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SUBVARIETIES OF GENERAL HYPERSURFACES IN PROJECTIVE SPACE

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0. Introduction

We are interested in the following question: If C is an irreducible curve (possibly singular) on a generic surface of degree d in a projective 3-space $\mathbf{P}^3$, can the geometric genus of C (the genus of the desingularization of C) be bound from below in terms of d? Bogomolov and Mumford [14] have proved that there is a rational curve and a family of elliptic curves on every K-3 surface. Since a smooth quartic surface in $\mathbf{P}^3$ is a K-3 surface, there are rational and elliptic curves on a generic quartic surface in $\mathbf{P}^3$. On the other hand, Harris conjectured that on a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$ there are neither rational nor elliptic curves.

Now let C be a curve on a surface S of degree d in $\mathbf{P}^3$. By the Noether-Lefschetz Theorem, if $d \geq 4$ and S is generic, then C must be a complete intersection of S with another surface S_1 of degree k. In this case we say that C is a type (d, k) curve on S. Clemens [4] has proved that there is no type (d, k) curve with geometric genus $g \leq \frac{1}{2}dk(d-5)$ on a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$; in particular, there is no curve with geometric genus $g \leq \frac{1}{2}d(d-5)$ on a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$.

Our first main result is the following.

Theorem 1. *On a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$, there is no curve with geometric genus $g \leq \frac{1}{2}d(d-3)-3$, and this bound is sharp. Moreover this sharp bound can be achieved only by a tritangent hyperplane section if $d \geq 6$.*

We immediately conclude that the above conjecture of Harris is true. Meanwhile it is not hard to see that for a generic surface S of degree d in $\mathbf{P}^3$, there is a tritangent hyperplane H and thus $C = H \cap S$ has three double points. Since $\pi(C) = \frac{1}{2}(C\cdot C + K_S \cdot C) + 1 = \frac{1}{2}d(d-3)+1$, and an ordinary double point drops the genus of a curve by 1, the above bound is sharp.

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Let C be a curve on a generic surface S of degree d in $\mathbf{P}^3$. The main point of the proof of Theorem 1 is to see how bad the singularities of such a curve C can be. We first study the deformation of C at the singular points of C, and obtain that if there is a type (d, k) curve C with certain geometric genus g on a generic surface S of degree d, then there are some homogeneous polynomials vanishing at the singular points of C to a certain expected order. By a Koszul type of argument, we can reduce the degree of these homogeneous polynomials. From these we get control over the singularities of C and obtain Theorem 2.1 which is just a slight improvement of Clemens' results (cf. [3], [4]). Then to prove Theorem 1 in the case $d \geq 6$, it remains only to see what kind of singularities a hyperplane section of S can afford.

We can generalize the above result in $\mathbf{P}^3$ to higher dimensions.

Theorem 2. *Let V be a generic hypersurface of degree $d \geq n+3$ in $\mathbf{P}^{n+1}$ $(n \geq 3)$, $M \subset V$ a reduced and irreducible divisor, and $p_g(M)$ the geometric genus of the desingularization of M. Then*

$$(0.1)\quad p_g(M) \geq \min\left\{\binom{d-2}{n+1} - \binom{d-4}{n+1} + 1,\ \binom{d}{n+1} - \binom{d-1}{n+1}\right\}.$$

Moreover if

$$(0.2)\qquad \binom{d-2}{n+1} - \binom{d-4}{n+1} + 1 \geq \binom{d}{n+1} - \binom{d-1}{n+1},$$

then the bound

$$(0.3)\qquad p_g(\mathbf{M}) \geq \binom{d}{n+1} - \binom{d-1}{n+1}$$

is sharp, and this sharp bound can be achieved only by a hyperplane section for the case where the inequality holds in (0.2).

Remark. The inequality (0.2) is true when $d \geq C(n)$. For example, $C(3) = 14$, $C(4) = 19$.

If $M \subset V$ as in Theorem 2, then it is well known that M is a complete intersection of V with another hypersurface of degree k. Ein (cf. [5], [6]) has proved that

$$p_g(M) \geq \binom{d-2}{n+1} - \binom{d-2-k}{n+1}$$

in this case, and his results have generalized to varieties of higher codimensions. Therefore the improvement we make here is in the case $k = 1$.

When $n = 3$ Theorem 2 implies that $p_g(M) \geq 2$ if $d \geq 6$. In case $d = 5$, there is a very interesting conjecture.

Clemens' Conjecture. On a generic quintic 3-fold in a projective 4-space $\mathbf{P}^4$, there are only finite number of rational curves in each degree.

This assertion has been proved by Katz for degree up to 7 (cf. [7], [13], [15]). Mark Green has asked the following:

Question. Does every surface on a generic quintic 3-fold in $\mathbf{P}^4$ have positive geometric genus?

If V is a generic quintic 3-fold, since any one-parameter family of rational curves on V sweeps out a surface of geometric genus 0, an affirmative answer to Green's question will imply Clemens' conjecture.

This paper is organized as follows. We introduce a certain type of singularity in §1. In §2 we state and prove Theorem 2.1, which will be used in the next section. In §3 we prove Theorem 1. Section 4 is devoted to the proof of Theorem 2. In the last section we outline a proof of Proposition 4 which states that a hyperplane section of a generic hypersurface can only have very mild singularities.

Throughout this paper we work over the complex number field $\mathbb{C}$.

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1. Weak type δ singularities

In this section, we introduce a type of singularity, establish some of its elementary properties,and show its relationship with the canonical divisor.

Let V be an n-dimensional smooth variety, and $M \subset V$ be an irreducible codimension-1 singular subvariety. According to Hironaka [11], there is a desingularization of M: $V_{m+1} \xrightarrow{\pi_{m+1}} V_m \xrightarrow{\pi_m} \cdots \xrightarrow{\pi_2} V_1 \xrightarrow{\pi_1} V_0 = V$, so that the proper transform $\widetilde{M}$ of M in V_{m+1} is smooth. Here $V_j \xrightarrow{\pi_j} V_{j-1}$ is the blow-up of V_{j-1} along a ν_{j-1}-dimensional submanifold X_{j-1} with $E_{j-1} \subset V_j$ the exceptional divisor. If X_{j-1} is a μ_{j-1}-fold singular submanifold of the proper transform of M in V_{j-1}, we say that M has a *type* $\mu = (\mu_j, \mathbf{X}_j, \mathbf{E}_j | \mathbf{j} \in \{0, 1, \cdots, \mathbf{m}\})$ *singularity.*

If $M \subset V$ has a type $\mu = (\mu_j, X_j, E_j | j \in \Gamma)$ singularity, and $\Omega \subset V$ is an open set, then we localize our definition by saying that M has a type $\mu_\Omega = (\mu_j, X_j, E_j | j \in \Gamma_\Omega = \{j | \exists q \in E_j, \ q$ is an infinitely near point of some $p \in \Omega\})$ singularity on Ω.

Given any resolution of the singularity of $M \subset V$ as above, if $Z \subset V$ is a codimension-1 subvariety, such that

$$\pi_j^*(\cdots(\pi_2^*(\pi_1^*(Z) - \delta_0 E_0) - \delta_1 E_1) - \cdots) - \delta_{j-1} E_{j-1}$$

is an effective divisor for $j = 1, 2, \cdots, m+1$, then we say that Z has a *weak type* $\delta = (\delta_j, \mathbf{X}_j, \mathbf{E}_j | \mathbf{j} \in \{0, 1, \cdots, m\})$ *singularity.* It is easy to see that a type μ singularity implies a weak type μ singularity.

In terms of local coordinates, we assume that M has a type $\mu_\Omega = (\mu_j, X_j, E_j | j \in \Gamma_\Omega = \{0, 1, \cdots, m\})$ singularity on Ω, and $\{z_1, \cdots, z_n\}$ are coordinates on Ω with X_0 defined by $z_{s+1} = \cdots = z_n = 0$. Let

$$z_1' = z_1, \cdots, z_s' = z_s, \quad z_{s+1}' = \frac{z_{s+1}}{z_n}, \cdots, z_{n-1}' = \frac{z_{n-1}}{z_n}, \quad z_n' = z_n$$

be coordinates on the blow-up of Ω along X_0, and $h(z_1, \cdots, z_n)$ be a holomorphic function defined on Ω. Setting

$$\begin{aligned} h(z_1, \cdots, z_n) &= h(z_1', \cdots, z_s', z_{s+1}' z_n', \cdots, z_{n-1}' z_n', z_n') \\ &= (z_n')^\rho h^\sharp(z_1', \cdots, z_n'), \end{aligned}$$

then we say that the variety $\{h(z_1, \cdots, z_n) = 0\}$ on Ω has a weak type $\delta_\Omega = (\delta_j, X_j, E_j | j \in \Gamma_\Omega = \{0, 1, \cdots, m\})$ singularity, if $\rho \geq \delta_0$, $h^\sharp$ is holomorphic, and $\{(z_n')^{\rho - \delta_0} h^\sharp(z_1', \cdots, z_n') = 0\}$ has a weak type $(\delta_j, X_j, E_j | j \in \{1, \cdots, m\})$ singularity on the blow-up of Ω along X_0.

The property of having a weak type δ singularity is additive in the following sense: if two varieties $\{h_1(z_1, \cdots, z_n) = 0\}$ and $\{h_2(z_1, \cdots, z_2) = 0\}$ have weak type $\delta_\Omega = (\delta_j, X_j, E_j | j \in \Gamma_\Omega)$ singularities on Ω, then so does the variety $\{h_1 + h_2 = 0\}$. This holds because

$$\begin{aligned} h_1(z_1, \cdots, z_n) &= (z_n')^{l_1} h_1^\sharp(z_1', \cdots, z_n'), \\ h_2(z_1, \cdots, z_n) &= (z_n')^{l_2} h_2^\sharp(z_1', \cdots, z_n') \end{aligned}$$

with $l_1, l_2 \geq \delta_0$, so $\min(l_1, l_2) \geq \delta_0$, and

$$\begin{aligned} &(h_1 + h_2)(z_1, \cdots, z_n) \\ &= (z_n')^{\min(l_1, l_2)}((z_n')^{l_1 - \min(l_1, l_2)} h_1^\sharp(z_1', \cdots z_n') \\ &\quad + (z_n')^{l_2 - \min(l_1, l_2)} h_2^\sharp(z_1', \cdots, z_n')) \\ &= (z_n')^{\delta_0}((z_n')^{l_1 - \delta_0} h_1^\sharp(z_1', \cdots, z_n') + (z_n')^{l_2 - \delta_0} h_2^\sharp(z_1', \cdots, z_n')). \end{aligned}$$

Since both $\{(z_n')^{l_1-\delta_0}h_1^\sharp(z_1', \cdots, z_n') = 0\}$ and $\{(z_n')^{l_2-\delta_0}h_2^\sharp(z_1', \cdots, z_n') = 0\}$ have weak type $(\delta_j, X_j, E_j | j \in \{1, \cdots, m\})$ singularities on the blow-up of Ω along X_0, by induction

$$\{(z_n')^{l_1-\delta_0}h_1^\sharp(z_1', \cdots, z_n') + (z_n')^{l_2-\delta_0}h_2^\sharp(z_1', \cdots, z_n') = 0\}$$

also has a weak type $(\delta_j, X_j, E_j | j \in \{1, \cdots, m\})$ singularity. Then $\{h_1(z_1, \cdots, z_n) + h_2(z_1, \cdots, z_n) = 0\}$ has a weak type $\delta_\Omega = (\delta_j, X_j, E_j) | j \in \Gamma_\Omega = \{0, 1, \cdots, m\})$ singularity on Ω.

If $M \subset V$ has a type $\mu = (\mu_j, X_j, E_j | j \in \{0, 1, \cdots, m\})$ singularity, and $\widetilde{M}_j$ is the proper transform of M in V_j, then by the adjunction formula,

$$\begin{aligned} K_{\widetilde{m}} &= K_{\widetilde{M}_{m+1}} \\ &= K_{V_{m+1}} + \widetilde{M}_{m+1} \\ &= \pi_{m+1}^*(K_{V_m}) + (n - \nu_m - 1)E_m + \pi_{m+1}^*(\widetilde{M}_m) - \mu_m E_m \\ &= \pi_{m+1}^*(K_{V_m} + \widetilde{M}_m) - (\mu_m - (n - \nu_m - 1))E_m \qquad (1.1) \\ &= \cdots \\ &= \pi_{m+1}^*(\cdots(\pi_2^*(\pi_1^*(K_V + M) - (\mu_0 - (n - \nu_0 - 1))E_0) \\ &\qquad\qquad - (\mu_1 - (n - \nu_1 - 1))E_1 \cdots) \\ &\quad - (\mu_m - (n - \nu_m - 1))E_m. \end{aligned}$$

Since $n - \nu_j - 1 \geq 1$, we get

Proposition 1.1. *A section of $K_V \otimes M$ with a weak type $\mu - 1 = (\mu_j - 1, X_j, E_j | j \in \{0, 1, \cdots, m\})$ singularity induces a section of $K_{\widetilde{M}}$.*

Definition. Let $T \subset \mathbb{C}^N$ be an open neighborhood of the origin $0 \in T$. Assuming that $\sigma: M \to T$ is a family of reduced equidimensional algebraic varieties, $M_t = \sigma^{-1}(t)$, then we say that the family M_t is μ-equisingular at $t = 0$ in the sense that we can resolve the singularity of M_t simultaneously, that is, there is a proper morphism $\pi: \widetilde{M} \to M$, so that $\sigma \circ \pi: \widetilde{M} \to T$ is a flat map and $\sigma \circ \pi: \widetilde{M}_t = (\sigma \circ \pi)^{-1}(t) \to M_t$ is a resolution of the singularities of M_t. Moreover, if M_t has a type $\mu(t) = (\mu_j(t), X_j(t), E_j(t) | j \in \Gamma(t))$ singularity with the above resolution, then $\mu_j(t) = \mu_j$ and $\Gamma(t) = \Gamma$ are independent of t, and the exceptional divisors and the singular loci of the desingularization $\widetilde{M}_t \to M_t$ have the same configuration for all t (cf. [16], [17], [18]).

