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HOMOLOGY OF MODULI SPACES OF INSTANTONS ON ALE SPACES. I

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1. Introduction

In [13] P. B. Kronheimer and the author introduced a new class of hyper-Kähler manifolds which arise as moduli spaces of anti-self-dual connections on a certain class of 4-dimensional noncompact manifolds, the so-called ALE spaces. The ALE space is diffeomorphic to the minimal resolution of the simple singularity $\mathbb{C}^2/\Gamma$ for a finite subgroup Γ of $\mathrm{SU}(2)$, and was constructed by P. B. Kronheimer [12]. In [17] we studied the geometry of the moduli space, and showed that, under a certain topological condition on the vector bundle (cf. (5.1)), its middle cohomology group is isomorphic to a weight space of an irreducible finite dimensional representation of a simple Lie algebra. The key geometric property of the moduli space is the existence of an S^1-action.

The aim of the present paper and its sequel is to compute the homology of the moduli spaces. The method is to use the moment map for the S^1-action as a Morse function. In this paper we treat the case where the group Γ is a cyclic group (i.e., the base ALE space is of type A_n). One of the main results in this paper is the following:

Theorem 1.1. *Let $\mathfrak{M}$ be the moduli space of anti-self-dual connections on a vector bundle over the ALE space of type A_n (see §2 for more precise definitions). Suppose that the hyper-Kähler metric on $\mathfrak{M}$ is complete. Then $\mathfrak{M}$ has a nondegenerate Morse function F which has only critical points of even index. In particular, the homology of $\mathfrak{M}$ has no torsion and vanishes in odd degrees, and every component of $\mathfrak{M}$ is simply connected.*

In fact, we can say more: we have a precise description of the critical points and an algorithm to compute the Betti numbers (Theorem 3.2, equation (3.4), Proposition 4.3).

It was shown that the homology group of the moduli space is isomorphic to that of Spaltenstein's variety in [17, 8.7] when the ALE space is

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of type A_n and the above mentioned topological condition holds. Spaltenstein's varieties were studied extensively in conjunction with the representations of the Weyl groups. Borho and MacPherson gave formulae of their Poincaré polynomials in terms of Green polynomials [3]. More recently Terada gave their partitions into affine spaces and another description of the Poincaré polynomials in terms of Young tableaux [21], inspired by the works on similar varieties by Spaltenstein [19] and Hotta-Shimomura [18], [10]. Their approaches are totally different from ours, but we can show that our algorithm also has a combinatorial description in terms of Young tableaux (Theorem 5.14), which is essentially equivalent to Terada's description.

Together with the result in [17], our result gives a formula for the dimension of the weight spaces in terms of Young tableaux (see (5.16)). This is nothing but the classically well-known correspondence between the number of semistandard tableaux and the Kostka coefficient (see e.g., [6]).

Goto gave an inductive formula for the Betti numbers for the very specific moduli spaces (i.e., $\dim V_0 = 0, \dim V_k = 1 \quad (1 \le k \le n)$; see §2 for the notation) [8]. Under his assumption, the moduli space $\mathfrak{M}$ has an action of the $m/4$-dimensional torus $(m = \dim \mathfrak{M})$. His approach has some similarities in the homology calculations for toric varieties, but his condition is too restrictive for our purpose.

2. The ALE spaces and the ADHM description

The main result of [13] states that the moduli spaces of anti-self-dual connections can be identified with spaces of solutions of the ADHM equation for the representations of the quiver associated with the corresponding extended Dynkin diagram. We shall review the construction of ALE spaces [12] and the ADHM description [13].

Let Γ be a nontrivial finite subgroup of SU(2) and let Q be the 2-dimensional Γ-module obtained from the inclusion $\Gamma \subset \mathrm{SU}(2)$. Suppose that we are given a pair of unitary Γ-modules, V and W. Define a Hermitian vector space $\mathbf{M}$ by

$$\mathbf{M} \stackrel{\text{def.}}{=} \mathrm{Hom}_\Gamma(V, Q \otimes V) \oplus \mathrm{Hom}_\Gamma(W, V) \oplus \mathrm{Hom}_\Gamma(V, W),$$

where Hom_Γ means the space of Γ-morphisms. If we choose an orthonormal basis for Q so as to represent an element of $\mathbf{M}$ as a quadruple of endomorphisms (B_1, B_2, i, j), we can define an antilinear map $J: \mathbf{M} \to \mathbf{M}$

by

$$J(B_1, B_2, i, j) = (-B_2^\dagger, B_1^\dagger, -j^\dagger, i^\dagger)$$
$$\text{for } B_1, B_2 \in \operatorname{End}(V), i \in \operatorname{Hom}_\Gamma(W, V), j \in \operatorname{Hom}_\Gamma(V, W).$$

Together with the original complex structure I, J gives a quaternion-module structure on $\mathbf{M}$. We put a Hermitian metric on $\mathbf{M}$ induced from those on V and W. In particular, $\mathbf{M}$ is a hyper-Kähler manifold.

Let $\mathrm{U}(V)^\Gamma$ be the group of unitary transformations of V commuting with the Γ-action. It acts on $\mathbf{M}$ preserving the metric and the quaternion structure. Define maps $\mu_{\mathbb{R}}: \mathbf{M} \to \mathfrak{u}(V)^\Gamma$, $\mu_{\mathbb{C}}: \mathbf{M} \to \mathfrak{gl}(V)^\Gamma$ by

$$\mu_{\mathbb{R}}(B_1, B_2, i, j) = \frac{\sqrt{-1}}{2}\left([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j\right),$$
$$\mu_{\mathbb{C}}(B, i, j) = [B_1, B_2] + ij.$$

This is a hyper-Kähler moment map of the $\mathrm{U}(V)^\Gamma$-action in the sense of [9]. Combining the two components, we write

$$\mu = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}}): \mathbf{M} \to (\mathbb{R} \oplus \mathbb{C}) \otimes \mathfrak{u}(V)^\Gamma.$$

To gain more information, we decompose V, W into irreducible representations. Let $R_0, \cdots, R_n$ be the irreducible representations of Γ, with R_0 the trivial representation. Then V and W decompose as

$$V = \bigoplus V_k \otimes R_k, \qquad W = \bigoplus W_k \otimes R_k.$$

Now

$$\operatorname{Hom}_\Gamma(V, Q \otimes V) = \bigoplus \operatorname{Hom}(V_l, V_k) \otimes \operatorname{Hom}_\Gamma(R_l, Q \otimes R_k).$$

If $a_{kl} = \dim_{\mathbb{C}} \operatorname{Hom}_\Gamma(R_l, Q \otimes R_k)$, then $\widetilde{\mathbf{C}} = (2\delta_{kl} - a_{kl})_{0 \le k, l \le n}$ is an extended Cartan matrix of type A_n, D_n, E_6, E_7, or E_8. The McKay correspondence [15] states that this gives a bijection beteen finite subgroups of $\mathrm{SU}(2)$ and simple Lie algebras of type ADE.

