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A HANDLE DECOMPOSITION OF AN EXOTIC $\mathbb{R}^4$

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Abstract

In [4] S. DeMichelis and M. Freedman constructed uncountably many exotic $\mathbb{R}^4$'s that can be embedded in S^4. Such an exotic $\mathbb{R}^4$ can be constructed in the boundary of a smooth compact 5-dimensional h-cobordism that is not diffeomorphic to the product cobordism. An example of nonproduct h-cobordism was explicitly described by S. Akbulut [1]. In the present article Akbulut's description is used to construct a compact handlebody which contains two copies of an eight-level Casson tower and which has the following property. Any open handlebody that is obtained from this compact handlebody by replacing each Casson tower by a Casson handle embedded in it and by removing the boundary is an exotic $\mathbb{R}^4$. A concrete example of such an exotic $\mathbb{R}^4$ is obtained by an application of the Reimbedding algorithm from [2]. This exotic $\mathbb{R}^4$ is an open handlebody fully described by an infinite link calculus picture.

0. Introduction. It is known that there are smooth, compact, simply connected h-cobordisms between nondiffeomorphic 4-dimensional closed manifolds. If $(W^5; X_0^4, X_1^4)$ is such an h-cobordism, then it clearly cannot be diffeomorphic to the product cobordism, $X_0 \times I$. However, by work of M. Freedman [6], the h-cobordism theorem holds in the category of 4-dimensional, simply connected topological manifolds, so any h-cobordism between closed, simply connected 4-manifolds is homeomorphic to the product cobordism. An analysis of Freedman's proof of the h-cobordism theorem shows that the "product structure" always smoothly exists over the complement of a flat cell in X_0 (see [6], [4], [9] or [7]). Moreover, this flat cell contains an exotic $\mathbb{R}^4$ that is embedded in both X_0 and X_1, and X_1 can be reconstructed from X_0 by changing the embedding of this exotic $\mathbb{R}^4$. A detailed exposition can be found in [4] or [9], but here we will restrict ourselves to the slightly less general situation coming from the h-cobordism in [1]. Starting with Akbulut's explicit description of this h-cobordism, we extract a handle decomposition for an exotic $\mathbb{R}^4$.

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Akbulut's paper [1] uses Kirby's link calculus to describe handle decompositions of 4-dimensional manifolds and diffeomorphisms of their boundaries. We assume some familiarity with the link calculus on the part of the reader (see [8] or [9]). Also, Casson handles and Casson towers [3] are used in our construction, but only their link calculus pictures are described. Only some of the figures from [1] are reproduced and, for example, [1, Figure 5] will refer to Figure 5 from [1]. All presented figures appear at the end of this article (pp. 501–508).

1. Nonproduct h-cobordisms, Casson handles and exotic $\mathbb{R}^4$'s. Before we start a construction of an exotic $\mathbb{R}^4$, we review an argument due to Casson and Freedman that predicts the existence of exotic $\mathbb{R}^4$'s in the boundary of a nonproduct smooth h-cobordism (see [4] or [9]). Let $(W^5; X_0^4, X_1^4)$ be a smooth simply connected h-cobordism between two nondiffeomorphic closed 4-manifolds, X_0 and X_1. Furthermore, we assume that W has a handle decomposition with only one 2-handle and one 3-handle, as in the case of the h-cobordism from [1]. So we have:

$$W^5 \cong (X_0 \times I) \cup h^2 \cup h^3.$$

Let $X_{1/2}$ denote the middle level between the 2- and 3-handles, that is, $X_{1/2} = \partial\left((X_0 \times I) \cup h^2\right) - (X_0 \times 0)$. It is easy to see that the manifold $X_{1/2}$ is diffeomorphic to $X_0 \sharp (S^2 \times S^2)$. The second homology of the summand $S^2 \times S^2$ of $X_{1/2}$ is generated by the attaching sphere of the 3-handle, 'A', and by the dual (or "belt") sphere of the 2-handle, 'B' (see [10, §6] for their definitions).

The algebraic intersection number of the 2-spheres A and B is equal to ± 1 but, geometrically, there can be additional $\pm$ pairs of intersection points of A and B. Note that if there was just one intersection point between A and B, then the handles h^2 and h^3 would form a complementary pair of handles, and W would be diffeomorphic to the product cobordism $X_0 \times I$ [10, p. 78]. The h-cobordism from [1] is the simplest possible nonproduct cobordism: there is only one additional $\pm$ pair of intersections between A and B.

Recall that the standard proof of the h-cobordism theorem for manifolds of dimensions greater than four uses the Whitney trick [10, §6]. Namely, a "Whitney loop" (defined in [10, p. 71]) for such an extra pair of intersections between an attaching sphere A and a belt sphere B bounds a 2-handle embedded in the complement of these spheres in the middle level of the given h-cobordism. The existence of such an embedded "Whitney disc" is guaranteed by a general position argument. This embedded 2-handle is used to construct an ambient isotopy of A, such that the new attaching sphere intersects B at only one point.

For 4-dimensional manifolds, the general position produces only an immersed 2-handle. Freedman's proof of the topological h-cobordism theorem uses a construction of A. Casson [3]. Instead of a Whitney disc, Casson's construction produces an embedded "flexible" or Casson handle [3, Lecture I]. Freedman proved that every Casson handle is homeomorphic to the standard open 2-handle, $(D^2 \times \mathbb{R}^2, S^1 \times \mathbb{R}^2)$, [6, Theorem 1.1]. In particular, in the obtained Casson handle there is a topologically embedded 2-handle that caps the same framed Whitney circle as the Casson handle. Such an embedded handle can be used in the Whitney trick. In contrast to this topological result, any such Casson handle obtained as a Whitney disc in the h-cobordism we are considering does not contain a *smoothly* embedded 2-handle with the same framed boundary. Otherwise the Whitney trick would also work in the smooth category which would contradict the assumption that the manifolds X_0 and X_1 are not diffeomorphic.

