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LOCAL PROPERTIES OF FAMILIES OF PLANE CURVES

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Introduction

Let $\mathbf{P}^N$ be the projective space parametrizing all projective plane curves of degree n $(N = n(n+3)/2)$. For $d \geq 1$, we let $\Sigma_{n,d} \subset \mathbf{P}^N \times \mathrm{Sym}^d(\mathbf{P}^2)$ be the closure of the locus of pairs $(E, \Sigma_{i=1}^d P_i)$, where E is an irreducible nodal curve and $P_1, \cdots, P_d$ are its nodes. The purpose of this paper is to prove the following theorem.

Theorem. *The variety $\Sigma_{n,d}$ is unibranch everywhere.*

The variety $\Sigma_{n,d}$ plays an important role in the study of the family of irreducible plane curves of degree n with d nodes and no other singularities as well as the locus $V(n, g) \subset \mathbf{P}^N$ of reduced and irreducible curves of genus g, where $g = (n-1)(n-2)/2 - d$. We mention two corollaries.

Corollary 1 (Harris [5]). *The variety $\overline{V(n, g)} \subset \mathbf{P}^N$ is irreducible.*

Corollary 2. *The locus $V(n, g)$ is unibranch everywhere.*

It is well known that $\overline{V(n, g)}$ is not unibranch everywhere [3], [5, §1], [6, Lecture 3], [10, §11]. We now prove the corollaries. Recall a result of Arbarello and Cornalba [1] and Zariski [13]: *the general members of $V(n, g)$ have $d = (n-1)(n-2)/2 - g$ nodes and no other singularities.* It follows that the projection of $\Sigma_{n,d}$ to $\mathbf{P}^N$ coincides with $\overline{V(n, g)}$. Every component of $\Sigma_{n,d}$ contains a pair of the form $(\Sigma_{r=1}^n L_r, dP)$, where the lines L_r $(1 \leq r \leq n)$ meet only at P, and by the deformation theory, $\Sigma_{n,d}$ contains all such pairs [6, Lecture 3, §2], [10, §11]. It is clear that these pairs form an irreducible family. Hence $\Sigma_{n,d}$ is irreducible by our theorem. It follows that $\overline{V(n, g)}$ is also irreducible.

We now prove Corollary 2. Let C be an arbitrary member of $V(n, g)$. For a point $P \in C$, we set $\delta_P = \dim_{\mathbf{C}} \widetilde{O}_P / O_P$, where O_P is the local ring of C at P, and $\widetilde{O}_P$ its normalization. By the genus formula, $\Sigma_{Q \in C} \delta_Q = d$ [7, Theorem 2]. Therefore if a nodal member of $V(n, g)$ specializes to C, then exactly δ_P of its nodes specialize to $P \in C$ [12, §3.4]. Hence C

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is the projection of a unique pair $(C, \Sigma_{i=1}^d Q_i) \in \Sigma_{n,d}$. Since $V(n, g)$ is open in $\overline{V(n, g)}$ [7, Theorem 5], Corollary 2 follows from the theorem.

The proof of the theorem relies on the result of Arbarello and Cornalba and Zariski and its generalization by Harris [5, §2].

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Proof of the theorem

We fix $d \geq 1$ and prove the theorem by decreasing induction on n. For each n, we consider the projections $\pi: \Sigma_{n,d} \to \mathbf{P}^N$ and $\pi_d: \Sigma_{n,d} \to \mathrm{Sym}^d(\mathbf{P}^2)$ (by abuse of notation we omit the index n).

For $n \gg d$, the theorem is elementary. Indeed, $\mathrm{Sym}^d(\mathbf{P}^2)$ is obviously unibranch everywhere. For $n \gg d$, π_d is surjective and its general fiber is a linear system of curves with d assigned singularities. Let $S \subset \mathrm{Sym}^d(\mathbf{P}^2)$ denote the singular locus. Outside S, π_d is a bundle whose fibers are canonically isomorphic to *linear subspaces* of $\mathbf{P}^N$. Hence $\Sigma_{n,d} \backslash \pi_d^{-1}(S)$ is unibranch everywhere.

