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IRREDUCIBILITY OF MODULI OF RANK-2 VECTOR BUNDLES ON ALGEBRAIC SURFACES

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Let X be a smooth algebraic surface over $\mathbb{C}$, let I be a fixed line bundle on X, and let H be a very ample line bundle on X. We recall that a sheaf E is H-stable (resp. H-semistable) if it is coherent, torsion free and so that for any proper subsheaf $F \subset E$, we have $p_F \prec p_E$ (resp. $p_F \preceq p_E$), where $p_E = (1/\operatorname{rank} E)\chi_E$ and $\chi_E(n) = \chi(E \otimes H^{\otimes n})$. Here by $p_F \prec p_E$ we mean $p_F(n) < p_E(n)$ for all sufficiently large n. There is a coarse moduli space $\mathfrak{M}_X^{d,I}$ parameterizing all rank 2 H-semistable sheaves E with $\det E = I$ and $c_2(E) = d$ (modulo a certain equivalence). $\mathfrak{M}_X^{d,I}$ is a projective scheme [8]. For small d, $\mathfrak{M}_X^{d,I}$ can have rather wild behavior, e.g., the dimension of $\mathfrak{M}_X^{d,I}$ may be larger than expected [9]. However, S. Donaldson [4], later generalized by R. Friedman [6] and K. Zhu [34] showed that for large d, every component of $\mathfrak{M}_X^{d,I}$ is reduced and has the expected dimension. $\mathfrak{M}_X^{d,I}$ is also normal [20] for $d \gg 0$.

Our purposes of this paper is twofold. The first is to develop a method of studying $\mathfrak{M}_X^{d,I}$ by degeneration. The second is to use this method to prove

Main Theorem. *Let X be any smooth algebraic surface over $\mathbb{C}$, and I be a fixed ample divisor. Then there is a constant A depending on (X, H, I) such that whenever $d \geq A$, then $\mathfrak{M}_X^{d,I}$ is irreducible.*

The proof of the theorem is based on the following well-known observation: Let A be large so that for $d \geq A$, $\mathfrak{M}_X^{d,I}$ is smooth at a dense subset. Take $\mathbf{M} \subseteq \mathfrak{M}_X^{d,I}$ be any irreducible component and take $E \in \mathbf{M}$ be a smooth point. Let $\mathbb{C}_x$ be the skyscraper sheaf over $x \in X$ and let $E \to \mathbb{C}_x$ be a general surjective homomorphism. The kernel E' of $E \to \mathbb{C}_x$ is a stable sheaf with $c_2(E') = d + 1$ and $\operatorname{Ext}^2(E', E')^0 = \{0\}$. Thus E' belongs to a unique irreducible component of $\mathfrak{M}_X^{d+1,I}$. Now if we let $\Lambda(d)$ be the set of irreducible components of $\mathfrak{M}_X^{d,I}$, then this construction

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provides us a map $f_d\colon \Lambda(d) \to \Lambda(d+1)$, $d \geq A$. Let $f_d^l = f_{d+l-1} \circ \cdots \circ f_d$. We have the following theorem due to C. Taubes [32].

Theorem 0.1 (Taubes). *For any $d \geq A$, there is an integer $l(d)$ so that for any $l \geq l(d)$, $f_d^l(\Lambda(d))$ is a single point set.*

We will give an elementary proof of Theorem 0.1 in the end of §7. The main theorem will be a consequence of the following theorem.

Theorem 0.2. *With the notation as in the main theorem, there is a constant A depending on (X, H, I) such that for any $d \geq A$ and any irreducible component $M \subseteq \mathfrak{M}_X^{d,I}$, the set*

$$\mathbf{M}^{\mathrm{sh}} = \{s \in \mathbf{M} \mid s \textit{ corresponds to nonlocally free stable sheaf}\}$$

is a codimension 1 subset of $\mathbf{M}$. In particular, it is nonempty.

A corollary of Theorem 0.2 is that when d is sufficiently large, $f_d\colon \Lambda(d) \to \Lambda(d+1)$ is surjective. Therefore combined with Theorem 0.1, $\Lambda(d)$ has exactly one element for $d \gg 0$. This proves the main theorem.

The proof of Theorem 0.2 is by studying a degeneration of moduli space which we describe now: Let C be a smooth curve which will function as parameter space for our deformation and let $0 \in C$. Pick a smooth divisor $\Sigma \in |H|$ and blow up $X \times C$ along $\Sigma \times \{0\}$ to obtain a threefold Z. Let $\pi\colon Z \to C$ be the projection and let $C^* = C\backslash\{0\}$. Note that $\pi^{-1}(C^*) = X \times C^*$ and $\pi^{-1}(0) = X \cup \Delta$, where Δ is a ruled surface over Σ. Let $\mathfrak{M}_X^{d,I} \times C^*$ be the constant family of schemes over C^*. We intend to construct a flat family (denoted by $\mathfrak{M}_{Z/C}^{d,\alpha}$) that extends (the normalization of) the family $\mathfrak{M}_X^{d,I} \times C^*$ over C^* to C. This extension depends on the choice of α, where α is a pair of rational numbers. It turns out that the closed point of the special fiber of $\mathfrak{M}_{Z/C}^{d,\alpha}$ over 0 has a rather nice description: For any coherent torsion free sheaf F on $\pi^{-1}(0) = X \cup \Delta$ and for any choice of α, we introduce the concept of α-stability of F. (This concept was originally introduced by Seshadri in the context of sheaves on reducible curves [31].) The family $\mathfrak{M}_{Z/C}^{d,\alpha}$ constructed has the property that all closed points $s \in \mathfrak{M}_{Z/C}^{d,\alpha}$ over $0 \in C$ are canonically associated to α-semistable sheaves $\mathscr{E}$ on $X \cup \Delta$. Moreover, $\mathscr{E}$ can be constructed by "gluing" torsion free sheaves E_1 on X and E_2 on Δ along $\Sigma^- = X \cap \Delta$. The merit of α-stability is that though E_1 and E_2 are in general not stable, they are not far from being stable. Thus we can define a rational map Ψ from $\mathfrak{M}_0^{d,\alpha}$ = fiber of $\mathfrak{M}_{Z/C}^{d,\alpha}$ over $0 \in C$ to $\bigcup_c \mathfrak{M}_\Delta^{c,I'}$, the union of moduli spaces of H' semistable sheaves on Δ, where I' and H' are appropriate divisors on Δ.

To utilize this rational map, we need to locate those c so that $\mathfrak{M}_\Delta^{c,I'}$ contains image of $\Psi(\mathfrak{M}_0^{d,\alpha})$ and to understand the geometry of $\Psi(\mathfrak{M}_0^{d,\alpha})$ in $\mathfrak{M}_\Delta^{c,I'}$. We show that we can choose α_d such that there is at least one c so that

(1) $\mathfrak{M}_\Delta^{c,I'} \cap \Psi(\mathfrak{M}_0^{d,\alpha})$ has codimension bounded independently of d (in $\mathfrak{M}_\Delta^{c,I'}$) and

(2) $d-c$ is bounded independently of d.

We will see that (1) will relate the properties of $\mathfrak{M}_\Delta^{c,I'} \cap \Psi(\mathfrak{M}_0^{d,\alpha})$ to that of $\mathfrak{M}_\Delta^{c,I'}$ and (2) will allow us to deduce properties of $\mathfrak{M}_0^{d,\alpha}$ (and then $\mathfrak{M}_X^{d,I}$) from that of $\mathfrak{M}_\Delta^{c,I'} \cap \Psi(\mathfrak{M}_0^{d,\alpha})$. Thus we have an effective tool for reducing questions about $\mathfrak{M}_X^{d,I}$ to questions about $\mathfrak{M}_\Delta^{c,I'}$. Note that $\mathfrak{M}_\Delta^{c,I'}$ is much better understood than $\mathfrak{M}_X^{d,I}$.

We show that $d-c$ is bounded independently of d by first showing that the assumption $\mathfrak{M}_\Delta^{c,I'} \cap \Psi(\mathfrak{M}_0^{d,\alpha}) \neq \varnothing$ and $c \gg d$ violate the α-stability of $\mathscr{E} \in \mathfrak{M}_0^{d,\alpha}$. To simplify our explanation, we assume $\mathscr{E} \in \mathfrak{M}_0^{d,\alpha}$ with $\Psi(\mathscr{E}) \in \mathfrak{M}_\Delta^{c,I'}$ locally free on $Z_0 = X \cup \Delta$. Then $c \gg d$ forces the second Chern class of its restriction to X, $\mathscr{E}_{|X}$, to be quite negative. However, the bundle $\mathscr{E}_{|X}$ is not far from being stable, so we can use argument of Bogomolov to show that $c_2(\mathscr{E}_{|X})$ cannot be too negative which contradicts to the assumption $c \gg d$. Next, if we can show that there is a c so that the image of $\Psi(\mathfrak{M}_0^{d,\alpha})$ in $\mathfrak{M}_\Delta^{c,I'}$ has dimension equal to $\dim \mathfrak{M}_X^{d,I}$, then $c-d$ cannot be too negative, since $\dim \mathfrak{M}_\Delta^{c,I'} - \dim \mathfrak{M}_0^{d,\alpha} = 4(c-d)+O(1)$.

To show that the dimension of $\Psi(\mathfrak{M}_0^{d,\alpha})$ is equal to the dimension of $\mathfrak{M}_X^{d,I}$, we use an extension of Donaldson's line bundle $\mathscr{L}_X$ on $\mathfrak{M}_X^{d,I}$ [4]. Recall that $\mathscr{L}_X$ is constructed as a determinant line bundle of a complex of sheaves on $\mathfrak{M}_X^{d,I}$, that $\mathscr{L}_X^{\otimes m}$ is generated by global sections when $m \gg 0$ and that the associated map is birational. One creates sections of $\mathscr{L}_X^{\otimes m}$ by the following process: Let D_r be a smooth curve linearly equivalent to rH. By restricting sheaves $E \in \mathfrak{M}_X^{d,I}$ to D_r, one obtains a rational map $\mathbf{F}_X\colon \mathfrak{M}_X^{d,I} \to \mathfrak{M}_{D_r}$, where $\mathfrak{M}_{D_r}$ is the moduli space of semistable bundles on D_r. If $r \gg 0$, Bogomolov showed that for any $E \in \mathfrak{M}_X^{d,I}$, $\mathbf{F}_X(E)$ is defined if $E_{|D_r}$ is locally free. In [4], Donaldson showed that $\mathscr{L}_X$ is the pullback of the ample line bundle $\mathscr{L}_{D_r}$ on $\mathfrak{M}_{D_r}$. Thus, after a careful study of the pullback sections, we get a map $H^0(\mathfrak{M}_{D_r}, \mathscr{L}_{D_r}^{\otimes m}) \to H^0(\mathfrak{M}_X^{d,I}, \mathscr{L}_X^{\otimes m})$. Then one proves the base point freeness of $H^0(\mathfrak{M}_X^{d,I}, \mathscr{L}_X^{\otimes m})$ by using sections obtained from $H^0(\mathfrak{M}_{D_r}, \mathscr{L}_{D_r}^{\otimes m})$ as one varies D_r in its linear equivalent class [4], [19]. Thus there is a morphism $\mathbf{F}_X\colon \mathfrak{M}_X^{d,I} \to \mathbf{P}^L$ that is one-to-one at the generic points of $\mathfrak{M}_X^{d,I}$.

This setup can be easily extended to $\mathfrak{M}_0^{d,\alpha}$. Let $\mathscr{L}$ be the Donaldson's line bundle on $\mathfrak{M}_0^{d,\alpha}$. Choose $D_r \subseteq \Delta$ with D_r linearly equivalent to $r\Sigma^+$. Here, we use Σ^+ (resp. Σ^-) to denote sections of $\Delta \to \Sigma$ with positive (resp. negative) self-intersection. We show that if α_d is correctly chosen, then $\mathscr{E}_{|D_r}$, $\mathscr{E} \in \mathfrak{M}_0^{d,\alpha}$, is semistable for general D_r. Again we create enough sections of $\mathscr{L}^{\otimes m}$ to show that $\mathscr{L}^{\otimes m}$ is base point free by pullback sections of $\mathscr{L}_{D_r}^{\otimes m}$ for various D_r. Thus we get a map $\mathbf{F}_0\colon \mathfrak{M}_0^{d,\alpha} \to \mathbf{P}^N$. By construction, $\mathbf{F}_0$ is a degeneration of $\mathbf{F}_X$. Therefore by semicontinuity of dimension, $\dim \mathbf{F}_0(\mathfrak{M}_0^{d,\alpha}) = \dim \mathbf{F}_X(\mathfrak{M}_X^{d,I}) = \dim \mathfrak{M}_0^{d,\alpha}$. Finally, there is a similar rational map $\mathbf{F}_\Delta\colon \mathfrak{M}_\Delta^{c,I'} \to \mathbf{P}^N$ so that $\mathbf{F}_0$ is identical to $\mathbf{F}_\Delta \circ \Psi$. Hence $\dim \Psi(\mathfrak{M}_0^{d,\alpha})$ must be equal to $\dim(\mathfrak{M}_X^{d,I})$.

The proof of Theorem 0.2 is based on a careful study of deformation theory once we have constructed the degeneration. Take any irreducible component $\mathbf{M} \subseteq \mathfrak{M}_X^{d,I}$. We intend to show that $\mathbf{M}$ contains at least one non-locally-free sheaf. Let $\mathfrak{M}$ be the corresponding irreducible component in $\mathfrak{M}_{Z/C}^{d,\alpha}$ and let $\mathfrak{M}_0$ be the fiber of $\mathfrak{M}$ over $0 \in C$. Take $\mathbf{\Theta} \subseteq \mathfrak{M}_\Delta^{c,I'}$ be the image $\Psi(\mathfrak{M}_0)$ in $\mathfrak{M}_\Delta^{c,I'}$ with the mentioned property. By studying the vector bundles on the ruled surface Δ, we first prove that there is a closed point $v \in \mathfrak{M}_0$ so that the corresponding α-stable sheaf $\mathscr{E}$ on $X \cup \Delta$ is not locally free on Δ. Then by studying the deformation problem, we show that v is the limit of non-locally-free sheaves in $\mathbf{M}$. Thus $\mathbf{M}$ has to contain at least one non-locally-free sheaf. We sketch the idea of the argument briefly. First of all, we choose $\mathscr{E} \in \mathfrak{M}_0$ so that $\Psi(\mathscr{E})$ is generic in $\mathbf{\Theta}$. For simplicity, let us assume that $\mathscr{E}$ is locally free. Let $4\Sigma^-$ be the subscheme of Δ defined by $\mathscr{I}_{\Sigma^-}^{\otimes 4}$, where $\mathscr{I}_{\Sigma^-}$ is the ideal sheaf of Σ^-. Denote

$$\mathbf{\Theta}' = \{\mathscr{E}' \in \mathfrak{M}_\Delta^{c,I'} \mid \mathscr{E}' \text{ is a torsion free sheaf on } \Delta \text{ with } \mathscr{E}'_{|4\Sigma^-} \cong \mathscr{E}_{|4\Sigma^-}\}.$$

The codimension of $\mathbf{\Theta}'$ in $\mathfrak{M}_\Delta^{c,I'}$ is bounded independently of d. Next, we show that $\mathbf{\Theta}'$ is almost contained in $\mathbf{\Theta}$. The idea here is that if $\mathscr{F}_s'$ is a family of bundles in $\mathbf{\Theta}'$ parameterized by S (irreducible), $0 \in S$, with $\mathscr{F}_0' \cong \mathscr{E}_{|\Delta}$, then we can form a new family of bundles $\mathscr{E}_s'$, $s \in S$, on $X \cup \Delta$ by gluing $\mathscr{F}_s'$ with $\mathscr{E}_{|X}$ along $\Sigma^- = X \cap \Delta$. This is possible because $(\mathscr{F}_s')_{|\Sigma^-} \cong (\mathscr{E}_{|\Delta})_{|\Sigma^-} \cong (\mathscr{E}_{|X})_{|\Sigma^-}$. If we can show that the family $\mathscr{E}_s'$ on $Z_0 \times S$ can be lifted to a family on $Z \times S$, then by the openness of the semistability condition, $\mathscr{F}_s'$ will be in Θ for generic $s \in S$. The problem with this approach is that we have no control over the deformation theory of $\mathscr{E}_s'$—in general the obstruction to the existence of such lifting is nonvanishing. However, since for $\mathscr{F} \in \Theta'$, $\mathscr{F}_{|4\Sigma^-} \cong \mathscr{E}_{|4\Sigma^-}$, we can consider analogous family of bundles $\mathscr{E}_s$ on the *scheme* $4X\cup\Delta$ constructed by gluing sheaves along $4\Sigma^- = 4X \cap \Delta$. We show that these $\mathscr{E}_s$ can be lifted to $5X \cup 2\Delta$, $6X \cup 3\Delta$, etc. (cf. [9] for a similar idea.) Since $\mathfrak{M}$ is projective over C, $\mathscr{E}_{s|Z_0}$ lifts for generic s. In particular, if we choose S so that for some closed $s_1 \in S$, $\mathscr{F}_{s_1}$ is not locally free (and satisfies some additional technical conditions), then the gluing of $\mathscr{E}_{|X}$ with $\mathscr{F}_{s_1|\Delta}$, i.e., $\mathscr{E}_{s_1}$, belongs to $\mathfrak{M}_0$. Thus we know $\mathfrak{M}_0$ has to contain a non-locally-free sheaf. Finally, because the subsets of $\mathfrak{M}$ and $\mathfrak{M}_0$ of non-locally-free sheaves are codimension-1 subsets, $\mathfrak{M}_t = \mathbf{M}$, $t \in C^*$, has to contain a non-locally-free sheaf.

It remains to show that we can find a family (of sheaves over Δ) in Θ' containing $\mathscr{E}_{|\Delta}$ so that this family also contains some non-locally-free sheaves. The idea is first to find a non-locally-free sheaf in the closure $\overline{\Theta'} \subseteq \mathfrak{M}_\Delta^{c,I'}$. We do this by finding a product of projective spaces $T \subseteq \overline{\Theta'}$ with $\mathscr{E}_{|\Delta} \in T$ so that the set of non-locally-free sheaves is an ample divisor on T. It can be seen as follows: Let $P \subseteq \Delta$ be a fiber of Δ over Σ. Any sheaf E belongs to the exact sequence

$$0 \to E \to \mathscr{O}_\Delta \oplus \mathscr{O}_\Delta \xrightarrow{\sigma} \mathscr{O}_P(1) \to 0$$

has $c_2(E) = 1$. The (surjective) homomorphisms σ are parameterized by a subset $U \subseteq \mathbf{P}^3$. Clearly, the compliment of $\mathbf{P}^3 \backslash U$ is an ample divisor of $\mathbf{P}^3$. Thus any nonconstant complete family of sheaves belonging to the preceding exact sequence must contains at least one non-locally-free sheaf that corresponds to closed point in $\mathbf{P}^3 \backslash U$. The idea of attacking higher $c_2(E)$ is basically the same. Using the fact that the deformation of generic sheaves on Δ are unobstructed, we then argue that we can indeed deform $\mathscr{E}_{|\Delta}$ within Θ' to a non-locally-free sheaf. The proof of Theorem 0.2 thus is completed.

Our degeneration scheme is an algebrogeometric analogue to cutting and pasting construction of Donaldson [4], Morgan [24], and Taubes [33] in studying ASD connections on four-manifolds. In his paper, Taubes removes a tubular neighborhood Δ' of S (a Riemann surface) from X and endows the two open manifolds Δ' and $X' = X\backslash S$ with complete metrics having cylindrical metric at their ends. Certain questions on connections on bundles over X can then be reduced to studying connections on Δ' and X' with finite total curvature and their gluing problem. Compare to the degeneration scheme we carried out, Δ' corresponds to our $\Delta\backslash\Sigma$ and X' corresponds to $X\backslash\Sigma$ $(\Sigma = X\cap\Delta)$. Analogously, our approach reduces questions of the moduli scheme $\mathfrak{M}_X^{d,I}$ to that of vector bundles over Δ and X.

It is clear from this paper that we can give a different proof of Donaldson's general smoothness result on rank-2 bundles based on the degeneration theory. In a future paper we will show that same technique can be applied to prove the general smoothness theorem for moduli space of semistable sheaves of higher rank. We mention that K. O'Grady has proved Theorem 0.2 independently [28].

0. Conventions and preliminaries

All schemes are defined over the field $\mathbb{C}$ of complex numbers and are of finite type. We shall always identify a vector bundle with its sheaf of sections. If D is a divisor on a variety X, then we denote by $|D|$ the complete linear system associated to D. We will use p_X and p_Y to denote the projections from the product $X\times Y$ to X and Y respectively. Occasionally, we will also use p_1 and p_2 to denote the projections onto the first and second factor respectively. When F is a coherent sheaf supported on finite points, then we denote by $l(F)$ the length of F. When X is a smooth surface and E is a torsion free sheaf on X, then we abbreviate $l(E^{\vee\vee}/E)$ to $\operatorname{col}(E)$, where $E^{\vee\vee}$ is the double dual of E. If $x\in X$ is any closed point, then $\operatorname{col}(E)_x = l((E^{\vee\vee}/E)\otimes\mathscr{O}_{X,x})$. Note that $\operatorname{col}(E)_x = 0$ if and only if E is locally free at x. If p and q are two polynomials with real coefficients, we say $p\succ p$ (resp. $p\succeq q$) if $p(n) > q(n)$ (resp. $p(n)\geq q(n)$) for all n sufficiently large.

Let X be a smooth variety and let H be an ample divisor on X. Besides the H-stability introduced at the beginning of the introduction, there is the concept of e-stability first appeared in [21].

Definition. Let e be a real number and let W be a smooth variety with very ample divisor H. Then a torsion free sheaf E on W is said to be e-stable if one of the following two equivalent conditions hold:

(1) Whenever $L \to E$ is a subsheaf, $0 < \operatorname{rank}(L) < \operatorname{rank}(E)$, then

$$\frac{1}{\operatorname{rank}(L)}\deg(L) < \frac{1}{\operatorname{rank}(E)}\deg(E) + \frac{1}{\operatorname{rank}(L)}e.$$

(2) Whenever $E \to Q$ is a quotient sheaf, $0 < \operatorname{rank}(Q) < \operatorname{rank}(E)$, then

$$\frac{1}{\operatorname{rank}(E)}\deg(E) < \frac{1}{\operatorname{rank}(Q)}\deg(Q) + \frac{1}{\operatorname{rank}(Q)}e.$$

We say E is μ-stable if E is e-stable with $e = 0$. When the strict inequality is replaced by $\leq$, then we say E is e-semistable.

Starting from §2, we will mainly be concerned with schemes flat over a smooth curve C. (Later, C will specifically be a Zariski open subset of $0 \in \operatorname{Spec}\mathbb{C}[t]$.) If Z and U are two schemes over C, we will denote the product $Z \times_C U$ by Z_U. The convention is that we will use subscript to specify the base scheme unless the base scheme is C. Also, we will reserve t as the uniformizing parameter of C. Assume $u \in U$ is any point. Then we will denote by Z_u the fiber of Z_U over $u \in U$; that is, $Z_u = Z_U \times_U \operatorname{Spec} k(u)$.

Let $Z \to S$ be a flat morphism and let $E \to Z$ be any sheaf. We will use E_s to denote the restriction of E to the fiber Z_s; that is, $E_s = E \otimes_{\mathscr{O}_Z} \mathscr{O}_{Z_s}$. In case $Z = X \times S$ and E is flat over S, we call E a flat family of sheaves on X parameterized by S, and sometimes will use the subscript E_S to emphasize this. Let $D \subseteq X$ be any subscheme, and, by abuse of notation, we will denote by $E_{S|D}$ the restriction of E_S to $D \times S$. $E_{S|D}$ is a family of sheaves on D parameterized by S (not necessarily flat). If we assume E_S is a flat family of torsion free sheaves and $D \subseteq X$ is a local complete intersection (l.c.i.) codimension one subscheme, then $E_{S|D}$ is flat over S [23, Theorem 22.6].

By a (length m) locally free resolution of E we mean an exact sequence $0 \to B^m \to \cdots \to B^1 \to E \to 0$, where B^i are locally free. One fact which we need is the following: Let $Z \to S$ be flat and projective, and let E be any coherent sheaf on Z flat over S. Assume E_s, $s \in S$ closed, admits a length m locally free resolution. Then E admits a length m locally free resolution near Z_s. We only sketch the proof when $m = 2$. Since the question is local, we can assume S is affine. Let Q be a locally free sheaf on Z so that $H^1(Z, \mathscr{H}om(Q, E)(-Z_s)) = \{0\}$ and there is a surjective homomorphism $Q_s \xrightarrow{f_0} E_s$. Since E_s admits a length two locally free resolution, $\ker(f_0)$ is locally free. On the other hand, the vanishing of $H^1(Z, \mathscr{H}om(Q, E)(-Z_s))$ implies that f_0 extends to $Q \xrightarrow{f} E$. Let $R = \ker(f)$. Since both Q and E are flat over S, R is flat over S also.

Further, $R_s = \ker(f_0)$ is locally free, so R is locally free near Z_s. Then $0 \to R \to Q \to E \to 0$ is a length 2 locally free resolution of E near Z_s.

1. Semistable sheaves on singular surfaces

In this section, we will study the moduli of vector bundles over singular surfaces. The class of surfaces which we will consider in this section will be reduced, complete algebraic surfaces X with normal crossing singularities. For simplicity, we assume X has only two smooth components X_1 and X_2 that intersect along a smooth divisor Σ. The result of this section can be generalized to the case of many components. In the first part of this section, we will introduce the concept of H-stability of torsion free sheaves on X. Most of the properties enjoyed by stable sheaves on smooth surfaces are still valid for our situations, though for the sake of the length of this paper, we will only mention those that are related to our study. The main body of this section is to establish some technical results regarding the embedding of the Grothendieck's Quot-scheme to a projective space after [8] that are essential to our construction of the degeneration in the next section. In the end, we will discuss how to construct the moduli of stable sheaves over X. We begin with the following definitions.

Definition 1.1. Let S be any reduced quasi-projective scheme and let E be a coherent sheaf on S. For $x \in S$, E is said to be torsion free at x if f is a zero divisor of the $\mathscr{O}_{S,x}$-module $\mathscr{O}_{S,x}$, whenever $f \in \mathscr{O}_{S,x}$ is a zero divisor of the $\mathscr{O}_{S,x}$-module E_x. The sheaf E is said to be torsion free if E_x is torsion free for any $x \in S$.

Let E be any coherent sheaf on x, then an easy argument shows that there is a torsion subsheaf T of E so that E/T is torsion free, and such T and E/T are unique. By abuse of notation, we will call T the torsion part of E, and E/T the torsion free part of E. We define the rank of E to be an integer pairs, $\mathbf{rk}(E) = (\mathbf{rk}(E)_1, \mathbf{rk}(E)_2)$ where $\mathbf{rk}(E)_i = \operatorname{rank}(E \otimes_{\mathscr{O}_X} \mathscr{O}_{X_i})$. When $\mathbf{rk}(E)_1 = \mathbf{rk}(E)_2 = r$, we call E a rank r sheaf.

Let H be a very ample line bundle on X and let I be an invertible sheaf on X. In this section we will fix H and I once and for all. We denote the sheaf $E \otimes H^{\otimes n}$ by $E(n)$ and denote by χ_E the Hilbert polynomial given by $\chi_E(n) = \chi(E(n))$. If E is a torsion free sheaf of rank r, then

$$\chi_E(n) = \frac{r}{2}n^2(H \cdot H) + n(\tau(E) - \frac{r}{2}\omega_X \cdot H) + \chi(E),$$

where ω_X is the canonical sheaf of X, and $\tau(E)$ is an integer. As usual, $\tau(E)$ is called the degree of E and denoted by $\deg E$.

Next we shall make sense of the determinant line bundle of a torsion free coherent sheaf E on X. In case E has a locally free resolution of finite length, [17] showed that there is an invertible sheaf $\det E$ on X, which is the determinant line bundle of E. In general, a torsion free sheaf E on X does not necessarily admit a locally free resolution of finite length, and the existence of $\det E$ is not obvious even if we allow $\det E$ to be a rank 1 sheaf on X. In this paper, we will use the following ad hoc definition.

Definition 1.2. Let E be a torsion free sheaf on X of rank r, and let I be an invertible sheaf on X. We say $\det E \approx I$, if in addition $\deg E = I \cdot H$, there are isomorphisms $\det E_{|X_i \setminus \Sigma} \cong I_{|X_i \setminus \Sigma}$ for $i = 1, 2$.

For the very ample line bundle H on X, we denote by H_1 (resp. H_2) the restriction of H to X_1 (resp. X_2). Let $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i = (H_i \cdot H_i)/(H \cdot H)$. For any torsion free sheaf E we define the polynomial

$$p_E(n) = \chi_E(n)/\alpha \cdot \mathbf{rk}(E).$$

Note that since $\alpha_1 + \alpha_2 = 1$, if $\operatorname{rank}(E) = r$, then $p_E(n) = \chi_E(n)/r$.

Definition 1.3. A sheaf E on X is said to be H-stable (resp. H-semistable) if E is a coherent, torsion free sheaf, and $p_F \prec p_E$ (resp. $p_F \preceq p_E$) whenever F is a proper coherent subsheaf of E.

Lemma 1.4. *A sheaf E on X is H-stable (resp. H-semistable) if E is a coherent, torsion free sheaf, and $p_E \prec p_Q$ (resp. $p_E \preceq p_Q$) whenever Q is a quotient sheaf of E with* $\mathbf{rk}(Q) \neq (0, 0)$.

Proof. The proof is similar to the proof given for semistable sheaves on smooth variety [3, p. 153]. q.e.d.

Let d be any integer and let χ be the polynomial depending on d and r:

$$(1.1)\quad \chi(n) = \frac{r}{2}n^2(H\cdot H) - n[(H\cdot I) - \frac{r}{2}(\omega_X\cdot H)] + (r-1)\chi(\mathscr{O}_X) + \chi(I) - d.$$

Note that when E is any sheaf of Hilbert polynomial (1.1), E is of rank r and degree $\deg I$.

Definition 1.5. $\mathscr{E}^{\chi}(n)$ will be the set of all H-semistable sheaves $E(n)$ of rank r with $\det E \approx I$ and $\chi_E = \chi$.

Let E be any coherent sheaf on X. We denote by $E^{(1)}$ (resp. $E^{(2)}$) the torsion free part of $E_{|X_1}$ (resp. $E_{|X_2}$). By abuse of notation, we will view $E^{(i)}$ as a sheaf of $\mathscr{O}_{X_i}$-modules. Then the surjections $\sigma_i\colon E \to E^{(i)}$ induce a homomorphism $\sigma\colon E \to E^{(1)} \oplus E^{(2)}$. In general, it is neither injective nor surjective. On the other hand, since X has only normal crossing singularities along Σ, we have inclusion $E^{(1)}(-\Sigma) \oplus E^{(2)}(-\Sigma) \to E$.

Lemma 1.6. *Let E be a torsion free coherent sheaf on X of* rank (r_1, r_2). *Then $\sigma\colon E \to E^{(1)} \oplus E^{(2)}$ is injective. Moreover, if we denote the cokernel of σ by $E^{(0)}$, then $E^{(0)}$ is a sheaf of $\mathscr{O}_\Sigma$-module and at the generic point $x \in \Sigma$, $E_x^{(0)} \cong \mathscr{O}_{\Sigma,x}^{\oplus r_0}$ with $0 \le r_0 \le \min(r_1, r_2)$.*

Proof. Assume $\ker\{\sigma\} \ne \{0\}$. Then $\operatorname{supp}(\ker\{\sigma\}) \subseteq \Sigma$ which violates the torsion freeness of E. Therefore, σ is injective. Let $E^{(0)}$ be the cokernel of σ. The exact sequence

$$E \to E \otimes_{\mathscr{O}_X} \mathscr{O}_{X_1} \oplus E \otimes_{\mathscr{O}_X} \mathscr{O}_{X_2} \to E \otimes_{\mathscr{O}_X} \mathscr{O}_\Sigma \to 0$$

yields the commutative diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, equal] & E \otimes_{\mathscr{O}_X} \mathscr{O}_{X_1} \oplus E \otimes_{\mathscr{O}_X} \mathscr{O}_{X_2} \arrow[r] \arrow[d, "\gamma_1"] & E \otimes_{\mathscr{O}_X} \mathscr{O}_\Sigma \arrow[r] \arrow[d, "\gamma_2"] & 0 \\ E \arrow[r] & E^{(1)} \oplus E^{(2)} \arrow[r, "\rho"] & E^{(0)} \arrow[r] & 0 \end{tikzcd} \tag{1.2}$$

Since γ_1 is surjective, γ_2 is surjective. Thus $E^{(0)}$ is a sheaf of $\mathscr{O}_\Sigma$-modules. Now we denote $\rho = (\rho_1, \rho_2)\colon E^{(1)} \oplus E^{(2)} \to E^{(0)}$. Since $E \to E^{(1)}$ is surjective, the exactness of (1.2) implies that both ρ_1 and ρ_2 are surjective. To show that $E_x^{(0)} \cong \mathscr{O}_{\Sigma,x}^{\oplus r_0}$ at the generic point $x \in \Sigma$, we use the result from [31, p. 166] which states that if E is torsion free, then at generic point $x \in \Sigma$,

$$E_x \cong \mathscr{O}_{X,x}^{\oplus a} \oplus \mathscr{O}_{X_1,x}^{\oplus b} \oplus \mathscr{O}_{X_2,x}^{\oplus c}. \tag{1.3}$$

Thus $E_x^{(1)} \cong \mathscr{O}_{X_1,x}^{\oplus(a+b)}$ and $E_x^{(2)} \cong \mathscr{O}_{X_2,x}^{\oplus(a+c)}$ while $E_x^{(0)} \cong \mathscr{O}_{\Sigma,x}^{\oplus a}$. This completes the proof of the lemma. q.e.d.

For the remainder of this section, we let E be a rank-r torsion free coherent sheaf on X. If we assume $\det E \approx I$, then since $\det E_{|X_i \setminus \Sigma} = I_{|X_i \setminus \Sigma}$ for $i = 1, 2$, there are integers a_1 and a_2 such that $\det E^{(i)} = I_{|X_i}(a_i\Sigma)$. Let $\operatorname{rank}_\Sigma(E^{(0)}) = r_0$. Then by comparing the Hilbert polynomials of E, $E^{(0)}$, $E^{(1)}$, and $E^{(2)}$, we obtain

$$a_1 + a_1 + r_0 - r = 0. \tag{1.4}$$

Indeed, if we assume that E is H-semistable, then the tuples (a_1, a_2, r_0) is very much limited.

Lemma 1.7. *There is a constant A such that if $E \in \mathscr{E}^\chi$ is an H-semistable sheaf with* $\det(E^{(i)}) = I(a_i\Sigma)$ *for $i = 1, 2$, then $|a_1|, |a_2| \le A$.*

Proof. The proof can be carried out easily by noting that both $E^{(1)}$ and $E^{(2)}$ are quotient sheaves of E. Then the lemma follows from Lemma 1.4. We leave the proof to the readers. q.e.d.

To construct the moduli scheme of H-semistable sheaves on X, we need to know that the set $\mathscr{E}^{\chi}$ is bounded. More precisely, $\mathscr{E}^{\chi}$ is bounded if there is a scheme S of finite type over $\mathbb{C}$ and a coherent sheaf F on $X \times S$ flat over S so that whenever $E \in \mathscr{E}^{\chi}$, then there is a closed point $s \in S$ so that the sheaf F_s, which is the restriction of F to the fiber $X \times \{s\}$ over s, is isomorphic to E. We will use the following characterization due to Kleiman [16].

Proposition 1.8 (Kleiman). *Let X be any projective surface and let H be a very ample divisor. Assume that p is a polynomial and $\mathscr{R}$ is a set of sheaves E with $\chi_E = p$. If there is a constant K such that whenever $E \in \mathscr{R}$, then $h^0(E) \le K$ and $h^0(E \otimes \mathscr{O}_{H'}) \le K$ for some divisor H' linearly equivalent to H. Then $\mathscr{R}$ is bounded.*

We need a technical lemma similar to [8, Lemma 1.2]. We adopt the convention that if p is a polynomial, then $\Delta p(n) = p(n) - p(n-1)$.

Lemma 1.9. *Suppose a_1 and a_2 are rational numbers, and r, A_1 and l_0 are positive integers. Let $p(n) = \frac{1}{2}n^2(H \cdot H) + a_1 n + a_2$. Then there are integers K and N_0 so that if E is a torsion free coherent sheaf on X of* rank (r_1, r_2) *with $r_1, r_2 \le r$ satisfying*

(i) *every nontrivial subsheaf F of E has $\Delta p_F \le \Delta p$,*

(ii) *$\Delta p_E \le \Delta p$ and*

(iii) *$h^0(E(n)) \ge (\alpha \cdot \mathbf{rk}(E))p(n)$ for some $n \ge N_0$,*

then the following hold:

(1) *$h^0(E \otimes \mathscr{O}_{H'}) \le K$ for some divisor H' linearly equivalent to H,*

(2) *$h^2(E(m)) = 0$ if $m \ge N_0$,*

(3) *$\Delta p = \Delta p_E$ and*

(4) *if $h^1(E(-l_0)) \le A$, then $h^1(E(m)) = 0$ for $m \ge N_0$.*

Proof. The proof is very much similar to that of [8]; we will give a sketch of it.

