Let $S'$ be a covering of $S$ with covering map $p: S' \to S$. ($S$ is not necessarily compact.)

**Lemma.** *Suppose that $\alpha$ and $\beta$ are essential imbedded closed curves freely homotopic to each other in $S'$. Suppose that $\alpha \neq \beta$ and that $p(\alpha)$ and $p(\beta)$ are imbedded closed curves. Then $p(\alpha)$ and $p(\beta)$ are not identical.*

*Proof.* We may assume without loss of generality that $p(\alpha)$ is a closed geodesic. Suppose that $p(\alpha) = p(\beta)$. The covering map $p$ induces a generic hyperbolic structure on $S'$, where $\alpha$ and $\beta$ are imbedded closed geodesics in $S'$ distinct from each other. However, it is well known that an essential simple closed curve in a hyperbolic surface with convex boundary is freely homotopic to at most one imbedded closed geodesic (see Lemma 2.3 in [3]). This is a contradiction.

**6.** Given a topologically imbedded submanifold $S_1$ of $S$, let $S_1'$ be a component of $p^{-1}(S_1)$. Then $p|S_1'$ is a covering map onto $S_1$.

Suppose that $\alpha$ and $\beta$ are essential simple closed curves in $S'$ disjoint from and freely homotopic to each other. Suppose that $p|\alpha$ and $p|\beta$ are imbeddings onto $p(\alpha)$ and $p(\beta)$ respectively, and that $p(\alpha)$ and $p(\beta)$ are disjoint.

Now, $p(\alpha)$ and $p(\beta)$ bound a unique annulus, say $A$, and $\alpha$ and $\beta$ also bound a unique annulus, say $A'$. Each component of $p^{-1}(A)$ is a cover of $A$ and, hence, is homeomorphic to either $\mathbf{S}^1 \times I$ or $\mathbf{R} \times I$ for a compact interval $I$. Let $F$ be the component containing $\alpha$. Then $\alpha$ is a component of $\delta F$. It follows that $F$ is homeomorphic to $\mathbf{S}^1 \times I$, and the covering map $p|F$ is an imbedding onto $A$. Let $\beta'$ be the other component of $\delta F$. Then $p(\beta') = p(\beta)$. By the preceding Lemma 5, $\beta' = \beta$. Since $\alpha$ and $\beta$ bound a unique compact annulus, $F = A'$. It follows that $p|A'$ is an imbedding onto $A$.

## Appendix B: The annulus decomposition theorem

**1.** In this section, we will prove the annulus decomposition theorem (Corollary 3.6 of [14]). The version proved here is more general than that of Goldman since we do not require the boundary components to be principal geodesic and the holonomy to be hyperbolic. We will base our argument on the articles of Goldman [13], [14] (see [9] also) except for Proposition 2, which is not a step used by Goldman. (The author benefited greatly from a conversation with W. Goldman in February 1993 in constructing the proof.) For examples of projective annuli, refer to Sullivan and Thurston [26] and §3 of Goldman [14].

Note that we will need only the materials in §1, the introduction of §2, and §2.1 in the main text. The results in this appendix are independent of other parts of the main text and Appendix A.

**Annulus decomposition theorem.** *Let $A$ be a hyperbolic or quasi-hyperbolic projective annulus with convex boundary. Then $A$ decomposes into*

*subsurfaces that are elementary annuli or trivial annuli.*

Let $p: \tilde{A} \to A$ be the universal covering map. Let $A$ have a complete metric $d$. We may assume that $d$ is a hyperbolic metric with respect to which $\delta A$ is convex. We denote also by $d$ the induced complete metric on $\tilde{A}$. Let $\mathbf{dev}: \tilde{A} \to \mathbf{S}^2$ be a developing map, and let $\vartheta$ be the projective automorphism corresponding to a generator $\varphi$ of the deck transformation group of $\tilde{A}$. Let $\mu$ be the spherical Riemannian metric on $\tilde{A}$ induced from $\mathbf{S}^2$ by $\mathbf{dev}$, and let $\mathbf{d}$ be the induced metric on $\tilde{A}$. Let $\check{A}$ be the completion. (See [18] and the introduction of [5].)

