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COMPUTATIONS ON THE TRANSVERSE MEASURED FOLIATIONS ASSOCIATED WITH A PSEUDO-ANOSOV AUTOMORPHISM

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The following is a brief summary of the work. Let g be a pseudo-Anosov diffeomorphism of a compact surface S of genus p $(p > 1)$. First, we show how to make a partition of the lift of the unstable foliation Φ_U associated with g to the universal covering space (the unit disk U) into a countable number of layers approximating inaccessible points for Φ_U at infinity (∂U). We prove the following theorem.

Theorem 1. *Let $g^*(z_0) = z_0$. Then there exists an ascending sequence of Cantor sets of measure zero on ∂U invariant under g^*: $\overline{F}_1 \subseteq \overline{F}_2 \cdots \subseteq \overline{F}_n \subseteq \cdots$, such that $E = \cup \overline{F}_n \backslash A$, where E is the set of the endpoints of leaves of Φ_U and A is a countable set.*

The regular step lines will be studied in the second paragraph, both in U and on S. The idea of step lines belongs to Strebel (see [8]). We add one more requirement that each step end at a singularity of ϕ_u. In many aspects the regular step lines are similar to geodesics on a surface. For example, we prove the following theorem.

Theorem 2. *Let z_0, $z \in S$. Then there exists a unique regular step curve from z_0 to z in each homotopy class of curves with fixed points at z_0 and z.*

(Actually, z_0 in Theorem 2 is either a singularity of Φ_U or does not lie on any horizontal or vertical leaf. But this requirement can be easily lifted.) The regular step lines, also, minimize the total variations, both of the first and second coordinates, in a homotopy class of curves with fixed points at z_0 and z. However, the regular step line from z_0 to z is different from the one from z to z_0. At the end of the second section we use the regular step lines to parameterize $\overline{U}$ by sequences of real numbers, where g^* has especially simple form. The above parameterization induces a lexicographical order on points of $\overline{U}$ which agrees with the natural order on ∂U. We notice, here, that all results in the first two sections,

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except those related to the action of g, are automatically true for any pair of transverse measured foliations which does not have any horizontal or vertical connections between its singularities.

In the third section we obtain formulas for computing the fixed points of both g and g^* on S and in U, respectively, in terms of regular step lines. In particular, a criterion will be given for a point $z \in S$ to be a fixed point of g in the case where g does not rotate the direction from z. This criterion can also be used for determining the periodic points of g.

Finally, the above results will be applied for constructing algorithms for determining $g(z)$, $g^*(z)$, and the fixed points of g and g^*.

1. Partition of Φ_U

Some of the results in this section have appeared in [7].

Let S be a smooth oriented compact surface of genus p $(p > 1)$, and g a pseudo-Anosov diffeomorphism of S. Consider $\Phi = (\Phi_U, \Phi_S)$, the pair of transverse measured foliations associated with g, which increases lengths along Φ_U and decreases them along Φ_S by the same factor. Let U, the unit disk, be the universal covering space of S. Consider g^*, a lift of g to U. Then Φ_U and Φ_S may be lifted to a pair of transverse measured foliations in U, for which we will keep the same notation. Bers [1] made an observation that Φ can be viewed as the union of the horizontal and vertical geodesics associated with a quadratic differential on a conformal structure on S. Thus we can apply the results from [5] to study the boundary behavior of leaves of Φ. We recall, now, some properties of the leaves of Φ_U in U (see Marden and Strebel [5]).

(a) Every leaf γ of Φ_U from a point in U tends to its endpoint γ_e, a uniquely determined point on ∂U.

(b) Let γ_1, γ_2 be two leaves of Φ_U from the same point in U. Then γ_1, γ_2 have different endpoints on ∂U.

(c) *Divergence principle for the endpoints.* If α is a closed segment on a leaf of Φ_U, and β_1, β_2 are the leaves of Φ_S stemming from the endpoints of α, then β_1 and β_2 converge neither inside U nor on its boundary.

A leaf which passes through a singularity of g^* is said to be *critical.* Let $z \in U$. Then $n(z)$ denotes the number of leaves of Φ_U from z.

Definition. $z_0 \in U$ is called a *nice point* if it is either a singularity of Φ or it does not lie on any critical leaf of Φ_U or Φ_S.

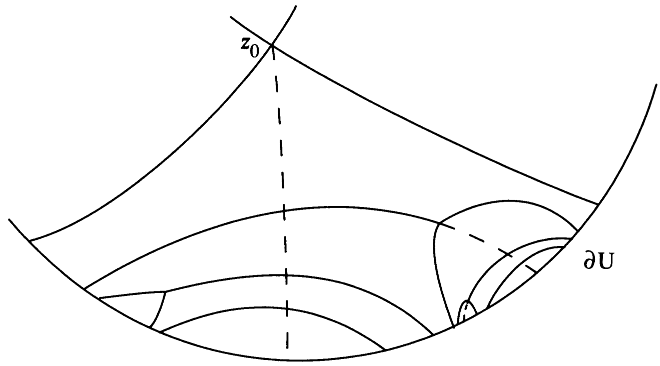

FIGURE 1

We now show how to make a partition of Φ_U into an infinite countable number of layers (see Figure 1). This partition is uniquely determined by an arbitrary nice point $z_0 \in U$. The collection of finitely many leaves of Φ_U from z_0 form Φ_0, *the zero layer of* Φ_U *with respect to* z_0. Then there exist $n(z_0)$ leaves of Φ_S from z_0: $\beta_1, \beta_2, \cdots, \beta_{n(z_0)}$. The collection of all leaves of $\Phi_U \backslash \Phi_0$ through the points on $\beta_1, \beta_2, \cdots, \beta_{n(z_0)}$ form Φ_1, *the first layer of* Φ_U *with respect to* z_0. Consider $\gamma \in \Phi_1$. Then there are two possibilities: either γ is not a critical leaf or there is a singular point z which lies on γ. In the latter case there are $n(z) - 1$ leaves of Φ_U from z which do not intersect any of $\beta_1, \beta_2, \cdots, \beta_{n(z_0)}$. They comprise $n(z) - 2$ sectors around z. No point inside these sectors belongs to any leaf from Φ_1. In each of these sectors we consider the leaves of Φ_U of the first layer with respect to z. We repeat the same procedure at each singular point of Φ_1. The collection of all leaves of Φ_U obtained in this manner form Φ_2. When we iterate the process we obtain $\Phi_3, \Phi_4, \cdots, \Phi_n, \cdots$. On each step of iteration Φ_n $(n > 0)$ is composed of leaves of the first layer with respect to singular points of Φ_{n-1}.