Let E be the coherent torsion free sheaf satisfying (i), (ii), and (iii). Let F_n be the smallest subsheaf of $E(n)$ so that $H^0(F_n) = H^0(E(n))$ and $E(n)/F_n$ is torsion free. Note that by (i), for some constant c depending on p and r only, $F_{-c} = 0$. Without loss of generality, by replacing E with $E(-c)$ and changing the polynomial p accordingly, we can assume $F_0 = 0$. We remark that $H^0(F_n(-1)) = H^0(E(n-1))$. Let C be linearly equivalent to H. Then C is a union of two smooth curves, so $C = C_1 \cup C_2$ with $C_i \subseteq X_i$. By letting C be in generic position, we can assume that the

F_n's are locally free on $C\backslash\Sigma$ and such that for any $x \in C \cap \Sigma$, $F_{n,x}$ is of the form (1.3). We choose C so that for two distinct points $x_1 \in C_1$ and $x_2 \in C_2$, the stalk of F_n at x_1 and x_2 are generated by global sections. Since $0 \to F_n(-1) \to F_n \to F_{n|C} \to 0$ is exact, our remark shows that

$$h^0(E(n)) - h^0(E(n-1)) \leq h^0(C, F_n).$$

Let $n_1, n_2, \cdots, n_k$ be the integers so that $F_{n_k} \neq F_{n_{k-1}}(1)$. Clearly $k \leq r_1 + r_1$. If we let (a_n, b_n) be the rank of F_n, then there is an exact sequence

$$0 \to \mathscr{O}_{C_1}^{\oplus a_n}(-\Sigma) \oplus \mathscr{O}_{C_2}^{\oplus b_n}(-\Sigma) \to F_n \otimes \mathscr{O}_C \to Q_n \to 0,$$

where Q_n is supported at a finite number of points. Hence

$$h^1(F_n(l) \otimes \mathscr{O}_C) \leq a_n h^1(\mathscr{O}_{C_1}(-\Sigma)(l)) + b_n h^1(\mathscr{O}_{C_2}(-\Sigma)(l)).$$

There are constants A_2 and A_3 so that for $i = 1, 2$, $h^1(\mathscr{O}_{C_i}(-\Sigma)(l)) \leq A_2$ for $l \geq 0$ and $h^1(\mathscr{O}_{C_i}(-\Sigma)(l)) = 0$ for $l \geq A_3$. Thus

$$h^0(F_n \otimes \mathscr{O}_C) \leq_\chi (F_n \otimes \mathscr{O}_C) + (a_n + b_n)A_2$$

and

$$h^0(F_n \otimes \mathscr{O}_C) = \chi(F_n \otimes \mathscr{O}_C),$$

if $n_j > n > n_{j-1} + A_3$ for some j. Therefore,

$$h^0(F_n) \leq \sum_{k=0}^{n} h^0(F_k \otimes \mathscr{O}_C) \leq \sum_{k=0}^{n} \chi(F_k \otimes \mathscr{O}_C) + A_4, \tag{1.5}$$

where A_4 is a constant depending on r and H. Because of assumption (i), there is a constant A_5 such that $\deg(F_n^{(i)}(-n), C_i) < A_5$ for $i = 1, 2$. Then since the cokernel of $F_{n|C} \to F_n^{(1)}{}_{|C_1} \oplus F_n^{(2)}{}_{|C_2}$ is supported at a finite number of points, we have for $n < n_k$,

$$\begin{aligned}\chi(F_n \otimes \mathscr{O}_C) = {} & \deg((F_n^{(1)})_{|C_1}) + a_n(1 - g(C_1)) + \deg((F_n^{(2)})_{|C_2}) \\ & + b_n(1 - g(C_2)) - c_n\end{aligned}$$

with $0 \leq c_n$ since C being generic implies $F_{n|C}$ is torsion free. Hence

$$\begin{aligned}\chi(F_n \otimes \mathscr{O}_C) & \leq \deg((F_n^{(1)})_{|C_1}) + \deg((F_n^{(2)})_{|C_2}) \\ & \leq a_n n \cdot (H_1 \cdot H_1) + A_5 + b_n n \cdot (H_2 \cdot H_2) + A_5 \\ & \leq (\alpha \cdot \mathbf{rk}(E) - \varepsilon)(H \cdot H) \cdot n + 2A_5\end{aligned}$$

for some $\varepsilon > 0$ depending on r and H only. In the following, we denote $\mathbf{r}' = \mathbf{rk}(E)$. Let

$$\begin{aligned} g(l) &= \sum_{m=0}^{l}((\alpha \cdot \mathbf{r}' - \varepsilon)(H \cdot H) \cdot m + 2A_5) \\ &= (\alpha \cdot \mathbf{r}' - \varepsilon) \cdot \tfrac{1}{2}(H \cdot H) \cdot l(l+1) + 2(l+1)A_5. \end{aligned}$$

If $n < n_k$, then

$$\sum_{m=0}^{n} \chi(F_m \otimes \mathcal{O}_C) \leq g(n). \tag{1.6}$$

If $n \geq n_k$, then

$$\chi(F_n \otimes \mathcal{O}_C) = \Delta\chi_E(n) = (\alpha \cdot \mathbf{r}')\Delta p(n) - \beta,$$

where $\beta = (\alpha \cdot \mathbf{r}')\Delta p(n) - \Delta\chi_E(n) \geq 0$ by (ii). Thus

$$\begin{aligned} \sum_{m=n_k}^{n} \chi(F_m \otimes \mathcal{O}_C) &\leq \sum_{m=n_k}^{n} (\alpha \cdot \mathbf{r}')\Delta p(m) - \beta \\ &= (\alpha \cdot \mathbf{r}')(p(n) - p(n_k - 1)) - \beta \cdot (n - n_k + 1). \end{aligned} \tag{1.7}$$

Since the leading coefficient of $g(n) - (\alpha \cdot \mathbf{r}')p(n)$ is negative, there is an A_6 such that if $n \geq A_6$, then

$$g(n) - (\alpha \cdot \mathbf{r}')p(n) < -A_4. \tag{1.8}$$

Assume for some $\overline{n}_0 \geq A_6$ we have $h^0(E(\overline{n}_0)) \geq (\alpha \cdot \mathbf{r}')p(\overline{n}_0)$. We claim that then $\overline{n}_0 \geq n_k$. Otherwise, by (1.5) and (1.6),

$$h^0(E(\overline{n}_0)) < g(\overline{n}_0) + A_4 \leq (\alpha \cdot \mathbf{r}')p(n),$$

contradicts our assumption. Thus $\overline{n}_0 \geq n_k$. Next we claim that $A_6 \geq n_k$. Assume not, that is $n_k \geq A_6 + 1$. Then by (1.5), (1.6), and (1.7),

$$\begin{aligned} &h^0(E(\overline{n}_0)) - (\alpha \cdot \mathbf{r}')p(\overline{n}_0) \\ &\leq \sum_{m=0}^{n_k-1} \chi(F_m \otimes \mathcal{O}_C) + \sum_{m=n_k}^{\overline{n}_0} \chi(F_m \otimes \mathcal{O}_C) + A_4 - (\alpha \cdot \mathbf{r}')p(\overline{n}_0) \\ &\leq g(n_k - 1) - (\alpha \cdot \mathbf{r}')p(n_k - 1) - \beta \cdot (\overline{n}_0 - n_k + 1) + A_4 < 0. \end{aligned} \tag{1.9}$$

The last inequality is the result of applying (1.8) with $n = n_k - 1$ and noting that $\overline{n}_0 - n_k + 1 \geq 1$. This violates the assumption (iii). Therefore we must have $A_6 \geq n_k$.

Now assume β is positive. Since n_k is bounded by A_6, we can find $A_7 > A_6$ so that for any $n \geq A_7$,

$$g(n_k - 1) - (\alpha \cdot \mathbf{r}')p(n_k - 1) - \beta \cdot (n - n_k - 1) + A_4 < 0. \tag{1.10}$$

If $h^0(E(\overline{n}_0)) \geq (\alpha \cdot \mathbf{r}')p(\overline{n}_0)$ for some $\overline{n}_0 \geq A_7$, we can use (1.5), (1.6), and (1.8) again to see that (1.9) is impossible. Therefore β must be zero and thus (3) is established if we let $N_0 = A_7$. Since (1), (2) and (4) follow easily from the arguments of [8], we will omit the proofs. q.e.d.

Theorem 1.10. *$\mathscr{E}^\chi$ is bounded.*

Proof. Assume $E \in \mathscr{E}^\chi$. Apply Lemma 1.9, we see that there is a K such that $h^0(E) \leq K$ and $h^0(E \otimes \mathscr{O}_{H'}) \leq K$ for some H' linearly equivalent to H. Then by Proposition 1.8, we conclude that $\mathscr{E}^\chi$ is bounded. q.e.d.

Corollary 1.11. *For any polynomial χ of (1.1), there is an N_0 so that if $n \geq N_0$ and $E \in \mathscr{E}^\chi(n)$, then $h^i(E) = 0$ for $i = 1, 2$ and E is generated by the global sections of $H^0(E)$.*

For any χ of (1.1), we let N be the integer provided by Corollary 1.11. Let $E \in \mathscr{E}^\chi(n)$, $n \geq N_0$, be a fixed H-semistable sheaf. For $i = 1, 2$, let

$$\lambda_E^i \colon \det E_{|X_i \backslash \Sigma} \to I(rn)_{|X_i \backslash \Sigma} \tag{1.11}$$

be isomorphisms. Since Σ is irreducible, λ_E^i are unique up to scalars. Let $N = h^0(E)$ and let $V^i = V^i(E)$ be the image of

$$\bigwedge^r H^0(X, E) \to H^0\left(X_i \backslash \Sigma, \bigwedge^r E\right) \to H^0(X_i \backslash \Sigma, I(rn)). \tag{1.12}$$

We remark that V^i do depend on the choice of $E \in \mathscr{E}^\chi(n)$. Then any isomorphism $\varphi \colon \mathbb{C}^N \to H^0(E)$ induces homomorphisms

$$\mu_{E,\varphi}^i = \lambda_E^i \circ \bigwedge^r \varphi \in \operatorname{Hom}\left(\bigwedge^r \mathbb{C}^N, V^i\right). \tag{1.13}$$

It is clear that $\mu_{E,\varphi}^i$ are nontrivial and that they are unique up to scalars. Let

$$\overline{\mu}_{E,\varphi}^i \in \mathbf{P}\left(\operatorname{Hom}\left(\bigwedge^r \mathbb{C}^N, V^i\right)\right) \tag{1.14}$$

be the corresponding points ($\mathbf{P}(\mathbb{C}^l)$ is the space of lines in $\mathbb{C}^l$) and let

$$\mu(E, \varphi) = [\overline{\mu}_{E,\varphi}^1, \overline{\mu}_{E,\varphi}^2] \in \mathbf{P}(W^1) \times \mathbf{P}(W^2), \tag{1.15}$$

where W^i is the space $\mathrm{Hom}(\bigwedge^r \mathbb{C}^N, V^i)$. Assume that $\psi: \mathbb{C}^N \to H^0(E)$ is another identification, $\psi = \varphi \circ g$ with $g \in GL(N, \mathbb{C})$. If we denote by $[g]$ the dual action of g on $\mathbf{P}(W^i)$, then we have $\mu(E, \psi) = [g]\cdot\mu(E, \varphi)$. Thus $E \in \mathscr{E}^\chi(n)$ corresponds to an $SL(N, \mathbb{C})$ orbit in $\mathbf{P}(W^1) \times \mathbf{P}(W^2)$. It is nowadays standard to use the geometric invariant theory developed by Mumford to study the space of these orbits. We first recall the theory very briefly. Let N_1, N_2 be positive integers so that

$$N_1/N_2 = (H_1 \cdot H_1)/(H_2 \cdot H_2) \quad (= \alpha_1/\alpha_2). \tag{1.16}$$

Let $L(N_1, N_2)$ be the very ample line bundle on $\mathbf{P}(W^1) \times \mathbf{P}(W^2)$ that corresponds to the invertible sheaf $p_1^*\mathscr{O}(N_1) \otimes p_2^*\mathscr{O}(N_2)$. There is a canonical $SL(N, \mathbb{C})$ linearization on $L(N_1, N_2)$ induced from the canonical $SL(W^i, \mathbb{C})$ linearization on $\mathscr{O}(N_i)$ over $\mathbf{P}(W^i)$. For any $x \in \mathbf{P}(W^1) \times \mathbf{P}(W^2)$, let $\overline{x} \in L(N_1, N_2)^\vee$ be a lifting of x. Let $\omega_1, \omega_2, \cdots, \omega_l$ be a basis of $H^0(L(N_1, N_2))$. Denote $G = SL(N, \mathbb{C})$.

Proposition 1.12. *The following conditions are equivalent*:

(1) *Some G invariant sections of $L(N_1, N_2)$ do not vanish at x.*

(2) *The closure of the orbit of $\overline{x}$ is disjoint from the zero section of $L(N_1, N_2)^\vee$.*

(3) *Let $\lambda: \mathbb{G}_m \to G$ be any one parameter subgroup. Then at least one of the rational functions of α, say $\omega_i(\overline{x}^{\lambda(\alpha)})$, does not vanish at 0. That is, $\lim_{\alpha\to 0} \overline{x}^{\lambda(\alpha)} \neq 0$.*

If these conditions are satisfied, we say x (or $\overline{x}$) is semistable under G. Further, if the orbit of x is closed and the stabilizer G_x of x is finite, then we say x (or $\overline{x}$) is stable under G. In [8], it is shown that when X is smooth, E is H-semistable if and only if $\mu(E, \varphi)$ is $SL(N, \mathbb{C})$-semistable. In the following, we will show that the same result holds for the singular surface X.

We first introduce the concept of weighted basis. A weighted basis (e_i, t_i) of $\mathbb{C}^N$ is an ordered basis of $\mathbb{C}^N$ together with integers t_i with $t_1 \le t_2 \le \cdots \le t_N$ and $t_1 + \cdots + t_N = 0$. Clearly, any weighted basis corresponds to a one-parameter subgroup $\lambda: \mathbb{G}_m \to G$ by $\lambda(s)e_j = s^{t_j}e_j$, and vice versa. The weighted basis (e_i, t_i) induces a weighted basis of $S^{N_i}\mathrm{Hom}(\bigwedge^r \mathbb{C}^N, V^i)^\vee$ as follows: Let $u_1^i, u_2^i, \cdots, u_s^i$ be a basis of $(V^i)^\vee$. Then

$$\varepsilon_J^i = u_{j_0}^i \otimes (e_{j_1} \wedge \cdots \wedge e_{j_r}), \qquad J = (j_0, j_1, \cdots, j_r), \tag{1.17}$$

form a weighted basis of $\mathrm{Hom}(\bigwedge^r \mathbb{C}^N, V^i)^\vee$ of weight $\delta(\varepsilon_J^i, \lambda) = t_{j_1} + \cdots + t_{j_r}$, and

$$\varepsilon^i_{[J]} = \bigotimes_{k=1}^{N_i} \varepsilon^i_{J_k}, \qquad [J] = (J_1, \cdots, J_{N_i}), \tag{1.18}$$

is a weighted basis of $S^{N_i}\operatorname{Hom}(\bigwedge^r \mathbb{C}^N, V^i)^\vee$ of weight $\sum_{k=1}^{N_i} \delta(\varepsilon^i_{J^i_k}, \lambda)$ (cf. [10]). For any $x = [x_1, x_2] \in \mathbf{P}(W^1)\times\mathbf{P}(W^2)$ with $x_i \in \operatorname{Hom}(\bigwedge^r \mathbb{C}^N, V^i)$, we define

$$\delta(x_i, \lambda) = \min_{j_1<\cdots<j_r} \{t_{j_1} + \cdots + t_{j_r} | x_i(e_{j_1} \wedge \cdots \wedge e_{j_r}) \neq 0\}. \tag{1.19}$$

Lemma 1.13. *The point* $x = [x_1, x_2] \in \mathbf{P}(W^1) \times \mathbf{P}(W^2)$ *is stable* (*resp. semistable*) *under* G *if and only if for any one-parameter subgroup* $\lambda\colon \mathbb{G}_m \to G$,

$$\alpha_1\delta(x_1, \lambda) + \alpha_2\delta(x_2, \lambda) < 0 \quad (\textit{resp.} \ \leq 0). \tag{1.20}$$

Proof. Assume $\overline{x} = (\overline{x}_1, \overline{x}_2) \in L(N_1, N_2)^\vee$ is any stable (resp. semistable) point. Then there is an $\omega \in H^0(\mathscr{O}(N_1, N_2))$ of the form

$$\omega = \left(\bigotimes_{k=1}^{N_1} \varepsilon^1_{J^1_k}\right) \otimes \left(\bigotimes_{k=1}^{N_2} \varepsilon^2_{J^2_k}\right) \tag{1.21}$$

such that $\omega(\overline{x}) \neq 0$ and $\omega(\lambda(\alpha)\overline{x}) = \alpha^T\omega(\overline{x})$ with $T < 0$ (resp. $T \leq 0$). By the previous argument, we have

$$T = \sum_{k=1}^{N_1} \delta(\varepsilon^1_{J^1_k}, \lambda) + \sum_{k=1}^{N_2} \delta(\varepsilon^2_{J^2_k}, \lambda). \tag{1.22}$$

Since $\delta(x_i, \lambda) = \min_{J=(j_0, \cdots, j_r)}\{t_{j_1}+\cdots+t_{j_r} | \varepsilon^i_J(x_i) \neq 0\}$, and since $\omega(\overline{x}) \neq 0$ guarantees that $\varepsilon^i_{J^i_k}(x_i) \neq 0$ for $k = 1, \cdots, N_i$, we have $\delta(\varepsilon^i_{J^i_k}, \lambda) \geq \delta(x_i, \lambda)$. Thus

$$\begin{aligned} T &= \sum_{k=1}^{N_1} \delta(\varepsilon^1_{J^1 k}, \lambda) + \sum_{k=1}^{N_2} \delta(\varepsilon^2_{J^2_k}, \lambda) \\ &\geq N_1\delta(x_1, \lambda) + N_2\delta(x_2, \lambda) = \frac{1}{N}(\alpha_1\delta(x_1, \lambda) + \alpha_2\delta(x_2, \lambda)). \end{aligned}$$

Therefore that x is stable (resp. semistable) implies (1.20). Similarly, if for a one-parameter group $\lambda\colon \mathbb{G}_m \to G$, we have

$$\alpha_1\delta(x_1, \lambda) + \alpha_2\delta(x_2, \lambda) < 0 \text{ (resp. } \leq 0),$$

then we can find an $\omega \in H^0(\mathscr{O}(N_1, N_2))$ of the form (1.21) such that $\omega(\lambda\overline{x}) = \alpha^T\omega(\overline{x})$ with $T < 0$ (resp. $\leq$). So we have established the lemma. q.e.d.

Now we denote by $(\mathbf{P}(W^1) \times \mathbf{P}(W^2))^s$ (resp. $(\mathbf{P}(W^1) \times \mathbf{P}(W^2))^{ss}$) the space of G-stable (resp. G-semistable) points in $\mathbf{P}(W^1) \times \mathbf{P}(W^2)$. The rest of the section is devoted to proving

Theorem 1.14. *There is an N_0 so that the following hold for $n \geq N_0$:*

(i) *If $E \in \mathscr{E}^\chi(n)$, then $h^j(E) = 0$ for $j > 0$, and E is generated by global sections.*

(ii) *For any $E \in \mathscr{E}^\chi(n)$ and isomorphism $\varphi: \mathbb{C}^N \to H^0(E)$, $\mu(E, \varphi) \in (\mathbf{P}(W^1) \times \mathbf{P}(W^2))^{ss}$. Furthermore, $\mu(E, \varphi)$ is stable under G if and only if E is H-stable.*

(iii) *Let E be a torsion free sheaf of rank r with $\det E \approx I(rn)$ and $\chi_E(\cdot) = \chi(\cdot + n)$. Suppose there is a homomorphism $\varphi: \mathbb{C}^N \to H^0(E)$ so that $\mu(E, \varphi)$ is semistable under G. Then E is H-semistable.*

Proof. The proof is similar to that of [8]. We will give a sketch of the proof of (ii) and leave that of (iii) to readers. Since $\mathscr{E}^\chi$ is bounded, there is an A_1 such that $h^1(E) \leq A_1$ for any $E \in \mathscr{E}^\chi$. Now fix a large N_0 that is provided by Lemma 1.9 with data r, A_1 and $p = \Delta\chi/r$. Let $n \geq N_0$. Assume that $E \in \mathscr{E}^\chi(n)$ is H-stable and that $\mu(E, \varphi) = [x_1, x_2]$ is not stable. Then there is a one-parameter subgroup $\lambda: \mathbb{G}_m \to G$ such that

$$\alpha_1\delta(x_1, \lambda) + \alpha_2\delta(x_2, \lambda) \geq 0. \tag{1.23}$$

Let (e_i, t_i) be the weighted basis of $\mathbb{C}^N$ associated to λ. Note $t_1 \leq t_2 \leq \cdots \leq t_N$ and $\sum t_j = 0$.

We fix the basis $\{e_1, \cdots, e_N\}$. We seek to find a better choice of weight $t_1 \leq t_2 \leq \cdots \leq t_N$ so that we can indeed compute the number in (1.23). Let $V \subseteq \mathbb{R}^N$ be the set

$$V = \{(v_1, v_2, \cdots, v_N) | \sum_{k=1}^N v_k = 0, v_1 \leq v_2 \leq \cdots \leq v_N \text{ and } v_N = 1\}.$$

V is a closed bounded convex subset of $\mathbb{R}^N$. Given any $v \in V \cap \mathbb{Q}^N$, there associates a one-parameter subgroup $\lambda_v: \mathbb{G}_m \to G$, $\lambda_v(\alpha)e_j = \alpha^{Mv_j}e_j$, where M is the least common denominator of $v_1, v_2, \cdots, v_N$. For any $1 \leq k \leq N$, let $F_{i,k}$ be the subsheaf of $E_{|X_i}$ spanned by sections $\varphi(e_1), \cdots, \varphi(e_k) \in H^0(X, E)$. Let $\beta_i(k) = \operatorname{rank} F_{i,k}$. Clearly, $\beta_i(\cdot)$ is an increasing function and $\beta_i(N) = r$. Let $1 \leq j_1^i \leq \cdots \leq j_r^i \leq N$ be tuples of integers so that

$$\beta_i(j^i_k) = \beta_i(j^i_k - 1) + 1. \tag{1.24}$$

Here we agree that $\beta_i(0) = 0$. Clearly, the subsheaf of $E_{|X_i}$ generated by $\varphi(e_{j^i_1}), \cdots, \varphi(e_{j^i_r})$ has rank r. Thus

$$x_i(e_{j^i_1} \wedge \cdots \wedge e_{j^i_r}) \neq 0. \tag{1.25}$$

Further, one checks directly that if $1 \leq k^i_1 \leq \cdots \leq k^i_r \leq N$ is any tuple so that

$$x_i(e_{k^i_1} \wedge \cdots \wedge e_{k^i_r}) \neq 0,$$

then necessarily $k^i_l \geq j^i_l$ for all $l = 1, \cdots, r$. Now based on (1.19), for any $v \in V \cap \mathbb{Q}^N$,

$$\begin{aligned} \delta(x_i, \lambda_v) &= \min_{j_1 < \cdots < j_r} \{M(v_{j_1} + \cdots + v_{j_r}) | x_i(e_{j_1} \wedge \cdots \wedge e_{j_r}) \neq 0\} \\ &= M \sum_{k=1}^{r} v_{j^i_k}. \end{aligned} \tag{1.26}$$

Next, we define $f \colon V \to \mathbb{R}$,

$$f(v) = \sum_{k=1}^{r} (\alpha_1 v_{j^1_k} + \alpha_2 v_{j^2_k}).$$

Then for $v_0 = (1/t_N)(t_1, t_2, \cdots, t_N)$, $f(v_0) \geq 0$. Let $T = \max_{v \in V} f(v)$. Then $T \geq f(v_0) \geq 0$. We claim that T is attained at a point $\overline{v} = (\overline{v}_1, \cdots, \overline{v}_N)$ where $\overline{v}_1 = \cdots = \overline{v}_l$ and $\overline{v}_{l+1} = \cdots = \overline{v}_N = 1$ for some $1 \leq l \leq N$. Indeed, since the functional f is linear and V is a compact convex set, the maximum of f is always attained at a vertex of V. Clearly all vertices of V are those $v = (v_1, \cdots, v_N)$ with $v_1 = \cdots = v_k$ and $v_{k+1} = \cdots = v_N = 1$ for some $1 \leq k \leq N$. Since $\sum_{j=1}^{n} \overline{v}_j = 0$ and $\overline{v}_N = 1$, we have

$$\overline{v}_1 = \cdots = \overline{v}_l = -(N-l)/l, \qquad \overline{v}_{l+1} = \cdots = \overline{v}_N = 1.$$

Now we can assume that $t_i = \overline{v}_i$ and that (1.23) still holds. Let $F \subset E$ be the subsheaf generated by $\varphi(e_1), \cdots, \varphi(e_l) \in H^0(E)$. Let $\mathbf{rk}(F) = (r_1, r_2)$. We claim that $j^1_{r_1}, j^2_{r_2} \leq l$ and $j^1_{r_1+1}, j^2_{r_2+1} > l$. By (1.24), the sheaf generated by $\varphi(e_{j^i_1}), \cdots, \varphi(e_{j^i_{r_i+1}})$ has rank $r_i + 1$ over X_i. Thus $j^i_{r_i+1} > l$. Similarly, $j^i_{r_i} \leq l$ follows from (1.24). So $t_{j^i_1}, \cdots, t_{j^i_{r_i}} =$

$-(N-l)/l$, $t_{j^i_{r_i+1}}, \cdots, t_{j^i_r} = 1$,

$$\sum_{k=1}^{r} t_{j^i_k} = \sum_{k=1}^{r_i} \frac{-(N-l)}{l} + \sum_{k=r_i+1}^{r} 1 = r_i\left(\frac{-(N-l)}{l}\right) + (r - r_i)$$

and

$$\begin{aligned} 0 &\le \alpha_1\delta(x_1, \lambda) + \alpha_2\delta(x_2, \lambda) \\ &= M\left[\alpha_1\left(r_1\left(-\frac{N-l}{l}\right) + (r-r_1)\right) + \alpha_2\left(r_2\left(-\frac{N-l}{l}\right) + (r-r_2)\right)\right] \\ &= \frac{M}{l}[lr - (\alpha_1 r_1 + \alpha_2 r_2)N]. \end{aligned}$$

Therefore

$$\frac{1}{\alpha \cdot \mathbf{rk}(F)} h^0(F) \ge \frac{1}{\alpha \cdot \mathbf{rk}(F)} l \ge \frac{1}{r} N = \frac{1}{r}\chi(E).$$

Since E is H-stable, for any subsheaf $F' \subseteq F \subseteq E$ we have $\Delta p_{F'(-n)} \le \Delta\chi/r$. Clearly, $\Delta p_{F(-n)} \le \Delta\chi/r$ for the same reason. Now we apply Lemma 1.9 to the sheaf $F(-n)$. By our special choice of N_0, we conclude that $\Delta p_{F(-n)} = \Delta\chi/r$. We claim that there is an l_0 independent of particular choice of $E \in \mathscr{E}^\chi(n)$ so that $h^0((E/F)(-l_0 - n)) = 0$. Otherwise, $(E/F)(-n)$ will have a subsheaf $J(-n)$ such that

$$\Delta p_{J(-n)} > \Delta\chi/r = \Delta p_{E/F(-n)}.$$

We can further assume $Q = (E/F)/J$ is torsion free. Thus Q as a quotient sheaf of E has $p_Q \prec p_E$ which contradicts Lemma 1.4 since E is H-semistable. Therefore $h^0(F(-l_0 - n)) = 0$. Since l_0 is bounded independent of $E \in \mathscr{E}^\chi(n)$, we can assume $h^1(E(-l_0 - n)) \le A_1$. Thus $h^1(F(-l_0 - n)) \le A_1$. So by applying Lemma 1.9 again, we can assume $h^j(F) = 0$ for $j = 1, 2$ by enlarging N_0 if necessary. Since

$$p_{F(-n)}(n) = \frac{1}{\alpha \cdot rk(F)} h^0(F) \ge p_{E(-n)}(n)$$

and $\Delta p_{F(-n)} = \Delta p_{E(-n)}$, we have $p_{F(-n)} \succeq p_{E(-n)}$. This violates the H-stability of E. Therefore $\mu(E, \varphi)$ must be a stable point. The situation of semistability can be treated similarly. q.e.d.

For the sake of completeness, we now discuss how to construct the moduli space of semistable sheaves E on X with $\chi_E = \chi$ and $\det E \approx I$. As one will see, what we will construct is a reduced scheme whose closed points are in one-one correspondence with the set $\mathscr{E}^\chi$ (modulo certain equivalence relation). However, it is unlikely that the functor from the category of separable finite type schemes to the category of subsets in $\mathscr{E}^\chi$ (cf. [8]) will be represented by the moduli space that we will construct. The main difficulty here is due to the lack of the *Picard scheme* of open surfaces $X_i\backslash\Sigma$. Since the following discussion will not be needed in the future discussion of this paper, we will only sketch the idea of the construction and leave the details of the proof to the readers.

We first choose n large so that the conclusions of Theorem 1.14 all hold. We then let $\mathscr{Q}_n(\chi)$ be the Grothendieck's Quot-scheme parameterizing all quotient sheaves $\bigoplus^N \mathscr{O}_X \to E(n)$ with $N = \chi(n)$ such that $\chi_E = \chi$. Then $\mathscr{Q}_n(\chi)$ is projective. Next, we let $\widetilde{\mathscr{Q}}_n(\chi)$ be the normalization of $\mathscr{Q}_n(\chi)$ and let $\mathscr{E}$ be the sheaf on $X \times \widetilde{\mathscr{Q}}_n(\chi)$ that is the pullback of the universal family on $X \times \mathscr{Q}_n(\chi)$. Since $\mathscr{E}$ admits a finite length locally free resolution on $(X_i\backslash\Sigma) \times \widetilde{\mathscr{Q}}_n(\chi)$, we can define the determinant line bundles

$$\det \mathscr{E}^{(i)} = \det \mathscr{E} | (X_i\backslash\Sigma) \times \widetilde{\mathscr{Q}}_n(\chi).$$

Since $\widetilde{\mathscr{Q}}_n(\chi)$ is normal, similar to the proof in Lemma 2.2, one can show that there is a reduced closed subscheme $\widetilde{\mathscr{Q}}_n(\chi, I) \subseteq \widetilde{\mathscr{Q}}_n(\chi)$ such that $\xi \in \widetilde{\mathscr{Q}}_n(\chi)$ is a closed point in $\widetilde{\mathscr{Q}}_n(\chi, I)$ if and only if $\det \mathscr{E}_\xi \approx I$. Indeed, more is true: There are line bundles L_1 and L_2 on $\widetilde{\mathscr{Q}}_n(\chi, I)$ so that as line bundles on $(X_i\backslash\Sigma) \times \widetilde{\mathscr{Q}}_n(\chi, I)$, $\det \mathscr{E}^{(i)} = p_1^* I | X_i\backslash\Sigma \otimes p_2^* L_i$. Now let $\mathscr{V}^i \subseteq H^0(X_i\backslash\Sigma, I)$ be the least subspace containing all $V^i(E)$ of $E \in \mathscr{E}^\chi(n)$ (cf. (1.12)). Since $\mathscr{E}^\chi(n)$ is bounded, $\mathscr{V}^i$ are of finite dimension. Parallel to the discussion after Corollary 1.11 (see also §2), we have a morphism

$$\Phi: \widetilde{\mathscr{Q}}_n(\chi, I) \to \mathbf{P}\left(\operatorname{Hom}\left(\bigwedge^r \mathbb{C}^N, \mathscr{V}^1\right)\right) \times \mathbf{P}\left(\operatorname{Hom}\left(\bigwedge^r \mathbb{C}^N, \mathscr{V}^2\right)\right).$$

By Theorem 1.14, if we let $\widetilde{\mathscr{Q}}_n(\chi, I)^{ss}$ be the space of all semistable quotient sheaves, then the induced morphism

$$\Phi': \widetilde{\mathscr{Q}}_n(\chi, I)^{ss} \to \left(\prod_{i=1}^2 \mathbf{P}\left(\operatorname{Hom}\left(\bigwedge^r \mathbb{C}^N, \mathscr{V}^i\right)\right)\right)^{ss}$$

is finite (see proof of Proposition 2.8). Thus similar to the discussion at the end of §2, $\widetilde{\mathscr{Q}}_n(\chi, I)^{ss}//G$ is projective and

$$\Phi'//G: \widetilde{\mathscr{Q}}_n(\chi, I)^{ss}//G \to \left(\prod_{i=1}^2 \mathbf{P}\left(\operatorname{Hom}\left(\bigwedge^r \mathbb{C}^N, \mathscr{V}^i\right)\right)\right)^{ss} //G$$

is finite. We let $\mathfrak{M}_X^{\chi, I}$ be the image scheme of $\widetilde{\mathscr{Q}}_n(\chi, I)^{ss}//G$ with the reduced scheme structure. It is easy to see that the set of closed points of $\mathfrak{M}_X^{\chi, I}$ are in one-one correspondence with the set of semistable sheaves (modulo the equivalence relation specified in [8]) of Hilbert polynomial χ and determinant $\approx I$.

2. The construction of the degeneration of moduli schemes

In this section, we are going to construct a degeneration of moduli scheme of semistable sheaves (of arbitrary rank) over smooth algebraic surfaces. We will consider the following situation: Let C be a smooth (irreducible) affine curve, and $0 \in C$ a closed point, and let t be a uniformizing parameter at 0. By abuse of notation, we will use t $(\neq 0)$ to denote closed points of $C^* = C\backslash 0$. We consider a family of algebraic surfaces $\pi: Z \to C$ with smooth total space such that $Z_t = \pi^{-1}(t)$, $t \neq 0$, are smooth connected, and Z_0 is reduced with normal crossing singularities. For simplicity, we assume $Z_0 = X_1 \cup X_2$ intersect along a smooth (connected) divisor Σ. (The result of this section can be generalized to the case where $Z_0 = X_1 \cup \cdots \cup X_k$ such that $H^0(\mathscr{O}^*_{X_i^0}) = \mathbb{C}^*$, where $X_i^0 = X_i\backslash$ singular loci of Z_0.) We fix a relative ample line bundle H of $Z \to C$ and an invertible sheaf I on Z. As we mentioned at the beginning of this paper, our immediate goal is (for a fixed χ) to construct a degeneration $\mathfrak{M}_0$ of the family of schemes $\mathfrak{M}_{Z_t}^{\chi, I}$, where $\mathfrak{M}_{Z_t}^{\chi, I}$ is the moduli of H_t-semistable sheaves E over Z_t of Hilbert polynomial χ and determinant I_t, such that closed points of $\mathfrak{M}_0$ associates canonically to semistable sheaves over Z_0. Here we adopt the convention that $H_t = H_{|Z_t}$.

We begin our discussion with some remark about family of sheaves on Z. Let $\rho: S \to C$ be a scheme of finite type over C and let E_S be a rank-r sheaf on $Z_S = Z \times_C S$. Let $\pi_S: Z_S \to S$ be the projection. We say E_S is a flat family of torsion free sheaves over S if E_S is flat over S and if for every closed $s \in S$, the restriction of E_S to the fiber $Z_s = Z \times_C \{s\}$ (we denote it by E_s) is torsion free. Let $Z^{(1)}$ (resp. $Z^{(2)}$) be the smooth variety $Z\backslash X_2$ (resp. $Z\backslash X_1$), and let $Z_S^{(1)}$ (resp. $Z_S^{(2)}$) be the scheme $Z^{(1)} \times_C S$ (resp. $Z^{(2)} \times_C S$). For $i = 1, 2$, $\pi_S: Z_S^{(i)} \to S$ are smooth. Thus any sheaf E_S on Z_S that is flat over S locally admits a bounded locally free resolution on $Z_S^{(i)}$. Hence we can define determinant line bundles $\det E_{S|Z_S^{(i)}}$. We say that $\det E_S \approx p_Z^* I$, if there are isomorphisms

$$\lambda_i\colon \det E_{S|Z_S^{(i)}} \cong (\pi_S^* L_i \otimes p_Z^* I)_{|Z_S^{(i)}}, \tag{2.1}$$

where $L_i \in \mathrm{Pic}(S)$ and $p_Z\colon Z_S \to Z$. Note that if S is flat over C, then $\det E_S \approx p_Z^* I$ implies that for any closed $q \in S$ over $0 \in C$, we have $\det E_q \approx I_0$ according to Definition 1.3, where $I_0 = I_{|Z_0}$.