Suppose that $\vartheta$ is hyperbolic (leaving aside the case where $\vartheta$ is quasi-hyperbolic until the end). Let $s$, $m$, $w$, $-s$, $-m$, and $-w$ denote the fixed points of $\vartheta$ as in §1.4 of the main text. There are three invariant great circles $l_1$, $l_2$, and $l_3$. We use the decompositions used in Goldman's article [14, §2.1]. (See [12] also.) For each $i$, since $l_i$ is $\vartheta$-invariant, $\mathbf{dev}^{-1}(l_i)$ is $\varphi$-invariant. Let $L_i = p(\mathbf{dev}^{-1}(l_i))$. Then $p^{-1}(L_i) = \mathbf{dev}^{-1}(l_i)$, $L_i$ is a compact set, and each component of $L_i$ is a one-dimensional compact submanifold of $A$ or the set of a point of $\delta A$. Each one-dimensional manifold component is either an imbedded closed geodesic or a maximal segment. Either the component is a subset of $\delta A$, or its manifold interior is a subset of $A^o$ by Lemma 3.4 in [5]. Note that $\mathbf{dev}^{-1}(l)$ is also $\varphi$-invariant. Let $L = p(\mathbf{dev}^{-1}(l))$. Then $L$ is the union of one-dimensional submanifolds transversal to one another and the finite set of points of $\delta A$. Notice also that $p^{-1}(L) = \mathbf{dev}^{-1}(l)$.

2. Our first step is as follows:

**Proposition.** *There exists a component of $L$ that is an imbedded tight curve freely homotopic to a component of $\delta A$.*

*Proof.* Suppose that $A$ is convex. Then $\mathbf{dev}: \tilde{A} \to \mathbf{S}^2$ is an imbedding onto a convex domain $\Omega$ by Lemma 1.5 of [5]. $A$ is projectively homeomorphic to $\Omega/\langle\vartheta\rangle$. Since $\Omega$ is $\vartheta$-invariant, an attractor and a repeller of $\vartheta$ are in $\mathrm{bd}\,\Omega$. We may assume without loss of generality that they are $s$ and $w$ respectively. Suppose that $\overline{sw}^o \not\subset \Omega$. Then $\overline{sw} \subset \mathrm{bd}\,\Omega$ and $\overline{sw}^o \cap (\mathrm{bd}\,\Omega - \Omega) \neq \varnothing$, which imply that $\Omega/\langle\vartheta\rangle$ is not compact. Hence, $\overline{sw}^o$ is a subset of $\Omega$ and corresponds to an imbedded tight curve. Clearly, it is a component of $L$.

Suppose that $A$ is not convex. Then as in §5.1 of [5], $\check{A}$ includes a triangle $T$ such that $T \cap \tilde{A}_\infty \subset \eta^o \cap \tilde{A}_\infty \neq \varnothing$ holds for an edge $\eta$ of $T$. Let $\alpha$ be a maximal line in $\eta - \tilde{A}_\infty$ containing an endpoint of $\eta$. Let $\alpha$ be oriented away from the endpoint of $\eta$. Since $p|\alpha$ is semi-infinite in $d$, there is a point $x$ in $A$ and a $d$-unit vector $\mathbf{v}$ at $x$ such that

$p|\alpha$ passes arbitrarily close to $x$ and such that its $d$-unit direction vector passes arbitrarily near $\mathbf{v}$ (see Fried [11]). This implies that there is a sequence $\{\alpha_i\}$ of lines in $\widetilde{A}$ such that the following properties hold:

(1) For each $i$, there is an integer $n(i)$ such that $\alpha_i = \varphi^{n(i)}(\alpha)$. We have $\{|n(i)|\} \to \infty$ as $i \to \infty$.

(2) There is a sequence $\{x_i\}$, $x_i \in \alpha_i$ for each $i$, converging to a point $\tilde{x}$ of $\widetilde{A}$ corresponding to $x$.