2. Curves on generic surfaces in $\mathbf{P}^3$

Our starting point is the following (cf. [2], [8], [9]).

Noether-Lefschetz Theorem. *Every curve on a generic surface of degree $d \geq 4$ in $\mathbf{P}^3$ is a complete intersection.*

Let C be an irreducible curve on a generic surface $S = \{F = 0\}$ of degree $d \geq 5$ in $\mathbf{P}^3$. Then C is a complete intersection of S with another surface $S_1 = \{G = 0\}$ of degree k, i.e., C is a type (d, k) curve on S. Here we always assume that the generic surface S is smooth, and both $\{F = 0\}$ and $\{F = 0\} \cap \{G = 0\}$ are reduced. First of all, we have the following lower bound estimate on the geometric genus $g(C)$ of C.

Theorem 2.1. *If C is a curve on a generic surface S of degree $d \geq 5$ in $\mathbf{P}^3$, and C is a complete intersection of S with another surface of degree k, then $g(C) \geq \frac{1}{2}dk(d-5)+2$.*

Before we go into the proof of Theorem 2.1, let us first set down our notation.

For P a singular point of $C \subset S$, we use $\mathbf{e}(\mathbf{P}, \mathbf{C})$ to denote the multiplicity of C at P (cf. [12, Chap. 9]), that is, if $\pi: W \to S$ is the blow-up of S at P, and E is the exceptional divisor, then $\pi^* C = C^* + e(P, C)E$. Here C^* is the proper transform of C by π. If $\{q_1, \cdots, q_s\} = C^* \cap E$, then the points q_i are said to be the *infinitely near points of* $\mathbf{P}$ on $\mathbf{C}$ *of the first order.* Inductively, infinitely near points of q_i $(i = 1, 2, \cdots, s)$ on C^* of the jth order are said to be the *infinitely near points of* $\mathbf{P}$ *on* $\mathbf{C}$ *of the* $(j+1)$*th order.* We define $e(q_i, C) = e(q_i, C^*)$, and so on.

If P_{0j} $(j = 0, 1, \cdots, n_0)$ are all the singular points on C, P_{ij} $(j = 0, 1, \cdots, n_i)$ are all the infinitely near points on C of the ith order $\mu_{ij} = e(P_{ij}, C)$, and E_{ij} is the exceptional divisor resulting from the blowing up at P_{ij}, then C has a type $\mu = (\mu_{ij}, P_{ij}, E_{ij} | (i, j) \in \Gamma)$ singularity with $\Gamma = \{(i, j) | \mu_{ij} > 1\}$, and

$$g(C) = \pi(C) - \sum_{i,j} \frac{1}{2}\mu_{ij}(\mu_{ij} - 1)$$

$$\frac{1}{2}dk(d+k-4) + 1 - \sum_{i,j} \frac{1}{2}\mu_{ij}(\mu_{ij} - 1).$$

Therefore the key to the proof of Theorem 2.1 is to see how bad the singularities of C may be.

Lemma 2.2. *If $F(z_1, z_2)$ is an analytic function on an open set $\Omega \subset \mathbb{C}^2$ defining a curve C, $P_{00} \in \Omega$ is the only singular point of C, and C has a type $\mu_\Omega = (\mu_{ij}, P_{ij}, E_{ij} | (i, j) \in \Gamma_\Omega)$ singularity at P_{00}, then the curves*

$\{\partial F/\partial z_1 = 0\}$ and $\{\partial F/\partial z_2 = 0\}$ in Ω have weak type $\mu_\Omega - 1 = (\mu_{ij} - 1, P_{ij}, E_{ij} | (i, j) \in \Gamma_\Omega)$ singularities at P_{00}.

Proof. First of all, we note that the conclusion of Lemma 2.2 is independent of the choice of the local coordinates on Ω. Without loss of generality, we may assume $P_{00} = (0, 0) \in \Omega$, and

$$\xi = z_1, \qquad \eta = z_2/z_1$$

are the new coordinates after blowing up at P_{00}; therefore

$$F(z_1, z_2) = z_1^{\mu_{00}} F^*(\xi, \eta).$$

Here $F^* = 0$ is the equation of the proper transform of the curve $\{F = 0\}$ after blowing up at P_{00}. Now

$$\frac{\partial F}{\partial z_1} = z_1^{\mu_{00}-1}\left(\mu_{00} F^* + \xi \frac{\partial F^*}{\partial \xi} - \eta \frac{\partial F^*}{\partial \eta}\right).$$

Since $\{F^* = 0\}$ has a singularity with fewer steps to resolve at P_{1j}, then by induction, both $\{\partial F^*/\partial \xi = 0\}$ and $\{\partial F^*/\partial \eta = 0\}$ have weak type $(\mu_{ij} - 1, P_{ij}, E_{ij} | (i, j) \in \Gamma_\Omega - (0, 0))$ singularities. Therefore by additivity $\{\partial F/\partial z_1 = 0\}$ has a weak type $\mu_\Omega - 1 = (\mu_{ij} - 1, P_{ij}, E_{ij} | (i, j) \in \Gamma_\Omega)$ singularity at P_{00}. On the other hand,

$$\frac{\partial F}{\partial z_2} = z_1^{\mu_{00}-1} \frac{\partial F^*}{\partial \eta}.$$

Again we see that $\{\partial F/\partial z_2 = 0\}$ has a weak type $\mu_\Omega - 1 = \mu_{ij} - 1, P_{ij}, E_{ij} | (i, j) \in \Gamma_\Omega)$ singularity at P_{00}. q.e.d.

Lemma 2.2 is a special case of the following.

Lemma 2.3. *If $C_t = \{F_t(z_1, z_2) = 0\}$ is an analytic μ-equisingular family of curves in an open set $\Omega \subset \mathbb{C}^2$, C_t has only one singular point $P_{00}(t)$ in Ω, and C_t has a type $\mu(t)_\Omega = (\mu_{ij}, P_{ij}(t), E_{ij}(t) | (i, j) \in \Gamma_\Omega)$ singularity, then the curve $\{dF_t/dt|_{t=0} = 0\}$ in Ω has a weak type $\mu_\Omega - 1 = (\mu_{ij}(0) - 1, P_{ij}(0), E_{ij}(0) | (i, j) \in \Gamma_\Omega)$ singularity at $P_{00}(0)$.*

Proof. Let $P(t) = (c_1(t), c_2(t))$, and

$$F_t(z_1, z_2) = \sum_{i+j \ge \mu_{00}} a_{ij}(t)(z_1 - c_1(t))^i (z_2 - c_2(t))^j.$$

Then

$$\left.\frac{dF_t}{dt}\right|_{t=0} = -\left.\left\{\frac{dc_1(t)}{dt}\frac{\partial F_0}{\partial z_1} + \frac{dc_2(t)}{dt}\frac{\partial F_0}{\partial z_2}\right\}\right|_{t=0} + \left.\frac{d}{dt}\left\{\sum_{i+j \ge \mu_{00}} a_{ij}(t)(z_1 - c_1(0))^i (z_2 - c_2(0))^j\right\}\right|_{t=0}.$$

By Lemma 2.2, both $\{\partial F_0/\partial z_1 = 0\}$ and $\{\partial F_0/\partial z_2 = 0\}$ have weak type $\mu_\Omega - 1$ singularities at $P_{00}(0)$.

If we move the singular point $P_{00}(t)$ of $F_t = 0$ to $P_{00}(0)$, we get

$$F_t^* = \sum_{i+j\geq\mu_{00}} a_{ij}(t)(z_1 - c_1(0))^i(z_2 - c_2(0))^j.$$

Now we can blow up simultaneously at $P_{00}(0)$. If we let

$$\xi = z_1 - c_1(0), \qquad \eta = (z_2 - c_2(0))/(z_1 - c_1(0))$$

be the new local coordinates after blowing up, then

$$F_t^* = (z_1 - c_1(0))^{\mu_{00}} F_t^\sharp(\xi, \eta),$$

$$\left.\frac{dF_t^*}{dt}\right|_{t=0} = (z_1 - c_1(0))^{\mu_{00}} \left.\frac{dF_t^\sharp(\xi, \eta)}{dt}\right|_{t=0}.$$

Here $F_t^\sharp$ is still a μ-equisingular family, but has improved singularities. By induction, $\{dF_t^\sharp(\xi, \eta)/dt|_{t=0} = 0\}$ has a weak type $(\mu_{ij}(0) - 1, P_{ij}(0), E_{ij}(0)|(i, j) \in \Gamma_\Omega - (0, 0))$ singularity. By additivity we conclude that $\{dF_t/dt|_{t=0} = 0\}$ has a weak type $\mu_\Omega - 1$ singularity at $P_{00}(0)$.

Lemma 2.4. *Let $F_t \in H^0(\mathbf{P}^3, \mathscr{O}(d))$, $G_t \in H^0(\mathbf{P}^3, \mathscr{O}(k))$, and $C_t = \{F_t = 0\} \cap \{G_t = 0\}$ be a μ-equisingular family of curves with a type $\mu(t) = (\mu_{ij}, P_{ij}(t), E_{ij}(t)|(i, j) \in \Gamma)$ singularity. Set $dF_t/dt|_{t=0} = F'$, and $dG_t/dt|_{t=0} = G'$. If all the surfaces $F_t = 0$ are smooth, and $\partial F_0(P)/\partial Z_i \neq 0$, $Z_i(P) \neq 0$ $(i = 0, 1, 2, 3)$ at every singular point P of $C = \{F_0 = 0\} \cap \{G_0 = 0\} = \{F = 0\} \cap \{G = 0\}$, where $\{Z_0, Z_1, Z_2, Z_3\}$ are homogeneous coordinates, then the curve $\{(\partial F/\partial Z_i)G' - (\partial G/\partial Z_i)F' = 0\}$ on $S = \{F = 0\}$ has a weak type $\mu - 1 = (\mu_{ij} - 1, P_{ij}(0), E_{ij}(0)|(i, j) \in \Gamma)$ singularity.*

Proof. We fix $P = P_{0s}(0)$ for some s, and assume that C_t has a type $\mu_s(t) = (\mu_{ij}, P_{ij}(t), E_{ij}(t)|(i, j) \in \Gamma_s)$ singularity at $P(t) = P_{0s}(t)$. Denoting $\{z_1, z_2, z_3\} = \{Z_1/Z_0, Z_2/Z_0, Z_3/Z_0\}$, if we solve the equation $F_t(1, z_1, z_2, z_3) = 0$ near the point $P(t)$, and get $z_3 = \varphi_t(z_1, z_2)$, then we can view C_t as a μ-equisingular family of curves locally defined by the equation $G_t(1, z_1, z_2, \varphi_t(z_1, z_2)) = 0$ in an open set $\Omega \subset \mathbb{C}^2$. By Lemma 2.3, the curve locally defined by the equation

$$\frac{dG_t}{dt}(1, z_1, z_2, \varphi_t(z_1, z_2))|_{t=0} = 0$$

on the surface $S = \{F = 0\}$ has a weak type $\mu_s(0) - 1 = (\mu_{ij} - 1, P_{ij}(0), E_{ij}(0)|(i, j) \in \Gamma_s)$ singularity at $P(0) = P_{0s}(0)$.

From the equation $F_t(1, z_1, z_2, \varphi_t(z_1, z_2)) = 0$, we get

$$F'(1, z_1, z_2, \varphi_0(z_1, z_2)) + \frac{\partial F}{\partial Z_3}(1, z_1, z_2, \varphi_0(z_1, z_2))\frac{d\varphi_t}{dt}(z_1, z_2)|_{t=0} = 0,$$

and thus

$$\frac{d\varphi_t}{dt}|_{t=0} = -\left(\frac{\partial F}{\partial Z_3}\right)^{-1} F'.$$

We also have

$$\begin{aligned}\frac{dG_t}{dt}(1, z_1, z_2, \varphi_t(z_1, z_2))|_{t=0} &= G' + \frac{\partial G}{\partial Z_3}\frac{d\varphi_t}{dt}|_{t=0} \\ &= G' - \left(\frac{\partial F}{\partial Z_3}\right)^{-1}\left(\frac{\partial G}{\partial Z_3}\right)F'.\end{aligned}$$

Thus the curve $\{(\partial F/\partial Z_3)G' - (\partial G/\partial Z_3)F' = 0\}$ on the surface S has a weak type $\mu_s(0) - 1 = (\mu_{ij} - 1, P_{ij}(0), E_{ij}(0)|(i, j) \in \Gamma_s)$ singularity at $P(0) = P_{0s}(0)$. Since s is arbitrary, we conclude that the curve $\{(\partial F/\partial Z_3)G' - (\partial G/\partial Z_3)F' = 0\}$ on surface $S = \{F = 0\}$ has a weak type $\mu - 1 = (\mu_{ij} - 1, P_{ij}(0), E_{ij}(0)|(i, j) \in \Gamma)$ singularity.

