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LOOP SPACES AS COMPLEX MANIFOLDS

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1. Introduction

Given a topological space M, its loop space consists of mappings of the circle S^1 into M. Depending on what conditions we impose on the mappings we get several loop spaces associated with M. If M has more structure than just topological, the loop spaces tend to inherit this structure: for example if M is a (Riemannian) manifold, the space of smooth loops is also a (Riemannian) manifold, albeit infinite dimensional. There is nothing surprising about this. In some cases, however, it happens that the *interaction* of the structures of M and S^1 gives rise to a structure on a loop space. For example with G a compact Lie group, the space of smooth loops in G modulo the action of G is a complex manifold (see [20]). Similarly, the manifold $\mathrm{Diff}\, S^1/S^1$ is also a complex manifold (see [3], [10]). Here $\mathrm{Diff}\, S^1$ stands for the space of smooth, orientation preserving diffeomorphisms of the circle, hence can be thought of as a space of embedded smooth loops in S^1.

More recently J. Brylinski observed that the manifold of smooth, oriented, unparametrized knots in an oriented Riemannian manifold (M, g) of dimension 3 also has a complex structure; see [4]. We shall now describe this complex structure, which naturally lives on the space of immersed rather than embedded loops (knots).

Thus, let $\mathfrak{M}$ denote the set of equivalence classes of smooth (meaning C^∞) immersions $f: S^1 \to M$. Two immersions $f_1, f_2: S^1 \to M$ are equivalent if $f_1 = f_2 \circ \varphi$, with φ an orientation preserving diffeomorphism of S^1. Elements of $\mathfrak{M}$ are called immersed loops. First we endow $\mathfrak{M}$ with a topology as follows. Fix an immersed loop $\Gamma \in \mathfrak{M}$ represented by $f: S^1 \to M$. Let $\nu \to S^1$ denote the normal bundle of f:

$$\nu = \bigcup_{t \in S^1} \{v \in T_{f(t)}M : v \perp f_* T_t S^1\},$$

and exp the (partially defined) exponential map $\nu \to M$. ν inherits a Riemannian metric and a connection from TM, and so it makes sense

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to speak of C^k-norms $|s|_k$ of its sections $s \in C^\infty(\nu)$ $(k = 1, 2, \dots)$. Given now a small positive ϵ and a positive integer k, put

$$\mathfrak{U}(f, k, \epsilon) = \{[\exp \circ s] : s \in C^\infty(\nu), |s|_k < \epsilon\}, \tag{1.1}$$

where $[g]$ denotes the class of g. By declaring the sets $\mathfrak{U}(f, k, \epsilon)$ a neighborhood basis of Γ we obtain a topology on $\mathfrak{M}$.

In fact, since $\mathfrak{U}(f, k, \epsilon)$ can be identified with open subsets in Fréchet spaces (e.g. in the space of smooth mappings $S^1 \to \mathbb{R}^2$), we obtain a Fréchet manifold structure on $\mathfrak{M}$. The tangent space $T_\Gamma\mathfrak{M}$ can be identified with $C^\infty(\nu)$. Define an endomorphism $\mathfrak{J} : T_\Gamma\mathfrak{M} \to T_\Gamma\mathfrak{M}$ by $\mathfrak{J}s_1 = s_2$ if for every $t \in S^1$, $s_1(t)$ and $s_2(t)$ are orthogonal and have the same length, and their vector product $s_1(t) \times s_2(t) \in f_* T_t S^1$ points in the direction of the orientation of Γ. Then $\mathfrak{J}^2 = -\mathrm{id}$, and so $\mathfrak{J}$ defines an almost complex structure on $\mathfrak{M}$ (see [16]). The complexified tangent bundle $\mathbb{C} \otimes T\mathfrak{M}$ splits as $T^{1,0}\mathfrak{M} \oplus T^{0,1}\mathfrak{M}$, with $T^{1,0}\mathfrak{M}$ $(T^{0,1}\mathfrak{M})$ the eigenspaces of $\mathfrak{J}$ corresponding to the eigenvalue i (resp. $-i$). Brylinski proves that $\mathfrak{J}$ is formally integrable, that is for sections $\mathfrak{X}$, $\mathfrak{Y}$ of $T^{1,0}\mathfrak{M}$ $[\mathfrak{X}, \mathfrak{Y}]$ is again a section of $T^{1,0}\mathfrak{M}$. (Strictly speaking, [4] proves integrability only on the subset of $\mathfrak{M}$ consisting of [singular] knots, but this subset being dense, integrability on $\mathfrak{M}$ follows.)

Now the question arises whether this formally integrable almost complex structure is locally integrable. As a matter of fact, local integrability can be understood in several different ways. In this context the most natural (and most restrictive) concept is that of a bona fide complex manifold. This asks for the existence of a neighborhood $\mathfrak{U}$ of an abitrary point $\Gamma \in \mathfrak{M}$ and a holomorphic C^1 diffeomorphism (biholomorphism) $F : \mathfrak{U} \to V$ with V some open subset of a Fréchet space over $\mathbb{C}$. Here a C^1 mapping F_0 of an open subset of $\mathfrak{M}$ into a Fréchet space is holomorphic if any local section $\mathfrak{X}$ of $T^{0,1}\mathfrak{M}$ annihilates it: $\mathfrak{X}F_0 = 0$.

On the other end of the scale, one can consider the following weak notion of local integrability (as in [4]): given any $\Gamma \in \mathfrak{M}$ and $\mathfrak{X} \in T_\Gamma\mathfrak{M}$, $\mathfrak{X} \neq 0$, there is a neighborhood $\mathfrak{U} \subset \mathfrak{M}$ of Γ and a holomorphic function $F : \mathfrak{U} \to \mathbb{C}$ such that $\mathfrak{X}F \neq 0$.

In finite dimensions formal integrability and the above versions of local integrability are all equivalent, the difficult implication being the content of the Newlander-Nirenberg Theorem; see [18]. On Fréchet manifolds, where even real vector fields may fail to be integrable, the Newlander-Nirenberg Theorem does not hold. This leaves the question of local integrability of the complex structure of $\mathfrak{M}$ open.

Momentarily we shall abandon $\mathfrak{M}$ for another type of loop space. Independently of Brylinski's work, in 1988–89 I observed that certain loop spaces associated with Cauchy-Riemann (CR for short) manifolds (for definitions see §§2,3) also carry formally integable almost complex structures. I hoped to use these structures to study the tangential Cauchy-Riemann equations; however, S. Baouendi pointed out that in my approach I was tacitly assuming that the structures in question are locally integrable in the strong (complex manifold) sense.

It soon turned out that this assumption is untenable and, indeed, the loop spaces I considered are generally speaking not locally biholomorphic to open subsets of Fréchet spaces. The proof, which will be given in §5 (see also §6), uses a theorem of Hans Lewy about analytic continuation of solutions of the tangential Cauchy-Riemann equations. But here is a pleasant twist! The careful reader of Lewy's paper will notice that his proof, in turn, revolves around the complex structure of CR loop spaces, even though the space of all loops never actually appears there (let alone the complex structure). We shall say a little more about that in §3.

Returning to Brylinski's loop space $\mathfrak{M}$, we will see that it can be holomorphically embedded into some CR loop space $\mathfrak{N}$. The construction of the embedding uses ideas from twistor theory, as advanced by LeBrun in [13]. That twistor theory should be of use in the study of $\mathfrak{M}$ was first observed by Drinfeld and LeBrun, who, however, restricted their attentions to spaces of real analytic loops. Anyway, we can use the embedding to deduce that $\mathfrak{M}$ is not locally biholomorphic to open subsets of Fréchet spaces, either (see §10).

On the other hand, the almost complex structure of both types of loop spaces can be shown to be locally integrable in the weak sense, when the CR manifold in question is embeddable in a complex manifold or when the Riemannian manifold is real analytic. For the loop spaces associated with CR manifolds this is quite straightforward (see §4); for $\mathfrak{M}$ it will then follow since it embeds in a CR loop space; see §9.

2. Cauchy-Riemann manifolds

Let us start with a complex manifold Q of finite dimensions $n \geq 2$, and $N \subset Q$ a piece of a smooth real hypersurface (of real dimension $2n-1$). The complexified tangent bundle $\mathbb{C} \otimes TQ$ splits as $T^{1,0}Q \oplus T^{0,1}Q$; in local coordinates $T^{1,0}Q$ ($T^{0,1}Q$) is spanned by $\partial/\partial z_j$ (resp. $\partial/\partial \bar{z}_j$).

Put

$$H^{1,0}N = T^{1,0}Q|_N \cap (\mathbb{C} \otimes TN)\,; \qquad H^{0,1}N = T^{0,1}Q|_N \cap (\mathbb{C} \otimes TN).$$

Then we have $\overline{H^{1,0}N} = H^{0,1}N$, $H^{1,0}N \cap H^{0,1}N = (0)$, and

$$(2.1) \qquad [H^{1,0}N, H^{1,0}N] \subset H^{1,0}N,$$

meaning that the Lie bracket of any two sections of $H^{1,0}N$ is again a section of $H^{1,0}N$.