We now digress with a short description of Casson's construction [3]. Any of [3], [6] and [9] can serve as a good reference on Casson handles. Instead of definition, we present a link calculus picture of a Casson tower, a finite part of a Casson handle.

The framed link in Figure 1 is in the boundary of a 4-ball. This ball is the unique 0-handle in the presented handlebody. Recall that a component of a link that has a number next to it is the attaching circle of a 2-handle, and the number specifies the framing of the attaching area with respect to the 0-handle. The components with a dot represent 1-handles and are the attaching circles of the replacing 2-handles that are scooped out from the 0-handle. Following a customary abuse of notation, we call such replacing 2-handles "1-handles". The component denoted by 'a' in Figure 1 is the attaching circle of the handlebody itself, and its framing is assumed to be the 0-framing. The building blocks of Casson towers and handles are "kinky handles" that can be defined as relative regular neighborhoods of transversal immersions of (D^2, ∂) into (D^4, ∂) that have only double singular points. The diffeomorphism type of a kinky handle is determined by its "core", i.e., by an immersion of a 2-disc in a 4-ball that defines the given kinky handle. The core, up to diffeomorphism, is determined by the number and signs of the double points, or "kinks". There are three levels of kinky handles in the Casson tower in Figure 1: the first level has a single kinky handle, the second one has two, and on the third level there are three kinky handles. The five dashed circles in Figure 1 represent the "standard loops", and the assumed framings are again the 0-framing. Standard loops generate the fundamental group of a Casson tower and show where the kinky handles of a fourth level should be attached. Each 1-handle (circle with a dot) represents a kink on the core of a kinky handle.

When infinitely many levels are added and the boundary is removed, except for an open solid torus around the 'a'-circle, then the resulting infinite (open) handlebody is a Casson handle. Information sufficient to restore a Casson handle up to diffeomorphism can be stored in an infinite based signed tree, where vertices of a tree correspond to kinks on a Casson handle and the signs of vertices correspond to the signs of kinks. For each pair of Casson handles, CH_1 and CH_2, there is always a Casson handle CH that embeds in both of them; such a Casson handle can be constructed by taking the union of corresponding trees and then, if necessary, by adding kinks to the cores of CH_1 and CH_2, as prescribed by this union of trees. Any Casson handle can be embedded in the standard 2-handle (that corresponds to the tree with one vertex) by an embedding that preserves the attaching area. In a situation like ours, when a Whitney circle does not span a smoothly embedded 2-handle in $X_{1/2} - (A \cup B)$, one can embed by hand a six-level Casson tower that caps the Whitney circle. Then, as Freedman has shown [6, Theorem 5.1], in any six-level Casson tower there is an embedded Casson handle with the same first level. A modification of this result is the Reimbedding algorithm from [2] which actually describes an embedded Casson handle in a given six-level tower [2, Theorem 3.1].

The three intersection points of A and B form two $\pm$ pairs of intersections, and we cap each pair by a Casson handle in $X_{1/2} - \mathcal{N}(A \cup B)$. Here $\mathcal{N}(A \cup B)$ is an open regular neighborhood of A and B. We denote these Casson handles by CH_i $(i = 1, 2)$, and by U the open manifold $\mathcal{N}(A \cup B) \cup CH_1 \cup CH_2$. A piece of the h-cobordism W that contains U as its middle level is a proper h-cobordism $(V^5; R_0, R_1)$, where R_0 and R_1 are obtained from U by surgering a 2-sphere B or A, respectively. Note that V also has a handle decomposition consisting of a 2- and a 3-handle. Furthermore R_i $(i = 0, 1)$, is homeomorphic to $\mathbb{R}^4$. To prove this, we will use a link calculus description of the cobordism V.

Figures 2–4 contain link calculus pictures of R_0, U and R_1, respectively. These are all open manifolds, so it is assumed that the boundaries of these handlebodies are removed. In the topological category, Casson handles are equivalent to the standard (open) 2-handles, so in Figures 2–4 the Casson handles are treated as (open) 2-handles and their attaching circles are drawn by dashed circles. So each pair consisting of a 1-handle (= scooped-out 2-handle) and a Casson handle CH_i is a complementary pair and can be deleted from the picture. The remaining open handlebodies in R_0 and R_1 consist of an open 4-ball with a pair of complementary handles, and therefore R_i $(i = 0, 1)$ is homeomorphic to $\mathbb{R}^4$.

Remark. As mentioned above, there is a Casson handle CH embedded in both CH_1 and CH_2, so whenever it is convenient we may assume $CH_1 = CH_2 (= CH)$. Note that this implies that R_1 is diffeomorphic to R_0 by a diffeomorphism that exchanges CH_1 and CH_2 for the 1- and 2-handle in Figure 2.

An argument that R_0 and R_1 are not diffeomorphic to $\mathbb{R}^4$ is presented in [9, §XIV]. A short outline of the argument is presented next. The embeddings of CH_i $(i = 1, 2)$ in the standard 2-handle are extended to an embedding of the open handlebody R_i in S^4 and of U into $S^2 \times S^2$. In other words, these Casson handles are replaced by the 2-handles in Figures 2 and 4, and so the Whitehead trick can be performed smoothly. So the open cobordism $(V; R_0, R_1)$ embeds into a cobordism of S^4 to itself whose handle decomposition consists of a complementary pair of 2- and 3-handles, and therefore it is diffeomorphic to $S^4 \times I$.