Lemma 2.5. *Assume $C = \{F = 0\} \cap \{G = 0\}$ is a curve on a smooth surface $S = \{F = 0\}$ in $\mathbf{P}^3$, $\deg F = d$, $\deg G = k$, and C has a type $\mu = (\mu_{ij}, P_{ij}, E_{ij}|(i, j) \in \Gamma)$ singularity. If $Q \in H^0(\mathbf{P}^3, \mathscr{O}(m))$ is not in the homogeneous polynomial ideal (F, G) generated by F and G, and the curve $\{Q = 0\}$ on S has a weak type $\mu - 1 = (\mu_{ij} - 1, P_{ij}, E_{ij}|(i, j) \in \Gamma)$ singularity, then*

$$\sum_{(i,j)\in\Gamma} \mu_{ij}(\mu_{ij} - 1) \le dkm.$$

Proof. By Bezout's Theorem, the intersection number $I(Q, G)_F$ of the divisors $\{Q = 0\}$ and $\{G = 0\}$ on $S = \{F = 0\}$ is equal to dkm. Let $P_{0s} = P_{os}(0)$ $(s = 0, 1, \cdots, n_0)$ be all the singular points of C on S, $S_{0,1} \xrightarrow{\pi_{0,1}} S_{0,0} = S$ be the blow-up of S at $P_{0,0}$ with $\widetilde{C}_{0,1}$ the proper transform of $C = \{G = 0\} \cap S$ in $S_{0,1}$ and inductively $S_{0,s+1} \xrightarrow{\pi_{0,s+1}} S_{0,s}$ be the blow-up of $S_{0,s}$ at $P_{0,s}$ with $\widetilde{C}_{0,s+1}$ the proper transform of $\widetilde{C}_{0,s}$ in $S_{0,s+1}$. Then $\pi^*_{0,1}C = \mu_{00}E_{00} + \widetilde{C}_{0,1}$. Since $Q = \{Q = 0\}$ has a weak type $\mu - 1$ singularity, $\pi^*_{0,1}Q - (\mu_{00} - 1)E_{00}$ is an effective divisor in $S_{0,1}$, so

$$\begin{aligned}\widetilde{C}_{0,1}(\pi_{0,1}^{*}Q-(\mu_{00}-1)E_{00})\\ &=(\pi_{0,1}^{*}C-\mu_{00}E_{00})(\pi_{0,1}^{*}Q-(\mu_{j00}-1)E_{00})\\ &=C\cdot Q-\mu_{00}(\mu_{00}-1).\end{aligned}$$

Therefore

$$\begin{aligned}I(Q,G)_F&=C\cdot Q\\ &=\widetilde{C}_{0,1}\cdot(\pi_{0,1}^{*}Q-(\mu_{00}-1)E_{00})+\mu_{00}(\mu_{00}-1)\\ &=\cdots\\ &=\widetilde{C}_{0,n_0+1}\cdot(\pi_{0,n_0+1}^{*}(\cdots\pi_{0,2}^{*}(\pi_{0,1}^{*}Q-(\mu_{00}-1)E_{00})\\ &\qquad -(\mu_{01}-1)E_{01})-\cdots-(\mu_{0n_0}-1)E_{0n_0})\\ &\quad+\sum_{s=0}^{n_0}\mu_{0s}(\mu_{0s}-1).\end{aligned}$$

If we continue the above process on all the infinitely near points on C of the first order, and so on, finally we will get

$$I(Q,G)_F\geq\sum_{(i,j)\in\Gamma}\mu_{ij}(\mu_{ij}-1).\quad \text{q.e.d.}$$

After these four lemmas, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. We first fix an integer $d\geq 5$. Let g be the minimum integer so that on a generic surface of degree d in $\mathbf{P}^3$ there is a curve C with geometric genus $g(C)\leq g$. Setting

$$\begin{aligned}H_{m,g}=\{F\in\mathbf{P}H^0(\mathbf{P}^3,\mathcal{O}(d))|\ &\text{there is a degree } m \text{ curve}\\ &C\subset\{F=0\}\text{ with } g(C)\leq g\},\end{aligned}$$

it is well known that $H_{m,g}\subset\mathbf{P}H^0(\mathbf{P}^3,\mathcal{O}(d))$ is an algebraic subvariety. By our assumption on g and the Noether-Lefschetz Theorem, the natural map

$$\bigcup_{k=1}^{\infty}H_{dk,g}\to\mathbf{P}H^0(\mathbf{P}^3,\mathcal{O}(d))$$

is surjective, so $H_{dk,g}\to\mathbf{P}H^0(\mathbf{P}^3,\mathcal{O}(d))$ is surjective for some positive integer k, and the image of $H_{dk,g-1}\to\mathbf{P}H^0(\mathbf{P}^3,\mathcal{O}(d))$ is a proper algebraic subvariety. Let

$$W_{d,k,g} = \{F \in \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d)) | \exists G \in \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(k)) \text{ such that the curve } C=\{F=0\}\cap\{G=0\} \text{ is reduced, irreducible and } g(C)\le g\},$$
$$\widetilde{W}_{d,k,g} = \{\{F, G\} \in \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d)) \times \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(k)) | \text{ the curve } C=\{F=0\}\cap\{G=0\} \text{ is reduced, irreducible and } g(C)\le g\}.$$

Since the natural map $H_{dk,g} - W_{d,k,g} \to \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d))$ is not dominant by Noether-Lefschetz Theorem, the image of the map $\sigma_2\colon W_{d,k,g} \to \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d))$ contains a Zariski open set. By our assumption, $\sigma_2\colon W_{d,k,g-1} \to \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d))$ is not dominant. Since the two natural maps $\sigma_1\colon \widetilde{W}_{d,k,g} \to W_{d,k,g}$, $\sigma_3\colon \widetilde{W}_{d,k,g} \to \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d))$ satisfy $\sigma_3 = \sigma_2 \circ \sigma_1$, there are two sets $W \subset W_{d,k,g} - W_{d,k,g-1}$ and $\widetilde{W} \subset \widetilde{W}_{d,k,g}$, so that the image of the map $\sigma_2\colon W \to \mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d))$ contains a Zariski open set of $\mathbf{P}H^0(\mathbf{P}^3, \mathscr{O}(d))$, and $\sigma_1\colon \widetilde{W} \to W$ is dominant. Therefore at some regular point of W, we can find a smooth section of $\sigma_1\colon \widetilde{W} \to W$, that is, there is a pair $\{F, G\} \in \widetilde{W}$, such that for any deformation F_t of F with $F = F_0$ in W, there is an unique deformation G_t of G with $G = G_0$ so that $\{F_t, G_t\} \in \widetilde{W}$. Moreover, we can assume the family of curves $C_t = \{F_t = 0\} \cap \{G_t = 0\}$ is μ-equisingular, and C_t has a type $\mu(t) = (\mu_{ij}, P_{ij}(t), E_{ij}(t) | (i, j) \in \Gamma)$ singularity.

Since the surface $S = \{F = 0\}$ is smooth, we may choose homogeneous coordinates $\{Z_0, Z_1, Z_2, Z_3\}$ for $\mathbf{P}^3$, so that

$$\frac{\partial F}{\partial Z_i}(P_{0j}(0)) \neq 0, \quad Z_i(P_{0j}(0)) \neq 0, \qquad \forall i, (0, j) \in \Gamma.$$

By Lemma 2.4, for any $F' \in H^0(\mathbf{P}^3, \mathscr{O}(d))$, there is a unique deformation $G' \in H^0(\mathbf{P}^3, \mathscr{O}(k))$ of G constructed above, such that the curve $\{(\partial F/\partial Z_3)G' - (\partial G/\partial Z_3)F' = 0\}$ on S has a weak type $\mu - 1 = (\mu_{ij} - 1, P_{ij}(0), E_{ij}(0) | (i, j) \in \Gamma)$ singularity.

Consider the case $F' = Z_i U$ with $U \in H^0(\mathbf{P}^3, \mathscr{O}(d-1))$, and let $G' = G'(Z_i U) \in H^0(\mathbf{P}^3, \mathscr{O}(k))$ be the corresponding deformation of G. Since

(2.1)
$$\begin{aligned}\frac{\partial F}{\partial Z_3}&(Z_i G'(Z_j U) - Z_j G'(Z_i U)) \\ &= Z_i\left(\frac{\partial F}{\partial Z_3} G'(Z_j U) - \frac{\partial G}{\partial Z_3} Z_j U\right) - Z_j\left(\frac{\partial F}{\partial Z_3} G'(Z_i U) - \frac{\partial G}{\partial Z_3} Z_i U\right),\end{aligned}$$

we find that the curve $\{\partial F/\partial Z_3(Z_iG'(Z_jU)-Z_jG'(Z_iU))=0\}$ on S has a weak type μ-1 singularity. But $(\partial F/\partial Z_3)(P_{0s}(0))\neq 0$ for all s by our assumption, so the curve $\{K_{ij}(U)=0\}=\{Z_iG'(Z_jU)-Z_jG'(Z_iU)=0\}$ on S has a weak type $\mu-1$ singularity.

Since $\{F=0\}\cap\{G=0\}$ is reduced and irreducible, it is well known that the polynomial ideal (F, G) generated by F and G satisfies $(F, G)=\sqrt{(F, G)}$. Let K_{k+1} be the space of homogeneous polynomials of degree $k+1$ generated by $K_{ij}(U)$ with $i, j=0, 1, 2, 3$ and

$$U\in H^0(\mathbf{P}^3, \mathscr{O}(d-1)).$$

Case 1. If $\dim(K_{k+1}/(F, G))\geq 2$, we can choose $0\neq Q\in K_{k+1}/(F, G)$ so that the curve $\{Q=0\}$ on S passes through an extra smooth point of $C=\{F=0\}\cap\{G=0\}$. Lemma 2.5 gives

$$dk(k+1)=I(Q, G)_F\geq\sum_{(i,j)\in\Gamma}\mu_{ij}(\mu_{ij}-1)+1,$$

$$\begin{aligned} g(C)&=\frac{1}{2}dk(d+k-4)+1-\sum_{(i,j)\in\Gamma}\frac{1}{2}\mu_{ij}(\mu_{ij}-1)\\ &\geq\frac{1}{2}dk(d+k-4)+1-\frac{1}{2}dk(k+1)+\frac{1}{2}, \end{aligned}$$

that is, $g(C)\geq\frac{1}{2}dk(d-5)+2$.

Case 2. If $\dim(K_{k+1}/(F, G))=1$, let Q be a generator of $K_{k+1}/(F, G)$. Then $K_{ij}(U)\equiv A_{ij}(U)Q \bmod (F, G)$, where $A_{ij}(U)$ are complex numbers. We may assume $A_{ij}(U)\neq 0$ for some i, j, U. From the construction of $K_{ij}(U)$, we get

$$\begin{gathered} Z_hK_{ij}(U)+Z_iK_{jh}(u)+Z_jK_{hi}(U)=0,\\ (Z_hA_{ij}(U)+Z_iA_{jh}(U)+Z_jA_{hi}(U))Q\equiv 0 \mod (F, G). \end{gathered}$$

Since $\{F=0\}\cap\{G=0\}$ is reduced and irreducible, and Q is nontrivial, we must have

$$Z_hA_{ij}(U)+Z_iA_{jh}(U)+Z_jA_{hi}(U)\equiv 0 \mod (F, G).$$

But $\deg F=d\geq 5$, so $\deg G=k=1$. We may assume that $(i, j)=(0, 1)$, i.e., $A_{01}(U)\neq 0$. Then

$$\begin{gathered} G|A_{01}(U)Z_2+A_{12}(U)Z_0+A_{20}(U)Z_1,\\ G|A_{01}(U)Z_3+A_{13}(U)Z_0+A_{30}(U)Z_1, \end{gathered}$$

and this is impossible.

Case 3. If $\dim(K_{k+1}/(F, G)) = 0$, then

$$K_{ij}(U) = B_{ij}(U)F + C_{ij}(U)G.$$

Here $B_{ij}(U)$ and $C_{ij}(U)$ are homogeneous polynomials. From the equation

$$Z_hK_{ij}(U) + Z_iK_{jh}(U) + Z_jK_{hi}(U) = 0,$$

it follows that

$$\begin{aligned}&(Z_hB_{ij}(U) + Z_iB_{jh}(U) + Z_jB_{hi}(U))F\\ &\quad + (Z_hC_{ij}(U) + Z_iC_{jh}(U) + Z_jC_{hi}(U))G = 0.\end{aligned}$$

Since F and G are relative prime, $\deg C_{ij}(U) = 1$, and $\deg F = d \geq 5$, it is easy to see that

$$\begin{aligned}Z_hC_{ij}(U) + Z_iC_{jh}(U) + Z_jC_{hi}(U) = 0,\\ Z_hB_{ij}(U) + Z_iB_{jh}(U) + Z_jB_{hi}(U) = 0,\end{aligned}$$

so that

$$\begin{aligned}C_{ij}(U) = Z_iC_j(U) - Z_jC_i(U),\\ B_{ij}(U) = Z_iB_j(U) - Z_jB_i(U)\end{aligned}$$

for some homogeneous polynomials $B_i(U)$, $C_i(U)$. Therefore

$$\begin{aligned}Z_iG'(Z_jU) - Z_jG'(Z_iU) &= K_{ij}(U)\\ &= (Z_iB_j(U) - Z_jB_i(U))F\\ &\quad + (Z_iC_j(U) - Z_jC_i(U))G,\\ Z_i(G'(Z_jU) - B_j(U)F - C_j(U)G)&\\ - Z_j(G'(Z_iU) - B_i(U)F - C_i(U)G) &= 0,\\ G'(Z_jU) - B_j(U)F - C_j(U)G &= Z_jV\end{aligned}$$

for some $V \in H^0(\mathbf{P}^3, \mathscr{O}(k-1))$. The curve $\{(\partial F/\partial Z_3)G'(Z_jU) - (\partial G/\partial Z_3)Z_jU = 0\}$ on S has a weak type $\mu - 1$ singularity, $Z_j(P_{0s}(0)) \neq 0$, so we conclude that for any $U \in H^0(\mathbf{P}^3, \mathscr{O}(d-1))$, there is a corresponding $V \in H^0(\mathbf{P}^3, \mathscr{O}(k-1))$, so that the curve $\{(\partial F/\partial Z_3)V - (\partial G/\partial Z_3)U = 0\}$ on S has a weak type $\mu - 1$ singularity. Note that $V = V(U)$ is unique $\bmod (F, G)$.