Theorem 1.1 (Slutskin [6]). $\Phi_0, \Phi_1, \cdots, \Phi_n, \cdots$ *form a partition of* Φ_U, *i.e.*, $\Phi_U = \bigcup \Phi_n$ *and* $\Phi_i \cap \Phi_j = \varnothing$, *where* $i, j = 0, 1, 2, \cdots$, *and* $i \neq j$.

Let z be a singular point of Φ_n $(n > 0)$. Consider the sectors $S_1, S_2, \cdots, S_{n(z)-2}$ around z, which do not have inside leaves of Φ_n. We call $B(z) = S_1 \cup S_2 \cup \cdots \cup S_{n(z)-2}$ the *bud centered at* z *of order* n (*with respect to* z_0).

Lemma 1.1. *Let $B(z)$ be the bud centered at z of order n $(n > 0)$. Then there exists a uniquely determined descending sequence of n buds: $B(z_1) \supseteq \cdots \supseteq B(z_{n-1}) \supseteq B(z)$, where $B(z_i)$, $0 < i < n$ is a bud of order i.*

Proof. The existence follows from the definition of the partition. The uniqueness is implied by the fact that two different buds of the same order do not intersect. q.e.d.

We say that $x \in \partial U$ is *an inaccessible point for* Φ_U if it is not the endpoint of any leaf of Φ_U. Let E_{in} denote the set of inaccessible points for Φ_U. Then $E = \partial U - E_{in}$ is the set of the endpoints of leaves of Φ_U.

Lemma 1.2. *The endpoint of any leaf of Φ_S is an inaccessible point for Φ_U.*

Proof. Let $x \in \partial U$ be the endpoint of a leaf $\beta \in \Phi_S$. If β is a critical leaf, then we consider the partition of Φ_U with respect to the singularity on β. Otherwise, we consider the partition of Φ_U with respect to a point on β, which does not lie on a critical leaf of Φ_U. In either case it follows from the construction of the partition that x is an inaccessible point for Φ_U. q.e.d.

We need the following well-known fact from the theory of discrete groups (see Ford [4]).

Let $G = G(S)$ be a Fuchsian group corresponding to S in U. If $A \subset \partial U$ is invariant under G $(G(A) = A)$, then A is everywhere dense in ∂U.

Corollary. *E and E_{in} are everywhere dense subsets of ∂U.*

Theorem 1.2 (a criterion for a point on ∂U to be an inaccessible point). *$x \in \partial U$ is an inaccessible point for Φ_U iff either x is the endpoint of leaf of Φ_S from z_0 or the endpoint of a critical leaf of Φ_S, or there exists an infinite descending sequence of buds with respect to z_0:*

$$B(y_1) \supseteq B(y_2) \supseteq \cdots \supseteq B(_n) \supseteq \cdots \tag{1.1}$$

where $B(y_n)$ is a bud of order n, such that $x = \cap B(y_n)$. In this case the sequence (1.1) *is uniquely determined by x.*

Proof. 1. Let $x \partial U$ be an inaccessible point for Φ_U. We showed in [6] that in this case, either x is the end point of a critical leaf from z_0 or x is inside a bud of the first order. By continuing in the same manner we either find the critical leaf with x as its end point or there exists an infinite descending sequence of buds $B(y_1) \supseteq B(y_2) \supseteq \cdots \supseteq B(y_n) \supseteq \cdots$, such that $x \in \cap B(y_n)$.

2. Let $x \in \bigcap B(y_n)$. Since $B(y_{n+1}) \cap \Phi_n = \varnothing$, it follows from Theorem 1.1 that $(\bigcap B(y_n)) \cap U = \varnothing$, or the same, $\bigcap B(y_n) \subset \partial U$. First we prove

that $\bigcap B(y_n)$ may contain only inaccessible points for Φ_h. Indeed, let us assume that $x \in \bigcap B(y_n)$ is the end point of a horizontal leaf γ. Then $\gamma \in B(y_n)$, $n = 1, 2, \cdots$. It implies that $\gamma \in \bigcap B(y_n)$ and this contradicts the fact that $\bigcap B(y_n) \subset \partial U$.

Now, we show that $\bigcap B(y_n)$ is a point on ∂U. $B(y_n) \cap \partial U$ is a closed arc for any $n > 0$. It follows that $\bigcap B(y_n)$ is either an arc or a point on ∂U. If $\bigcap B(y_n)$ is an arc, then it contains, by Lemma 1.2, the points of E, a contradiction.

The unique representation of x by a sequence of type (1.1) follows from the fact that two buds of the same order do not intersect. q.e.d.

"Either" and "or" in the statement of the theorem are not mutually exclusive. As it can be seen from the proof, only the end points of the leaves of Φ_S which stem from z_0 and from the centers of the buds and inside them cannot be represented as descending sequences of buds of type (1.1).

Corollary 1. *Let $x \in \partial U$ not be the common point of a sequence of buds of type* (1.1). *Then x is the end point, either of a leaf of Φ_U or of a critical leaf of Φ_S which is inside the bud centered at its singularity.*

Corollary 2. *With the exception of a countable subset of ∂U each inaccessible point is the common point of a descending sequence of buds of type* (1.1).

Let $E_n = E_n(z_0)$ $(n > 0)$ be the locus of the end points of the leaves of Φ_n. It follows that $E = \bigcup E_n$. The isolated points of E_n are the end points of those critical leaves of Φ_n which are inside the buds of order n. Let $E'_n = E_n \backslash \{\text{its isolated points}\}$. Take $F_n = E_1 \cup \cdots \cup E_{n-1} \cup E'_n$. It follows that $F_n \subseteq F_{n+1}$. Consider $\overline{F}_n$. $\overline{F}_n$ is obtained from F_n by adding the end points of the leaves of Φ_S which stem from z_0 and from the centers of the buds of the first $n-1$ orders and inside them. It follows that $\overline{F}_n \backslash F_n$ is, at most, a countable set, and it implies that $E = \bigcup \overline{F}_n \backslash A$, where A, $A \subset \bigcup \overline{F}_n$, is a countable subset of ∂U. From Corollary 1 to Theorem 1.2 we can conclude that $x \in \partial U$ may be represented as a sequence of buds of type (1.1) iff $x \in \partial U \backslash \bigcup \overline{F}_n$.