Now suppose that R_0 is diffeomorphic to $\mathbb{R}^4$. Then there is a standard 4-ball, B_0^4 embedded in R_0 and a compact set K in its interior such that the smooth product structure on W exists over $X_0 - K$ [7, Theorem 7.1C]. The product structure over $S_0^3 = \partial B_0^4$ gives an embedding of a 3-sphere, S_1^3 in R_1. If S_1^3 bounds a ball in R_1, then the product structure over $X_0 - K$ extends over B_0^4 and so $W \cong X_0 \times I$. Therefore, either R_0 is not diffeomorphic to $\mathbb{R}^4$, or S_1^3 does not bound in R_1. It is easy to see that if R_0 is diffeomorphic to $\mathbb{R}^4$, then S_1^3 bounds a 4-ball in R_1: The cobordism V is smoothly embedded in $S^4 \times I$ and B_0^4 in $S^4 \times 0$. The complement $S^4 - \mathrm{int} B_0^4$ is a 4-ball, and because over $R_0 - K$ the product structures $S^4 \times I$ and $(M_0 - K) \times I$ coincide, S_1^3 bounds "outside" in $S^4 \times 1$. Again, a complement of a smoothly embedded 4-ball in S^4 is a 4-ball, so S_1^3 does bound a 4-ball in R_1.

Let N be the handlebody from Figure 5. It contains two copies of T_8, the eight-level Casson tower that has only one positive kink on each level. Let CH_i for $i = 1, 2$ be any Casson handles embedded in T_8. We construct an open handlebody in N: each of the two copies of T_8 in N is replaced by one of CH_i, and the remaining boundary is removed.

The obtained open manifold R is the same as R_0 or R_1 in Figure 2 and Figure 4, and, indeed, R has the same property:

Theorem A. *If R and N are as above, then R is an exotic $\mathbb{R}^4$.*

A proof of Theorem A is the content of the remaining sections of this article. Now we state our main result—a handle decomposition of an exotic $\mathbb{R}^4$. The Reimbedding algorithm from [2] produces an explicitly described Casson handle in the first six levels of the Casson tower T_8 (compare [2, Theorem 3.1]). The embedded Casson handle CH has a single positive kink on the first level, and each kinky handle on the nth level has y_n positive and y_n negative kinks where:

(a) $y_2 = y_3 = y_4 = 1$, $y_5 = 100$,

(b) y_6 is $10^{10^{10^{10}}}$ and

(c) for $n > 6$, $y_n = 10^{10^{10^{10^{10^{10^{10^{10^{(y_{n-1})}}}}}}}}$

If in the construction of R we set both CH_1 and CH_2 to be equal to CH, then we can complete the handlebody description of R from Figure 5 by adding to the picture an infinite handle decomposition of the two copies of CH in R. Such a handlebody is described in Figure 6.

Theorem B. *Let y_n, $2 \leq \infty$, be a sequence of numbers defined by* (a), (b) *and* (c). *Then the open handlebody described in Figure* 6 *is an exotic* $\mathbb{R}^4$.

2. Constrained decomposition of E_n. We present a standard proof that "E_n" linked once with a copy of $\overline{\mathbb{C}P}^2$ decomposes into $\mathbb{C}P^2$ and n copies of $\mathbb{C}P^2$. The proof is based on the link calculus operations that preserve the diffeomorphism type of a handlebody. However, in the presence of other components of a framed link that are linked with E_n and $\overline{\mathbb{C}P}^2$, the resulting $\mathbb{C}P^2$ and $\overline{\mathbb{C}P}^2$'s will be also linked with the other components. So in Figures 10–18 we will follow two such arcs through the applied link calculus operations.

Figures 7–9 show the standard sliding-off in a chain of 2-handles. Notice that the -1-framed component in Figure 7 is linked with an arc. In Figure 9 the other component is unlinked from the first one but is now linked with the arc.

In Figure 10 there is an E_9 linked with an arc 'A' and a $\overline{\mathbb{C}P}^2$. The $\overline{\mathbb{C}P}^2$ is linked with an arc 'B'. Figure 11 is obtained by applying the handle slides from Figures 7–9 five times. Then, as indicated in the figures by dashed arrows, slides are performed until E_9 completely decomposes into a $\mathbb{C}P^2$ and nine $\overline{\mathbb{C}P}^2$'s, Figures 11–18. However, in the presence of the arcs 'A' and 'B', the resulting $\overline{\mathbb{C}P}^2$'s are linked once with B and the unique copy of $\mathbb{C}P^2$ ends up being linked once with A and 3 times with B (Figure 18). This process obviously works for any E_n, $n > 5$; the difference is in the number of $\overline{\mathbb{C}P}^2$'s in Figure 18.

3. Construction of the handlebody N. The h-cobordism from [1] contains a relative h-cobordism of Mazur's manifold to a different smooth structure. Then the Mazur manifold is embedded in a homology ball, denoted by Q [1, Figure 7], and a closed manifold $\tilde{M}$ is formed by gluing $-Q$ to a manifold "M_1" [1, Figure 33] by a diffeomorphism of its boundaries, denoted by h. The relative h-cobordism of Mazur's manifold is a product on the boundary so it can be extended to an h-cobordism over $\tilde{M}$. The other end of the h-cobordism is denoted by M, and it is a direct sum of $\overline{\mathbb{C}P}^2$ and S, a homotopy Kummer surface. As shown in [1], $\tilde{M} = M_1 \cup_h (-Q) \cong (3\,\mathbb{C}P^2)\sharp(20\,\overline{\mathbb{C}P}^2)$.

The manifold with boundary, M_1, contains an embedded Brieskorn homology sphere $\Sigma(2, 3, 7)$, and M_1 itself is contained in the K3 surface S. From [5] it follows that S has a nontrivial Donaldson invariant. Because Donaldson invariants persist under connected summing with $\overline{\mathbb{C}P}^2$, $M \cong S\sharp\overline{\mathbb{C}P}^2$ has a nontrivial Donaldson invariant. Since $\tilde{M}$ decomposes to a sum of complex projective planes, it has no nontrivial Donaldson invariant. Therefore $\tilde{M}\sharp(k\,\overline{\mathbb{C}P}^2)$ and $(M\sharp\overline{\mathbb{C}P}^2)\sharp(k\,\overline{\mathbb{C}P}^2)$ are not diffeomorphic for any $k \geq 0$.

