J. DIFFERENTIAL GEOMETRY
38 (1993) 545–558

CLOSED HYPERBOLIC 3-MANIFOLDS WHOSE CLOSED GEODESICS ALL ARE SIMPLE

TED CHINBURG & ALAN W. REID

1. Introduction

Let M be a complete orientable hyperbolic n-manifold of constant curvature -1 and finite volume (in the sequel this will be abbreviated to simply "hyperbolic n-manifold").

A closed geodesic in M is *simple* if it has no self-intersections, and *nonsimple* otherwise. When $n = 2$, there are always nonsimple closed geodesics. Indeed much effort has been made in the case $n = 2$ to algorithmically describe simple geodesics; see for example [3], [4]. Moreover there are hyperbolic 2-manifolds where no closed geodesic is simple. For example, since the Teichmüller space of a twice-punctured disc is a point, it has a unique hyperbolic structure, and it is not difficult to see that all closed geodesics in this case are nonsimple. Here we prove the following result:

Theorem 1. *There exist infinitely many noncommensurable closed hyperbolic* 3*-manifolds all of whose closed geodesics are simple.*

Whether such examples exist has been frequently asked. Our examples are arithmetic, and are constructed via the theory of quaternion algebras. More precisely, suppose Γ is a torsion-free arithmetic Kleinian group which is derived from a quaternion algebra B over a number field, in the sense of §2.2. If B is a division algebra, then $M = \mathbf{H}^3/\Gamma$ is a closed hyperbolic 3-manifold. We show in Proposition 5 that if M has a nonsimple closed geodesic, then B must have a Hilbert symbol $\{a, b\}$ of a particular kind. In §4 we show that there are infinitely many nonisomorphic B which do not have Hilbert symbols of the above kind; this implies Theorem 1.

While Hilbert symbols thus provide a nontrivial obstruction to the existence of nonsimple closed geodesics, we do not know if this is the only obstruction for arithmetic M as above. We should also mention that the

Received August 28, 1992. The first author acknowledges the support of NSF Grant number DMS 92-01016, and the second author NSF Grant numbers DMS-850550 administered through M.S.R.I. and DMS-910850.

Hilbert symbol obstruction also produces nonarithmetic examples having only simple closed geodesics (see Theorem 11). In this case one considers the invariant quaternion algebra and invariant trace field defined in §2.3.

As noted just before the statement of Theorem 1, every hyperbolic 2-manifold has nonsimple closed geodesics. Thus the examples of manifolds exhibited by Theorem 1 can have no totally geodesic surfaces. In [11] and [13] those arithmetic hyperbolic 3-manifolds that contain an immersion of a totally geodesic surface are completely classified, and in some sense the methods used here are a refinement of those in [11] and [13].

We remark that every finite volume hyperbolic 3-manifold has a simple closed geodesic. In the closed case any shortest one is simple. However this need not be the case in the presence of cusps since it is possible to construct finite volume hyperbolic 3-manifolds containing an embedded twice-punctured disc (which is necessarily totally geodesic) such that the shortest closed geodesic in the 3-manifold is the closed geodesic in the twice-punctured disc that forms a "figure-eight" round the two punctures. See [1] for more on this and related topics.

2. Preliminaries on quaternion algebras and arithmetic hyperbolic 3-manifolds

2.1. For convenience we recall some standard facts from the theory of quaternion algebras that we shall require. See [18] for details.

Let k be a field of characteristic different from 2. The standard notation for a quaternion algebra over k is the following. Let a and b be nonzero elements of k. Then $(\frac{a,b}{k})$ denotes the quaternion algebra over k with basis $\{1, i, j, ij\}$ subject to $i^2 = a$, $j^2 = b$, and $ij = -ji$.

Now assume k is a number field, i.e., a finite extension of $\mathbf{Q}$. By a place ν of k we will mean one of the canonical absolute values of k defined in [10, pp. 34–35]. The finite places of k correspond bijectively to the prime ideals of the ring of integers R_k of k. An infinite place of k is either real, corresponding to an embedding of k into $\mathbf{R}$, or complex, corresponding to a pair of distinct complex conjugate embeddings of k into $\mathbf{C}$. We refer the reader to [10, p. 36] for the definition of the completion k_ν of k at a place ν. When ν is an infinite place, k_ν is isomorphic to $\mathbf{R}$ or $\mathbf{C}$ depending on whether ν is real or complex.

The classification of quaternion algebras $B_\nu = B \otimes_k k_\nu$ over the fields k_ν is quite simple. If ν is complex, then B_ν is isomorphic to $M(2, k_\nu)$ over k_ν. Otherwise there is up to isomorphism over k_ν a unique quaternion division algebra over k_ν, and B_ν is isomorphic over k_ν to either this division algebra or $M(2, k_\nu)$.

Let B be a quaternion algebra over the number field k. B is *ramified* at a place ν of k if B_ν is a division algebra. Otherwise we say B is *unramified* at ν. The classification theorem for quaternion algebras over number fields (see [18, Chapter 3]) implies that set $\mathrm{Ram}(B)$ of places of k which ramify in B is finite and of even cardinality and contains no complex places. Conversely, suppose S is a finite set of places of k which has even cardinality and contains no complex places. Then there is a quaternion algebra B over k with $\mathrm{Ram}(B) = S$, and this B is unique up to isomorphism over k. By [18, Chapter 2], a place ν of k is ramified in $B = (\frac{a,b}{k})$ exactly when $ax^2 + by^2 = 1$ has no solution (x, y) in $k_\nu \times k_\nu$.