Theorem 1.3. *$\overline{F}_n$ is a Cantor subset of ∂U.*

Proof. Since $\overline{F}_n \backslash A \subset E$, it follows that $\overline{F}_n$ is totally disconnected. Now let us show that F_n does not contain isolated points. Let $x \in \partial U$ be an isolated point of E_i, $i < n$. It implies that x is the end point of a critical leaf from the singularity which is the center of a bud of order i. Then x is an accumulation point of E_{i+1}. q.e.d.

Consider the Riemannian metric on S given by the formula $ds^2 = dx^2 + dy^2$, where dx, dy are the linear elements determined by the

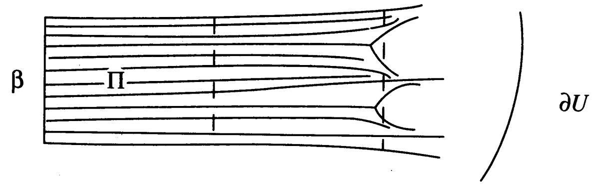

FIGURE 2

transverse measures on Φ_S and Φ_U, respectively. For this reason, we will call leaves and segments of Φ_U horizontal and those of Φ_S vertical. We keep the same notation ds for the lift of ds to U. Let $x, y \in S$. Then there exists a unique ds-geodesic on S which connects x and y (see Strebel [8]).

Theorem 1.4. *E has a zero measure on ∂U.*

Proof. It is enough to prove the theorem for the end points of the leaves of Φ_U passing through a rectangle Π which does not contain singularities inside it. When we move along the leaves transverse to the vertical sides of Π towards ∂U (Figure 2), the total sum of segments transverse to them remains unchanged in the ds-metric and equal to the length of β, the vertical side of Π. It follows that their total Euclidean length approaches zero uniformly.

Corollary. *$\overline{F}_n$ has a zero measure on ∂U for any $n > 0$.*

2. Regular step lines

In order to understand how a lift of a pseudo-Anosov diffeomorphism acts on $\overline{U}$, we have to introduce a notion of a regular step line from a nice point z_0.

Definition 1. A curve in U consisting of a vertical and a horizontal segments which intersect at their end points is called a *step.* We always assume, if it is not said otherwise, that the vertical segment comes first.

Definition 2. A connected subset in U composed of a finite sequence of steps is called a (*finite*) *regular step line from* z_0 (*without remainder*) if the following three conditions are satisfied:

(i) the first step begins at z_0;

(ii) each step ends at a singularity;

(iii) (*the angle condition*) the angle between the horizontal segment of the preceding step and the vertical segment of the succeeding step is greater than $\pi/n(z)$, where z is the joint singular point, i.e., the vertical segment of the succeeding step does not belong to any of two sectors formed by the

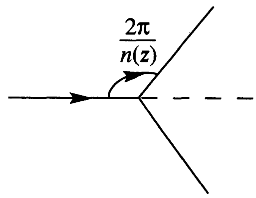

FIGURE 3. The angle condition

horizontal segment of the preceding step and by the adjacent to it critical horizontal segment from the joint point of the preceding and succeeding steps. See Figure 3.

Definition. The curve on S satisfying conditions (i)–(iii) is called a *regular step curve on* S.

It follows that the projection of a regular step line on the underlying surface S is a regular step curve.

If we allow the number of steps to be infinite we obtain an *infinite regular step line from* z_0. Let L be a regular step line. Then $\mathrm{ord}(L)$ denotes the number of steps of L. It follows from how we defined the partition of Φ_U with respect to z_0 that a regular step line of order n from z_0 determines in a unique way a descending sequence of n buds of type (1.1), such that the end of each step is the center of a bud in the sequence. When L is an infinite regular step line, then by Theorem 1.2, L approaches an inaccessible point on ∂U which is the point of intersection of the corresponding infinite descending sequence of buds. Hence we have the following lemma.

Lemma 2.1. (a) *Let* z_0 *be a nice point of* U *and* z, $z \in \Phi_n$, *a singular point of* Φ. *Then there exists a unique regular step line* L *from* z_0 *to* z *such that* $\mathrm{ord}(L) = n$.

(b) *There is the one-to-one correspondence between infinite regular step lines from* z_0 *and descending sequences of buds of type* (1.1) *in* U, *such that if* $x \in \partial U$ *is the common point of a descending sequence of buds, then the corresponding infinite regular step line approaches* x.

Corollary 1. *An infinite regular step line approaches an inaccessible point of* ∂U.

Corollary 2. *Any inaccessible point of* ∂U, *except a countable number of them, is uniquely determined by the infinite regular step line approaching it.*

Definition. Let L be a regular step line from z_0 to z_1. $L' = L \cup R$ is called a *regular step line* (*with remainder*) if $R \subset B(z_1)$ (the *remainder*) is one of the following:

(i) a horizontal segment or leaf from z_1;

(ii) a vertical segment or leaf from z_1;

(iii) a union of a vertical segment from z_1 and a horizontal noncritical segment or a leaf from the other end point of the vertical segment.

The following theorem follows from Theorem 1.1, Corollary 1 to Theorem 1.2, and Lemma 2.1.

Theorem 2.1. *Let z_0 be a nice point of U.*

1. *Let $z \in U$. Then there exists a unique regular step line L from z_0 to z, and L has a remainder iff z is a regular point of Φ.*

2. *Suppose that $z \in \partial U$ and z is not the common point of a sequence of buds of type* (1.1). *Then there exists a unique regular step line with remainder from z_0 to z.*

3. *Suppose that $z \in \partial U$ and z can be represented as the common point of a sequence of buds of type* (1.1). *Then there exists a unique infinite regular step line from z_0 to z.*

If we consider the projection of the regular step line from z_0 to z not S, we obtain the following corollary.

Corollary. *Let z_0 be a nice point on S, $z \in S$, and α a path from z_0 to z. Then there exists a unique regular step curve* (*with or without remainder*) *leading from z_0 to z and homotopic to α.*

Lemma 2.2. (a) *Let L be a regular step line from z_0. Then $g^*(L)$ is a regular step line from $g^*(z_0)$ of the same order.*

(b) *Let L be a regular step line with remainder. Then $g^*(L)$ is a regular step line with a remainder of the same type as L.*

Proof. g^* leaves invariant Φ_U and Φ_S and interchanges the singularities of Φ. The angle condition is, obviously, satisfied.