Let χ be the polynomial depending on d and r:

$$\begin{aligned}\chi(n) = \frac{r}{2} n^2 (H_t \cdot H_t) + n((H_t \cdot I) - \frac{r}{2}(H_t \cdot K_{Z_t})) \\ + (r-1)\chi(\mathscr{O}_{Z_t}) + \chi(I_t) - d, \qquad t \neq 0.\end{aligned} \tag{2.2}$$

Let $\mathscr{E}_{Z_t}^\chi$ be the set of all H_t-semistable sheaves E on Z_t satisfying $\det E \approx I_t$ and $\chi_E = \chi$. (We agree that $\det E \approx I_t$ means $\det E = I_t$ when $t \neq 0$.) We fix an N_0 sufficiently large so that the conclusions of Theorem 1.14 hold for $\mathscr{E}_{Z_t}^\chi$ for all $t \in C$ (cf. [8, Lemma 1.2]). Let $n \geq N_0$ and let $N = \chi(n)$. Following Grothendieck [12], we define $\mathfrak{Quot}_{Z/C}^{\chi, \mathscr{O}^N}$ to be the functor sending any scheme S of finite type over C to the set of all quotient sheaves E of $\mathscr{O}_{Z_S}^{\oplus N}$ on Z_S flat over S so that $\chi_{E_s}(m) = \chi(n+m)$ for any closed $s \in S$. $\mathfrak{Quot}_{Z/C}^{\chi, \mathscr{O}^N}$ is represented by a scheme $\mathscr{Q}\mathrm{uot}_{Z/C}^{\chi, \mathscr{O}^N}$ projective over C, called Grothendieck's Quot-scheme. Next, we let $\mathfrak{U}_{C^*}^{ss}$ be the set of all closed points $\xi \in \mathscr{Q}\mathrm{uot}_{Z/C}^{\chi, \mathscr{O}^N}$ over $t \in C^*$ such that the corresponding quotient sheaf E_ξ (over Z_t) is H_t-semistable and such that $\det E_\xi = I_t(rn)$. Since being semistable on smooth surface is an open condition and having determinant $= I_t$ is a closed condition, $\mathfrak{U}_{C^*}^{ss}$ is a locally closed subscheme of $\mathscr{Q}\mathrm{uot}_{Z/C}^{\chi, \mathscr{O}^N}$. Since $\mathscr{Q}\mathrm{uot}_{Z/C}^{\chi, \mathscr{O}^N}$ is projective over C, by shrinking C if necessary, we can assume $\mathfrak{U}_{C^*}^{ss}$ is flat over C^*. Now we define $\mathscr{Q}\mathrm{uot}_{Z/C}^{\chi, I, \mathscr{O}^N} \subseteq \mathscr{Q}\mathrm{uot}_{Z/C}^{\chi, \mathscr{O}^N}$ to be the closure of $\mathfrak{U}_{C^*}^{ss}$ in $\mathscr{Q}\mathrm{uot}_{Z/C}^{\chi, \mathscr{O}^N}$ endowed with reduced scheme structure, and let $\mathfrak{R}_{Z/C}^{\chi, I, \mathscr{O}^N}$ be the normalization of $\mathscr{Q}\mathrm{uot}_{Z/C}^{\chi, I, \mathscr{O}^N}$. $\mathfrak{R}_{Z/C}^{\chi, I, \mathscr{O}^N}$ will be the basic object in our construction of degeneration of moduli.

As is clear from the construction, $\mathfrak{R}_{Z/C}^{\chi, I, \mathscr{O}^N}$ depends on the parameters d, r, I and H (χ depends on d and r as in (2.2)) and also on the choice of $n \geq N_0$ (or $N = \chi(n)$) that is always chosen large enough (cf. Theorem 1.14). Since I is fixed throughout the paper, we will not build I into the notation. Also, the choice of n is not important as long as it is large enough. In this and the next sections, we work on arbitrary

$r \geq 2$. Starting from §4, we will study the case where $r = 2$. If all these are understood, we will abbreviate $\mathfrak{M}^{\chi, I, \mathscr{O}^N}_{Z/C}$ to $\mathfrak{M}^{d,H}$. More frequently we shall write just $\mathfrak{M}^d$ if the context makes the choice of H either clear or unimportant. We will also use $\mathfrak{M}^d_t$ to denote the fiber of $\mathfrak{M}^d$ over $t \in C$. We will adopt the following convention throughout the paper. Unless otherwise is stated, all schemes are over C. In case the set is related to the surface Z_t, we then will build the subscript Z_t into the notation. Finally, the subscript 0 (resp. t) is reserved to denote the fiber over $0 \in C$ (resp. $t \in C$) of the respective scheme.

Returning to the scheme $\mathfrak{M}^d$, because $\mathfrak{M}^d$ is constructed as the closure of a scheme over C^*, no component of $\mathfrak{M}^d$ is contained entirely in $\mathfrak{M}^d_0$. Thus by [13, III.9.7], by shrinking C (still containing 0) if necessary, we can assume $\mathfrak{M}^d$ is flat over C. Thus we have

Proposition 2.1. *$\mathfrak{M}^d$ is a normal scheme that is flat and projective over C. Further, for any closed $v \in \mathfrak{M}^d$, there is a curve S, $s_0 \in S$ and a morphism $\varphi: S \to \mathfrak{M}^d$ such that $\varphi(s_0) = v$, $\pi_C \circ \varphi: S \to C$ is dominant and $\varphi(s)$ is H-semistable for general $s \in S$.*

Following [8], we intend to find a C-morphism μ from $\mathfrak{M}^d$ to a big projective space $\mathbf{P}^L_C$. We first study the following situation. Let **Sch** be the set of all normal C-schemes of finite type that are flat over C. Let $S \in$ **Sch** be an affine scheme and let E_S be a flat family of sheaves (not necessarily torsion free) on Z_S over S. We assume that there is a homomorphim $h_S: \mathscr{O}^{\oplus N}_{Z_S} \to E_S$ such that at any closed point $s \in S$, the restriction $h_s: \mathscr{O}^{\oplus N}_{Z_s} \to E_s$ is surjective at the generic points of Z_s.

Lemma 2.2. *With the notation as above, and suppose on Z_{S^0}, where $S^0 = \rho^{-1}(C^*)$ with $\rho: S \to C$, we have $\det E_{S|Z_{S^0}} = p_Z^* I(rn)|_{Z_{S^0}}$. Then $\det E_S \approx p_Z^* I(rn)$.*

Proof. We need to prove that there are isomorphisms

$$\lambda_i: \det E_{S|Z^{(i)}_S} \xrightarrow{\cong} p_Z^* I(rn)|_{Z^{(i)}_S}. \tag{2.3}$$

By assumption, $\det E_{S|Z^{(1)}_S}|Z_{S^0} = p_Z^* I(rn)|Z_{S^0}$. So there is a canonical section $v \in H^0(Z_{S^0}, (p_Z^* I(rn))^{-1} \otimes \det E_{S|Z^{(1)}_S})$. We will show that there is a Cartier divisor D of S contained in S_0 such that v extends to a nonvanishing section in

$$H^0(Z^{(1)}_S, (p_Z^* I(rn))^{-1}(D) \otimes \det E_{S|Z^{(1)}_S}). \tag{2.4}$$

Since $Z^{(1)}_S$ is normal, v extends to a meromorphic section $\overline{v}$ of

$(p_Z^*I(rn))^{-1}\otimes \det E_{S|Z_S^{(1)}}$. Knowing that the zero and pole divisors of $\overline{v}$ are contained in S_0 and that $Z_S^{(1)}\to S$ is smooth with connected fiber, there is a divisor D of S contained in S_0 such that $\overline{v}$ is a nonvanishing section of (2.4). Thus $\lambda_1=\overline{v}$ is an isomorphism between $\det E_{S|Z_S^{(1)}}(D)$ and $p_Z^*I(rn)$, and (2.3) holds for $i=1$ because S is affine. The same reason shows that λ_2 exists also. q.e.d.

Now suppose we have isomorphisms (2.3) for the family E_S. By composing the isomorphisms λ_i with the homomorphism $\bigwedge^r h_S\colon \bigwedge^r \mathscr{O}_{Z_S}^{\oplus N}\to \det E_{S|Z_S^{(i)}}$, we get

$$\lambda_i\circ \bigwedge^r h_S\colon \bigwedge^r \mathscr{O}_{Z_S}^{\oplus N}\to p_Z^*I(rn)_{|Z_S^{(i)}}.$$

Thus $\lambda_i\circ\bigwedge^r h_S$ induce

$$\delta_i\colon \pi_{S*}\left(\bigwedge^r \mathscr{O}_{Z_S}^{\oplus N}\right)\to \pi_{S*}(p_Z^*I(rn)_{|Z_S^{(i)}}). \tag{2.5}$$

Considering the square

$$\begin{array}{ccc} Z_S & \xrightarrow{p} & Z \\ \downarrow{\scriptstyle \pi_S} & & \downarrow{\scriptstyle \pi_C} \\ S & \xrightarrow{\rho} & C. \end{array}$$

Since $(p_Z^*I(rn))_{|Z_S^{(i)}}=p_Z^*(I(rn)_{|Z_C^{(i)}})$ and $\rho\colon S\to C$ is flat, by [13, III.9.3] we have

$$\pi_{S*}(p_Z^*I(rn)_{|Z_S^{(i)}})=\rho^*\pi_{C*}(I(rn)_{|Z_C^{(i)}}). \tag{2.6}$$

Finally, let $\mathscr{V}^i=\pi_{C*}(I(rn)_{|Z_C^{(i)}})$. Compared to the V^i defined in (1.12), it is easy to see that if $\mathscr{O}_{Z_S}^{\oplus N}\to E_S$ is a quotient sheaf, then for any closed $s\in S$, the subspace $V^i(E_s)\subseteq H^0(X_i\backslash\Sigma, I(rn))$ corresponding to E_s is contained in $\mathscr{V}^i\otimes k(t_0)$, where t_0 lies under s. Clearly, $\mathscr{V}^i$ are torsion free sheaves on C. We claim that $\mathscr{V}^i$ are coherent. Indeed, let p_0 be a positive integer so that for any $p\ge p_0$, $H^0(X_2, I(rn)(-pX_1)_{|X_2})=\{0\}$. There is an obvious injective homomorphism

$$\varphi\colon \pi_{C*}(I(rn)(p_0X_2))\to \pi_{C*}(I(rn)_{|Z^{(1)}}). \tag{2.7}$$

Let $s\in\pi_{C*}(I(rn)_{|Z^{(1)}})$. Let q_0 be the smallest integer such that there is an integer $a\ge 0$ so that s/t^a extends to section $\beta\in\pi_{C*}(I(rn)(q_0X_2))$

with $\beta_{|X_1} \neq 0$. By the minimality of q_0, $\beta_{|X_2} \neq 0$. Thus

$$H^0(X_2, I(rn)(-q_0X_1)_{|X_2}) \neq \{0\},$$

which implies that $q_0 < p_0$ and $\beta \in \pi_{C*}(I(rn)(p_0X_2))$, so that

$$s \in \pi_{C*}(I(rn)(p_0X_2)),$$

and (2.7) is surjective. Therefore, $\mathscr{V}^1$ is coherent. Similarly, $\mathscr{V}^2$ is coherent. Hence $\mathscr{V}^i$ are locally free.

The homomorphisms δ_i considered as sections

$$\delta_i\colon S \to \mathrm{Hom}_S\left(\rho^*\left(\bigwedge^r \mathscr{O}_C^{\oplus N}\right), \rho^*(\pi_{C*}(I(rn)_{|Z_C^{(i)}}))\right)$$

induce C-morphisms $\tilde{\mu}_i\colon S \to \mathrm{Hom}_C(\bigwedge^r \mathscr{O}_C^{\oplus N}, \mathscr{V}^i)$. If we can show that for any closed $s \in S$ over $t \in C$, $\tilde{\mu}_i(s) \in \mathrm{Hom}(\bigwedge^r \mathscr{O}_C^{\oplus N}, \mathscr{V}^i) \otimes k(t)$ is nontrivial, then we can associate to E_S morphisms

$$\mu_i\colon S \to \mathbf{P}\left(\mathrm{Hom}_C\left(\bigwedge^r \mathscr{O}_C^{\oplus N}, \mathscr{V}^i\right)\right) \tag{2.8}$$

and the C-morphism

$$\begin{aligned}\mu = [\mu_1, \mu_2]\colon S \to \mathbf{P}&\left(\mathrm{Hom}_C\left(\bigwedge^r \mathscr{O}_C^{\oplus N}, \mathscr{V}^1\right)\right)\\ &\times \mathbf{P}\left(\mathrm{Hom}_C\left(\bigwedge^r \mathscr{O}_C^{\oplus N}, \mathscr{V}^2\right)\right).\end{aligned} \tag{2.9}$$

We need the following technical lemmas.

Lemma 2.3. *Let $S \in$ **Sch**. Let $s \in S$ be any closed point, and $t_0 = \rho(s)$. Suppose that $v \in \bigwedge^r \mathbb{C}^N$ and that $\overline{v} \in H^0(\bigwedge^r \mathscr{O}_{Z_s}^{\oplus N})$ is the corresponding section. Then $\tilde{\mu}_i(s)(v) \in \mathscr{V}^i \otimes k(t_0)$ is trivial if and only if the section $(\lambda_i \circ \bigwedge^r h(s))(\overline{v}) \in H^0(Z_s^{(i)}, p_Z^* I(rn)_{|Z_s^{(i)}})$ is also.*

Proof. It suffices to show that for any closed $t_0 \in C$, the restriction homomorphism $r_{t_0}\colon \mathscr{V}^i \otimes k(t_0) \to H^0(Z_{t_0}^{(i)}, I(rn)_{|Z_{t_0}^{(i)}})$ is injective. Assume that $v \in \mathscr{V}^i \otimes k(t_0)$ with $r_{t_0}(v) = 0$. Let $w \in H^0(C, \mathscr{V}^i)$ be the section with $w \otimes k(t_0) = v$. Then $r_{t_0}(v) = 0$ is equivalent to $w_{|Z_{t_0}^{(i)}} = 0$, and the latter implies that $w = (t - t_0)w'$ for some $w' \in H^0(C, \mathscr{V}^i)$. Hence $v = w \otimes k(t_0) = 0$. q.e.d.

The lemma guarantees that $\tilde{\mu}_i(s)$ is never trivial if we assume that $h_s\colon \mathscr{O}_{Z_s}^{\oplus N} \to E_s$ is surjective at the generic points of $Z_s^{(i)}$. Next we show that the morphisms μ_i are canonical.

Lemma 2.4. *For $S \in \mathbf{Sch}$, the morphisms*

$$\mu_i\colon S \to \mathbf{P}(\mathrm{Hom}_C(\bigwedge^r \mathscr{O}_C^{\oplus N}, \mathscr{V}^i))$$

are canonical.

Proof. Clearly, the definition of μ_i depends (only) on the choice of the isomorphisms λ_i of (2.3). We claim that if λ_i and λ_i' are two pairs of isomorphisms,

$$\lambda_i, \lambda_i'\colon \det E_{S|Z_S^{(i)}} \cong p_Z^* I(rn)_{|Z_S^{(i)}},$$

then there are $f_i \in H^0(S, \mathscr{O}_S^*)$ such that $\lambda_i' = \pi_S^*(f_i)\cdot \lambda_i$. Indeed, since both λ_i and λ_i' are isomorphisms,

$$g_i = \lambda_i' \circ \lambda_i^{-1}\colon p_Z^* I(rn)_{|Z_S^{(i)}} \xrightarrow{\cong} p_Z^* I(rn)_{|Z_S^{(i)}}$$

are isomorphisms. So $q_i \in H^0(Z_S^{(i)}, \mathscr{O}_{Z_S^{(i)}}^*)$. Since $\rho\colon S \to C$ is flat and $\pi_{C*}\mathscr{O}_{Z^{(i)}} = \mathscr{O}_C$, we have

$$H^0(Z_S^{(i)}, \mathscr{O}_{Z_S^{(i)}}^*) = H^0(S, \pi_{S*}\mathscr{O}_{Z_S^{(i)}}^*) = H^0(S, \rho^*\pi_{C*}\mathscr{O}_{Z^{(i)}}^*) = H^0(S, \mathscr{O}_S^*).$$

Thus, the $g_i \in H^0(Z_S^{(i)}, \mathscr{O}_{Z_S^{(i)}}^*)$ are pullback sections of $f_i \in H^0(Z_S, \mathscr{O}_S^*)$. It follows that $\lambda_i' = \pi_S^*(f_i)\cdot\lambda_i$, so that the induced homomorphisms δ_i and δ_i' differ by $\pi_S^*(f_i)$. Hence the morphisms μ_i and μ_i' are identical. q.e.d.

Combining Lemmas 2.3 and 2.4, we have proved

Proposition 2.5. *For any affine $S \in \mathbf{Sch}$ and flat family of sheaf E_S over S with homomorphism $h_S\colon \mathscr{O}_{Z_S}^{\oplus N} \to E_S$ having the property that at each closed point $s \in S$, $h_s\colon \mathscr{O}_{Z_s}^{\oplus N} \to E_s$ is surjective at the generic points of Z_s and that $\det E_{S^0} = \pi_S^* L \otimes p_Z^* I(rn)$, $S^0 = \rho^{-1}(C^*)$, there is a canonical C-morphism*

$$\mu_S = [\mu_1(S), \mu_2(S)]\colon S \to \mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2),$$

where $W_C^i = \mathrm{Hom}_C(\bigwedge^r \mathscr{O}_C^{\oplus N}, \mathscr{V}^i)$.

Now we are ready to construct the C-morphism μ_R from $\mathfrak{R}^d$ to $\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2)$. We first cover $\mathfrak{R}^d$ by affine open sets $S_1, S_2, \cdots, S_l$ in $\mathbf{Sch}$.

On each S_j, we have a C-morphism

$$\mu_j\colon S_j \to \mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2), \tag{2.10}$$

provided by Proposition 2.5. Since the morphisms $\mu_{(\cdot)}$ are canonical, $\mu_k \equiv \mu_j|_{S_k \cap S_j}$. Therefore, $\{\mu_j\}_{j=1}^l$ patches together to give a C-morphism

$$\mu_R\colon \mathfrak{R}^d \to \mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2). \tag{2.11}$$

Let $\mathcal{G} = \mathcal{S}l(N, C) = SL(N, \mathbb{C}) \otimes_{\mathbb{C}} C$ be the special linear group scheme over C. For any scheme $S \in \mathbf{Sch}$, assume that $g \in \mathcal{G}_S = \mathcal{S}l(N, S)$ and that $\sigma\colon \mathcal{O}_{Z_S}^{\oplus N} \to E_S$ is a quotient sheaf homomorphism. Then $g(\sigma) = \sigma \circ g\colon \mathcal{O}_{Z_S}^{\oplus N} \to E_S$ is a quotient sheaf also. Hence $\mathcal{Q}\mathrm{uot}_{Z/C}^{\chi, \mathcal{O}^N}$ is naturally a $\mathcal{G}$-scheme. Since $\mathcal{G}$ is an irreducible smooth group scheme and $\mathcal{Q}\mathrm{uot}_{Z/C}^{\chi, I, \mathcal{O}^N}$ is invariant under $\mathcal{G}$, $\mathcal{Q}\mathrm{uot}_{Z/C}^{\chi, I, \mathcal{O}^N}$ as well as its normalization $\mathfrak{R}^d$ are natural $\mathcal{G}$-schemes. One checks easily that through the dual action $g(v) = v \circ g$, where $g \in \mathcal{G}$ and $v \in \mathrm{Hom}_C(\bigwedge^r \mathcal{O}_C^{\oplus N}, \mathcal{V}^i)$, $\mathbf{P}(\mathrm{Hom}_C(\bigwedge^r \mathcal{O}_C^{\oplus N}, \mathcal{V}^i))$ are also $\mathcal{G}$-schemes.

Proposition 2.6. *Through the dual action of $\mathcal{G}$ on $\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2)$, the morphism $\mu_R\colon \mathfrak{R}^d \to \mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2)$ is a $\mathcal{G}$-morphism.*

Proof. One needs to check that for $S \in \mathbf{Sch}$, if $\sigma_S\colon \mathcal{O}_{Z_s}^{\oplus N} \to E_S$ is a flat family of quotient sheaves on Z_S over S, and $g_S \in \mathcal{G}l(N, S)$ is a section over S, then

$$\mu(g_S(\sigma_S)) \equiv g_S(\mu(\sigma_S))\colon S \to \mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2).$$

This is straightforward from our construction of the map $\mu_R\colon \mathbf{R}^d \to \mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2)$. (See also [21, p. 114].) q.e.d.

Let $(\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2))^{ss}$ be the open subset of $\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2)$ consisting of all semistable points under $\mathcal{G}$, and let $\mathfrak{U}^{ss} \subseteq \mathfrak{R}^d$ be the set consisting of all semistable quotient sheaves. We prove the following.

Proposition 2.7. $\mu_R^{-1}((\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2))^{ss}) = \mathfrak{U}^{ss}$. *In particular, $\mathfrak{U}^{ss} \subset \mathfrak{R}^d$ is open.*

Proof. We first show that $\mu_R(\mathfrak{U}^{ss}) \subseteq (\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2))^{ss}$. Let $z \in \mathfrak{U}^{ss}$ be any closed point over $t \neq 0 \in C$. Then $\mu_R(z) \in (\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2))^{ss}$ follows from [8, Theorem 0.7]. Now assume that $z \in \mathfrak{U}^{ss}$ is a closed point over $0 \in C$, and that E is the corresponding semistable sheaf over Z_0 with isomorphisms (λ_1, λ_2) of (2.3). Then by using Lemma 2.3 and the fact that $\mathfrak{N}^d$ is flat over C, we conclude

$$\begin{array}{ccccc}
\bigwedge^r H^0(Z_0, \mathscr{O}_{Z_0}^{\oplus N}) & \longrightarrow & & & V^i(E) \\
\downarrow \cong & & & & \downarrow j \\
\bigwedge^r \mathscr{O}_C^{\oplus N} \otimes k(0) & \longrightarrow & (\pi_{C*} I(rn)_{|Z_C^{(i)}}) \otimes k(0) & \longrightarrow & \mathscr{V}^i \otimes k(0)
\end{array}$$

is commutative up to scalars in $\mathbb{C}^*$, where j is the obvious injection, and $V^i(E)$ is defined in (1.12). By Theorem 1.14, $[E] \in \mathbf{P}(W^1) \times \mathbf{P}(W^2)$ is $SL(N, \mathbb{C})$ semistable, where W^i is the space associated to E defined in (1.15). Since $W^i \subseteq W_C^i \otimes k(0)$ (Lemma 2.3), $\mu_R(z)$ must be semistable also. Thus we have proved that $\mu_R(\mathfrak{U}^{ss}) \subseteq (\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2))^{ss}$.

Now we prove the other direction: $\mu_R^{-1}((\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2))^{ss}) \subseteq \mathfrak{U}^{ss}$. Let $w \in \mu_R^{-1}((\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2))^{ss})$ be any closed point corresponding to the quotient sheaf $\mathscr{O}_{Z_t}^{\oplus N} \to E$, where w is over t. In case E is torsion free, $w \in \mathfrak{U}^{ss}$ by [8] and Theorem 1.14. We now prove the general situation. There are two possible situations: Either $t = 0$ or $t \neq 0$. In the following, we will prove the case where $t = 0$ and leave the easier case $t \neq 0$ to the readers. We assume $w \in \mathfrak{N}_0^d$. By Proposition 2.1, we can find a smooth curve S flat over C, $s \in S$ over $0 \in C$ and a C-morphism

$$\varphi\colon (s, S) \to (w, \mathfrak{N}^d) \tag{2.12}$$

such that $\varphi(S\backslash s) \subseteq \mathfrak{U}^{ss}$. We can assume $S\backslash s$ is over $C\backslash 0$. Let E_S be the pullback quotient sheaf on Z_S via φ, and let $h_S\colon \mathscr{O}_{Z_S}^{\oplus N} \to E_S$ be the quotient homomorphism. Consider the homomorphism $h_{S^0}\colon \mathscr{O}_{Z_{S^0}}^{\oplus N} \to E_{S^0}$ that is the restriction of h_S to Z_{S^0} where $S^0 = S\backslash s$. Following the proof of Lemma 3.2 and the remark after Lemma 3.2 in the next section (see also [8]), we can find a flat family of torsion free sheaves F_S with $F_{S|Z_{S^0}} = E_{S^0}$ and a homomorphism $h_S'\colon \mathscr{O}_{Z_S}^{\oplus N} \to F_S$ on Z_S which extends h_{S^0} such that $h_s'\colon \mathscr{O}_{Z_s}^{\oplus N} \to F_s$ is surjective at the generic points of Z_s. We then have the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(h_S) & \longrightarrow & \mathscr{O}_{Z_S}^{\oplus N} & \xrightarrow{h_S} & E_S & \longrightarrow & 0 \\
 & & & & \| & & & & \\
0 & \longrightarrow & \ker(h_S') & \longrightarrow & \mathscr{O}_{Z_S}^{\oplus N} & \xrightarrow{h_S'} & F_S & \longrightarrow & 0.
\end{array} \tag{2.13}$$

By assumption, $h'_S(\ker(h_S))$ is supported on Z_s. Thus it must be zero since F_S is flat over S. E_S is a quotient sheaf of $\mathcal{O}_{X\times S}^{\oplus N}$, so there is a homomorphism $g_S\colon E_S \to F_S$ induced from $\mathcal{O}_{Z_S}^{\oplus N} \xrightarrow{\mathrm{id}} \mathcal{O}_{Z_S}^{\oplus N}$ such that $g_S \circ h_S = h'_S$. Because $\mathcal{O}_{Z_s}^{\oplus N} \to F_s$ is surjective at the generic points of Z_s, g_S induces isomorphisms $\det(g_S)_i\colon \det E_{S|Z_S^{(i)}} \to \det F_{X|Z_S^{(i)}}$. Thus the morphisms $\mu(E_S)$, $\mu(F_S)\colon S \to \mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2)$ induced by quotient homomorphisms h_S and h'_S respectively are identical. So $\mu(F_S)(s)$ is semistable. Since F_s is torsion free, by Theorem 1.14, F_s is semistable and then $h_s\colon \mathcal{O}_{Z_s}^{\oplus N} \to F_s$ is surjective. Thus $\mathcal{O}_{Z_S}^{\oplus N} \to F_S$ is also a family of quotient sheaves in the Quot-scheme $\mathfrak{R}^d$. Because $\mathfrak{R}^d$ is separated, $E_s = F_s$, which completes the proof of the proposition. q.e.d.

Let $\mu_U\colon \mathfrak{U}^{ss} \to (\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2))^{ss}$ be the restriction of μ_R to $\mathfrak{U}^{ss}$. Then we have

Proposition 2.8. $\mu_U\colon \mathfrak{U}^{ss} \to (\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2))^{ss}$ *is finite.*

Proof. We first check that μ_U is proper. Since $\mathfrak{R}^d$ is projective over C, $\mu_R\colon \mathfrak{R}^d \to \mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2)$ is proper. Thus the restriction of μ_R to $\mu_R^{-1}((\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2))^{ss}) = \mathfrak{U}^{ss}$ is a proper morphism to $(\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2))^{ss}$.

Next we check that for any closed point $z \in (\mathbf{P}(W_C^1)\times\mathbf{P}(W_C^2))^{ss}$, $\mu_U^{-1}(z)$ is finite. We first prove that if η_1 and η_2 are two closed points of $\mathfrak{U}^{ss}$ in $\mu_R^{-1}(z)$, and $h_i\colon \mathcal{O}_{Z_t}^{\oplus N} \to E_i$ are corresponding quotient sheaves on Z_t where t lies under η_1 and η_2, then $E_1 = E_2$ as quotient sheaves of $\mathcal{O}_{Z_t}^{\oplus N}$. Indeed, according to [8, Lemma 4.3], there is a dense open subset $V \subset Z_t$ such that $E_{1|V} = E_{2|V}$ as quotient sheaves of $\mathcal{O}_V^{\oplus N}$. Now let Q_i be the kernel of $h_i\colon \mathcal{O}_{Z_t}^{\oplus N} \to E_i$. We have the diagram

$$\begin{tikzcd} 0 \arrow[r] & Q_1 \arrow[r, "f_1"] & \mathcal{O}_{Z_t}^{\oplus N} \arrow[r, "h_1"] \arrow[d, equal] & E_1 \arrow[r] & 0 \\ 0 \arrow[r] & Q_2 \arrow[r, "f_2"] & \mathcal{O}_{Z_t}^{\oplus N} \arrow[r, "h_2"] & E_2 \arrow[r] & 0. \end{tikzcd}$$

Since $E_{1|V} = E_{2|V}$ as quotient sheaves of $\mathcal{O}_{Z_t|V}^{\oplus N}$, $h_2(f_1(Q_1))$ is a torsion subsheaf of E_2 supported on $Z_t \backslash V$. Thus $h_2(f_1(Q_1)) = 0$ since E_2 is torsion free. So $Q_1 \subseteq Q_2$. Similarly, $Q_2 \subseteq Q_1$. Therefore $Q_1 = Q_2$ and $E_1 = E_2$ as quotient sheaves of $\mathcal{O}_{Z_t}^{\oplus N}$.

Thus we have proved that the set $\mu_U^{-1}(z) \subset \mathfrak{U}^{ss} \subseteq \mathfrak{R}^d$ is over a single closed point in $\mathcal{Q}\mathrm{uot}_{Z/C}^{\chi,I,\mathcal{O}^H N}$. Since $\mathfrak{R}^d$ is the normalization of $\mathcal{Q}\mathrm{uot}_{Z/C}^{\chi,I,\mathcal{O}^N}$, $\mathfrak{R}^d \to \mathcal{Q}\mathrm{uot}_{Z/C}^{\chi,I,\mathcal{O}^N}$ is finite. Hence $\mu_U^{-1}(z)$ is finite, and the proposition has been established. q.e.d.

Let $\Pi_C = (\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2))^{ss}//\mathcal{G}$. By the geometric invariant theory, Π_C is a good quotient of $(\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2))^{ss}$ by $\mathcal{G}$. Further, Π_C is projective over C. The following lemma (for its proof see [8]) shows that there is a good quotient $\mathfrak{U}^{ss}//\mathcal{G}$.

Lemma 2.9. *Let G be a reductive group, and M_1 and M_2 be two G-schemes. Suppose that $\mu\colon M_1 \to M_2$ is a finite G-morphism and that a good quotient of M_2 by G, say $M_2//G$, exists. Then a good quotient of M_1 by G exists, and the induced morphism $M_1//G \to M_2//G$ is finite.*

We summarize our result as follows.

Theorem 2.10. *For any integer d, let $n \geq N_0$ be a sufficiently large integer, and let $\mathfrak{U}^{ss} \subseteq \mathfrak{R}^{d,H}$, $N = \chi(n)$, be the set of closed points corresponding to H-semistable quotient sheaves. Then the C-morphism $\mu_U\colon \mathfrak{U}^{ss} \to (\mathbf{P}(W_C^1) \times \mathbf{P}(W_C^2))^{ss}$ so constructed is finite and $\mathcal{G}$ equivariant. Further, the good quotient $\mathfrak{M}^{d,H} = \mathfrak{U}^{ss}//\mathcal{G}$ exists. Finally, $\mathfrak{M}^{d,H}$ is projective and flat over C.*

The following theorem shows that the scheme $\mathfrak{M}^{d,H}$ is a degeneration of the moduli $\mathfrak{M}_{Z_t}^{d,H_t}$ in the weak sense, where $\mathfrak{M}_{Z_t}^{d,H_t}$ is the moduli of rank r H_t-semistable sheaves on Z_t of $\det = I_t$ and $c_2 = d$.

Theorem 2.11. *$\mathfrak{M}^{d,H}$ is normal. Further, for any closed $t \neq 0 \in C$, there is a finite morphism from $\mathfrak{M}_t^{d,H}$ to $\mathfrak{M}_{Z_t}^{d,H_t}$.*

Proof. By the universal mapping property, it is easy to check that $\mathfrak{M}^{d,H}$ is normal if $\mathfrak{R}^{d,H}$ is normal [26, p.5]. For $\mathfrak{M}_t^{d,H}$, we note that $\mathfrak{M}_t^{d,H}$ is the good quotient of $\mathfrak{U}_t^{ss}$ that is finite over the open set of semistable sheaves in $\mathfrak{Quot}_X^{\chi,\mathcal{O}^N}$ (see the construction at the beginning of this section). Therefore, the quotient $\mathfrak{M}_t^{d,H} = \mathfrak{U}_t^{ss}//SL(N)$ is finite over $\mathfrak{M}_{Z_t}^{d,H_t}$. q.e.d.

3. Geometric realization of sheaves on Z_0

In this section, we will keep the notation developed in §2 and still work on the moduli of sheaves of arbitrary rank.

Let $\mathcal{X}$ be any irreducible component of $\mathfrak{U}_0^{ss}$, and let $\overline{m}$ be the corresponding multiplicity. Since $\mathfrak{M}^d$ is normal, it is smooth at the generic point of $\mathscr{X}$. Let $U \subset \mathfrak{U}^{ss}$ be a smooth open set so that $\varnothing \neq U \cap \mathfrak{M}_0^d \subset \mathscr{X}$. Set $p: U \to C$. We can and do assume that the reduction of $U - 0 = p^{-1}(0)$ is smooth. Let $f_m: C^m \to C$ be a branch covering with $0 \in C^m$ as its only ramification point of index m over $0 \in C$. We define U^m as the normalization of the cartesian product

$$\begin{array}{ccc} C^m \times_C U & \longrightarrow & U \\ \big\downarrow{\scriptstyle p_m} & & \big\downarrow{\scriptstyle p} \\ C^m & \xrightarrow{f_m} & C. \end{array} \tag{3.1}$$

We assume $\overline{m}|m$. Then U^m is smooth and further, the fiber $p_m^{-1}(0) \subset U^m$ has multiplicity 1 because the fiber $p^{-1}(0) \subset U$ has multiplicity $\overline{m}$. Therefore, p_m is a smooth morphism at $p_m^{-1}(0)$. By shrinking U if necessary, we can assume $p_m: U^m \to C^m$ is smooth.

Similarly, we can form the Cartesian product $Z^m_{C^m}$:

$$\begin{array}{ccc} Z^m_{C^m} & \longrightarrow & Z \\ \big\downarrow{\scriptstyle \pi_m} & & \big\downarrow{\scriptstyle \pi} \\ C^m & \xrightarrow{f_m} & C \end{array} \tag{3.2}$$

$Z^m_{U^m} = Z^m_{C^m} \times_{C^m} U^m$, where $Z^m_{C^m}$ and U^m are schemes over C^m via π_m and p_m respectively. Obviously, $Z^m_{C^m}$ is not smooth when $m > 1$. Let $\gamma: \widetilde{Z}^m_{C^m} \to Z^m_{C^m}$ be the desingularization of $Z^m_{C^m}$, and let $\Xi \subseteq \widetilde{Z}^m_{C^m}$ be the exceptional divisor of γ. Clearly, Ξ is the union of $(m-1)$ copies of the ruled surface Δ, and $\widetilde{Z}^m_0$ is the union of X_1, X_2 and these $(m-1)$ copies of Δ. In the following, we denote $\widetilde{Z}^m_0 = \Delta_0 \cup \cdots \cup \Delta_m$, where $\Delta_0 = X_1$, $\Delta_1 = \cdots = \Delta_{m-1}$ are the ruled surfaces, and $\Delta_m = X_2$ such that $\Sigma_i = \Delta_{i-1} \cap \Delta_i$ is nonempty. Next, we let $\widetilde{Z}^m_{U^m} = \widetilde{Z}^m_{C^m} \times_{C^m} U^m$, and let $\Xi_{U^m} = \Xi \times_{C^m} U^m$ be the exceptional divisor of $\widetilde{Z}^m_{U^m} \to Z^m_{U^m}$. Let $\gamma_{U^m}: \widetilde{Z}^m_{U^m} \to Z_U$. Let

$$\Psi: \bigoplus^N \mathscr{O}_{Z_U} \to F_U \tag{3.3}$$

be the restriction to Z_U of the universal quotient family. F_U is a flat family of torsion free sheaves. By pullback via γ_{U^m}, we get

$$\gamma^*_{U^m}(\Psi): \bigoplus^N \mathscr{O}_{\widetilde{Z}^m_{U^m}} \to \gamma^*_{U^m} F_U. \tag{3.4}$$

Since $\widetilde{Z}^m_{U^m}\backslash\Xi_{U^m} \to Z_U\backslash\Sigma\times_C U$ is finite and flat, the restriction of (3.4) to $\widetilde{Z}^m_{U^m}\backslash\Xi_{U^m}$ is a flat family of torsion free quotient sheaves. We seek to find a new sheaf E_{U^m} that is a "modification" of $\gamma^*_{U^m}F_U$ along Ξ_{U^m}.

Definition 3.1. A sheaf E_{U^m} on $\widetilde{Z}^m_{U^m}$ is called a good modification of $\gamma^*_{U^m}F_U$ along Ξ_{U^m} if there is a homomorphism

$$\widetilde{\Psi}\colon \bigoplus^N \mathscr{O}_{\widetilde{Z}^m_{U^m}} \to E_{U^m} \tag{3.5}$$

such that the following hold:

1. The restrictions of (3.5) and (3.4) to $\widetilde{Z}^m_{U^m}\backslash\Xi_{U^m}$ are isomorphic.
2. E_{U^m} is a family of torsion free sheaves on $\widetilde{Z}^m_{U^m}$ flat over U^m.
3. For any closed $u\in\widetilde{U}^m$ over $0\in C$, $\Psi_u\colon \bigoplus^N\mathscr{O}_{\widetilde{Z}^m_u}\to E_u$ is surjective at the generic points of $\widetilde{Z}^m_0$.
4. E_{U^m} admits a length-two locally free resolution at each point.

The main goal of this section is to construct good modification of $\gamma^*_{U^m}F_U$. Intuitively, the modification E_{U^m} will be constructed as follows: We take $\mathscr{F}$ as the restriction of $\gamma^*_{U^m}F_U$ to the compliment of the exceptional divisor of $\widetilde{Z}^m_{U^m}\to Z^m_{U^m}$. We extend $\mathscr{F}$ to an open $V\xrightarrow{i}\widetilde{Z}^m_{U^m}$ so that V contains the generic points of the exceptional divisor Ξ_{U^m} and so that $\gamma^*_{U^m}(\Psi)$ extends to V and $\mathscr{F}$ is generated by the image of $\bigoplus^N\mathscr{O}_{\widetilde{Z}^m_{U^m}}$ on V. Since $\widetilde{Z}^m_{U^m}$ is smooth, we can continue to extend $\mathscr{F}$ to all $\widetilde{Z}^m_{U^m}$ via i_*. Then by restricting to a smaller U if necessary, we obtain the desired modification. In the following, we give the details of this construction.