2.2. Arithmetic hyperbolic 3-manifolds are obtained as follows (cf. [5] and [18, Chapter 4]).

Let k be a number field having exactly one complex place. Let B be a quaternion algebra over k which ramifies at all real places of k. Let $\mathscr{O}$ be an order of B, and $\mathscr{O}^1$ the group of elements of reduced norm 1 in $\mathscr{O}$. Over an embedding $k \hookrightarrow \mathbf{C}$ inducing the complex place of k one may choose an algebra embedding $\rho\colon B \hookrightarrow M(2, \mathbf{C})$ which restricts to an injection $\rho\colon \mathscr{O}^1 \hookrightarrow \mathrm{SL}(2, \mathbf{C})$. Let $\mathrm{P}\colon \mathrm{SL}(2, \mathbf{C}) \to \mathrm{PSL}(2, \mathbf{C})$ be the natural projection. Then $\mathrm{P}\rho(\mathscr{O}^1)$ is a Kleinian group of finite covolume. An *arithmetic Kleinian group* Γ is a subgroup of $\mathrm{PSL}(2, \mathbf{C})$ commensurable with a group of the type $\mathrm{P}\rho(\mathscr{O}^1)$. We say Γ *is derived from a quaternion algebra* if Γ is actually a subgroup of some $\mathrm{P}\rho(\mathscr{O}^1)$. We say the orbifold $M = \mathbf{H}^3/\Gamma$ is arithmetic or derived from a quaternion algebra if Γ is arithmetic or derived from a quaternion algebra. Here M is a manifold if and only if Γ is torsion-free.

It is shown in [11] that a Kleinian group of finite covolume is arithmetic if and only if the group $\Gamma^{(2)} = \mathrm{gp}\{\gamma^2 : \gamma \in \Gamma\}$ is derived from a quaternion algebra.

2.3. As described in [12], to any Kleinian group Γ of finite covolume we can associate a quaternion algebra which is an invariant of the commensurability class of Γ. In the arithmetic case this coincides with the algebra described above. Let $\Gamma^{(2)}$ be as above. The field $k\Gamma = \mathbf{Q}(\mathrm{tr}(\gamma) : \gamma \in \Gamma^{(2)})$ was shown in [14] to be an invariant of the commensurability class of Γ. Following [12], we will call $k\Gamma$ the *invariant trace-field* of Γ. The algebra

$$A\Gamma = \left\{\sum a_i \gamma_i : a_i \in k\Gamma,\ \gamma_i \in \Gamma^{(2)}\right\},$$

where all sums are finite, is a quaternion algebra over $k\Gamma$ (see for instance

[2] or [8]) and is an invariant of the commensurability class of Γ. In [12] $A\Gamma$ is called the *invariant quaternion algebra* of Γ. In the arithmetic case it is a complete invariant; cf. [5] and [11]; however there are many examples of noncommensurable nonarithmetic groups with the same invariant quaternion algebra; cf. [12].

We now indicate how to reconstruct the invariant quaternion algebra from essentially any pair of noncommuting elements of Γ. The proof of the following lemma is completely analogous to that of Proposition 2 of [16]; see also [8].

Lemma 2. *Let Γ be a Kleinian group of finite covolume, and let γ and δ be a pair of noncommuting elements of Γ such that γ and δ are not of order 2. Then,*

$$A\Gamma \cong \left(\frac{\operatorname{tr}(\gamma)^2(\operatorname{tr}(\gamma)^2-4),\ (\operatorname{tr}(\gamma^2)+2)(\operatorname{tr}(\delta^2)+2)(\operatorname{tr}([\gamma,\delta])-2)}{k\Gamma}\right).$$

In the case where the trace field $\mathbf{Q}(\operatorname{tr}(\gamma)\colon \gamma \in \Gamma)$ of Γ coincides with the centre of $A\Gamma$ the situation is simplified. Here we need not exclude either (or both) of γ and δ being elliptic of order 2, and the algebra is simply

$$\left(\frac{\operatorname{tr}(\gamma)^2-4,\ \operatorname{tr}([\gamma,\delta])-2}{k\Gamma}\right);$$

again see [16] or [8] for example. In particular, this simplification applies to the case of arithmetic Kleinian groups derived from a quaternion algebra.

3. Proof of Theorem 1

Let $M = \mathbf{H}^3/\Gamma$ be a closed hyperbolic 3-manifold, and assume g is a nonsimple geodesic in M. We begin with a few basic geometric observations.

There exist a loxodromic element $\gamma \in \Gamma$, and a geodesic in $\mathbf{H}^3$, namely the axis A of γ, such that under the canonical projection map to M, the image of A is freely homotopic to g. As g is nonsimple, there is at least one element δ_0 of Γ such that $\delta_0 A \neq A$ and $\delta_0 A \cap A \neq \varnothing$. Then $\delta_0 A$ is the axis of the element $\delta = \delta_0\gamma\delta_0^{-1}$.

Let the fixpoints of γ be a_1 and a_2; these are just the endpoints in $\mathbf{C}\cup\infty$ of the geodesic A in $\mathbf{H}^3$. Let the images of a_1 and a_2 under δ_0 be b_1 and b_2, ordered so that reading anticlockwise we have a_1, b_1, a_2, b_2.