Corollary 1. *Let $z \in \Phi_n(z_0)$. Then $g^*(z) \in \Phi_n(g^*(z_0))$.*

Corollary 2. *Let $g^*(z_0) = z_0$. Then g^* leaves invariant the layers of Φ_U.*

Corollary 3. *Let $g^*(z_0) = z_0$. Then $g^*(E_n) = E_n$ and $g^*(F_n) = F_n$.*

Theorem 2.2. *Let $g^*(z_0) = z_0$. Then there exists an ascending sequence of Cantor sets of measure zero on ∂U invariant under g^*: $\overline{F}_1 \subseteq \overline{F}_2 \cdots \subseteq \overline{F}_n \subseteq \cdots$, such that $E = \bigcup \overline{F}_n \backslash A$, where A is a countable set.*

Proof. It follows from Theorems 1.3, 1.4, and Corollary 3 to Lemma 2.2. q.e.d.

We show, now, how to parameterize points in $\overline{U}$ by using regular step lines from z_0. First of all, we number the angles about z_0 by integers from 1 to $n(z_0)$ moving in the positive direction. Let z be a singular point of Φ different from z_0. Then we number all angles about z, except

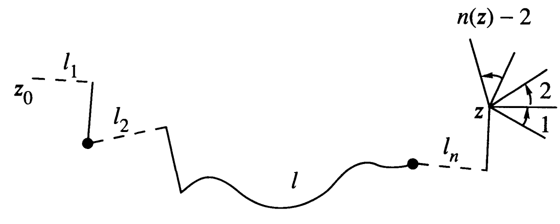

FIGURE 4

the two adjacent to the last horizontal segment of the regular step line l from z_0 to z, by integers from 1 to $n(z) - 2$, moving in the positive direction. See Figure 4.

Let $y_0, y_1, \cdots, y_n$, where $y_0 = z_0$ and $y_n = z$, be the set of singularities of Φ on l. Let S_k, $0 < k \leq n$, be a step of l from y_{k-1} to y_k. Then l and z are both uniquely determined by a sequence of n 4-tuples of real numbers: $[i_1, s_1, l_1^{-s_1}, \ell_1]$, $[i_2, s_2, l_2^{-s_2}, \ell_2], \cdots, [i_n, s_n, l_n^{-s_n}, \ell_n]$, where for any k, $0 < k \leq n$, i_k is the number assigned to the angle about y_{k-1} which contains the vertical segment of S_k, s_k is equal to 1 or -1, depending on whether the horizontal segment of S_k is to the left or to the right from its vertical one. Finally, l_k and ℓ_k are the lengths of the vertical and the horizontal segments of S_k respectively. In the same manner an infinite sequence of 4-tuples: $[i_1, s_1, l_1^{-s_1}, \ell_1]$, $[i_2, s_2, l_2^{-s_2}, \ell_2], \cdots, [i_n, s_n, l_n^{-s_n}, \ell_n], \cdots$ can be assigned to the limit point of an infinite regular step line on ∂U.

Now, let $z \in \overline{U}$ be such that the regular step line l from z_0 to z has a remainder R. Let $z_0, y_1, \cdots, y_n$ be the set of singularities of Φ on l. If R is a horizontal segment or leaf from y_n, then we assign a 4-tuple $(i, 2, 0, \ell)$ to R, where i is the number assigned to the angle about y_n which is to the right from R ($i = 0$, when R is to the right from the first angle about y_n), and ℓ is the length of R (when R is a leaf $\ell = \infty$). Let R be a vertical segment or leaf from y_n. Then the 4-tuple $(i, 0, l, 0)$ is assigned to R, where i is the number assigned to the angle about y_n which contains R, and l is the length of R (when R is a leaf $l = \infty$). Finally, let R be a union of a vertical segment β from y_n and a horizontal noncritical segment or a leaf α. Then R is parameterized by the 4-tuple (i, s, l^{-s}, ℓ), where i is the number assigned to the angle about y_n which contains R, s is equal 1 or -1 according as α is to the left or to the right from β, l is the length of β, and ℓ is the length of α (when α is a leaf $\ell = \infty$). We have obtained a sequence $([i_1, s_1, l_1^{-s_1}, \ell_1], [i_2, s_2, l_2^{-s_2}, \ell_2], \cdots, [i_n, s_n, l_n^{-s_n}, \ell_n], [a, b, c, d])_{z_0}$

assigned to z, where $([i_1, s_1, l_1^{-s_1}, \ell_1], [i_2, s_2, l_2^{-s_2}, \ell_2], \cdots, [i_n, s_n, l_n^{-s_n}, \ell_n])_{z_0}$ is the sequence of n 4-tuples assigned to y_n, and $[a, b, c, d]$ is the 4-tuple assigned to R. Thus we have obtained a parameterization of $\overline{U}$. We can consider the lexicographical order on points of $\overline{U}$ induced by this parameterization. The following theorem is self-evident.

Theorem 2.3. *The lexicographical order on points of ∂U coincides with their natural order moving in the counterclockwise direction within each sector about z_0.*

Let $\mathfrak{I}(z_0)$, $\mathfrak{I}(g^*(z_0))$ be parameterization of $\overline{U}$ corresponding to z_0 and $g^*(z_0)$, respectively, and let A_i be an angle about z_0 in $\mathfrak{I}(z_0)$. Then we define $g^*(i)$ as the number assigned to $g^*(A_i)$ in $\mathfrak{I}(g^*(z_0))$. When $g^*(z_0) = z_0$ and g^* preserves directions from z_0, it implies that $g^*(i) = i$. Since ℓ_k is uniquely determined by i_k, s_k and l_k, we can omit it, i.e., $[i_i, s_k, l_k^{-s_k}] = [i_k, s_k, l_k^{-s_k}, \ell_k]$.