Now the above argument can be repeated again. We construct the space K_k. If $\dim(K_k/(F, G)) \geq 2$, then as before we get the estimate $g(C) \geq \frac{1}{2}dk(d-4) + 2 \geq \frac{1}{2}dk(d-5) + 2$, while otherwise we may continue on.

If $k \geq d$ and $\dim(K_j/(F, G)) = 0$ for $j = k+1, k, \cdots, k-d+2$, then the above argument will end with a homogeneous polynomial R_3 of degree $k-d$, such that the curve $\{(\partial F/\partial Z_3)R_3 - \partial G/\partial Z_3 \cdot 1 = 0\}$ on S has a weak type $\mu - 1$ singularity. If we replace Z_3 by Z_i $(i = 0, 1, 2)$ and repeat the same argument, then either we get the right estimate for $g(C)$, or we have homogeneous polynomials R_0, R_1, R_2 of degree $k - d$, such that the curve $\{(\partial F/\partial Z_i)R_i - \partial G/\partial Z_i \cdot 1 = 0\}$ $(i = 0, 1, 2)$ on S has a weak type $\mu - 1$ singularity. By our construction $R_0 \equiv R_1 \equiv R_2 \equiv R_3 \bmod (F, G)$ and $\deg R_i = k - d < k$, so $R_0 \equiv R_1 \equiv R_2 \equiv R_3 \bmod (F)$. If $(\partial F/\partial Z_i)R_i - \partial G/\partial Z_i \equiv 0 \bmod (F, G)$ for all i, then $\deg \partial G/\partial Z_i = k - 1 < k$ implies that $(\partial F/\partial Z_i)R_i - \partial G/\partial Z_i \equiv 0$ $(\bmod F)$, so that the Euler relation will give us $G \equiv 0 \bmod (F)$. Therefore one of $(\partial F/\partial Z_i)R_i - \partial G/\partial Z_i \not\equiv 0 \bmod (F, G)$, hence $\sum \mu_{ij}(\mu_{ij} - 1) \leq dk(k-1)$ as before, i.e.,

$$g(C) \geq \frac{dk(d-3)}{2} + 1 \geq \frac{dk(d-5)}{2} + 2.$$

If $k < d$ and $\dim(K_j/(F, G)) = 0$ for $j = k+1, k, \cdots, 2$, the above three steps of the argument will end with the following situation: for any $U \in H^0(\mathbf{P}^3, \mathscr{O}(d-k))$, there is a corresponding constant $V = V(U)$, such that the curve $\{(\partial F/\partial Z_3)V - (\partial G/\partial Z_3)U = 0\}$ on S has a weak type $\mu - 1$ singularity. Now we define K_1, and we only need to consider the case $\dim(K_1/(F, G)) = 0$. Take $U = Z_i U'$, and let $V = V(Z_i U')$ be the corresponding constant. Then

$$Z_i V(Z_j U') - Z_j V(Z_i U') = A_{ij}(U')G$$

in K_1, thanks to the fact $\deg F = d \geq 5$. Now

$$(Z_h A_{ij}(U') + Z_i A_{jh}(U') + Z_j A_{hi}(U'))G = 0,$$

and forces $A_{ij}(U') = 0$ for any U', that is $V = V(U') = 0$. Then the curve $\{(\partial G/\partial Z_3)U' = 0\}$ on S has a weak type $\mu - 1$ singularity for any $U' \in H^0(\mathbf{P}^3, \mathscr{O}(d-k-1))$, i.e., the curve $\{\partial G/\partial Z_3 = 0\}$ on S has a weak type $\mu - 1$ singularity. Since $k < d$ and one of the $\partial G/\partial Z_i$ $(i = 0, 1, 2, 3)$ is nontrivial, we get $\sum \mu_{ij}(\mu_{ij} - 1) \leq dk(k-1)$, and

$$g(C) \geq dk(d-5)/2 + 2.$$

This completes the proof of Theorem 2.1.

3. Hyperplane sections of generic surfaces and the proof of Theorem 1

Before we go into the proof of Theorem 1, let us first have a look at the special case $k = 1$. Namely, if C is a hyperplane section of a generic surface in $\mathbf{P}^3$, what kind of singularities can C have?

Proposition 3. *Every hyperplane section of a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$ has at most either* (1) 3 *ordinary double points,* (2) *an ordinary double point and a simple cusp* (*locally defined by $x^2 = y^3$*), *or* (3) *a tacnode* (*locally defined by $x^2 = y^4$*).

Proof. We follow the notations in the proof of Theorem 2.1. Let $\{F, G\} \in \widetilde{W}$, and assume $C = \{F = 0\} \cap \{G = 0\}$ has a type $\mu = (\mu_{ij}, P_{ij}, E_{ij})$ singularity. Since for any deformation $F' \in H^0(\mathbf{P}^3, \mathscr{O}(d))$ of F, there is a deformation $G' \in H^0(\mathbf{P}^3, \mathscr{O}(1))$ of G, such that the curve $\{(\partial F/\partial Z_3)G' - (\partial G/\partial Z_3)F' = 0\}$ on $S = \{F = 0\}$ has a weak type $\mu - 1 = (\mu_{ij} - 1, P_{ij}, E_{ij})$ singularity, we have

$$\left(\frac{\partial G}{\partial Z_3}F' - \frac{\partial F}{\partial Z_3}G'\right)(P_{0s}) = 0 \tag{3.1}$$

on S for all the singular points P_{0s} on C. If C has at least one double point, then there will be a nontrivial condition imposed on G'. Because of the fact $\deg G = 1$, we may choose homogeneous coordinates $\{Z_0, Z_1, Z_2, Z_3\}$ such that $\partial G/\partial Z_i \neq 0$ for $i = 0, 1, 2, 3$. Note that $P_{0s} \in \{G = 0\}$, $h^0(\mathbf{P}^2, \mathscr{O}(1)) = h^0(\{G = 0\}, \mathscr{O}(1)) = 3$, and that it is well known that any four distinct points of $\mathbf{P}^3$ impose independent conditions on homogeneous polynomials of degree ≥ 3. Thus (3.1) implies that C can be singular at most at three different points.

We show next that there is no point $P \in C$ such that its multiplicity $e(P, C) \geq 3$, i.e., $\mu_{0s} \leq 2$ for all s. Assuming there is one, then for any deformation F_t of $F = F_0$, there is a deformation G_t of $G = G_0$, such that the family of curves $C_t = \{F_t = 0\} \cap \{G_t = 0\}$ is μ-equisingular and C_t has a singular point $P(t)$ with multiplicity $e(P(t), C_t) \geq 3$. Because $k = 1$ and the surface $\{G_t = 0\}$ is smooth, solving $G_t(1, z_1, z_2, z_3) = 0$, we get $z_3 = \psi_t(z_1, z_2)$, where ψ_t is linear in z_1, z_2. Let

$$f_t(z_1, z_2) = F_t(1, z_1, z_2, \psi_t(z_1, z_2)),$$
$$P(t) = [1, c_1(t), c_2(t), \psi_t(c_1(t), c_2(t))].$$

Then

$$f_t(z_1, z_2) = \sum_{i+j\geq 3} a_{ij}(t)(z_1 - c_1(t))^i(z_2 - c_2(t))^j,$$

$$\begin{aligned}\left.\frac{df_t}{dt}(z_1, z_2)\right|_{t=0} &= -\left.\frac{\partial f_0}{\partial z_1}(z_1, z_2)\frac{dc_1(t)}{dt}\right|_{t=0} - \left.\frac{\partial f_0}{\partial z_2}(z_1, z_2)\frac{dc_2(t)}{dt}\right|_{t=0} \\ &+ \sum_{i+j\geq 3}\left\{\left.\frac{da_{ij}(t)}{dt}\right|_{t=0}\right\}(z_1 - c_1(0))^i(z_2 - c_2(0))^j.\end{aligned}$$

As in the proof of Lemma 2.4,

$$\left.\frac{df_t}{dt}(z_1, z_2)\right|_{t=0} = F' - \left(\frac{\partial G}{\partial Z_3}\right)^{-1}\frac{\partial F}{\partial Z_3}G'; \tag{3.2}$$

thus

$$\left(F' - \left(\frac{\partial G}{\partial Z_3}\right)^{-1}\frac{\partial F}{\partial Z_3}G'\right)(1, z_1, z_2, \psi_0(z_1, z_2)) + \left.\frac{\partial f_0}{\partial z_1}\frac{dc_1(t)}{dt}\right|_{t=0} + \left.\frac{\partial f_0}{\partial z_2}\frac{dc_2(t)}{dt}\right|_{t=0} = O(3)$$

at $P(0)$ on $\{G = 0\}$. Since $h^0(\mathbf{P}^2, \mathscr{O}(1)) = 3$, $h^0(\mathbf{P}^2, \mathscr{O}(d)) \geq 6$ for $d \geq 5$, and the set

$$\begin{aligned}A_2 = \{1, z_1 - c_1(0), z_2 - c_2(0), (z_1 - c_1(0))^2, \\ (z_1 - c_1(0))(z_2 - c_2(0)), (z_2 - c_2(0))^2\}\end{aligned}$$

has six elements, so we can choose F', such that the above equation is not true for any choices of $G' \in H^0(\{G = 0\}, \mathscr{O}(1))$ and the two numbers $dc_1(t)/dt|_{t=0}$, $dc_2(t)/dt|_{t=0}$. Therefore C has only double points.

Now we look at the case where C has a simple cusp. Let C_t be a μ-equisingular deformation of C, and $P(t)$ be the simple cusp of C_t. Using the notation of the last paragraph, we have

$$\begin{aligned}f_t(z_1, z_2) = (a(t)(z_1 - c_1(t)) + b(t)(z_2 - c_2(t)))^2 \\ + \sum_{i+j\geq 3} a_{ij}(t)(z_1 - c_1(t))^i(z_2 - c_2(t))^j,\end{aligned}$$

$$\begin{aligned}\frac{df_t}{dt}(z_1, z_2)|_{t=0} = &-\frac{\partial f_0}{\partial z_1}\frac{dc_1(t)}{dt}|_{t=0} - \frac{\partial f_0}{\partial z_2}\frac{dc_2(t)}{dt}|_{t=0} \\ &+ \sum_{i+j\geq 3}\left\{\frac{da_{ij}(t)}{dt}|_{t=0}\right\}(z_1 - c_1(0))^i(z_2 - c_2(0))^j \\ &+ 2(a(0)(z_1 - c_1(0)) + b(0)(z_2 - c_2(0))) \\ &\cdot\left(\frac{da(t)}{dt}|_{t=0}(z_1 - c_1(0)) + \frac{db(t)}{dt}|_{t=0}(z_2 - c_2(0))\right),\end{aligned}$$

and also, by (3.2),

$$\begin{aligned}&\left(F' - \left(\frac{\partial G}{\partial Z_3}\right)^{-1}\frac{\partial F}{\partial Z_3}G'\right)(1, z_1, z_2, \psi_0(z_1, z_2)) \\ &\quad + \frac{\partial f_0}{\partial z_1}\frac{dc_1(t)}{dt}\bigg|_{t=0} + \frac{\partial f_0}{\partial z_2}\frac{dc_2(t)}{dt}\bigg|_{t=0} \\ &= 2(a(0)(z_1 - c_1(0)) + b(0)(z_2 - c_2(0))) \\ &\quad \cdot\left(\frac{da(t)}{dt}\bigg|_{t=0}(z_1 - c_1(0)) + \frac{db(t)}{dt}\bigg|_{t=0}(z_2 - c_2(0))\right) + O(3)\end{aligned}$$

at $P = P(0)$ on $\{G = 0\}$. The set A_2 just defined above contains six elements, and we are free to choose $dc_1(t)/dt|_{t=0}$, $dc_2(t)/dt|_{t=0}$, $da(t)/dt|_{t=0}$, and $db(t)/dt|_{t=0}$, so having a simple cusp imposes at least two conditions on G'. Now if D_1 and D_2 are two distinct points of C, one can find hyperplanes H_i $(i = 1, 2)$ so that $H_i = 0$ at D_i and $H_i \neq 0$ at D_j for $j \neq i$. Writing $F' = H_1^3F_1 + H_2^3F_2$, because $F' \in H^0(\mathbf{P}^3, \mathscr{O}(d))$ and $d \geq 5$, we can choose F_1, F_2 so that the Taylor expansion of $F'|_{G=0}$ has prescribed coefficients up to the second order at any two distinct points $D_1, D_2 \in C$ simultaneously. However $G' \in H^0(\{G = 0\}, \mathscr{O}(1)) = H^0(\mathbf{P}^2, \mathscr{O}(1))$, and $h^0(\mathbf{P}^2, \mathscr{O}(1)) = 3$, so C could not afford two simple cusps. Similarly, writing $F' = H_1F_1 + H_2F_2 + H_1H_2F_3$, we can choose F_1, F_2, F_3 such that $F'|_{G=0}$ has prescribed values at D_1, D_2 and simultaneously its Taylor expansion has prescribed coefficients up to the second order at a point $D_3 \in C$. By (3.1) and above, we see that C cannot have two ordinary double points D_1, D_2 and a simple cusp D_3. So we conclude that if C has no infinitely near point P_{1j} of the first order such that $e(P_{ij}, C) = \mu_{1j} > 1$, then C has at most three nodes or a node and a simple cusp.