Lemma 3. *a_1, a_2, b_1, and b_2 lie on a circle in $\mathbf{C} \cup \infty$, and the cross-ratio $[a_1, a_2, b_1, b_2]$ is a real number lying in the interval $(0, 1)$.*

Proof. By an element of $\mathrm{PSL}(2, \mathbf{C})$ we can map $a_1 \to 0$, $b_1 \to 1$, and $b_2 \to \infty$. Assume that a_2 maps to w. Because $\delta_0 A \neq A$ and $\delta_0 A \cap A \neq \varnothing$, w must be a real number greater than 1. Since elements of $\mathrm{PSL}(2, \mathbf{C})$ map circles to circles, this proves the first statement. Cross-ratio is also preserved by elements of $\mathrm{PSL}(2, \mathbf{C})$. Therefore the cross-ratio we require is $[0, w, 1, \infty]$, which is simply $1/w$, and hence real and in $(0, 1)$. q.e.d.

Expanding on the proof of Lemma 3, note that γ and δ have the same trace since they are conjugate. The mapping described in the proof has the effect of conjugating Γ so that

$$\gamma = \begin{pmatrix} \lambda & 0 \\ r & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} \lambda & (\lambda^{-1} - \lambda) \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Let $t = (\lambda^{-1} - \lambda)$. With this notation, the fixpoint w of Lemma 3 is $-t/r$. Thus by Lemma 3, $-t/r$ is real and greater than 1.

Lemma 4. *Let k denote the trace-field of Γ. Then $t/r = -[a_1, a_2, b_1, b_2]^{-1}$ lies in k.*

Proof. This follows by inspection of certain traces of elements of Γ. We leave the verification of the following to the reader.

(1) $t^2 = \mathrm{tr}(\gamma)^2 - 4 = \mathrm{tr}(\delta)^2 - 4$.

(2) $\mathrm{tr}(\gamma\delta^{-1}) = 2 - \mathrm{rt}$.

(3) $\mathrm{tr}([\delta, \gamma^{-1}]) = 2 + r^2 t^2 + \mathrm{rt} \cdot t^2$.

By (1), $t^2 \in k$ and by (2), $rt \in k$. In the proof of Lemma 3 we showed $-r/t = [a_1, a_2, b_1, b_2]$ lies in $(0, 1)$, so $t \neq 0 \neq r$. Thus, (3) implies that r^2 lies in k. Finally $t/r = rt/r^2 \in k$ as required.

Proposition 5. *Let Γ be torsion-free and derived from a quaternion algebra. Assume that the quaternion algebra B is a division algebra, so that $M = \mathbf{H}^3/\Gamma$ is a closed hyperbolic 3-manifold. Fix an embedding $\tau\colon k \hookrightarrow \mathbf{C}$ corresponding to the unique complex place of k. The subfield $k \cap \mathbf{R}$ of k is then a proper subfield of k which does not depend on the choice of τ. If M has a nonsimple closed geodesic, then $B \cong (\frac{a, b}{k})$ for some $a \in k$ and $b \in k \cap \mathbf{R}$.*

Proof. From [18, Theorem 4.1.1] one knows that M is compact if B is a division algebra. The embedding τ is unique up to complex conjugation, and $\tau(k)$ is not contained in $\mathbf{R}$ since τ determines a complex place of k. Hence $k \cap \mathbf{R}$ is a proper subfield of k independent of the choice of τ. Assuming now that M has a nonsimple geodesic g, we shall compute the expression for B resulting from (the remark after) Lemma 2 using the elements δ and γ^{-1} of Γ. These elements are noncommuting loxodromic elements because they have different axes.

By the remark after Lemma 2, $B \cong (\frac{a,b'}{k})$ where $a = \operatorname{tr}(\delta)^2 - 4$ and $b' = \operatorname{tr}[\delta, \gamma^{-1}] - 2$. From the calculations in Lemma 4,

$$a = t^2 \quad \text{and} \quad b' = r^2t^2 + rt \cdot t^2 = t^2r^2(1 + t/r).$$

Lemmas 3 and 4 show $x = r/t$ lies in $k \cap \mathbf{R}$ and that r, t and x are nonzero. Thus $b' = (t^2x)^2(1 + t/r)$, where $t^2x = ax \in k^*$. By standard properties of the Hilbert symbol, (see [18, Chapter 2]), the isomorphism class of the algebra is unchanged by removing squares. Hence, we can conclude that $B \cong (\frac{a,b}{k})$ where $b = 1 + t/r$ lies in $k \cap \mathbf{R}$.

Corollary 6. *With the notation of Proposition* 5, *suppose that there do not exist nonzero* $a \in k$ *and* $b \in k \cap \mathbf{R} \neq k$ *such that* B *is isomorphic over* k *to the quaternion algebra* $(\frac{a,b}{k})$. *Then all of the closed geodesics of the closed hyperbolic* 3*-manifold* $M = \mathbf{H}^3/\Gamma$ *are simple. To prove Theorem* 1 *it will suffice to construct infinitely many nonisomorphic quaternion algebras* B *of the above kind.*

Proof. By [18, Chapter 1], the only quaternion algebra over k which is not a division algebra is the matrix algebra $M(2, k)$, which is isomorphic to $(\frac{1,1}{k})$. Hence B must be a division algebra, and Proposition 5 shows M has no nonsimple closed geoedesics. By [5] or [11], the commensurability class of M determines the isomorphism class of B, so the last statement of Corollary 6 is now clear. q.e.d.