We first study a special situation. Let $k\supseteq\mathbb{C}$ be any field, and let $R\supseteq k$ be a discrete valuation ring with maximal ideal $\mathfrak{m}$ generated by a uniformizing parameter t. Let K be the field of fractions of R. We assume that there is a flat morphism $\operatorname{Spec}R\to C^m$. Let $\widetilde{Z}^m_R=\widetilde{Z}^m_{C^m}\times_C\operatorname{Spec}R$ be the product scheme. We denote the generic fiber by $\widetilde{Z}^m_K$, and the closed fiber of $\widetilde{Z}^m_R$ over $\operatorname{Spec}R$ by $\widetilde{Z}^m_k$. We also let i be the open immersion $\widetilde{Z}^m_K\to\widetilde{Z}^m_R$, and let j be the closed immersion $\widetilde{Z}^m_k\to\widetilde{Z}^m_R$. We first prove the following lemma.

Lemma 3.2. *Let E_K be a torsion free sheaf on $\widetilde{Z}^m_K$, and let L_R be a locally free sheaf on $\widetilde{Z}^m_R$ so that there is a surjective homomorphism*

$$\Psi_K\colon \bigoplus^N i^*L_R\to E_K \tag{3.6}$$

over $\widetilde{Z}^m_K$. Then there exist a unique coherent sheaf $E_R\subseteq i_*E_K$ on $\widetilde{Z}^m_R$

flat over $\operatorname{Spec} R$ *and a unique extension of* Ψ_K,

$$\Psi_R\colon \bigoplus^N L_R \to E_R \tag{3.7}$$

such that the following hold:

(1) $i^* E_R = E_K$, *and* $E_{k|\Delta_i} = (j^* E_R)_{|\Delta_i}$ *are torsion free.*

(2) $j^*\Psi_R\colon \bigoplus^N j^* L_R \to E_k$ *is surjective at the generic points of* $\widetilde{Z}_k^m$.

(3) E_R *admits length-*2 *locally free resolution everywhere.*

Proof. We first assume k is algebraically closed. Let $\Sigma_1, \cdots, \Sigma_m \subseteq \widetilde{Z}_k^m$ be the smooth curves in the singular locus of $\widetilde{Z}_k^m$. Then by [18, p. 100], there is a unique coherent sheaf E_R' on $\widetilde{Z}_R^m \setminus \bigcup^m \Sigma_i$ flat over $\operatorname{Spec} R$ and a unique extension of Ψ_K:

$$\Psi_R'\colon \bigoplus^N L_{R|\widetilde{Z}_R^m \setminus \cup \Sigma_i} \to E_R' \tag{3.8}$$

such that (1), (2), and (3) of the lemma hold.

Now let η be the open immersion $\widetilde{Z}_R^m \setminus \bigcup \Sigma_i \hookrightarrow \widetilde{Z}_R^m$ and let $E_R = \eta_* E_R'$. Clearly E_R is torsion free. Because $\widetilde{Z}_R^m$ is smooth and $\bigcup^m \Sigma_j \subset \widetilde{Z}_R^m$ is a codimension 2 subvariety, $E_R = \eta_* E_R'$ is coherent. Next, the existence of Ψ_R follows easily from the fact that $L_R = \eta_*(L_{R|\widetilde{Z}_R^m \setminus \bigcup \Sigma_i})$. Now we show that $(j^* E_R)_{|\Delta_i}$ is torsion free. Assume $E_{k|\Delta_i}$ has torsion elements at $x \in \Delta_i$. Let w be (local) parameter so that $\{w = 0\}$ defines Δ_i at x. Since E_R'/wE_R' is torsion free, $x \in \Sigma_{i-1} \cup \Sigma_i$. We assume $x \in \Sigma_i$. Then there is $f \in \mathfrak{m}_{\widetilde{Z}_R^m, x}$ such that the germ $\{f = 0, w = 0\}$ is contained in Σ_i as set and such that there are $u, v \in E_{R,x}$, $[v] \neq 0 \in E_{R,x}/wE_{R,x}$, and $fv = wu$. Thus $(u/f)_{|Z_R \setminus \Sigma_i} = (v/t)_{|Z_R \setminus \Sigma_i}$ as elements in $E_{R,x}'$. Therefore,

$$u/f = v/w \in \eta_* E_R' = E_R$$

and $v = w \cdot v/w = 0 \in E_{R,x}/wE_{R,x}$. So $(j^* E_R)_{|\Delta_i}$ is torsion free, and (1) is established. In particular, this argument shows that $\operatorname{depth} E_{R,x} \geq 2$ for any $x \in \widetilde{Z}_R^m$. Thus E_R admits length-2 locally free resolution everywhere [22]. Finally, the uniqueness follows from the uniqueness of E_R' and (1).

Now we assume k is any field. Let $\overline{k}$ be the algebraic closure of k, let $\overline{R} = R \times_k \overline{k}$, and let $\overline{K}$ be the fractional field of $\overline{R}$. Let $E_{\overline{K}} = E_K \otimes_K \overline{K}$ and let $L_{\overline{R}} = L_R \otimes_k \overline{k}$. If we apply the previous argument to

$$\Psi_{\overline{K}} = \Psi_K \otimes_K \overline{K}\colon \bigoplus^N L_{\overline{R}} \to E_{\overline{K}}, \tag{3.9}$$

then there is a unique extension sheaf $E_{\overline{R}}$ on $\tilde{Z}^m_{\overline{R}}$ flat over $\operatorname{Spec}\overline{R}$ and unique extension homomorphism $\Psi_{\overline{R}}$ such that (1), (2), and (3) of the lemma hold. We claim that there is a sheaf E_R on $\tilde{Z}^m_R$ and a homomorphism $\Psi_R\colon \bigoplus^N L_R \to E_R$ such that

$$E_{\overline{R}} = E_R \otimes_k \overline{k}, \qquad \Psi_{\overline{R}} = \Psi_R \otimes_k \overline{k}. \tag{3.10}$$

Indeed, let $\sigma \in \operatorname{Gal}(\overline{k}/k)$, and let $\sigma_z = 1_{\tilde{Z}^m_R} \otimes \phi\colon \tilde{Z}^m_R \times_k \operatorname{Spec}\overline{k} \to \tilde{Z}^m_R \times_k \operatorname{Spec}\overline{k}$ be the morphism, where $\phi\colon \operatorname{Spec}\overline{k} \to \operatorname{Spec}\overline{k}$ is defined by taking σ^{-1} as the homomorphism $\phi^*\colon \overline{k} \to \overline{k}$. Then there are canonical isomorphisms $\sigma_z^* E_{\overline{K}} = E_{\overline{K}}$ and $\sigma_z^* \Psi_{\overline{K}} = \psi_{\overline{K}}$. By the uniqueness of the extension,

$$\sigma_z^* E_{\overline{R}} = E_{\overline{R}} \quad \text{and} \quad \sigma_z^* \Psi_{\overline{R}} = \Psi_{\overline{R}}.$$

Since $E_{\overline{R}} \subseteq (i_* E_K) \otimes_k \overline{k}$, by descent theory, there is an E_R on Z_R and a $\Psi_R\colon \bigoplus^N L_R \to E_R$ such that (3.10) holds. Finally, (1), (2), and (3) hold for E_R and Ψ_R since they hold for $E_R \otimes_k \overline{k}$ and $\Psi_R \otimes_k \overline{k}$. The uniqueness follows from the flatness of E_R, (1) and the fact that $\Psi_R\colon \bigoplus^N L_R \to E_R$ is surjective at generic points of Z_k. We shall leave the details to the reader. q.e.d.

Remark. If we replace $\tilde{Z}^m_R$ by $Z_R = Z \times_C \operatorname{Spec} R$, then since Z_R has only normal singularities along $\Sigma \subseteq Z_k$, the argument is valid and the resulting sheaf $\eta_* E'_R$ still satisfies (1) and (2) of the lemma.

Now we are ready to prove the main result of this section.

Proposition 3.3. *Let $\mathscr{X} \subseteq \mathfrak{U}_0^{ss}$ be any irreducible component, let $\overline{m}$ be the multiplicity of $\mathscr{X} \subseteq \mathfrak{U}_0^{ss}$, and let $m > 0$ be divisible by $\overline{m}$. Then there is a smooth open subset $U \subseteq \mathfrak{U}^{ss}$, $U_0 = U \cap \mathfrak{U}_0^{ss} \neq \varnothing \subseteq \mathscr{X}$, having the following properties: Let $\Psi\colon \bigoplus^N \mathscr{O}_{Z_U} \to F_U$ be the pullback of the universal quotient sheaf on Z_U, and let $U^m \to U$ be the branched covering as in (3.2). Then there is a good modification $\tilde{\Psi}\colon \bigoplus^N \mathscr{O}_{\tilde{Z}^m_{U^m}} \to E_{U^m}$ of the pullback $\gamma^*_{U^m}(\Psi)\colon \bigoplus^N \mathscr{O}_{\tilde{Z}^m_{U^m}} \to \gamma^*_{U^m} F_U$ along Ξ_{U^m}. Further, for any closed $u \in U^m$ over $0 \in C^m$, the restriction of $\tilde{\Psi}\colon \bigoplus^N \mathscr{O}_{\tilde{Z}^m_{U^m}} \to E_{U^m}$ to the generic point of $\Sigma_m \subseteq \tilde{Z}^m_{U^m}$ is surjective. Finally, there is a large m, $\overline{m}|m$, so that E_u is locally free at $\Sigma_m \times_{C_m} U^m \subseteq \tilde{Z}^m_{U^m}$.*

Proof. We let $U^m \to U$ be the base change so that $\rho_m\colon U^m \to C^m$ is smooth. Let K be the field of rational functions over U^m. There is a canonical discrete valuation $v\colon K\setminus\{0\} \to \mathbb{Z}$ so that $v(\tilde{t}) = 1$, where $\tilde{t} = \rho_m^*(t_m)$, and t_m is the uniformizing parameter of C^m. Let $R = \{v \geq 0\}$

and let $k = \{v = 0\}$. v is a valuation of K/k.

Clearly, $\operatorname{Spec} R$ is flat over C^m. Consider $\operatorname{Spec} R \to U^m$, $\eta_R\colon \widetilde{Z}_R^m = \widetilde{Z}_{C^m}^m \times_{C^m} \operatorname{Spec} R \to \widetilde{Z}_{U^m}^m$, and $\eta_K\colon \widetilde{Z}_K^m \to \widetilde{Z}_{U^m}^m \backslash \Xi_{U^m}$. We obtain homomorphism of sheaves

$$\Psi_K\colon \bigoplus^N \mathscr{O}_{\widetilde{Z}_K^m} \to E_K, \tag{3.11}$$

where each of them is a pullback via η_K. Clearly, (3.11) satisfies the condition of Lemma 3.2. Therefore there is a unique flat extension E_R of E_K and an extension Ψ_R of Ψ_K such that (1), (2), and (3) of Lemma 3.1 hold.

Now, by general tautology, we can find an open set $\widetilde{U} \subseteq U$, where $\widetilde{U} \cap \mathfrak{U}_0^{ss}$ is dense in $U \cap \mathfrak{U}_0^{ss}$ such that over $\widetilde{Z}_{C^m}^m \times_{C^m} \widetilde{U}^m$, there is an extension sheaf $E_{\widetilde{U}_m}$, an extension homomorphism

$$\widetilde{\Psi}\colon \bigoplus^N \mathscr{O}_{\widetilde{Z}_{\widetilde{U}^m}^m} \to E_{\widetilde{U}^m} \tag{3.12}$$

with the desired properties. Indeed, if we denote by $\eta_R'\colon \widetilde{Z}_R^m \to \widetilde{U}^m$ the obvious morphism, then $\eta_R'^* E_{\widetilde{U}_m} = E_R$ and $\eta_R'^* \widetilde{\Psi} = \Psi_R$. The statement that $\widetilde{\Psi}_u\colon \bigoplus^N \mathscr{O}_{\widetilde{Z}_u^m} \to E_u$, $u \in U^m$, is surjective at the generic points of Σ_m will be proved in Proposition 4.1. Here we remark that in doing so, we may have to further shrink $\widetilde{U}$.

To prove the last statement, we first consider the following situation. Suppose that E_m is a family of torsion free sheaves on $\widetilde{Z}_{C^m}^m$ flat over C^m, and that $\Psi\colon \bigoplus^N \mathscr{O}_{\widetilde{Z}_{C^m}^m} \to E_m$ is surjective at $\widetilde{Z}_{C^m}^m \backslash \widetilde{Z}_{C^m,0}^m$, surjective at the generic point of $\widetilde{Z}_{C^m,0}^m$ and surjective at the generic points of $\Sigma_m \subseteq \widetilde{Z}_{C^m,0}^m$. Let $C^{lm} \to C^m$ be a branched covering with only ramification point $0 \in C^{lm}$ ramified over $0 \in C^m$ of index l and $\phi_l\colon \widetilde{Z}_{C^{lm}}^{lm} \to \widetilde{Z}_{C^m}^m$. We claim that for some large l, the modification of $\phi_l^*(\Psi)\colon \bigoplus^N \mathscr{O}_{\widetilde{Z}_{C^{lm}}^{lm}} \to \phi_l^*(E_m)$ (along the exceptional divisor of ϕ_l) constructed by Lemma 3.2, say E_{lm}, is locally free at $\Sigma_{lm} \subseteq \widetilde{Z}_{C^{lm},0}^{lm}$.

We first study the case when $l = 2$. Let $p' \in \Sigma_{2m}$ be any closed point and let $p \in \Sigma_m$ be the closed point under p' via ϕ_2. We will prove that

$$\operatorname{col}(E_{2m|\Delta_{2m}})_{p'} \le \max\{\operatorname{col}(E_{m|\Delta_m})_p - 1, 0\}. \tag{3.13}$$

(For definition of col, see §0.) Indeed, let (x, y, z) be a local coordinate of an analytic neighborhood $V \subseteq \tilde{Z}^m_{C^m}$ of p so that

$$xy = t_m, \quad \{y=0\} \subseteq \Delta_m, \quad \{x=0\} \subseteq \Delta_{m-1},$$
$$\{x=0, y=0, z=0\} = \{p\}.$$

Then there is a local coordinate (x', y', z') of an analytic neighborhood $V' \subseteq \tilde{Z}^{2m}_{C^{2m}}$ of p', $V' \subseteq \phi_2^{-1}(V)$, such that

$$x'y' = t_{2m}, \quad \{y'=0\} \subseteq \Delta_{2m}, \quad \{x'=0\} \subseteq \Delta_{2m-1},$$
$$\{x'=0, y'=0, z'=0\} = \{p'\},$$

and such that the map $\phi_2: \tilde{Z}^{2m}_{C^{2m}} \to \tilde{Z}^m_{C^m}$ is given locally by $\phi_2(x', y', z') = (x', x'y'^2, z')$ [2]. Without loss of generality, we can assume $E_{m|\Delta_m \cap V}$ is locally free away from p. Let $V_- = V \backslash \{p\}$ and $V'_- = \phi_2^{-1}(V_-) \cap V'$. Since Ψ_m is surjective at the generic points of Σ_m, we can assume Ψ_m is surjective on $V_- \cap \Delta_m$. Thus $E_{2m|V'_-} = \phi_2^*(E_m)_{|V'_-}$, and by the proof of Lemma 3.2, $E_{2m|V'} = \eta_*(\phi_2^*(E_m))$, where η is the inclusion $V'_- \hookrightarrow V'$. Clearly, if E_m is locally free at p, then $E_{2m|V'} = \phi_2^*(E_m)_{|V'}$ is locally free at p'. Now we assume E_m is not locally free at p. Let

$$0 \to \mathscr{O}_V^{\oplus h} \xrightarrow{A} \mathscr{O}_V^{\oplus(h+r)} \xrightarrow{B} E_{m|V} \to 0$$

be a locally free resolution of $E_{m|V}$ provided by Lemma 3.2. Then over V'_-, $E_{2m|V'_-}$ has a locally free resolution

$$0 \to \mathscr{O}_{V'_-}^{\oplus h} \xrightarrow{\phi^*(A)} \mathscr{O}_{V'_-}^{\oplus(h+r)} \xrightarrow{\phi^*(B)} E_{2m|V'_-} \to 0,$$

where $\phi_2^*(A) = A \circ \phi_2$. Since $E_{m,p}$ is not locally free, there is $f \in \mathscr{O}^{\oplus h}_{V,p}$, $f \notin \mathfrak{m}^{\oplus h}_{V,p}$, such that $A(f) \in \mathfrak{m}^{\oplus(h+r)}_{V,p}$. We write $A(f) = xM_1 + yM_2 + zM_3$ where $M_1, M_2, M_3 \in \mathscr{O}^{\oplus(h+r)}_{V,p}$. Notice

$$(\phi_2^* A)(f) = x'M_1 + x'y'^2 M_2 + z'M_3 = x'(M_1 + y'^2 M_2) + z'M_3.$$

Since $\{x'=0, z'=0\} \subseteq \phi_2^{-1}(p)$, the image of the meromorphic section

$$(M_1 + y'^2 M_2)/z' = -M_3/x'$$

via $\phi_2^*(B)$ in E_{2m} is regular over V'_-. Because $E_{2m|V} = \eta_*(E_{2m|V'_-})$, this section extends to a regular section in $E_{2m|V'}$. Now we define F_{2m} on V' as the cokernel

$$\mathscr{O}_{V'}^{\oplus h} \oplus \mathscr{O}_{V'} \oplus \mathscr{O}_V \xrightarrow{\tilde{A}} \mathscr{O}_{V'}^{\oplus(h+r)} \oplus \mathscr{O}_{V'} \xrightarrow{\tilde{B}} F_{2m} \to 0, \tag{3.14}$$

where

$$\widetilde{A} = \begin{pmatrix} \phi^* A & 0 \\ M_1 + y'^2 M_2 & z' \\ M_3 & -x' \end{pmatrix}.$$

It is straightforward to check that there is a commutative diagram

(3.15)

$$\begin{tikzcd} \mathcal{O}_{V',p'}^{\oplus h} \arrow[r, "\phi_2^*(A)"] \arrow[d, "(1,0,0)"] & \mathcal{O}_{V',p'}^{\oplus(h+r)} \arrow[r, "\phi_2^*(B)"] \arrow[d, "(1,0)"] & \phi_2^*(E_m)_{p'} \arrow[r] \arrow[d, phantom, "\cap"] & 0 \\ \mathcal{O}_{V',p'}^{\oplus h} \oplus \mathcal{O}_{V',p'}^{\oplus 2} \arrow[r, "\widetilde{A}"] & \mathcal{O}_{V',p'}^{\oplus(h+r)} \oplus \mathcal{O}_{V',p'} \arrow[r, "\widetilde{B}"] & F_{2m,p'} \arrow[r] & 0. \end{tikzcd}$$

Clearly, $\phi_2^*(E_m)_{p'} \subsetneq F_{2m,p'} \subseteq E_{2m,p'}$ and further, by studying the restriction to Δ_{2m} of (3.15), we have

$$(\phi_2^*(E_m)_{|\Delta_{2m}})_{p'} \cong (E_{m|\Delta_m})_p, \qquad (\phi_2^*(E_m)_{|\Delta_{2m}})_{p'} \subsetneq (F_{2m|\Delta_{2m}})_{p'}.$$

Therefore we must have (3.13).

To finish the proof of the proposition, we argue as follows. Let l be the integer so that for any closed $\xi \in U_0^m$ and any $p \in \Sigma_m \times_{C^m} \{\xi\}$, we have $\operatorname{col}(E_{U^m|\Delta_m \times_{C^m}\{\xi\}})_p \leq l$. By shrinking U if necessary, we can assume $E_{U^{2^l m}}$ is a good modification on $\widetilde{Z}^{2^l m}_{U^{2^l m}}$. Then by induction on l, we can show that by further shrinking U if necessary, $E_{U^{2^l m}}$ is locally free at $\Sigma^{2^l m} \times_{C^{2^l m}} U^{2^l m}$. We leave the details to the reader. q.e.d.

We will call E_{U^m} a proper transform of sheaves over $U \subseteq \mathfrak{U}^{ss}$. In general, a sheaf E on $\widetilde{Z}_0^m$ is said to be a proper transform of an H-semistable sheaf F on Z_0 if there is a smooth curve S flat over C, $s_0 \in S$ over $0 \in C$, a C-morphism $g\colon S \to \mathfrak{M}^d$, $g(s_0) = \{F\}$ so that if we let F_S be the sheaf on Z_S that is the pullback of the universal quotient family and let E_S be the good modification on $\widetilde{Z}^m_{C^m} \times_C S$ constructed in Proposition 3.3, then $E = E_S \otimes k(s_0)$.

Corollary 3.4. *Every closed point of $\mathfrak{M}_0^d$ has proper transforms.*

In the next section, we will concentrate on studying the geometry of the proper transform E_{U^m}. More specifically, we will give a quite explicit description of the distribution of the first and second Chern classes of the sheaves $E_{|\Delta_j}$ for $j = 0, \cdots, m$ and will study the stability of the restriction of E to Δ_m.

4. The Chern classes of $E_{|\Delta_i}$

From now on, we will work with a special degeneration Z that suits our purpose of studying the degeneration of moduli of vector bundles over X. First we explain how to construct our Z. Let C be a Zariski neighborhood of $0 \in \operatorname{Spec}(\mathbb{C}[t])$. During our discussion, we will feel free to replace C by a smaller neighborhood $0 \in C' \subseteq \operatorname{Spec}(\mathbb{C}[t])$. We form a smooth threefold Z_C over C by blowing up $X \times C$ along the subvariety $\Sigma \times \{0\}$, where $\Sigma \in |H|$ is a smooth divisor and H is a very ample line bundle on X. Let $\pi_C\colon Z_C \to C$ be the projection. Then $Z_0 = \pi^{-1}(0) = X \cup \Delta$, where Δ is a ruled surface. Next, we need to choose an appropriate ample divisor on Z. Let $p_X\colon Z \to X$ be the projection. For any rational number $\varepsilon = p/q$, $0 < \varepsilon < 1/2$, we form a divisor $p_X^* H^{\otimes q}(-(q-p)\Delta)$ on Z. For convenience, we will abbreviate the $\mathbb{Q}$-divisor $p_X^* H(-\frac{q-p}{q}\Delta)$ to $H(\varepsilon)$. Clearly $H(\varepsilon)$ is relative ample over C. In the sequel, we will write $H(\varepsilon)^{\otimes n}$ freely, and understand that by $H(\varepsilon)^{\otimes n}$ we always mean $n \in \mathbb{Z}^+$ so that $n \cdot \varepsilon \in \mathbb{Z}$. Let I be a line bundle over X. Following §2, we construct the degeneration $\mathfrak{M}^{d, H(\varepsilon)}$ $(= \mathfrak{M}^{d,\varepsilon})$ based on the threefold $Z \to C$, ample divisor $H(\varepsilon)$, line bundle $p_X^* I$ over Z and the Hilbert polynomial (depending on r and d)

$$\chi(n) = \frac{r}{2} n^2 (H \cdot H) + n((H \cdot I) - \frac{r}{2}(H \cdot K_X)) + (r-1)\chi(\mathscr{O}_X) + \chi(I) - d. \tag{4.1}$$

In the following, we will restrict ourselves to $r = 2$. We first study the distribution of the first Chern classes of a proper transform E along Δ_i. Let $F(n) \in \mathfrak{U}_0^{ss}$ be any $H(\varepsilon)$-semistable sheaf on Z_0 and let E be a sheaf on $\widetilde{Z}_0^m$ that is a proper transform of F. By Lemma 1.6, there is an exact sequence over Z_0,

$$0 \to F \to F^{(1)} \oplus F^{(2)} \to F^{(0)} \to 0, \tag{4.2}$$

where $F^{(1)}$ (resp. $F^{(2)}$) is the torsion free part of $F_{|X}$ (resp. $F_{|\Delta}$), and $F^{(0)}$ is a sheaf of $\mathscr{O}_\Sigma$-modules with $(F^{(0)})_\xi = \mathscr{O}_{\Sigma,\xi}^{\oplus r_0}$, $0 \le r_0 \le 2$, at the generic point $\xi \in \Sigma$. Recall that $\det F \approx I_0$, $I_0 = p_X^* I_{|Z_0}$. Thus there are integers a_1, a_2 such that $\det F^{(1)} = I(a_1\Sigma)$ and $\det F^{(2)} = I_{0|\Delta}(a_2\Sigma)$. Since F is semistable and $F^{(1)}$ is the quotient sheaf of F, we have $p_F \preceq p_{F^{(1)}}$ (resp. $p_F \preceq p_{F^{(2)}}$). Note that with the choice of $H(\varepsilon)$, $\alpha = (\varepsilon^2, 1 - \varepsilon^2)$. So $p_F = \chi / r$ and

$$p_{F^{(1)}} = \frac{1}{2\varepsilon^2}[n^2 \varepsilon^2 (H \cdot H) + n\varepsilon \cdot H(a_1\Sigma + I - K_X) + \chi(F^{(1)})].$$

Therefore by $H(\varepsilon)$-semistability, we have

$$a_1 \geq (1-\varepsilon)\frac{H\cdot(K_X - I)}{(H\cdot H)}. \tag{4.3}$$

Similarly, since $F^{(2)}$ is the quotient sheaf of F, we have $p_F \preceq p_{F^{(2)}}$. Thus we have

$$a_2 \geq -(1-\varepsilon)\frac{H\cdot(K_X - I)}{(H\cdot H)} - 2. \tag{4.4}$$

Further by (1.4), $a_1 + a_2 + 2 = r_0$. Since H is a very ample divisor on X, we may and will assume throughout this paper that

$$(H\cdot H) \geq 18|(K_X\cdot H)| + 18|(I\cdot H)|. \tag{4.5}$$

Therefore, $2 \geq a_1 \geq 0$ and $0 \geq a_2 \geq -2$. The sheaf F is said to be of type I if $r_0 = 2$, and type II (resp. III) if $r_0 = 1$ (resp. $r_0 = 0$). We remark that type III may possibly occur only when $H\cdot(K_X - I) = 0$.

Now we will use this information to analyze the geometry of the sheaf E. Recall that $\Delta_0, \Delta_1, \cdots, \Delta_m$ are irreducible components of $\widetilde{Z}_0^m$ and that $\Sigma_i = \Delta_{i-1}\cap\Delta_i$. Let P_i be the generic fiber of Δ_i over Σ. We say that $E_{|\Delta_i}$ has generic fiber type (a_i, b_i) if the restriction sheaf $E_{|P_i} = \mathscr{O}(a_i)\oplus\mathscr{O}(b_i)$. Then $\det E_{|P_i} = \mathscr{O}(a_i+b_i)$. Clearly, the line bundle $\gamma^* H(\varepsilon)$, $\gamma\colon \widetilde{Z}_0^m \to Z$, satisfies

$$\gamma^* H(\varepsilon)_{|P_i} \cong \mathscr{O}_{P_i}, \qquad i = 1, \cdots, m-1. \tag{4.6}$$

Since E admits length-2 locally free resolution, $\det E$ is locally free and further by assumption on the determinant line bundle, there are integers $\alpha_1, \cdots, \alpha_m$ so that

$$\det E = I_0\left(\sum_{i=0}^{m}\alpha_i\Delta_i\right). \tag{4.7}$$

Here by abuse of notation, we denote by $I_0 = p_X^* I_{|\widetilde{Z}_0^m}$ the pullback line bundle on $\widetilde{Z}_0^m$ as well as on Z_0. We claim that

$$n_i = \alpha_{i-1} + \alpha_{i+1} - 2\alpha_i \geq 0 \quad \text{for } i = 1, \cdots, m-1. \tag{4.8}$$

Indeed, if we let $E(n) = E\otimes\gamma^* H(\varepsilon)^{\otimes n}$, then the homomorphism φ: $\bigoplus^N \mathscr{O}_{\widetilde{Z}_0^m} \to E(n)$, that is the restriction to $\widetilde{Z}_0^m$ of the homomorphism constructed in Proposition 3.3, is surjective at the generic points of Δ_i for $i = 0, \cdots, m$. In particular,if E has generic fiber type (a_i, b_i) on

Δ_i, then by (4.6), $a_i, b_i \geq 0$, $i = 1, \cdots, m-1$. Since $\det E(n)_{|P_i} = \mathcal{O}_{P_i}(a_i + b_i)$ and $a_i + b_i = a_{i-1} + \alpha_{i+1} - 2\alpha_i$, we have established (4.8).

Since E is a proper transform of F, $F_{|X\setminus\Sigma} \cong E_{|\Delta_0\setminus\Sigma}$ (resp. $F_{|\Delta\setminus\Sigma} \cong E_{|\Delta_m\setminus\Sigma_m}$). In fact, more is true. Since $\bigoplus^N \mathcal{O}_{Z_0}$ generates the sheaf $F(n)$, $\bigoplus^N \mathcal{O}_X$ (resp. $\bigoplus^N \mathcal{O}_\Delta$) generates the sheaf $F(n)^{(1)}$ (resp. $F(n)^{(2)}$). Hence the sheaf $F(n)^{(1)}$ (resp. $F(n)^{(2)}$) considered as sheaf on $\Delta_0 \subset \tilde{Z}_0^m$ (resp. $\Delta_m \subset \tilde{Z}_0^m$) is generated by $\bigoplus^N \mathcal{O}_{\tilde{Z}_0^m}$. By the way E was constructed, we conclude that

$$F^{(1)} \subseteq E_{|\Delta_0} \quad (\text{resp. } F^{(2)} \subseteq E_{|\Delta_m}) \tag{4.9}$$

as subsheaf on Δ_0 (resp. Δ_m).

Let $j_0\colon F^{(1)} \to E_{|\Delta_0}$ (resp. $j_m\colon F^{(2)} \to E_{|\Delta_m}$) be the inclusion. In the following, we will show that $\operatorname{coker}(j_0)$ (resp. $\operatorname{coker}(j_m)$) is a torsion sheaf supported at discrete point set of Δ_0 (resp. Δ_m). We will prove this by showing that $\det F^{(1)} = \det E_{|\Delta_0}$ and $\det F^{(2)} = \det E_{|\Delta_m}$.

Assume F is of type I, that is $\det F^{(1)} = I_0(\bar{n}_0\Delta)_{|X}$ and $\det F^{(2)} = I_0(\bar{n}_m X)_{|\Delta}$ with $\bar{n}_0 + \bar{n}_m = 0$. Further assume $\det E = I_0(\Sigma\alpha_i\Delta_i)$. Then by (4.9), we have

$$n_0 = \alpha_1 - \alpha_0 \geq \bar{n}_0, \qquad n_m = \alpha_{m-1} - \alpha_m \geq \bar{n}_m. \tag{4.10}$$

Therefore from (4.8) it follows that

$$0 = (\alpha_1 - \alpha_0) + \sum_{i=1}^{m-1}(\alpha_{i-1} - \alpha_{i+1} - 2\alpha_i) + (\alpha_{m-1} - \alpha_m) \geq \bar{n}_0 + \bar{n}_m = 0.$$

Hence all equalities in (4.8) and (4.10) hold. Thus, $\det F^{(1)} = \det E_{|\Delta_0}$, $\det F^{(2)} = \det E_{|\Delta_m}$, and $E_{|\Delta_i}$ are of generic fiber type $(0,0)$ for $i = 1, \cdots, m-1$.

Now assume F is of type II. Then $\det F^{(1)} = I_0(\bar{n}_0\Delta)_{|X}$ and $\det F^{(2)} = I_0(\bar{n}_m X)_{|\Delta}$ with $\bar{n}_0 + \bar{n}_m = -1$. Let n_0, n_m be defined by (4.10) and $n_1, n_2, \cdots, n_{m-1}$ be defined by (4.8). Then $\sum_{j=0}^m n_j = 0$, $n_i \geq 0$ for $i = 1, \cdots, m-1$ and $n_0 \geq \bar{n}_0$, $n_m \geq \bar{n}_m$. Since $\bar{n}_0 + \bar{n}_m = -1$, there is an $i_0 \in \{0, \cdots, m\}$ such that

$$n_i = \begin{cases} \bar{n}_i, & i \neq i_0, \\ \bar{n}_i + 1, & i = i_0. \end{cases}$$

Here we agree $\overline{n}_i = 0$ for $i = 1, \cdots, m-1$. We claim that $i_0 \neq 0$, m. Assume $i_0 = m$. Since F is of type II, there is a section $v \in H^0(Z_0, F(n))$ so that the restriction of v to Σ is nontrivial while the restriction of v to Δ is trivial. Indeed, the generic sections of the kernel of $H^0(Z_0, F(n)) \to H^0(\Delta, F(n)^{(2)})$ satisfy this condition (such sections exist follows from Theorem 1.10 by assuming n large). Then the section v will provide a section $\overline{v} \in H^0(\widetilde{Z}_0^m, E(n))$. Since $i_0 = m$, $\det F(n)^{(1)} = \det E(n)_{|\Delta_0}$. So $j_0 \colon F(n)^{(1)} \to E(n)_{|\Delta^0}$ is surjective at the generic point of Σ_1. Thus $\overline{v}_{|\Sigma_1}$ is nontrivial. On the other hand, since $i_0 = m$, $E(n)_{|\Delta_i}$ are of generic fiber type $(0, 0)$ for $i = 1, \cdots, m-1$. Hence $\overline{v}_{|\Sigma_i}$ are nontrivial for $i = 1, \cdots, m$. Thus $\overline{v}_{|\Delta_m}$ is nontrivial. This contradicts the fact that $v_{|\Delta}$ is trivial and $j_m \colon F^{(2)} \to E_{|\Delta_m}$ is isomorphic at generic point of Δ_m. So $i_0 \neq m$. Similarly, we can prove $i_0 \neq 0$. Hence $\det F^{(1)} = \det E_{|\Delta_0}$ and $\det F^{(2)} = \det E_{|\Delta_m}$.

The situation where F is of type III can be analyzed similarly. We will omit its detail but state it as a proposition.

Proposition 4.1. *Let F be any $H(\varepsilon)$-semistable sheaf on Z_0 and let E be a proper transform of F on $\widetilde{Z}_0^m$. Then there are canonical inclusions $j_0 \colon F^{(1)} \to E_{|\Delta_0}$ and $j_m \colon F^{(2)} \to E_{|\Delta_m}$ satisfying the following:*

(1) *j_0 (resp. j_m) is an isomorphism except at finitely many points of Δ_0 (resp. Δ_m).*

(2) *Assume $E_{|\Delta_i}$ are of generic fiber type (a_i, b_i) for $i = 1, \cdots, m-1$, then (a_i, b_i) can only be of the forms $(0, 0)$, $(0, 1)$, or $(1, 1)$. Moreover,*

$$\sum_{j=1}^{m-1}(a_i + b_i) = \begin{cases} 0, & F \text{ is of type I}, \\ 1, & F \text{ is of type II}, \\ 2, & F \text{ is of type III}. \end{cases}$$

Remark. Since $\bigoplus^N \mathscr{O}_{Z_0} \to F(n)$ is surjective, conclusion (1) of the proposition proves what we need in Proposition 3.3 when $r = 2$. For $r > 2$, the same argument works without any change. Since we only need $r = 2$ in the rest of this paper, we will leave the proof of the general case to the readers.

In the remainder of this section, we will study the distribution of the second Chern classes of E along Δ_i. Assume $\det E = I_0(\sum a_i \Delta_i)$. Without loss of generality, we can assume $a_0 = 0$. Then since $E_{|\Delta_i}$ are torsion free (Definition 3.1), we have the exact sequence

$$0 \to E \to \bigoplus_{i=0}^{m} E_{|\Delta_i} \to \bigoplus_{i=1}^{m} E_{|\Sigma_i} \to 0.$$

By using the Riemann-Roch theorem, we get

$$\begin{aligned}\chi(E) &= \sum_{i=0}^{m} \chi(E_{|\Delta_i}) - \sum_{i=1}^{m} \chi(E_{|\Sigma_i}) \\ &= \chi(E_{|\Delta_0}) + \sum_{i=1}^{m} (\chi(E_{|\Delta_i}) - \chi(E_{|\Sigma_i})) = \chi(\mathscr{O}_X) + \chi(I) - \sum_{i=0}^{m} c_2(E_{|\Delta_i}).\end{aligned}$$

Hence we have proved

Lemma 4.2. *Let E on $\tilde{Z}_0^m$ be a proper transform of F with $F(n) \in \mathfrak{U}_0^{ss}$. Then the sum of the second Chern classes $\sum_{i=0}^{m} c_2(E_{|\Delta_i})$ is equal to d.*

Now, we are going to derive an upper bound of $c_2(E_{|\Delta_m})$. We will show that there is a constant A_2 so that $c_2(E_{|\Delta_m}) \le d + A_2$. Here and in the following context, unless the contrary is mentioned, by a constant we always mean a constant that depends only on (X, H, I). In particular, it is independent of the choice of d and $0 < \varepsilon < \frac{1}{2}$. Our approach is as follows. Since $F^{(2)} \subseteq E_{|\Delta_m}$ has cokernel supported at a discrete point set, $c_2(E_{|\Delta_m}) \le c_2(F^{(2)})$. On the other hand, by (4.2), we have

$$\chi(F^{(1)}) + \chi(F^{(2)}) = \chi(F) + \chi(F^{(0)}). \tag{4.11}$$

Since $F^{(1)}$ is not far from being stable, we should have an upper bound of $\chi(F^{(1)})$. Thus if we can find a manageable lower bound of $\chi(F^{(0)})$, the bound of $c_2(F^{(2)})$ then follows immediately.