In general, a smooth manifold N of odd dimension $2n-1$ equipped with two smooth subbundles $H^{1,0}N$, $H^{0,1}N \subset \mathbb{C} \otimes TN$ of rank $n-1$ is called a Cauchy-Riemann manifold if the bundles $H^{1,0}N$, $H^{0,1}N$ have the properties listed above. Of course, to define a CR manifold, it suffices to specify $H^{1,0}N$ or $H^{0,1}N$, and the other bundle is then determined. Thus hypersurfaces in complex manifolds are CR manifolds (but not every CR manifold can be embedded into a complex manifold).

A C^1 function $u : N \to \mathbb{C}$ is a CR function if

$$(2.2) \qquad Xu = 0 \qquad \text{whenever} \quad X \in H^{0,1}N.$$

Equations (2.2) are called tangential Cauchy-Riemann equations. When N is embedded in a complex manifold Q, traces of holomorphic functions defined on Q are examples of CR functions on $N \subset Q$. Lewy's theorem is a converse to this; it applies when N is not very degenerate in the following sense:

Definition 2.1. A point $p \in N$ is said to be *Levi flat* if for any smooth section X of $H^{1,0}N$, defined near p,

$$(2.3) \qquad [X, \bar{X}](p) \in H_p^{1,0}N \oplus H_p^{0,1}N.$$

We remark here that whether for a given X (2.3) holds or not depends only on (N and) the value of X at p.

This definition applies to abstractly defined CR manifolds, but let us now assume that N is a hypersurface in a complex manifold Q.

Lewy's Theorem (See [14],[22]). *Any non-Levi flat point $p \in N$ has a neighborhood basis consisting of open sets $G \subset Q$ such that $G \setminus N$ has two components and one of them, say G_1, has the following property. Given a CR function u on $G \cap N$ there is a function $\tilde{u} \in C^1(G_1 \cup (G \cap N))$, holomorphic on G_1, which agrees with u on $G \cap N$.*

3. Loop spaces associated with CR manifolds

Let N be a CR manifold as in §2, and with notation as there, put $HN = TN \cap (H^{1,0}N \oplus H^{0,1}N)$. This is a real vector bundle of rank

$2n-2$ over N. Levi flatness at $p \in N$ is equivalent to $[X, Y](p) \in HN$ whenever X, Y are local sections of HN. Thus $p \in N$ is not Levi flat if HN defines a contact structure near p; the converse is also true when $\dim N = 3$. As real vector bundles, HN, $H^{1,0}N$, and $H^{0,1}N$ are isomorphic, for example $H^{1,0}N \ni X \mapsto \operatorname{Re} X \in HN$ establishing an isomorphism between the first two. HN comes equipped with an endomorphism J that maps $\operatorname{Re} X \in HN$ to $-\operatorname{Im} X \in HN$ for any $X \in H^{1,0}N$. Obviously, $J^2 = -\mathrm{id}$. In particular, we see that the bundle HN has a canonical orientation.

By a transverse loop Γ in N we shall mean an equivalence class of smooth immersions $f: S^1 \to N$ such that for every $t \in S^1$, $f(S^1)$ is transverse to $H_{f(t)}N$. Two immersions f_1, f_2 are equivalent if $f_1 = f_2 \circ \varphi$, with $\varphi : S^1 \to S^1$ a smooth orientation preserving diffeomorphism. Somewhat abusively, we shall also denote by Γ the image $f(S^1) \subset N$ of an f representing $\Gamma = [f]$. Denote the set of transverse loops in N by $\mathfrak{N}$.

$\mathfrak{N}$ is a topological space, indeed a Fréchet manifold. To define these structures, endow N with a Riemannian metric. It will be convenient to assume the metric is complete, although this is not essential; the stucture on $\mathfrak{N}$ will be independent of the metric.

Let $\Gamma \subset N$ be a transverse loop in N, represented by $f : S^1 \to N$. Denote by exp the exponential map $f^*HN \to N$. This is an immersion when restricted to a sufficiently small neighborhood of the zero section. Again, f^*HN inherits a metric and a connection from the metric on N, so we can introduce C^k norms $|s|_k$ for sections $s \in C^\infty(f^*HN)$. Given now a small positive ϵ and a positive integer k, put

$$(3.1) \qquad \mathfrak{U}(f, k, \epsilon) = \{[\exp \circ s] : s \in C^\infty(f^*HN), \quad |s|_k < \epsilon\}.$$

By declaring the sets (3.1) a neighborhood basis of Γ we obtain a topology on $\mathfrak{N}$. Again, there is an obvious way to identify $\mathfrak{U}(f, k, \epsilon)$ with open sets in Fréchet spaces (namely $C^\infty(f^*HN) \cong$ space of smooth mappings $S^1 \to \mathbb{R}^{2n-2}$), and this identification turns $\mathfrak{N}$ into a Fréchet manifold. The tangent space $T_\Gamma\mathfrak{N}$ is simply $C^\infty(f^*HN)$.

We can also introduce an endomorphism $\mathfrak{J}$ of $T_\Gamma\mathfrak{N}$ by $\mathfrak{J}s_1 = s_2$ if $s_1, s_2 \in C^\infty(f^*HN)$ are such that $Js_1(t) = s_2(t)$ for every point $t \in S^1$. Then $\mathfrak{J}^2 = -\mathrm{id}$, so $\mathfrak{J}$ defines an almost complex structure. The $\pm i$ eigenspaces of $\mathfrak{J}$ determine a splitting $C \otimes T\mathfrak{N} = T^{1,0}\mathfrak{N} \oplus T^{0,1}\mathfrak{N}$, and so again we have the notion of holomorphicity of a C^1 mapping from an open set $\mathfrak{U} \subset \mathfrak{N}$ into a, say, Fréchet space, and also the notion of holomorphicity of a C^1 mapping of an (almost) complex manifold into $\mathfrak{N}$. For later use let us record the fact that for a loop $\Gamma = [f] \in \mathfrak{N}$ tangent vectors $\mathfrak{X} \in T^{1,0}\mathfrak{N}$ can be identified with sections $s \in C^\infty(f^*H^{1,0}N)$.

At this point we could discuss formal integrability, but we would first like to exhibit examples of holomorphic mappings into and from $\mathfrak{N}$. For this purpose we shall assume N is a hypersurface in a complex manifold Q. Suppose furthermore that with some domain $A \subset \mathbb{C}$ we are given a holomorphic immersion $g : A \to Q$ that is transverse to N, and moreover the smooth curve $\gamma = g^{-1}(N)$ is compact and connected. In this case g restricted to γ defines a transverse loop $\Gamma \in \mathfrak{N}$. (Observe that, as a simple closed curve in $\mathbb{C}$, γ has a canonical orientation.) Conversely, any real analytic transverse loop $\Gamma \in \mathfrak{N}$ comes from a holomorphic immersion of some annulus $A \subset \mathbb{C}$ as above.

Assume now that we are given a complex manifold M and a holomorphic mapping $G : M \times A \to Q$ such that $G(p_0, \zeta) = g(\zeta)$ for some $p_0 \in M$ and all $\zeta \in A$. Put $g_p(\zeta) = G(p, \zeta)$. Then for $p \in M$ close to p_0, g_p define transverse loops $\Gamma_p \subset N$ (essentially $\Gamma_p = N \cap g_p(A)$). It is straightforward to check that the map $p \mapsto \Gamma_p \in \mathfrak{N}$ is holomorphic. In particular, suppose N is a real analytic hypersurface in Q, $\Gamma = [f]$ is a real analytic transverse loop in N, and s is a real analytic section of $f^*H^{1,0}N$, which determines a tangent vector $\mathfrak{X} \in T^{1,0}_\Gamma\mathfrak{N}$. Then there is a holomorphic curve in $\mathfrak{N}$ through Γ in direction $\mathfrak{X}$; i.e., there is a neighborhood V of $0 \in \mathbb{C}$ and a holomorphic mapping $\varphi : V \to \mathfrak{N}$ with $\varphi(0) = \Gamma$ and $\varphi_*(0)(\partial/\partial\zeta) = \mathfrak{X}$. We shall see later that without the assumption of real analyticity such holomorphic curves need not exist.

Next we will produce (local) holomorphic functions on $\mathfrak{N}$. To this end fix a holomorphic $(1, 0)$ form α on Q.

Proposition 3.1. *The formula*

$$F(\Gamma) = \int_\Gamma \alpha \tag{3.2}$$

defines a holomorphic function $F : \mathfrak{N} \to \mathbb{C}$.

Proof. Let $\mathfrak{X} \in T_\Gamma\mathfrak{N}$ be given by a section $s \in C^\infty(f^*HN)$ with $f : S^1 \to N$ representing Γ. For brevity, put $\tau = f_*\partial/\partial t$, t denoting the coordinate on S^1. We claim

$$\begin{aligned} \mathfrak{X}F &= \int_{S^1} (\langle d\alpha, s(t) \wedge \tau(t)\rangle\, dt + d\langle \alpha, s(t)\rangle) \\ &= \int_{S^1} \langle d\alpha, s(t) \wedge \tau(t)\rangle\, dt. \end{aligned} \tag{3.3}$$

This would prove the proposition, since then

$$\mathfrak{J}\mathfrak{X}F = \int_{S^1} \langle d\alpha\,,\, Js(t) \wedge \tau(t)\rangle\, dt = i \int_{S^1} \langle d\alpha\,,\, s(t) \wedge \tau(t)\rangle\, dt = i\mathfrak{X}F\,,$$

(for $d\alpha$ is a $(2, 0)$ form).