Consider the corresponding extended Dynkin diagram. We number the vertices so that we get the ordinary Dynkin diagram if we remove the vertex 0. Let define $H \subset \{0, 1, \cdots, n\} \times \{0, 1, \cdots, n\}$ so that $(k, l) \in H$ if and only if the vertex k and l are joined by an edge, or equivalently $a_{kl} = 1$. Note that $(k, l) \in H$ is equivalent to $(l, k) \in H$. Our space $\mathbf{M}$ has the following description:

$$\mathbf{M} = \left(\bigoplus_{l \to k \in H} \operatorname{Hom}(V_l, V_k)\right) \oplus \left(\bigoplus_m \operatorname{Hom}(W_m, V_m) \oplus \operatorname{Hom}(V_m, W_m)\right).$$

For an element of $\mathbf{M}$ we denote its components by $B_{k,l}$, i_m, j_m. We write B, i, j for the collection $(B_{k,l})_{l\to k\in H}$, etc. When we want to emphasize V and W, we use the notation $\mathbf{M}(\mathbf{v}, \mathbf{w})$, where

$$\mathbf{v} = (\dim V_0, \cdots, \dim V_n)^t, \qquad \mathbf{w} = (\dim W_0, \cdots, \dim W_n)^t \in (\mathbb{Z}_{\geq 0})^{n+1}.$$

Choose and fix an orientation Ω of the diagram, that is a collection of arrows $k \to l$, one for each edge $k — l$ in the diagram. We denote by $\overline{\Omega}$ the reversed orientation. Hence $H = \Omega \cup \overline{\Omega}$. Define a function ε by $\varepsilon(k, l) = 1$ if $k \to l \in \Omega$, $\varepsilon(k, l) = -1$ if $k \to l \in \overline{\Omega}$.

The group $\mathrm{U}(V)^\Gamma$ can be written $\prod \mathrm{U}(V_k)$. So its Lie algebra $\mathfrak{u}(V)^\Gamma$ is given by $\bigoplus_k \mathfrak{u}(V_k)$. The hyper-Kähler moment map μ is explicitly given by

$$\mu_{\mathbb{R}}(B, i, j) = \frac{\sqrt{-1}}{2}\left(\sum_{l\,:\,k\to l\in H} B_{k,l}B_{k,l}^\dagger - B_{l,k}^\dagger B_{l,k} + i_k i_k^\dagger - j_k^\dagger j_k\right)_k \in \bigoplus_k \mathfrak{u}(V_k),$$

$$\mu_{\mathbb{C}}(B, i, j) = \left(\sum_{l\,:\,k\to l\in H} \varepsilon(k, l) B_{k,l}B_{l,k} + i_k j_k\right)_k \in \bigoplus_k \mathfrak{gl}(V_k).$$

(The choice of the orientation is not essential. The orientation depends on the choice of the basis for $\mathrm{Hom}_\Gamma(R_l, Q\otimes R_k)$.)

Let $Z_{\mathbf{v}} \subset \mathfrak{u}(V)^\Gamma$ denote the center. Choose an element $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in Z_{\mathbf{v}} \oplus (Z_{\mathbf{v}} \otimes \mathbb{C})$, and define a hyper-Kähler quotient $\mathfrak{M}$ of $\mathbf{M}$ by $\mathrm{U}(V)^\Gamma$ as follows:

$$\mathfrak{M} = \mathfrak{M}(\mathbf{v}, \mathbf{w}) \stackrel{\text{def.}}{=} \{(B, i, j) \in \mathbf{M} \mid \mu(B, i, j) = -\zeta\} / \mathrm{U}(V)^\Gamma.$$

We denote by $[(B, i, j)]$ the $U(V)^\Gamma$-orbit of (B, i, j) considered as a point in $\mathfrak{M}$. We call $\mu_{\mathbb{R}}(B, i, j) = -\zeta_{\mathbb{R}}$ (resp. $\mu_{\mathbb{C}}(B, i, j) = -\zeta_{\mathbb{C}}$) the *real* (resp. *complex*) ADHM equation.

In general, $\mathfrak{M}$ has singularities. We take a subset

$$\mathfrak{M}^{\mathrm{reg}} \stackrel{\text{def.}}{=} \{(B, i, j) \in \mu^{-1}(-\zeta) \mid \text{the stabilizer of } (B, i, j) \text{ in } U(V)^\Gamma \text{ is trivial}\} / \mathrm{U}(V)^\Gamma,$$

which is a nonsingular hyper-Kähler manifold of dimension (over $\mathbb{R}$)

$$\dim M - 4\dim U(V)^\Gamma = 2\mathbf{v}^t(2\mathbf{w} - \tilde{\mathbf{C}}\mathbf{v}), \tag{2.1}$$

where $\tilde{\mathbf{C}}$ is the extended Cartan matrix. (See [13, §9].)

The ALE spaces can also be described as $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ for specific data $\mathbf{v}$, $\mathbf{w}$. Let $R = L^2(\Gamma)$ be the regular representation of Γ, which decomposes as

$$R = \bigoplus \mathbb{C}^{n_k} \otimes R_k,$$

where $n_k = \dim_{\mathbb{C}} R_k$. We consider the case $V = R$, $W = 0$. Then the group $\mathrm{U}(1)$ of scalars acts trivially, so we consider the action of the quotient group $\mathrm{PU}(R)^\Gamma = \mathrm{U}(R)^\Gamma / \mathrm{U}(1)$. Choose $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in Z \oplus (Z \otimes \mathbb{C})$, where $Z \subset \mathrm{u}(R)^\Gamma$ is the trace-free part of the center. Now define

$$X \stackrel{\text{def.}}{=} \{y \in \mathbf{M}(\mathbf{n}, 0) \mid \mu(y) = \zeta\} / \mathrm{PU}(R)^\Gamma,$$

where $\mathbf{n} = (n_0, \cdots, n_n)^t$. Then the main result of [12] can be summarized as follows.

Proposition 2.2. *If $\zeta \in Z \oplus (Z \otimes \mathbb{C})$ is generic (see [12, p. 666] for more precise information), the action of $\mathrm{PU}(R)^\Gamma$ on $\mu^{-1}(\zeta)$ is free, and the quotient X is a smooth 4-dimensional hyper-Kähler manifold, then ζ is diffeomorphic to the minimal resolution of $\mathbb{C}^2/\Gamma$, and the metric is ALE.*

The ALE condition means there exists a coordinate system at infinity $\mathfrak{X}: X \setminus K \to (\mathbb{C}^2 \setminus \overline{B_R})/\Gamma$ for some compact set K, and the metric approximates the Euclidean metric on $\mathbb{C}^2/\Gamma$.

The ALE spaces are fundamental among the spaces $\mathfrak{M}(\mathbf{v}, \mathbf{w})$; they are 4-dimensional, i.e., they have the lowest possible positive dimension. They are fundamental for another reason. The other spaces are all obtained as moduli spaces of anti-self-dual connections on ALE spaces, as we explain below.

The construction of ALE spaces gives a natural principal $\mathrm{PU}(V)^\Gamma$-bundle $\mu^{-1}(\zeta) \to X$. This bundle has a natural connection [7]; the horizontal subspace is the orthogonal complement to the fiber direction. Identifying $\mathrm{PU}(R)^\Gamma$ with $\prod_{k \neq 0} \mathrm{U}(n_k)$ (recall $n_0 = 1$), we consider an associated vector bundle

$$\mathscr{R}_l = \mu^{-1}(\zeta) \times_{\mathrm{PU}(R)^\Gamma} \mathbb{C}^{n_l} \qquad (l = 0, 1, \cdots, n),$$

where $\mathrm{U}(n_k)$ acts trivially on $\mathbb{C}^{n_l}$ when $k \neq l$, and $\mathrm{U}(n_l)$ acts naturally on $\mathbb{C}^{n_l}$. From the definition of X there is also a *tautological* vector bundle endomorphism

$$\xi = (\xi_{k,l}) \in \bigoplus_{l \to k \in H} \mathrm{Hom}(\mathscr{R}_l, \mathscr{R}_k).$$

The parameter $\zeta \in Z \oplus (Z \otimes \mathbb{C})$ determines an element, for which we use the same symbol ζ, in $Z_{\mathbf{v}} \oplus (Z_{\mathbf{v}} \otimes \mathbb{C})$ as follows. In the decomposition

$u(R)^\Gamma = \bigoplus_k u(n_k)$, the kth component of ζ is a scalar matrix of size n_k. Multiplying the identity matrix of size $\dim V_k$ by the same scalar, we consider ζ as an element of $Z_{\mathbf{v}} \oplus (Z_{\mathbf{v}} \otimes \mathbb{C})$. We shall treat only the parameters given in this way throughout this paper.