We will use e-stability defined in §0.

Lemma 4.3. *There is a constant e such that the sheaf $F^{(1)}$ is e-stable.*

Proof. Let Q be any rank-one quotient sheaf of $F^{(1)}$. Then Q is also a rank $(1, 0)$ quotient sheaf of F. Hence by semistability of F, $p_Q(n) \succeq p_E(n)$. Thus

$$(1/\varepsilon^2) \cdot \varepsilon H \cdot (c_1(Q) - \tfrac{1}{2} K_X) \ge \tfrac{1}{2}((I \cdot I) - (K_X \cdot H))$$

and we have

$$\det(Q) \ge \tfrac{1}{2}(1 - \varepsilon)(K_X \cdot H) + (\varepsilon/2)(I \cdot H). \tag{4.12}$$

On the other hand, by (4.4), $\det F^{(1)}$ can only possibly be I, $I(H)$, or $I(2H)$. Hence

$$\begin{aligned}\deg(Q) - \tfrac{1}{2}\deg(F^{(1)}) &\geq \tfrac{1}{2}(1-\varepsilon)(H \cdot K_X) + \tfrac{\varepsilon}{2}(I \cdot H) - \tfrac{1}{2}(I \cdot H) - (H \cdot H) \\ &> -3(H \cdot H).\end{aligned}$$

Thus for $e = 6(H \cdot H)$, $F^{(1)}$ is e-stable. q.e.d.

$e = 6(H \cdot H)$ will be fixed in the rest of this section.

Lemma 4.4. *There is a constant A such that for any e-stable rank-two locally free sheaf V to X with $\det V = I(aH)$, $|a| \leq 2$ and any sheaf of $\mathscr{O}_\Sigma$-modules Q that is a quotient sheaf of $V_{|\Sigma}$, we have*

$$\chi(Q) \geq -c_2(V) + A.$$

Proof. The proof is a modification of Bogomolov's theorem showing that the restriction of a μ-stable rank-2 vector bundle to any hyperplane curve of high degree is stable. We will prove the only nontrivial case where rank $Q = 1$.

First assume Q is a locally free $\mathscr{O}_\Sigma$-module. Then there is an exact sequence of sheaves $0 \to W \to V \to Q \to 0$ with W locally free on X. By using Riemann-Roch, one checks directly that $c_1(W) = c_1(V) - [\Sigma] = I + (a-1)[\Sigma]$ and

$$c_2(W) = c_2(V) + \chi(Q) + (g(\Sigma) - 1) - a(H \cdot H) - (I \cdot H). \tag{4.13}$$

Thus

$$\begin{aligned}4c_2(W) - c_1(W)^2 = 4\chi(Q) &+ 4c_2(V) + 4(g(\Sigma) - 1) - (a+1)^2(H \cdot H) \\ &- 2(a+1)(I \cdot H) - (I \cdot I).\end{aligned}$$

Assume that

$$\begin{aligned}\chi(Q) < &-c_2(V) + \tfrac{1}{4}(a+1)^2(H \cdot H) + \tfrac{1}{2}(a+1)(I \cdot H) \\ &+ \tfrac{1}{4}(I \cdot I) - (g\Sigma) - 1).\end{aligned}$$

Then W violates Bogomolov's inequality for stable bundle and is therefore unstable. So there is a rank-one destabilizing subsheaf $L \subseteq W$, W/L torsion free, such that

$$L \cdot H - \tfrac{1}{2}(a-1)(H \cdot H) - \tfrac{1}{2}(I \cdot H) > 0. \tag{4.14}$$

By the e-stability of V, we have

$$L \cdot H - \tfrac{1}{2}a(H \cdot H) - \tfrac{1}{2}(I \cdot H) < \tfrac{1}{2}e. \tag{4.15}$$

Finally, W belongs to the following exact sequence

$$0 \to L \to W \to I \otimes L^{-1}((a-1)\Sigma) \otimes \mathcal{J}_z \to 0,$$

where z is some zero scheme of X. Thus

$$c_2(W) = -L \cdot L + L \cdot I + (a-1)L \cdot H + l(z) \geq -L \cdot L + L \cdot I + (a-1)L \cdot H.$$

Combined with (4.13), we obtain

$$\chi(Q) \geq -c_2(V) - (g(\Sigma) - 1) + (H \cdot H) - L \cdot L + (I \cdot H) + (a-1)L \cdot H. \tag{4.16}$$

Next, by Hodge index theorem, $L \cdot L \leq (L \cdot H)^2/(H \cdot H)^2$. Thanks to (4.14), (4.15), $|L \cdot H|$ is bounded by a constant A'. Therefore, by (4.16) for some constant A, we have $\chi(Q) \geq -c_2(V) + A$. In general, when Q is not locally free, the previous argument shows that the locally free part Q^{fr} of Q has $\chi(Q^{fr}) \geq -c_2(V) + A$. Thus

$$\chi(Q) \geq \chi(Q^{fr}) \geq -c_2(V) + A. \quad \text{q.e.d.}$$

Corollary 4.5. *There is a constant A_1 so that $c_2(F^{(1)}) \geq A_1$.*

Proof. Let V be the double dual of $F^{(1)}$ and let $\det V = I(aH)$. Then $|a| \leq 2$. Assume $4c_2(V) - c_1(V)^2 < 0$. Then by Bogomolov's inequality, V is not stable. Let $L \subseteq V$ be the destabilizing subsheaf so that V/L is torsion free. Then $\deg V \leq \deg L \leq \deg V + \frac{1}{2}e$. Thus by the Hodge index theorem, $c_1(L) \cdot c_1(L)$ is bounded from above by a constant. Therefore,

$$c_2(V) \geq c_1(L)(I + aH - c_1(L)) \geq A_1$$

for some constant A_1. q.e.d.

Proposition 4.6. *There is a constant A_2 such that for any proper transform E of F with $F(n) \in \mathfrak{U}_0^{ss}$,*

$$c_2(E_{|\Delta_m}) \leq d + A_2.$$

Proof. It suffices to show that $c_2(F^{(2)}) \leq d + A_2$. Let S be the kernel of $(F^{(1)})_{|\Sigma} \to F^{(0)}$ and let F' be the cokernel of the composition

$$S \to (F^{(1)})|\Sigma \to (F^{(1)})^{\vee\vee}|\Sigma.$$

We have the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_1 \arrow[r] \arrow[d] & T_1 \arrow[r] \arrow[d] & T_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S \arrow[r] \arrow[d] & F^{(1)}|\Sigma \arrow[r] \arrow[d] & F^{(0)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_2 \arrow[r] \arrow[d] & (F^{(1)})^{\vee\vee}|\Sigma \arrow[r] \arrow[d] & F' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_1' \arrow[r] \arrow[d] & T_1' \arrow[r] \arrow[d] & T_2' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where S_2 is the kernel of the corresponding row exact sequence, and T_1, T_2, and S_1 (resp. T_1', T_2' and S_1') are the corresponding kernels (resp. cokernels) of the column sequences. Then the whole diagram is a commutative diagram, and each column as well as each row is exact. Hence

$$\begin{aligned}
\chi(F^{(0)}) &= \chi(F') + \chi(T_2) - \chi(T_2') \\
&= \chi(F') + (\chi(T_1) - \chi(S_1)) - (\chi(T_1') - \chi(S_1')) \\
&\geq (-c_2((F^{(1)})^{\vee\vee}) + A) + (\chi(S_1') - \chi(S_1)) \geq -c_2(F^{(1)}) + A.
\end{aligned}$$

Here the third inequality holds because of Lemma 4.4 and $\chi(T_1') = \chi(T_1)$ which can be seen as follows: $\chi(T_1') = \chi(T_1)$ if $\chi((F^{(1)})_{|\Sigma}) = \chi((F^{(1)})^{\vee\vee}|\Sigma)$. But the later two only depend on the degree of $F^{(1)}$ and $F(F^{(1)})^{\vee\vee}$ since both are torsion free. The last inequality holds since $c_2(F^{(1)}) \geq c_2((F^{(1)})^{\vee\vee}) + l(T_1')$, and $l(T_1') = l(T_1) \geq \chi(S_1)$. Also $\chi(S_1') \geq 0$. Therefore,

$$\begin{aligned}
-\chi(F^{(2)}) &= -\chi(0) - \chi(F^{(0)}) + \chi(F^{(1)}) \\
&\leq (d - \chi(\mathcal{O}_X) - \chi(I)) + (c_2(F^{(1)}) - A) + \chi(F^{(1)}) \leq d + A_2
\end{aligned}$$

for some constant A_2, and the proposition is established.

5. Irreducible components of $\mathfrak{M}^d$

In [19], the second author constructed a line bundle $\mathscr{L}_{\mathrm{m}}(H)$ on $\mathfrak{M}_X^{d,I}$, and showed that $\mathscr{L}_{\mathrm{m}}(H)$ is nef. and its top self-intersection number is positive. In particular, when $\mathfrak{M}_X^{d,I}$ has the expected dimension $c_d = 4d - 3\chi(\mathscr{O}_X) - I \cdot I$, the self-intersection number is positive, i.e.,

$$[\mathscr{L}_{\mathrm{m}}(H)]^{c_d}(\mathfrak{M}_X^{d,I}) > 0. \tag{5.1}$$

S. Donaldson first introduced this number (5.1) in a different category and showed that this intersection number is an invariant of the underlining smooth structure of the four manifold X when $p_g \geq 1$. In this paper, we will use (5.1) to deduce some properties of the degeneration $\mathfrak{M}^{d,\varepsilon}$. In short, we will show that there is a constant A_2 having the property that for any irreducible component $\mathfrak{N}^{d,\varepsilon} \subseteq \mathfrak{M}^{d,\varepsilon}$, there is an irreducible component $\mathfrak{X}$ of $\mathfrak{N}_0^{d,\varepsilon}$ such that for generic $F \in \mathfrak{X}$ and proper transforms E of F, $E_{|\Delta_m}$ is μ-stable and further, all such sheaves over Δ_m form an algebraic subset of dimension c_d in an appropriate moduli space of vector bundles over Δ_m.

We first construct the line bundle on $\mathfrak{M}^d$. We will briefly sketch the construction of this line bundle and outline some useful properties enjoyed by this line bundle. The full proof of these results can be found in [17], [19].

For any integer $r > 0$, let $D^r \subseteq Z$ be a divisor such that $\pi\colon D^r \to C$ is smooth, $D_t^r = \pi^{-1}(t) \in |rH|$ for $t \neq 0$ and $D_0^r \subseteq \Delta\backslash\Sigma$. We call such D^r good divisors in $|rH_C(-r\Delta)|$, where $H_C = p_X^* H$. Since H is very ample, the set of good divisors in $|rH_C(-r\Delta)|$ is base point free. Let $\mathrm{Pic}(D^r/C)$ be the relative Picard scheme. On $\mathrm{Pic}(D^r/C)$, there is a curve $\widetilde{C} \subseteq \mathrm{Pic}(D^r/C)$ consisting of line bundles L on D_t^r such that $L^{\otimes 2} \cong K_{D_t^r} \otimes p_X^* I^{-1}_{|D_t^r}$, which exists if $\Sigma \cdot I$ is even which can certainly be arranged in advance. Clearly, $\widetilde{C}$ is an étale covering of C. Then there is a line bundle $\widetilde{\theta}^r$ on $D_{\widetilde{C}}^r = \widetilde{C} \times_C D^r$ so that $(\widetilde{\theta}_v^r)^{\otimes 2} \cong K_{D_v^r} \otimes p_X^* I^{-1}_{|D_t^r}$ for any closed $v \in \widetilde{C}$, where $\widetilde{\theta}_v^r = (\widetilde{\theta}^r)_{|D_v^r}$ and D_v^r is the fiber of $D_{\widetilde{C}}^r$ over $v \in \widetilde{C}$.

We now construct a line bundle on $\mathfrak{U}_{\widetilde{C}}^{ss} = \widetilde{C} \times_C \mathfrak{U}^{ss}$ as follows. Let $\mathscr{F}(n)$ be the universal quotient family on $Z \times_C \mathfrak{U}^{ss}$. Since $\mathscr{F}$ is a family of torsion free sheaves on $Z \times_C \mathfrak{U}^{ss}$ flat over $\mathfrak{U}^{ss}$ and $D^r \cap \Sigma = \emptyset$, $\mathscr{F}|_{D^r}$ is flat over $\mathfrak{U}^{ss}$ also (see §0), where $\mathscr{F}_{|D^r}$ is the restriction of $\mathscr{F}$ to $D^r \times_C \mathfrak{U}^{ss}$. Let p_{12} (resp. p_{13}; resp. p_{23}) be projection $D^r \times_C \mathfrak{U}_{\widetilde{C}}^{ss} \to D_{\widetilde{C}}^r$ (resp. $D^r \times_C \mathfrak{U}_{\widetilde{C}}^{ss} \to$

$D^r \times_C \mathfrak{U}^{ss}$; resp. $D^r \times_C \mathfrak{U}^{ss}_{\widetilde{C}} \to \mathfrak{U}^{ss}_{\widetilde{C}})$. Note that p_{23} is smooth. Hence

$$R^{\cdot} p_{23*}(p_{13}^{*}(\mathscr{F}_{|D^r}) \otimes p_{12}^{*}\widetilde{\theta}^r) \tag{5.2}$$

is a perfect complex on $\mathfrak{U}^{ss}_{\widetilde{C}}$ [17]. Following [17], we can form the determinant line bundle

$$\det(R^{\cdot} p_{23*}(p_{13}^{*}(\mathscr{F}_{|D^r}) \otimes p_{12}^{*}\widetilde{\theta}^r)) \tag{5.3}$$

on $\mathfrak{U}^{ss}_{\widetilde{C}}$ of the complex (5.2). We have the following useful result.

Lemma 5.1. *Let $\mathscr{P}$ be a Poincaré line bundle on $D^r \times_C \mathrm{Pic}^{g_r-1}(D^r/C)$ and let $\mathscr{K} = \det(Rq_{2*}\mathscr{P})$, where q_2 is the projection onto $\mathrm{Pic}^{g_r-1}(D^r/C)$. Then there is a line bundle $\mathscr{L}$ on $\mathfrak{U}^{ss}$ such that*

$$\det(R^{\cdot} p_{23*}(p_{13}^{*}(\mathscr{F}_{|D^r}) \otimes p_{12}^{*}\widetilde{\theta}^r)) = p_1^{*}\mathscr{K}^{\otimes 2} \otimes p_2^{*}\mathscr{L}^{-1}, \tag{5.4}$$

where p_1 and p_2 are projections from $\mathfrak{U}^{ss}_{\widetilde{C}}$ onto $\widetilde{C} \subseteq \mathrm{Pic}^{g_r-1}(D^r/C)$ and $\mathfrak{U}^{ss}$ respectively, and g_r is the genus of the fibers of $D^r \to C$. We denote the line bundle $\mathscr{L}$ by $\mathscr{L}_U(D^r)$. Further, suppose D^r and $\widetilde{D}^r$ are two good divisors in $|rH_C(-r\Delta)|$. Then there is an isomorphism $\mathscr{L}_U(D^r) \cong \mathscr{L}_U(\widetilde{D}^r)$.

Proof. For technical reasons, we will show that the line bundle $\mathscr{L}_U(D^r)$ is the restriction to $\mathfrak{U}^{ss}_{\widetilde{C}}$ of a similar line bundle on $\mathfrak{R}^d_{\widetilde{C}}$. Let $\mathscr{F}'(n)$ be the restriction of the universal family to $(Z\backslash X) \times_C \mathfrak{R}^d$. Then there is a global length-3 locally free resolution of $\mathscr{F}'$. Namely, there are locally free sheaves $\mathscr{Q}_1$, $\mathscr{Q}_2$ and $\mathscr{Q}_3$ on $(Z\backslash X) \times_C \mathfrak{R}^d$ such that $\mathscr{F}'$ belongs to the exact sequence

$$0 \to \mathscr{Q}_3 \to \mathscr{Q}_2 \to \mathscr{Q}_1 \to \mathscr{F}' \to 0. \tag{5.5}$$

Thus similar to (5.3), we define line bundle

$$\left(\bigotimes_{i=1}^{3}(\det(R^{\cdot} p_{23*}(p_{13}^{*}(\mathscr{Q}_{i|D^r_C}) \otimes p_{12}^{*}\widetilde{\theta}^r)))^{-1})^i\right)^{-1} \tag{5.6}$$

on $\widetilde{C} \times_C \mathfrak{R}^d$. Since the determinant line bundle preserves triangle of perfect complexes and is independent of choice of resolution (5.5) [17], one concludes that the restriction of (5.6) to $\mathfrak{U}^{ss}_{\widetilde{C}}$ is canonically isomorphic to (5.3). Then by mimicking the proof of [19, Lemma 2.1], one shows that there is a line bundle $\mathscr{L}_{\mathbf{R}}(D^r)$ over $\mathfrak{R}^d$ so that (5.6) is isomorphic to $p_1^{*}\mathscr{K}^{\otimes 2} \otimes p_2^{*}\mathscr{L}_{\mathbf{R}}(D^r)^{-1}$. Thus the first part of the lemma is established.

Next, let D^r and $\widetilde{D}^r$ be any two good divisors in $|rH_C(-r\Delta)|$, and let $\mathscr{L}_{\mathbf{R}}(D^r)$ and $\mathscr{L}_{\mathbf{R}}(\widetilde{D}^r_C)$ be the corresponding line bundles just constructed.

In [19, Lemma 2.2], it was shown that for any closed $t \neq 0 \in C$,

$$\mathscr{L}_{\mathbf{R}}(D^r) \otimes k(t) \cong \mathscr{L}_{\mathbf{R}}(\tilde{D}^r) \otimes k(t). \tag{5.7}$$

Then by applying [13, III.12.9] to the locally free sheaf $\mathscr{L}_{\mathbf{R}}(D^r) \otimes \mathscr{L}_{\mathbf{R}}(\tilde{D}^r)^{-1}$ over $\mathfrak{R}^d \to C$ and adding the fact that $\mathfrak{R}^d$ is normal, projective and flat over C, we infer that

$$\mathscr{L}_{\mathbf{R}}(D^r) \cong \mathscr{L}_{\mathbf{R}}(\tilde{D}^r)(\mathbf{D}), \tag{5.8}$$

where $\mathbf{D} \subseteq \mathfrak{D}^d$ is a divisor contained in $\mathfrak{R}^d_0$. Finally, since the set of good divisors in $|rH_C(-r\Delta)|$ is an irreducible set, when $D^r = \tilde{D}^r$, the divisor $\mathbf{D}$ in (5.8) satisfies $\mathscr{O}(\mathbf{D}) = \mathscr{O}$. Thus, for any choice of D^r and $\tilde{D}^r$, $\mathscr{O}(\mathbf{D}) = \mathscr{O}$, and the lemma has been established. q.e.d.

In the sequel, we will denote the unique line bundle $\mathscr{L}_{\mathrm{U}}(D^r)$ by $\mathscr{L}_{\mathrm{U}}(r)$. Our next task is to analyze when the line bundle $\mathscr{L}_{\mathrm{U}}(r)$ descends to a line bundle on $\mathfrak{M}^d$. We need the following descent lemma of Kempf.

Lemma 5.2 (Descent Lemma). *Let $\mathscr{L}$ be a $\mathscr{G}$-line bundle on $\mathfrak{U}^{ss}$. $\mathscr{L}$ descends to $\mathfrak{M}^d$ if and only if for every closed point $w \in \mathfrak{U}^{ss}$ with closed orbit $\mathscr{G} \cdot \{w\}$, the stabilizer* $\operatorname{stab}(w) \subset \mathscr{G}$ *of w acts trivially on $\mathscr{L}_w = \mathscr{L} \otimes k(w)$.*

Proof. We can invoke Theorem 2.3 of [5] since $\mathfrak{M}^d$ is normal and is a good quotient of $\mathfrak{U}^{ss}$ by $\mathscr{G}$. q.e.d.

Proposition 5.3. *There is a function $\kappa : \mathbb{Z}^+ \to (0, 1/2)$ and a constant A having the following properties: For any $d \geq A$, there is a large r such that whenever $\varepsilon \in (0, \kappa(d))$, then the line bundle $\mathscr{L}_{\mathrm{U}}(r)$ descends to a line bundle on $\mathfrak{M}^d$.*

Proof. The proof is very much the same as [19, Proposition 1.7]. We give a sketch here. First note that with $\mathscr{G} = SL(N, \mathbb{C}) \times_{\mathbb{C}} C$, $\mathfrak{U}^{ss}$ is a $\mathscr{G}$-scheme. The action of $\mathscr{G}$ on $\mathscr{O}_Z^{\oplus N}$ induces a $\mathscr{G}$ action on the universal quotient sheaf $\mathscr{F}(n)$, and thus induces a $\mathscr{G}$ action on the line bundle $\mathscr{L}_{\mathrm{U}}(D^r)$.

To show the existence of the descent line bundle, thanks to the descent lemma, we only need to check that for any closed $w \in \mathfrak{U}^{ss}$ with closed orbit $SL(N, \mathbb{C}) \cdot \{w\}$, the stabilizer $\operatorname{stab}(w)$ acts trivially on $\mathscr{L}_{\mathrm{U}}(D^r)_w$. By [19], it suffices to check the case when $\mathscr{F}(n)_w$ is a quotient sheaf on Z_0. First, note that $\mathbb{Z}/N\mathbb{Z} \subseteq \operatorname{stab}(w)$ for any closed $w \in \mathfrak{U}^{ss}_0$. By using the fact that $\chi(\mathscr{F}_{w|D^r_v} \otimes \tilde{\theta}^r_v) = 0$ for any closed $w \in \mathfrak{U}^{ss}$ (and $v \in \tilde{C}$) over $t \in C$, one concludes that the $\mathbb{Z}/N\mathbb{Z}$ action on $\mathscr{L}_{\mathrm{U}}(D^r) \otimes k(w)$ is trivial [19]. Next, note that the only closed $w \in \mathfrak{U}^{ss}$ over $0 \in C$ that have closed orbits $SL(N, \mathbb{C}) \cdot \{w\}$ and stabilizers $\mathrm{stab}(w) \subsetneq \mathbb{Z}/N\mathbb{Z}$ are those $F(n)$'s where F are nontrivial direct sums of $H(\varepsilon)$-stable sheaves. Assume $F = J_1 \oplus \cdots \oplus J_r$, $r \geq 2$. By base change property [17],

$$\mathscr{L}_{\mathrm{U}}(D^r) \otimes k(w) = \left(\bigotimes_{i=0}^{1} \left(\bigwedge^{\mathrm{top}} H^i(D_0^r, (J_1 \oplus \cdots \oplus J_r)_{|D_0^r} \otimes \theta_0^r) \right)^{(-1)^i} \right)^{-1}, \tag{5.9}$$

where $(\theta_0^r)^{\otimes 2} = K_{D_0^r} \otimes I^{-1}_{0|D_0^r}$. Then following the proof of [19], for any $g \in \mathrm{stab}(w)$,

$$\det(g_{|D_0^r}) \colon \mathscr{L}_{\mathrm{U}}(D^r) \otimes k(w) \to \mathscr{L}_{\mathrm{U}}(D^r) \otimes k(w)$$

is the identity if $\chi(J_{j|D_0^r} \otimes \theta_0^r) = 0$ for all $j = 1, \cdots, r$. But this follows from Proposition 5.8.

It remains to show that the descended line bundle $\mathscr{L}_{\mathrm{m}}(D^r)$ is independent of the choice of $D^r \in |rH_C(-r\Delta)|$. It suffices to show that for two different D^r and $\tilde{D}^r_C$, the isommorphism (5.8) is $\mathscr{G}$ equivariant. But this follows immediately from the fact that the isomorphism (5.7) is $\mathscr{G}$ equivariant [19]. q.e.d.

We denote the descended line bundle on $\mathfrak{M}^d$ by $\mathscr{L}_{\mathrm{m}}(r)$, and denote its restriction to $\mathfrak{M}^d_t$ by $\mathscr{L}_{\mathrm{m}}(r)_t$. We remark that by [19], for $t \neq 0$, $\mathscr{L}_{\mathrm{m}}(r)_t$ exists for any $r \geq 1$.

In [4], S. Donaldson (see also [6, 34]) showed that for the algebraic surface X and the fixed divisors I and H, there is a constant A such that $\mathfrak{M}^{d,I}_X$ is smooth at the generic points when $d \geq A$. We fix such an A and assume that $d \geq A$ throughout the rest of this paper. Let $c_d = \dim \mathfrak{M}^{d,I}_X$. In the sequel, we fix an irreducible component $\mathfrak{N}^d \subseteq \mathfrak{M}^d$.

Lemma 5.4 (Donaldson). *Let $\mathfrak{N}^d$ be any irreducible component of $\mathfrak{M}^d$. Let $r \geq 1$. Then the self-intersection number is positive, i.e.,*

$$[\mathscr{L}_{\mathrm{m}}(r)_t]^{c_d}(\mathfrak{N}^d_t) > 0 \tag{5.10}$$

for any closed $t \neq 0 \in C$.

Proof. In [19], the second author of this paper showed that for sufficiently large q, $H^0(\mathfrak{N}^d_t, \mathscr{L}_{\mathrm{m}}(r)^{\otimes q}_t)$ is base point free and the morphism

$$\gamma : \mathfrak{N}_t^d \to \mathbf{P}(H^0(\mathfrak{N}_t^d, \mathscr{L}_{\mathrm{m}}(r)_t^{\otimes q})^\vee) \tag{5.11}$$

is a birational morphism onto the image $\gamma(\mathfrak{N}_t^d)$. So (5.10) follows immediately. q.e.d.

Now we intend to generalize (5.11) to a C-morphism $\gamma_C : \mathfrak{M}^d \to \mathbf{P}_C$. To accomplish this, we need to construct many $\mathscr{G}$ invariant sections of $\mathscr{L}_{\mathrm{U}}(r)^{\otimes q}$ over $\mathfrak{U}^{ss}$. (We always assume r to be so chosen so that $\mathscr{L}_{\mathrm{U}}(r)$ exists.) This will be done by using the technique of restricting sheaves to D^r. For any good divisor $D^r \in |rH_C(-r\Delta)|$, let $\mathfrak{M}(D^r/C)$ be the relative moduli scheme of rank-2 semistable vector bundles E on D^r/C with $\det E = p_X^* I_{|D^r}$. Since D^r is a family of smooth curves, $\mathfrak{M}(D^r/C) \to C$ is a flat family. Now let $\mathfrak{U}^{ss}[D^r]$ be the open subset of $\mathfrak{U}^{ss}$ consisting of closed points $u \in \mathfrak{U}^{ss}$ such that $\mathscr{F}_{u|D_t^r}$ is locally free and semistable, where u is over $t \in C$. By restricting $F \in \mathfrak{U}^{ss}[D^r]$ to D^r, we obtain a morphism

$$\Phi_{D^r} : \mathfrak{U}^{ss}[D^r] \to \mathfrak{M}(D^r/C). \tag{5.12}$$

If we view $\mathfrak{M}(D^r/C)$ as a $\mathscr{G}$ scheme with trivial $\mathscr{G}$ action, Φ_{D^r} is $\mathscr{G}$-equivalent.

Proposition 5.5 [Donaldson]. *There is a relative ample line bundle $\mathscr{L}_{D^r}$ on $\mathfrak{M}(D^r/C) \to C$ such that the following isomorphism holds over $\mathfrak{U}^{ss}[D^r]$:*

$$\Phi_{D^r}^*(\mathscr{L}_{D^r}) \cong \mathscr{L}_{\mathrm{U}}(D^r)|\mathfrak{U}^{ss}[D^r]. \tag{5.13}$$

Further, this isomorphism is $\mathscr{G}$-equivariant.

Proof. See [4, 6, 19]. q.e.d.

Take q large so that $\mathscr{L}_{D^r}^{\otimes q}$ is very ample and let $v \in H^0(\mathfrak{M}(D^r/C), \mathscr{L}_{D^r}^{\otimes q})$ be any section. Then $\Phi_{D^r}^*(v)$ is a $\mathscr{G}$ invariant section of $\mathscr{L}_{\mathrm{U}}(r)^{\otimes q}$ over $\mathfrak{U}^{ss}[D^r]$. We will show that the section $\Phi_{D^r}^*(v)$ can be extended to the scheme $\mathfrak{U}^{ss}$. We first state a technical lemma whose proof can be found in [20] (see also [19]).

Lemma 5.6. *Let S be a smooth curve over C, $s \in S$ over $t \in C$. Assume we have the following: a map $\rho : S \to \mathfrak{U}^{ss}$, $\rho(S^0) \subset \mathfrak{U}^{ss}[D^r]$, $S^0 = S \backslash s$, with a sheaf F_S on $Z \times_C S$ which is the pullback of the universal quotient sheaf by ρ, a section $v \in H^0(\mathfrak{M}(D^r/C), \mathscr{L}_{D^r}^{\otimes q})$, and v_{S^0} which is the pullback section $\rho^* \Phi_{D^r}^*(v)$ on S^0. Then v_{S^0} extends uniquely to a section $\tilde{v}_S \in H^0(S, \rho^* \mathscr{L}_{\mathrm{U}}(r)^{\otimes q})$. Further, $w \in \tilde{v}_S^{-1}(0)$ if and only if either $F_{w|D_t^r}$ is not semistable or $v(V_{w|D_t^r}) = 0$.*

Proposition 5.7. *With the notation as above, let $D^r \in |rH_C(-r\Delta)|$ be any good divisor and let $v \in H^0(\mathfrak{M}(D^r/C), \mathscr{L}_{D^r}^{\otimes q})$ be any section. Then the* *pullback section* $\Phi^*_{D^r}(v) \in H^0(\mathfrak{U}^{ss}[D^r], \mathscr{L}_U(r)^{\otimes q})$ *extends canonically over* $\mathfrak{U}^{ss}$ *to a* $\mathscr{G}$ *invariant section. We shall denote this extension by* $\Phi^*_{D^r}(v)_{\mathrm{ex}}$. *Further,*

$$\Phi^*_{D^r}(v)^{-1}_{\mathrm{ex}}(0) = (\mathfrak{U}^{ss}\backslash\mathfrak{U}^{ss}[D^r]) \cup \{F \in \mathfrak{U}^{ss}[D^r] | v(F_{|D^r}) = 0\}. \tag{5.14}$$

Proof. Let $\Phi^*_{D^r}(v)$ be the pullback section. Since $\mathfrak{U}^{ss}$ is normal, by Lemma 5.6, $\Phi^*_{D^r}(v)$ extends uniquely to a section v' on the closure of $\mathfrak{U}^{ss}[D_r]$ in $\mathfrak{U}^{ss}$, and also

$$\overline{(\mathfrak{U}^{ss}\backslash\mathfrak{U}^{ss}[D_r])} \bigcap \overline{\mathfrak{U}^{ss}[D_r]} \subseteq (v')^{-1}(0). \tag{5.15}$$

Thus, the extension of v' by zero is regular. We let $\Phi^*_{D^r}(v)_{\mathrm{ex}}$ be such an extension. Clearly, $\Phi^*_{D^r}(v)_{\mathrm{ex}}$ is canonical and is $\mathscr{G}$ invariant. The characterization of $\Phi^*_{D^r}(v)^{-1}_{\mathrm{ex}}(0)$ follows from the construction and Lemma 5.6. q.e.d.

To construct the C-morphism γ_C promised earlier, we need to show that the space of $\mathscr{G}$-invariant sections, say $H^0(\mathfrak{U}^{ss}, \mathscr{L}_{\mathrm{U}}(r)^{\otimes q})^{\mathscr{G}}$, is base point free. It is certainly base point free on the set $\mathfrak{U}^{ss}\backslash\mathfrak{U}^{ss}_0$ [19, Theorem 3]. To attack the general situation, we need the following technical proposition:

Proposition 5.8. *There is a function* $\kappa: \mathbb{Z}^+ \to (0, 1/2)$ *depending on* (X, H, I) *having the following properties: For any* $d_0 \geq A$, *there is an* $r \geq 1$ *such that for any* $\varepsilon \in (0, \kappa(d_0))$, *whenever* $d \leq d_0$ *and* $F \in \mathfrak{U}^{ss}$ *is an* $H(\varepsilon)$*-semistable sheaf over* $t \in C$, $F_{|D^r_t}$ *is semistable for generic good divisor* $D^r \in |rH_C(-r\Delta)|$.

Proposition 5.8 will be proved shortly.

By Proposition 5.8, for d_0 and r, if we choose ε small enough, then there are good divisors $D^r_1, \cdots, D^r_k \in |rH_C(-r\Delta)|$ such that for any closed $s \in \mathfrak{U}^{ss}$ and at least one $j \in \{1, \cdots, k\}$, $\mathscr{F}_{s|D^r_{j,t}}$ is semistable, where t lies under s. Thus by choosing q large so that all $\mathscr{L}^{\otimes q}_{D^r_j}$ are very ample, we have:

Proposition 5.9. *Let* $V_j = \pi_{j*}(\mathscr{L}^{\otimes q}_{D^r_j})$, *where* $\pi_j: \mathfrak{M}(D^r_j/C) \to C$, *let* $\mathscr{V}_j = \{\Phi^*_{D^r_j}(v)_{\mathrm{ex}} | v \in V_j\}$ *and let* $\mathscr{V}_C$ *be the span of* $\mathscr{V}_1, \cdots, \mathscr{V}_k$ *in* $\pi_{C*}(\mathscr{L}_{\mathrm{U}}(r)^{\otimes q})^{\mathscr{G}}$, *where* $\pi_C: \mathfrak{U}^{ss} \to C$. *Then* $\mathscr{V}_C$ *is base point free.*

Let $\Upsilon: \mathfrak{U}^{ss} \to \mathbf{P}(\mathscr{V}_C^\vee)$ be the induced C-morphism. Since $\mathscr{V}_C \subseteq \pi_{C*}(\mathscr{L}_{\mathrm{U}}(r)^{\otimes q})^{\mathscr{G}}$, Υ factors through a C-morphism $\gamma: \mathfrak{M}^d \to \mathbf{P}(\mathscr{V}_C^\vee)$.

Theorem 5.10. *Let $\mathfrak{N}^d \subseteq \mathfrak{M}^d$ be any irreducible component. Then*

$$\dim \gamma(\mathfrak{N}_0^d) = c_d. \tag{5.16}$$

Proof. Clearly, $\gamma(\mathfrak{N}^d)$ is a scheme dominant over C and $\dim \gamma(\mathfrak{N}_t^d) = c_d$. Since $\gamma(\mathfrak{N}^d)$ is irreducible and projective over C, it is flat over C [13, III.9.7]. Thus $\gamma(\mathfrak{N}^d)_0 = \gamma(\mathfrak{N}_0^d)$ has dimension c_d. q.e.d.

Corollary 5.11. *With the notation as above, let $\mathfrak{U} \subseteq \mathfrak{U}^{ss}$ be the irreducible component corresponding to $\mathfrak{N}^d$, and let $X_1, \cdots, X_l$ be the irreducible components of $\mathfrak{U}_0$. For any good divisor $D^r \in |rH_C(-r\Delta)|$ and $v \in H^0(\mathfrak{M}(D^r/C), \mathscr{L}_{D^r}^{\otimes q})$, if we let $W(v) \subseteq \mathfrak{N}^d$ be the zero locus of the extension section $\Phi_{D^r}^*(v)_{\mathrm{ex}}$, then there is at least one $i \in \{1, \cdots, l\}$ such that for any tuple $\{D_j^r, v_j\}_{j=1}^{c_d}$,*

$$\left(\bigcap_{j=1}^{c_d} W(v_j)\right) \cap X_i \neq \varnothing.$$

Proof. Without loss of generality, we can assume $\{D_j^r, v_j\}_{j=1}^{c_d}$ is contained in the collection used in Proposition 5.9. Then the corollary follows since $W(v_j)$ are divisors cut out by ample divisotrs in $\mathbf{P}(\mathscr{V}_C^\vee)$. q.e.d.

Let X be the irreducible component of $\mathfrak{U}_0$ satisfying Corollary 5.11, and let $U \subseteq \mathfrak{U}$ be the (irreducible) open subset provided by Proposition 3.3. Namely, $\varnothing \neq U \cap \mathfrak{U}_0 \subseteq X$ and there is a family of torsion free sheaves $\mathscr{E}$ on $\tilde{Z}_{C^m}^m \times_{C^m} U^m$ such that $\mathscr{E}$ is a good modification of the restriction to $Z \times_C U$ of the universal family on $Z \times_C \mathfrak{U}^{ss}$. Note that $\mathscr{E}$ is flat over U^m and locally free along $\Sigma_m \times_{C^m} U^m$. Then $\mathscr{E}_{|\Delta_m \times_{C^m} U^m}$ is a flat family to torsion free sheaves on $\Delta_m \times_{C^m} U^m$. Let (α, k) be integers so that for any closed $u \in U_0^m$, $\det E_{u|\Delta_m} = I_{\Delta_m}(\alpha\Sigma^-)$ and $c_2(E_{u|\Delta_m}) = k$, where we denote the line bundle $p_X^* I_{|\Delta_m}$ by I_{Δ_m}. By Proposition 4.6, $|\alpha| \leq 2$ and $k \leq d + A_2$.