Theorem 2.4 (Computational formulas for $g^*(z)$). *Let $z \in \overline{U}$.*

1. *Suppose that $z \in U$ is a singularity of Φ. Let*

$$z = ([i_1, s_1, l_1^{-s_1}], [i_2, s_2, l_2^{-s_2}], \cdots, [i_n, s_n, l_n^{-s_n}])_{z_0}.$$

Then

$$g^*(z) = ([g^*(i_1), s_1, (\lambda^{-1}l_1)^{-s_1}], [i_2, s_2, (\lambda^{-1}l_2)^{-s_2}], \cdots, [i_n, s_n, (\lambda^{-1}l_n)^{-s_n}])_{g^*(z_0)}.$$

2. *Suppose that $z \in \overline{U}$, such that the regular step line from z_0 to z has a remainder. Let*

$$z = ([i_1, s_1, l_1^{-s_1}], [i_2, s_2, l_2^{-s_2}], \cdots, [i_n, s_n, l_n^{-s_n}], [a, b, c, d])_{z_0}.$$

(a) *If $abcd = 0$, then*

$$g^*(z) = ([g^*(i_1), s_1, (\lambda^{-1}l_1)^{-s_1}], [i_2, s_2, (\lambda^{-1}l_2)^{-s_2}], \cdots, [i_n, s_n, (\lambda^{-1}l_n)^{-s_n}], [a, b, \lambda^{-1}c, \lambda d])_{g^*(z_0)};$$

(b) *If $abcd \neq 0$, then*

$$g^*(z) = ([g^*(i_1), s_1, (\lambda^{-1}l_1)^{-s_1}], [i_2, s_2, (\lambda^{-1}l_2)^{-s_2}], \cdots, [i_n, s_n, (\lambda^{-1}l_n)^{-s_n}], [a, b, \lambda^{b}c, \lambda d])_{g^*(z_0)};$$

3. *Suppose that $z \in \partial U$ is the limit point of an infinite regular step line. Let*

$$z = ([i_1, s_1, l_1^{-s_1}], [i_2, s_2, l_2^{-s_2}], \cdots, [i_n, s_n, l_n^{-s_n}], \cdots)_{z_0}.$$

Then

$$g^*(z) = ([g^*(i_1), s_1, (\lambda^{-1}l_1)^{-s_1}], [i_2, s_2, (\lambda^{-1}l_2)^{-s_2}], \cdots, [i_n, s_n, (\lambda^{-1}l_n)^{-s_n}], \cdots)_{g^*(z_0)}.$$

Proof. Let z_1 be a singular point of Φ different from z_0. Consider l, the regular step line from z_0 to z_1. Then g^* preserves the angles and orientations between corresponding segments of l and $g^*(l)$. At the same time g^* increases the lengths of horizontal segments and decreases the lengths of vertical ones in λ times. It implies 1 and 3. Since g^* preserves a type of remainder, we have 2.

3. Computational formulas for the fixed point of g^*

Φ defines a structure of a Riemann surface on S in the following way. The open rectangles with the horizontal sides on leaves of Φ_U, and the vertical sides on leaves of Φ_S, which do not have inside singularities of Φ, define local coordinates on S outside critical points. If we add to them the interiors of the unions of the closed rectangles around singular points, we obtain a Riemann surface X. Now, we define a quadratic differential q on X. $q = dz^2$ in any rectangle which does not contain a singular point, and $q = [1/4(n+2)^2]z^n\,dz^2$ in a neighborhood of a singular point, where $n+2$ is equal to the number of the leaves of $\Phi_U(\Phi_S)$ stemming from the singular point. It follows that the horizontal and vertical geodesics of q coincide with the leaves of Φ_U and Φ_S, respectively. We call q a *quadratic differential associated with* Φ on S. We notice that $ds = |\sqrt{q}|$.

Let γ be a curve on S which connects z_0 with z. Then $\int_\gamma \Phi_U$, $\int_\gamma \Phi_S$ denote the total variations of the second and first coordinates along γ, respectively. Thus for a regular step line L, $\int_L \Phi_U = \Sigma b_i$ and $\int_L \Phi_S = \Sigma a_i$, where Σa_i, Σb_i are the sums of the lengths of horizontal and vertical segments of L, respectively.

Theorem 3.1. *$\int_\gamma \Phi_U$, $\int_\gamma \Phi_S$ attain their minimum on the regular step line L from z_0 to z among all curves on S in the same homotopy class as L with fixed z_0 and z.*

Proof. It is enough to prove the theorem for $\int_L \Phi_U$.

Lemma. *Let $z_1 \in U$ be a singularity of Φ which does not lie on L^*, where L^* is a regular step line in U, which is projected on L. Then at most one horizontal leave from z_1 may intersect L^* with the intersection being a single point.*

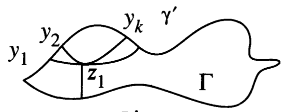

FIGURE 5

Proof. It follows from how we defined the partition of Φ that all horizontal leaves from z_1 belong to the layer of the same order, say Φ_n. Since different steps of L belong to different layers, the horizontal leaves from z_1 may intersect L only at inner points of one of its vertical segments, say β. Thus no two horizontal leaves from z_1 may intersect β, as it would contradict the divergence principle. q.e.d.

Let γ be a curve in U between z_0 and z. We can assume that γ is a simple curve. We show that $\int_L \Phi_U \leq \int_\gamma \Phi_U$. By substituting, if necessary, a step line (not regular) for γ, with the same total sum of the lengths of vertical segments, we can achieve that L would break into a finite number of pieces, each one having only its end points common with γ. Let L' be such a piece and γ' a piece of γ which connects the endpoints of L'. Let Γ be the Jordan domain bounded by γ' and L'. We can assume that there are no singularities inside Γ. Indeed, let z_1 be such a singularity. Consider $y_1, y_2, \cdots, y_n$ the first points of intersection of horizontal leaves from z_1 with $\partial\Gamma$. We have obtained n Jordan domains bounded by pieces of γ', L' and horizontal segments from z_1. It follows from the lemma that each of them has on its boundary a piece of γ'. Thus it is enough to give a proof for each domain separately, as the horizontal segments from z_1 do not affect the total sum of the lengths of vertical segments of γ'. In this manner we get rid of all singularities inside Γ. See Figure 5.

Now, we draw the α, the horizontal leaf from each singularity y of L', except z_0, which is adjacent to the horizontal segment of L' leading to y and inside Γ. It may be shown, in the same way as we did it in the lemma, that α does not intersect L', but at y. Again, we have a subdivision of Γ into a finite number of Jordan domains with the unchanged total sum of the lengths of vertical segments of γ'. Thus it is enough to prove the theorem for the case where L' is a step. $\sqrt{q}$ maps Γ conformally onto a Jordan domain in the complex plane, with L' mapped onto a right angle. q.e.d.

In the rest of the section we obtain formulas for determining the fixed points of both g^* and g in U and S, respectively. Let $z_0 \in U$ be the fixed point of g^*. We can assume that g^* does not rotate stable (unstable) directions from z_0. Otherwise we consider $f = (g^*)^n$ for some $n > 1$. It follows from Thurston's theorem (see §4) that z_0 is the fixed point of g^* iff z_0 is the fixed point of f.