We define $X_0 = (M_1\sharp\overline{\mathbb{C}P}^2) \cup_h (-Q)$. An h-cobordism W is obtained by trivially extending the relative cobordism over $-Q$ to one over X_0. So $X_0 = \tilde{M}\sharp\overline{\mathbb{C}P}^2 \cong (3\,\mathbb{C}P^2)\sharp(21\,\overline{\mathbb{C}P}^2)$ and $X_1 = M\sharp\overline{\mathbb{C}P}^2 = S\sharp(2\,\overline{\mathbb{C}P}^2)$. The obtained h-cobordism has the same pair of 2-handles and a 3-handle as the cobordism between $\tilde{M}$ and M from [1]. By the argument above, X_0 is not diffeomorphic to X_1.

We start with a description of the relative h-cobordism over the Mazur manifold. Figures 19–29 all represent Mazur's manifold; Figures 19–25 are ambient isotopies of S^3; and Figure 26 is obtained by introducing a pair of 1- and 2-handles. In Figure 27 another pair is introduced. Figure 28 is a result of a slide of a 1-handle over the other one, as indicated by the dashed curve in Figure 27. The 0-framed 2-handle can slide over the $+1$ framed handle (Figure 29).

The manifold 'Q' is the handlebody from Figure 30 (also [1, Figure 7]), and it has a 0-handle and a 2-handle. This handlebody is diffeomorphic to the manifolds in Figures 31 and 32. Q obviously contains Mazur's manifold. The relative h-cobordism is shown in Figures 32–34 (compare with Figures 2–4). Figure 35 is obtained by a handle slide from Figure 34, and Figure 36 is obtained by the diffeomorphism described by Figures 20–26. Figure 37 is obtained by blowing down the -1-framed circle in Figure 36, and represents a closed 3-dimensional manifold diffeomorphic to the boundary of the handlebody in Figure 36. So we have an extension of the relative h-cobordism over the Mazur's manifold to a relative h-cobordism over Q. The other end of the extended h-cobordism (denoted by W_1 in [1]) contains a copy of $\mathbb{C}P^2$.

In Figure 33, one can see the spheres 'A' and 'B' with an extra pair of intersections. The Casson handles should be added to 0-framed meridians of 1-handles, as indicated in Figure 38, which shows a relevant piece of Figure 28. Note that these two meridians are isotopic: one meridian can slide over the 0-handle that is linked with the two 1-handles to the other meridian. These two framed circles will be capped by a Casson tower, but outside of Q, so these circles are isotoped into the boundary of Q. They pass the levels that contain the two added complementary pairs of 1- and 2-handles (Figures 26 and 27). In Figure 39 only one meridian is shown; the other one is its parallel copy. Figures 40–44 are pieces of Figure 25. In Figure 42 the two circles are now linked and have the framing $+1$ (only one of them is drawn). In Figure 43 the circles bound twice-punctured disjoint discs. Their framed punctures are visible in Figure 44. By an obvious isotopy, we may assume that Figure 44 shows a piece of the boundary of Q.

Akbulut has described a diffeomorphism of the boundary of Q onto the 3-manifold from Figure 45 [1, Figures 8–31]. Figure 45 is [1, Figure 31] together with a missing piece visible in [1, Figure 23]. A blowup with a -1-framing produces the boundary of the 4-dimensional manifold in Figure 46. The $+1$-framed circle from Figure 45 has the 0-framing in Figure 46, where it bounds a scooped-out 2-handle. M_1 is the manifold in Figure 46. In [1], the manifold M_1 is defined slightly differently [1, Figure 33]. These two manifolds with boundary are obviously diffeomorphic: Figure 46 differs from [1, Figure 33] in having a complementary pair of 1- and 2-handles. The diffeomorphism h described by [1, Figures 8–31] and by the change from Figure 45 (or [1, Figure 31]) to Figure 46 maps a 0-framed meridian of the knot in Figure 30 (= [1, Figure 8]) onto the 0-framed meridian of the dotted circle in Figure 46. In [1] the manifold $\tilde{M}$ is obtained by gluing $-Q$ to M_1 by the diffeomorphism h of their boundary. An inverse orientation of Q is obtained by taking the dual decomposition that starts from the collar on the 3-manifold from Figure 30. An equivalent way is to glue $-Q$ and M_1 with a composition of h with the orientation-reversing diffeomorphism that maps Figure 30 to its mirror image. Note that this orientation-reversing diffeomorphism changes the signs of framings. Consequently, the correct figures of $-Q$ in $\tilde{M}$ are obtained by taking the mirror image of Figures 30–44, but in the boundary of M_1 (after reversing the orientations and framings again) the four framed punctures from Figure 44 are exactly as drawn and are linked with the dotted circle in Figure 46.

Note that A and B from Figure 33 are mirror images of those from Figure 3, and so the pairs of 1- and 2-handles from Figures 2, 4, 5 or 6 are mirror images of the corresponding pairs from Figures 32 and 34.

Figures 47–50 show isotopes of the 0-framed 2-handle from Figure 46. Note that there are two copies of the boundary of $E_{10} = \Sigma(2, 3, 7)$ in M_1: one is clearly visible and the other is obtained by sliding the -1-framed 2-handle linked with E_9 over the other -1-framed 2-handle that is linked with E_{10}. Figure 51 is obtained by adding a $\overline{\mathbb{C}P}^2$ to M_1 and by performing the "constrained decompositions" of E_9 and E_{10} from Figures 7–18.

Figures 53 to 55 form an inverse sequence of embeddings, i.e., some of the 2-handles are not drawn, and other changes are the result of handle slides. The reason for forming the connected sum of $\tilde{M}$ with $\overline{\mathbb{C}P}^2$ is now obvious from Figure 54: a copy of $\mathbb{C}P^1$ from the added $\overline{\mathbb{C}P}^2$ is used in Figure 54 to correct the framing of a 2-handle. In Figure 56 the attaching circle of the 0-framed 2-handle bounds a punctured torus. The puncture is a meridian of the 1-handle (arc with a dot), and the remaining piece of the torus looks exactly as the standard picture of the characteristic torus for a kink in a Casson tower. A symplectic basis of the first homology also consists of the (0-framed) meridians to the 1-handle. The torus can be surgered by a pair of $\pm$ copies of the $\mathbb{C}P^1$ in one of 17 $\overline{\mathbb{C}P}^2$'s linked with the 1-handle. This surgery amounts to running two pipes around the 1-handle (the circle with a dot) that connect symplectic generators on the torus with a $\pm$ pair of copies of a punctured $\pm\mathbb{C}P^1$ (these punctures come from their intersections with the scooped-out 2-handle). Figure 57 is a link calculus description of this surgery and is obtained from Figure 56 by two handle slides. Figures 58 and 59 are isotopic to Figure 57.