(3) The sequence $\{\mathbf{v}_i\}$, where $\mathbf{v}_i$ is the $d$-unit direction vector of $\alpha_i$ at $x_i$ for each $i$, converges to a vector $\tilde{\mathbf{v}}$ at $\tilde{x}$ corresponding to $\mathbf{v}$.

Let $\alpha_\infty$ be the maximal line in $\widetilde{A}$ passing through $\tilde{x}$ in the direction of $\tilde{\mathbf{v}}$. *We will prove that $\alpha_\infty$ corresponds to an imbedded tight curve in $L$ that we seek.*

First, let $A'$ be a real projective open annulus including $A$. Clearly, $\widetilde{A}$ is a closed proper subset of $\widetilde{A'}$, an open disk. The metric $d$ on $\widetilde{A}$ extends to a complete hyperbolic metric on $\widetilde{A'}$ such that the inclusion map from $\widetilde{A}$ to $\widetilde{A'}$ is isometric. Let us denote the complete metric on $\widetilde{A'}$ by $d$ also. The developing map $\mathbf{dev}$ extends to a developing map on $\widetilde{A'}$, which is denoted by $\mathbf{dev}$ also. Thus, the spherical metric $\mu$ also extends to a spherical metric on $\widetilde{A'}$ induced by $\mathbf{dev}$. Let us denote the Riemannian metric by $\mu$ also.

Considering $\widetilde{A'}$ as a Riemannian manifold with the spherical metric $\mu$, we obtain a differentiable exponential map for projective geodesics defined on an open domain in $T(\widetilde{A'})$ and mapping to $\widetilde{A'}$. We may reparametrize every projective geodesic so that it is $d$-length parametrized. It is easy to see that there is an associated continuous map $\exp': T(\widetilde{A'}) \to \widetilde{A'}$ where $\exp'|\{t\mathbf{u}|t \in \mathbf{R}\}$ is a $d$-length parametrized projective geodesic for each $d$-unit vector $\mathbf{u}$ in $T(\widetilde{A'})$.

Next, we claim that $\alpha_\infty$ is infinitely long with respect to $d$. Suppose not. Then $\alpha_\infty$ is a segment of finite $d$-length. Let $y$ be the endpoint of $\alpha_\infty$ in the direction of $\tilde{\mathbf{v}}$. Let $\tilde{l}$ be the $d$-length of the segment on $\alpha_\infty$ with endpoints $\tilde{x}$ and $y$. Let us choose a point $y_i$ in the infinite component of $\alpha_i - \{x_i\}$ for each $i$ such that the $d$-length of the segment on $\alpha_i$ with endpoints $x_i$ and $y_i$ is $\tilde{l}$. The above paragraph implies that $\{y_i\} \to y$ in $\widetilde{A'}$. Since $\alpha_i$ is semi-infinite, $\alpha_i$ includes a segment $\zeta_i$ with $\zeta_i^o \ni y_i$ such that the $d$-distance along $\alpha_i$ from $y_i$ to each point of $\delta\zeta_i$ is equal to $\tilde{l}$. Again, the above paragraph shows that $\{\zeta_i\}$ converges to a segment $\zeta$ in $\widetilde{A'}$ such that $\zeta^o \ni y$, and $\zeta$ given an orientation is in the same direction as $\alpha_\infty$ at $y$. Since $\zeta_i \subset \widetilde{A}$ for each $i$, it follows

that $\zeta \subset \widetilde{A}$, which gives a contradiction that $\alpha_\infty$ is not maximal in $\widetilde{A}$. Therefore, $\alpha_\infty$ is infinite with respect to $d$.

For some $k$, $k = 1, 2, 3$, $l_k$ includes $\mathbf{dev}(\alpha_\infty)$. Thus, $p|\alpha_\infty$ maps into $L_k$. Since $p|\alpha_\infty$ is infinite in $d$, it maps onto an imbedded closed geodesic that is a component of $L_k$. Since $\alpha_\infty$ is a maximal line, it follows that $p|\alpha_\infty$ is a covering map onto the imbedded closed geodesic.