Lemma 3.1. *Let z_0 be the fixed point of g^* in U, such that g^* does not rotate stable (unstable) leaves from z_0, and let L be the regular step line from a nice point $z_1 \in \Phi_1(z_0)$ to $g^*(z_1)$. Let d, d' be, respectively, the lengths of β and α_1, the vertical and horizontal segments of the regular step from z_0 to z_1, respectively. Then*

$$d = \frac{\lambda \Sigma b_i}{\lambda - 1}, \qquad d' = \frac{2a_n - \Sigma a_i}{\lambda - 1}, \tag{3.1}$$

where Σb_i is the total sum of vertical segments of L, Σa_i is the total sum of horizontal segments of L, and a_n is the length of the horizontal segment of the last step of L.

All singularities of L belong to $\Phi_1(z_0)$. They are all in the same sector about z_0 as z_1 and on the same side of β as z_1. Let $z_2, \cdots, z_n$ be the singularities of L moving from z_1 to $g^*(z_1)$. Then the corresponding points of intersection of the horizontal leaves $\alpha_1, \alpha_2, \cdots, \alpha_n$ from $z_1, z_2, \cdots, z_n$ with β monotonically approach z_0. The horizontal segment of each step of L, except the last one, is on the same side of its vertical segment as the horizontal leaf from the beginning of the step which intersects β. For each i, $1 \le i \le n$, α_i composes the angle equal to $2\pi/n(z)$ with the corresponding horizontal segment of L, and $\pi/n(z)$ with the vertical one.

Proof. We will actually construct L. Let c be the length of the vertical segment on β between the points of intersection β with α_1 and α', the horizontal leaf from $g^*(z_1)$. Now, we draw β_1, a vertical leaf from z_1 in the sector which borders α_1 and contains z_0. β_1 either intersects α' or a bud of the first order with respect to z_0. Indeed, otherwise, there are points on β_1 at the distance more than c from z_1 which belong to leaves of Φ_1. Then the points of intersection of the horizontal leaves from them with β are closer to z_0 than $g^*(z_1)$, contradiction.

Let β_1 intersect α'. Then L consists of one vertical segment on β_1 and one horizontal segment on α'. Assume that β_1 does not intersect α'. Then β_1 intersects a bud of the first order with respect to z_0. Let $B(z_2)$ be the first bud that β_1 intersects, and y_1 is the point of intersection of β_1 with a horizontal leaf from y_1. Then the first step of L begins at z_1, ends at z_2, and consists of the vertical segment on β_1 and the horizontal segment between y_1 and z_2. Let a_1, b_1 be the lengths of the corresponding horizontal and the vertical segments. It follows that the

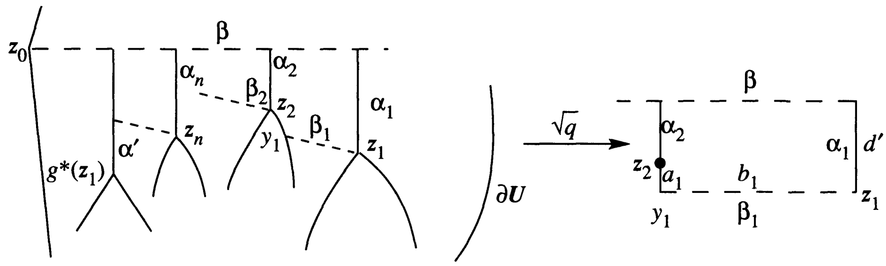

FIGURE 6

length of α_2 is equal to $d' - a_1$, and the length of the vertical segment on β between α_1 and α_2 is equal b_1. Indeed, look at Π, the five-sided polygon composed, successively, of the segments on α_1, β_1, the horizontal segment between y_1 and z_2, the horizontal segment from z_2, and the segment on β. Π does not contain any singularity within it, since the horizontal leaves from such a singularity might intersect Π only at z_1 or z_2, which is impossible. $\sqrt{q}$ maps Π onto a rectangle in the complex plane (see Figure 6).

We continue, now, the same procedure from z_2. In this way we get singularities of Φ_1: $z_2, \cdots, z_i, \cdots$, all at distance less than d' along horizontal leaves to the intersection with β. It follows that there is only a finite number of them, so that we have constructed a regular step line L from z_1 to $g^*(z_1)$. Then $\operatorname{ord}(L) = n$, and therefore the length of the horizontal segment from the juncture point of the last segment of L to β is equal to $d' - a_1 - a_2 - \cdots - a_{n-1}$. From the other side the same distance is equal to $\lambda d' - a_a$. Thus we have obtained the first formula in (3.1). By the same token the length of the vertical segment on β between α_1 and α' is $b_1 + b_2 + \cdots + b_n$. From the other side it is $d(1 - \lambda^{-1})$. It yields the second formula in (3.1). The description of L in the lemma follows from Figure 6. q.e.d.

Consider $(g^*)^{-1}$. Then Φ_U becomes Φ_S and vice versa. Let $\Phi_1'(z_1)$ denote the first layer of $\Phi' = (\Phi_S, \Phi_U)$ with respect to z_1. Consider a nice point $z' \in U$. It follows that $z' \in \Phi_1'(z_1)$, iff $z_1 \in \Phi_1(z')$. Let A be the step from z_1 to z' in $\Phi_1'(z_1)$. Thus z' is uniquely determined by the following four parameters: the length of the vertical segment of A, the length of the horizontal segment of A, the side of the vertical segment of A that its horizontal segment lies on, and the sector about z_1 that A belongs to.

Theorem 3.1 (A criterion for a point in $\Phi_1'(z_1)$ to be the fixed point of g^*). *Let $z_1 \in U$ be a nice point not fixed by g^*, and L the regular* *step line from* z_1 *to* $g^*(z_1)$. *Then* $z_0 \in \Phi_1'(z_1)$ *is the fixed point of* g^*, *such that* g^* *does not rotate the direction from* z_0, *iff* A, *the step from* z_1 *(first the horizontal segment* α_1 *of length* d', *then the vertical segment* β *of length* d), *such that* α_1 *composes the angle* $\pi/n(z_1)$ *with the vertical segment of the first step of* L *and lies on the same side of it as its horizontal segment, and* β *is on the same side of* α_1 *as the vertical segment of the first step of* L, *where*

$$d' = \frac{2a_n - \Sigma a_i}{\lambda - 1}, \qquad d = \frac{\lambda \Sigma b_i}{\lambda - 1},$$

Σa_i *is the total sum of horizontal segments of* L, Σb_i *the total sum of vertical segments of* L, *and* a_n *the length of the horizontal segment of the last step of* L, *leads to* z_0, *and the following hold*:

1. *Both the singularities of* L *and* $g^*(z_1)$ *belong to* $\Phi_1(z_0)$, *and they are in the same sector about* z_0 *as* z_1 *and on the same side of* β *as* z_1.