Finally, we consider the case that the proper transform of C after blowing up at P_{00} is singular at P_{10}. Let $\{z_1, z_2, z_3\} = \{Z_1/Z_0, Z_2/Z_0, Z_3/Z_0\}$ be local coordinates, and $C_t = \{F_t = 0\} \cap \{G_t = 0\}$ be a

μ-equisingular deformation of C. Keeping f_t, g_t, ψ_t as before, and denoting $\xi = z_1 - c_1(0)$, $\eta = z_2 - c_2(0)/z_1 - c_1(0)$, $P_{00}(t) = [1, c_1(t), c_2(t), \psi_t(c_1(t), c_2(t))]$, $P_{10}(t) = (0, c_3(t))$, we then have

$$f_t(z_1, z_2) = \sum_{i+j\geq 2} a_{ij}(t)(z_1 - c_1(t))^i (z_2 - c_2(t))^j,$$

$$\begin{aligned}\sum_{i+j\geq 2} a_{ij}(t)(z_1 - c_1(0))^i (z_2 - c_2(0))^j &= (z_1 - c_1(0))^2 \left(\sum_{i+j\geq 2} b_{ij}(t)\xi^i (\eta - c_3(t))^j \right) \\ &= (z_1 - c_1(0))^2 f_t^\sharp(\xi, \eta),\end{aligned}$$

$$\begin{aligned}\frac{df_t}{dt}(z_1, z_2)\bigg|_{t=0} &= -\frac{\partial f_0}{\partial z_1}(z_1, z_2)\,\frac{dc_1(t)}{dt}\bigg|_{t=0} - \frac{\partial f_0}{\partial z_2}(z_1, z_2)\,\frac{dc_2(t)}{dt}\bigg|_{t=0} \\ &\quad + \frac{d}{dt}\left\{ \sum_{i+j\geq 2} a_{ij}(t)(z_1 - c_1(0))^i (z_2 - c_2(0))^j \right\}\bigg|_{t=0} \\ &= -\frac{\partial f_0}{\partial z_1}\frac{dc_1(t)}{dt}\bigg|_{t=0} - \frac{\partial f_0}{\partial z_2}\frac{dc_2(t)}{dt}\bigg|_{t=0} \\ &\quad + \frac{d}{dt}((z_1 - c_1(0))^2 f_t^\sharp(\xi, \eta))|_{t=0},\end{aligned}$$

$$\frac{d}{dt} f_t^\sharp(\xi, \eta)\bigg|_{t=0} = -\frac{\partial f_0^\sharp}{\partial \eta}\frac{dc_3(t)}{dt}\bigg|_{t=0} + \sum_{i+j\geq 2} \frac{db_{ij}(t)}{dt}\bigg|_{t=0} \xi^i (\eta - c_3(0))^j,$$

and also, by (3.2),

$$\begin{aligned}&\left(F' - \left(\frac{\partial G}{\partial Z_3}\right)^{-1}\left(\frac{\partial F}{\partial Z_3}\right) G'\right)(1, z_1, z_2, \psi_0(z_1, z_2)) \\ &\quad + \frac{\partial f_0}{\partial z_1}\frac{dc_1(t)}{dt}\bigg|_{t=0} + \frac{\partial f_0}{\partial z_2}\frac{dc_2(t)}{dt}\bigg|_{t=0} \\ &= (z_1 - c_1(0))^2 \left(-\frac{\partial f_0^\sharp}{\partial \eta}\frac{dc_3(t)}{dt}\bigg|_{t=0} + O(2) \right).\end{aligned} \tag{3.3}$$

If we take the Taylor expansion of the left side of (3.3) at $z_1 = c_1(0)$, $z_2 = c_2(0)$, then its coefficients of 1, $z_1 - c_1(0)$, $z_2 - c_2(0)$ must be zero.

As we noted early, this imposes at least one condition on G' due to the free choices of $dc_1(t)/dt|_{t=0}$ and $dc_2(t)/dt|_{t=0}$. Since the set $\{1, \xi, \eta - c_3(0)\}$ has three elements, and we are free to choose the number $dc_3(t)/dt|_{t=0}$, if the proper transform of C in the blow-up of S at P_{00} has a double point P_{10}, then at least two more conditions will be imposed on G'. Altogether at least three conditions are imposed on G'. However, $\dim H^0(\{G = 0\}, \mathscr{O}(1)) = 3$, thus it is not hard to see that P_{10} must be an ordinary double point. If P_{10} is a simple cusp, then at least one more condition will be imposed on G' as we have seen in the last paragraph. If we have a worse singularity than a node or a simple cusp at P_{10}, we can go on one more step up as we will do in the proof of Proposition 4 to see that it will impose extra conditions on G'. Therefore P_{00} is a tacnode of C. q.e.d.

Finally we give the

Proof of Theorem 1. Let C be a curve on a generic surface S of degree $d \geq 5$ in $\mathbf{P}^3$. Then C is a complete intersection of S with another surface of degree k. By Theorem 2.1, the geometric genus $g(C) \geq \frac{1}{2}dk(d-5)+2$. For $d \geq 6$, we have

$$g(C) \geq \frac{dk(d-5)}{2} + 2 > \frac{d(d-3)}{2} - 2$$

when $k \geq 2$. We conclude that the sharp lower bound of $g(C)$ can be achieved only by a hyperplane section. When $k = 1$,

$$\begin{aligned} g(C) &= \pi(C) - \sum \frac{\mu_{ij}(\mu_{ij} - 1)}{2} \\ &= \frac{d(d-3)}{2} + 1 - \sum \frac{\mu_{ij}(\mu_{ij} - 1)}{2} \\ &\geq \frac{d(d-3)}{2} - 2 \end{aligned}$$

by Proposition 3.

It only remains to consider the case $d = 5$. By Theorem 2.1, $g(C) \geq 2$. Our goal is to show that actually we have $g(C) \geq 3$.

Now we assume there is a type $(5, k)$ curve of geometric genus $g(C) = 2$ on a generic quintic surface S. By Proposition 3, we must have $k > 1$. Again we follow the notation in the proof of Theorem 2.1. Let $\{F, G\} \in \widetilde{W}$, and let $C = \{F = 0\} \cap \{G = 0\}$ have a type $\mu = (\mu_{ij}, P_{ij}, E_{ij})$ singularity, such that for any $F' \in H^0(\mathbf{P}^3, \mathscr{O}(5))$, there is a unique $G' = G'(F') \in H^0(\mathbf{P}^3, \mathscr{O}(k))$, so that the curve $\{(\partial F/\partial Z_3)G' - (\partial G/\partial Z_3)F' = 0\}$ on S has a weak type $\mu - 1$ singularity. Let $F_1', F_2' \in H^0(\mathbf{P}^3, \mathscr{O}(5))$.

Then the curve $\{G'(aF_1'+bF_2')-aG'(F_1')-bG'(F_2')=0\}$ on S has a weak type $\mu-1$ singularity. We may assume that $G'(aF_1'+bF_2')-aG'(F_1')-bG'(F_2')\equiv 0 \bmod (F, G)$ for all a, b, F_1', F_2'; otherwise we will get $\sum \mu_{ij}(\mu_{ij}-1)\leq dkk$ by Lemma 2.5, and $g(C)\geq \frac{1}{2}dk(d-4)\geq 3$. Therefore the map $H^0(\mathbf{P}^3, \mathscr{O}(5))\to H^0(\mathbf{P}^3, \mathscr{O}(k))/(F, G)$, $F'\to G'=G'(F')$ is linear.

Recall that we use K_{k+1} to denote the linear space of homogeneous polynomials of degree $k+1$ generated by $K_{ij}(U)=Z_iG'(Z_jU)-Z_jG'(Z_iU)$ with $i, j=0, 1, 2, 3$, and $U\in H^0(\mathbf{P}^3, \mathscr{O}(4))$. From the proof of Theorem 2.1 it is easy to see that $\dim(K_{k+1}/(F, G))\leq 1$ implies $g(C)\geq 3$. So we need only to consider the case where $\dim(K_{k+1}/(F, G))\geq 2$. As we noted in (1.1), a section of $K_S\otimes C=\mathscr{O}(d+k-4)=\mathscr{O}(k+1)$ with a weak type $\mu-1$ singularity induces a section of the canonical bundle of the desingularization of C. But $\deg K_{ij}(U)=k+1$, and the curve $\{K_{ij}=0\}$ on S has a weak type $\mu-1$ singularity, so $\dim(K_{k+1}/(F, G))=2$ because of $g(C)=2$.

If we fix some $U\in H^0(\mathbf{P}^3, \mathscr{O}(4))$, so that $K_{ij}(U)$ is nontrivial in $K_{ij}/(F, G)$ for some i, j, then the linear span of the set $\{K_{ij}(U)|i, j=0, 1, 2, 3\}$ is the whole space $K_{k+1}/(F, G)$, as we noted in case 2 of the proof of Theorem 2.1. Let Q_1, Q_2 be two generators of $K_{k+1}/(F, G)$, and

$$\begin{aligned} Z_iG'(Z_jU)-Z_jG'(Z_iU) &= K_{ij}(U)\\ &\equiv a_{ij}Q_1+b_{ij}Q_2 \mod (F, G). \end{aligned}$$

Then the 4×4 matrices $A=(a_{ij})$ and $B=(b_{ij})$ are skewsymmetric and nontrivial. If we take a linear transformation $Z_i'=\sum_j h_{ij}Z_j$ of the homogeneous coordinates $\{Z_i\}$, and use the linearity of $F'\to G'=G'(F')$, then

$$Z_i'G'(Z_j'U)-Z_j'G'(Z_i'U)\equiv (HAH^t)_{ij}Q_1+(HBH^t)_{ij}Q_2 \mod (F, G)$$

with $H=(h_{ij})$. It is well known that we can choose new homogeneous coordinates, still denoted by $\{Z_0, Z_1, Z_2, Z_3\}$, so that the alternative form B has the following standard form:

Case 1:

$$B=\begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since

$$(3.4)\qquad Z_hK_{ij}(U)+Z_iK_{jh}(U)+Z_jK_{hi}(U)=0,$$

we have

$$(a_{ij}Z_h+a_{jh}Z_i+a_{hi}Z_j)Q_1+(b_{ij}Z_h+b_{jh}Z_i+b_{hi}Z_j)Q_2 \equiv 0 \mod (F, G).$$

Setting $\{i, j, h\}=\{1, 2, 3\}$ in (3.4), we get

$$(a_{ij}Z_h+a_{jh}Z_i+a_{hi}Z_j)Q_1 \equiv 0 \mod (F, G),$$
$$a_{ij}Z_h+a_{jh}Z_i+a_{hi}Z_j \equiv 0 \mod (F, G).$$

Because $k>1$, $a_{ij}=0$ for $i, j=1, 2, 3$.

Similarly, $a_{ij}=0$ for $i, j=0, 2, 3$. Setting $\{i, j, k\}=\{0, 1, 2\}$ in (3.4), we obtain

$$a_{01}Z_2Q_1+Z_2Q_2 \equiv 0 \mod (F, G),$$

which contradicts the fact that $\deg G=k>1$.

Case 2.

$$B=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Setting $\{i, j, h\} = \{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}$ in (3.4), we get

$$M_1Q_1+Z_2Q_2 \equiv 0 \mod (F, G),$$
$$M_2Q_1+Z_3Q_2 \equiv 0 \mod (F, G),$$
$$M_3Q_1+Z_0Q_2 \equiv 0 \mod (F, G),$$
$$M_4Q_1+(Z_3+Z_1)Q_2 \equiv 0 \mod (F, G).$$

A linear combination of the above will lead to

$$(3.5)\qquad L_1Q_1+L_2Q_2 \equiv 0 \mod (F, G),$$

where the line $L_2=aZ_0+bZ_1+cZ_2+dZ_3$ with free choices of a, b, c, d. Now we may choose L_2 so that $L_2\cap C$ does not contain any singular points of C, and the intersection number $I_P(L_2, C)_S=1$ at any point P of $L_2\cap C$. By Bezout's Theorem, $L_2\cap C$ contains $5k$ points with at most 2 points in $\{Q_1=0\}\cap C$, because $\deg K_{\tilde{C}}=2g-2=2$ and Q_1 induces a section of $K_{\tilde{C}}$. From $L_1Q_1=-L_2Q_2$ it follows that at least $5k-2$ points of $L_2\cap C$ are on $L_1=0$, so they are on $L_1\cap L_2\cap S$. Since Q_1

and Q_2 are linear independent, (3.5) implies that $L_1 \neq L_2$. We conclude again by Bezout's Theorem that $5k - 2 \leq 5$, i.e., $k = 1$, a contradiction.

This completes the proof of Theorem 1.

4. Subvarieties of higher dimensional hypersurfaces

By the Noether-Lefschetz Theorem, we know that every curve on a generic surface of degree $d \geq 4$ in $\mathbf{P}^3$ is a complete intersection. In higher dimensions we have a better situation, thanks to the Lefschetz Theorem, which states that if V is a hypersurface in $\mathbf{P}^{n+1}$ with $n \geq 3$, then $\operatorname{Pic} V = \mathbb{Z}$, and it is generated by $\mathscr{O}_V(1)$. Now if $M \subset V$ is a codimension-1 subvariety, then it is a complete intersection of V with another hypersurface.

Almost the whole proof of Theorem 1 can be generalized to prove Theorem 2, except we cannot apply intersection theory in higher dimensions; instead we need the following theorem of Hopf (cf. [1, pp. 108]).

Lemma 4.1 (Hopf). *Given any setup of a linear map $\nu: A \otimes B \to C$, where A, B, C are complex vector spaces and ν is injective on each factor separately, then*

$$\dim \nu(A \otimes B) \geq \dim A + \dim B - 1.$$

The analogy of Theorem 2.1 in higher dimensions is the following.

Theorem 4.2. *If M is a codimension-1 subvariety of a generic hypersurface V of degree $d \geq n+3$ in $\mathbf{P}^{n+1}$ $(n \geq 3)$, and M is a complete intersection of V with another hypersurface of degree k, then*

$$p_g(M) \geq \binom{d-2}{n+1} - \binom{d-k-2}{n+1} + 1.$$

Again the proof of Theorem 4.2 is based on the following three lemmas.