Take $(B, i, j) \in \mu^{-1}(-\zeta)$ and consider vector bundle homomorphisms

$$\bigoplus_l V_l \otimes \mathscr{R}_l \xrightarrow{\sigma} \left(\bigoplus_{l \to k \in H} V_k \otimes \mathscr{R}_l\right) \oplus \left(\bigoplus_l W_l \otimes \mathscr{R}_l\right) \xrightarrow{\tau} \bigoplus_l V_l \otimes \mathscr{R}_l, \tag{2.3}$$

where

$$\sigma = (B_{k,l} \otimes 1_{\mathscr{R}_l} + \varepsilon(k, l) 1_{V_l} \otimes \xi_{k,l}) \oplus j_l \otimes 1_{\mathscr{R}_l},$$
$$\tau = (\varepsilon(l, k) B_{l,k} \otimes 1_{\mathscr{R}_l} - 1_{V_l} \otimes \xi_{l,k}, i_l \otimes 1_{V_l}).$$

Here $1_\bullet$ is the identity map. Then the complex ADHM equation $\mu_{\mathbb{C}}(B, i, j) = -\zeta_{\mathbb{C}}$ implies that $\tau\sigma = 0$, so (2.3) is a complex. The condition of the trivial stabilizer is equivalent to saying that σ is injective and τ is surjective [13, 9.2]. Now the main theorem of [13] states that the induced connection A on the bundle

$$E = \operatorname{Coker}(\sigma, \tau^\dagger) \subset \left(\bigoplus_{k \to l \in H} V_k \otimes \mathscr{R}_l\right) \oplus \left(\bigoplus_l W_l \otimes \mathscr{R}_l\right)$$

is anti-self-dual (the real ADHM equation is used here) and has the following asymptotic behavior:

$$A - A_0 = O(r^{-3}), \quad \nabla_{A_0}(A - A_0) = O(r^{-4}), \cdots,$$

where A_0 is a flat connection defined on $E|(X_\zeta \setminus K)$, ∇_{A_0} denotes the covariant derivative associated with A_0, and r is the absolute value $|\mathfrak{X}|$ of the coordinate system at infinity. Conversely any such a connection is obtained by this ADHM description.

Denote by $\mathscr{A}^{\mathrm{asd}}$ the set of anti-self-dual connections A satisfying the above asymptotic behavior and having a fixed topological charge (the L^2-norm of the curvature). It has an action of the group $\mathscr{G}_0$ of gauge transformations γ satisfying

$$\gamma - 1_E = O(r^{-2}), \qquad \nabla_{A_0}(\gamma - 1_E) = O(r^{-3}), \cdots,$$

where 1_E is the identity transformation of E. We call the quotient space $\mathfrak{M}(E) = \mathscr{A}^{\mathrm{asd}}/\mathscr{G}_0$ the moduli space of anti-self-dual connections.

The main result of [13] can be stated as follows.

Proposition 2.4. *When ζ is generic, the ADHM description gives a one-to-one correspondence between $\mathfrak{M}(E)$ and $\mathfrak{M}^{\mathrm{reg}}(\mathbf{v}, \mathbf{w})$ for some $\mathbf{v}$, $\mathbf{w}$. (The data $\mathbf{v}$, $\mathbf{w}$ are determined from the topological data of E and A_0.)*

The one point compactification $\overline{X} = X \cup \{\infty\}$ has a structure of an orbifold (or V-manifold). On a neighborhood U of the singular point $\{\infty\}$ we can take coordinates $\mathfrak{Y}: U \to B_r/\Gamma$. Changing the hyper-Kähler metric conformally, we get an orbifold metric on $\overline{X}$.

Let $\overline{E}$ be an orbifold vector bundle over $\overline{X}$ with a Hermitian metric. This means, by definition, $(\mathfrak{Y}^{-1})^*(\overline{E}|U)$ is a quotient of a vector bundle over B_r with a Γ-action. In particular, $\overline{E}_\infty$ is a Γ-module. Our moduli space $\mathfrak{M}$ can be identified with the framed moduli space of anti-self-dual connections on $\overline{E}$. This is the set of gauge equivalence classes of the pair $[(A, \varphi)]$ of an anti-self-dual connection A and the trivialization $\varphi: \overline{E}_\infty \to W$ of the fiber over ∞. Here φ is a Γ-equivariant isomorphism.

It is known [13, 2.2] that $\mathscr{R}_l$ has an anti-self-dual connection and extends to $\overline{X}$. The fiber over infinity is the irreducible Γ-module R_l. It is also known that the first Chern classes $c_1(\mathscr{R}_l)$ $(l = 1, 2, \cdots, n)$ form a basis for $H^2(X; \mathbb{R})$. ($\mathscr{R}_0$ is the trivial bundle with the fiber $\mathbb{C}$.)

In this paper, we shall put the following assumption:

(2.5) the hyper-Kähler metric on $\mathfrak{M}(E) \cong \mathfrak{M}^{\mathrm{reg}}(\mathbf{v}, \mathbf{w})$ is complete.

We have the following proposition [13, Remark after Proposition 9.2]:

Proposition 2.6. *The assumption* (2.5) *is equivalent to*

$$\mathfrak{M}^{\mathrm{reg}}(\mathbf{v}, \mathbf{w}) = \mathfrak{M}(\mathbf{v}, \mathbf{w}).$$

For the convenience of the reader, we explain (2.5) and (2.6) in the gauge theory language. Let $[A_i]$ be a sequence in $\mathfrak{M}(E)$. By Uhlenbeck's compactness theorem we can take a subsequence $[A_j]$ such that

(1) there exists a finite set $S = \{x_1, \cdots, x_n\} \subset \overline{X}$ such that A_j converges to an anti-self-dual connection A_∞ outside S after gauge transformations,

(2) there exist constants a_k $(k = 1, \cdots, n)$ such that the curvature densities $|R_{A_j}|^2 dV$ converge to

$$|R_{A_\infty}|^2 dV + \sum_k a_k \delta_{x_k}.$$

The above constant a_k relates to the curvature integral of an anti-self-dual connection bubbling out around x_k. If x_k is a regular point of $\overline{X}$ (i.e., $x_k \in X$), a_k is an integer multiple of $8\pi^2$. On the other hand, if $x_k = \infty$, a_k is an integer multiple of $8\pi^2/\#\Gamma$, where $\#\Gamma$ is the order of Γ.

By [13] Proposition 2.6 can be restated in the following way:

Proposition 2.7. *The metric on the moduli space $\mathfrak{M}$ is complete if and only if we have $S = \emptyset$ or $S = \{\infty\}$ for any sequence $[A_i]$ as above.*

This proposition gives us examples of moduli spaces $\mathfrak{M}$ which do not satisfy the condition (2.5). For example, if we take a rank-2 vector bundle E with $c_1(E) = 0$, $c_2(E) = 1$ and a trivial connection A_0 on the restriction of E to the end $X_\zeta \setminus K$, Taubes existence theorem [20] gives us a sequence $[A_i]$ with the singular set S consisting of a single point x in X_ζ. On the other hand, if we have $c_1(E) = 0$, $c_2(E) < 1$, then we have $S = \emptyset$ or $S = \{\infty\}$. Anyhow there are infinitely many examples with (2.5), as we shall see.