As to (3.3), only the first equality needs to be proved. Assume first Γ is an *embedded* loop, and construct a vector field X on a neighborhood of $\Gamma \subset N$ such that $X_{f(t)} = s(t)$. Denote by h_θ $(|\theta| < \epsilon)$ the local flow of X. Then

$$\begin{aligned}\mathfrak{X}F &= \frac{d}{d\theta}\Big|_{\theta=0} \int_{h_\theta(\Gamma)} \alpha = \frac{d}{d\theta}\Big|_{\theta=0} \int_\Gamma h_\theta^* \alpha \\ &= \int_\Gamma L_X \alpha = \int_\Gamma (X \rfloor d\alpha + d(X \rfloor \alpha)),\end{aligned}$$

which is equivalent to (3.3). If Γ is not embedded, we can represent it as a union of embedded arcs $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$; applying the above argument to the subarcs Γ_j, we obtain (3.3) for an arbitrary immersed loop.

Remark 3.2. There is an alternative way to reduce the case of a nonembedded $\Gamma = [f]$ to embedded loops. Endow Q with a (complete) Riemannian metric, and let $\nu = f^*TQ \ominus TS^1$ denote the normal bundle of f. Using the exponential map from ν to Q, we can pull back the complex structure of Q to a neighborhood $\tilde{Q} \subset \nu$ of the zero section. We can also pull back α to get a holomorphic $(1, 0)$ form $\tilde{\alpha}$ on $\tilde{Q}$. As a result we can work on the complex manifold $\tilde{Q}$ with embedded loops. This argument also shows that even if α is a multivalued holomorphic $(1, 0)$ form, formula (3.2) (understood as $F([f]) = \int_{S^1} f_*\alpha$) still defines a holomorphic function on the open set of those loops $\Gamma = [f]$ for which $f_*\alpha$ (as a form on f^*TQ) can be made single valued.

It is clear that in Proposition 3.1 only properties of $\alpha|_N$ matter. In particular, (3.2) defines a holomorphic function F if $\alpha = u\,dv$ with u, v smooth CR functions on N. Putting together this construction of holomorphic functions on $\mathfrak{N}$ with our previous construction of holomorphic mappings into $\mathfrak{N}$, we see that for certain families $\{\gamma_\zeta\}$ of closed curves in N, parametrized by points ζ in a complex manifold M, the integrals $\int_{\gamma_\zeta} u\,dv$ depend holomorphically on ζ, if u, v are smooth CR functions. This circumstance has been known and exploited for a long time, first by Lewy, and then by others; see [1], [8], [14], [21]; the idea can be traced back to F. John; see [9].

Formula (3.2) can be thought of as an infinite dimensional Radon transformation. It follows that F will satisfy an infinite system of second order partial differential equations. Consequently (3.2) cannot describe all holomorphic functions on $\mathfrak{N}$. More holomorphic functions can be constructed by a generalization of (3.2) in the spirit of Chen's iterated integrals [5].

For this purpose choose a positive integer k and consider those k-tuples $(t_1, t_2, \cdots, t_k)$ of distinct points in S^1 that follow each other according to the orientation of S^1. The set of these k-tuples will be denoted $\Delta_k \subset (S^1)^k$; thus Δ_k is a generic orbit of the diagonal action of $\operatorname{Diff} S^1$ on $(S^1)^k$. In particular, $\Delta_1 = S^1$, $\Delta_2 = S^1 \times S^1 \setminus$ diagonal, but when $k > 2$, $(S^1)^k \setminus \Delta_k$ has nonempty interior. Let α be a holomorphic $(k, 0)$ form on $Q \times \cdots \times Q = Q^k$ (k factors). If $f : S^1 \to Q$ is any mapping, define $f_k : (S^1)^k \to Q^k$ by $f_k(t_1, \cdots, t_k) = (f(t_1), \cdots, f(t_k))$. Then we can construct a holomorphic function F on $\mathfrak{N}$ by the formula

$$F(\Gamma) = \int_{\Delta_k} f_k^* \alpha, \qquad \Gamma = [f].$$

Finite sums of functions thus constructed form an algebra; we conjecture that locally on $\mathfrak{N}$ this algebra is dense in the space of holomorphic functions.

There is another way of generalizing the construction of holomorphic functions on $\mathfrak{N}$ as given in Proposition 3.1. This is based on a remark by Brylinski to the effect that an integral like the one in (3.2) should be thought of as a holonomy. Accordingly, instead of a holomorphic $(1, 0)$ form on Q, consider a holomorphic principal G-bundle $P \to Q$ with a holomorphic connection. Here G is a complex Lie group. If $f : S^1 \to N$ is a transverse immersion, horizontal lift along f defines a holonomy $g(f) \in G$. If $\varphi \in \operatorname{Diff} S^1$, $g(f \circ \varphi) = g(f)$, so g in fact descends to a group valued function $F : \mathfrak{N} \to G$; and it is not hard to check that F is holomorphic.

As a matter of fact, this construction can be made more intrinsic to N, if, instead of a holomorphic principal bundle over Q, one takes a CR principal G-bundle over N (G still complex Lie group) with a CR connection. Holonomy again will define a holomorphic function $F : \mathfrak{N} \to G$. In this fashion one can even construct holomorphic functions on loop spaces of nonembeddable CR manifolds, at least in principle. The difficulty of course lies in finding CR bundles with CR connections over N. We hope to return to this question in a later publication.

4. Weak integrability

Again we shall assume that our CR manifold N is a hypersurface in a complex manifold Q. We shall prove that the almost complex structure

$\mathfrak{J}$ on $\mathfrak{N}$ is locally integrable in a weak sense, hence it is also formally integrable.

Theorem 4.1. *Any $\Gamma \in \mathfrak{N}$ has a neighborhood $\mathfrak{N}_0$ such that for any nonzero $\mathfrak{X} \in T_\Gamma \mathfrak{N}$ there is a holomorphic function F on $\mathfrak{N}_0$ with $\mathfrak{X}F \neq 0$.*

We will prove this theorem after some preparation.

Proposition 4.2. *Let A be a doubly connected Riemann surface, $\gamma \subset A$ a smooth Jordan curve, not null homotopic. Given a continuous 1-form φ on γ such that $\int_\gamma \varphi = 0$ and a positive ϵ, there exists a holomorphic function h on A such that $|dh - \varphi| < \epsilon$ on γ. Here the uniform norm $|\ |$ of a 1-form is measured using some fixed Riemannian metric on A.*

Proof. By the uniformization theorem we can assume that $A = \{\zeta \in \mathbb{C} : r < |\zeta| < R\}$. Write $\varphi = \varphi_1\, d\zeta$, with φ_1 a continuous function on γ. We can uniformly approximate φ_1 by Laurent polynomials of form $\psi_1(\zeta) = \sum_{-k}^{k} a_j \zeta^j$ (see, e.g., [15]). Here

$$a_{-1} = \frac{1}{2\pi i} \int_\gamma \psi_1\, d\zeta = \frac{1}{2\pi i} \int_\gamma (\psi_1 - \varphi_1)\, d\zeta,$$

which is small, so that $\psi_1 - a_{-1}\zeta^{-1}$ is also close to φ_1 on γ. Hence $h(\zeta) = \int (\psi_1(\zeta) - a_1 \zeta^{-1})\, d\zeta$ will do.

Proposition 4.3. *Any smooth Jordan curve $\gamma \subset \mathbb{C}^m$ has a neighborhood U with the following property. Given a continuous 1-form $\varphi = \sum_1^m \varphi_j\, dz_j$ along γ such that $\int_\gamma \varphi_j\, dz_j = 0$ $(j = 1, \cdots, m)$, and $\epsilon > 0$, there is a holomorphic function h on U such that $|dh - \varphi| < \epsilon$ along γ.*

Proof. By a generic linear change of coordinates we can achieve that the coordinate projections $\pi_j : \mathbb{C}^n \to \mathbb{C}$ restrict to immersions on γ. Then the fibers of π_j are transverse to γ, so we can find a neighborhood U of γ such that any connected component of $U \cap \pi_j^{-1}(\zeta)$ intersects γ at one point at most $(j = 1, \cdots, m)$. Pick now a j. Let A_j denote the leaf space of the holomorphic foliation of U determined by the fibers of π_j. This is a Riemann surface (spread over $\mathbb{C}$). After shrinking U, we can assume A_j is doubly connected. With $\sigma_j : U \to A_j$ denoting the canonical projection, $\sigma_j(\gamma) \subset A_j$ is a smooth Jordan curve, not null homotopic. By Proposition 4.2 there is a holomorphic function h_j on A_j such that dh_j approximates (the push forward to $\sigma_j(\gamma)$ of) $\varphi_j\, dz_j$. Then $h = \sum h_j \circ \sigma_j$ will do.