Lemma 4.3. *Let M be a codimension-1 subvariety of a smooth variety V of dimension n, and assume that M has a type $\mu = (\mu_j, X_j, E_j)$ singularity. If $\Omega \subset V$ is an open neighborhood of some point of M, $\{z_1, \cdots, z_n\}$ are local coordinates on Ω, and M is defined by $g(z_1, \cdots, z_n) = 0$ and has a type $\mu_\Omega = (\mu_j, X_j, E_j | j \in \{0, \cdots, m\})$ singularity on Ω, then the subvariety $\{\partial g(z_1, \cdots, z_n)/\partial z_i = 0\}$ $(i = 1, \cdots, n)$ has a weak type $\mu_\Omega - 1 = (\mu_j - 1, X_j, E_j | j \in \{0, \cdots, m\})$ singularity on Ω.*

Proof. Since the statement of the conclusion is independent of the choice of the local coordinates, we may assume that X_0 is defined locally by $z_{h+1} = \cdots = z_n = 0$. Let

$$z'_1 = z_1, \cdots, z'_h = z_h, z'_{h+1} = \frac{z_{h+1}}{z_n}, \cdots, z'_{n-1} = \frac{z_{n-1}}{z_n}, z'_n = z_n$$

be coordinates on the blow-up of Ω along X_0. Then

$$\begin{aligned} g(z_1, \cdots, z_n) &= g(z_1', \cdots, z_h', z_{h+1}'z_n', \cdots, z_{n-1}'z_n', z_n') \\ &= (z_n')^{\mu_0} g^\sharp(z_1', \cdots, z_n'), \\ \frac{\partial g}{\partial z_i} &= (z_n')^{\mu_0} \frac{\partial g^\sharp}{\partial z_i'}, \qquad i = 1, 2, \cdots, h, \\ \frac{\partial g}{\partial z_i} &= (z_n')^{\mu_0 - 1} \frac{\partial g^\sharp}{\partial z_i'}, \qquad i = h+1, \cdots, n-1, \\ \frac{\partial g}{\partial z_n} &= \mu_0 (z_n')^{\mu_0 - 1} g^\sharp + (z_n')^{\mu_0} \sum \frac{\partial g^\sharp}{\partial z_i'} \frac{\partial z_i'}{\partial z_n} \\ &= \mu_0 (z_n')^{\mu_0 - 1} g^\sharp + (z_n')^{\mu_0 - 1} \left(- \sum_{i=h+1}^{n-1} z_i' \frac{\partial g^\sharp}{\partial z_i'} + z_n' \frac{\partial g^\sharp}{\partial z_n'} \right). \end{aligned}$$

Since $\{g^\sharp = 0\}$ has improved singularities, by induction, $\{\partial g^\sharp / \partial z_i' = 0\}$ $(i = 1, \cdots, n)$ has a weak type $(\mu_j - 1, X_j, E_j | j \in \{1, \cdots, m\})$ singularity on the blow-up of Ω along X_0, so $\{\partial g / \partial z_i = 0\}$ $(i = 1, \cdots, n)$ has a weak type $\mu_\Omega - 1$ singularity on Ω.

Lemma 4.4. *If $M_t = \{g_t(z_1, \cdots, z_n) = 0\}$ is a μ-equisingular family of varieties defined in an open set $\Omega \subset \mathbb{C}^n$, and M_t has a type $\mu(t)_\Omega = (\mu_j, X_j(t), E_j(t) | j \in \{0, \cdots, m\})$ singularity on Ω, then the variety $\{dg_t/dt|_{t=0} = 0\}$ has a weak type $\mu(0)_\Omega - 1 = (\mu_j - 1, X_j(0), E_j(0) | j \in \{0, \cdots, m\})$ singularity on Ω.*

Proof. Since $X_0(t)$ is a smooth manifold, we may assume that $X_0(t)$ is locally defined by

$$z_{h+1} = c_{h+1}(z_1, \cdots, z_h, t), \cdots, \qquad z_n = c_n(z_1, \cdots, z_h, t).$$

Then

$$\begin{aligned} g_t(z_1, \cdots, z_n) = &\sum_{i_{h+1} + \cdots + i_n \geq \mu_0} A_{i_{h+1}, \cdots, i_n}(z_1, \cdots, z_h, t) \\ &\cdot (z_{h+1} - c_{h+1}(z_1, \cdots, z_h, t))^{i_{h+1}} \cdots (z_n - c_n(z_1, \cdots, z_h, t))^{i_n}. \end{aligned}$$

By replacing Lemma 2.2 by Lemma 4.3, the proof goes exactly in the same way as that of Lemma 2.3.

Lemma 4.5. *Let $F_t \in H^0(\mathbf{P}^{n+1}, \mathscr{O}(d))$, $G_t \in H^0(\mathbf{P}^{n+1}, \mathscr{O}(k))$, and $M_t = \{F_t = 0\} \cap \{G_t = 0\}$ be a μ-equisingular family of varieties with a type $\mu(t) = (\mu_j, X_j(t), E_j(t) | j \in \Gamma)$ singularity. Set $dF_t/dt|_{t=0} = F'$, $dG_t/dt|_{t=0} = G'$, and assume that all the hypersurfaces $F_t = 0$ are* *smooth for* t *in a neighborhood of* 0. *Then the subvariety* $\{(\partial F_0/\partial Z_i)G' - (\partial G_0/\partial Z_i)F' = 0\}$ $(i = 0, 1, \cdots, n+1)$ *on* $V = \{F_0 = 0\}$ *has a weak type* $\mu(0) - 1 = (\mu_j - 1, X_j(0), E_j(0) | j \in \Gamma)$ *singularity, where* $\{Z_0, Z_1, \cdots, Z_{n+1}\}$ *are homogeneous coordinates.*

Proof. For any point $P \in M_0$, we can find an open set $\Omega \ni P$ of V, and generic homogeneous coordinates $\{Z_i'\}$ with $Z_i' = \sum_{j=0}^{n+1} l_{ij} Z_j$ $(i = 0, 1, \cdots, n+1)$, so that $\partial F_0/\partial Z_i' \neq 0$ on Ω for all i. Assuming M_0 has a type $\mu_\Omega(0) = (\mu_j, X_j(0), E_j(0) | j \in \Gamma_\Omega)$ singularity on Ω, and proceeding as in the proof of Lemma 2.4 except using Lemma 4.4 instead of Lemma 2.3, we conclude that the subvariety $\{(\partial F_0/\partial Z_i')G' - (\partial G_0/\partial Z_i')F' = 0\}$ has a weak type $\mu_\Omega(0) - 1$ singularity on Ω. Since $(\partial F_0/\partial Z_i)G' - (\partial G_0/\partial Z_i)F'$ is a linear combination of the $(\partial F_0/\partial Z_j')G' - (\partial G_0/\partial Z_j')F'$ $(j = 0, 1, \cdots, n+1)$, and the property of having a weak type $\mu_\Omega(0) - 1$ singularity is additive by §1, we see that $\{(\partial F_0/\partial Z_i)G' - (\partial G_0/\partial Z_i)F' = 0\}$ has a weak type $\mu_\Omega(0) - 1$ singularity on Ω. Selecting a covering of V with open sets, we deduce that the subvariety $\{(\partial F_0/\partial Z_i)G' - (\partial G_0/\partial Z_i)F' = 0\}$ on V has a weak type $\mu(0) - 1$ singularity.

Proof of Theorem 4.2. As we noted at the beginning of this section, every codimension-1 subvariety of V is a complete intersection. As in $\mathbf{P}^3$, we can find a pair $\{F, G\} \in H^0(\mathbf{P}^{n+1}, \mathscr{O}(d)) \times H^0(\mathbf{P}^{n+1}, \mathscr{O}(k))$, which has the following property: both $\{F = 0\}$ and $\{F = 0\} \cap \{G = 0\}$ are reduced and irreducible, and for any deformation F_t of F with $F = F_0$, there is a unique deformation G_t of G with $G = G_0$, so that the family $M_t = \{F_t = 0\} \cap \{G_t = 0\}$ is μ-equisingular, and M_t has a type $\mu(t) = (\mu_j, X_j(t), E_j(t) | j \in \Gamma)$ singularity.

Now using Lemma 4.5, we may repeat the argument in the proof of Theorem 2.1. We construct the space K_{k+1}, so that for any $K \in K_{k+1}$, $\deg K = k+1$, and the subvariety $\{K = 0\}$ on $V = \{F = 0\}$ has a weak type $\mu - 1 = (\mu_j - 1, X_j(0), E_j(0))$ singularity. By (1.1), a section of $K_V \otimes M = K_V \otimes M_0 = \mathscr{O}(k + d - n - 2)$ with a weak type $\mu - 1$ singularity gives a section of $K_{\widetilde{M}}$. Since

$$\dim(H^0(\mathbf{P}^{n+1}, \mathscr{O}(d - n - 3))/(F, G) = \binom{d-2}{n+1} - \binom{d-k-2}{n+1},$$

if $\dim K_{k+1} \geq 2$, then by Lemma 4.1, we conclude

$$p_g(M) = h^0(\widetilde{M}, K_{\widetilde{M}}) \geq \binom{d-2}{n+1} - \binom{d-k-2}{n+1} + 1.$$

If $\dim K_{k+1} \leq 1$, we may follow the argument in the proof of Theorem 2.1 and get the same estimate on $p_g(M)$. q.e.d.

In the special case $k = 1$, we have

Proposition 4. *Let M be a hyperplane section of a generic hypersurface V of degree $d \geq n+3$ in $\mathbf{P}^{n+1}$ $(n \geq 3)$. Then M has at most $n+1$ singular points, all of which are double points, and the singularity does not affect the geometric genus of M, i.e.,*

$$p_g(M) = \binom{d}{n+1} - \binom{d-1}{n+1}.$$

We postpone the proof of Proposition 4 until the next section. Now Theorem 2 is an easy consequence of Theorem 4.2 and Proposition 4.

Proof of Theorem 2. Let M be a complete intersection of V with another hypersurface of degree k. Then by Theorem 4.2, we have

$$p_g(M) \geq \binom{d-2}{n+1} - \binom{d-k-2}{n+1} + 1.$$

If $k \geq 2$, then

$$p_g(M) \geq \binom{d-2}{n+1} - \binom{d-4}{n+1} + 1;$$

if $k = 1$, then by Proposition 4, we obtain

$$p_g(M) = \binom{d}{n+1} - \binom{d-1}{n+1}.$$

So

$$p_g(M) \geq \min\left\{ \binom{d-2}{n+1} - \binom{d-4}{n+1} + 1,\ \binom{d}{n+1} - \binom{d-1}{n+1} \right\}.$$

This completes the proof of Theorem 2.

5. Hyperplane sections of generic hypersurfaces in $\mathbf{P}^{n+1}$

In the last section, we saw that if a codimension-1 subvariety $M = \{F = 0\} \cap \{G = 0\}$ of a generic hypersurface has a type $\mu = (\mu_j, X_j, E_j)$ singularity, then for any deformation F' of F, there is a deformation G' of G, such that the subvariety $\{(\partial G/\partial Z_{n+1})F' - (\partial F/\partial Z_{n+1})G' = 0\}$ on $\{G = 0\}$ has a weak type $\mu - 1$ singularity. Now we are free to choose $F' \in H^0(\mathbf{P}^{n+1}, \mathscr{O}(d))$ arbitrarily, and if $\deg G = 1$, then G' must stay in $H^0(\{G = 0\}, \mathscr{O}(1))$ with $\dim H^0(\{G = 0\}, \mathscr{O}(1)) = n+1$. Thus M cannot afford very bad singularities. Here is a sketch of the

Proof of Proposition 4. We first take a pair

$$\{F, G\} \in H^0(\mathbf{P}^{n+1}, \mathscr{O}(d)) \times H^0(\mathbf{P}^{n+1}, \mathscr{O}(1))$$

as in the proof of Theorem 4.2, and assume that the codimension-1 subvariety $M = \{F = 0\} \cap \{G = 0\}$ of the generic hypersurface $V = \{F = 0\}$ has a type $\mu = (\mu_j, X_j, E_j | j \in \{0, \cdots, m\})$ singularity. Since the hyperplane $\{G = 0\}$ is smooth, we can find homogeneous coordinates $\{Z_0, \cdots, Z_{n+1}\}$ such that $\partial G/\partial Z_i \neq 0$ for $i \in \{0, \cdots, n+1\}$. By Lemma 4.5, we conclude that for any $F' \in H^0(\mathbf{P}^{n+1}, \mathscr{O}(d))$, there is a $G' \in H^0(\mathbf{P}^{n+1}, \mathscr{O}(1))$ so that the variety $\{(\partial G/\partial Z_{n+1})F' - (\partial F/\partial Z_{n+1})G' = 0\}$ on $\{G = 0\}$ has a weak type $\mu - 1 = (\mu_j - 1, X_j, E_j)$ singularity. If P is a singular point of M, we must have

$$\left(\frac{\partial G}{\partial Z_{n+1}} F' - \frac{\partial F}{\partial Z_{n+1}} G'\right)(P) = 0 \tag{5.1}$$

on $\{G = 0\}$. It is well known that homogeneous polynomials of degree $d \geq n+1$ take independent values on any $n+2$ distinct points in $\mathbf{P}^{n+1}$. But $G' \in H^0(\{G = 0\}, \mathscr{O}(1))$, and $h^0(\mathbf{P}^n, \mathscr{O}(1)) = h^0(\{G = 0\}, \mathscr{O}(1)) = n+1$; thus (5.1) implies that M has at most $n+1$ singular points. The same argument as in the proof of Proposition 3 shows that M has no triple points, that is, $\mu_j = 2$ for every j.