In the next section we determine which quaternion algebras over a number field K are isomorphic to $(\frac{a,b}{K})$ for some $a \in K$ and some b lying in a specified subfield F of K. This then leads to the construction of infinitely many nonisomorphic B satisfying the conditions of Corollary 6, which will complete the proof of Theorem 1.

4. Existence of suitable quaternion algebras

4.1. In this section we allow K to be an arbitrary number field. For simplicity, the quaternion algebra $B = (\frac{a,b}{K})$ over K will be denoted by $\{a, b\}$. This is the *Hilbert symbol* of B; cf. [18, Chapter 2].

Theorem 7. *Suppose* K/F *is an extension of number fields and that* B *is a quaternion algebra over* K. *Then the following are equivalent:*

(1) B *can be realized by a Hilbert symbol* $\{a, b\}$ *for which* $a \in K$ *and* $b \in F$.

(2) B *contains a quadratic extension* $K(\sqrt{b})/K$ *for some* b *in* F.

(3) *For all finite places* ω *of* F *which are ramified in* K, *there is an element* $a(\omega)$ *of* F_ω *which is not a square in* K_ν *for all places* ν *of* K *lying above* ω *which ramify in* B.

(4) *For all finite places* ω *of* F *which are ramified in* K, *there is a quadratic extension* L *of* F_ω *which is not contained in* K_ν *for all places* ν *of* K *over* ω *which ramify in* B.

We remark that if ω is a finite place of F which is unramified in K, then a uniformizer in F_ω is a nonsquare at every place ν of K over ω. In particular, the conclusion of condition (3) will be satisfied automatically. If ω is an infinite place of K, then condition (3) also holds automatically for the following reason. If ω is complex, no place ν of K above ω can ramify in B, while if ω is real we may choose $a(\omega) = -1$.

Proof. We will show the implications $(1) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$.

To show that (1) implies (4), we may assume B is a division algebra, since otherwise no places of K ramify in B. Let B have Hilbert symbol $\{a, b\}$ with $b \in F$. Then $K_\nu(\sqrt{b})$ is a quadratic extension of K_ν inside the division algebra $B \otimes_K K_\nu$ for all places ν of K which ramify in B. Hence if ω is the place of F under ν, then $L = F_\omega(\sqrt{b})$ is a quadratic extension of F_ω which is not contained in K_ν; this proves (1) implies (4).

The equivalence of (4) and (3) is clear from the fact that each quadratic extension L of F_ω has the form $L = F_\omega(\sqrt{a(\omega)})$ for some $a(\omega) \in F_\omega^*$, and L is not contained in K_ν if and only if $a(\omega)$ is not a square in K_ν.

Suppose now that (3) holds. Let S be the union of the set of places of K over which B ramifies together with one finite place of K, which is unramified over F and does not ramify in B. Let $S(F)$ be the set of all places of F lying below places in S. By the remark following the statement of Theorem 7, we know that for each ω in $S(F)$ there is an element $a(\omega)$ of F_ω such that $a(\omega)$ is not a square in K_ν for all places $\nu \in S$ which lie over ω. Thus $K_\nu(\sqrt{a(w)})/K_\nu$ is a quadratic extension for all such ν and ω. By [10, Theorem II.1] we can find an element b of F which is ω-adically close to $a(\omega)$ for all ω in $S(F)$. Then $F_\omega(\sqrt{b})$ will equal $F_\omega(\sqrt{a(\omega)})$ for all ω in $S(F)$. Now $K_\nu(\sqrt{b})$ is the compositum of K_ν and $F_\omega(\sqrt{b}) = F_\omega(\sqrt{a(\omega)})$, so $K_\nu(\sqrt{b})$ is the quadratic extension $K_\nu(\sqrt{a(\omega)})$ of K_ν for ν in S over the place ω of F. Therefore since S is nonempty, $K(\sqrt{b})$ is a quadratic extension of K which does not split over any place ν in S. By the criteria for embedding quadratic extensions in quaternion algebras, [18, Theorems 1.2.8 and 3.3.8], $K(\sqrt{b})$ embeds into B because S contains all of the places of K over which B ramifies. This shows that (3) implies (2).

Let us now suppose that (2) holds, so that $K(\sqrt{b})/K$ is a quadratic extension in B for some $b \in F$. To show that (1) holds, we can reduce to the case in which B is a division algebra, since otherwise B has symbol $\{1, 1\}$. Conjugation by $\sqrt{b}$ has eigenvalues 1 and -1 on B, and the -1 eigenspace V is of dimension 2 over K. Let c be any nonzero element of V. Then $K(c)/K$ is a quadratic extension in B, and conjugation by $\sqrt{b}$ in B sends c to $-c$ and induces the nontrivial automorphism of $K(c)$ over K. Thus $c^2 = a$ is in K, and B has Hilbert symbol $\{a, b\}$. We have now shown (2) implies (1), which completes the proof of Theorem 7.