Let $\mathrm{Sym}^d(\mathbf{P}^2) \subset \mathbf{P}^M$ be a closed imbedding. For a point $v \in \pi_d^{-1}(S)$, we consider a fundamental system of *polycylinders* $\{U_\gamma\}$ in $\mathbf{P}^N \times \mathbf{P}^M$ containing v. Set $U'_\gamma = U_\gamma \cap \Sigma_{n,d}$. We get a fundamental system of neighborhoods $\{U'_\gamma\}$ of v in $\Sigma_{n,d}$. Let $\eta, \xi \in \Sigma_{n,d}$ be two distinct points such that $\pi(\eta)$ and $\pi(\xi)$ are nodal curves and $\pi_d(\eta) = \pi_d(\xi)$. Then $\pi(\Sigma_{n,d})$ contains the line in $\mathbf{P}^N$ passing through $\pi(\eta)$ and $\pi(\xi)$.

We consider a decomposition of $U'_\gamma \backslash (U'_\gamma \cap \pi_d^{-1}(S))$ in a union of its connected components. Projecting these components to $\mathrm{Sym}^d(\mathbf{P}^2)$, we obtain a decomposition of $\pi_d(U'_\gamma) \backslash S$ in a disjoint union of *open* subsets. Since $\mathrm{Sym}^d(\mathbf{P}^2)$ is unibranch, the latter decomposition must be trivial. Hence $U'_\gamma \backslash (U'_\gamma \cap \pi_d^{-1}(S))$ is connected, and $\Sigma_{n,d}$ is unibranch at v.

We now suppose that $\Sigma_{n+1,d}$ is unibranch everywhere. Let $(C, \Sigma_{i=1}^d Q_i)$ be an arbitrary point of $\Sigma_{n,d}$. Let $l \subset \mathbf{P}^2$ be a fixed line in *general* position with respect to $(C, \Sigma_{i=1}^d Q_i)$, and $p \in l \backslash C$ a fixed point. We set $\alpha = (C + l, \Sigma_{i=1}^d Q_i)$. To get rid of l we need two general lemmas.

Recall that a noetherian topological space W is connected in codimension 1 if and only if for every closed subspace $K \subset W$ of codimension ≥ 2, the set $W \backslash K$ is connected.

Lemma 1. *Let A be a complete local noetherian domain. Let $h_1, \cdots, h_m$ be elements of A, and $B = A/(h_1, \cdots, h_m)$. If* $\dim A = \dim B + m$, *then* $\operatorname{Spec}(B)$ *is connected in codimension* 1.

Proof of Lemma 1. See [4, Exp. XIII, Theorem 2.1].

Following Harris [5, §2], for $m \le n$, we let $\Sigma_{n,d,m} \subset \Sigma_{n,d}$ be the closure of the locus of pairs $(F, \Sigma_{i=1}^d R_i)$, where F is an irreducible nodal curve having smooth contact of order at least m with l at p. Let $\mathbf{P}^{N_1}$ be the projective space parametrizing all projective plane curves of degree $n+1$. We consider a small open analytic neighborhood $\mathscr{A} \subset \Sigma_{n+1,d}$ of α. Let $(E, \Sigma_{i=1}^d P_i)$ be a point of $\mathscr{A}$, and let

$$f_E(X, Y, Z) = \Sigma a_{jk} X^j Y^k Z^{n+1-j-k} = X(\cdots) + \Sigma a_{0k} Y^k Z^{n+1-k}$$

be an equation of E. We have chosen our coordinate system in $\mathbf{P}^2$ such that $l = \{X = 0\}$ and $p = (0:1:0)$. For $m \ge 1$, the condition $a_{0n+1} = \cdots = a_{0n+2-m} = 0$ means that E has contact of order at least m with l at p (if $E \supset l$, then by definition, they have contact of order ∞ at p). We set

$$\Sigma_{n+1,d,n+2} = \{(D+l, \Sigma_{i=1}^d R_i) | (D, \Sigma_{i=1}^d R_i) \in \Sigma_{n,d}\}.$$

For sufficiently small $\mathscr{A}$, the curves of $\pi(\mathscr{A})$ have *no* singularities at p and the supports of the cycles of $\pi_d(\mathscr{A})$ do *not* intersect l. For each m, $0 \le m \le n+2$, the general points of $\Sigma_{n+1,d,n+2}$ belong to $\Sigma_{n+1,d,m}$ [5, §2], hence $\alpha \in \Sigma_{n+1,d,m}$. It follows from the semistable reduction theorem for families of curves and dimension counts that the locus of nonreduced curves has codimension strictly greater than 1 in $\pi(\Sigma_{n+1,d,m} \cap \mathscr{A})$ for $0 \le m \le n+2$, [2, §1(a)], [5, §2], [9].