The result is a kinky handle with one positive kink. The framed boundary and the "standard loop" of this kinky handle are ambiently isotopic: they are 0-framed meridians to the 1-handle. We form 16 copies of this kinky handle: take 16 copies of the core of the 0-framed 2-handle in Figure 48 and slide each of them twice over a *different* $\overline{\mathbb{C}P}^2$. Equivalently, each of 16 copies of the core bounds a punctured torus, as described before. The tori, and also the added pipes and punctured $\mathbb{C}P^1$'s, can all be made disjoint from each other. In both descriptions, we have 16 kinky handles that can be connected by disjoint pipes to form two eight-level Casson towers: start with a meridian from Figure 52, pipe it to one of the kinky handles. A two-level Casson tower is formed by attaching one more kinky handle over the standard loop of the first one. This standard loop is again a 0-framed meridian to the 1-handle, and it can be piped to the boundary of another copy of the constructed kinky handle, and so on. It is straightforward to keep all pipes disjoint.

This process works because each kink, or clasp in the figures, is formed from a separate $\overline{\mathbb{C}P}^2$. Consequently, one can produce as many disjoint kinky handles as there are $\overline{\mathbb{C}P}^2$'s linked with the same 1-handle.

The two punctures that form the $+1$-framed Hopf link in Figure 44 are connected by pipes to the -1-framed Hopf link that is boundary of two copies of the punctured $\mathbb{C}P^1$ in the remaining $\overline{\mathbb{C}P}^2$ that is linked with the 1-handle from M_1 (see Figure 60). Equivalently, the $+1$ Hopf link can be slid over a 2-handle with -1 framing. These slides unlink the components of the Hopf link from each other and from the 1-handle. The resulting trivial link with 0-framings is, of course, slice and bounds 2-handles in the 0-handle of M_1.

All punctures from Figure 44 are now capped, and the resulting submanifold N of X_0 is shown in Figure 5. Note that the two Casson towers T_8 are completely contained in $M_1 \# \overline{\mathbb{C}P}^2$, and so N is obtained from the handlebody $\mathcal{N}(A \cup B) \cup T_8 \cup T_8$ in $X_{1/2}$ by surgering the 2-sphere B. It follows that for any pair of Casson handles CH_1 and CH_2 embedded in T_8, the open handlebody R is obtained by surgering B in $U = \mathcal{N}(A \cup B) \cup CH_1 \cup CH_2$ and therefore is an exotic $\mathbb{R}^4$.