4.2. Here we discuss in more detail the case of quartic fields in order to construct examples which will demonstrate Theorem 1. As a corollary of Theorem 7 we have

Corollary 8. *Suppose K is a quartic extension of $\mathbf{Q}$ such that*

(i) *K has exactly one complex place,*
(ii) *the Galois closure of K over $\mathbf{Q}$ is of degree 24 over $\mathbf{Q}$, and*
(iii) *there is a finite place ν of K lying over an odd prime p such that K_ν is a biquadratic extension of $\mathbf{Q}_p$.*

Embed K into $\mathbf{C}$ by one of the two complex conjugate nonreal embeddings corresponding to the complex place of K. Then $F = K \cap \mathbf{R}$ equals $\mathbf{Q}$. Furthermore, no quaternion algebra B over K which ramifies over ν has a Hilbert symbol $\{a, b\}$ for which $b \in F$. In particular, there are infinitely many nonisomorphic quaternion division algebras B over K which ramify at each real place of K and cannot be represented by a Hilbert symbol $\{a, b\}$ with $b \in F$.

Proof. The field F is not equal to K, so F is either equal to $\mathbf{Q}$ or a real quadratic field. Let N be the Galois closure of K over $\mathbf{Q}$. Then $G = \mathrm{Gal}(N/\mathbf{Q})$ is isomorphic to the symmetric group S_4 of order 24. Since K has exactly one complex place, complex conjugation defines an odd permutation σ in G which fixes F. If F is real quadratic, then $\mathrm{Gal}(N/F)$ must be isomorphic to the alternating group A_4 in S_4. Since A_4 contains no odd permutations, this is a contradiction; hence $F = \mathbf{Q}$.

Choose $c \in \mathbf{Z}_p^*$ such that the image of c mod p generates the cyclic group $(\mathbf{Z}/p)^*$. Suppose $\gamma \in \mathbf{Q}_p^*$. Then $\gamma = p^\alpha c^\beta d$ for some $\alpha, \beta \in \mathbf{Z}$ and some $d \in 1 + p\mathbf{Z}_p$. Because p is odd, we know by Hensel's Lemma (cf. [10, Proposition II.2]) that $d = e^2$ for some $e \in 1 + p\mathbf{Z}_p$. Hence $\mathbf{Q}_p(\sqrt{p}, \sqrt{c})$ contains $\sqrt{\gamma} = \pm(\sqrt{p})^\alpha(\sqrt{c})^\beta e$. Thus $\mathbf{Q}_p(\sqrt{p}, \sqrt{c})$ contains the square roots of every element of $\mathbf{Q}_p$, and therefore the biquadratic extension K_ν of $\mathbf{Q}_p$. Because $\mathbf{Q}_p(\sqrt{p}, \sqrt{c})$ has degree at most 4 over

$\mathbf{Q}_p$, $\mathbf{Q}_p(\sqrt{p}, \sqrt{c})$ must equal K_ν. Hence K_ν contains the square roots of every element of $\mathbf{Q}_p$, so B does not fulfill condition (3) of Theorem 7 if B ramifies over ν. The statement of Corollary 8 now follows from Theorem 7 and the fact that the isomorphism class over K of a quaternion algebra is determined by its set of ramified places, which can be any finite set of noncomplex places of K which has even cardinality. q.e.d.

We note without proof some refinements of Corollary 8 which can be deduced from Theorem 7 and will not be used in what follows. First, any quaternion algebra over a number field of degree less than four over $\mathbf{Q}$ can be represented by a Hilbert symbol $\{a, b\}$ for which $b \in \mathbf{Q}$. Suppose now that K is a quartic field with exactly one complex place. Then conditions (ii) and (iii) of Corollary 8 are both necessary and sufficient for there to exist a quaternion algebra over K which has no Hilbert symbol of the form $\{a, b\}$ with $b \in K \cap \mathbf{R}$.

We now determine precisely when the quartic field K generated over $\mathbf{Q}$ by a root of a monic quartic irreducible polynomial $f(x) \in \mathbf{Z}[x]$ satisfies the conditions of Corollary 8. Write

$$f(x) = x^4 + bx^3 + cx^2 + dx + e$$

with $b, c, d, e \in \mathbf{Z}$. The roots $\theta_1, \cdots, \theta_4$ of $f(x)$ are algebraic integers. The resolvent cubic

$$h(y) = y^3 - cy^2 + (bd - 4e)y - b^2e + 4ce - d^2$$

of $f(x)$ has roots $\theta_1\theta_2 + \theta_3\theta_4$, $\theta_1\theta_3 + \theta_2\theta_4$ and $\theta_1\theta_4 + \theta_3\theta_2$ (cf. [9, pp. 51–52]; this definition differs from the one used in [7, pp. 528–529]). The disciminant of $f(x)$ is (cf. [7, p. 529])

$$\begin{aligned}\Delta = {} & -128c^2e^2 - 4b^3d^3 + 16c^4e - 4c^3d^2 - 27b^4e^2 + 18bcd^3 \\ & + 144b^2ce^2 - 192bde^2 + b^2c^2d^2 - 4b^2c^3e - 6b^2d^2e \\ & + 144cd^2e + 256e^3 - 27d^4 - 80bc^2de + 18b^3cde.\end{aligned}$$

Lemma 9. *The quartic field K generated over $\mathbf{Q}$ by a root of $f(x)$ satisfies the conditions of Corollary 8 if and only if the following are true:*

(a) *$f(x)$ has exactly two real roots.*

(b) *No root of the resolvent cubic $h(y)$ of $f(x)$ is an integer.*

(c) *There is an odd prime p such that $f(x)$ is irreducible in $\mathbf{Q}_p[x]$ and Δ is a square in $\mathbf{Q}_p^*$.*

Condition (b) *can be replaced by*

(b′) *There is a prime l such that $f(x)$ mod l has a cubic irreducible factor.*

A quartic field K with these properties ramifies over p.