Proof of Theorem 4.1. Assume first that Γ is an embedded loop represented by $f : S^1 \to N$. With an arbitrary Riemannian metric on Q put $r(q) = \text{dist}^2(q, \Gamma)$ $(q \in Q)$. This is a strictly plurisubharmonic function in a neighborhood of $\Gamma \subset Q$, whence for $\epsilon > 0$ sufficiently small

$Q_1 = \{q \in Q : r(q) < \epsilon\}$ is a Stein manifold and so embeds into some Euclidean space $\mathbb{C}^m$. (For these matters, see [2], [6], [17]). Accordingly, we shall think of Q_1 and $N_1 = N \cap Q_1$ as submanifolds of $\mathbb{C}^m$. In addition we can assume that the coordinate projections restrict to immersions on Γ.

The vector $\mathfrak{X} \in T_\Gamma \mathfrak{N}$ is represented by a section $s \in C^\infty(f^*HN)$, $[f] = \Gamma$. There is an interval on which s does not vanish, which we can take to be $[\pi/2, 3\pi/2] \subset S^1$ (now we think of S^1 as $\mathbb{R} \bmod 2\pi$). Choose a real valued function $\rho \in C^\infty(S^1)$, $\operatorname{supp} \rho \subset [\pi/2, 3\pi/2]$, $\rho(\pi) = 1$. Put $\tau(t) = f_*(t)\partial/\partial t \in T\Gamma$ as before. Since for $t \in [\pi/2, 3\pi/2]$, $\tau(t), s(t) \in T_{f(t)}\mathbb{C}^m$ are independent, we can construct smooth $(1, 0)$ forms $\varphi = \sum \varphi_j dz_j$, $\psi = \sum \psi_j dz_j$ on $\mathbb{C}^m$ so that we have

$$\text{(4.1)} \quad \langle \varphi(f(t)), s(t)\rangle = \rho(t), \quad \langle \varphi(f(t)), \tau(t)\rangle = 0 \qquad (0 \le t \le 3\pi/2),$$

$$\text{(4.2)} \quad \langle \psi(f(t)), s(t)\rangle = 0, \quad \langle \psi(f(t)), \tau(t)\rangle = \rho(t) \qquad (\pi/2 \le t \le 2\pi).$$

Conditions (4.1), (4.2) do not restrict $\varphi(f(t))$, $\psi(f(t))$ for $3\pi/2 < t < 2\pi$, resp. $0 < t < \pi/2$. We can use this freedom to arrange that in addition to (4.1), (4.2) also

$$\int_\Gamma \varphi_j dz_j = \int_\Gamma \psi_j dz_j = 0 \qquad (j = 1, \cdots, m)$$

hold.

By virtue of Proposition 4.3 on Γ we can approximate φ (resp. ψ) by du (resp. dv) with u, v holomorphic functions on a fixed neighborhood $U \subset \mathbb{C}^m$ of Γ. Put $\alpha = u\,dv$. If $\gamma \in \mathfrak{N}$ is sufficiently close to Γ, then its image in N (also denoted γ) lies entirely in U, so we can define a holomorphic function F in a neighborhood $\mathfrak{N}_0$ of $\Gamma \in \mathfrak{N}$ by $F(\gamma) = \int_\gamma \alpha$, as in Proposition 3.1.

According to (3.3), we have

$$\mathfrak{X}F = \int_{S^1} (\langle du, s(t)\rangle\langle dv, \tau(t)\rangle - \langle du, \tau(t)\rangle\langle dv, s(t)\rangle)\, dt,$$

which, by appropriate choice of u, v can be made arbitrarily close to

$$\int_{S^1} (\langle \varphi(f(t)), s(t)\rangle\langle \psi(f(t)), \tau(t)\rangle - \langle \varphi(f(t)), \tau(t)\rangle\langle \psi(f(t)), s(t)\rangle)\, dt$$
$$= \int_{S^1} \rho^2(t)\, dt > 0.$$

Thus, with an appropriate choice of u, v $\mathfrak{X}F \neq 0$.

The above argument can be enhanced to take care of general immersed loops, as explained in Remark 3.2.

Corollary 4.4. *The almost complex structure $\mathfrak{J}$ on $\mathfrak{N}$ is formally integrable.*

Proof. Let $\mathfrak{Y}$, $\mathfrak{Z}$ be local sections of $T^{1,0}\mathfrak{N}$, and write $[\mathfrak{Y}, \mathfrak{Z}] = \mathfrak{W} + \mathfrak{X}$ with $\mathfrak{W}$ a section of $T^{1,0}\mathfrak{N}$, $\mathfrak{X}$ a section of $T\mathfrak{N}$. For any local holomorphic function F we have

$$\mathfrak{X}F = \overline{\mathfrak{Y}}\,\overline{\mathfrak{Z}}F - \overline{\mathfrak{Z}}\,\overline{\mathfrak{Y}}F - \overline{\mathfrak{W}}F = 0.$$

Hence $\mathfrak{X} = 0$ by Theorem 4.1.

Remark 4.5. With a little more work Theorem 4.1 can be strengthened as follows. Any $\Gamma \in \mathfrak{N}$ has a neighborhood $\mathfrak{N}_0$ such that, in addition to Theorem 4.1, for any $\Gamma_1, \Gamma_2 \in \mathfrak{N}_0$ $(\Gamma_1 \neq \Gamma_2)$ there is a holomorphic $F: \mathfrak{N}_0 \to \mathbb{C}$ with $F(\Gamma_1) \neq F(\Gamma_2)$.

Remark 4.6. We do not know if Theorem 4.1 remains true without the assumption that N can be embedded in a complex manifold. On the other hand Corollary 4.4 is true for any CR manifold N, as can be checked by direct computations (or by approximating the CR structure of N, locally near a loop, by embeddable CR structures).

5. Failure of strong integrability

Again assume N is a hypersurface in a complex manifold Q, $\dim_{\mathbb{C}} Q = n$.

Theorem 5.1. *Suppose N has a non-Levi flat point. Then $\mathfrak{N}$ is not locally biholomorphic to open sets in Fréchet spaces.*

It is not hard to show that when N is everywhere Levi flat, it is locally CR equivalent to $S^1 \times \mathbb{C}^{n-1}$ and hence $\mathfrak{N}$ is locally biholomorphic to $C^\infty(S^1 \to \mathbb{C}^{n-1})$. We remark that in this case (and only in this case, cf. Proposition 5.5) $\mathfrak{N}$ can be empty.

The proof of Theorem 5.1 is based on the following observation, which is true even when N is not embedded in a complex manifold.

Proposition 5.2. *Let $D \subset \mathbb{C}^{n-1}$ be an open set, and $\Phi : D \to \mathfrak{N}$ a holomorphic mapping (of class C^1). Suppose that for points q in some open set $U \subset N$ there is a unique $\zeta = u(q) \in D$ such that the (range of the) loop $\Phi(\zeta)$ contains q. Assume furthermore that $u : U \to \mathbb{C}^{n-1}$ is of class C^1. Then u is a CR function and a submersion.*

Proof. Let $q_0 \in U$, $\zeta_0 = u(q_0)$. After possibly shrinking D, we can construct a C^1 mapping $f: D \times S^1 \to N$ such that the mappings $f_\zeta = f(\zeta, \cdot)$ are smooth immersions and represent the loops $\Phi(\zeta)$. Put also

$f^t = f(\cdot, t)$ $(t \in S^1)$. Let t_0 be such that $f(\zeta_0, t_0) = q_0$. For (ζ, t) close to (ζ_0, t_0) we have

$$u(f(\zeta, t)) = \zeta, \tag{5.1}$$

whence $\operatorname{rk} u_* = 2n-2$, so u is a submersion. By reparametrization we can arrange that $f_*^{t_0}(T_{\zeta_0}D) \subset H_{q_0}N$. Then (5.1) implies that $f_*^{t_0}|_{T_{\zeta_0}D}$ and $u_*|_{H_{q_0}N}$ are inverses of one another. Since by virtue of the holomorphicity of Φ the former intertwines the almost complex endomorphisms J_D of D and J of N, the same holds for the latter, whence u indeed satisfies the tangential Cauchy-Riemann equations.

In the following proposition we shall think of S^1 as $\mathbb{R} \bmod 2\pi$. Henceforward we shall need that N is a hypersurface in a complex manifold Q.