Finally, we claim that $\alpha_\infty$ is a convex line. Suppose not. Then the **d**-length of $\alpha_\infty$ is greater than $\pi$. In fact, since $p|\alpha_\infty$ covers an imbedded closed geodesic, a closed one-dimensional real projective manifold, it is straightforward to show that $\alpha_\infty$ is infinitely long in **d** in both directions. Each $\alpha_i$ is a convex line and a geodesic of **d**. Hence, $\alpha_\infty$ includes a convex segment $\beta$ such that $\mathbf{dev}(\beta)$ is the geometric limit of a subsequence of $\{\mathrm{Cl}(\mathbf{dev}(\alpha_i))\}$ and such that $\beta \ni \tilde{x}$. Since $\beta$ is convex, $\mathbf{dev}|\beta$ is injective; thus, $\widetilde{A}'$ includes a $d$-bounded open neighborhood $\mathscr{U}$ of $\beta$ such that $\mathbf{dev}|\mathscr{U}$ is an imbedding. For infinitely many $i$, it is easy to see that $\mathbf{dev}(\alpha_i) \subset \mathbf{dev}(\mathscr{U})$ and, hence, $\alpha_i \subset \mathscr{U}$. Since this contradicts the fact that $\alpha_i$ for each $i$ is infinite with respect to $d$, it follows that $\alpha_\infty$ is convex, and, therefore, $p|\alpha_\infty$ is a covering map onto the imbedded tight curve. This implies the conclusion of Proposition 2.

**3.** To continue, we need several topological facts. The first collection of facts are as follows: By a *simple piecewise-regular curve*, we mean a topologically imbedded curve that is the union of finitely many regular curves transversal to one another. Let $S$ be a compact real projective surface with convex boundary whose components are simple piecewise-regular curves. Let $F$ be a compact subset consisting of components $\zeta_1, \cdots, \zeta_n$ satisfying the following conditions:

(1) Each $\zeta_i$ is an imbedded geodesic curve unless it is the set of a point of $\delta S$.

(2) Each $\zeta_i \cap \delta S$ is either the set of endpoints of $\zeta_i$ or $\zeta_i$ itself or the empty set.

Suppose that $S$ includes no imbedded closed curve intersecting $F$ at a point transversally and intersecting $F$ at no other point. Then it is straightforward to show that given a component $R$ of $S - F$, the closure $\mathrm{Cl}(R)$ is a compact surface with convex boundary whose components are simple piecewise-regular curves and that $\mathrm{Cl}(R)^o = R^o$.

The second collection of facts that we need are as follows: Suppose that $B$ and its closure $B_1$ are topologically imbedded subsurfaces of $S$ such that $B_1^o = B$. Let $S'$ be a covering of $S$ with covering map $q: S' \to S$. Let $\mathscr{B}$ be a component of $q^{-1}(B)$. Then $q|\mathscr{B}$ is a covering map onto

$B$. Let $\mathscr{B}_1$ be a component of $q^{-1}(B_1)$ including $\mathscr{B}$. Also, $q|\mathscr{B}_1$ is a covering map onto $B_1$. Clearly, $\mathscr{B} \subset \mathscr{B}_1^o$. Since $q(\mathscr{B}_1^o) \subset B$, we have $\mathscr{B}_1^o \subset q^{-1}(B)$. Since $\mathscr{B}_1^o$ is path-connected, we conclude $\mathscr{B} = \mathscr{B}_1^o$ and $\mathrm{Cl}(\mathscr{B}) = \mathscr{B}_1$.

The third set of facts are as follows: Let $K$ be a closed subset of $S$, and let $H$ be a component of $S^o - K$. Let $K' = q^{-1}(K)$, and let $H'$ be a component of $q^{-1}(H)$. Then $H'$ is a subset of $S'^o - K'$. Let $H''$ be a component of $S'^o - K'$ including $H'$. Since $q(H'')$ is path-connected and disjoint from $K$, we have $q(H'') \subset H$. Thus, $H'' \subset q^{-1}(H)$. Since $H''$ is path-connected, $H' = H''$. We conclude that $H'$ is a component of $S'^o - K'$.