Proof. The quartic $f(x)$ will have a single pair of nonreal complex conjugate roots if and only if $f(x)$ has exactly two roots. Since the complex places of K correspond to pairs of complex conjugate nonreal roots of $f(x)$, we see that condition (a) of Lemma 9 is equivalent to condition (i) of Corollary 8. We assume in what follows that (a) holds.

Let N be the Galois closure of K over $\mathbf{Q}$. The Galois group $G = \mathrm{Gal}(N/\mathbf{Q})$ is then the Galois group of $f(x)$. Hence G is a subgroup of S_4, and G is all of S_4 if and only if condition (ii) of Corollary 8 holds. Since we are assuming (a), G contains a two-cycle (complex conjugation), and G has order divisible by 4 since K is quartic. Because $f(x)$ is irreducible, G is a transitive subgroup of S_4. Hence $G = S_4$ if G contains a three-cycle and otherwise G is isomorphic to the dihedral group of order 8. Therefore [9, p. 52] shows that $h(y)$ has no root in $\mathbf{Q}$ if and only if $G = S_4$. Since the roots of $h(y)$ are algebraic integers, any root of $h(y)$ in $\mathbf{Q}$ lies in $\mathbf{Z}$. We have now shown that assuming (a) holds, condition (b) is equivalent to condition (ii) of Corollary 8.

If $G = S_4$, then by the Cebotarev density theorem, there are infinitely many rational primes l relatively prime to Δ such that the Frobenius in G of a prime over l (see [10, p. 17]) in N has order 3. For such l, $f(x)$ mod l must have a cubic irreducible factor by [10, Proposition I.25]. Hence condition (ii) of Corollary 8 implies condition (b') of Lemma 9. Conversely, suppose that l is a rational prime such that $f(x)$ mod l has a cubic irreducible factor. Since N contains the roots of $f(x)$ and these roots are algebraic integers, there is a prime ideal $\mathscr{L}$ over l in N and an element of the residue field $k(\mathscr{L})$ of $\mathscr{L}$ which has degree 3 over $\mathbf{Z}/l$. Hence 3 divides the residue field degree of $\mathscr{L}$, so G has order divisible by 3. Thus G cannot be dihedral, so $G = S_4$ and condition (ii) of Corollary 8 holds.

Suppose now that as in condition (iii) of Corollary 8, p is an odd rational prime for which there is a place ν over p such that K_ν is a biquadratic extension of $\mathbf{Q}_p$. Since K_ν is generated over $\mathbf{Q}_p$ by a root of $f(x)$ and K_ν is quartic, we conclude that $f(x)$ is irreducible in $\mathbf{Q}_p[x]$. Since unramified extensions of $\mathbf{Q}_p$ are cyclic over $\mathbf{Q}_p$, K must ramify over p. By [7, p. 529], Δ must be a square in $\mathbf{Q}_p^*$ because $\mathrm{Gal}(K_\nu/\mathbf{Q}_p)$ is noncyclic of order 4.

Conversely, let us now suppose that $f(x)$ is irreducible in $\mathbf{Q}_p[x]$ and that $\Delta \in (\mathbf{Q}_p^*)^2$. Then there is a unique place ν over p in K, and K_ν is generated over $\mathbf{Q}_p$ by a root of $f(x)$. Let N_η be the Galois closure of K_ν over $\mathbf{Q}_p$. Define $G_\eta = \mathrm{Gal}(N_\eta/\mathbf{Q}_p)$. Let $I_\eta = (G_\eta)_0$ and $P_\eta = (G_\eta)_1$

be the inertia subgroup and the wild inertia subgroup of G_η, respectively (cf. [15, Chapter IV, §1]). By [7, p. 529], $\Delta \in (\mathbf{Q}_p^*)^2$ implies that either $N_\eta = K_\nu$ is biquadratic over $\mathbf{Q}_p$ or G_η is isomorphic to A_4. Hence to complete the proof of Lemma 9, it will suffice to rule out the possibility that $G_\eta \cong A_4$. By [15, pp. 32, 70, and 75], $P_\eta \subset I_\eta$, P_η and I_η are normal subgroups of G_η, G_η/I_η is cyclic, I_η/P_η is cyclic of order prime to p, and P_η is a p-group. Since $p > 2$, we see that there are no such I_η and P_η if $G_\eta \cong A_4$; this completes the proof.

In view of Corollaries 6 and 8, the following result completes the proof of Theorem 1.

Proposition 10. *There are infinitely many nonisomorphic quartic fields K fulfilling the conditions of Corollary* 8.

Proof. Let $p > 3$ be a rational prime for which $p \equiv 3 \bmod 4$. Define $f(x) = f_p(x) = x^4 + 2x^2 + px + 1$. We will show that $f(x)$ satisfies conditions (a), (b), and (c) of Lemma 9. Lemma 9 then implies that the quartic field K generated over $\mathbf{Q}$ by a root of $f(x)$ fulfills the conditions of Corollary 8 and ramifies over p. Varying p thus produces infinitely many nonisomorphic fields as in Corollary 8.