3. Torus action on the moduli space

From now on, we assume that the group Γ is the cyclic group of order $n + 1$. Then the corresponding extended Dynkin diagram is of type A_n.

We assume that the parameter ζ satisfies

$$\zeta_{\mathbb{C}} = 0, \tag{3.1}$$

which means that the ALE space X is *biholomorphic* to the minimal resolution of quotient variety $\mathbb{C}^2/\Gamma$. Since the differentiable structure of the moduli space $\mathfrak{M}$ is independent of ζ [17, 4.2], the assumption (3.1) is not essential for studying the homology.

Under (3.1) the equation $\mu(y) = \zeta$ is preserved under the S^1 action $y \to t^{-1}y$. This action commutes with the $\mathrm{PU}(R)^\Gamma$ action, so induces an S^1 action on X which preserves the natural metric and the chosen complex structure I, but *not* J and K. In fact, this is the *pullback* of the following action on $\mathbb{C}^2/\Gamma$:

$$(z_1, z_2) \mod \Gamma \mapsto (t^{-1}z_1, t^{-1}z_2) \mod \Gamma.$$

If a gauge equivalence class of an anti-self-dual connection A with a frame $\varphi: E_\infty \to W$ is given, we can pull back the bundle E with them by the above map $t: X \to X$. Since E and t^*E have the same topological data, we can find a bundle isomorphism $\gamma: E \to t^*E$, and therefore define an S^1-action by

$$[(A, \varphi)] \to [(\gamma^* t^* A, (t^*\varphi) \lim_{x\to\infty} \gamma(x))].$$

The action on the corresponding ADHM data is given by $(B, i, j) \to (tB, ti, tj)$. The ADHM equation $\mu(B, i, j) = -\zeta$ is preserved by this action.

We use the method due to Frankel [5] to compute the Betti numbers of $\mathfrak{M}$. The moment map of the S^1 action on $\mathfrak{M}$ is given by $F_0 = \|B\|^2 + \|i\|^2 + \|j\|^2 = \int_X m|R_A|^2 dV$, where $m([y]) = \|y\|^2$ (cf. [14]). This is a nondegenerate perfect Morse function on $\mathfrak{M}$ in the sense of Bott (see [1], [11]). If V is the generating vector field, we have $\operatorname{grad} F_0 = IV$. Thus the critical points are the fixed point sets of the S^1 action.

Unfortunately, it is not so easy to determine the fixed point sets explicitly in general cases. So we consider another torus action on $\mathfrak{M}$ and perturb F_0 by adding the corresponding moment map as follows. Fixing a maximal torus T^{w_k} of the unitary group $\mathrm{U}(W_k)$ $(w_k = \dim W_k)$, we define the action by

$$(B, i, j) \to (B, ih^{-1}, hj) \quad \text{for } h \in T^r = \prod T^{w_k}.$$

Note that $r = \operatorname{rank} E$ since $\dim R_k = 1$ for all k for the cyclic group Γ. Geometrically this action means the change of the frame at infinity. It preserves the metric and the hyper-Kähler structure, not only the complex structure. We take a unitary basis $\langle e_1, \cdots, e_r\rangle$ of W so that the maximal torus T^r is a set of diagonal matrices. We take a diagonal matrix ε and make a pairing with the moment map (the real component of the hyper-Kähler moment map): $\langle \varepsilon, jj^\dagger - i^\dagger i\rangle$. This can be written in terms of connections: if we regard ε as a section of $\operatorname{End}(E)$ over the end which is parallel with respect to the flat connection A_0, it is equal to

$$\lim_{s\to\infty} \int_{S_s} i\left(\frac{\partial}{\partial s}\right) \operatorname{tr}(\sqrt{-1}\varepsilon R_A) \wedge \omega,$$

where $s = |\mathfrak{X}|$ is an absolute value of the coordinates at infinity, $S_s = \{x \mid |\mathfrak{X}(x)| = s\}$, i denotes the interior product, and ω is the Kähler form.

We consider the function

$$F = F_0 + \langle \varepsilon, jj^\dagger - i^\dagger i\rangle.$$

Theorem 3.2. *Assume the condition* (2.5). *Then for sufficiently small and generic* ε *the function* F *satisfies the following properties*:

(1) F *is proper.*

(2) *The gauge equivalence class* $[(A, \varphi)]$ *is a critical point of* F *if and only if there exists a decomposition* $E = L_1 \oplus \cdots \oplus L_r$ *into a sum of line bundles such that* A *decomposes accordingly and* φ *maps the component* $(L_a)_\infty$ *into* $\mathbb{C}e_a$.

(3) F *is a perfect Morse function, and the index of a critical point is an even number between* 0 *and* $\dim_{\mathbb{C}} \mathfrak{M}_\zeta$.

In particular, the homology of $\mathfrak{M}$ has no torsion and vanishes in odd degrees, and every component of $\mathfrak{M}$ is simplyconnected.

Proof. (1) If ε is sufficiently small, $F \le c$ implies a bound on $\|B\|^2 + \|i\|^2 + \|j\|^2$. This shows that $F \le c$ is compact.

(2) Since F is a moment map of the torus $S^1 \times T^r$ action coupling with ε, the critical points of F are precisely the fixed points if ε is generic. Take a gauge equivalence class of a pair (A, φ). It is fixed by T^r if and only if for each $h \in T^r$ there exists a gauge transformation γ such that $\gamma^* A = A$ and

$$\varphi \lim_{x\to\infty} \gamma(x) = h\varphi.$$

Then A decomposes as the bundle decomposes into eigenspaces of γ. If the eigenvalues of h are all distinct, the bundle is a direct sum of line bundles, that is $L_1 \oplus \cdots \oplus L_r$. On the ALE space X, anti-self-dual connections on line bundles are classified by their first Chern class. In particular, the framed moduli space on L_a consists of one point, so the point must be fixed by the S^1 action. Therefore the direct sum is also a fixed point.

(3) This statement holds for a general function arising from a moment map (see [1], [11]). But we give the proof for our situation since it will be nesessary in the next section.

Take a fixed point $[(A, \varphi)]$ in $\mathfrak{M}$. Let d_A, d_A^*, and d_A^+ denote the exterior differential operator acting on forms with values in $\operatorname{End}(E)$ (twisted by the connection A), its formal adjoint and its projection to the space of self-dual 2-forms respectively. The complexified tangent space of $\mathfrak{M}$ at $[(A, \varphi)]$ is identified with the L^2 kernel of the operator

$$d_A^* \oplus d_A^+ : \Omega^1(\operatorname{End}(E)) \to \Omega^0(\operatorname{End}(E)) \oplus \Omega^+(\operatorname{End}(E))$$

(see [16, §2]). The complex structure I on X induces an almost complex structure on this space. Since x corresponds to the sum of line bundles $L_1 \oplus \cdots \oplus L_r$, the L^2 kernel has a $\mathbb{C}$-vector space decomposition

$$\bigoplus_{a\neq b} (L^2 \text{ kernel of } d_A^* \oplus d_A^+) \cap \Omega^1(L_a^* \otimes L_b).$$