Final facts are as follows: Let $M$ be a surface, $K$ a closed subset of $M$, and $H$ a component of $M^o - K$. Then $(\mathrm{Cl}(H) \cap M^o) - H \subset K$. The proof is as follows: Let $x$ be a point of a component $H'$ of $M^o - K$ different from $H$. Then $H'$ is an open subset of $M^o - K$ and is an open subset of $M^o$. Since $H$ and $H'$ are disjoint, $x$ does not belong to $\mathrm{Cl}(H)$. Hence, if a point $x$ belongs to $\mathrm{Cl}(H) \cap M^o$, then $x$ belongs to $H$ or $K$. Thus the claim follows.

**4.** Let $B$ be a component of $A - L$. We will prove properties of $B_1$ where $B_1 = \mathrm{Cl}(B)$. (Recall the facts on $L$ in §1 of this appendix.)

**Proposition.** *Suppose that $B_1$ includes an imbedded tight curve. Then $B_1$ is a trivial annulus or an elementary annulus, where $B_1 \cap L$ equals the union of all geodesic components of $\delta B_1$.*

*Proof.* We claim that $B_1$ is a compact surface such that $B^o = B_1^o$. To prove this, we apply the results of §3 to $B_1$. First consider $A - L_1$. Choose a component $R_1$ including $B$. Note that $R_1$ is disjoint from $L_1$. Then by the above argument, $\mathrm{Cl}(R_1)$ is a compact surface with convex boundary whose components are simple piecewise-regular curves, and $\mathrm{Cl}(R_1)^o = R_1^o$. Consider $\mathrm{Cl}(R_1) - L_2$. We obtain a component $R_2$ including $B$. Similarly, $\mathrm{Cl}(R_2)$ is a compact surface with convex boundary whose components are simple piecewise-regular curves, and $\mathrm{Cl}(R_2)^o = R_2^o$. Since $R_2 \subset \mathrm{Cl}(R_1) - L_2$, it follows that $R_2^o \subset \mathrm{Cl}(R_1)^o - L_2 = R_1^o - L_2$. Hence, $R_2^o$ is disjoint from $L_1$ and $L_2$ since $R_2^o$ is a subset of $R_1^o$ and $R_2$. Finally, we obtain a component $R_3$ of $\mathrm{Cl}(R_2) - L_3$ including $B$. Similarly, $\mathrm{Cl}(R_3)$ is a compact surface with convex boundary, and $\mathrm{Cl}(R_3)^o = R_3^o$. Since $R_3 \subset \mathrm{Cl}(R_2) - L_3$, we have $R_3^o \subset R_2^o - L_3$. Hence, $R_3^o$ is disjoint from $L_1$, $L_2$, and $L_3$ since $R_3^o$ is a subset of $R_2^o$ and $R_3$. Clearly, $B^o \subset R_3^o$. Since $B^o$ is a component of $S^o - L_1 - L_2 - L_3$, we have $B^o = R_3^o$. Hence, $B_1 = \mathrm{Cl}(R_3)$ and $B_1^o = B^o$. Thus, our claim follows.

Next, let $B'$ be a component of $p^{-1}(B^o)$ for the universal covering map $p \colon \tilde{A} \to A$, and let $\tilde{B}_1$ be the component of $p^{-1}(B_1)$ that includes $B'$. By Lemma 5, $\mathbf{dev}|B'$ is an imbedding onto a convex open domain $\mathscr{O}$ in an open $\vartheta$-invariant triangle $T$. Let $\mathrm{Cl}(B')$ denote the closure of $B'$ in $\check{A}$. Since $B'$ is convex, §1.4 of [5] implies that $\mathbf{dev}|\mathrm{Cl}(B')$ is an imbedding onto a convex compact subset $\mathrm{Cl}(\mathscr{O})$ of $\mathrm{Cl}(T)$. Section 3 shows that $\tilde{B}_1 = \mathrm{Cl}(B') \cap \tilde{A}$. Therefore, $\mathbf{dev}|\tilde{B}_1$ is an imbedding onto a dense domain in $\mathrm{Cl}(\mathscr{O})$, which is a subsurface denoted by $\Omega$.