We will first check condition (c) of Lemma 9, since this will show that $f(x)$ is irreducible in $\mathbf{Q}[x]$, as required in the hypothesis of the lemma. If $f(x) \in \mathbf{Z}_p[x]$ is not irreducible in $\mathbf{Q}_p[x]$, then $f(x) = g_1(x)g_2(x)$ for some monic nonconstant polynomials $g_1(x)$ and $g_2(x)$ in $\mathbf{Z}_p[x]$. Reducing mod p gives $f(x) \equiv (x^2+1)^2 \equiv g_1(x)g_2(x) \bmod p$. Since $p \equiv 3 \bmod 4$, x^2+1 is irreducible $\bmod p$. Hence each $g_i(x)$ must be quadratic and must have the form $g_i(x) = (x^2+1) + ph_i(x)$ for some $h_i(x) \in \mathbf{Z}_p[x]$ of degree at most 1. Therefore

$$f(x) = (x^2+1)^2 + px \equiv (x^2+1)^2 + p(x^2+1)(h_1(x) + h_2(x)) \quad \bmod p^2,$$

which we see is impossible. Hence $f(x)$ is irreducible in $\mathbf{Q}_p[x]$. The discriminant of $f(x)$ is $\Delta = p^2((16)^2 - 27p^2)$. Since p is odd, $\Delta/p^2 = (16)^2 - 27p^2$ is a square in $\mathbf{Z}_p$ because it is a square $\bmod p$. Thus Δ is a square in $\mathbf{Q}_p^*$, so we have verified condition (c) of Lemma 9.

To check (a) of Lemma 9, note that $f(-1) = 3 - p < 0$ and that $f''(x) = 12x^2 + 4$ has no real zeroes. Thus $f(x)$ has exactly two real zeroes.

Finally, to check (b) of Lemma 9, let $h(y) = y^3 - 2y^2 - 4y + 8 - p^2$ be the resolvent cubic of $f(x)$. Since $p > 3$, we have $p^2 \equiv 1 \bmod 3$.

A simple calculation shows $h(y)$ has no zeros in $\mathbf{Z}/3$, so $h(y)$ has no integral zeroes. This verifies (b) and completes the proof of Proposition 10.

Example. Let $p = 3$ and consider the polynomial $f(x) = x^4 - x^2 + 3x - 2$. The discriminant of $f(x)$ is $-2151 = -3^2 \cdot 239$, which is a square in $\mathbf{Q}_3^*$, and $f(x)$ has exactly two real roots. As in the proof of Proposition 10, one checks that $f(x)$ is irreducible in $\mathbf{Q}_3[x]$ by checking that $f(x)$ is irreducible $\bmod 9$.

Modulo 5, $f(x)$ has an irreducible cubic factor. Hence $f(x)$ satisfies conditions (a), (b'), and (c) of Lemma 9, so the quartic field K generated over $\mathbf{Q}$ by a root of $f(x)$ has the properties in Corollary 8.

By reducing $f(x) \bmod 2$ one sees that there is a unique place ν_2 in K of norm 2. Let ν be the unique place of K over $p = 3$, and let B be the quaternion algebra over K ramified at the two real places together with ν and ν_2. Let $\mathscr{O}$ be a maximal order in B. The group $\mathrm{P}\rho(\mathscr{O}^1) = \mathscr{O}^1/\{\pm 1\}$ has no elements of order two, since $\mathbf{Q}_3(\sqrt{-1}) \subseteq K_\nu$ implies $K(\sqrt{-1})$ does not embed in B over K. Hence $\mathrm{P}_\rho(\mathscr{O}^1)$ is torsion-free; see Corollary 12. The covolume of $\mathrm{P}\rho(\mathscr{O}^1)$ can be reasonably estimated by using the Euler product expansion for $\zeta_k(2)$ in the formula of [5] and calculating norms of small primes in k. With this we get a volume for the manifold $\mathbf{H}^3/\mathrm{P}\rho(\mathscr{O}^1)$ of approximately $18.75039923\ldots$.

4.3. Here we note some geometric restrictions that are forced on our manifolds by the above construction.

First observe that, as discussed in §2.3, to any finite covolume Kleinian group Γ we can associate a pair $(A\Gamma, k\Gamma)$, where $k\Gamma$ is the invariant trace-field of Γ, and $A\Gamma$ is a quaternion algebra over $k\Gamma$.

Arguing as in §4.1, Theorem 1 has the following generalization:

Theorem 11. *Let $M = \mathbf{H}^3/\Gamma$ be a finite volume hyperbolic 3-manifold. Let $B = A\Gamma$ and $K = k\Gamma$. Suppose that there is a nonreal complex embedding $K \hookrightarrow \mathbf{C}$ for which B fails to satisfy the (equivalent) conditions of Theorem 7 relative to the fields K and $F = K \cap \mathbf{R}$. Then there are manifolds commensurable with M, all of whose closed geodesics are simple.*

If M satisfies the hypothesis of Theorem 11, then M is necessarily closed. For if M is not closed, then Γ contains parabolic elements, and it is shown in [12] that

$$A\Gamma \cong M(2, k\Gamma) \cong \left(\frac{1, 1}{k\Gamma}\right).$$

This is impossible if $A\Gamma$ fulfills the hypotheses of Theorem 11.

We also have

Corollary 12. *Let $M = \mathbf{H}^3/\Gamma$ be as in Theorem* 11 *and suppose $Q = \mathbf{H}^3/\Delta$ is an orientable orbifold which is commensurable to M. Then all nontrivial elements of Δ which have finite order are of order* 2.