Lemma 2. *For an integer m, $0 \le m \le n+2$, let E be a general point of an arbitrary codimension* 1 *subfamily of $\pi(\Sigma_{n+1,d,m} \cap \mathscr{A})$. Then E has at most one non-nodal singularity which is a cusp, a tacnode, or an ordinary triple point. Furthermore, $\Sigma_{n+1,d,m} \cap \mathscr{A}$ is smooth at all points corresponding to E.*

Proof of Lemma 2. For $m = 0$ or $n+2$, the lemma is known; see [2, §1(a)]. Since $\dim \Sigma_{n+1,d,m} = \dim \Sigma_{n,d} + n + 2 - m$ [5, §2], by taking the corresponding hyperplane sections, we reduce the proof of the first part of the lemma to the case $m = n+2$.

We now assume that our E is a member of $\pi(\Sigma_{n+1,d,m} \backslash \Sigma_{n+1,d,m+1})$ of genus $g(E) = n(n-1)/2 - d$; the remaining cases are similar only easier. We apply a general argument of Harris [5, §2]. For $i = 0, 1, \cdots, m$, we blow up the plane i times at p in the direction of l; let $S_i \to \mathbf{P}^2$ be the corresponding morphism, and K_{S_i} the canonical divisor on S_i. Let E_i be the proper transform of E in S_i, and $\varphi_i: \widetilde{E} \to E_i$ the normalization morphism. We have $-E_m \cdot K_{S_m} = 3(n+1) - m$ [5, p. 451]. Therefore, for $i = 0$ or m, the deformations of the pair $(\widetilde{E}, \varphi_i)$ are parametrized by a germ $\mathscr{D}_i$ of a smooth manifold of dimension

$$3(n+1) + g(E) - 1 - i = N_1 - d - i = \dim \Sigma_{n+1,d} - i,$$

and there is a natural immersion $\mathscr{D}_m \hookrightarrow \mathscr{D}_0$ [8], [11, 1.3–1.6]. On the other hand, $\Sigma_{n+1,d}$ is smooth at $\pi^{-1}(E)$ and, in a neighborhood of E, π^{-1} is a one-to-one map [2, §1(a)]. Hence there is a natural analytic isomorphism between $\mathscr{D}_0$ at $(\widetilde{E}, \varphi_0)$ and $\Sigma_{n+1,d}$ at $\pi^{-1}(E)$. The image of $\mathscr{D}_m$ in $\Sigma_{n+1,d}$ lies in $\Sigma_{n+1,d,m}$. Thus $\Sigma_{n+1,d,m}$ is smooth at $\pi^{-1}(E)$. This proves the lemma.

We now finish the proof of the theorem. Let $\mathscr{B}$ denote the locus in $\mathscr{A}$ of the solutions of $n+2$ equations corresponding to the $n+2$ elements: $a_{0n+1}, \cdots, a_{00}$. It is clear that $\Sigma_{n+1,d,n+2} \cap \mathscr{A} \subset \mathscr{B}_{\text{red}}$.

To compute $\dim \mathscr{B}$, we apply [5, §2]. For $1 \leq m \leq n+1$, let $(D, \Sigma_{i=1}^d R_i)$ be a general point of the locus in $\mathscr{A}$ of the solutions of m equations corresponding to the m elements: $a_{0n+1}, \cdots, a_{0n+2-m}$. Since $R_1, \cdots, R_d \notin l$, $l \not\subset D$ and D has contact of order m with l at p, provided D is reduced; moreover, D is reduced, as before, by the semistable reduction theorem. So

$$\dim \mathscr{B} = \dim \Sigma_{n+1,d} - n - 2 = \dim \Sigma_{n+1,d,n+2}.$$

By Lemma 1, $\mathscr{B}$ is connected in codimension 1 at α. Hence, by Lemma 2 (with $m = n+2$), $\mathscr{B}_{\text{red}} = \Sigma_{n+1,d,n+2} \cap \mathscr{A}$ and $\Sigma_{n+1,d,n+2}$ is unibranch at α. Therefore $\Sigma_{n,d}$ is unibranch at $(C, \Sigma_{i=1}^d Q_i)$. This proves the theorem.

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