Put $U_0^m[\Delta_m] = \{u \in U_0^m | \mathscr{E}_{u|\Delta_m}$ is $H(\varepsilon)$-semistable$\}$ and let $\Psi\colon U_0^m[\Delta_m] \to \mathfrak{M}_\Delta^{\alpha,k}$ be the map sending $u \in U_0^m[\Delta_m]$ to $\mathscr{E}_{u|\Delta_m}$. Here we denote by $\mathfrak{M}_\Delta^{\alpha,k}$ the moduli scheme of $H(\varepsilon)_\Delta$-semistable rank-two sheaves E on Δ satisfying $\det E = I_{\Delta_m}(\alpha\Sigma^-)$ and $c_1(E) = k$. We now state and prove the main result of this section:

Theorem 5.12. *With the notation as before, there is a constant A and a function $\kappa\colon \mathbb{Z}^+ \to (0, 1/2)$ such that for any $d \geq A$, any $\varepsilon \in (0, \kappa(d))$*

and any irreducible component X of $\mathfrak{U}_0$ satisfying Corollary 5.11, we have

$$\dim \Psi(U_0^m) = c_d. \tag{5.17}$$

We need the following technical lemmas.

Lemma 5.13. *There is a constant e such that for any $F \in \mathfrak{U}_0^{ss}$, $F^{(2)}$ is $e\varepsilon$-stable.*

Proof. Let Q be any rank-one quotient sheaf of $F^{(2)}$. Then Q is a quotient sheaf of F also. Thus by comparing the linear coefficients of $p_Q(N) \preceq p_F(n)$, we get

$$\frac{1}{1-\varepsilon^2}\left[H(\varepsilon)_\Delta\left(c_1(Q) - \frac{1}{2}K_\Delta\right)\right] \geq \frac{1}{2}H(I - K_X).$$

Let $\deg Q = H(\varepsilon)_\Delta \cdot c_1(Q)$. Then

$$\deg Q \geq -\tfrac{1}{2}\varepsilon(1-\varepsilon)(K_X \cdot H) - 2\varepsilon(H\cdot H) + \tfrac{1}{2}(1-\varepsilon^2)(I\cdot H).$$

On the other hand, we have $c_1(F^{(2)}) = I_{|\Delta}(\alpha\Sigma^-)$ with α possibly be 0, -1, or -2. Since $\Sigma^- \cdot H(\varepsilon)_\Delta = \varepsilon(H\cdot H)$ and $I_\Delta \cdot H(\varepsilon) = (1-\varepsilon)(I\cdot H)$, for $e = 6(H\cdot H)$,

$$\deg Q - \tfrac{1}{2}\deg F^{(2)} > \tfrac{1}{2}e\varepsilon. \qquad \text{q.e.d.}$$

Lemma 5.14. *Given the constants e and A_2, we can find a function $\kappa\colon \mathbb{Z}^+ \to (0, 1/2)$ having the following properties: For any $d \geq A$, there is a large r such that whenever $\varepsilon \in (0, \kappa(d))$ and V is an $e\varepsilon$-stable (with respect to $H(\varepsilon)_\Delta$) rank-2 vector bundle on Δ with $\det V = I_\Delta(\alpha\Sigma^-)$, $|\alpha| \leq 2$ and $c_2(V) \leq d + A_2$, the following hold:*

(1) *V either is μ-stable or belongs to the exact sequence*

$$0 \to I'(\Gamma + \beta\Sigma^-) \to V \to I'(-\Gamma - (\beta-\alpha)\Sigma^-) \otimes \mathscr{I}_z \to 0 \tag{5.18}$$

for some zero scheme $z \subseteq \Delta$, numerical zero divisor $\Gamma \subseteq \Delta$ and $2I' = I_\Delta$.

(2) *For generic $D_r \in |r\Sigma^+|$, $V_{|D_r}$ is semistable.*

Lemma 5.14 will be proved shortly.

Proof of Theorem 5.12. Recall $\mathscr{F}(n)$ is the universal family on $Z \times_C \mathfrak{U}^{ss}$, and for any closed $u \in \mathfrak{U}_0^{ss}$, $\mathscr{F}_u^{(2)}$ is the torsion free part of $\mathscr{F}_{u|\Delta}$. We first show that for generic $u \in \mathscr{X}$, $\mathscr{F}_u^{(2)}$ is μ-stable. Assume not. Then by Lemma 5.14, $\mathscr{F}_u^{(2)}$ belongs to the exact sequence

$$0 \to I'(\Gamma + \beta\Sigma^-) \otimes \mathscr{I}_{z_1} \to \mathscr{F}_u^{(2)} \to I'(-\Gamma - (\beta-\alpha)\Sigma^0) \otimes \mathscr{I}_{z_2} \to 0, \tag{5.19}$$

where z_1 and z_2 are two zero schemes, and Γ is a numerical zero divisor. Since $c_2(\mathscr{F}_u^{(2)}) \leq d+A_2$ (Proposition 4.7), by calculating the second Chern class of $\mathscr{F}_u^{(2)}$ based on (5.19), we get

$$l(z_1)+l(z_2) \leq d+(A_2+4(H\cdot H)). \tag{5.20}$$

Clearly, then $\mathscr{X}[D_0^r]$ $(=\{u \in \mathscr{X} | \mathscr{F}_{u|D_0^r}$ is semistable$\})$ is the set of all $u \in \mathscr{X}$ such that in (5.19), $z_1, z_2 \subseteq \Delta \backslash D_0^r$. We define

$$\Pi\colon \mathscr{X} \to \operatorname{Pic}^0(\Sigma)/\{1,-1\} \tag{5.21}$$

to be the map sending $u \in \mathscr{X}$ to $\mathscr{O}(\Gamma)_{|\Sigma^-}$. Π is well-defined. Further, for any $v \in H^0(\mathfrak{M}(D^r/C), \mathscr{L}_{D^r}^{\otimes q})$, the zero locus of $\Phi_{D^r}^*(v)_{\text{ex}}$ (denoted by $W(v)\cap\mathscr{X}[D_0^r]$) is mapped (under Π) into a divisor in $\operatorname{Pic}^0(\Sigma)/\{1,-1\}$. We denote the image set $\Pi(W(v)\cap\mathscr{X}[D_0^r])$ by $B(v)$.

Now we choose c_d generic good divisors $D_j^r \in |rH_C(-f\Delta)|$ in the sense that no two D_j^r's coincide along Z_0 and no three D_j^r's share common intersections along Z_0. We also choose sections $v_j \in H^0(\mathfrak{M}(D_j^r/C), \mathscr{L}_{D_j^r}^{\otimes q})$, $j=1,\cdots,c_d$, such that for any $\{i_1,\cdots,i_{g+1}\} \subset \{1,\cdots,c_d\}$,

$$\bigcap_{j=1}^{g+1} B(v_{i_j}) = \varnothing. \tag{5.22}$$

The tuple $\{(D_j^r, v_j)\}_{j=1}^{c_d}$ always exists because of the ampleness of H and $\mathscr{L}_{D_j^r}$. Let

$$u \in \left(\bigcap_{j=1}^{c_d} W(v_j)\right) \bigcap \mathscr{X} \neq \varnothing,$$

which exists, thanks to Corollary 5.11. Since $c_d = 4d+O(1)$, when $d \gg 0$, (5.20) implies that there are at least $g+1$ of D_j^r's, say $D_1^r,\cdots,D_{g+1}^r$, so that $u \in \bigcap_{j=1}^{g+1}\mathscr{X}[D_j^r]$. Then $\Pi(u) \in B(v_j)$ for $j=1,\cdots,g+1$, which contradicts to (5.22). So we have proved that for generic $u \in \mathscr{X}$, $\mathscr{F}_u^{(2)}$ is μ-stable and thus $U_0^m[\Delta_m] \neq \varnothing$.

It remains to show that (5.17) is true. Let $\varphi_m\colon U_0^m \to U$ be induced by projection. Without loss of generality, we assume for any closed $u \in U_0^m$, $\mathscr{E}_{u|\Delta_m}$ is μ-stable. By (5.14), for any D^r and $v \in H^0(\mathfrak{M}(D^r/C), \mathscr{L}_{D^r}^{\otimes q})$, $\varphi_m^{-1}(W(v)) \cap U_0^m$ is a union of fibers of Ψ. Thus for generic D^r and generic section v, by Proposition 5.8, $\Psi(\varphi_m^{-1}(W(v)\cap U))$ is a proper divisor in $\Psi(U_0^m)$. Now suppose $\dim \Psi(U_0^m) < c_d$. Then by using Proposition 5.8 and the ampleness of $\mathscr{L}_{D^r}$, for generic $D_1^r, \cdots, D_{c_d}^r$ and sections $v_1, \cdots, v_{c_d}$ so that

$$\bigcap_{i=1}^{c_d} \Psi(\varphi_m^{-1}(W(v_i))) = \varnothing.$$

On the other hand, $\dim(\mathscr{X}\backslash U_0) < c_d$. So by choosing $\{D_i^r, v_i\}$ generic, we also have

$$(\mathscr{X}\backslash U_0) \bigcap \left(\bigcap_{i=1}^{c_d} W(v_i) \right) = \varnothing.$$

Thus

$$\mathscr{X} \bigcap \left(\bigcap_{i=1}^{c_d} W(v_i) \right) = \varnothing, \tag{5.23}$$

which contradicts the assumption. So we must have $\dim \Psi(U_0^m) = c_d$.

In the remainder of this section, we will prove Proposition 5.8 and Lemma 5.14. Assume that V is a rank-2 vector bundle on Δ such that $\det V = \mathscr{O}(\alpha\Sigma^-)$, $|\alpha| \le 2$ and $c_2(V) \le d + A_2$, that V is $e\varepsilon$-stable and that V is not μ-stable. Let $L \subseteq V$ be a destabilizing subsheaf, $L = \mathscr{O}(\Gamma + b\Sigma^-)$ where Γ is a pullback divisor from Σ. We assume V/L is torsion free. Then we have the following exact sequence

$$0 \to \mathscr{O}(\Gamma + b\Sigma^-) \to V \to \mathscr{O}(-\Gamma - (b-\alpha)\Sigma^-) \otimes \mathscr{I}_z \to 0, \tag{5.24}$$

where $\mathscr{I}_z$ is the ideal sheaf of a zero subscheme $z \subseteq \Delta$. Thus

$$d + A_2 \ge c_2(V) \ge -(\Gamma + b\Sigma^-)(\Gamma + (b-\alpha)\Sigma^-). \tag{5.25}$$

Since V is $e\varepsilon$-stable and L is the destabilizing subsheaf,

$$\tfrac{1}{2}e\varepsilon \ge (\Gamma + b\Sigma^-) \cdot H(\varepsilon)_\Delta - \tfrac{1}{2}(\alpha\Sigma^-) \cdot H(\varepsilon)_\Delta \ge 0.$$

Setting $a = (\Gamma \cdot \Sigma^0)/(H \cdot H)$, we obtain

$$a(2b - \alpha) + b(b - \alpha) \le \frac{d + A_2}{(H \cdot H)}, \tag{5.26}$$

$$0 \le b\varepsilon + a - \frac{1}{2}\varepsilon\alpha \le \frac{e\varepsilon}{2(H \cdot H)}. \tag{5.27}$$

One checks easily that there is a function $\kappa \colon \mathbb{Z}^+ \to (0, 1/2)$ so that for any $\varepsilon \in (0, \kappa(d))$, the only solutions (a, b) of (5.26) and (5.27) with $a \cdot (H \cdot H)$ and b integers are $a = 0$ and $0 \le b \le e/(H \cdot H)$. Thus we have proved part 1 of Lemma 5.14.

Proof of Lemma 5.14. By tensoring V with $(I')^{-1}$, we can assume V satisfies $\det V = \mathscr{O}(\alpha\Sigma^-)$ and $c_2(V) \le d + A_2$. Then (1) follows from the previous argument. For (2), it is clear that the sheaves V of (5.18) have semistable restriction $V_{|D_r}$ for general D_r. We now show that the same conclusion holds for all μ-stable sheaves V. Assume that $V_{|D_r}$ is not semistable. Then there is a quotient line bundle Q of $V_{|D_r}$ over D_r, $V_{|D_r} \to Q$ with $\deg Q < 0$. Let W be the kernel of $V \to Q$. Then W is a locally free sheaf on Δ with $c_1(W) = \alpha[\Sigma^-] - r[\Sigma^+]$ and $c_2(W) = c_2(V) + \deg_\Sigma Q < c_2(V)$. Hence

$$\begin{aligned} 4c_2(W) - c_1(W)^2 &= 4c_2(V) + 4\deg Q - (r^2 - \alpha^2)(H \cdot H) \\ &< 4c_2(V) - (r^2 - \alpha^2)(H \cdot H). \end{aligned}$$

Since r can be made large, we assume $(r^2 - 4) \ge 4(d + A_2)/(H \cdot H) \ge 4c_2(V)/(H \cdot H)$. Then Bogomolov's inequality shows that W is Bogomolov unstable. That is, there is a rank-1 locally free sheaf $L \subset W$, W/L is torsion free, so that $L^{\otimes k}(-(k\alpha/2)[\Sigma^-] + (kr/2)[\Sigma^+])$ is effective for $k \gg 0$. Write $\mathscr{O}(L) = \mathscr{O}(a\Sigma^- + \Gamma)$ where Γ is a divisor in Δ supported on the fibers of $q\colon \Delta \to \Sigma$. Denote $b = (\Gamma \cdot \Sigma)/(H \cdot H)$. Then

$$a - \alpha/2 + r/2 \ge 0, \qquad b + r/2 \ge 0. \tag{5.28}$$

By computing the second Chern class of W in terms of L similar to (5.25) and then comparing it with $c_2(V)$, we get

$$(a\Sigma^- + \Gamma)((\alpha - a)\Sigma^- - r\Sigma^+ - \Gamma) \le c_2(W) \le c_2(V).$$

So

$$a(a - \alpha) - b(r - \alpha + 2a) \le \frac{d + A_2}{(H \cdot H)}. \tag{5.29}$$

Since V is semistable, $(L - (\alpha/2)\Sigma^-) \cdot H(\varepsilon)_\Delta \le 0$. Thus

$$\varepsilon(a - \alpha/2) + (1 - \varepsilon)b \le 0. \tag{5.30}$$

We will show that $b = 0$ by showing, after an appropriate choice of r and ε, that $b > 0$ and $b < 0$ are impossible. We first assume $b < 0$, so $b \le -1/(H \cdot H)$. By (5.28) and (5.29),

$$\frac{d + A_2}{(H \cdot H)} \ge a(a - \alpha) + (-b)(r - \alpha + 2a) \ge a(a - \alpha).$$

Thus $|a| \le \sqrt{d + A_2} + 2$ and then $r - \alpha + 2a \ge r - 2\sqrt{d + A_2} - 4$. So by (5.29) again,

$$\frac{d + A_2}{(H \cdot H)} \ge \frac{1}{(H \cdot H)}(r - 2\sqrt{d + A_2} - 4). \tag{5.31}$$

If $b > 0$, thanks to (5.28) and (5.30), we have

$$(5.32) \qquad 0 \leq \left(a - \frac{\alpha}{2}a\right) + \frac{r}{2} \leq \frac{1-\varepsilon}{\varepsilon}(-b) + \frac{r}{2} \leq \frac{1-\varepsilon}{\varepsilon} \cdot \frac{(-1)}{(H \cdot H)} + \frac{r}{2}.$$

Now for any d, we can choose r large so that (5.31) does not hold. Then let $\varepsilon = \kappa(d, r)$ be small enough so that the right-hand side of (5.32) is negative. Since neither (5.29) nor (5.32) holds by our choice of r and ε, b has to be zero. Therefore, for any rational $\varepsilon \in (0, \kappa(d, r))$, $V_{|D_r}$ is not semistable only if the line bundle $L \subseteq W$ satisfies $c_1(L) \cdot [\Sigma^+] = 0$. Let $D \subseteq \Delta$ be a divisor so that $L(D)$ is a subsheaf of V with $V/L(D)$ torsion free. Since V/L is torsion free over $\Delta \backslash D_r$, we have $D = \varnothing$ or $D = D_r$. We claim that $D = \varnothing$. Otherwise, since L is a destabilizing subsheaf of W, $L(D) \subset V$ will be a destabilizing subsheaf of V. This would violate the μ-stability of V. Therefore, V belongs to the exact sequence

$$0 \to \mathscr{O}(a\Sigma^- + \Gamma) \to V \to \mathscr{O}((\alpha - a)\Sigma^- - \Gamma) \otimes \mathscr{I}_z \to 0$$

with a numerical zero divisor Γ. Therefore, $V_{|D_r}$ is semistable for generic $D_r \in |r\Sigma^+|$. This completes the proof of Proposition 5.8 and Theorem 5.12.

6. Vector bundles over the ruled surface Δ

In previous sections, we have succeeded in constructing a degeneration of the moduli scheme $\mathfrak{M}^{d,\varepsilon}$. By an appropriate choice of ε, we also showed that for any irreducible component $\mathfrak{N}^d$ of the degeneration $\mathfrak{M}^d$, there is an irreducible component $\mathscr{X}$ of $\mathfrak{N}_0^d$ so that the restriction to Δ_m of proper transforms E of generic $F \in \mathscr{X}$ are μ-stable. Further, the set of all such $E_{|\Delta_m}$ is an algebraic subset of $\mathfrak{M}_\Delta^{\alpha,k}$ of dimension equal to c_d. In this section, we will concentrate on studying this subset of $\mathfrak{M}_\Delta^{\alpha,k}$.

First let us fix some notation. We still denote by Σ^+ (resp. Σ^-) a section of $q\colon \Delta \to \Sigma$ such that $(\Sigma^+ \cdot \Sigma^+) > 0$ (resp. $(\Sigma^- \cdot \Sigma^-) < 0$). Note that Σ^- is unique while $|\Sigma^+|$ is base point free. We use $l\Sigma^-$ to denote the nonreduced curve that is the lth infinitesimal neighborhood of $\Sigma^- \subseteq \Delta$. Let $H(\varepsilon)_\Delta$ be the ample $\mathbb{Q}$-divisor $q^*H_\Sigma((1-\varepsilon)\Sigma^-)$, where H_Σ is the restriction of H to $\Sigma \subseteq X$. We understand that ε is a small positive rational number. We will work on the Grothendieck's Quot-scheme instead of on the moduli space since the previous space has a universal family. Let α be either 0 or 1, k a positive integer and let $\mathfrak{N}_k^\alpha$ be the Quot-scheme parameterizing all quotient sheaves E of $\bigoplus^N H(\varepsilon)_\Delta^{\otimes(-n)}$ with

$\det E = \mathscr{O}(\alpha\Sigma^-)$ and $c_2(E) = k$. Here $N = \chi(E(n))$, where n will be chosen large enough (depending on d and ε). For any constant $e \geq 0$, we let $\mathfrak{M}_k^{\alpha,\varepsilon} \subseteq \mathfrak{M}_k^\alpha$ be the subset of all e-stable quotient sheaves. When the choice of α is made, we will suppress the index α and denote $\mathfrak{M}_k^\alpha$ and $\mathfrak{M}_k^{\alpha,e}$ by $\mathfrak{M}_k$ and $\mathfrak{M}_k^e$ respectively. Let $g = g(\Sigma)$ and let A_2 be a constant that remains fixed throughout this section.

Let V be a fixed locally free sheaf on $4\Sigma^-$. We define $\Theta_V^e \subseteq \mathfrak{M}_k^e$ to be the subset consisting of all quotient sheaves $E \in \mathfrak{M}_k^e$ with $E_{|4\Sigma^-} \cong V$. Clearly, Θ_V^e is a constructible set (Proposition 6.5). Then Θ_V^e is the finite union of locally closed subsets of $\mathfrak{M}_k^e$. We call $\Theta \subseteq \Theta_V^e$ a maximal irreducible component if Θ is locally closed, irreducible, and $\Theta_1 \subseteq \Theta$ whenever Θ_1 is a locally closed irreducible component of Θ_V^e such that $\Theta \subseteq \overline{\Theta_1}$. The goal of this section is to prove the following theorem.

Theorem 6.1. *For constants A_2 and e, there are constant A and function $\kappa\colon \mathbb{Z}^+ \to (0, 1/2)$ having the following property: For any $k \geq A$, any $\varepsilon \in (0, \kappa(k))$, any locally closed irreducible $S \subseteq \Theta_V^e$ with codimension at most A_2 in $\mathfrak{M}_k^e$ and a general closed $s \in S$, there is an irreducible curve $T \subseteq \Theta_V^e$, $s \in T$ such that there is a quotient sheaf $E \in T$ with $h^2(\Delta, \mathscr{E}nd^0(\mathscr{E}_{t_1})(-4\Sigma^-)) = 0$ and $col(E)_p = 1$ for some $p \in \Delta\backslash\Sigma^-$.*

Proof. Clearly, there is a maximal irreducible component $\mathfrak{S} \subseteq \Theta_V^e$ such that $S \subseteq \overline{\mathfrak{S}}$. Of course, $\operatorname{codim}\mathfrak{S} \leq A_2$. Thus Theorem 6.1 follows immediately from

Theorem 6.2. *For constants A_2 and e, there are constant A and function $\kappa\colon \mathbb{Z}^+ \to (0, 1/2)$ having the following property: For any $k \geq A$, any $\varepsilon \in (0, \kappa(k))$, and any maximal irreducible component $\mathfrak{S} \subseteq \Theta_V^e$ with codimension at most A_2 in $\mathfrak{M}_k^e$, there are non-locally-free sheaves $F \in \mathfrak{S}$ such that $h^2(\Delta, \mathscr{E}nd^0(F)(-r\Sigma^-)) = 0$ and $col(F)_p = 1$ for some $p \in \Delta\backslash\Sigma^-$.*

We now sketch the idea of the proof of Theorem 6.2. First, as we explained in the introduction, we can deform an $E \in \mathfrak{S} \subseteq \Theta_V^e$ to $F \in \mathfrak{M}_k^e$ so that F is not locally free at some $p \in \Delta\backslash\Sigma$. If $F_{|4\Sigma^-} \cong V$, then we are done. Otherwise, we let $\mathfrak{M}_k^{\text{sh}} \subseteq \mathfrak{M}_k^e$ be the subset of non-locally-free sheaves. Assume $\mathfrak{M}_k^{\text{sh}} \cap \mathfrak{S} \neq \varnothing$. Then since $\mathfrak{M}_k^{\text{sh}} \subseteq \mathfrak{M}_k^e$ is a divisor, some component T of $\overline{\mathfrak{S}}\backslash\mathfrak{S}$ which contains F is contained in $\mathfrak{M}_k^{\text{sh}} \cap \overline{\mathfrak{S}}$. On the other hand, by studying the deformation problem, we can show that we can deform a general sheaf $E \in T$ within T to a non-locally-free sheaf. Thus T cannot be contained in $\mathfrak{M}_k^{\text{sh}} \cap \overline{\mathfrak{S}}$, a contradiction. In the rest of this section, we will provide the detail of this argument. First we state some results regarding the set $\mathfrak{S}$.

Lemma 6.3. *With the notation as above, there is a constant A such that whenever $k \geq A$, any generic sheaf $E \in \mathfrak{S}$ has generic fiber type $(\alpha, 0)$.*

Proof. Let $\mathscr{B}_l$ be the subset of $\mathfrak{M}_k^d$ consisting of locally free sheaves that are of generic fiber type $(\alpha + l, -l)$, $l \geq 1$. By [3], $\dim \mathscr{B}_l \leq 3k + N^2 + O(1)$. Since $\dim \mathfrak{M}_k^e = 4k + N^2 + O(1)$, there is a constant A such that for $k \geq A$, $\operatorname{codim}(\bigcup_{l \geq 1} \mathscr{B}_l) \geq A_2 + 1$. Thus $\mathfrak{S} \setminus \bigcup_{l \geq 1} \mathscr{B}_l$ is nonempty. q.e.d.

Next, let D be any fixed divisor on Δ. We will estimate the dimension of the set $\mathscr{A}_k$ of all sheaves $F \in \mathfrak{M}_k^e$ with $h^2(\Delta, \mathscr{E}nd^0(F)(-D)) \neq 0$. Here $\mathscr{E}nd^0(\cdot)$ is the traceless part of the endomorphism sheaf. We have the following estimate.

Lemma 6.4. *There is a constant A_3 independent of k and ε such that*

$$\operatorname{codim} \mathscr{A}_k \geq k - A_3.$$

Proof. Since F is torsion free,

$$h^2(\mathscr{E}nd^0(F)(-D)) = h^2(\mathscr{E}nd^0(F^{\vee\vee})(-D)),$$

which is equal to $h^0(\mathscr{E}nd^0(F^{\vee\vee})(K_\Delta + D))$ by Serre duality. For $\rho \in H^0(\mathscr{E}nd^0(F^{\vee\vee})(K_\Delta + D))$, let $\det \rho \in H^0(2K_\Delta + 2D)$. In the following, we denote by $\mathscr{A}_k^{\mathrm{vb},1}$ (resp. $\mathscr{A}_k^1$) the set of all rank-2 locally free (resp. arbitrary) sheaves F in $\mathfrak{M}_k^e$ having $\rho \in H^0(\mathscr{E}nd^0(F^{\vee\vee})(K_\Delta + D))$ with nontrivial $\det \rho$. Then by [4], [6], [34], there is a constant $A_4(D)$ such that

$$\dim \mathscr{A}_k^{\mathrm{vb},1} \leq 3k + N^2 + A_4(D).$$

Assume $F \in \mathscr{A}_k^1$. Then $F^{\vee\vee} \in \mathscr{A}_{k'}^{\mathrm{vb},1}$, where $k' = c_2(F^{\vee\vee})$. Further, F is a subsheaf of $F^{\vee\vee}$ with cokernel Q of length $k - k'$. It is shown in [19] that the number of moduli of the set of F with $c_2(F) = k$ and $F^{\vee\vee}$ fixed ($c_2(F^{\vee\vee}) = k'$) is $3(k - k')$. Therefore,

$$\dim \mathscr{A}_k^1 \leq \max_{j \geq 0}(\dim \mathscr{A}_{k-j}^{\mathrm{vb},1} + 3j) \leq 3k + N^2 + A_4(D).$$

Now let $\mathscr{A}_k^{\mathrm{vb},0}$ (resp. $\mathscr{A}_k^0$) be the set of locally free (resp. arbitrary) sheaves $F \in \mathfrak{M}_k^e$ so that there is a $\rho \neq 0 \in H^0(\mathscr{E}nd^0(F^{\vee\vee})(K_\Delta + D))$ such that $\det \rho = 0$. We claim that there is a constant $A_5(D)$ so that

$$\dim \mathscr{A}_k^{\mathrm{vb},0} \leq 3k + N^2 + A_5(D). \tag{6.1}$$

Let $F \in \mathscr{A}_k^{\mathrm{vb},0}$, and let $\rho \colon F \to F \otimes K_\Delta(D)$ be a traceless homomorphism with $\det \rho = 0$. Let $L = \ker(\rho)$. Then we have the exact sequence

$$0 \to L \to F \to L^{-1} \otimes \det F \otimes \mathscr{I}_z \to 0. \tag{6.2}$$

Following [4, p. 296], ρ is induced by a homomorphism $L^{-1}(\alpha\Sigma^-) \to L(K_\Delta + D)$. Combined with the $H(\varepsilon)_\Delta$-stability of F, we get a uniform bound of the degree of L. Thus a standard argument of counting the number of moduli of all possible extensions of (6.2) yields (6.1). Finally, using argument similar to those used in deriving estimate of $\dim \mathscr{A}_k^1$ from that of $\dim \mathscr{A}_k^{\mathrm{vb},1}$, we obtain the main estimate of the lemma. q.e.d.

As a first step toward establishing Theorem 6.2, we have the following proposition that will be proved eventually.

Proposition 6.5. *With the notation as in Theorem 6.2, there is a constant A such that if $k \geq A$, and $\mathfrak{S}$ is a maximal irreducible component of $\mathbf{\Theta}_V^e$ of codimension less than A_2, then there is a stable sheaf $E \in \overline{\mathfrak{S}}$ so that $h^2(\Delta, \mathscr{E}nd^0(E)(-4\Sigma^-)) = 0$ and $\operatorname{col}(E)_p = 1$ for at least one closed point $p \in \Delta \backslash \Sigma^-$. Here, $\overline{\mathfrak{S}}$ is the closure of $\mathfrak{S}$ in $\mathfrak{M}_k^e$.*

We next show that we can find an $E \in \mathfrak{S}$ so that E satisfies the conclusion of the Proposition 6.5. We will accomplish this by studying the deformation of family of sheaves on Δ. We first state some known facts that we shall need.

Lemma 6.6. *Let S be any quasi-projective scheme and let $\mathscr{E}$ be a family of torsion free sheaves on $\Delta \times S$ flat over S. Let F be any sheaf on $4\Sigma^-$. We define S_F to be the set $\{s \in S | \mathscr{E}_{s|4\Sigma^-} \cong F\}$. Then S_F is a constructible set. Take $\overline{S_F} \subseteq S$ be the closure of S_F. Then there is a closed subscheme structure on $\overline{S_F}$ with the following property: For any morphism $f: T \to S$, suppose there is an isomorphism $(\mathrm{id}_\Delta \times f)^*(\mathscr{E})_{|r\Sigma^- \times T} \cong p_{4\Sigma^-}^* F$, where $p_{4\Sigma^-}: 4\Sigma^- \times T \to 4\Sigma^-$. Then f factors through $\overline{S_F}$.*

Proof. Let L^{-1} be a very ample line bundle on $4\Sigma^-$ so that $h^1(F \otimes L^{-1}) = 0$ and $H^0(F \otimes L^{-1}) = \mathbb{C}^m$ generates $F \otimes L^{-1}$. Fix a homomorphism $g: \oplus^m L \to F$ so that the induced homomorphism $\mathbb{C}^m \to H^0(F \otimes L^{-1})$ is an isomorphism.

Consider the Grothendieck's Quot-scheme $\mathfrak{R}_{mL}^{\chi_F}$ parameterizing all quotient sheaves of $\bigoplus^m L$ having Hilbert polynomial χ_F. g defines a closed point $[g] \in \mathfrak{R}_{mL}^{\chi_F}$. Put $\pi_{4\Sigma^-}$ (resp. π_S): $4\Sigma^- \times S \to 4\Sigma^-$ (resp. $\to S$) and denote $\mathscr{V} = \mathscr{H}om(\bigoplus^m \pi_{4\Sigma^-}^* L, \mathscr{E}_{|4\Sigma^-})$. Consider the sheaf $\pi_{S*}(\mathscr{V})$ on S. Since S is quasi projective, by choosing L^{-1} sufficiently ample, we can assume $\pi_{S*}(\mathscr{V})$ is locally free. Here we have used the fact that since $\mathscr{E}$ is a flat family of torsion free sheaves, $\mathscr{E}_{|r\Sigma^-}$ is still flat over S (cf. §0). Let $V \subseteq \mathbf{P}(\mathscr{V})$ be the open subset consisting of all closed points corresponding to surjective homomorphisms. Over $4\Sigma^- \times V$, there is a flat family of quotient sheaves

$$\mathbf{h}\colon \pi_{4\Sigma^-}^*\left(\bigoplus^m L\right) \to p^*(\mathscr{E}_{|4\Sigma^-}) \otimes \pi_V^*\mathscr{O}_{\mathbf{P}}(1)_{|V}, \tag{6.3}$$

where p is the projection $4\Sigma^- \times V \to 4\Sigma^- \times S$. Thus there is an induced morphism $\Psi\colon V \to \mathfrak{R}_{mL}^{\chi_F}$. We define $\mathfrak{B} \subseteq V$ to be the closed subscheme defined by the ideal sheaf that is the pullback via Ψ of the ideal sheaf defining $[g] \in \mathfrak{R}_{mL}^{\chi_F}$.

We then define $\overline{S_F}$ to be the scheme theoretic image of $\pi\colon \mathfrak{B} \subseteq \mathbf{P}(\mathscr{V}) \to S$. Set-theoretically, $\overline{S_F}$ is identical to the closure of S_F in S. Now we check that for any morphism $f\colon T \to S$ satisfying the condition of the lemma, f will factor through $\overline{S_F}$. Since $(\mathrm{id}_\Delta \times f)^*(\mathscr{E})_{|4\Sigma^- \times T} \cong p_{4\Sigma^-}^* F$, the homomorphism $g\colon \bigoplus^m L \to F$ induces

$$\eta = p_{4\Sigma^-}^*(g)\colon \bigoplus^m p_{4\Sigma^-}^* L \to (\mathrm{id}_\Delta \times f)^*(\mathscr{E})_{|4\Sigma^- \times T} \cong p_{4\Sigma^-}^* F. \tag{6.4}$$

Clearly, η induces a morphism $[\eta]\colon T \to \mathfrak{R}_{mL}^{\chi_F}$ that factors through $\zeta\colon T \to [g] \in \mathfrak{R}_{mL}^{\chi_F}$. On the other hand, $[\eta]$ induces a morphism $j\colon T \to V$ [13, II.7.12] such that the pullback of the (6.3) via j is identical to (6.4). Therefore, we have the commutative diagram

$$\begin{array}{ccc} & & V \\ & & \downarrow \Psi \\ T & \xrightarrow{\ \zeta\ } & [g] \subseteq \mathfrak{R}_{mL}^{\chi_F} \end{array}$$

with $\Psi \circ j = \zeta$. So, $T \xrightarrow{j} V$ factors through $\mathfrak{B}$. Since $\overline{S_F}$ is the scheme-theoretic image of $\mathfrak{B}$, $f\colon T \xrightarrow{j} S$ factors through $\overline{S_F}$. q.e.d.

Now, suppose $s \in \mathfrak{R}_k^d$ is a closed point so that for a closed point $p \in \Delta\backslash\Sigma^-$, $\mathrm{col}(\mathscr{E}_s)_p = 1$. Let $U \subseteq \Delta \times \mathfrak{R}_k^e$ be a classical neighborhood of (p, s) so that there is a locally free resolution

$$0 \to \mathscr{O}_U \xrightarrow{f} \mathscr{O}_U^{\oplus 3} \to \mathscr{E}_{|U} \to 0. \tag{6.5}$$

Since general sheaves $E \in \mathfrak{R}_k^e$ are locally free for large k [20], $f^{-1}(0) \subseteq \Delta \times \mathfrak{R}_k^e$ is a codimension three analytic subvariety of U, where we think of f as a map of U to $\mathbb{C}^3$. Let $\mathfrak{R}^{\mathrm{sh}}(s, p, U)$ be the image set $\pi_2(f^{-1}(0))$. Since $\mathscr{E}_s$ is torsion free, $\pi_2(f^{-1}(0))$ is a codimension one subset.

Lemma 6.7. *With the notation as above, there is a classical neighborhood U of (p, s) so that near s, $\mathfrak{R}^{\mathrm{sh}}(\mathscr{E}_s, p, U)$ is an analytic hypersurface of $\mathfrak{R}_k^e$. Further, if we assume $\mathfrak{R}_k^e$ is smooth at s, then there is a* *scheme structure of* $\mathfrak{M}^{\mathrm{sh}}(\mathscr{E}_s, p, U)$ *so that for any Artin ring of the form* $A_n = \operatorname{Spec}\mathbb{C}[t]/(t^n)$ *and* $\varphi: A_n \to \mathfrak{M}^{\mathrm{sh}}(s, p, U)$ *with* $\varphi(0) = s$, *there is a codimension* 2 *subscheme* $Z_n \subseteq \Delta \times_{\mathbb{C}} A_n$ *flat over* A_n *and a classical neighborhood* U' *of* $\{p\} \times_{\mathbb{C}} A_n \subseteq \Delta \times_{\mathbb{C}} A_n$ *such that* $\varphi^*(\mathscr{E})_{|U'}$ *belongs to the exact sequence*

$$0 \to \varphi^*(\mathscr{E})_{|U'} \to \mathscr{O}_{U'}^{\oplus 2} \to \mathscr{O}_{Z_n} \to 0. \tag{6.6}$$

Proof. The first statement follows from the previous argument. Now we assume $\mathfrak{M}_k^e$ is smooth at s. Since $\operatorname{col}(\mathscr{E}_s)_p = 1$, there are analytic coordinates $\mathbf{y} = (y_1, y_2)$ of $p \in \Delta$ and $\mathbf{z} = (z_1, \cdots, z_k)$ of $s \in \mathfrak{M}_k^e$ respectively so that the resolution (6.5) has the form $f = (y_1 + h_1(\mathbf{y}, \mathbf{z}), y_2 + h_2(\mathbf{y}, \mathbf{z}), h_3(\mathbf{z}))$ with

$$\frac{\partial}{\partial y_1} h_1(0, 0) = 0, \qquad \frac{\partial}{\partial y_2} h_2(0, 0) = 0. \tag{6.7}$$

Now we give $\mathfrak{M}^{\mathrm{sh}}(\mathscr{E}_s, p, U)$ the scheme structure $\{h_3(\mathbf{z}) = 0\}$. The lemma follows immediately from the explicit expression of f and (6.7). q.e.d.