Proof. Suppose Δ has an element g of order n different from 2. By using Lemma 2 taking g and some conjugate of g in Δ, it is clear that the invariant quaternion algebra can be written $(\frac{a,b}{k\Gamma})$, where at least one of a and b lies in $\mathbf{Q}(\cos\pi/n)$ if n is odd or in $\mathbf{Q}(\cos 2\pi/n)$ if n is even. This contradicts the hypotheses of Theorem 11.

5. Final comments

1. The proof of Theorem 1 actually shows the existence of closed hyperbolic 3-manifolds all of whose closed geodesics are simple and moreover, those of the same complex length are disjoint. Recall the complex length of the closed geodesic g in a hyperbolic 3-manifold M is $l + i\phi$, where l is the length of g and ϕ the holonomy angle incurred in traveling once round g. It is well known that the complex length determines the trace of the corresponding loxodromic element up to sign, and vice versa.

2. In [11] and [13] those arithmetic Kleinian groups that contain nonelementary Fuchsian subgroups are classified. As part of the classification, the associated algebra of any arithmetic Kleinian group containing such a subgroup has the form $(\frac{a,b}{k})$, where a and b are both nonzero elements of $k \cap \mathbf{R}$. Notice that this is a consequence of the proof of Theorem 1 (see the calculations of Lemma 4) as one merely chooses a pair of noncommuting hyperbolic elements in the nonelementary Fuchsian subgroup (i.e., the elements have real trace with absolute value greater than 2) and their commutator also has real trace as the group is Fuchsian.

3. It is interesting to consider the construction used above to prove Theorem 1 in light of the Theorem 2 of [6]. In [6], Clozel constructs finite covers of arithmetic hyperbolic 3-manifolds with positive first Betti number. More precisely, let B be a quaternion algebra over a number field k having exactly one complex place, and suppose that B ramifies at all of the real places of k. Suppose further that if ν is a finite place of k which ramifies in B, then k_ν contains no quadratic extension of $\mathbf{Q}_p$ when p is the place of $\mathbf{Q}$ under ν. Clozel's Theorem says that under these conditions, every arithmetic hyperbolic 3-manifold in the commensurability class defined by B has a finite cover with positive first Betti number. Condition 4 of Theorem 7 implies that every quaternion algebra B satisfying Clozel's conditions has a Hilbert symbol of the form $\{a, b\}$ with $b \in \mathbf{Q}$. In particular, none of the quaternion algebras which were used in Corollary 6 to prove Theorem 1 satisfy Clozel's conditions.

Acknowledgments

The second author thanks C. Adams for some useful conversations.

References

[1] C. Adams, J. Hass & P. Scott, to appear.
[2] H. Bass, *Groups of integral representation type*, Pacific J. Math. **86** (1980) 15–51.
[3] J. S. Birman & C. Series, *An algorithm for simple curves on surfaces*, J. London Math. Soc. (2) **29** (1984) 331–342.
[4] ——, *Geodesics with multiple self-intersections and symmetries on Riemann surfaces*, Low-dimensional Topology and Kleinian Groups (D. B. A. Epstein, ed.), London Math. Soc. Lecture Note Ser., No. 112, 1986, 3–12.
[5] A. Borel, *Commensurability classes and volumes of hyperbolic* 3*-manifolds*. Ann. Scuola Norm. Sup. Pisa **8** (1981) 1–33.
[6] L. Clozel, *On the cuspidal cohomology of arithmetic subgroups of* $\mathrm{SL}(2n)$ *and the first Betti number of arithmetic* 3*-manifolds*, Duke Math. J. **55** (1987) 475–486.
[7] D. S. Dummit & R. M. Foote, *Abstract algebra*, Prentice-Hall, Englewood Cliffs, NJ, 1991.
[8] H. M. Hilden, M. T. Lozano & J-M. Montesinos, *A characterization of arithmetic subgroups of* $\mathrm{SL}(2, \mathbf{R})$ *and* $\mathrm{SL}(2, \mathbf{C})$, preprint.
[9] I. Kaplansky, *Fields and rings*, 2nd ed., Univ. of Chicago Press, Chicago, 1972.
[10] S. Lang, *Algebraic number theory*, Addison-Wesley, Reading, MA, 1970.
[11] C. Maclachlan & A. W. Reid, *Commensurability classes of arithmetic Kleinian groups and their Fuchsian subgroups*, Math. Proc. Cambridge Philos. Soc. **102** (1987) 251–257.
[12] W. D. Neumann & A. W. Reid, *Arithmetic of hyperbolic* 3*-manifolds*, TOPOLOGY '90, Proc. Research Semester in Low-Dimensional Topology at Ohio State University (B. Apanasov, W. D. Neumann, A. W. Reid, and L. Siebenmann, eds.), De Gruyter-Verlag, 1992, 273–310.
[13] A. W. Reid, Ph.D. thesis, University of Aberdeen, 1987.
[14] ——, *A note on trace-fields of Kleinian groups*, Bull. London Math. Soc. **22** (1990) 349–352.
[15] J. P. Serre, *Corps locaux*, Hermann, Paris, 1968.
[16] K. Takeuchi, *Commensurability classes of arithmetic triangle groups*, J. Fac. Sci. Univ. Tokyo Sect. IA **24** (1977) 201–212.
[17] W. P. Thurston, *The geometry and topology of* 3*-manifolds*, Lecture Notes, Princeton University, 1978.
[18] M-F. Vignéras, *Arithmétique des algèbres de quaternions*, Lecture Notes in Math., Vol. 800, Springer, Berlin, 1980.

University of Pennsylvania

University of Texas, Austin