2. *Let* $z_2, \cdots, z_n$ *be the singularities of* L *moving from* z_1 *towards* $g^*(z_1)$. *Then the corresponding points of intersection of the horizontal leaves* $\alpha_1, \alpha_2, \cdots, \alpha_n$, $\alpha' = g^*(\alpha_1)$, *from* $z_1, z_2, \cdots, z_n$, $g^*(z_1)$ *with* β *monotonically approach* z_0.

3. *The horizontal segment of each step of* L, *except the last one, is on the same side of its vertical segment as the horizontal leaf from the initial point of the step which intersects* β.

Proof. Necessity follows from Lemma 3.1. In order to prove sufficiency we consider the Jordan domain D bounded by L, α_1, α', and β. D does not contain a singularity. Indeed, let $z \in D$ be a singularity of Φ. $\alpha_2, \cdots, \alpha_n$ divide D into n Jordan domains. No horizontal leaf from z may intersect any of $\alpha_1, \alpha_2, \cdots, \alpha_n, \alpha'$. It implies that at least two of them must intersect either β or the same vertical segment of L. But this contradicts the divergence principle.

Let d'' denote the length of α'. From the same argument as in Lemma 3.1 it follows that $d'' = d' + 2a_n - \Sigma a_i$. By (3.1), $d'' = \lambda d'$. Thus β is left invariant by g^*. Now, let $z' = g^*(z_0)$. $g^*(\beta)$ is the vertical segment of length λd between x, the point of intersection of α' with β, and z'. The distance between x and z_0 is $d - \Sigma b_i$. By (3.1) we hence have that $z' = z_0$. q.e.d.

The following theorem is an analogue of Theorem 3.1 for surface S.

Theorem 3.2 (A criterion for a point of S to be a fixed point of g). *Let* z_1 *be a nice point of* S. *Then* z_0 $(\neq z_1)$ *is a fixed point of* g, *such that* g *does not rotate the directions from* z_0, *iff there exists* L, *a regular step line from* z_1 *to* $g(z_1)$, *such that* A, *the step from* z_1 *(first the horizontal* *segment α_1 of length d', in the same direction from z_1 as the preimage of the horizontal segment of the last step of L. then the vertical segment β of length d), where α_1 composes the angle $\pi/n(z_1)$ with the vertical segment of the first step of L and lies on the same side of it as its horizontal segment, and β is on the same side of α_1 as the vertical segment of the first step of L, where*

$$d' = \frac{2a_n - \Sigma a_i}{\lambda - 1}, \qquad d = \frac{\lambda \Sigma b_i}{\lambda - 1},$$

Σa_i being the total sum of horizontal segments of L, Σb_i the total sum of vertical segments of L, and a_n the length of the horizontal segment of the last step of L, leads to z_0, and the following hold:

1. *Let $z_2, \cdots, z_n$ be the singularities of L moving from z_1 towards $g^*(z_1)$. Then the regular step lines from z_0 to $z_2, \cdots, z_n, g(z_1)$ homotopic to the curves composed of A^{-1} and the parts of L from z_1 to the corresponding singularities are all the steps whose vertical segments belong to β, and their horizontal segments are on the same side of β as α_1.*
2. *The juncture points of the steps in 1 from z_0 to $z_1, z_2, \cdots, z_n$, $g(z_1)$ on β monotonically approach z_0.*
3. *The horizontal segment of each step A_1 of L, except the last one, is on the same side of the vertical segment of A_1 as the horizontal segment of the step from z_0 to the initial point of A_1.*

Proof. First, we notice that the last three conditions of the theorem are equivalent to the corresponding conditions of Theorem 3.1. The sufficiency now follows from Theorem 3.1. Since α is dense in S (see [3] or [8]), there exists step A from z_1 to z_0 (first a horizontal, then a vertical segment). Let z_0^* be the end point of the lift of A to U. Then $\pi(z_0^*) = z_0$. Consider the lift g^* of g to U, such that $g^*(z_0^*) = z_0^*$. Now, we can apply Lemma 3.1 to the regular step line L from z_1^* to $g^*(z_1^*)$.

Corollary. *The set of the endpoints of the steps from z_1 satisfying conditions* (3.1) *contains all fixed points of g.*

Note. Consider g^n, $n > 0$. g^n is a pseudo-Anosov diffeomorphism of S with a stretching factor λ^n, and Φ is the pair of transverse measured foliations associated with g^n. Then Theorem 3.2 implies a criterion for $z_0 \in S$ to be a periodic point of g, if we substitute λ^n for λ in (3.1), as g^n leaves invariant the directions from z_0 for some $n > 0$. In this case we need not assume any a priori knowledge about g, as we can choose a singularity of g as z, and take $n \geq (4p-2)!$ (see the next section). In particular, Theorem 3.2 for $n = (4p - 2)!$ yields, among other periodic points of g, all its fixed points.

4. Algorithms

We assume, till the end of the section, that we "know" Φ either in U or on S (or both). We discuss here two algorithms: the first one how to determine $g^*(z)(g(z))$, and the second how to find the fixed points of $g^*(g)$.

1. *An algorithm for finding* $g^*(z)$, *for any* $z \in U$, *when* $g^*(z_1)$ *and* g^**-image of at least one direction from* z_1 *are known for* $z_1 \in U$.

Note. We assume that z_1 is a nice point, though the same construction can be carried out for any z_1 if in the definition of regular step line we allow a critical segment, horizontal or vertical, to precede its first step.

Let L be a regular step line with or without remainder which leads from z_1 to z. Then $g^*(L)$ can be drawn from $g^*(z_1)$ by increasing horizontal segments and decreasing vertical segments in λ times, where λ is the stretching factor of g. We note that while constructing $g^*(L)$ we must preserve the orientation of segments within each step and the angle between two adjacent segments from different steps.

Now, we discuss how to construct a regular step line L from z_1 to z.

(a) Let $z_1 \in S$. Then move along a vertical leaf β from z_1 until we reach a small neighborhood of z, where we can connect β with z by a horizontal segment.