Note that $d_A^* \oplus d_A^+$ has trivial L^2 kernel on $\Omega^1(L_a^* \otimes L_a)$, because the moduli space on L_a is zero dimensional. Since $[(A, \varphi)]$ is a fixed point, there exists a lift $\tilde{t}$ to E of $t: X \to X$ which respects the connection A, preserves the decomposition $E = L_1 \oplus \cdots \oplus L_r$ and acts as the identity on $E_\infty = \bigoplus (L_a)_\infty$. Hence $T_x \mathfrak{M}_\zeta$ becomes an S^1 module and decomposes into the sum

$$\bigoplus_{a\neq b}\bigoplus_{m\in\mathbb{Z}} H^m_{a,b} \tag{3.3}$$

of complex subspaces where S^1 acts on $H^m_{a,b}$ with weight m. Then the Hessian of F_0 acts on $H^m_{a,b}$ as multiplication by m. Suppose ε, regarded as an element in $\operatorname{End}(W)$, acts on $(L_a)_\infty$ as the multiplication by ε_a. Then the Hessian of $\langle \varepsilon, jj^\dagger - i^\dagger i\rangle$ acts on $H^m_{a,b}$ as multiplication by $\varepsilon_b - \varepsilon_a$. So the Hessian of F is nondegenerate, if all ε_a's are distinct, as we have been assuming. The index is given by

$$\sum_{a\neq b}\left(\sum_{m<0}\dim_{\mathbb{R}} H^m_{a,b} + \sum_{m=0,\,\varepsilon_a>\varepsilon_b}\dim_{\mathbb{R}} H^m_{a,b}\right). \tag{3.4}$$

Since $H^m_{a,b}$ is a complex vector space, the index is even. In particular, the Morse function F is perfect. Finally, the vanishing of cohomology in degree $> \dim_{\mathbb{C}} \mathfrak{M}$ follows from the fact "$\mathfrak{M}$ is diffeomorphic to an affine algebraic variety" [17, 4.2].

Remark 3.5. For a general Γ, we can define a function F on $\mathfrak{M}$ by the same construction. But it is no longer a Morse function; it may have critical submanifolds of positive dimension.

4. Lefschetz formula

By (3.2),(3.3) the calculation of the homology of the framed moduli space $\mathfrak{M}$ is reduced to two steps: (1) the classification of decompositions into the sum of line bundles, (2) the calculation of the torus action on the L^2 kernel of $d_A^* \oplus d_A^+$ when A is the sum of connections on line bundles. We shall carry out step (2) in this section. Since a line bundle is classified by its first Chern class on the ALE space of type A_n, step (1) is reduced to the classification of decompositions of a vector into a sum of vectors (see the next section for the special case). Step (2) will be carried out by determining the fixed points of the S^1 action on the ALE space X explicitly, and applying the Lefschetz formula.

Suppose that the orbit $[y]\in X$ is a fixed point of the S^1 action. Then there exists a one-parameter subgroup $\lambda\colon S^1 \to \mathrm{PU}(R)^\Gamma = \prod_{k\neq 0}\mathrm{U}(n_k)$ such that $t^{-1}y = \lambda(t).y$. Since $n_k = 1$ for all k, we can write

$$\lambda(t) = (t^{r_1}, \cdots, t^{r_n}) \in \prod_{k\neq 0}\mathrm{U}(n_k)$$

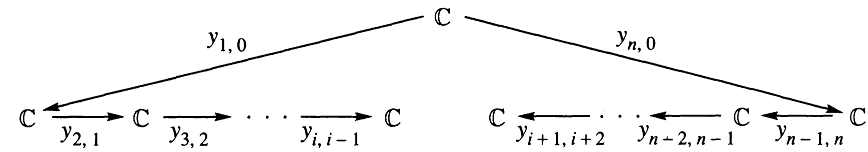
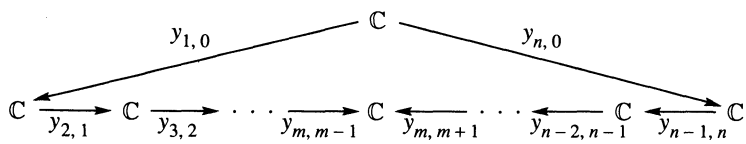


FIGURE 1. p_i.



$(m = \frac{n+1}{2})$

FIGURE 2. Σ.

for some integers $r_1, \cdots, r_n$. We put $r_0 = 0$. We then have:

(1) if $y_{k,k+1} \neq 0$, $r_k = r_{k+1} - 1$,
(2) if $y_{k+1,k} \neq 0$, $r_k = r_{k+1} + 1$,

where $n+1$ is understood as 0. Thus it is not too difficult to determine the fixed point set explicitly. For simplicity, we put the following assumption on $\zeta_{\mathbb{R}} = (\frac{i}{2}\zeta_{\mathbb{R}}^{(0)}, \cdots, \frac{i}{2}\zeta_{\mathbb{R}}^{(n)})$: $\zeta_{\mathbb{R}}^{(k)} > 0$ for $k = 1, 2, \cdots, n$. Let p_i be the point whose ADHM data are given by Figure 1, where $i = 0, 1, \cdots, n$. The linear maps $y_{k,l}$'s such that the corresponding arrows are not written in Figure 1 are zero. When n is odd, let Σ be a submanifold given by Figure 2.

Lemma 4.1. *When n is even, the fixed points set consists of $p_0, p_1, \cdots, p_n$, and S^1 acts on the tangent space $T_{p_i}X$ with weights $2i-1-n$ and $n-1-2i$. When n is odd, it consists of $p_0, p_1, \cdots, p_{(n-3)/2}, p_{(n+3)/2}, \cdots, p_n$ and Σ. The manifold Σ is isomorphic to $\mathbb{C}P^1$. If $p \in \Sigma$, S^1 acts on the tangent space T_pX with weights 0 and -2.*

We can lift the S^1 action to the tautological bundle

$$\mathscr{R}_k = \mu^{-1}(\zeta) \times_{\mathrm{PU}(R)^\Gamma} R_k$$

by letting it act trivially on the R_k factor. The induced action on the fiber at infinity R_k is trivial. Using the above explicit description, we can calculate the weight of the action on the fiber $(\mathscr{R}_k)_{p_i}$ over the fixed point p_i. It is given by

$$\begin{cases} n+1-k, & \text{if } i < k, \\ k, & \text{if } i \geq k. \end{cases} \tag{4.2}$$

Similarly the weight of the action on $(\mathscr{R}_k)_\Sigma$ is given by the same formula with $i = (n+1)/2$.

Suppose that $[(A, \varphi)]$ is a critical point of the Morse function F as in Theorem 3.2, and let $E = L_1 \oplus \cdots \oplus L_r$ be the corresponding decomposition into the sum of line bundles. The index of F at $[(A, \varphi)]$ is given by the formula (3.4). At each fixed point p_i, the fiber of the line bundle $L_a^* \otimes L_b$ becomes an S^1-module. Let m_i denote its weight. Similarly let m_Σ be the weight of the S^1-action on the fiber on Σ.

Choose k so that $(L_a^* \otimes L_b)_\infty$ is isomorphic to R_k as a Γ-module. Let r_i (resp. r_Σ) be the weight of the S^1-action on the fiber $(\mathscr{R}_k)_{p_i}$ (resp. $(\mathscr{R}_k)_\Sigma$) over the fixed point p_i (resp. Σ). These are given by the formula (4.2).

We now give the Lefschetz type formula. We assume that n is even for simplicity. (When n is odd, the formula contains the term coming from the action on Σ.)