Proposition 5.3. *Suppose $f : S^1 \to N$ is a smooth immersion such that for some $t_0 \in S^1$ $p = f(t_0) \in N$ is a non-Levi flat point, and $s_1, \cdots, s_{n-1} \in C^\infty(f^*H^{1,0}N)$ are such that $s_1(t_0), \cdots, s_{n-1}(t_0)$ form a basis of $H_f^{1,0}N$. f defines a loop $\Gamma \in \mathfrak{N}$ and $s_1, \cdots, s_{n-1}$ determine tangent vectors $\mathfrak{X}_1, \cdots, \mathfrak{X}_{n-1} \in T_\Gamma^{1,0}\mathfrak{N}$. If there is a holomorphic mapping $\Phi : D \to \mathfrak{N}$ of class C^1, with D some neighborhood of $0 \in \mathbb{C}^{n-1}$, such that $\Phi(0) = \Gamma$, and $\Phi_*(0)\partial/\partial\zeta_j = X_j$ $(j = 1, \cdots, n-1)$, then*

(a) *there is a Riemann surface $\Sigma \subset Q$ with C^1 boundary such that $\partial\Sigma$ contains an arc $f(t_0 - \epsilon, t_0 + \epsilon)$ of Γ, for some $\epsilon > 0$;*

(b) *there are continuous sections X_j of $T^{1,0}Q|_\Sigma$, holomorphic on $\operatorname{int}\Sigma$, such that*

$$X_j(f(t)) \equiv s_j(t) \mod T^{1,0}_{f(t)}\Sigma, \quad t_0 - \epsilon < t < t_0 + \epsilon, \tag{5.2}$$

$j = 1, 2, \cdots, n-1$.

Proof. (a) Assume f is an embedding (we already know how to lift merely immersed loops to embedded ones). The implicit function theorem (on N) implies that for q in a neighborhood $U \subset N$ of p there is a unique $\zeta = u(q)$ in a neighborhood $D_0 \subset D$ of 0 such that the loop $\Phi(\zeta)$ passes through q; furthermore $u : U \to \mathbb{C}^{n-1}$ is of class C^1. By Proposition 5.2 u is CR, and so by Lewy's theorem there are a neighborhood $G \subset Q$ of p cut in two by N, and a holomorphic mapping $\tilde{u}$ on one of the components, say G_1, C^1 on $G_1 \cup (N \cap G)$, such that $\tilde{u} = u$ on $N \cap G$. Obviously (if G is sufficiently small) $\tilde{u}$ is still a submersion, whence $\Sigma = \tilde{u}^{-1}(0) \subset Q$ is a Riemann surface with C^1 boundary. The boundary near $p \in \Sigma$ agrees with an arc $f(t_0 - \epsilon, t_0 + \epsilon)$ of the loop Γ.

(b) Fix j. For $q \in \Sigma$ the vectors in $T_q^{1,0}Q$ that $u_*(q)$ maps to $\partial/\partial\zeta_j \in T_0^{1,0}\mathbb{C}^{n-1}$ form an equivalence class in $T_q^{1,0}Q/T_q^{1,0}\Sigma$. From (5.1) one can read off that when $q = \tilde{f}(t) \in \Sigma \cap N$, $s_j(t)$ is in this equivalence class. When we let q vary on Σ, these equivalence classes define a continuous section of the normal bundle of Σ in Q, holomorphic on $\operatorname{int}\Sigma$. This section can be lifted to a continuous section X_j of $T^{1,0}Q|_\Sigma$, holomorphic on $\operatorname{int}\Sigma$. Clearly X_j has the required property.

Remark 5.4. When N is strictly pseudoconvex, a converse to Proposition 5.3 is also true. Suppose $f : S^1 \to N$ defines a loop $\Gamma \in \mathfrak{N}$ and f has a holomorphic continuation $\tilde{f}$ to $\Pi_\epsilon = \{t \in \mathbb{C} : 0 < \operatorname{Im} t < \epsilon\}$ such that $\tilde{f}(\Pi_\epsilon)$ lies on the pseudoconvex side of N. Let $\mathfrak{X}_1, \cdots, \mathfrak{X}_{n-1} \in T^{1,0}\mathfrak{N}$ be determined by sections $s_1, \cdots, s_{n-1} \in C^\infty(f^*H^{1,0}N)$ that have, mod $\tilde{f}_*T^{1,0}\Pi_\epsilon$, holomorphic continuations to Π_ϵ (as sections of $\tilde{f}^*T^{1,0}Q$). Then there is a smooth holomorphic mapping $\Phi : D \to \mathfrak{N}$ with $D \subset \mathbb{C}^{n-1}$ a neighborhood of 0 such that $\Phi(0) = \Gamma$ and $\Phi_*(0)(\partial/\partial\zeta_j) = \mathfrak{X}_j$.

Indeed (when Γ is an embedded loop), the loops $\Phi(\zeta)$ can be obtained as fibers $N \cap \tilde{u}^{-1}(\zeta)$, with $\tilde{u}$ a suitable holomorphic submersion of the pseudoconvex side of N, into $\mathbb{C}^{n-1}$, which is smooth up to N.

To prove Theorem 5.1, we shall need one more result.

Proposition 5.5. *Suppose N has a non-Levi flat point p_0. Then there is a loop $[f] = \Gamma \in \mathfrak{N}$ and sections $s_j \in C^\infty(f^*H^{1,0}N)$ defining tangent vectors $\mathfrak{X}_j \in T_\Gamma^{1,0}\mathfrak{N}$ $(j = 1, \cdots, n-1)$ such that for every $t \in S^1$, $f(t) \in N$ is not Levi flat, $s_1(t), \cdots, s_{n-1}(t)$ are independent and for no $t_0 \in S^1$ and $\epsilon > 0$ are* (a) *and* (b) *of Proposition* 5.3 *simultaneously satisfied.*

Proof. First transversely intersect N with a two-dimensional complex manifold $Q_1 \subset Q$, $p_0 \in Q_1$, so that $N \cap Q_1 = N_1$ still be non-Levi flat at p_0. Since $\dim N_1 = 3$, this in fact means p_0 is a strictly pseudoconvex point of N_1. Therefore by shrinking N and Q we can assume that $N \subset \mathbb{C}^n$,

$$N_1 = \{z \in N : \; z_3 = \cdots = z_n = 0\}$$

is a strictly *convex* hypersurface in $\mathbb{C}^2 \times \{0\}$ (see, e.g., [11]), and indeed N_1 is given by equations

$$\operatorname{Im} z_1 = C(z_1, \operatorname{Re} z_2), \qquad z_3 = \cdots = z_n = 0,$$

where C is a smooth, nonnegative, strictly convex function defined on some ball $\{(z_1, x_2) : |z_1|^2 + x_2^2 < r\}$, $C(0) = 0$, $\operatorname{grad} C(0) = 0$.

If $\delta > 0$ is sufficiently small, the line $L = \{z \in \mathbb{C}^n : z_2 = i\delta, z_3 = \cdots = z_n = 0\}$ intersects N_1, hence N, in a smooth Jordan curve γ, which is transverse to the plane field $\{H_p N_1\}$, and so also to the plane field $\{H_p N\}$. Therefore γ is the image of some embedded loop $[f] = \Gamma \in \mathfrak{N}$. The projection of γ on the z_1-axis is a smooth Jordan curve γ_1. Choose a nowhere vanishing function $h \in C^\infty(\gamma_1)$ which cannot be holomorphically continued to any one-sided neighborhood of any point of γ_1. For every $j = 1, \cdots, n-1$ there is a unique smooth function $h_j \in C^\infty(\gamma_1)$ such that the vector field Y_j defined along γ by

$$(5.3) \qquad Y_j(z_1, i\delta, 0, \cdots, 0) = h_j(z_1)\frac{\partial}{\partial z_1} + h(z_1)\frac{\partial}{\partial z_{j+1}} \qquad (z_1 \in \gamma_1)$$

becomes a section of $C^\infty(H^{1,0}N|_\gamma)$. $s_j(t) = Y_j(f(t))$ now defines $s_j \in C^\infty(f^* H^{1,0}N)$ and so determines $\mathfrak{X}_j \in T_\Gamma \mathfrak{N}$. Clearly $s_1(t), \cdots, s_{n-1}(t)$ are independent for every $t \in S^1$. We claim that with these $\mathfrak{X}_1, \cdots, \mathfrak{X}_{n-1}$ and arbitrary $t_0 \in S^1$, $\epsilon > 0$, both (a) and (b) of Proposition 5.3 cannot be satisfied.

Indeed, suppose Σ and $X_j = \sum_k h_{jk}(\partial/\partial z_k)$ are as in that Proposition. Since $L \cap \partial\Sigma$ contains an arc, it follows that $\Sigma \subset L$, hence $h_{jk} = h_{jk}(z_1, i\delta, 0, \cdots, 0)$ are holomorphic functions of z_1 for z_1 in a one-sided neighborhood of a point $w \in \gamma_1$. Since $T^{1,0}\Sigma$ is spanned by $\partial/\partial z_1$, (5.2), (5.3) imply $h_{j,j+1}(z_1, i\delta, 0, \cdots, 0) = h(z_1)$ for $z_1 \in \gamma_1$ close to w, which contradicts the impossibility of analytic continuation of h.

Proof of Theorem 5.1. Let $\Gamma, \mathfrak{X}_1, \cdots, \mathfrak{X}_{n-1}$ as in Proposition 5.5. If a neighborhood of $\Gamma \in \mathfrak{N}$ were biholomorphic to an open set in a Fréchet space, there would exist a neighborhood D of $0 \in \mathbb{C}^{n-1}$ and a holomorphic mapping $\Phi : D \to \mathfrak{N}$ such that $\Phi(0) = \Gamma$ and $\Phi_*(0)(\partial/\partial\zeta_j) = \mathfrak{X}_j$. This, however contradicts Proposition 5.3, and the choice of Γ, $\mathfrak{X}_j$.