By formula (1.1), in order to conclude that the singularity of M does not affect its geometric genus, it suffices to show that $\dim X_j < n-2$ for each j.

Now assume that $\dim X_j = n-2$ for some j. For simplicity, we may assume that M has one double point $P = X_0$, $\dim X_j < n-2$ for $j < m$, $\dim X_m = n-2$, and all points of X_i $(i = 1, \cdots, m)$ are infinitely near points of P.

Given any deformation F_t of F, there is a deformation $M_t = \{F_t = 0\} \cap \{G_t = 0\}$ of $M = \{F = 0\} \cap \{G = 0\}$, so that the family M_t is μ-equisingular and M_t has a type $\mu(t) = (\mu_j, X_j(t), E_j(t) | j \in \{0, 1, \cdots, m\})$ singularity with $\mu_j = 2$ for all j. Let the point $X_0(t) = [1, c_1(t), \cdots, c_{n+1}(t)]$, $z_{0i} = Z_i/Z_0$ for $i = 1, \cdots, n+1$. Solving the equation $G_t = 0$, we get $z_{0(n+1)} = \psi_t(z_{01}, \cdots, z_{0n})$. Set

$$f_{0,t}(z_{01}, \cdots, z_{0n}) = F_t(1, z_{01}, \cdots, z_{0n}, \psi_t(z_{01}, \cdots, z_{0n})),$$

$$\frac{dF_t}{dt}(Z_0, \cdots, Z_{n+1})|_{t=0} = F'(Z_0, \cdots, Z_{n+1}),$$

$$\frac{dG_t}{dt}(Z_0, \cdots, Z_{n+1})|_{t=0} = G'(Z_0, \cdots, Z_{n+1}).$$

Then

$$\frac{df_{0,t}}{dt}\Big|_{t=0} = F' - \left(\frac{\partial G}{\partial Z_{n+1}}\right)^{-1} \frac{\partial F}{\partial Z_{n+1}} G'. \tag{5.2}$$

Since $X_0(t)$ is a double point of $M_t = \{f_{0,t} = 0\}$, we have
(5.3)

$$f_{0,t} = \sum_{i_1+\cdots+i_n \geq 2} a_{i_1\cdots i_n}(t)(z_{01} - c_1(t))^{i_1} \cdots (z_{0n} - c_n(t))^{i_n},$$

$$\frac{df_{0,t}}{dt}\Bigg|_{t=0} = -\sum_{i=1}^{n} \frac{\partial f_{0,0}}{\partial z_{0i}} \cdot \frac{dc_i(t)}{dt}\Bigg|_{t=0} + \left\{ \sum_{i_1+\cdots+i_n \geq 2} \frac{d}{dt} a_{i_1\cdots i_n}(t)(z_{01} - c_1(0))^{i_1} \cdots (z_{0n} - c_n(0))^{i_n} \right\}\Bigg|_{t=0}.$$

Let

$$f_0^*(z_{01}, \cdots, z_{0n}) = \frac{df_{0,t}}{dt}\Bigg|_{t=0} + \sum_{i=1}^{n} \frac{\partial f_{0,0}}{\partial z_{0i}} \cdot \frac{dc_i(t)}{dt}\Bigg|_{t=0}. \tag{5.4}$$

If we write down the Taylor polynomial of f_0^* at the point $X_0(0)$, then its coefficients of 1, $z_{01} - c_1(0), \cdots, z_{0n} - c_n(0)$ must all be 0. Since

$$\begin{aligned} &F'(1, z_{01}, \cdots, z_{0n}, \psi_0(z_{01}, \cdots, z_{0n})) \\ &\quad = \sum_{d \geq i_1+\cdots+i_n \geq 0} b_{i_1\cdots i_n}(z_{01} - c_1(0))^{i_1} \cdots (z_{0n} - c_n(0))^{i_n} \end{aligned} \tag{5.5}$$

with free choices of all its coefficients $b_{i_1\cdots i_n}$, the set $\{dc_i(t)/dt|_{t=0} | i = 1, \cdots, n\}$ contains n elements, and f_0^* depends linearly on F', we see that (5.2) and (5.4) imply that there will be at least one condition imposed on G' if M has one double point.

We may move the point $X_0(t) \in V_{0,t} = \{G_t = 0\}$ to $X_0(0) \in \{G = 0\}$ and blow up simultaneously at $X_0(0)$. Let $V_{1,t} \to V_{0,t}$ be the blow-up, $M_{1,t}$ be the proper transform of M_t in $V_{1,t}$, and

$$z_{11} = z_{01} - c_1(0), \quad z_{12} = \frac{z_{02} - c_2(0)}{z_{01} - c_1(0)}, \cdots, \quad z_{1n} = \frac{z_{0n} - c_n(0)}{z_{01} - c_1(0)}$$

be the new coordinates after the blowing up. Then $M_{1,t}$ is defined by $f_{1,t}(z_{11}, \cdots, z_{1n}) = 0$. Here

$$f_{1,t} = \sum_{i_1+\cdots+i_n \geq 2} a_{i_1\cdots i_n}(t) z_{11}^{i_1+\cdots+i_n-2} z_{12}^{i_2} \cdots z_{1n}^{i_n}.$$

By (5.3) and (5.4),

$$(5.6)\quad \begin{aligned}&\frac{df_{1,t}}{dt}|_{t=0}\\ &= (z_{01}-c_1(0))^{-2} f_0^*(z_{01}, \cdots, z_{0n})\\ &= z_{11}^{-2} f_0^*(z_{11}+c_1(0), z_{11}\cdot z_{12}+c_2(0), \cdots, z_{11}\cdot z_{1n}+c_n(0)).\end{aligned}$$

If we let

$$F_1' = \sum_{d\geq i_1+\cdots+i_n\geq 2} b_{i_1\cdots i_n} z_{11}^{i_1+\cdots+i_n-2} z_{12}^{i_2}\cdots z_{1n}^{i_n},$$

then by (5.5) we can choose $b_{i_1\cdots i_n}$ freely. Furthermore $df_{1,t}/dt|_{t=0}$ depends linearly on F_1' because of (5.2), (5.4), and (5.6). Since $G' \in H^0(\{G=0\}, \mathscr{O}(1))$ and $h^0(\{G=0\}, \mathscr{O}(1)) = n+1$, the main point of rest of the proof is to see what condition

$$\frac{df_{0,t}}{dt}|_{t=0} = F' - \left(\frac{\partial G}{\partial Z_{n+1}}\right)^{-1} \frac{\partial F}{\partial Z_{n+1}} G'$$

must satisfy if M has a certain type of singularity; then we choose an appropriate F' so that there is no G' which satisfies the condition. We need to continue our discussion in the following cases.

Case a. $n=3$. We claim that the proper transform $M_{1,t}$ of M_t in $V_{1,t}$ cannot have more than one singular point on the exceptional divisor $E_0(t)$. Assume that $M_{1,t}$ has two distinct singular double points $P_1(t)$ and $P_2(t)$ on the exceptional divisor $E_0(t)$, and let $P_1(t) = (0, d_1(t), e_1(t))$ and $P_2(t) = (0, d_2(t), e_2(t))$ in the $\{z_{1i}\}$ coordinates. By generic choice of the homogeneous coordinates $\{Z_0, \cdots, Z_4\}$, we may further assume that $d_1(0)\neq d_2(0)$, $e_1(0)\neq e_2(0)$. Since $M_{1,t}$ is defined by $f_{1,t}=0$, we have

$$f_{1,t}(z_{11}, z_{12}, z_{13}) = \sum_{i_1+i_2+i_3\geq 2} c_{i_1i_2i_3}(t) z_{11}^{i_1}(z_{12}-d_1(t))^{i_2}(z_{13}-e_1(t))^{i_3},$$

$$(5.7)\quad \begin{aligned} f_1^* &= \left.\frac{df_{1,t}}{dt}\right|_{t=0} + \frac{\partial f_{1,0}}{\partial z_{12}}\left.\frac{dd_1(t)}{dt}\right|_{t=0} + \frac{\partial f_{1,0}}{\partial z_{13}}\left.\frac{de_1(t)}{dt}\right|_{t=0}\\ &= \frac{d}{dt}\left\{\sum_{i_1+i_2+i_3\geq 2} c_{i_1i_2i_3}(t) z_{11}^{i_1}(z_{12}-d_1(0))^{i_2}(z_{13}-e_1(0))^{i_3}\right\}\Bigg|_{t=0}.\end{aligned}$$

So the coefficients of 1, z_{11}, $z_{12} - d_1(0)$, $z_{13} - e_1(0)$ in the Taylor expansion of f_1^* at $P_1(0)$ must be 0. We have

$$\begin{aligned} F_1' &= \sum_{d\geq i_1+i_2+i_3\geq 2} b_{i_1i_2i_3} z_{11}^{i_1+i_2+i_3-2} z_{12}^{i_2} z_{13}^{i_3} \\ &= \sum_{2\geq i+j\geq 0} b'_{ij}(z_{12}-d_1(0))^i(z_{13}-e_1(0))^j \\ &\quad + z_{11} \sum_{3\geq i+j\geq 0} b''_{ij}(z_{12}-d_1(0))^i(z_{13}-e_1(0))^j + z_{11}^2\cdot(\cdots). \end{aligned}$$

Here we are free to choose b'_{ij}, b''_{ij}. By (5.7), f_1^* depends on the two numbers $dd_1(t)/dt|_{t=0}$, $de_1(t)/dt|_{t=0}$. Therefore (5.2), (5.5), and (5.6) imply that if $P_1(0)$ is a double point of $M_{1,0}$, then at least two more conditions will be imposed on G'. Similarly the coefficients of 1, $z_{12} - d_2(0)$, and $z_{13} - e_2(0)$ in the Taylor expansion of

$$\left.\frac{df_{1,t}}{dt}\right|_{t=0} + \left.\frac{\partial f_{1,0}}{\partial z_{12}}\frac{dd_2(t)}{dt}\right|_{t=0} + \left.\frac{\partial f_{1,0}}{\partial z_{13}}\frac{de_2(t)}{dt}\right|_{t=0}$$

at $P_2(0)$ must be 0. Moreover any change of the coefficients of $(z_{12}-d_1(0))^2$, $(z_{13}-e_1(0))^2$, $(z_{12}-d_1(0))(z_{13}-e_1(0))$, or $z_{11}(z_{12}-d_1(0))$ of F_1' does not affect the above situation at $P_1(0)$. Since

$$\begin{aligned} (z_{12}-d_1(0))^2 &= 2(d_2(0)-d_1(0))(z_{12}-d_2(0)) \\ &\quad + (z_{12}-d_z(0))^2 + (d_2(0)-d_1(0))^2, \\ (z_{13}-e_1(0))^2 &= 2(e_2(0)-e_1(0))(z_{12}-e_2(0)) \\ &\quad + (z_{13}-e_2(0))^2 + (e_2(0)-e_1(0))^2, \\ (z_{12}-d_1(0))(z_{13}-e_1(0)) &= (d_2(0)-d_1(0))(e_1(0)-e_1(0)) \\ &\quad + (d_2(0)-d_1(0))(z_{13}-e_2(0)) \\ &\quad + (e_2(0)-e_1(0))(z_{12}-d_2(0)) \\ &\quad + (z_{12}-d_2(0))(z_{13}-e_2(0)), \\ z_{11}(z_{12}-d_1(0)) &= (d_1(0)-d_1(0))z_{11} + z_{11}(z_{12}-d_2(0)), \end{aligned}$$

the conditions $d_2(0) \neq d_1(0)$ and $e_2(0) \neq e_1(0)$ imply that we are free to choose the coefficients of 1, z_{11}, $z_{12} - d_2(0)$, $z_{13} - e_2(0)$ of F_1'; thus we are free to choose the coefficients of 1, z_{11}, $z_{12} - d_2(0)$, $z_{13} - e_2(0)$ of f_1^*. Moreover, if $M_{1,0}$ has a second double point $P_2(0)$, then at least two extra conditions will be imposed on G'. But $1 + 2 + 2 > 4 =$

$h^0(\{G = 0\}, \mathscr{O}(1))$, so $M_{1,0}$ has at most one singular point. So far if M has a double point, there will be at least one condition imposed on G'. If $M_{1,0}$ has a double point, then two more conditions will be imposed on G'. Since $d \geq 5$, we are free to choose the coefficients of $z_{11}^2, z_{11}^3, z_{11}(z_{12} - d_1(0)), z_{11}(z_{13} - e_1(0))$ of F_1'. It is not hard to see that there will be at least two other conditions imposed on G' if the proper transform of $M_{1,0}$ after blowing up at $P_1(0)$ has a double point. Since $h^0(\{G = 0\}, \mathscr{O}(1)) = 4$, this is impossible. In conclusion, $\dim X_j = 0$ for every j in case $n = 3$.

Case b. $m = 1$, that is, $\dim X_1(t) = n - 2$, where $X_1(t)$ is a two-fold submanifold of $M_{1,t}$. Since $M_{1,t}$ is defined by $f_{1,t}(z_{11}, \cdots, z_{1n}) = 0$, by Lemma 4.3, $df_{1,t}/dt|_{t=0} = 0$ on $X_1(0)$. Now we can choose all the coefficients of the monomials $1, z_{12}, \cdots, z_{12}^2, z_{12}z_{13}, \cdots, z_{1n}^2$ of F_1' freely, $\dim X_1(0) = n - 2$, $h^0(\mathbf{P}^{n-2}, \mathscr{O}(2)) = \binom{n}{2}$, and $df_{1,t}/dt|_{t=0}$ depends linearly on F_1'. Thus the singularity of $M_{1,t}$ along $X_1(t)$ imposes at least $\binom{n}{2}$ conditions on G'. On the other hand, $h^0(\{G = 0\}, \mathscr{O}(1)) = n + 1 < \binom{n}{2}$ if $n \geq 4$. This is impossible.