Proposition 4.3. *If n is even, then*

$$\frac{1}{2}\sum_m t^m \dim_{\mathbb{R}} H^m_{a,b} = \sum_{0 \leq i \leq n} \frac{t^{-r_i} - t^{-m_i}}{(1 - t^{2i-1-n})(1 - t^{n-1-2i})}.$$

Proof. We use the Lefschetz formula for manifolds with boundary [4]. It is not difficult to make a suitable modification of such a theorem to the ALE manifold X via a conformal change of the metric and working on a manifold with a cylindrical end (cf. [2]). Since the L^2 condition differs from the boundary condition used in [2], [4], one must check the asymptotic behavior of the solutions of the Dirac equation. But it is contained, for example, in [16, Lemma 4.8 and Proposition 5.2]: Any solution which converges to zero at infinity, in fact, satisfies the decay condition $O(|\mathfrak{x}|^{-3})$.

The L^2 cokernel of $d_A^* \oplus d_A^+$ is zero, hence the dimension of the kernel is equal to the index. If n is even, we have

$$\frac{1}{2}\sum_m t^m \dim_{\mathbb{R}} H^m_{a,b} = -\sum_{i=0}^{n} \frac{t^{-m_i}}{(1 - t^{2i-1-n})(1 - t^{n-1-2i})} + \xi_t,$$

where ξ_t is an error term which depends only on the asymptotic behavior of the connection.

To calculate ξ_t, we consider the tautological vector bundle $\mathscr{R}_k$. The asymptotic behavior, including the S^1 action, is the same for $L_a^* \otimes L_b$ and $\mathscr{R}_k$. In particular, the error terms ξ_t are equal for $L_a^* \otimes L_b$ and $\mathscr{R}_k$. But we know that the L^2 kernel of $d_A^* \oplus d_A^+$ for $\mathscr{R}_k$ is zero by [13, Lemma 7.1]. Hence

$$\xi_t = \sum_{i=0}^{n} \frac{t^{-r_i}}{(1 - t^{2i-1-n})(1 - t^{n-1-2i})}.$$

This completes the proof.

5. Row increasing Young tableaux

We give a combinatorial algorithm to calculate the cohomology of the framed moduli spase $\mathfrak{M}$. We shall treat the case

$$V_0 = W_0 = 0. \tag{5.1}$$

By [13, remark after Proposition 9.2] the condition (5.1) implies (2.5).

Then we may assume that n is even and the fixed point set consists of finite points. (Otherwise, put $V_{n+1} = 0$ and replace n by $n + 1$.)

We modify the S^1 action as

$$\begin{aligned} &B_{k-1,k} \to B_{k-1,k}, \qquad B_{k,k-1} \to t^2 B_{k,k-1}, \\ &\quad i_k \to t^2 i_k, \qquad\qquad j_k \to j_k \qquad\qquad \text{for } k = 1, 2, \cdots, n, \end{aligned}$$

for a later purpose. If we set $g_k = t^{k+1} \in \mathrm{U}(V_k)$, $h_k = t^k \in \mathrm{U}(W_k)$, the new action relates to the previous one by

$$\begin{aligned} B_{k-1,k} &= g_{k-1}\, tB_{k-1,k}\, g_k^{-1}, & t^2 B_{k,k-1} &= g_k\, tB_{k,k-1}\, g_{k-1}^{-1}, \\ t^2 i_k &= g_k\, ti_k\, h_k^{-1}, & j_k &= h_k\, tj_k\, g_k^{-1}. \end{aligned}$$

Therefore the new action is a combination of the previous one and the change of frame at infinity caused by $\prod h_k \in \prod \mathrm{U}(W_k)$. It can also be checked that the corresponding moment map is proper.

We first give the classification of line bundles. Suppose that an anti-self-dual connection on a line bundle L is given. Let V_k, W_k, (B, i, j) be the corresponding ADHM data. Since the moduli space consists of one point, the data (B, i, j) are uniquely determined from V_k, W_k. So the classification of line bundles is reduced to the classification of $v_k = \dim_{\mathbb{C}} V_k$ and $w_k = \dim_{\mathbb{C}} W_k$. There exists a number w such that

$$w_k = \begin{cases} 1, & \text{if } k = w, \\ 0, & \text{otherwise.} \end{cases}$$

We assume that $v_0 = w_0 = 0$.

Lemma 5.1. *The data v_k, w correspond to a line bundle L with $L_\infty = R_w$ if and only if*

$$v_k - v_{k-1} = \begin{cases} 0 \text{ or } 1, & \text{if } k \le w, \\ 0 \text{ or } -1, & \text{if } k > w. \end{cases} \tag{5.2}$$

Proof. Suppose that the data correspond to a line bundle L. Since the dimension of the moduli space is 0, the formula (2.1) gives us

$$\begin{aligned} 0 &= 2\sum_{k=1}^{n} v_k v_{k-1} + 2v_w - 2\sum_{k=1}^{n} v_k^2 \\ &= -\sum_{k=1}^{n+1} (v_k - v_{k-1})^2 + 2v_w, \end{aligned} \tag{5.3}$$

where we set $v_{n+1} = 0$ in convention. Noticing that $q^2 \ge \pm q$ holds for an integer q and equality holds if and only if $q = 0$ or ± 1, we find

$$\begin{aligned} &\sum_{k=1}^{w} (v_k - v_{k-1})^2 + \sum_{k=w+1}^{n+1} (v_k - v_{k-1})^2 \\ &\quad \ge \sum_{k=1}^{w} (v_k - v_{k-1}) + \sum_{k=w+1}^{n+1} (v_{k-1} - v_k) = 2v_w. \end{aligned}$$

So (5.3) implies (5.2).

Conversely we can show that there exists a line bundle if v_k satisfies (5.2). In fact, let

$$L \overset{\text{def.}}{=} \bigotimes_{k=1}^{n} \mathscr{R}_k^{\otimes(v_{k-1}+v_{k+1}-2v_k)} \otimes \mathscr{R}_w,$$

where $\mathscr{R}_k^{\otimes m} = (\mathscr{R}_k^*)^{\otimes -m}$ when $m < 0$. The condition (5.2) ensures $L_\infty = R_w$. The first Chern class of L is given by

$$\sum_{k=1}^{n} (v_{k-1} + v_{k+1} - 2v_k) c_1(\mathscr{R}_k) + c_1(\mathscr{R}_w).$$

By [13, §9], the corresponding ADHM data should be

$$\dim_{\mathbb{C}} V_k = v_k + m \quad \text{for } k = 0, 1, \cdots, n$$

for some integer m. The moduli space must be 0-dimensional, therefore the dimension formula implies $m = 0$. q.e.d.

We now make a bijection between the set of line bundles and that of strictly increasing sequences of [b integers between 1 and $n+1$. In order to fit the notation used later, we enclose a figure by a box, e.g., $\boxed{2}\boxed{4}$. Suppose that a line bundle L with $V_0 = W_0 = 0$ is given. The line bundle is determined uniquely by the data v_k and w. We set the number of boxes (the length of the sequence) to be w. The number is always positive since $L_\infty \neq R_0$ by the assumption $W_0 = 0$. We define a sequence so that the figure k appears if and only if

$$\begin{cases} v_{k-1} - v_k = 0, & \text{for } k \leq w, \\ v_{k-1} - v_k = 1, & \text{for } k > w. \end{cases}$$

(Recall $v_{n+1} = 0$.) If the first Chern class of L is given by

$$c_1(L) = \sum_k u_k c_1(\mathscr{R}_k),$$

the above condition is equivalent to

$$v_n + \sum_{i \geq k} u_i = 1. \tag{5.4}$$

It can be checked easily that the number of figures is equal to w. Thus we get

Lemma 5.5. *There exists a bijection between isomorphism classes of line bundles with $V_0 = W_0 = 0$ and strictly increasing sequences of positive integers between 1 and $n+1$.*

For example, $\boxed{2}\boxed{4}$ corresponds to the line bundle L with $L_\infty = R_2$, $v_1 = v_2 = v_3 = 1$.