6. The case of nonembeddable CR manifolds

Theorem 5.1 remains true for not necessarily embeddable CR manifolds, if we require a little more regularity of local biholomorphic maps. For example we have

Theorem 6.1. *Suppose an arbitrary CR manifold N has a non-Levi flat point. Then $\mathfrak{N}$ is not locally biholomorphic to open sets in Fréchet spaces via smooth biholomorphisms.*

Sketch of proof. Construct an embedded loop $\Gamma = [f] \in \mathfrak{N}$ that passes through a non-Levi flat point $p \in N$. Suppose a neighborhood $\mathfrak{U}$ of Γ is smoothly biholomorphic to an open set in a Fréchet space. Then for given tangent vectors $\mathfrak{X}_1, \cdots, \mathfrak{X}_{n-1} \in T^{1,0}\mathfrak{N}$ there is a neighborhood D of $0 \in \mathbb{C}^{n-1}$ and a smooth holomorphic mapping $\Phi : D \to \mathfrak{N}$ such that $\Phi(0) = \Gamma$, $\Phi_*(0)(\partial/\partial\zeta_j) = \mathfrak{X}_j$ $(j = 1, \cdots, n-1)$. An appropriate choice of $\mathfrak{X}_j$ would then yield a neighborhood $U \subset N$ of p and a smooth CR submersion $u = (u_1, \cdots, u_{n-1}) : U \to \mathbb{C}^{n-1}$. Now perturb Γ slightly to get a loop $\Gamma' \in \mathfrak{U}$ that still passes through p but has different direction there. We again obtain a smooth CR submersion $u' = (u'_1, \cdots, u'_{n-1})$ near p. The kernel of $u_*(p)$ (resp. $u'_*(p)$) is given by the direction of Γ (resp. Γ') in p; it follows that for some k $(u'_k, u_1, \cdots, u_{n-1})$ CR embeds a neighborhood $N_0 \subset N$ of p as a smooth hypersurface in $\mathbb{C}^n$. By Theorem 5.1 the loop space $\mathfrak{N}_0$ of N_0 is not locally biholomorphic to open sets in Fréchet spaces. Since $\mathfrak{N}_0$ is open in $\mathfrak{N}$, Theorem 6.1 follows.

7. Brylinski's loop space

In this section we shall fix some notation and then describe holomorphic curves in Brylinski's loop space $\mathfrak{M}$, that is, holomorphic mappings of some open subset of $\mathbb{C}$ into $\mathfrak{M}$.

Let $\Gamma = [f]$ be an immersed loop in an oriented three-dimensional Riemannian manifold (M, g). On the normal bundle $\nu = \nu_f$ of f, as defined in the Introduction, construct an endomorphism $J = J_f$ by putting $Jv_1 = v_2$ if $v_1, v_2 \in T_{f(t)}M$ are orthogonal and of the same length, and their vector product $v_1 \times v_2 \in T_{f(t)}M$ points in the direction of the orientation of Γ. Then $J^2 = -\mathrm{id}$. The complexified bundle $\mathbb{C} \otimes \nu$ splits as $\nu^{1,0} \oplus \nu^{0,1}$, where $\nu^{1,0}$ (resp. $\nu^{0,1}$) consists of vectors of form $v - iJv$ (resp. $v + iJv$), $v \in \nu$. Both $\nu^{1,0}$ and $\nu^{0,1}$ are complex line bundles over S^1, and $T^{1,0}_\Gamma\mathfrak{M}$, $T^{0,1}_\Gamma\mathfrak{M}$ can be identified with $C^\infty(\nu^{1,0})$, $C^\infty(\nu^{0,1})$. Further, let $\nu^* = \nu^*_f$ denote the subbundle of f^*T^*M consisting of those one-forms $\alpha \in T^*M$ that annihilate $f_*\partial/\partial t$. Clearly ν^* is the dual of ν, but it also comes with a fixed embedding $\nu^* \subset f^*T^*M$. Again we have a splitting $\mathbb{C} \otimes \nu^* = \nu^{*1,0} \oplus \nu^{*0,1}$, where forms in $\nu^{*1,0}$ (resp. $\nu^{*0,1}$) annihilate vectors in $\nu^{0,1}$ (resp. $\nu^{1,0}$). The complexification of the metric g endows $\mathbb{C} \otimes \nu$ with a complex quadratic form, again denoted g. Vectors v in $\nu^{1,0} \cup \nu^{0,1}$ are isotropic vectors, i.e., $g(v, v) = 0$; moreover these are the only isotropic vectors in $\mathbb{C} \otimes \nu$.

Similarly, the dual metric g' on T^*M endows $\mathbb{C} \otimes \nu^*$ with a complex quadratic form, and the set of isotropic forms in $\mathbb{C} \otimes \nu^*$ coincides with $\nu^{*1,0} \cup \nu^{*0,1}$. Whether an isotropic form α is in $\nu_t^{*1,0}$ or $\nu_t^{*0,1}$ depends on whether $\operatorname{Re}\alpha$, $\operatorname{Im}\alpha$ and the dual of $f_*(t)\partial/\partial t$ constitute a positively or negatively oriented basis of $T^*_{f(t)}M$.

The following result parallels Proposition 5.2.

Proposition 7.1. *Let $D \subset \mathbb{C}$ be an open set, $\Phi : D \to \mathfrak{M}$ a holomorphic mapping. Suppose that for points q in some open set $U \subset M$ there is a unique $\zeta = u(q)$ such that the (range of the) loop $\Phi(\zeta)$ passes through q. Assume furthermore that $u : U \to \mathbb{C}$ is of class C^1. For some $\zeta_0 \in D$ let $\Phi(\zeta_0)$ be represented by $f : S^1 \to M$. If $f(t_0) \in U$ then $(du)(f(t_0)) \in \nu_f^{*1,0}$. Consequently,*

$$g(\operatorname{grad} u, \operatorname{grad} u) = 0. \tag{7.1}$$

The proof is analogous to that of Proposition 5.2 and will be omitted. We shall, however, state a converse result (and leave its proof to the interested reader):

Proposition 7.2. *Let $U \subset M$ be an open set, and $u : U \to \mathbb{C}$ a smooth submersion that satisfies* (7.1). *Assume that for points ζ in some open set $D \subset \mathbb{C}$ the curves $u^{-1}(\zeta) \subset M$ are simple and closed. Then these curves $u^{-1}(\zeta)$ can be oriented so that they represent embedded loops $\Phi(\zeta)$ with $\Phi : D \to \mathfrak{M}$ a smooth holomorphic mapping.*

When the metric of M is real analytic, we can apply the Cauchy-Kovalevskaya theorem to conclude equation (7.1) has many real analytic solution. This implies there are many holomorphic curves in $\mathfrak{M}$, which are, at the same time, real analytic. A few explicit solutions can also be given. As an example, let $M = S^3 \subset \mathbb{C}^2$, with the standard round metric, and $u(z) = z_2/z_1$. u satisfies (6.1) and so defines an entire curve $\Phi : \mathbb{C} \to \mathfrak{M}$. In fact, since u is a submersion onto $\mathbb{P}_1$, we get a rational curve $\hat{\Phi} : \mathbb{P}_1 \to \mathfrak{M}$.

By composing u with stereographic projection we get a solution of (7.1) in Euclidean space $\mathbb{R}^3$,

$$u(x_1, x_2, x_3) = \frac{x_1^2 + x_2^2 + (x_3 - i)^2}{x_1 + ix_2}.$$

In Euclidean space $\mathbb{R}^3$ (7.1) reduces to

$$\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2 = 0. \tag{7.2}$$

The following construction of smooth solutions of (7.2) we owe to L. Nirenberg. Fix a smooth closed curve γ in the plane $\mathbb{R}^2_{x_1, x_2}$. Let $v(x_1, x_2)$ denote signed distance to γ. This v is smooth in a neighborhood $V \subset \mathbb{R}^2$ of γ and satisfies $(\partial v/\partial x_1)^2 + (\partial v/\partial x_2)^2 = 1$. Hence $u(x_1, x_2, x_3) = v(x_1, x_2) + ix_3$ satisfies (7.2), and u is obviously a submersion. If $\Phi : D \to \mathfrak{M}$ is the corresponding holomorphic curve, then the range of the loop $\Phi(\zeta)$ $(\zeta \in D)$ is the curve

$$\{x \in \mathbb{R}^3 : x_3 = \operatorname{Im} \zeta,\ (x_1, x_2) \in \gamma_{\operatorname{Re} \zeta}\},$$

where γ_d denotes the set of points in $\mathbb{R}^2$ at (signed) distance d to γ. A similar construction is available when M is isometric to a product.