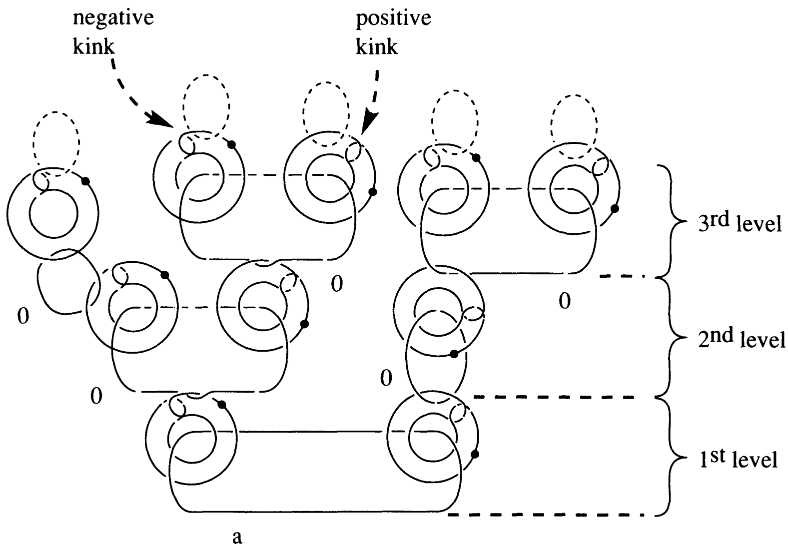


FIGURE 1

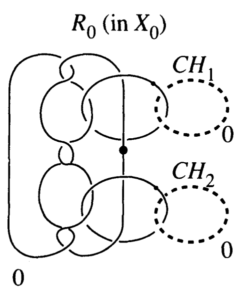


FIGURE 2

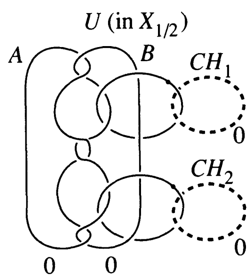


FIGURE 3

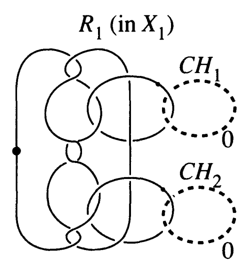


FIGURE 4

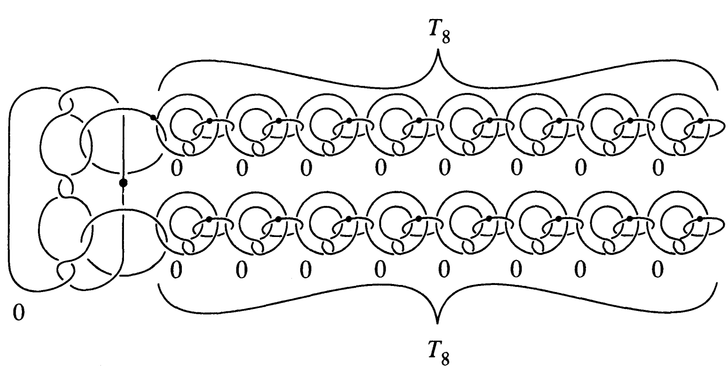


FIGURE 5

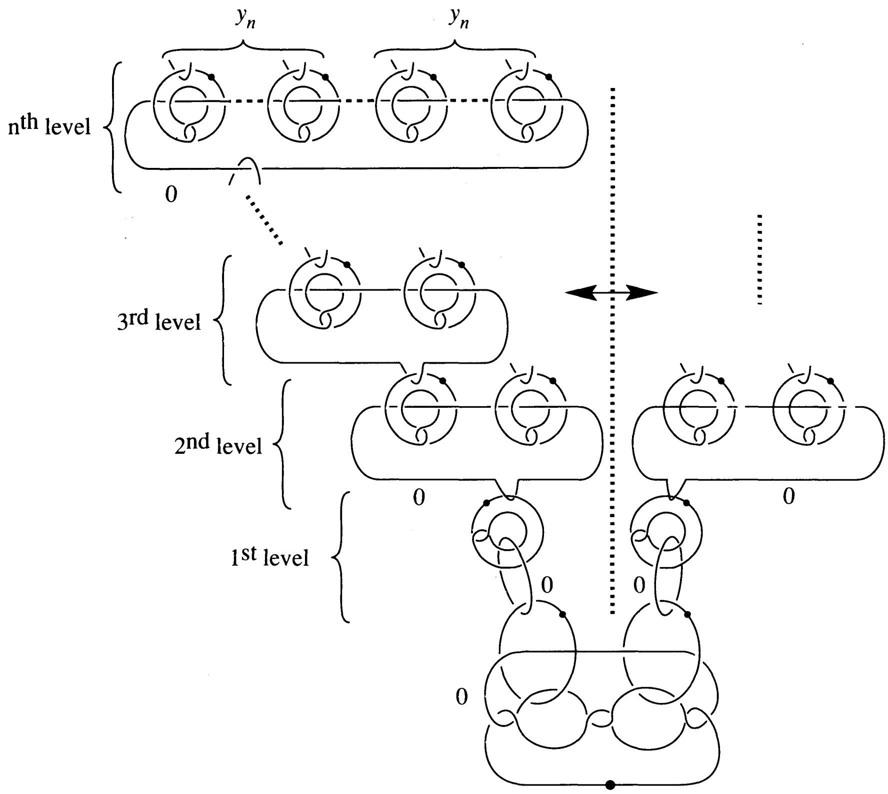


FIGURE 6

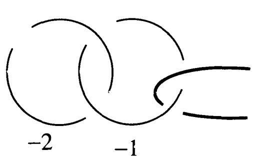
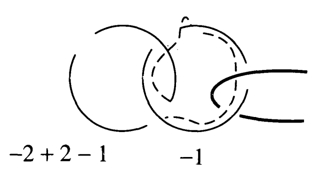


FIGURE 7



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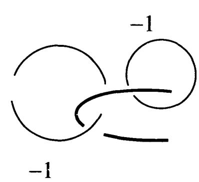

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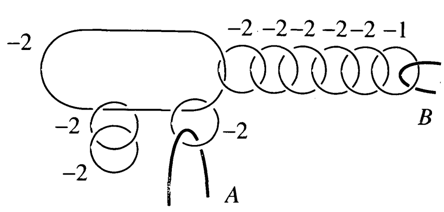

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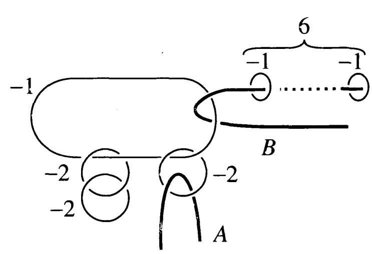

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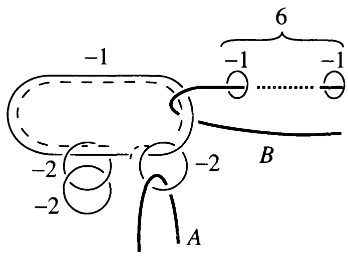

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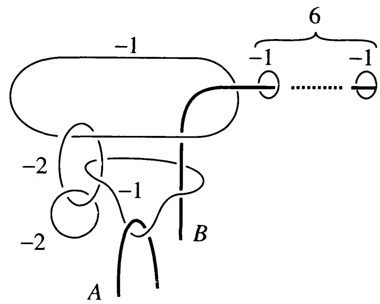

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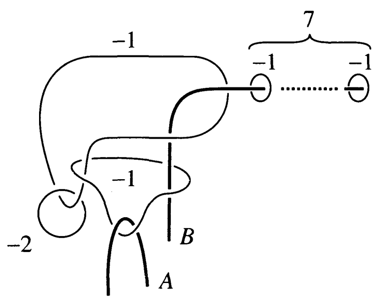

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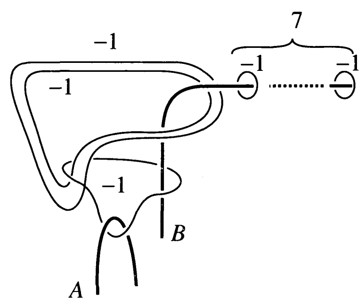

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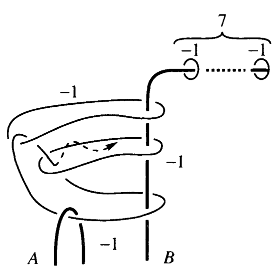

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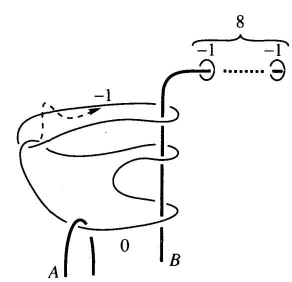