Set $n_k(L)$ to be 1 if the figure k appears in the sequence corresponding to the line bundle L, and to be 0 otherwise. Then

$$L = \bigotimes_{k=1}^{n} \mathscr{R}_k^{\otimes(n_k(L) - n_{k+1}(L))}.$$

We pull back the S^1 action to L so that the induced action on the fiber at infinity L_∞ is of weight w. Hence (4.2) shows that S^1 acts on the fiber L_{p_i} over the fixed point p_i with weight

$$\begin{aligned} &\sum_{k \leq i} k(n_k(L) - n_{k+1}(L)) + \sum_{k>i} (n+1-k)(n_k(L) - n_{k+1}(L)) + w \\ &\qquad = \sum_{k \leq i} 2n_k(L) + (n+1-2i)n_{i+1}(L). \end{aligned} \tag{5.6}$$

Now we return to the case of a moduli space on a vector bundle E. The representation at infinity E_∞, the first Chern class $c_1(E)$ and the instanton charge determine uniquely the data $v_k = \dim_{\mathbb{C}} V_k$ and $w_k = \dim_{\mathbb{C}} W_k$. We assume that $v_0 = w_0 = 0$, and let

$$u_k \stackrel{\text{def.}}{=} v_{k-1} + v_{k+1} + w_k - 2v_k \quad \text{for } 1 \le k \le n, \tag{5.7}$$

where we set $v_{n+1} = 0$ in convention. Then the first Chern class can be represented as $c_1(E) = \sum_{k\neq 0} u_k c_1(\mathscr{R}_k)$. We take a unitary basis $\langle e_1^k, \cdots, e_{w_k}^k \rangle$ of W_k for each k; collecting them, we have a basis of W:

$$\langle e_1^n, \cdots, e_{w_n}^n, e_1^{n-1}, \cdots, e_{w_{n-1}}^{n-1}, \cdots, e_1^1, \cdots, e_{w_1}^1 \rangle.$$

Then renumbering the vectors in the above order, we denote them by $\langle e_1, \cdots, e_r \rangle$ ($r = \operatorname{rank} E$). We consider the action of $S^1 \times T^r$ and the Morse function F on the framed moduli space $\mathfrak{M}$ of anti-self-dual connections on E as in §3.

We want to give a combinatorial description for critical points of F. Let $N = \sum_k k w_k$. To W, we associate λ, a partition of N (namely $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ with $\lambda_a \in \mathbb{N}$ such that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r$ and $\sum_{a=1}^r \lambda_a = N$) by the following rule: "The figure k appears w_k-times in the sequence λ". We also define a sequence $\mu = (\mu_1, \mu_2, \cdots, \mu_{n+1})$ by

$$\mu_k = v_n + \sum_{i \ge k} u_i \quad \text{for } k = 1, 2, 3, \cdots, n, \qquad \mu_{n+1} = v_n.$$

Note that $\sum_k \mu_k = N$. If $\mu_k < 0$ for some k, then the Morse function F has no critical points by the proof of Lemma 5.9. But this is impossible unless the moduli space $\mathfrak{M}$ is empty. Therefore F must attain the minimum at some point, and hence we may assume that $\mu_k \ge 0$ for all k.

Definition 5.8. Let λ, μ be as above.

(1) A μ-tableau of shape λ is a Young diagram of shape λ whose nodes are numbered with the figures from 1 to $n+1$ such that the cardinality of the nodes with figure k is μ_k.

(2) A μ-tableau of shape λ is said to be *row-increasing* if the entries in each row *strictly* increase from left to right.

Lemma 5.9. *There exists a one-to-one correspondence between the critical points of F and the set of row-increasing μ-tableaux of shape λ.*

Proof. If $[(A, \varphi)]$ is a critical point of F, there exists a line bundles decomposition $E = L_1 \oplus \cdots \oplus L_r$ with $\varphi: (L_a)_\infty \cong \mathbb{C} e_a$ (Theorem 3.2(2)). Each line bundle L_a corresponds to a sequence of boxed positive integers by Lemma 5.5. Place the sequence for L_a in the ath row to make a row-increasing tableau of shape λ. Let v_n^a be the dimension of the R_n-component of the ADHM data for the line bundle L_a. If

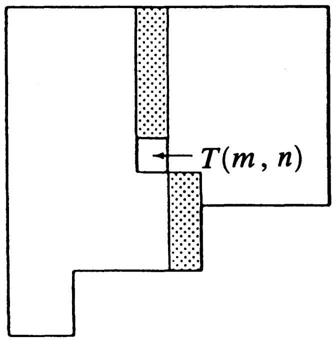


FIGURE 3

$c_1(L_a) = \sum_k u_k^a c_1(\mathscr{R}_k)$, the figure k appears in the kth row if and only if $v_n^a + \sum_{i \geq k} u_i^a = 1$ by (5.4). Since $v_n^a + \sum_{i \geq k} u_i^a = 0$ or 1, the figure k appears

$$\sum_{a=1}^{r} \left(v_n^a + \sum_{i \geq k} u_i^a \right) = v_n + \sum_{i \geq k} u_i = \mu_k$$

times in total. Thus we get a μ-tableau of shape λ. q.e.d.

Now we want to give a formula for the index (3.4) in terms of the Young tableau. We assume that $0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_r$. We need the following definition.

Definition 5.10. For a row-increasing μ-tableau T of shape λ we define an integer $l(T)$ as follows: Denote by $T(m, n)$ the figure in the mth row and the nth column. Define $l_{m,n}$ to be the number of nodes, sitting in the shaded region in Figure 3, such that the figure on the node is less than $T(m, n)$, and the figure $T(m, n)$ does not appear in the same row. Then set

$$l(T) = \sum_{m,n} l_{m,n}.$$

Lemma 5.11. *The index of the Morse function F at the critical point corresponding to T is given by $2l(T)$.*

Proof. The index is given by formula (3.4). We first calculate

$$\frac{1}{2} \sum_{m<0} \dim_{\mathbf{R}} H_{a,b}^m + \frac{1}{2} \sum_{m=0,\, \varepsilon_a > \varepsilon_b} \dim_{\mathbf{R}} H_{a,b}^m \tag{5.12}$$

for each a, b.

Define $w(a)$ so that the fiber at infinity $(L_a)_\infty$ is isomorphic to $R_{w(a)}$ as a Γ-module. As in Proposition 4.3, choose k so that $(L_a^* \otimes L_b)_\infty \cong R_k$.