Finally, we use the hypothesis that $B_1$ includes an imbedded tight curve. $B_1$ is projectively homeomorphic to the quotient surface of $\Omega$ by the action of a discrete group of projective automorphisms. Since $B_1$ is not simply connected and includes an imbedded tight curve, the discrete group is not trivial and equals $\langle \vartheta \rangle$. We may assume without loss of generality that $s$ and $w$ are the attractor and the repeller of $\vartheta$ in $\mathrm{Cl}(T)$ respectively. Since $\mathrm{Cl}(\mathscr{O})$ is a $\vartheta$-invariant convex compact subset of $\mathrm{Cl}(T)$, it follows that $\mathrm{Cl}(\mathscr{O})$ equals either $\mathrm{Cl}(T)$ or a convex compact subset $K$ whose boundary is the union of $\overline{sw}$ and an open arc $\gamma$ in $T$ connecting $s$ and $w$. In order that $\Omega/\langle \vartheta \rangle$ be compact, $\Omega$ must either equal $T \cup \overline{sw}^o \cup \eta^o$ for an edge $\eta$ of $T$ distinct from $\overline{sw}$, or equal $K - \{s, w\}$. Since the quotients of these sets are an elementary annulus or a primitive trivial annulus respectively, $B_1$ is an elementary annulus or a trivial annulus. Suppose that $B_1$ is a trivial annulus. The nongeodesic component of $\delta B_1$ is disjoint from $L$ since $\gamma \subset T$. Clearly, the geodesic component of $\delta B_1$ lies in $L$, and $B_1^o$ is disjoint from $L$. Hence, $B_1 \cap L$ is the geodesic component of $\delta B_1$. If $B_1$ is an elementary annulus, then a similar argument shows that $B_1 \cap L = \delta B_1$. This completes the proof of Proposition 4.

**5. Lemma.**  *The subset $B'$ is a convex subsurface of $\tilde{A}$, and $\mathbf{dev}|B'$ is an imbedding onto a convex subset of an open $\vartheta$-invariant triangle.*

*Proof.*  Since the connected set $B'$ lies in the complement of $\mathbf{dev}^{-1}(l)$, it follows that $\mathbf{dev}(B')$ is a subset of an open $\vartheta$-invariant triangle, say $T$. For each point $x$ of $B'$, we denote by $E_x$ the subset of the points of $B'$ reachable from $x$ by segments in $B'$. Similarly to Proposition 1.2 of Carrière [2], it follows that $\mathbf{dev}|E_x$ is an imbedding onto an open domain in $T$ for each point $x$ of $B'$. We let $E_x^*$ denote the image of $E_x$ under $\mathbf{dev}$. Similarly to Proposition 1.3.2 of Carrière or Lemma 3 of Koszul [19], $B'$ is convex if and only if $E_x^*$ is a convex subset of $T$ for each point $x$ of $B'$.

Suppose that $B'$ is not convex. Then $E_x^*$ is not convex for a point $x$ of $B'$. Hence, $T$ includes a compact triangle $R$ such that

$$R \cap (T - E_x^*) \subset \eta^o \cap (T - E_x^*) \neq \varnothing$$

holds for an edge $\eta$ of $R$ (see [2]). The injectivity of $\mathbf{dev}|E_x$ implies that $B'$ includes a convex open triangle imbedding onto $R^o$ under $\mathbf{dev}$. Taking the closure of the open triangle and using §1.4 of [5], we obtain a triangle $R'$ in $\mathrm{Cl}(B')$, a subset in $\check{A}$, such that

$$R' \cap (\check{A} - B') \subset \zeta^o \cap (\check{A} - B') \neq \varnothing$$

holds for an edge $\zeta$ of $R'$.