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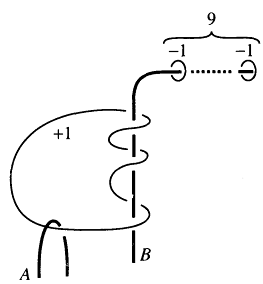

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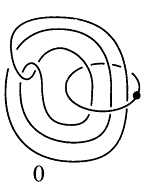

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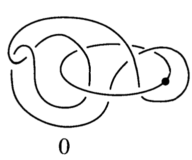

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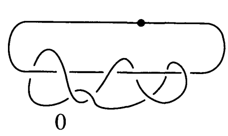

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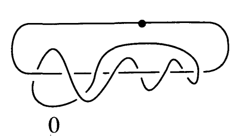

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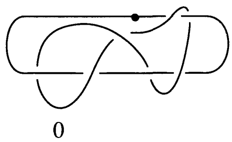

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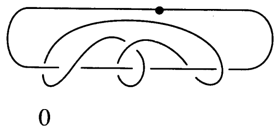

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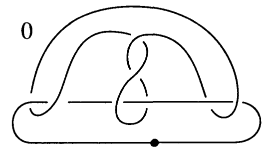

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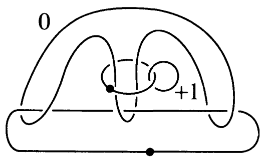

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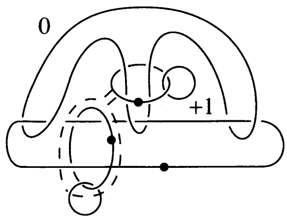

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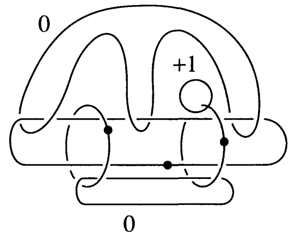

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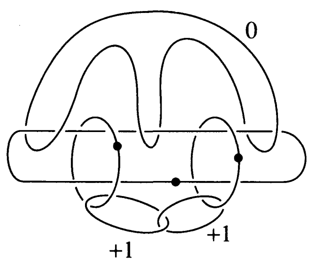

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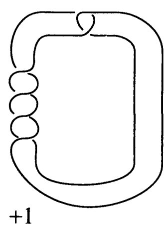

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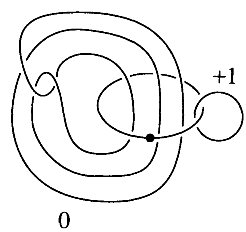

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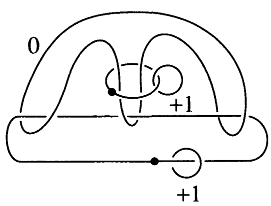

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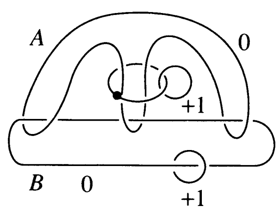

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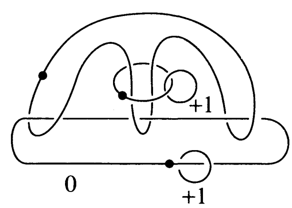

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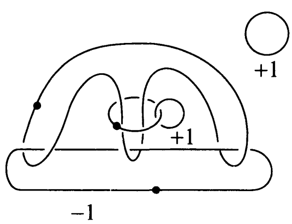

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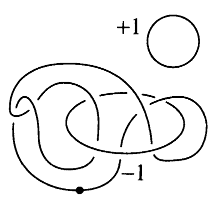

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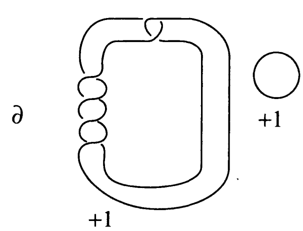

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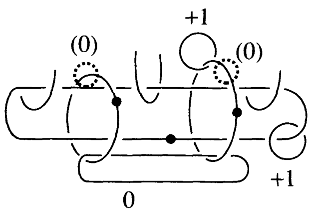

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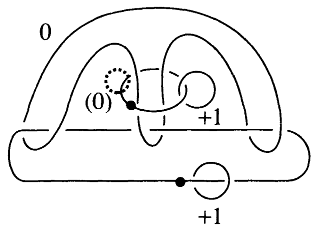

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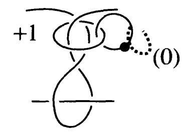

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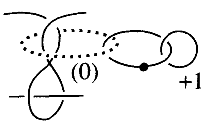

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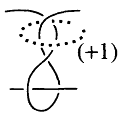

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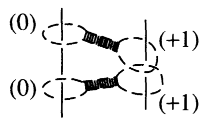

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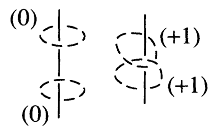

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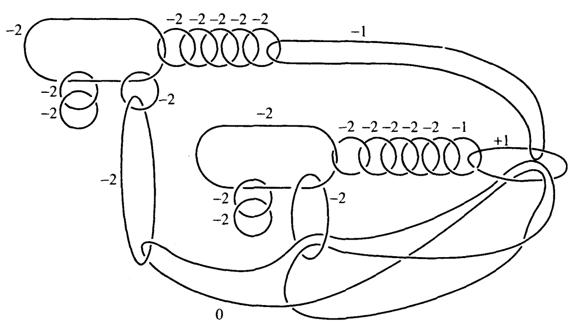

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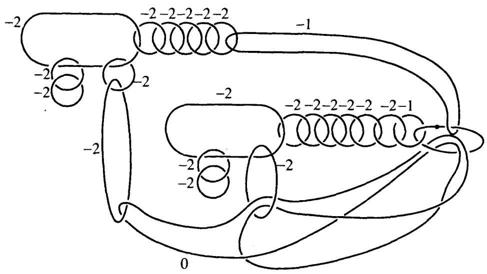

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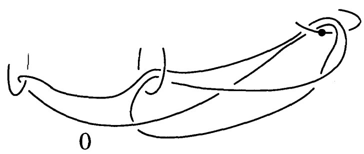