(b) Let $z_1 \in U$. Then a singularity of the first order with respect to z_1 characterized by the property that one of the horizontal leaves from it intersects a vertical leaf from z_1. Now, moving away from $z = 0$, we try all the singularities of Φ_1 which we come across until we find the one, say z_2, with the bud $B(z_2)$ containing z. The first step of L consists of the vertical segment from z_1 to x, the point of intersection of a vertical leaf from z_1 and a horizontal leaf from z_2, and the horizontal segment from x to z_2. We continue the procedure in $B(z_2)$ looking for the bud of the first order with respect to z_2 which contains z. If z is a singularity, we obtain, finally, the regular step line from z_1 to z. Otherwise, we reach the point where a horizontal leaf from z intersects a vertical leaf from the center of the last bud in the sequence. In this case we obtain the regular step line with remainder from z_1 to z.

2. *An algorithm for finding the fixed point of* g^* *in* U, *when* $g^*(z_1)$ *and* g^**-image of at least one direction from* z_1 *are known for* $z_1 \in U$.

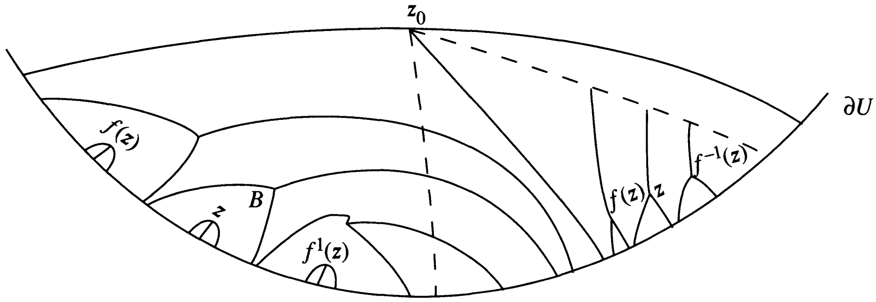

FIGURE 7

We recall Thurston's classification theorem for lifts of pseudo-Anosov diffeomorphisms.

Theorem ([2], [9]). 1. *Suppose that* g^* *does not have fixed points in* U. *Then* g^* *has exactly two fixed points on* ∂U; *one is an attracting fixed point, and the other is a repelling fixed point. All other points of* $\overline{U}$ *converge to the attracting fixed point.*

2. *Suppose that* $g^*(z_0) = z_0$, $z_0 \in U$. *Then* z_0 *is the only fixed point of* g^* *in* U. g^* *has* $2n$ *periodic points on* ∂U *which coincide with the endpoints of the leaves of* Φ_U *and* Φ_S *from* z_0. g^* *permutes the fixed points on* ∂U *in the same manner as* g^* *permutes the leaves of* Φ_U *and* Φ_S *from* z_0.

We are going to construct an algorithm for determining if g^* has the fixed point in U and locating it when it does. First of all, we substitute $f = (g^*)^{(4p-2)!}$, $4p-2$ being the upper bound for the number of leaves of $\Phi_U(\Phi_S)$ stemming from a point of U, for g^* in order to get rid of a possible rotation with a nonzero angle about the fixed point. It follows from Thurston's theorem that $z_0 \in U$ is the fixed point of g^* iff z_0 is the fixed point of f. We recall that $\Phi_0(z)$ denotes the union of leaves of Φ_U stemming from z.

Lemma 4.1. *Let* f *have a fixed point in* U. *Then* $f^{-1}(\Phi_0(z))$ *and* $f(\Phi_0(z))$ *are in different sectors about a nice point* z *iff* z *belongs to the layer of the first order with respect to the fixed point of* f.

Proof. Let z_0 be the fixed point of f. If $z \in \Phi_1(z_0)$, then $f^{-1}(\Phi_0(z))$ and $f(\Phi_0(z))$ are in two different sectors about z which border the horizontal leaf from z which intersects a vertical leaf from z_0. See Figure 7.

Let $z \in \Phi_n(z_0)$, $n > 1$. Then there exists B, the bud of the first order with respect to z_0 which contains $\Phi_0(z)$. Look at $f^{-1}(B)$ and $f(B)$.

They belong to the same sector about z which contains z_0. It follows that $f^{-1}(\Phi_0(z))$ and $f(\Phi_0(z))$ are in the same sector about z which contains z_0.

Lemma 4.2. *Let f not have the fixed point in U. Then there exists z, a singularity of Φ, such that $f^{-1}(\Phi_0(z))$ and $f(\Phi_0(z))$ belong to different sectors about z.*

Proof. Let z be a singularity of Φ such that the attracting and repelling fixed points of f belong to different sectors about z (see [6]). Then $f(\Phi_0(z))$ belongs to the sector which contains the attracting fixed point, and $f^{-1}(\Phi_0(z))$ belongs to the sector which contains the repelling fixed point.

Corollary. *There exists z, a singularity of Φ, such that $f^{-1}(\Phi_0(z))$ and $f(\Phi_0(z))$ belong to different sectors about z. If f has the fixed point in U, then z belongs to the layer of the first order with respect to the fixed point of f.*

(a) *The unit disk U.* Moving away from $z = 0$, we try all the singularities of Φ which we come across until we find the one, say z, having the property described in the corollary, i.e., such that $f^{-1}(\Phi_0(z))$ and $f(\Phi_0(z))$ belong to different sectors about z. Now, we connect z with $f(z)$ by the regular step line L, and, then construct a step A from z in the manner as it was described in Theorem 3.1. Let z_0 be the end point of A. Now, we can apply the criterion of Theorem 3.1 to decide if z_0 is the fixed point of f or not. If not, we can conclude from Lemma 4.1 and Theorem 3.1 that f does not have fixed points inside U.

(b) *The surface S.* Let $z \in S$ be a nice point. First, we check all singularities of Φ on being fixed points of g. Now, let $g_1 = g^2$. The remaining fixed points of g are among the fixed points of g_1. When we go through all regular step lines from z to $g_1(z)$ and use the criterion of Theorem 3.2 we obtain all fixed points of g. Let z_0 be a fixed point of g_1. To guarantee the stability of the solution $g(z_0) = z_0$ consider Π, a rectangular neighborhood about z_0. Take a rectangle $\Pi' \ni z_0$, $\Pi' \subset \Pi$, such that $g(\Pi') \subset \Pi$. Then z_0 is a fixed point of g iff $g(z_0) \subset \Pi'$.

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