Explicitly k is

$$k = \begin{cases} w(b) - w(a), & \text{if } w(b) \geq w(a), \\ n + 1 + w(b) - w(a), & \text{if } w(b) < w(a). \end{cases}$$

Then the weight r_i of the S^1-action on the fiber $(\mathscr{R}_k)_{p_i}$ over the fixed point p_i is given by

$$r_i = \begin{cases} n+1, & \text{for } i < w(b) - w(a). \\ 2(w(b) - w(a)), & \text{for } i \geq w(b) - w(a) \end{cases} \quad \text{if } w(b) \geq w(a)),$$

$$r_i = \begin{cases} 0, & \text{for } i < n + 1 + w(b) - w(a), \\ n + 1 + 2(w(b) - w(a)), & \text{for } i \geq n + 1 + w(b) - w(a), \end{cases}$$
$$(\text{if } w(b) \leq w(a)).$$

By Proposition 4.3, (5.12) is the sum of the coefficients of t^m $(m \leq 0$ or $m < 0)$ in

$$\sum_{i=0}^{n} \frac{t^{-r_i} - t^{-m_i}}{(1 - t^{2i-1-n})(1 - t^{n-1-2i})}, \tag{5.13}$$

where m_i is the weight of the S^1 action on the fiber $(L_a^* \otimes L_b)_{p_i}$ over the fixed point p_i and given by

$$m_i = \sum_{k \leq i} 2(n_k(L_b) - n_k(L_a)) + (n + 1 - 2i)(n_{i+1}(L_b) - n_{i+1}(L_a))$$

(see (5.6)). First consider the case $a > b$ (so $w(b) \geq w(a)$). Putting $V_{n+1} = V_{n+2} = \cdots = V_{n'} = 0$, $W_{n+1} = W_{n+2} = \cdots = W_{n'} = 0$ and replacing n by n', we may assume that n is sufficiently large compared with the maximum M of figures appeared in the ath and bth rows in the tableau and $w(a)$, $w(b)$. This procedure does not affect the index, since the ADHM data are unchanged essentially. If $i \geq \max(w(b) - w(a), M)$, we have

$$r_i = 2(w(b) - w(a)) = m_i.$$

Hence it is enough to sum up the terms from 0 to $\max(w(b) - w(a), M)$ in (5.13). In particular, we may assume that $n - 1 > 2i$. Then

$$\begin{aligned} &\frac{t^{-r_i} - t^{-m_i}}{(1 - t^{2i-1-n})(1 - t^{n-1-2i})} \\ &= (t^{-m_i+n+1-2i} - t^{-r_i+n+1-2i}) \sum_{k=0}^{\infty} t^{k(n+1-2i)} \sum_{k=0}^{\infty} t^{k(n-1-2i)}, \end{aligned} \tag{5.14}$$

where the summations are the formal power series. We may assume that $-m_i + 2(n - 2i)$, $-r_i + 2(n - 2i) > 0$. Therefore in order to calculate the sum of the terms of nonpositive power of t, it is enough to consider $t^{-m_i+n+1-2i} - t^{-r_i+n+1-2i}$. Then

$$-m_i+n+1-2i = \sum_{k\leq i} 2(n_k(L_a)-n_k(L_b))+(n+1-2i)(n_{i+1}(L_a)-n_{i+1}(L_b)+1)$$

is nonpositive if and only if

$$n_{i+1}(L_a)=0, \quad n_{i+1}(L_b)=1, \quad \sum_{k\leq i}(n_k(L_a)-n_k(L_b)) \leq 0.$$

The exponent of the second term on the right-hand side of (5.14) satisfies

$$-r_i+n+1-2i = \begin{cases} -2i, & \text{if } i < w(b)-w(a), \\ 2(w(a)-w(b))+n+1-2i, & \text{if } i \geq w(b)-w(a). \end{cases}$$

We have $w(b)-w(a)$ nonpositive terms for the first case, while we may assume that $2(w(a)-w(b))+n+1-2i$ is positive. Hence (5.12) is given by

$$\#\{m \mid T(b,m) < T(a,m) \text{ and } \text{“}b\text{th row does not contain the figure } T(a,m)\text{”}\}.$$

We have a similar description in the other case $a < b$. We replace the ADHM data by $V_1' = V_2' = \cdots = V_{n'}' = 0$, $V_{n'+k}' = V_k$, etc. Then we may assume that the minimum of figures in the ath and bth rows are sufficiently larger than $n/2$, and the remaining argument is similar to that in the first case.

Summing up with respect to a and b, we find that the index is equal to $2l(T)$. q.e.d.

Theorem 5.15. *The Poincaré polynomial of the moduli space $\mathfrak{M}$ is given by $\sum_T t^{2l(T)}$, where T runs over the set of row-increasing μ-tableaux of shape λ.*

This description is very manageable. For example, we can see

Corollary 5.16. *The moduli space is connected under the assumption* (5.1).

In fact, it is not so difficult to prove that there exists at most one tableau T with $l(T)=0$. Hence the moduli space is connected if it is nonempty.

Remark 5.17. By [17, 10.6] the middle cohomology group of $\mathfrak{M}$ is isomorphic to the $(\mathbf{w}-\mathbf{Cv})$-weight space of the irreducible representation of the simple Lie algebra $\mathfrak{su}(n+1)$ with highest weight $\mathbf{w}$. The vectors are related to λ, μ by

$$\mathbf{w} - \mathbf{C}\mathbf{v} = (\mu_1 - \mu_2, \mu_2 - \mu_3, \cdots, \mu_n - \mu_{n+1})^t,$$
$$\mathbf{w} = (\lambda'_1 - \lambda'_2, \lambda'_2 - \lambda'_3, \cdots, \lambda'_n - \lambda'_{n+1})^t,$$

where $\lambda' = (\lambda'_1, \lambda'_2, \cdots, \lambda'_{n+1})$ is the conjugate partition of λ defined by $\lambda'_k = \#\{l \mid \lambda_l \geq k\}$. A row-increasing μ-tableau T of shape λ has $2l(T) = \dim_{\mathbb{C}} \mathfrak{M}$ if and only if the entries in each column increase (may be stationary) from top to bottom. Such a tableau is said to be *semistandard*. (The roles of the column and the row are in reverse usually.) The fact that the number of a semistandard tableaux is equal to the the dimension of the weight space is classically known (see, e.g., [6]).

We give a few examples of the list of row-increasing μ-tableaux T of shape λ and their $l(T)$ (Figures 4–6).

$$\begin{array}{c|cccccc}
T & \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&4\\\hline 2&3\\\hline\end{array} & \begin{array}{|c|c|}\hline 2&3\\\hline 1&4\\\hline\end{array} & \begin{array}{|c|c|}\hline 3&4\\\hline 1&2\\\hline\end{array} & \begin{array}{|c|c|}\hline 2&4\\\hline 1&3\\\hline\end{array} \\
l(T) & 2 & 2 & 1 & 1 & 1 & 0
\end{array}$$

FIGURE 4. $\lambda = (2, 2)$, $\mu = (1, 1, 1, 1)$.

$$\begin{array}{c|cccccc}
T & \begin{array}{|c|c|}\hline 1&2\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&3\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&4\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&2\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&3\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1} \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&4\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} \\
l(T) & 3 & 3 & 3 & 2 & 2 & 2
\end{array}$$

$$\begin{array}{c|cccccc}
T & \begin{array}{|c|c|}\hline 2&3\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2&4\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2&3\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1} \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2&4\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 3&4\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1} \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 3&4\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1} \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array} \\
l(T) & 2 & 2 & 1 & 1 & 1 & 0
\end{array}$$

FIGURE 5. $\lambda = (2, 1, 1)$, $\mu = (1, 1, 1, 1)$.

T	$\begin{array}{cc} 1 & 2 \\ 3 & \\ 3 & \end{array}$	$\begin{array}{cc} 1 & 3 \\ 2 & \\ 3 & \end{array}$	$\begin{array}{cc} 1 & 3 \\ 3 & \\ 2 & \end{array}$	$\begin{array}{cc} 2 & 3 \\ 1 & \\ 3 & \end{array}$	$\begin{array}{cc} 2 & 3 \\ 3 & \\ 1 & \end{array}$
$l(T)$	2	2	1	1	0

FIGURE 6. $\lambda = (2, 1, 1)$, $\mu = (1, 1, 2)$.

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