Let $L' = \mathbf{dev}^{-1}(l)$. Since $p^{-1}(L) = L'$, and $B^o$ is a component of $A^o - L$, from §3 it follows that $B'$ is a component of $\widetilde{A}^o - L'$. Thus,

$$(\mathrm{Cl}(B') \cap \widetilde{A}^o) - B' \subset L', \qquad (\zeta^o \cap \widetilde{A}^o) - B' \subset L'.$$

Since $\check{A} = \widetilde{A}_\infty \cup \delta\widetilde{A} \cup \widetilde{A}^o$, we have

$$\zeta^o \cap (\check{A} - B') \subset \zeta^o \cap (\widetilde{A}_\infty \cup \delta\widetilde{A} \cup L').$$

Let $\alpha$ be the component of $\zeta \cap B'$ containing a vertex of $\zeta$. Let $y$ be the endpoint of $\alpha$ not in $\alpha$; that is, $y$ is the unique point of $\mathrm{Cl}(\alpha) - \alpha$ in $\check{A}$. By our choice, $y$ is a point of $\zeta^o \cap (\check{A} - B')$. Then $y$ is not an element of $\delta A$; otherwise $\alpha$ is tangent to a component of $\delta A$, and $\alpha \subset \delta A$. Also, $y$ is not an element of $\mathbf{dev}^{-1}(l)$ since $R \subset T$. Hence, $y$ is a point of $\widetilde{A}_\infty$. Now, this implies that $\alpha$ is semi-infinite in $d$. Again $p|\alpha$ is semi-infinite in $d$. We deduce as in the proof of Proposition 2 that there is a sequence $\{\alpha_i\}$ of $d$-infinite lines in $\widetilde{A}$ having the following properties:

(1) For each $i$, there is an integer $n(i)$ such that $\alpha_i = \varphi^{n(i)}(\alpha)$. We have $\{|n(i)|\} \to \infty$ as $i \to \infty$.

(2) There is a sequence $\{x_i\}$, where $x_i \in \alpha_i$ for each $i$, converging to a point $\tilde{x}$ of $\widetilde{A}$.

About the sequence we will now derive two facts that contradict each other. The first is as follows: There is a tiny disk $B(\tilde{x})$ of $\tilde{x}$. Clearly, $\mathbf{d}(\tilde{x}, \mathrm{bd}\, B(\tilde{x})) = \varepsilon$ for a positive constant $\varepsilon$. Let $N$ be an integer such that $\mathbf{d}(x_i, \tilde{x}) < \varepsilon/2$ for $i > N$. Since $\alpha_i$ is semi-infinite in $d$, it follows that for each $i$ greater than $N$, $\alpha_i \cap B(\tilde{x})$ includes a line whose $\mathbf{d}$-length is bounded below by $\varepsilon/2$. Thus, the $\mathbf{d}$-length of $\mathbf{dev}(\alpha_i)$ for each $i$ is bounded below by a positive constant independent of $i$.

Second, let us consider the image of $\alpha_i$ under $\mathbf{dev}$. For each $i$, there is the integer $n(i)$ such that $\mathbf{dev}(\alpha_i) = \vartheta^{n(i)}(\mathbf{dev}(\alpha))$. We may assume without loss of generality that $\{n(i)\} \to \infty$ or that $\{n(i)\} \to -\infty$ by extracting subsequences, if necessary. Since both endpoints of $\mathbf{dev}(\alpha)$ belong to $T$, the sequence $\{\mathrm{Cl}(\mathbf{dev}(\alpha_i))\}$ converges to the set of the attractor or the set

of the repeller of $\vartheta$ in $\mathrm{Cl}(T)$. Hence, $\{\mathbf{d}\text{-length}(\mathbf{dev}(\alpha_i))\}$ converges to $0$.

Since the above two paragraphs contradict each other, we conclude that $B'$ is convex. Since $B'$ is convex, it follows by [5] that $\mathbf{dev}|B'$ is an imbedding onto a convex open domain in $T$.

6. We can now prove the annulus decomposition theorem. By Proposition 2, there is a component $B$ of $A - L$ such that its closure $B_1$ includes an imbedded tight curve. By Proposition 4, $B_1$ is an elementary annulus or a trivial annulus. Each component of $\delta B_1$ is a subset of either $A^o$ or $\delta A$. Let $B'$ be any other component of $A - L$ sharing a boundary point with $B$. Since each component of $\delta B$ in $A^o$ is an imbedded tight curve that is a component of $L$, the closure of $B'$ also includes an imbedded tight curve. By Proposition 4, $\mathrm{Cl}(B')$ is also a trivial annulus or an elementary annulus. Thus, an induction shows that the closure of each component of $A - L$ is a trivial annulus or an elementary annulus. This implies the conclusion of the annulus decomposition theorem if $\vartheta$ is hyperbolic. We are left with proving the conclusion of the theorem when $\vartheta$ is quasi-hyperbolic. But it is absolutely clear that an entirely similar argument can be used to prove this.