We could now embark on an in depth study of equation (7.1), and the paucity of its solutions could be used to argue that $\mathfrak{M}$ is not locally biholomorphic to open sets in Fréchet spaces, as in the case of CR loop spaces $\mathfrak{N}$. However, we were able to do this only for flat metrics. Instead, we shall use "twistor theory" to connect $\mathfrak{M}$ with spaces of loops in CR manifolds. That twistors should play a role in the study of $\mathfrak{M}$ was first observed by Drinfeld and LeBrun (apparently, the idea, in a simpler context, goes back to Hitchin [7]). This should certainly not come as a surprise in light of equation (7.1). Indeed, if (M, g) is real analytic, we can complexify it to get a complex manifold $M^{\mathbb{C}}$ with a holomorphic quadratic form (still denoted g), and real analytic solutions u of (7.1) will extend to holomorphic solutions of the same equation (but now regarded on $M^{\mathbb{C}}$). Differentiation of (7.1) gives that the trajectories of the gradient field of u are isotropic (or: null) geodesics of the holomorphic Riemannian manifold $(M^{\mathbb{C}}, g)$ (as discussed in [12]), and level sets of u are null surfaces: surfaces on which g restricts to a degenerate quadratic form. At this point enters twistor theory, a science of isotropic geodesics, null surfaces, and such (see, e.g., [19]).

8. Twistor CR manifolds

In [13] LeBrun associates with a three-dimensional Riemannian manifold (M, g) a five-dimensional CR manifold $(N, H^{0,1}N)$ and, in view of some analogy of this construction with another one due to Penrose, calls N the twistor CR manifold of M. In this section we shall recall LeBrun's construction, and describe some properties of N.

The metric g defines a dual quadratic form on T^*M, and even on $E = \mathbb{C} \otimes T^*M$. We shall denote this latter complex quadratic form g'.

Let

$$\hat{N} = \{v \in E : g'(v, v) = 0\} \setminus \text{zero section.}$$

$\hat{N}$ is a seven-dimensional smooth manifold on which fiberwise multiplication by nonzero complex numbers $\lambda \in \mathbb{C}^*$ acts. Put $N = \hat{N}/\mathbb{C}^*$. Thus $N \subset \mathbb{P}(\mathbb{C}\otimes T^*M)$ consists of isotropic codirections. It is a five-dimensional manifold, indeed a locally trivial smooth fiber bundle over M, with fibers quadrics in $\mathbb{P}_2$, i.e., Riemann spheres. Let $\pi : N \to M$ denote the projection; for the projection $E = \mathbb{C} \otimes T^*M \to M$ we shall use the notation $\hat{\pi}$.

There is a canonical one-form $\hat{\theta}$ on E defined by

$$(8.1) \qquad \langle \hat{\theta}, v\rangle = \langle \alpha, \hat{\pi}_* v\rangle, \qquad v \in T_\alpha E.$$

In usual coordinates $x_1, x_2, x_3, p_1, p_2, p_3$ (with $p_j \in \mathbb{C}$) on $\mathbb{C} \otimes T^*M$, $\hat{\theta}$ becomes $\Sigma p_j dx_j$. $d\hat{\theta} = \hat{\omega}$ is then the (complexified) "symplectic" form $\Sigma dp_j \wedge dx_j$ on E. Let $H^{0,1}\hat{N}$ denote the kernel of $\hat{\omega}|_{\hat{N}}$, i.e.,

$$H^{0,1}\hat{N} = \{v \in \mathbb{C} \otimes T\hat{N} : \hat{\omega}(v, w) = 0 \text{ for every } w \in T\hat{N}\}.$$

This endows $\hat{N}$ with a CR structure, for $\mathrm{rk}_{\mathbb{C}} H^{0,1}\hat{N} = 3$, $H^{0,1}\hat{N} \cap \overline{H^{0,1}\hat{N}} = (0)$, and $[H^{0,1}\hat{N}, H^{0,1}\hat{N}] \subset H^{0,1}\hat{N}$ (because $\hat{\omega}$ is closed). The $\mathbb{C}^*$ action of multiplication in the fibers is a free CR action in the sense that $\mathbb{C}^* \times \hat{N} \to \hat{N}$ is a CR map, and so the CR structure of $\hat{N}$ projects down to define a CR structure $H^{0,1}N$ of N. This manifold $(N, H^{0,1}N)$ is the twistor CR manifold of M.

LeBrun proves that at no point is this CR structure Levi flat. In fact its Levi form is indefinite, of signature $(+, -)$; in particular HN defines a contact structure on N. This signature has the consequence that if N is a hypersurface in a complex manifold Q, Lewy's theorem implies that any CR function defined on a neighborhood of a $p \in N$ extends holomorphically to a neighborhood of p in Q.

We want to be a little more explicit about $(0, 1)$ vectors to N (resp. $\hat{N}$). Let $\alpha \in E$, $\hat{\pi}(\alpha) = q \in M$. Since $E_q = \mathbb{C} \otimes T_q M \subset E$ is a submanifold with a complex structure, $\mathbb{C} \otimes TE_q \subset \mathbb{C} \otimes TE$ has a splitting $T^{1,0}E_q \oplus T^{0,1}E_q$. In local coordinates as before the two subbundles are spanned by $\partial/\partial p_j$ (resp. $\partial/\partial \bar{p}_j$). From the local expressions it is clear that $T^{0,1}_\alpha E_q = \mathrm{Ker}\,\hat{\omega}$. If now $\alpha \in \hat{N}$, then complexified tangent vectors to $\hat{N}$ which are in $T^{0,1}_\alpha E_q$ (the space of such vectors will be denoted $T^{0,1}_\alpha \hat{N}_q$) are even more in $\mathrm{Ker}\,\hat{\omega}|_{\hat{N}}$. Thus we identified a rank-2 subbundle $\{T^{0,1}_\alpha \hat{N}_q\}$ of $H^{0,1}\hat{N}$; it consists of vertical $(0, 1)$ vectors.

A vector in $H^{0,1}_\alpha \hat{N} \setminus T^{0,1}_\alpha \hat{N}_q$ can be obtained as follows. Observe that $\hat{\omega}$ defines a nondegenerate pairing on $\mathbb{C} \otimes T_\alpha E / T^{0,1}_\alpha E_q$, hence also determines an isomorphism between this latter space and its dual. The dual in question consists of those covectors $c \in T^*_\alpha E$ that vanish on $T^{0,1}_\alpha E_q$. One such covector is $dg'|_\alpha$; by what has just been said, there is a vector $v \in \mathbb{C} \otimes T_\alpha E$ such that

$$\langle dg', w \rangle = \hat{\omega}(v, w), \qquad w \in T_\alpha E. \tag{8.2}$$

This v is determined mod $T^{0,1}_\alpha E_q$. Hence, we can choose v so that in addition to (8.2) $\langle dg', v \rangle = 0$, i.e.; $v \in \mathbb{C} \otimes T\hat{N}$. Comparing (8.2) with the definition of $H^{0,1}\hat{N}$ we find $v \in H^{0,1}_\alpha \hat{N}$. This v is not in $T^{0,1}_\alpha \hat{N}_q$ for we can show

$$\hat{\pi}_* v = 2a \neq 0, \tag{8.3}$$

where $a \in \mathbb{C} \otimes T_q M$ is the vector dual to $\alpha \in \mathbb{C} \otimes T^*_q M$ (under the duality determined by g). Indeed, choose normal coordinates x_1, x_2, x_3 centered at q, so that $g = \sum_1^3 (dx_j)^2 + O(|x|^2)$, whence

$$g'((x, p), (x, p)) = \sum_1^3 p_j^2 (1 + O(|x|^2)) \qquad (x \to 0);$$

hence $dg' = 2\sum_1^3 p_j \, dp_j$ at points of E_q. The dual of a covector $\alpha = \sum_1^3 p_j \, dx_j$ is $a = \sum_1^3 p_j (\partial / \partial x_j)$, and it is straightforward to check that, viewing $\sum_1^3 p_j(\partial/\partial x_j)$ as a tangent vector to E in α,

$$\hat{\omega}(v, w) = \langle dg', w \rangle = \hat{\omega}(2 \sum_1^3 p_j \frac{\partial}{\partial x_j}, w) \quad \text{for any } w \in T_\alpha E.$$

This proves (8.3). Since $H^{0,1}_\alpha \hat{N}$ is spanned by $T^{0,1}_\alpha \hat{N}_q$ and v, and $\hat{\pi}_* T^{0,1}_\alpha \hat{N}_q = \{0\}$, we have

Proposition 8.1. *For any* $\alpha \in \hat{N}$

$$\hat{\pi}_* H^{0,1}_\alpha \hat{N} = \{\lambda a : \lambda \in \mathbb{C}\},$$

where $a \in \mathbb{C} \otimes TM$ *is the dual of* $\alpha \in \mathbb{C} \otimes T^*M$. *Similarly, for any* $\beta \in N$,

$$\pi_* H^{0,1}_\beta N = b,$$

where the line $b \subset \mathbb{C} \otimes TM$ *is dual to the line* $\beta \subset \mathbb{C} \otimes T^*M$.

LeBrun also defines a so-called CR contact structure on N. We will not need the general definition of this notion; in the case at hand it is a subbundle $K \subset TN$ of rank 3. Its fiber at $\beta \in N$ is

$$K_\beta = \{v \in T_\beta N : \langle \beta, \pi_* v\rangle = 0\}, \tag{8.4}$$

where $\langle \beta, \pi_* v\rangle = 0$ means that for a covector α with codirection β we have $\langle \alpha, \pi_* v\rangle = 0$.