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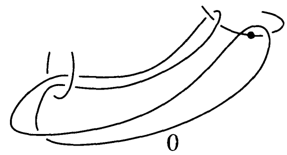

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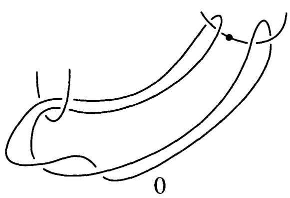

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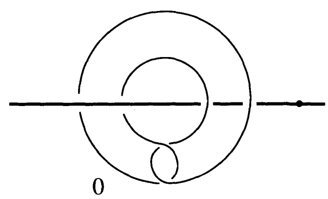

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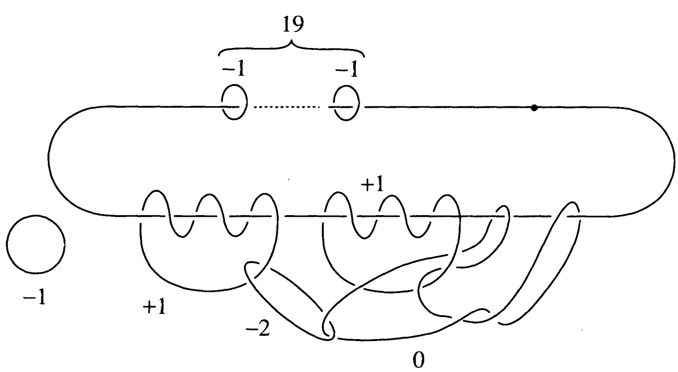

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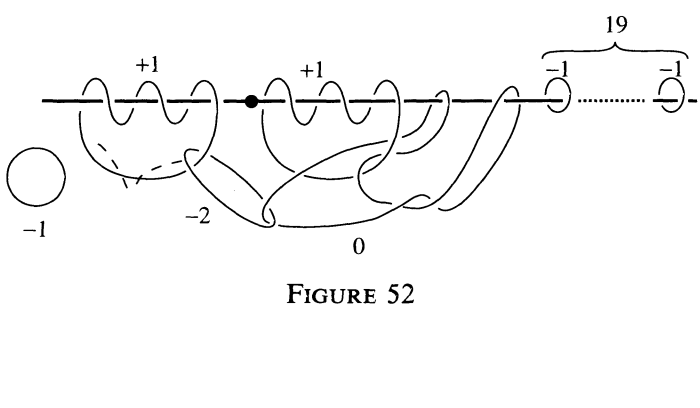

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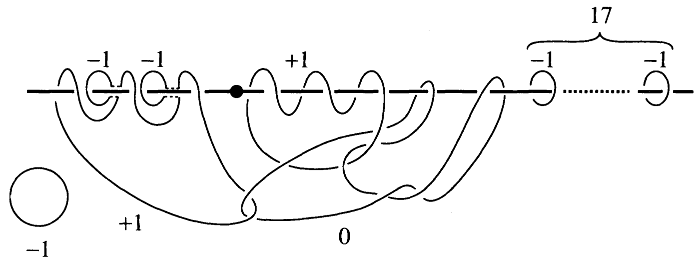

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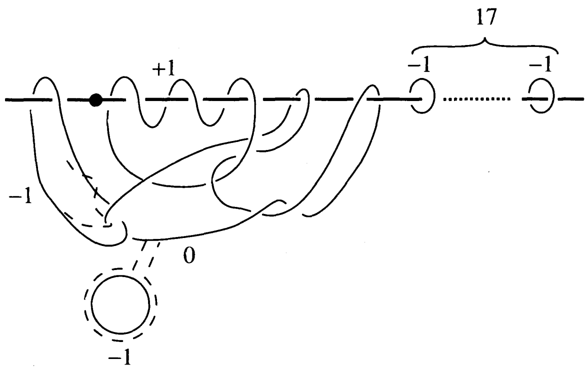

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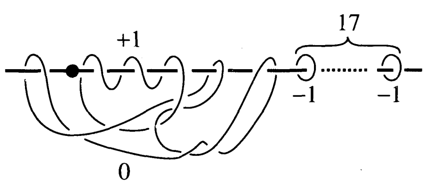

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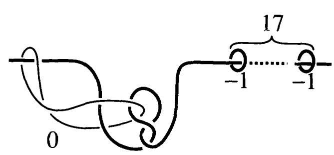

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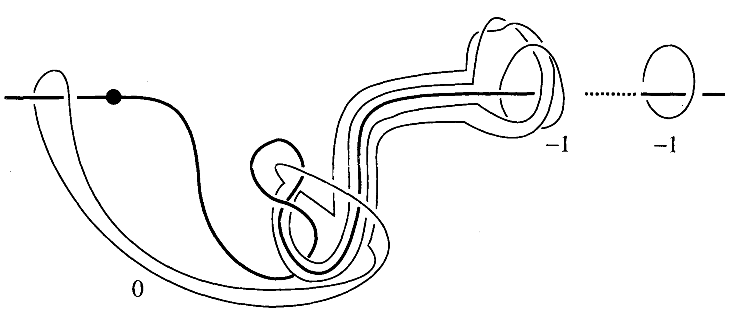

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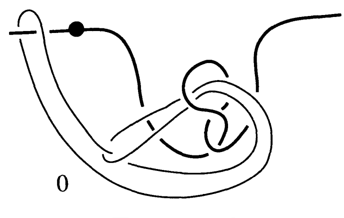

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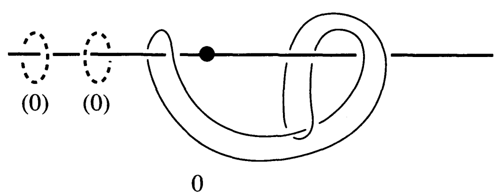

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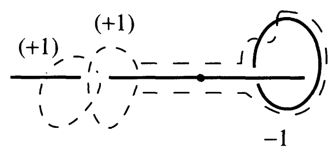

FIGURE 60

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