# References

[1] J. Benzécri, *Sur les variétés localement affines et projective*, Bull. Soc. Math. France **88** (1960) 229–332.

[2] Y. Carrière, *Autour de la conjecture de L. Markus sur les variétés affines*, Invent. Math. **95** (1989) 615–628.

[3] A. Casson & S. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, London Math. Soc. Student Texts, Vol. 9, Cambridge Univ. Press, Cambridge, 1988.

[4] S. Choi, *Real projective surfaces*, Ph.D. thesis, Princeton Univ., 1988.

[5] _____, *Convex decompositions of real projective surfaces*. I: $\pi$-*Annuli and convexity*, to appear.

[6] _____, *Convex decompositions of real projective surfaces*. III, to appear.

[7] _____, *Convex decompositions of real projective surfaces*. IV, to appear.

[8] S. Choi & W. Goldman, *Deformation spaces of real projective structures on compact surfaces*, to appear.

[9] G. Faltings, *Real projective structures on Riemann surfaces*, Compositio Math. **48** (1983) 223–269.

[10] M. Freedman, J. Hass & P. Scott, *Closed geodesics on surfaces*, Bull. London Math. Soc. **14** (1982) 385–391.

[11] D. Fried, *Closed similarity manifolds*, Comment. Math. Helv. **55** (1980) 709–719.

[12] D. Gallo, W. Goldman & R. Porter, *Projective structures with monodromy in* $\mathrm{PSL}(2, \mathbf{R})$, preprint.

[13] W. Goldman, *Affine manifolds with projective geometry on surfaces*, Senior thesis, Princeton Univ., 1977.

[14] _____, *Projective structures with Fuchsian holonomy*, J. Differential Geometry **25** (1987) 297–326.

[15] ——, *Projective geometry on manifolds*, Lecture notes, Univ. Maryland.

[16] ——, *Convex real projective structures on compact surfaces*, J. Differential Geometry **31** (1990) 791–845.

[17] V. Kac & E. B. Vinberg, *Quasi-homogeneous cones*, Mat. Zametki **1** (1967) 347–354, English transl., Math. Notes **1** (1967), 231–235.

[18] Y. Kamishima & S. Tan, *Deformation spaces on geometric structures,* Aspects of Low Dimensional Manifolds, (Y. Matsumoto & S. Morita, eds.), Advanced Studies in Pure Math., Vol. 20, Kinokuniya, Tokyo, 1922.

[19] J. L. Koszul, *Variétés localement plates et convexité*, Osaka J. Math. **2** (1965) 285–290.

[20] N. Kuiper, *On convex locally projective spaces*, Convegno Intenazionale di Geometria Differenziale, Rome, 1953, 200–213.

[21] B. Maskit, *Kleinian groups*, Grundlehren Math. Wiss., Vol. 287, Springer, Berlin, 1988.

[22] J. Milnor, *On the existence of a connection with curvature zero*, Comment. Math. Helv. **32** (1958) 215–223.

[23] P. Scott, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. (2) **17** (1978) 555–565.

[24] ——, *Correction to 'Subgroups of surface groups are almost geometric'*, J. London Math. Soc. (2) **32** (1985) 217–220.

[25] J. Simillie, *Affinely flat manifolds*, Ph.D. thesis, Univ. Chicago, 1977.

[26] D. Sullivan & W. Thurston, *Manifolds with canonical coordinate charts: Some examples*, Enseignement Math. **29** (1983) 15–25.

[27] W. Thurston, *The geometry and topology of three-manifolds*, preprint.

KYUNGPOOK NATIONAL UNIVERSITY, TAEGU, KOREA