Case c. $1 \leq \dim X_1(t) = s_1 < n-2$. Since $M_{1,t}$ has a type $(\mu_j, X_j(t), E_j(t)|j \in \{1, \cdots, m\})$ singularity with $\mu_j = 2$, and $M_{1,t}$ is defined by $f_{1,t} = 0$, by Lemma 4.3, $df_{1,t}/dt|_{t=0} = 0$ has a weak type $(1, X_j(0), E_j(0)|j \in \{1, \cdots, m\})$ singularity. Let us assume that $X_1(0)$ is locally defined by

$$z_{1i} = h_{1i}(z_{1(n-s_1+1)}, \cdots, z_{1n}), \qquad i = 1, \cdots, n - s_1.$$

Rewriting,

$$\begin{aligned} F_1' &= \sum_{d \geq i_1 + \cdots + i_n \geq 2} b_{i_1 \cdots i_n} z_{11}^{i_1 + \cdots + i_n - 2} z_{12}^{i_2} \cdots z_{1n}^{i_n} \\ &= \sum b_{i_1 \cdots i_n} ((z_{11} - h_{11}) + h_{11})^{i_1 + \cdots + i_n - 2} ((z_{12} - h_{12}) + h_{12})^{i_2} \\ &\quad \cdots ((z_{1(n-s_1)} - h_{1(n-s_1)}) + h_{1(n-s_1)})^{i_{n-s_1}} z_{1(n-s_1+1)}^{i_{n-s_1+1}} \cdots z_{1n}^{i_n} \\ &= F_{1*}'(z_{11} - h_{11}(\cdots), \cdots, z_{1(n-s_1)} - h_{1(n-s_1)}(\cdots), \\ &\qquad z_{1(n-s_1+1)}, \cdots, z_{1n}) + F_{1\sharp}'(z_{1(n-s_1+1)}, \cdots, z_{1n}). \end{aligned} \tag{5.8}$$

Here F_{1*}' is a polynomial of its variables and $F_{1*}'(0, \cdots, 0, z_{1(n-s_1+1)}, \cdots, z_{1n}) = 0$. Since we are free to choose $b_{i_1 \cdots i_n}$, we are free to choose the coefficients of the monomials

$$(z_{11} - h_{11}(\cdots))^{i_1} \cdots (z_{1(n-s_1)} - h_{1(n-s_1)}(\cdots))^{i_{n-s_1}} z_{1(n-s_1+1)}^{i_{n-s_1}+1} \cdots z_{1n}^{i_n}$$

of F'_{1*} provided that $i_1 + \cdots + i_n \leq 2$ and $i_1 + \cdots + i_{n-s_1} \neq 0$, and we are also free to choose the coefficients of the monomials $1, z_{1(n-s_1+1)}, \cdots, z_{1n}, z^2_{1(n-s_1+1)}, \cdots, z^2_{1n}$ of $F'_{1\sharp}$. Let

$$\frac{df_{1,t}}{dt}|_{t=0} = f'_{1*} + f'_{1\sharp}$$

as in (5.8). Then $df_{1,t}/dt|_{t=0} = 0$ on $X_1(0)$ implies that $f'_{1\sharp} \equiv 0$. Since $f_{1\sharp}$ depends linearly on $F'_{1\sharp}$, at least three conditions are imposed on G'. Altogether, we have imposed at least four conditions on G'; this makes up the difference between $h^0(\{G=0\}, \mathscr{O}(1)) = n+1$ and $\dim X_m(0) = n-2$.

Now let $M_{2,0}$ be the proper transform of $M_{1,0}$ after blowing up along $X_1(0)$, and

$$\begin{aligned} z_{21} &= z_{11} - h_{11}(z_{1(n-s_1+1)}, \cdots, z_{1n}), \\ z_{2i} &= \frac{z_{1i} - h_{1i}(z_{1(n-s_1+1)}, \cdots, z_{1n})}{z_{11} - h_{11}(z_{1(n-s_1+1)}, \cdots, z_{1n})}, \qquad i = 2, \cdots, n-s_1, \\ z_{2i} &= z_{1i}, \qquad i = n-s_1+1, \cdots, n, \end{aligned}$$

be the new local coordinates. Denoting

$$(5.9) \quad F'_2 = z_{21}^{-1} F'_{1*}(z_{21}, z_{21}z_{22}, \cdots, z_{21}z_{2(n-s_1)}, z_{2(n-s_1+1)}, \cdots, z_{2n}),$$

we have free choices of the coefficients of $1, z_{21}, \cdots, z_{2n}$ for F'_2. Set

$$(5.10) \quad \begin{aligned} f'_2 &= (z_{11} - h_{11}(z_{1(n-s_1+1)}, \cdots, z_{1n}))^{-1} \left.\frac{df_{1,t}}{dt}\right|_{t=0} \\ &= z_{21}^{-1} f'_{1*}(z_{21}, z_{21}z_{22}, \cdots, z_{21}z_{2(n-s_1)}, z_{2(n-s_1+1)}, \cdots, z_{2n}). \end{aligned}$$

Since $\{df_{1,t}/dt|_{t=0} = 0\}$ has a weak type $(1, X_j(0), E_j(0)|j \in \{1, \cdots, m\})$ singularity, by definition, $\{f'_2 = 0\}$ has a weak type $(1, X_j(0), E_j(0)|j \in \{2, \cdots, m\})$ singularity. Moreover, f'_2 depends linearly on F'_2.

From now on, we will continue our argument inductively. If $\dim X_2(0) = s_2$, we may assume that $X_2(0)$ is locally defined by

$$z_{2(s_2+1)} = h_{2(s_2+1)}(z_{21}, \cdots, z_{2s_2}), \cdots, z_{2n} = h_{2n}(z_{21}, \cdots, z_{2s_2}),$$

so that we get

$$F_2' = F_{2*}'(z_{21}, \cdots, z_{2s_2}, z_{2(s_2+1)} - h_{2(s_2+1)}, \cdots, z_{2n} - h_{2n})$$
$$+ F_{2\sharp}'(z_{21}, \cdots, z_{2s_2}),$$
$$f_2' = f_{2*}' + f_{2\sharp}'$$

as in (5.8). We are free to choose the coefficients of $z_{2(s_2+1)} - h_{2(s_2+1)}, \cdots, z_{2n} - h_{2n}$ of F_{2*}'. Since we can also choose the coefficients of $1, z_{21}, \cdots, z_{2s_2}$ for $F_{2\sharp}'$ freely, if $f_2' = 0$ holds on $X_2(0)$ (which is equivalent to $f_{2\sharp}' = 0$), then at least $s_2 + 1 = \dim X_2(0) + 1$ conditions will be imposed on G'.

Now if $m = 2$, we have already imposed $4 + \dim X_2(0) + 1 = n + 3$ conditions on G', then we are done. Otherwise, let M_{30} be the proper transform of M_{20} after blowing up along $X_2(0)$, and

$$z_{3i} = z_{2i}, \qquad i = 1, \cdots, s_2,$$
$$z_{3(s_2+1)} = z_{2(s_2+1)} - h_{2(s_2+1)},$$
$$z_{3i} = \frac{z_{2i} - h_{2i}}{z_{2(s_2+1)} - h_{2(s_2+1)}}, \qquad i = s_2 + 2, \cdots, n,$$

be the local coordinates. Denoting

$$f_3' = z_{3(s_2+1)}^{-1} f_{2*}'(z_{31}, \cdots, z_{3(s_2+1)}, z_{3(s_2+1)}z_{3(s_2+2)}, \cdots, z_{3(s_2+1)}z_{3n}),$$
$$F_3' = z_{3(s_2+1)}^{-1} F_{2*}'(z_{31}, \cdots, z_{3(s_2+1)}, z_{3(s_2+1)}z_{3(s_2+2)}, \cdots, z_{3(s_z+1)}z_{3n})$$

as in (5.9) and (5.10), we are free to choose the coefficients of $1, z_{3(s_2+2)}, \cdots, z_{3n}$ for F_3'. Moreover $\{f_3' = 0\}$ has a weak type $(1, X_j(0), E_j(0)|j \in \{3, \cdots, m\})$ singularity, and f_3' depends linearly on F_3'.

For simplicity, let us assume that $X_3(0)$ is locally defined by

$$z_{3i} = h_{3i}(z_{3(s+1)}, \cdots, z_{3(s+s_3)}), \qquad i \in \{1, \cdots, n\} - \{s+1, \cdots, s+s_3\}.$$

If we write down $f_3' = f_{3*}' + f_{3\sharp}'$, $F_3' = F_{3*}' + F_{3\sharp}'$ as before, then we are free to choose the coefficients of 1, $z_{3i} (i \in \{s_2+2, \cdots, n\} \cap \{s+1, \cdots, s+s_3\})$ for $F_{3\sharp}'$, and the coefficients of $z_{3i} - h_{3i}$ $(i \in \{s_2 + 2, \cdots, n\} - \{s + 1, \cdots, s + s_3\})$ for F_{3*}'. If $f_3' = 0$ holds on $X_3(0)$, then at least $\rho = 1 + \#\{\{s_2 + 2, \cdots, n\} \cap \{s + 1, \cdots, s + s_3\}\}$ conditions will be imposed on G'. If we construct F_4' inductively, then we are free to choose $(n - s_2 - 1) - (\rho - 1) = n + 1 - [(s_2 + 1 + \rho]$ coefficients of the zero and the first orders of F_4'.

We may continue this argument. Either we have already imposed more than $n+1$ conditions on G' before we have reached $X_m(0)$, or we have imposed $1+3+\lambda \leq n+1$ conditions on G', and we have a free choice of $n+1-\lambda$ coefficients of the zero and the first orders of F'_m (hence f'_m). Since $\dim X_m(0) = n-2$, if $X_m(0)$ is defined by $z_{m1} = h_{m1}(z_{m3}, \cdots, z_{mn})$, $z_{m2} = h_{m2}(z_{m3}, \cdots, z_{mn})$, then $f'_m = f'_{m*} + f'_{m\sharp} = 0$ on $X_m(0)$ implies that $f'_{m\sharp}(z_{m3}, \cdots, z_{mn}) = 0$. But we are free to choose at least $(n+1-\lambda)-2$ of the coefficients of $1, z_{m3}, \cdots, z_{mn}$ of F'_m. If $f'_m = 0$ holds on $X_m(0)$, then at least $n+1-\lambda-2$ conditions will be imposed on G'; this is impossible since $(1+3+\lambda)+(n+1-\lambda-2) = n+3 > h^0(\{G=0\}, \mathscr{O}(1)) = n+1$.

Case d. $\dim X_1(t) = 0$, that is, $X_1(t)$ is a double point of $M_{1,t}$. We see easily as in case (a) that this imposes two conditions on G'. Therefore if $X_0(0)$ is a double point of M_0 and $X_1(0)$ is a double point of $M_{1,0}$, there will be at least three conditions imposed on G'. Now we can construct F'_2 and f'_2 as above. Using the fact that $f'_2 = 0$ has a weak type $(1, X_j(0), E_j(0) | j \in \{2, \cdots, m\})$ singularity, we may repeat the argument of the second part of case (c). Finally this will impose at least $n+2$ (instead of $n+3$ in case (c)) conditions on G', a contradiction.

This completes the proof of Proposition 4.

References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths & J. Harris, *Geometry of algebraic curves, Vol. I*, Springer, Berlin, 1985.

[2] J. Carlson, M. Green, P. Griffiths & J. Harris, *Infinitesimal variation of Hodge structures.* I, Compositio Math. **50** (1983) 109–205.

[3] H. Clemens, *Curves on generic hypersurface*, Ann. Sci. École Norm. Sup. **19** (1986) 629–636.

[4] H. Clemens, J. Kollár & S. Mori, *Higher dimensional complex geometry*, Astérisque (1988).

[5] L. Ein, *Subvarieties of generic complete intersection*, Invent. Math. **94** (1988) 163–169.

[6] ——, *Subvarieties of generic complete intersections.* II, preprint.

[7] G. Ellingsrud & S. A. Stromme, *The number of twisted cubic curves on the general quintic threefold*, preprint.

[8] M. Green, *Koszul cohomology and geometry*, Lectures on Riemann Surfaces, World Sci. Publ., Singapore and River Edge, NJ, 1989.

[9] ——, *Koszul cohomology and the geometry of projective varieties.* I, II, J. Differential Geometry **19** (1984) 125–171; **20** (1984) 279–289.

[10] P. Griffiths & J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

[11] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. (2) **79** (1964) 109–203, 205–326.

[12] S. Iitaka, *Algebraic geometry*, Springer, Berlin, 1982.

[13] S. Katz, *On the finiteness of rational curves on quintic threefolds*, Compositio Math. **60** (1986) 151–162.

[14] S. Mori & S. Mukai, *The uniruledness of the moduli space of curves of genus* 11, Algebraic Geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., Vol. 1016, Springer, Berlin, 1983, 334–353.

[15] D. Morrison, *Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians*, preprint.

[16] B. Teissier, *Résolution simultanée. II: Résolution simultanée et cycles évanescents*, Sém. les Singularités des Surface (Palaiseau, 1976–1977), Lecture Notes in Math., Vol. 777, Springer, Berlin, 1980, 82–146.

[17] ——, *Variétés polaires. II: Multiplicités polaires, sections planes et conditions de Whitney*, Algebraic Geometry (La Rábida, 1981), Lecture Notes in Math., Vol. 961, Springer, Berlin, 1982, 314–491.

[18] J. Wahl, *Equisingular deformations of plane algeboid curves*, Trans. Amer. Math. Soc. **193** (1974) 143–170.

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