Now we return to the proof of Theorem 6.2. Let $x \in \overline{\mathfrak{S}}$, and let $p \in \Delta \backslash \Sigma^-$ be the closed point provided by Proposition 6.5. Clearly, Theorem 6.2 will be established if we can show that $\mathfrak{M}^{\mathrm{sh}}(s, p, U) \cap \mathfrak{S} \neq \varnothing$. Otherwise, since $\mathfrak{M}^{\mathrm{sh}}(s, p, U) \cap \overline{\mathfrak{S}}$ is a divisor (in $\overline{\mathfrak{S}}$) and $(\overline{\mathfrak{S}} \backslash \mathfrak{S}) \cap U$ has codimension at least one in $\overline{\mathfrak{S}} \cap U$, by choosing $s \in \overline{\mathfrak{S}} \cap \mathfrak{M}^{\mathrm{sh}}(s, p, U)$ generic and shrinking $U \ni (s, p)$ if necessary, we can assume $(\overline{\mathfrak{S}} \backslash \mathfrak{S}) \cap U \subseteq \mathfrak{M}^{\mathrm{sh}}(s, p, U)$ as sets and s is a smooth point of $\mathfrak{M}_k^e$. Let $\mathfrak{M}^{\mathrm{sh}}(s, p, U)$ has a scheme structure by Lemma 6.7. Now we endow $\overline{\mathfrak{S}} \subseteq \mathfrak{M}_k^e$ the closed scheme structure introduced in Proposition 6.5 with $S = \mathfrak{S}$ and $F = V$, and let $\overline{\mathfrak{S}_F}$ be the closure of $\{E \in \mathfrak{M}_k^e | E_{|4\Sigma^-} \cong F\}$ endowed with the scheme structure given by Proposition 6.5 with $F = \mathscr{E}_{s|4\Sigma^-}$. Then for some $\delta > 0$, the scheme $\overline{\mathfrak{S}_F} \cap \overline{\mathfrak{S}} \cap U$ ($\subseteq \overline{\mathfrak{S}} \backslash \mathfrak{S}$ as a set) must be a subscheme of $\delta\mathfrak{M}^{\mathrm{sh}}(s, p, U)$, where $\delta\mathfrak{M}^{\mathrm{sh}}(s, p, U)$ is defined by $f^\delta = 0$ when $\mathfrak{M}^{\mathrm{sh}}(s, p, U)$ is defined by $f = 0$. That is,

$$\overline{\mathfrak{S}_F} \cap \overline{\mathfrak{S}} \cap U \subseteq \delta\mathfrak{M}^{\mathrm{sh}}(s, p, U). \tag{6.8}$$

We will derive a contradiction by showing that (6.8) is impossible.

Let S be a smooth affine curve and $0 \in S$, and let $\varphi: (0, S) \to (s, \overline{\mathfrak{S}})$, $\varphi(S \backslash 0) \subseteq \mathfrak{S}$. Define E_S on $\Delta \times S$ to be the pullback of the universal family. By Lemma 6.3, without lose of generality we can assume that for any closed $s \in S$, $h^2(\mathscr{E}nd^0(E_s)(-4\Sigma^-)) = 0$. We will derive a contradiction to (6.11) by constructing deformation E_S^n of E_S over A_n so that

$E^n_{S|4\Sigma^-\times S\times A_n}$ is the constant deformation of $E_{S|4\Sigma^-\times S}$ while $E^n_{S|\Delta\times\{0\}\times A_n}$ is not contained in $(n-1)\mathfrak{R}^{\mathrm{sh}}(\mathscr{E}_s, p, U)$. We need the following deformation lemma.

Lemma 6.8 (Deformation lemma). *Let Y be either $\Delta\times S$ or $4\Sigma^-\times S$, where S is an affine scheme, and let M_n be a coherent sheaf on $Y\times A_n$ flat over A_n. Suppose that M_n admits length-2 locally free resolution and that there is a line bundle L on Y so that* $\det M_n = p_Y^*L$. *Then there is an obstruction class $\omega(M_n, L)\in \mathrm{Ext}^2_Y(M_0, M_0)^0$, $M_0 = M_{n|Y}$, such that $\omega(M_n, L) = 0$ if and only if there is a coherent sheaf M_{n+1} on $Y\times A_{n+1}$ flat over A_{n+1} so that $M_{n+1|Y\times A_n} = M_n$ and* $\det M_{n+1} = p_Y^*L$. *We call M_{n+1} a determinant fixing deformation of M_n over A_{n+1}. Further, the space of all determinant fixing deformations is a homogeneous space isomorphic to* $\mathrm{Ext}^1_Y(M_0, M_0)^0$.

Proof. The case where M_n is locally free was proved in [9], and that where Y is smooth was proved in [25][1]. In general, let $0\to Q_1\to Q_0\to M_n\to 0$ be a locally free resolution of M_n. Since M_n is flat over A_n,

$$0\to Q_{1|Y}\xrightarrow{d} Q_{0|Y}\to M_0\to 0 \tag{6.9}$$

is a locally free resolution of M_0 on Y [27, p. 296]. Thus (6.9) as a complex of sheaves gives rise to a complex of sheaves $\mathscr{H}\mathrm{om}(Q_{\cdot|Y}, Q_{\cdot|Y})$ whose associated hypercohomology is $\mathrm{Ext}_Y(M_0, M_0)$.

Now we fix an affine covering $\{U_\alpha\}$ of $Y\times A_{n+1}$, and let $Q_{1,\alpha}$ (resp. $Q_{0,\alpha}$; resp. $d_\alpha\colon Q_{1,\alpha}\to Q_{0,\alpha}$) be the restriction of Q_1 (resp. Q_0; resp. d) to $U_\alpha\cap Y\times A_n$. Since $Q_{0,\alpha}$ and $Q_{1,\alpha}$ are locally free on $Y\times A_n\cap U_\alpha$, we can find locally free sheaves $\tilde{Q}_{0,\alpha}$ and $\tilde{Q}_{1,\alpha}$ on U_α, that restrict to $Q_{0,\alpha}$ and $Q_{1,\alpha}$ respectively. Let $\tilde{d}_\alpha\colon \tilde{Q}_{1,\alpha}\to\tilde{Q}_{0,\alpha}$ be an extension of d_α, and let $\tilde{f}^i_{\alpha\beta}\colon \tilde{Q}_{i,\beta|U_{\alpha\beta}}\to\tilde{Q}_{i,\alpha|U_{\alpha\beta}}$ ($U_{\alpha\beta} = U_\alpha\cap U_\beta$) be a homomorphism that restricts to the identity of $Q_{i,\beta|U_{\alpha\beta}}\to Q_{i,\beta|U_{\alpha\beta}}\to Q_{i,\alpha|U_{\alpha\beta}}$. Since the restriction of $\tilde{f}^i_{\alpha\beta}\circ\tilde{f}^i_{\beta\gamma}\circ\tilde{f}^i_{\gamma\alpha}\colon\tilde{Q}_{i,\alpha|U_{\alpha\beta\gamma}}\to\tilde{Q}_{i,\alpha|U_{\alpha\beta\gamma}}$ to $U_{\alpha\beta\gamma}\cap Y\times A_n$ is the identity homomorphism,

$$[f]^i_{\alpha\beta\gamma} = \tilde{f}^i_{\alpha\beta}\circ\tilde{f}^i_{\beta\gamma}\circ\tilde{f}^i_{\gamma\alpha} - \mathrm{id}$$

vanishes along $Y\times A_n\cap U_{\alpha\beta\gamma}$. So by identifying $\mathscr{H}om(Q_i, Q_i)(-Y\times A_n)$ to $\mathscr{H}om(Q_{i|Y}, Q_{i|Y})$, we have

$$[f]^i_{\alpha\beta\gamma}\in\Gamma(U_{\alpha\beta\gamma}, \mathscr{H}om(Q_{i|Y}, Q_{i|Y})). \tag{6.10}$$

Similarly, one finds that

$$[f\odot d]_{\alpha\beta} = \tilde{d}_\alpha\circ\tilde{f}^1_{\alpha\beta} - \tilde{f}^0_{\alpha\beta}\circ\tilde{d}_\beta\in\Gamma(U_{\alpha\beta}, \mathscr{H}om(Q_{1|Y}, Q_{0|Y})). \tag{6.11}$$

It is straightforward to check [11, p. 724] that the triple $([f]^0, [f]^1; [f \odot d])$ is a cocycle in the hypercohomology group of $H^2(\underline{U}, \mathscr{H}om(Q_{|Y}, Q_{|Y}))$ and whose vanishing is equivalent to the existence of $\{(\tilde{f}_{\alpha\beta}, \tilde{d}_\alpha)\}$ so that the left-hand sides of (6.10) and (6.11) are zero cochains. Thus $(\widetilde{Q}_{i,\alpha}, \tilde{f}^i_{\alpha\beta})$ form locally free sheaves $\widetilde{Q}_i$ and $\{\tilde{d}_\alpha\}$ induces a homomorphism $\tilde{d}: \widetilde{Q}_1 \to \widetilde{Q}_0$. Let M_{n+1} be the cokernel of $\tilde{d}$. Since $\widetilde{Q}_i$ are locally free and $\tilde{d}_{|Y\times A_n}$ is identical to (6.9), one obtains easily that M_{n+1} is flat over A_{n+1} and $M_{n+1|Y\times A_n} = M_n$. We define $([f]^0, [f]^1; [f \odot d])$ to be the obstruction class $\omega(M_n, L)$. One further checks that if $\det M_n = p_Y^* L$, $\omega(M_n, L)$ is traceless [9]. Finally, following [9], when $\omega(M_n, L) = 0$, we can choose M_{n+1} so that $\det M_{n+1} = p_X^* L$. The last statement of the lemma follows from [15], [12].

Lemma 6.9. *With the notation as before and let* $4\Sigma_S^- = 4\Sigma^- \times S$. *Then there is a canonical exact sequence of* $\mathscr{O}_S$*-modules*:

$$\begin{aligned}(6.12)\qquad & \operatorname{Ext}^1_{\Delta\times S}(E_S, E_S(-4\Sigma_S^-))^0 \xrightarrow{i} \operatorname{Ext}^1_{\Delta\times S}(E_S, E_S)^0 \\ & \xrightarrow{\rho} \operatorname{Ext}^1_{4\Sigma^-\times S}(E_{S|4\Sigma_S^-}, E_{S|4\Sigma_S^-})^0 \to \operatorname{Ext}^2_{\Delta\times S}(E_S, E_S(-4\Sigma_S^-))^0.\end{aligned}$$

Further, if E_S^n *is a determinant fixing deformation of* E_S^{n-1} *over* A_n, *and* $g \in \operatorname{Ext}^1_{\Delta\times S}(E_S, E_S)^0$, *then the new deformation* $(E_S^n)^g$ *of* E_S^{n-1} *given by Lemma* 6.8 *satisfies* $(E_S^n)^g|4\Sigma_S^- = (E^n_{S|4\Sigma_S^-})^{\rho(g)}$.

Proof. Let $0 \to Q_2 \to Q_1 \to E_S \to 0$ be a locally free resolution. Then we have the following exact sequence of complexes of sheaves:

$$\begin{aligned}(6.13)\qquad & 0 \to \mathscr{H}om(Q_\cdot, Q_\cdot(-4\Sigma_S^-)) \to \mathscr{H}om(Q_\cdot, Q_\cdot) \\ & \to \mathscr{H}om(Q_{\cdot|4\Sigma_S^-}, Q_{\cdot|4\Sigma_S^-}) \to 0.\end{aligned}$$

Note that since $E_{S,v}$ is torsion free for closed $v \in S$,

$$(6.14)\qquad 0 \to Q_{1|4\Sigma_S^-} \to Q_{0|4\Sigma_S^-} \to E_{S|4\Sigma_S^-} \to 0$$

is still exact [19]. Thus by taking the corresponding long exact sequence of the hypercohomologies of the short exact sequence (6.13) and combining it with (6.14), we get

$$\begin{aligned}(6.15)\qquad & \operatorname{Ext}^1_{\Delta\times S}(E_S, E_S(-4\Sigma_S^-)) \to \operatorname{Ext}^1_{\Delta\times S}(E_S, E_S) \\ & \to \operatorname{Ext}^1_{4\Sigma^-\times S}(E_{S|4\Sigma_S^-}, E_{S|4\Sigma_S^-}) \to \operatorname{Ext}^2_{\Delta\times S}(E_S, E_S(-4\Sigma_S^-)).\end{aligned}$$

Finally, one checks directly that the traceless submodule $\operatorname{Ext}(\cdot, \cdot)^0 \subseteq \operatorname{Ext}(\cdot, \cdot)$ is preserved in (6.15). Thus (6.12) is exact. The last statement can be proved by using [12]. q.e.d.

Now we continue our construction of deformations. We first let F_S^2 be the constant deformation of E_S over A_2. By the deformation lemma, the space of all determinant fixing deformations is a homogeneous space with group $\operatorname{Ext}^1_{\Delta\times S}(E_S, E_S)^0$, and by Lemma 6.9 the subspace of deformations that induce constant deformation along $4\Sigma^- \times S$ is isomorphic to the image of $\operatorname{Ext}^1_{\Delta\times S}(E_S, E_S(-4\Sigma_S^-))^0 \xrightarrow{i} \operatorname{Ext}^1_{\Delta\times S}(E_S, E_S)^0$. Because $H^2(\Delta\times S, \mathscr{E}nd^0(E_S)(-4\Sigma_S^-)) = 0$, the long exact sequence coming from the spectral sequence $H^{\bullet}(\mathscr{E}xt(\;)) \Rightarrow \operatorname{Ext}^{\bullet}(\;)$ guarantees that

$$\operatorname{Ext}^1_{\Delta\times S}(E_S, E_S(-4\Sigma_S^-))^0 \xrightarrow{\zeta} H^0(\Delta\times S, \mathscr{E}xt^1(E_S, E_S(-4\Sigma_S^-)^0)) \to 0 \tag{6.16}$$

is exact [11]. Since $\operatorname{col}(E_{S,0})_p = 1$, there is a $g \in \operatorname{Ext}^1_{\Delta\times S}(E_S, E_S(-4\Sigma_S^-))^0$ so that $\zeta(g)$ generates the $\mathscr{O}_p$-module $\mathscr{E}xt^1(E_S, E_S(-4\Sigma_S^-))^0_p$. Thus by [29],

$$(F_S^2)^g|\Delta\times\{0\}\times A_2 \notin \mathfrak{R}^{\mathrm{sh}}(s, p, U). \tag{6.17}$$

We let $E_S^2 = (F_S^2)^g$. In general, since $\operatorname{Ext}^2(E_S, E_S)^0 = \{0\}$ by the vanishing of $H^2(\mathscr{E}nd^0(E_S)(-4\Sigma_S^-))$, we can successively apply Lemmas 6.8 and 6.9 to find determinant fixing deformation E_S^n of E_S^2 over A_n, $n \geq 3$, such that $E^n_{S|4\Sigma^-\times S\times A_n}$ is the constant deformation of $E_{S|4\Sigma^-\times S}$. Therefore by Lemma 6.6, $E^n_{S|\Delta\times\{0\}\times A_n} \in \overline{\mathfrak{S}_F}$ and $E^n_{S|\Delta\times(S\backslash 0)\times A_n} \subseteq \overline{\mathfrak{S}}$ which implies $E_S^n \subseteq \overline{\mathfrak{S}}$. Hence $E^n_{S|\Delta\times\{0\}\times A_n} \in \overline{\mathfrak{S}_F} \cap \overline{\mathfrak{S}}$. On the other hand, by (6.17), $E^n_{S|\Delta\times\{0\}\times A_n}$ is not contained in the $n-1)\mathfrak{R}^{\mathrm{sh}}(s, p, U)$. Since n can be arbitrary large, (6.8) is impossible. So Theorem 6.2 is established. q.e.d.

Now we prove Proposition 6.5. We first study the case where $\alpha = 0$. We state a useful result of [3] for constructing vector bundles on Δ.

Lemma 6.10. *Let E be any rank-2 locally free sheaf on Δ with $c_1 = 0$ of generic fiber type $(0, 0)$. Then there is a l.c.i. zero scheme $Z \subseteq \Delta$ such that E belongs to the exact sequence*

$$0 \to F \to E \to \mathscr{I}_{Z\subset Y} \to 0, \tag{6.18}$$

where $F = q^*q_*E$, and $\mathscr{I}_{Z\subset Y}$ is the ideal sheaf of Z in $Y = q^{-1}q(Z)$. Here $q(Z)$ is the scheme-theoretic image of Z. Further (6.18) *is unique, $c_1(F) = -[Y]$ and $c_2(E) = l(Z)$.*

For our purposes, we need to generalize this construction to a family of sheaves on Δ. Let $\mathscr{E}$ be the restriction to $\Delta\times\mathfrak{S}$ of the universal quotient family. Let $\mu\colon \Delta \times \mathfrak{S} \to \Sigma \times \mathfrak{S}$ be the obvious projection. Since for the generic points $s \in \mathfrak{S}$, $\mathscr{E}_s$ has generic fiber type $(0, 0)$ (Lemma 6.3) and

$$\mu_*(\mathscr{E}^\vee)_{|\Sigma\times\{s\}} \hookrightarrow q_*(\mathscr{E}^\vee_{|\Delta\times\{s\}}) \tag{6.19}$$

is an isomorphism, there is a Zariski open set $T \subseteq \mathfrak{S}$ such that $\mathscr{W} = \mu_*(\mathscr{E}^\vee)_{|\Sigma\times T}$ is locally free and that for any closed $s \in T$, (6.19) is an isomorphism. In the following, we shall view $\mathscr{W}$ as a family of locally free sheaves on Σ parameterized by T. Now according to Lemma 6.10, there is a l.c.i. codimension-two subscheme $Z \subset \Delta \times T$ so that for $Y = \mu^{-1}\mu(Z)$, $\mathscr{E}^\vee$ belongs to the exact sequence

$$0 \to \mu^*\mathscr{W} \to \mathscr{E}^\vee \to p_T^*\mathscr{L} \otimes \mathscr{I}_{Z\subset Y} \to 0, \tag{6.20}$$

where $\mathscr{L}$ is a line bundle on T. Since $l(Z_s)$ and $l(q(Z_s))$, $s \in T$ closed, are independent of $s \in T$ by Lemma 6.10, both Z and Y are flat over T. Now, by replacing T with a smaller open subset (which we still denote by T), we can assume $\#\operatorname{supp}(q(Z_s))$ is constant over T. We have the following estimate:

Lemma 6.11. *With the notation as above,*

$$\#\operatorname{supp}(q(Z_s)) \geq k - A_2', \qquad A_2' = A_2 + 2(g+1). \tag{6.21}$$

Proof. Let $s \in T$ be a general closed point. Assume $L \subseteq \mathscr{W}_s$ is a subline bundle so that no subline bundle of $\mathscr{W}_0$ has degree higher than the degree of L. Then $\mathscr{E}_s^\vee$ belongs to the exact sequence

$$0 \to q^*L \to \mathscr{E}_s^\vee \to q^*L^{-1} \otimes \mathscr{I}_{Z_s} \to 0. \tag{6.22}$$

The number of moduli of the set of vector bundles belonging to (6.22) (with L and Z_s fixed) is at most

$$\dim \operatorname{Ext}^1(q^*L^{-1} \otimes \mathscr{I}_{Z_s}, q^*L) - 1 \leq -2\deg L + 2g + l(Z_s).$$

(If $\deg L > -g$, the estimate is different but the conclusion of the lemma can be derived more easily.) It is easy to see that $\deg L \geq -\frac{1}{2}l(q(Z_s)) - g$ since $\deg \mathscr{W}_s = l(q(Z_s))$. Further, for fixed distinct points $x_1, \cdots, x_h \subseteq \Sigma$, the number of moduli of the set of all zero scheme $z \subseteq \Delta$ with $\operatorname{supp}(q(z)) \subseteq \{x_1, \cdots, x_h\}$ is at most $l(z)$ [14]. Therefore, the number of moduli of T can be at most

$$\begin{aligned} &\dim \operatorname{Pic}(\Sigma) + \#\text{moduli of } \{Z_s\} + \dim \operatorname{Ext}^1(q^*L \otimes \mathscr{I}_{z_s}, q^*) - 1 \\ &\quad \leq g + l(Z_s) + \#\operatorname{supp}(q(Z_s)) + (-2\deg L + 2g + l(Z_s)). \end{aligned}$$

However, we have assumed that the number of moduli of T is at least $4k + 3(g-1) - A_2$. Therefore, combined with $k = l(Z_s)$, we obtain (6.21).

Remark. The lemma implies that for generic $s \in T$, $\mathscr{W}_s$ is e_0-stable with $e_0 = A_2' + 1$, and Z_s is the union of z' and z'' where $q(z')$ and $q(z'')$ are disjoint, z' is a simple scheme with $l(z') \geq k - 2A_2'$. Here, we call $z \subseteq \Delta$ a simple scheme if z is reduced and the projection $\mathrm{supp}(z) \to \mathrm{supp}(q(z))$ is one-to-one.

For any rank-2 vector bundle W on Σ, we let $T_W = \{s \in T | \mathscr{W}_s \cong W\}$. Since the number of moduli of the set of rank-2 vector bundles on Σ is at most $4g - 3$, there is a W so that

$$\dim T_W \geq \dim T - (4g - 3). \tag{6.23}$$

We fix such a W. For any $E \in T_W$, by dualizing (6.20), E belongs to the exact sequence

$$0 \to E \to W^\vee \xrightarrow{\gamma} F \to 0. \tag{6.24}$$

By the remark after Lemma 6.11 and the assumption on the number of moduli of T, we can assume that there are m distinct points $x_1, \cdots, x_m \subseteq \Sigma$, $m \geq k - 2A_2'$, a sheaf F' supported on fibers of $q|\Delta$ other than $P_i = q^{-1}(x_i)$, $i = 1, \cdots, m$, with surjective homomorphism $\sigma' \colon W^\vee \to F'$ such that the number of moduli of the set

$$T([x], (\sigma', F')) = \{E \in T_W | E \text{ belongs to (6.27) with } F = \bigoplus_{i=1}^m \mathscr{O}_{P_i}(1) \oplus F'$$

$$\text{and } \gamma = \bigoplus_{i=1}^m \sigma_i \oplus \sigma' \text{ where } \sigma_i \colon W^\vee \to \mathscr{O}_{P_i}(1)\}$$

is at least $3m - A_4$, where A_4 is a constant depending on A_2 and Δ only. Put $R = \ker\{\sigma' \colon W^\vee \to F'\}$. R is locally free. Then all $E \in T([x], (\sigma', F'))$ belongs to the exact sequence

$$0 \to E \to R \xrightarrow{\sigma} \bigoplus_{i=1}^m \mathscr{O}_{P_i}(1) \to 0. \tag{6.25}$$

We choose the trivialization $R_{|P_i} \cong \mathscr{O}_{P_i} \oplus \mathscr{O}_{P_i}$, $P_i = q^{-1}(x_i)$, and basis $\{u_i, v_i\}$ of $H^0(\mathscr{O}_{P_i}(1))$ so that $u_i^{-1}(0) \in \Sigma^-$. We then identify $s^i = [s_1^i, \cdots, s_4^i] \in \mathbf{P}_i^3 \cong \mathbf{P}^3$ to homomorphism $s^i \colon R \to \mathscr{O}_{P_i}(1)$,

$$s^i(1, 0) = s_1^i u_i + s_2^i v_i, \qquad s^i(0, 1) = s_3^i u_i + s_4^i v_i. \tag{2.26}$$

For any $s = (s^1, \cdots, s^m) \in \Pi_{i=1}^m \mathbf{P}_i^3$, we define

$$\sigma_s = \bigoplus_{i=1}^m s^i \colon R \to \bigoplus_{i=1}^m \mathscr{O}_{P_i}.$$

Note that if $s^i \notin \mathbf{Q}_i \subseteq \mathbf{P}_i^3$ (resp. $s^i \notin \mathbf{M}_i \subseteq \mathbf{P}_i^3$), where $\mathbf{Q}_i = \{s_1^i s_4^i = s_2^i s_3^i\}$ (resp. $\mathbf{M}_i = \{s_4^i = s_2^i = 0\}$), then the kernel E of (6.28) with $s = \bigoplus^m s^i$ is locally free at P_i (resp. $\Sigma^- \cap P_i$). Now put

$$[\mathbf{Q}] = \{s \in \Pi^m \mathbf{P}_i^3 \mid \text{there is at least one } i \text{ so that } s^i \in \mathbf{Q}_i\}$$

and $[\mathbf{Q}]^- = \Pi^m \mathbf{P}_i^3 \backslash [\mathbf{Q}]$. By [3], there is a flat family of locally free sheaves $\mathscr{V}$ on $\Delta \times [\mathbf{Q}]^-$ such that for any closed $s \in [\mathbf{Q}]^-$, $\mathscr{V}_s$ belongs to (6.28) with $\sigma = \sigma_s$. Let $[\mathbf{Q}]_W^- = \{s \in [\mathbf{Q}]^- \mid \mathscr{V}_s \in T_W\}$. Since the number of moduli of $T([x], (\sigma', F'))$ is at least $3m - A_4$,

$$\text{(6.27)} \qquad \dim [\mathbf{Q}]_W^- \geq 3m - A_4.$$

Let $\mathscr{W}$ be the closure of $[\mathbf{Q}]_W^-$ in $\Pi^m \mathbf{P}_i^3$. Since $[\mathbf{Q}]$ is an ample divisor of $\Pi^m \mathbf{P}_i^3$, $[\mathbf{Q}] \cap \mathscr{W} \neq \varnothing$. Now let $s_0 \in [\mathbf{Q}] \cap \mathscr{W}$. We will show that s_0 corresponds to non-locally-free sheaf. Let $0 \in D$ be a small disk with parameter t and let $\varphi \colon (0, D) \to (s_0, \mathscr{W})$, $\varphi(D \backslash 0) \subseteq [\mathbf{Q}]_W^-$. φ is represented by families of homomorphisms $s^i(t) \in \operatorname{Hom}(R, \mathscr{O}_{P_i}(1))$. In this way we obtain a sheaf E_D which belongs to the exact sequence

$$0 \to E_D \to R \otimes_{\mathscr{O}_\Delta} \mathscr{O}_{\Delta \times D} \overset{\sigma_{s(\cdot)}}{\to} \bigoplus^m \mathscr{O}_{P_i \times D}(1).$$

Let $\mathscr{L}_D = \operatorname{Im}\{\sigma_{s(\cdot)}\} \subset \bigoplus^m \mathscr{O}_{P_i \times D}(1)$ and let $\mathscr{L}_D^i = \operatorname{Im}\{s^i(\cdot)\} \subset \mathscr{O}_{P_i \times D}(1)$. Clearly, $\mathscr{L}_D$ is flat over D. Assume $s^i(0) \in \mathbf{Q}_i$. Then $\mathscr{L}_D^i \otimes k(0) \cong \mathscr{O}_{P_i} \oplus \mathbb{C}_{p_i}$, $p_i \in P_i$. Consequently, $E_D \otimes k(0)$ is non-locally-free at p_i and $\operatorname{col}(E_{D,0})_{p_i} = 1$. We denote $E_{D,0} = E_{s_0}$, and remark that E_{s_0} does depend on the choice of the curve $\varphi \colon D \to \mathscr{W}$.

Proposition 6.12. *With the notation as above, there are constants A and e_3 such that whenever $m \geq A$, $[\mathbf{Q}] \cap \mathscr{W} \neq \varnothing$ and further, for generic $s \in [\mathbf{Q}] \cap \mathscr{W}$,*

(1) *there is a closed point $p \in \Delta \backslash \Sigma^-$ such that* $\operatorname{col}(E_s)_p = 1$,

(2) *E_s is e_3-stable.*

Proof. For (1), we only need to show that $[\mathbf{Q}] \cap \mathscr{W}$ is not contained in $\Pi^m \mathbf{M}_i$ when m is large. But this follows from the dimension comparison since $\dim \Pi_{i=1}^m \mathbf{M}_i = m$ while $\dim [\mathbf{Q}] \cap \mathscr{W} \geq 3m - A_4 - 1$.

Now let $s \in [\mathbf{Q}] \cap \mathscr{W}$ be a general closed point. Thanks to the remark after Lemma 6.11, we know that W is e_0-stable for a constant e_0. We fix a $\Sigma^+ \subseteq \Delta$ so that E_s (which is well-defined if we choose a curve D passing s) is locally free along Σ^+. Let $v_i \in H^0(\mathscr{O}_{P_i}(1))$ be such that $v_i^{-1}(0) \in \Sigma^+$.

By definition, $l(F'_{|\Sigma^+}) \le 2A'_2$. Thus $R_{|\Sigma^+}$ is e_1-stable with $e_1 \le e_0 + 2A'_2$. Now we define a birational map $f\colon \Pi^m \mathbf{P}^3_i \to \Pi^m \mathbf{P}^1$ by sending $s \in \Pi^m \mathbf{P}^3_i$ to $([s^1_2, s^1_4], \cdots, [s^m_2, s^m_4]) \in \Pi^m \mathbf{P}^1$. Since E_s is locally free along Σ^+, f is well defined near s. Since $\mathscr{W}$ has bounded codimension in $\Pi^m \mathbf{P}^3_i$, the images $f([\mathbf{Q}] \cap \mathscr{W})$ has bounded codimension also. In particular, since $s \in [\mathbf{Q}] \cap \mathscr{W}$ is general, for some universal constant e_2 (depending on A_2), we can assume $([s^1_2, s^1_4], \cdots, [s^{m-e_2}_2, s^{m-e_2}_4])$ is in general position of $\Pi^{m-e_2}\mathbf{P}^1$. Note that $E_{s|\Sigma^+}$ belongs to the exact sequence

$$0 \to E_{s|\Sigma^+} \to R_{|\Sigma^+} \xrightarrow{\gamma} \bigoplus^m \mathbb{C}_{\Sigma^+ \cap P_i} \to 0,$$

where $\gamma^i(a, b) = s^i_2 a + s^i_4 b$. Since when $k \gg 0$, $m - e_2 \ge 2e_1$, we can choose γ so that $\operatorname{Ker}\{\bigoplus_{i=1}^{m-e_2} \gamma^i\colon R_{|\Sigma^+} \to \bigoplus_{i=1}^{m-e_2} \mathbb{C}_{\Sigma^+ \cap P_i}\}$ is stable. Therefore, $E_{s|\Sigma^+}$ is e_2-stable. Then one checks directly that E_s is e_3-stable for some constant e_3 (depending on A_2 and Δ) because E_s is of generic fiber type $(0, 0)$ and ε is small.

Proof of Proposition 6.5 *in case* $\alpha = 0$. When $e_3 \le e$, the E_s constructed in Proposition 6.12 is contained in $\overline{\mathfrak{S}} \subseteq \mathfrak{M}^e_k$ already. Now assume $e_3 > e$. By the technique of [30], when $k \gg 0$,

$$\dim(\mathfrak{M}^{e_3}_k \backslash \mathfrak{M}^e_k) \le \dim \mathfrak{M}^{e_3}_k - 2A_2 - 2. \tag{6.28}$$

Therefore, there is a unique maximal irreducible component $\mathfrak{S}_3$ of $\mathbf{\Theta}^{e_3}_V \subseteq \mathfrak{M}^{e_3}_k$ such that $\mathfrak{S} \subseteq \mathfrak{S}_3$. Let $\overline{\mathfrak{S}_3}$ be the closure of $\mathfrak{S}_3$ in $\mathfrak{M}^{e_3}_k$. By (6.28),

$$\dim(\overline{\mathfrak{S}_3} \backslash \overline{\mathfrak{S}}) \le \dim \mathfrak{M}^{e_3}_k - 2A_2 - 2.$$

Now let E $(= E_s) \in \overline{\mathfrak{S}}$ be the sheaf provided by Proposition 6.12. Namely, for some $p \in \Delta \backslash \Sigma^-$, $\operatorname{col}(E)_p = 1$. We can apply Lemma 6.7 to the pair (E, p) to conclude that there is a classical neighborhood $U \subseteq \mathfrak{M}^{e_3}_k$ of E such that $\mathfrak{M}^{\mathrm{sh}}(E, U, p) \cap \overline{\mathfrak{S}_3}$ is a nonempty divisor. Then by (6.28), $\mathfrak{M}^{\mathrm{sh}}(E, U, p) \cap \overline{\mathfrak{S}} \neq \varnothing$. Therefore, Proposition 6.5 is true for $\alpha = 0$. q.e.d.

Our strategy of proving Proposition 6.5 when $\alpha = 1$ is by reducing it to the case $\alpha = 0$. Let $\mathfrak{S}$ be the maximal irreducible component of $\mathbf{\Theta}^e_V \subseteq \mathfrak{M}^{1,e}_k$ having codimension at most A_2. We fix a Σ^+. Since $d \gg 0$, for generic $V \in \mathfrak{S}$, $h^0(\mathscr{E}nd^0(V)(-\Sigma^+ - 4\Sigma^-)) = 0$. So

$$\operatorname{Ext}^1(V, V)^0 \to \operatorname{Ext}^1(V_{|4\Sigma^-}, V_{|4\Sigma^-})^0 \oplus \operatorname{Ext}^1(V_{|\Sigma^+}, V_{|\Sigma^+})^0$$

is surjective. Therefore, there is a deformation V_u of V such that $V_{u|4\Sigma^-}$ is a constant deformation while $V_{u|\Sigma^+}$ is stable for general u. In particular, we can assume $V_{|\Sigma^+}$ is stable already. Let $S \subseteq \mathfrak{S}$ be the set of all $V \in \mathfrak{S}$ such that $V_{|\Sigma_+}$ is stable. Let M be a line bundle on Σ^+ with $\deg M = -3g$. We consider the set

(6.29) $$\mathscr{C} = \{(V, \sigma) | V \in S, \sigma \text{ is a surjective map } V \to M^{-1}\}.$$

Let $p: \mathscr{C} \to S$ be the obvious projection. On $\Delta \times \mathscr{C}$, there is a family of sheaves $\mathscr{F}$ such that for any $w = (s, \sigma) \in \mathscr{C}$, $\mathscr{F}_w$ belongs to the exact sequence

(6.30) $$0 \to \mathscr{F}_w \to \mathscr{E}_s \xrightarrow{\sigma} M^{-1} \to 0,$$

where $\mathscr{E}$ is the universal family on $\Delta \times S$. We claim that for general $w \in \mathscr{C}$, $\mathscr{F}_w$ is of generic fiber type $(0, 0)$. Indeed, since $\mathscr{E}_s$ is of generic fiber type $(1, 0)$, by [3], $\mathscr{E}_s$ fits into the following exact sequence

(6.31) $$0 \to q^*L^{-1}(\Sigma^-) \to \mathscr{E}_s \to q^*L \otimes \mathscr{I}_z \to 0,$$

where $q: \Delta \to \Sigma$, and L is a line bundle on Σ. Since the number of moduli of S is at least $4k + 3(g-1) - A_2$, an argument similar to the proof of Lemma 6.11 shows that for generic $s \in S$, the line bundle L in (6.31) satisfies $\deg L = k + O(1)$. In particular, when k is large, $\deg L \geq 9g$. Thus,

$$q^*L^{-1}(\Sigma^-) \to \mathscr{E}_s \xrightarrow{\sigma} M^{-1}$$

cannot be trivial because otherwise we would have nontrivial $L \to M^{-1}$. Therefore, by shrinking S if necessary, we can assume the sheaves $\mathscr{F}_w$ in (6.30) have generic fiber type $(0, 0)$.

Next, we claim that the number of moduli of $\{\mathscr{F}_w | w \in \mathscr{C}\}$ is larger than $4k + 3(g-1) - A_2 - 9g$. Indeed, by (6.30), $\mathscr{F}_{w|\Sigma^+}$ belongs to the exact sequence

$$0 \to M' \to \mathscr{F}_{w|\Sigma^+} \to M \to 0,$$

where $M' = \mathscr{O}_{\Sigma^+}(-\Sigma^+ - M)$. In particular, $\deg M' \otimes M^{-1} \geq 2g$ since $9|K_X \cdot H| \leq |H \cdot H|$. Thus, $\mathscr{F}_{w|\Sigma^+} = M \oplus M'$. Therefore, for fixed $\mathscr{F}_w$, the number of moduli of the set of all locally free sheaves V that belong to the exact sequence

$$0 \to \mathscr{F}_w \to V \to M \to 0$$

is at most $9g$. Since the number of moduli of S is at least $4k+3(g-1) - A_2$, the number of moduli of $\{\mathscr{F}_w | w \in \mathscr{C}\}$ is at least $4k + 3(g-1) - A_2 - 9g$. Finally, since $\mathscr{F}_{w|\Sigma^+} = M \oplus M'$ and $\mathscr{F}_w$ is of generic fiber type $(0, 0)$, $\mathscr{F}_w$ is e_1-stable for $e_1 = 18g$. Notice $c_2(F_w) = k_1 = k - 3g$.

Now let $T \subseteq \mathfrak{R}_{k_1}^{0,e_1}$ be the set such that $z \in T$ if and only if $\mathscr{E}'_z \in \{\mathscr{F}_w |$ for some $w \in \mathscr{C}\}$, where $\mathscr{E}'$ is the universal family of $\mathfrak{R}_{k_1}^{0,e_1}$, and let

$$\Theta_{V,M}^{e_1} = \{E \in \mathfrak{R}_{k_1}^{0,e_1} | E_{|4\Sigma^-} \cong V \text{ and } E_{|\Sigma^+} \cong M \oplus M'\}.$$

Clearly, $T \subseteq \Theta_{V,M}^{e_1} \subseteq \mathfrak{R}_{k_1}^{0,e_1}$ and $\operatorname{codim}(T, \mathfrak{R}_{k_1}^{0,e_1}) \leq A_2 + 9g$. By the proof of Theorem 6.1 in case $\alpha = 0$, we can find a family of sheaves $E_D \subseteq \Theta_{V,M}^{e_1}$ on $\Delta \times D$, where D is an irreducible curve, having the following properties: There are $0, 1 \in D$ such that $E_0 \in T$ and $\operatorname{col}(E_1)_p = 1$ for some $p \in \Delta \backslash \Sigma^-$. (It is easy to see that Theorem 6.1 still holds when we replace Θ_V^e by $\Theta_{V,M}^e \subseteq \mathfrak{R}_k^{0,e}$.) Thus by reversing elementary transformation,

$$0 \to F_t \to E_t \to M^{-1} \to 0,$$

we obtain a family of sheaves F_t, $t \in D$, that are e_2-stable for a constant e_2 independent of k such that $F_0 \in S$ and $\operatorname{col}(F_1)_p = 1$. In particular, if we let $\mathfrak{S}_2 \subseteq \Theta_V^{1,e_2} \subset \mathfrak{R}_k^{1,e_2}$ be the maximal irreducible component containing $\mathfrak{S}$, then $\overline{\mathfrak{S}_2}$ contains a sheaf E with $\operatorname{col}(E)_p = 1$, $p \in \Delta \backslash \Sigma^-$. Thus by letting A large and assuming $k \geq A$, the estimate (6.31) will allow us to find $E \in \overline{\mathfrak{S}} \subseteq \mathfrak{R}_k^{1,e}$ that has the desired properties. This completes the proof of Proposition 6.5.

7. Proof of the main theorem

In this section, we will prove Theorems 0.1, 0.2, and the main theorem stated in the introduction. Let $\mathbf{M}$ be any irreducible component of $\mathfrak{M}_X^{d,I}$ and let $\mathfrak{N}^d \subseteq \mathfrak{M}^d$ be the corresponding irreducible component. Our strategy for proving that $\mathbf{M}$ contains non-locally-free stable sheaves is to show that there are non-locally-free stable sheaves in $\mathfrak{N}_0^d$ and that the generic such sheaf on Z_0 can be lifted to non-locally-free sheaves in $\mathfrak{N}_t^d = \mathbf{M}$. In producing the desired deformation, we need a vanishing theorem similar to [9]. Thus we will work on the scheme $\widetilde{Z}_{C^m}^m$ rather than on Z.

We assume that $d \geq A$ and in the subsequent discussion, A is large so that the previous requirements for the second Chern class d are all satisfied. We first prove the following theorem.

Theorem 7.1. *For any ample divisor H and any divisor I, there is a constant A such that whenever $d \geq A$, any irreducible component $\mathbf{M} \subseteq \mathfrak{M}_X^{d,I}$ (moduli of rank-2 H-semistable sheaves) contains at least one non-locally-free μ-stable sheaf. Further, if we let $\mathbf{M}^{(1)} \subseteq \mathbf{M}$ be the subset of*

μ-stable sheaves E with $\operatorname{col}(E) = 1$, *the* $\mathbf{M}^{(1)}$ *is a nonempty, codimension-1 subset of* $\mathbf{M}$.

Let $\mathfrak{N}^d \subseteq \mathfrak{M}^d$ be the irreducible component corresponding to $\mathbf{M} \subseteq \mathfrak{M}^d_X$. Let $\mathfrak{U}^{ss} \subseteq \mathfrak{R}^d$, and let $\pi\colon \mathfrak{U}^{ss} \to \mathfrak{M}^d$ be the quotient projection. Then by Proposition 3.3, there is an open subset

$$U \subseteq \mathfrak{U}^{ss} \cap \pi^{-1}(\mathfrak{N}^d), \tag{7.1}$$

an integer $m \geq 1$ and a flat family of sheaves $\mathscr{E}_{U^m}$ on $\tilde{Z}^m_{C^m} \times_{C^m} U^m$ having the following properties: $\mathscr{E}_{U^m}$ is a proper transform of the universal quotient family on $Z \times_C U$ such that it is flat over U^m and that the restriction of $\mathscr{E}_{U^m}$ to $\Delta_m \times U^m_0 \subseteq \tilde{Z}^m_{C^m} \times_{C_m} U^m$ satisfies the conclusion of Proposition 3.3. Choose a general closed point $E_0 \in U^m_0$. Let (α, k) be such that $\det E_{0|\Delta_m} = I_{\Delta_m}(\alpha\Sigma^-)$ and $c_2(E_{0|\Delta_m}) = k$. Here $\Sigma^- = \Delta_m \cap \Delta_{m-1}$. Let $4\Sigma^- \subseteq \Delta_m$ be as in §6; that is, $4\Sigma^- = 4\Delta_{m-1} \cap \Delta_m$. Then by Corollary 5.11 and Theorem 5.12, we can assume the set $\Psi(U^m_0) \subseteq \mathfrak{M}^{\alpha,k}_{\Delta_m}$ has dimension c_d, where $\Psi\colon U^m_0 \to \mathfrak{M}^{\alpha,k}_{\Delta_m}$ is the rational map induced by the family $\mathscr{E}_{U^m|\Delta_m}$. Further, by Proposition 4.6, k is bounded from above by $d + A_1$, where A_1 is a constant depending only on (X, H, I). Therefore, there is a constant A_2 such that $\operatorname{codim}\Psi(U^m_0) \leq A_2$.

Now let $\Theta = \{E \in \mathfrak{M}^{\alpha,k}_{\Delta_m} | E_{|4\Sigma^-} \cong E_{0|4\Sigma^-}\}$, and let A_3 be the number of moduli of the set of all rank-2 locally free sheaves on $4\Sigma^-$ (it is finite). Since E_0 is generic, we can assume

$$\operatorname{codim}(\Psi(U^m_0) \cap \Theta) \leq A_2 + A_3. \tag{7.2}$$

We remark that since $I_{\Delta_m} = 2I'$, by tensoring $\mathscr{E}_{U^m|\Delta_m}$ with $-I'([\alpha/2]\Sigma^-)$, the new family satisfies the condition of Theorem 6.1. Since when A is large and $d \geq A$, (7.2) ensures that k is large. We now fix $e = -(H \cdot H)$. By assuming d large, we can assume $E_{0|\Delta_m}$ is e-stable (see (6.31)). Then by Theorem 6.1, there is a smooth affine curve S, two closed points $s_0, s_1 \in S$ and a flat family of e-stable sheaves E_S on $\Delta_m \times S$ having the following properties: $E_{S|4\Sigma^-}$ is a constant family of locally free sheaves, $h^2(\Delta_m, \mathscr{E}nd^0(E_s)(-4\Sigma^-)) = 0$ for all closed $s \in S$, $E_{s_0} \cong E_{0|\Delta_m}$ and $\operatorname{col}(E_{s_1})_p = 1$ for some $p \in \Delta_m \backslash \Sigma^-$.

Let L be an ample line bundle on $\tilde{Z}^m_{C^m}$ such that the restriction of L to $\tilde{Z}^m_t$, $t \neq 0$, is isomorphic to H. Let $n \gg 0$ such that for any closed $u \in U^m$ (over $t \in C^m$), $h^i(\tilde{Z}^m_t, \mathscr{E}_u \otimes L_t^{\otimes n}) = 0$ for $i > 0$ and $H^0(\tilde{Z}^m_t, \mathscr{E}_u \otimes$

$L_t^{\otimes n}$) generates the sheaf $\mathscr{E}_u \otimes L_t^{\otimes n}$ on $\widetilde{Z}_t^m$. Put $W = \bigoplus^N L^{\otimes(-n)}$ and let $\mathfrak{R}^{\chi}_{\widetilde{Z}^m}$ be the Grothendieck's Quot-scheme of all quotient sheaves F of W on $\widetilde{Z}_t^m$, $t \in C^m$, with $\chi_F = \chi$. We let $\mathscr{V} \subseteq \mathfrak{R}^{\chi}_{\widetilde{Z}^m}$ be the set of all $v \in \mathscr{V}$ such that the quotient sheaf $\mathscr{F}_v$ ($\mathscr{F}$ is the universal quotient family of $\mathfrak{R}^{\chi}_{\widetilde{Z}^m}$) is isoorphic to $\mathscr{E}_u$ for some $u \in U^m$ with $\pi_m(u) = \pi_m(v) \in C^m$. Since U^m is irreducible, $\mathscr{V}$ is also so (notice $n \gg 0$).

Now let $\mathfrak{R}^{\chi,0}_{\widetilde{Z}^m} \subseteq \mathfrak{R}^{\chi}_{\widetilde{Z}^m}$ be the irreducible component that contains $\mathscr{V}$. In the following, we shall show that $\mathfrak{R}^{\chi,0}_{\widetilde{Z}^m}$ contains E_{s_1} as a quotient sheaf of W. Indeed, since the fiber of U^m over $0 \in C^m$ is reduced, the fiber of $\mathfrak{R}^{\chi,0}_{\widetilde{Z}^m}$ over $0 \in C^m$ is also so. Thus there is a smooth curve $T \to \mathfrak{R}^{\chi,0}_{\widetilde{Z}^m}$ flat over C^m and a closed point $v_0 \in T$ over $0 \in C^m$ such that the quotient sheaf $\mathscr{F}_{v_0}$ is isomorphic to E_0, that the induced morphism $T \to C^m$ is not branched near v_0 and that T is only contained in the irreducible component $\mathfrak{R}^{\chi,0}_{\widetilde{Z}^m} \subseteq \mathfrak{R}^{\chi}_{\widetilde{Z}^m}$. Let F_T be the pullback of the universal family on $\widetilde{Z}_T^m = \widetilde{Z}^m_{C^m} \times_{C^m} T$. Since T is not branched at v_0, $\widetilde{Z}_T^m$ is smooth near $\widetilde{Z}^m_{v_0} \subseteq \widetilde{Z}_T^m$. Further since $F_{v_0} = E_0$ has a length-2 locally free resolution, without loss of generality, we can assume F_T has a length-2 locally free resolution. Let Λ_0 be the fiber of $\widetilde{Z}_T^m$ over v_0 (i.e., $\widetilde{Z}^m_{v_0}$) and let Λ_1 be the subscheme of $\widetilde{Z}_T^m$ supported on Λ_0 defined by the ideal sheaf

$$\mathscr{I}_{\Lambda_1} = \mathscr{O}_{\widetilde{Z}_T^m}\left(-\sum_{i=0}^{m} h_i \Delta_i\right) \subseteq \mathscr{O}_{\widetilde{Z}_T^m}, \tag{7.3}$$

where $h_m = 1$, $h_{m-1} = 4$, and $h_j \geq 1$ will be specified shortly.

Now we construct a sheaf M^1 on $\Lambda_1 \times S$ as follows: We denote $\Delta_m^0 = \Delta_m \backslash \Sigma^-$. We let the restriction of M^1 to $(\Lambda_1 \backslash \Delta_m^0) \times S$ be the pullback of $F_{T|(\Lambda_1 \backslash \Delta_m^0)}$ via $p_1 \colon (\Lambda_1 \backslash \Delta_m^0) \times S \to (\Lambda_1 \backslash \Delta_m^0)$ (a constant family along S), and let the restriction of M^1 to $\Delta_m \times S$ be the family E_S. Since the restrictions of E_S and $p_1^*(F_{T|(\Lambda_1 \backslash \Delta_m^0)})$ to $4\Sigma^- \times S$ are isomorphic constant families on $4\Sigma^-$ (note that $E_{S|4\Sigma^-}$ is locally free), they can be glued together along $4\Sigma^- \times S$ to form a sheaf M^1 on $\Lambda_1 \times S$. Intuitively, M^1 is obtained by replacing the part of $p_1^*(F_{T|\Lambda_1})$ on $\Delta_m \times S$ by E_S. One special feature of M^1 is that the restriction of M^1 and F_T to $\Lambda_1 \times \{0\} = \Lambda_1 \times S \cap \widetilde{Z}_T^m \times \{0\}$ are isomorphic. Clearly, M^1 admits a length-2 locally free resolution and $\det M^1 = p_1^* \det F_{T|\Lambda_1}$. Let $\det F_T = I_T$ be the line bundle on $\widetilde{Z}_T^m$. Let $\Lambda_k = \Lambda_1 + (k-1)\Lambda_0$. We say a sheaf M^{k+1} on $\Lambda_{k+1} \times S$ is a preferred deformation of M^k (on $\Lambda_k \times S$) if (1) $(M^{k+1})_{|\Lambda_k \times S} = M^k$, (2) the length-two locally free solution of M^k can be extended to a locally free resolution of M^{k+1}, (3) $\det M^{k+1} = p_1^*(I_{T|\Lambda_{k+1}})$. We have the following deformation lemma:

Lemma 7.2. *Let* $M^0 = (M^1)_{|\Lambda_0}$. *Suppose*

$$\operatorname{Ext}^2_{\Lambda_0 \times S}(M^0, M^0(-\Lambda_1 \times S))^0 = 0.$$

Then there is a sequence of sheaves $\{M^k\}$, M^k *on* $\Lambda_k \times S$, *such that* M^{k+1} *is a preferred deformation of* M^k *and the restriction of* M^{k+1} *to* $\Lambda_{k+1} \times \{0\}$ *is isomorphic to* $F_{T|\Lambda_{k_1}}$.

Proof. If M^1 is locally free, the existence of a preferred deformation was proved in [9]. In general, we use the length-2 locally free resolution as we did in proving Lemma 6.8. We show that when $\det M^k = p_1^* I_{T|\Lambda_k}$, the obstruction class $\omega(M^k, I_T)$ of the existence of the preferred deformation M^{k+1} lies in the S-module $\operatorname{Ext}^2_{\Lambda_0 \times S}(M^0, M^0(-\Lambda_1 \times S))^0$. By assumption, $\operatorname{Ext}^2_{\Lambda_0 \times S}(M^0, M^0(-\Lambda_1 \times S))^0 = \{0\}$. So we know that there always exist preferred extensions M_-^{k+1} of M^k, and by [15], the set of all such extensions is a homogeneous space isomorphic to $\operatorname{Ext}^1_{\Lambda_0 \times S}(M^0, M^0(-\Lambda_1 \times S))^0$. Since

$$\operatorname{Ext}^1_{\Lambda_0 \times S}(M^0, M^0(-\Lambda_1 \times S))^0 \to \operatorname{Ext}^1_{\Lambda_0 \times \{0\}}(M^0_{|\Lambda_0 \times \{0\}}, M^0_{|\Lambda_0 \times \{0\}}(-\Lambda_1))^0$$

is surjective (see the proof of Lemma 6.9), there is a

$$g \in \operatorname{Ext}^1_{\Lambda_0 \times S}(M^0, M^0(-\Lambda_1 \times S))^0$$

so that $g(M_-^{k+1})_{|\Lambda_{k+1} \times \{0\}}$ is isomorphic to $F_{T|\Lambda_{k+1}}$. Thus we have constructed the desired extension.

One local property of M^{k+1} that follows from the existence of the extension of length-2 locally free resolution is the following: Let t be the uniformizing parameter of T. Then the homomorphism induced by multiplying t,

$$0 \to M^k \xrightarrow{\times t} M^{k+1} \to M^0 \to 0 \tag{7.4}$$

is exact. Indeed, let

$$0 \to Q_1 \xrightarrow{\varphi} Q_0 \to M^k \to 0 \tag{7.5}$$

be a locally free resolution on $\Lambda_k \times S$. Then near each closed point $p \in \Lambda_{k+1}$, there are locally free sheaves $\widetilde{Q}_1$, $\widetilde{Q}_0$ on $\Lambda_{k+1} \times S$ so that we have an exact sequence

$$0 \to \widetilde{Q}_1 \xrightarrow{\tilde{\varphi}} \widetilde{Q}_0 \to M^{k+1} \to 0$$

whose restriction to $\Lambda_k \times S$ is (7.5). Thus we have a commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Q_1 \arrow[r, "\times t"] \arrow[d, "\varphi"] & \widetilde{Q}_1 \arrow[r] \arrow[d, "\tilde{\varphi}"] & Q_{1|\Lambda_0 \times S} \arrow[r] \arrow[d, "\varphi_{|\Lambda_0 \times S}"] & 0 \\
0 \arrow[r] & Q_0 \arrow[r, "\times t"] \arrow[d] & \widetilde{Q}_0 \arrow[r] \arrow[d] & Q_{0|\Lambda_0 \times S} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M^k \arrow[r, "\times t"] \arrow[d] & M^{k+1} \arrow[r] \arrow[d] & M^0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the last row is exact because F_T is flat over T. q.e.d.

Next, we choose $\Lambda_1 = \Sigma_{i=0}^{m} h_i \Delta_i$ so that $\operatorname{Ext}^2_{\Lambda_0 \times S}(M^0, M^0(-\Lambda_1 \times S))^0$ vanishes automatically. By Proposition 4.1, $(M^0)_{|\Delta_j \times S}$, $1 \le j \le m-1$, has generic fiber type either $(0, 0)$, $(0, 1)$, or $(1, 1)$. Let $h_m = 1$, $h_{m-1} = 4$, and let h_{j-1} be such that $h_{j-1} = 2h_j - h_{j+1} - (a+b)$ when $(M^0)_{|\Delta_j}$ has generic fiber type (a, b). Notice that $\{h_j\}$ is strictly decreasing.

Lemma 7.3. *Assume* $h^2(\Delta_m, \mathscr{E}nd^0(M^0_{|\Delta_m \times \{s\}})(-4\Sigma^-)) = 0$ *for any closed* $s \in S$. *Then*

$$\operatorname{Ext}^2_{\Lambda_0 \times S}(M^0, M^0(-\Lambda_1 \times S))^0 = 0.$$

Proof. Let E be the sheaf $(M^0)_{|\Lambda_0 \times \{s\}}$ with $s \in S$ closed. Since S is affine, it suffices to show that $\operatorname{Ext}^2_{\Lambda_0}(E, E(-\Lambda_1))^0 = 0$ for all s. By Serre duality,

$$\operatorname{Ext}^2_{\Lambda_0}(E, E(-\Lambda_1))^0 = \operatorname{Ext}^0_{\Lambda_0}(E(-\Lambda_1), E \otimes \omega_{\Lambda_0})^0, \tag{7.6}$$

where ω_{Λ_0} is the canonical sheaf of Λ_0. (7.6) is

$$H^0(\Lambda_0, \mathscr{E}nd^0(E)\otimes\omega_{\Lambda_0}(\Lambda_1)).$$

Using the adjunction formula we obtain $\omega_{\Lambda_0} = \pi_X^* K_X(\sum_{j=1}^m j\Delta_j)$, where $\pi_X\colon \widetilde{Z}_0^m \to X$ is the projection. Therefore, ω_{Λ_0} has fiber type 0 on Δ_j, $1\le j\le m-1$. Since $K_{\Delta_m} = \pi_X^*K_X(-2\Delta_{m-1})_{|\Delta_m}$, by assumption we have

$$H^0(\Delta_m, \mathscr{E}nd^0(E)\otimes\omega_{\Lambda_0}(\Lambda_1)) = H^0(\Delta_m, \mathscr{E}nd^0(E)(K_{\Delta_m}+4\Sigma^-)) = \{0\}.$$

Therefore

$$H^0\left(\bigcup_{j=0}^{m-1}\Delta_j, \mathscr{E}nd^0(E)\otimes\omega_{\Lambda_0}(\Lambda_1-\Delta_m)\right) \to H^0(\Lambda_0, \mathscr{E}nd^0(E)\otimes\omega_{\Lambda_0}(\Lambda_1))$$

is surjective. For $1\le j\le m-1$, since $\omega_{\Lambda_0}(\Lambda_1-\Delta_{j+1})_{|\Delta_j}$ has fiber type -1 (resp. -2; resp. -3) when $E_{|\Delta_j}$ has generic fiber type $(0,0)$ (resp. $(0,1)$; resp. $(1,1)$), $h^0(\Delta_j, \mathscr{E}nd^0(E)\otimes\omega_{\Lambda_0}(\Lambda_1-\Delta_{j+1})) = 0$. Thus repeating the previous argument yields that

$$H^0(\Delta_0, \mathscr{E}nd^0(E)\otimes\omega_{\Lambda_0}(\Lambda_1-\Delta_1)) \to H^0(\Lambda_0, \mathscr{E}nd^0(E)\otimes\omega_{\Lambda_0}(\Lambda_1))$$

is surjective. The lemma will be established if we can show that

$$\begin{aligned} &H^0(\Delta_0, \mathscr{E}nd^0(E)\otimes\omega_{\Lambda_0}(\Lambda_1-\Delta_1)) \\ &\qquad = H^0(X, \mathscr{E}nd^0(E)(K_X-(3-\delta)H)) = \{0\}, \end{aligned}$$

where $\delta=0$ (resp. $\delta=1$; resp. $\delta=2$) when E is of type I (resp. type II; resp. type III). Now let $\rho\in H^0(X, \mathscr{E}nd^0(E)(K_X-(3-\delta)H))$. Then $\det\rho\in H^0(X, 2K_X-2(3-\delta)H)=\{0\}$ must be trivial. Let J be the kernel of ρ. Then $E_{|X}$ belongs to the exact sequence

$$0\to J\to E_{|X}\to J^{-1}\otimes I(aH)\otimes\mathscr{I}_z\to 0. \tag{7.7}$$

Since $\operatorname{tr}(\rho)=0$, ρ is induced from $J^{-1}\otimes I(aH)\otimes\mathscr{I}_z\to J(K_X-(3-\delta)H)$ and therefore

$$2(J\cdot H)\ge (3+a-\delta)(H\cdot H)-(H\cdot K_X)+(H\cdot I).$$

Finally recall that E is a proper transform of an $H(\varepsilon)$-semistable sheaf on Z_0, and by (4.12) the quotient sheaf $J^{-1}\otimes I(aH)\otimes\mathscr{I}_z$ of $E_{|X}$ satisfies

$$-(J\cdot H)+(I\cdot H)+a(H\cdot H)a\ge (1/2)(1-\varepsilon)(H\cdot K_X)+(\varepsilon/2)(I\cdot H).$$

Combining these two inequalities together, we obtain

(7.8)
$$2a(H\cdot H)+(2-\varepsilon)(I\cdot H)-(1-\varepsilon)(H\cdot K_X)\geq(3-\delta+a)(H\cdot H)-(H\cdot K_X)+(I\cdot H),$$

which holds only when $\delta+a\geq 3$. Therefore, we obtain a contradiction because by §4, $\det E_{|X}=I(aH)$ with $0\leq a\leq 2-\delta$, and hence the lemma is established. q.e.d.

We continue the proof of Theorem 7.1. By Lemmas 7.2 and 7.3, we can successively find preferred extensions $\{M^k\}_{k\geq 1}$ such that $(M^{k+1})_{|\Lambda_k\times S}=M^k$ and that $(M^k)_{|\Lambda_k\times\{0\}}=F_{T|\Lambda_k}$. Further, we remark that once we have chosen the family E_S, we can take n large so that with $W=\bigoplus^N L_T^{\otimes(-n)}$, there is a surjective homomorphism

$$g_1\colon p_1^*W_{|\Lambda_1\times S}\to M^1 \tag{7.9}$$

($p_1\colon\widetilde{Z}_T^m\times S\to\widetilde{Z}_T^m$) that coincides with the quotient sheaf $W\to F_T$ when restricted to $\Lambda_1\times\{0\}$. We can also assume $h^1(\widetilde{Z}_0^m,\mathscr{H}\mathrm{om}(L_0,E_s)(-\Lambda_1))=0$ for all closed $s\in S$. Then g_1 extends to

$$g_k\colon p_1^*W_{|\Lambda_k\times S}\to M^k, \tag{7.10}$$

that restricts to $W_{|\Lambda_k}\to F_{T|\Lambda_k}$ along $\Lambda_k\times\{0\}$.

Let $\mathscr{E}^k$ be the restriction of M^k to $\widetilde{Z}_{A_k}^m\times S\subseteq\widetilde{Z}_T^m\times S$, where $\widetilde{Z}_{A_k}^m=\widetilde{Z}_{C^m}^m\times_{C^m}A_k$, and $A_k=\operatorname{Spec}\mathbb{C}[t]/(t^k)\hookrightarrow T$ is the obvious immersion with closed point v_0 in its image. We claim that $\mathscr{E}^k$ is flat over $A_k\times S$. Indeed, since both T and S are smooth, and $\mathscr{E}^0=M^0=M^1_{|\Lambda_1\times S}$ is flat over S, by local criterion of flatness all we have to show is that $\mathscr{E}^k\xrightarrow{\times t}\mathscr{E}^{k+1}$ is injective. But that follows immediately from (7.4). So $\mathscr{E}^k$ is flat over $A_k\times S$. By the universality of the quotient scheme $\mathfrak{R}^{\chi}_{\widetilde{Z}^j}$, $\mathscr{E}^k$ induces a morphism

$$\varphi_k\colon A_k\times S\to\mathfrak{R}^{\chi}_{\widetilde{Z}^m}. \tag{7.11}$$

By definition, $\varphi_{k+1}|_{A_k\times S}=\varphi_k$.

Since $\mathfrak{R}^{\chi}_{\widetilde{Z}^m}$ is projective over C^m, there is an irreducible component $'\mathfrak{R}^{\chi}_{\widetilde{Z}^m}$ of $\mathfrak{R}^{\chi}_{\widetilde{Z}^m}$ so that φ_k factor through $'\mathfrak{R}^{\chi}_{\widetilde{Z}^m}$ for all $k\geq 1$. In particular, E_{s_1} belongs to $'\mathfrak{R}^{\chi}_{\widetilde{Z}^m}$ as a quotient sheaf. On the other hand, since $\varphi_k(A_k\times\{0\})$ is contained in the image of $T\to\mathfrak{R}^{\chi,0}_{\widetilde{Z}^m}$, the image of T is contained in $'\mathfrak{M}^{\chi}_{\widetilde{Z}^m}$. But we have assumed that $\mathfrak{M}^{\chi,0}_{\widetilde{Z}^m}$ is the only irreducible component of $\mathfrak{M}^{\chi}_{\widetilde{Z}^m}$ that contains T. So $'\mathfrak{M}^{\chi}_{\widetilde{Z}^m} = \mathfrak{M}^{\chi}_{\widetilde{Z}^m}$. In particular, $S \subseteq \mathfrak{M}^{\chi,0}_{\widetilde{Z}^m}$ and then $E_{s_1} \in \mathfrak{M}^{\chi,0}_{\widetilde{Z}^m}$.

Now we are ready to prove Theorem 7.1. By Lemma 6.7, there is a classical neighborhood $V_1 \subseteq \mathfrak{M}^{\chi,0}_{\widetilde{Z}^m}$ of s_1 and a classical neighborhood $V_2 \subseteq \widetilde{Z}^m_{C^m}$ of $p \in \Delta_m$, $V_2 \cap \widetilde{Z}^m_0 \subseteq \Delta_m$, such that if we let $\mathscr{F}$ be the universal quotient family of $\mathfrak{M}^{\chi,0}_{\widetilde{Z}^m}$, then the set

$$\mathfrak{M}^{(1)}(V_1, V_2) = \{u \in V_1 \mid \mathrm{col}(\mathscr{F}_u)_q = 1 \text{ for some } q \in \widetilde{Z}^m_t \cap V_2,\ t \text{ lies under } u\}$$

is a nonempty codimension 1 subset of V_1 containing s_1. Because of our construction, the curve $S \subseteq \mathfrak{M}^{\chi,0}_{\widetilde{Z}^m}$ is not contained in $\mathfrak{M}^{(1)}(V_1, V_2)$. Thus $\mathfrak{M}^{(1)}(V_1, V_2)$, is not contained in the fiber of $\mathfrak{M}^{\chi,0}_{\widetilde{Z}^m}$ over $0 \in C^m$. In other words, $\mathfrak{M}^{(1)}(V_1, V_2) \cap \mathfrak{M}^{\chi,0}_{\widetilde{Z}^m,t} \neq \emptyset$ for general $t \in C^m$. Therefore, we have proved the first part of the following proposition.

Proposition 7.4. *There is a curve $R \subseteq \mathfrak{M}^{\chi,0}_{\widetilde{Z}^m}$ containing s_1 and flat over C^m such that the following hold:*

(1) *For any closed $t \in R$ not lying over $0 \in C^m$, there is a closed point $q \in X$ such that for the pullback sheaf $\mathscr{F}$ on $\widetilde{Z}^m_{C^m} \times_{C^m} R$, $\mathrm{col}(\mathscr{F}_t)_q = 1$.*

(2) *For generic $t \in R$, $\mathscr{F}_t$ is μ-stable and further, $\mathrm{Ext}^2(\mathscr{F}_t, \mathscr{F}_t)^0 = 0$.*

Proof. We only need to prove (2). Let R be any curve $\subseteq \mathfrak{M}^{\chi,0}_{\widetilde{Z}^m}$ containing s_1, which is flat over C^m. Let $\overline{R}$ (over C^m) be the normalization of R. Let $\widetilde{Z}^{\overline{m}}_{\overline{R}}$ be the normalization of $\widetilde{Z}^m_{C^m} \times_{C^m} \overline{R}$, where $\overline{m} = m \cdot p$ and p is the branched order of $\overline{R}$ over C^m at 0. By Proposition 3.3, there is a good modification F (on $\widetilde{Z}^{\overline{m}}_{\overline{R}}$) of the pullback of $\mathscr{F}$ via $\widetilde{Z}^{\overline{m}}_{\overline{R}} \to \widetilde{Z}^m_R$. Further, since $\mathscr{F}_{s_1}$ is locally free along Σ_m, $\mathscr{F}_{s_1|\Delta_m} \cong F_{x_0|\Delta_{\overline{m}}}$, where $x_0 \in \overline{R}$ lies over s_1.

Now to prove (2), it suffices to show that F_x is μ-stable for generic $x \in \overline{R}$. Assume F_x are not μ-stable for all $x \in \overline{R}$. Then by the semicontinuity theorem, there is a line bundle $\mathscr{L}$ on $\widetilde{Z}^{\overline{m}}_{\overline{R}}$, a nontrivial homomorphism $F \to \mathscr{L}$ such that the degree of $\mathscr{L}_x$, $\mathscr{L}_x = \mathscr{L}_{|\widetilde{Z}^{\overline{m}}_x}$, is no bigger than $\frac{1}{2}(I \cdot H)$. Without loss of generality, we can assume $F_{x_0} \to \mathscr{L}_{x_0}$ is surjective at the generic points of all irreducible components $\Delta_i \subseteq \widetilde{Z}^{\overline{m}}_{x_0}$. Let $\mathscr{L}_i$ be the restriction of $\mathscr{L}$ to Δ_i. Since F_{x_0} has generic fiber type $(0, 0)$, $(0, 1)$, $(1, 1)$ along Δ_i, $1 \le i \le \overline{m}-1$, we must have $\beta_i = \mathscr{L}_i \cdot \gamma^* H(\varepsilon)_{|\Delta_i} \geq$

0, $\gamma\colon \widetilde{Z}^{\overline{m}}_{x_0} \to Z_0$. On the other hand, from (4.12) it follows that

$$\beta_0 \geq (1/2)(1-\varepsilon)(H \cdot K_X) + (\varepsilon/2)(I \cdot H).$$

Finally, by definition, $F_{x_0|\Delta_{\overline{m}}}$ is e-stable with $e = -(H \cdot H)$. Therefore,

$$\begin{aligned}\beta_{\overline{m}} &\geq \tfrac{1}{2}(H \cdot H) + \tfrac{1}{2}c_1(F_{x_0|\Delta_{\overline{m}}}) \cdot H(\varepsilon)_{|\Delta_{\overline{m}}} \\ &\geq \tfrac{1}{2}(H \cdot H) + \tfrac{1}{2}(1-\varepsilon)(I \cdot H) - \varepsilon(H \cdot H).\end{aligned}$$

So, we have

$$\sum_0^{\overline{m}} \beta_i > \frac{1}{2}(I \cdot H), \tag{7.12}$$

which certainly contradicts the assumption that $\mathscr{L}_x \cdot H \leq \frac{1}{2}(I \cdot H)$ for generic $x \in \overline{R}$. So we have proved that F_x are μ-stable for general $x \in \overline{R}$. Finally, $\operatorname{Ext}^2(F_x, F_x)^0 = \{0\}$ for general x because $H^0(\mathscr{E}nd^0(F_{x_0} \otimes \pi_X^* K_X) = 0$ which follows from the proof of Lemma 7.3 (cf. [9]).

Proof of Theorem 7.1. Fix a general closed $t \in C^m$. Let $U_t^0 \subseteq \mathfrak{R}^{\chi,0}_{\widetilde{Z}^m,t}$ be the open subset of all H-stable quotient sheaves. Since $\mathfrak{R}^{\chi,0}_{\widetilde{Z}^m,t}$ is irreducible, U_t^0 is also so. Thus all quotient sheaves $E \in U_t^0$ are represented by closed points in $\mathbf{M}$. In particular, Proposition 7.4 tells us that there is a stable sheaf $E \in \mathbf{M}$ with $\operatorname{Ext}^2(E, E)^0 = 0$ such that for some closed $p \in X$, $\operatorname{col}(E)_p = 1$. Now a straightforward deformation argument shows that there is an $E \in \mathbf{M}$ such that $\operatorname{Ext}^2(E, E)^0 = 0$ and $\operatorname{col}(E) = 1$.

Proposition 7.5. *There is a constant A depending on (X, H, I) such that whenever $d \geq A$, the number of irreducible components of $\mathfrak{M}_X^{d,I}$ is independent of d.*

Proof. Let Λ_d be the set of irreducible components of $\mathfrak{M}_X^{d,I}$. Since $\mathfrak{M}_X^{d,I}$ is projective, Λ_d is finite. Let A be the integer provided by Theorem 7.1. Then there are maps

$$\varphi_d\colon \Lambda_d \to \Lambda_{d+1}, \qquad d \geq A, \tag{7.13}$$

defined as follows: For any $\mathbf{M} \in \Lambda_d$, pick a μ-stable sheaf $E \in \mathbf{M}$ so that $\operatorname{Ext}^2(E, E)^0 = 0$. Since $d \geq A$, such E always exists. Let $x \in X$ be a general closed point and let F be a torsion free sheaf belonging to the exact sequence

$$0 \to F \to E \xrightarrow{\eta} \mathbb{C}_x \to 0, \tag{7.14}$$

where $\mathbb{C}_x$ is the skyscraper sheaf supported at x. Then F is still μ-stable and $\operatorname{Ext}^2(F, F)^0 = 0$. Thus F belongs to a unique irreducible component $\mathbf{M}' \in \Lambda_{d+1}$. We define $\varphi_d(\mathbf{M}) = \mathbf{M}'$. It is easy to see that φ_d are independent of the choice of E, η and x. Further, by Theorem 7.1, every irreducible component $\mathbf{M} \subseteq \mathfrak{M}_X^d$, $d \geq A$, contains μ-stable sheaf F such that $\mathrm{col}(E) = 1$ and $\mathrm{Ext}^2(E, E)^0 = 0$; that is, F belongs to the exact sequence (7.14) with $E = F^{\vee\vee}$. Thus, φ_d, $d \geq A$, are surjective. Therefore, $\{\#\Lambda_d\}_{d \geq A}$ is a decreasing sequence of positive integers. In particular, for some large A', $\#\Lambda_d$ is independent of d as long as $d \geq A'$. This proves Proposition 7.5.

Proposition 7.6 (Taubes). *With the notation as in Proposition* 7.4, *and let φ_d^β be the map*

$$\varphi_d^\beta = \varphi_{d+\beta-1} \circ \dots \varphi_{d+1} \circ \varphi_d \colon \Lambda_d \to \Lambda_{d+\beta}.$$

Then for any $d \geq A$, there is a large β_d such that $\varphi_d^{\beta_d}(\Lambda_d) =$ single point.

Proof. Let $\mathbf{M}_1$, $\mathbf{M}_2 \in \Lambda_d$, $d \geq A$, be any two irreducible components. Choose μ-stable sheaves $E_i \in \mathbf{M}_i$ so that $\mathrm{Ext}^2(E_i, E_i)^0 = 0$. Let m be large enough so that both E_i belong to the exact sequence

$$0 \to H^{\otimes(-m)} \to E_i \to H^{\otimes m} \otimes I \otimes \mathscr{I}_{z_i} \to 0$$

with z_1, z_2 zero subschemes of X. Without loss of generality, we can assume $\mathrm{supp}(z_1) \cap \mathrm{supp}(z_2) = \varnothing$. Let F_i be subsheaves of E_i belonging to the following diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{\otimes(-m)} & \longrightarrow & E_i & \longrightarrow & H^{\otimes m} \otimes I \otimes \mathscr{I}_{z_i} & \longrightarrow & 0 \\
 & & \| & & \cup & & \cup & & \\
0 & \longrightarrow & H^{\otimes(-m)} & \longrightarrow & F_i & \longrightarrow & H^{\otimes m} \otimes I \otimes \mathscr{I}_{z_1 \cup z_2} & \longrightarrow & 0.
\end{array}$$

Clearly $E_1/F_1 \cong \mathscr{O}_{z_2}$ and $E_2/F_2 \cong \mathscr{O}_{z_1}$. Thus $\{F_i\} \in \varphi_d^\beta(\{\mathbf{M}_i\})$, where $\beta = l(z_1) = l(z)$. On the other hand, this exact sequence tells us that both F_1 and F_2 correspond to closed points in the affine space $\mathrm{Ext}^1(H^{\otimes m} \otimes I \otimes \mathscr{I}_{z_1 \cup z_2}, H^{\otimes(-m)})$. Since being stable is an open condition, F_1 and F_2 are in the same irreducible component of $\Lambda_{d+\beta}$. Therefore, $\varphi_d^\beta(\{\mathbf{M}_1\}) = \varphi_d^\beta(\{\mathbf{M}_2\})$. So for sufficiently large integer β_d, $\varphi_d^{\beta_d}(\Lambda_d)$ consists of one point only.

Proof of the main theorem. By Proposition 7.4, there is a constant A so that for $d \geq A$, $\#\Lambda_d = \kappa$ is independent of d. Further, the map φ_d and subsequently φ_d^β are all isomorphisms. On the other hand, for any $d \geq A$, there is a large β_d such that $\varphi_d^{\beta_d}(\Lambda_d)$ is a single point set.

Therefore, κ has to be one; that is, $\mathfrak{M}_X^{d,I}$ will become irreducible when the second Chern class is sufficiently large.

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