Definition 8.2. If an immersion $\varphi : S^1 \to N$ is everywhere tangential to the distribution $\{K_\beta\}$, we shall say φ is Legendrean. Further, a vector field along a Legendrean immersion will be called Legendrean if it can serve as the variation of a one-parameter family of Legendrean immersions.

If M is oriented, Legendrean immersions $\varphi : S^1 \to N$ can be positive, negative, or singular according to whether for nonzero covectors $\hat{\varphi}(t) \in \varphi(t)$ $\operatorname{Re}\hat{\varphi}(t)$, $\operatorname{Im}\hat{\varphi}(t)$ and the dual of $(\pi \circ \varphi)_* \partial/\partial t \in TM$ constitute a positively or negatively oriented basis of $T^*_{\pi(\varphi(t))}M$, or for some t do not form a basis at all.

Observe that a Legendrean immersion φ is singular if and only if $\pi \circ \varphi$ is not an immersion. Indeed, isotropy implies that $\operatorname{Re}\hat{\varphi}(t)$, $\operatorname{Im}\hat{\varphi}(t)$ are independent, and both are orthogonal to the dual of $(\pi\circ\varphi)_*(\partial/\partial t)$ because of (8.4). The only case where the three do not constitute a basis is that where this latter covector is zero.

9. Embedding Brylinski's loop space into a CR loop space

From now on assume the smooth Riemannian three-manifold (M, g) is also oriented. Let $(N, H^{0,1}N)$ be its twistor CR manifold. With any smooth immersion $f : S^1 \to M$ we can associate an immersion $\tilde{f} : S^1 \to N$ as follows. For any $t \in S^1$ the fiber $(\nu_f^{*0,1})_t \subset \mathbb{C}\otimes T^*_{f(t)}M$ is an isotropic line (codirection), hence a point in $N_{f(t)}$. Let $\tilde{f}(t)$ be this point. Clearly $\pi \circ \tilde{f} = f$.

Proposition 9.1. *Let $\varphi : S^1 \to N$ be a smooth mapping. Then $\varphi = \tilde{f}$ for some immersion $f : S^1 \to M$ if and only if φ is negative Legendrean in the sense of Definition* 8.2.

Proof. If $\varphi = \tilde{f}$ then any covector in $\mathbb{C} \otimes \nu_f^*$ annihilates $f_*\partial/\partial t = \pi_*\varphi_*\partial/\partial t$ by definition of ν_f^*, so φ is Legendrean by (8.4). Also, the definition of $\nu_f^{*0,1}$ implies negativity (cf. §7).

Conversely, if φ is negative Legendrean, then $\pi \circ \varphi = f$ is an immersion. Further, the isotropic codirections $\varphi(t)$ annihilate $f_*(t)\partial/\partial t$, so

$\varphi(t)$ is either $(\nu_f^{*1,0})_t$ or $(\nu_f^{*0,1})_t$. By negativity it must be the latter.

We shall call $\tilde{f}$ the (Legendrean) lift of f.

Proposition 9.2. $\pi_* H^{0,1}_{\tilde{f}(t)} N = (\nu_f^{0,1})_t$.

Proof. This follows from Proposition 8.1 along with the observation that the dual of the line $\tilde{f}(t) = (\nu_f^{*0,1})_t \subset \mathbb{C} \otimes T^*M$ is $(\nu_f^{0,1})_t$.

Proposition 9.3. *The lift $\tilde{f}$ of any smooth immersion f is transverse to the CR structure, i.e., to the distribution $\{H_\beta N\}$.*

Proof. Taking real parts in Proposition 9.2 we obtain $\pi_* H_{\tilde{f}(t)} N = (\nu_f)_t$. Since $\pi_* \tilde{f}_*(t)\partial/\partial t = f_*(t)\partial/\partial t$ is transverse to $(\nu_f)_t$, the proposition follows.

Because the operation of lifting is $\operatorname{Diff} S^1$ equivariant, it induces a smooth mapping $\Theta : \mathfrak{M} \to \mathfrak{N}$ of the space of loops in M into the space of transverse loops in N. A smooth left inverse $\Pi : \mathfrak{N} \to \mathfrak{M}$ of Θ is obtained by associating with a transverse immersion $\varphi : S^1 \to N$ its projection $\pi \circ \varphi : S^1 \to M$. (Note that transverse immersions φ into N project down to M as immersions, for the fibers of $\pi : N \to M$, i.e., the manifolds N_q $(q \in M)$, are tangential to the distribution HN, hence transverse to φ; cf. Proposition 8.1). It follows from Proposition 9.1 that the image $\Theta(\mathfrak{M})$ is the set $\mathfrak{L} \subset \mathfrak{N}$ consisting of negative Legendrean loops.

Theorem 9.4. *$\mathfrak{L}$ is a smooth submanifold of $\mathfrak{N}$, and Θ is a diffeomorphism between $\mathfrak{M}$ and $\mathfrak{L}$.*

Proof. Let $\tilde{\Gamma} \in \mathfrak{L}$ be arbitrary, and suppose $\Gamma = \Pi(\tilde{\Gamma}) \in \mathfrak{M}$ is represented by $f : S^1 \to M$. Then $\tilde{\Gamma}$ is represented by the lift $\tilde{f}$ of f. Let $C^\infty(S^1)$ denote the Fréchet space of smooth complex valued functions on S^1. We shall construct neighborhoods $\mathfrak{U} \subset \mathfrak{M}$ of Γ, $\mathfrak{V} \subset C^\infty(S^1)$ of zero, and $\mathfrak{W} \subset \mathfrak{N}$ of $\tilde{\Gamma}$, and a smooth diffeomorphism $\Delta : \mathfrak{U} \times \mathfrak{V} \to \mathfrak{W}$ such that $\Delta(\mathfrak{U} \times \{0\}) = \mathfrak{L} \cap \mathfrak{W}$. This will show $\mathfrak{L}$ is a submanifold. Since the restriction of Δ to $\mathfrak{U} \times \{0\}$ will be Θ, we will conclude that Θ is a diffeomorphism.

Let us start by pulling back the bundle $N \to M$ along f to get a Riemann sphere bundle B_f over S^1. This is the bundle of isotropic codirections along f, of which $\nu^{*1,0} = \nu_f^{*1,0}$ and $\nu^{*0,1} = \nu_f^{*0,1}$ form two disjoint sections. Fix a smooth isotropic codirection field ω in a neighborhood of $f(S^1)$ such that for every $t \in S^1$ $\omega(f(t)) \neq \nu_t^{*0,1}$, $\nu_t^{*1,0} \cdot f^*\omega$ then defines a third section of B_f, disjoint from $\nu^{*0,1}$, $\nu^{*1,0}$.

Suppose μ is a fourth section of B_f. Since on a Riemann sphere the cross ratio of four points is well defined (as long as three of the four are distinct), the cross ratio $\mu^{\#} = (\mu, f^{*}\omega, \nu^{*0,1}, \nu^{*1,0})$ defines a smooth function $\mu^{\#} : S^1 \to \mathbb{C} \cup \{\infty\}$, and the correspondence $\mu \mapsto \mu^{\#}$ is one-to-one and onto. Furthermore, $\mu^{\#}$ has finite values if $\mu \neq \nu^{*1,0}$; it is zero when $\mu = \nu^{*0,1}$.

Similarly, if $h : S^1 \to M$ is close to f, and μ is a section of the pullback bundle B_h, we can define $\mu^{\#} = (\mu, h^{*}\omega, \nu_h^{*0,1}, \nu_h^{*1,0})$.

To construct the diffeomorphism Δ, let $s \in C^{\infty}(\nu)$ determine an immersion $h = \exp \circ s : S^1 \to M$ close to f (cf. (1.1)), and let $\rho \in C^{\infty}(S^1)$ be arbitrary. There is a unique smooth section μ of B_h such that $\mu^{\#} = \rho$. If $\rho \equiv 0$, then $\mu : S^1 \to N$ becomes a negative Legendrean immersion, hence it is transverse to the CR structure. It follows that for ρ in a neighborhood $\mathfrak{V}$ of $0 \in C^{\infty}(S^1)$ μ is still transverse, hence defines a transverse loop $[\mu] \in \mathfrak{N}$. Put $\Delta([h], \rho) = [\mu]$. It is a simple exercise to check that Δ is indeed a locally defined diffeomorphism $\mathfrak{M} \times C^{\infty}(S^1) \to \mathfrak{N}$, and its restriction to $\mathfrak{M} \times \{0\}$ is Θ.

Theorem 9.5. *Θ is holomorphic.*

Proof. Let $\Gamma \in \mathfrak{M}$ be represented by $f : S^1 \to M$; then $\tilde{\Gamma} = \Theta(\Gamma)$ is represented by the Legendrean lift $\tilde{f} : S^1 \to N$ of f. Since $T_{\Gamma}^{0,1}\mathfrak{M}$ (resp. $T_{\tilde{\Gamma}}^{0,1}\mathfrak{N}$) can be identified with $C^{\infty}(\nu_f^{0,1})$ (resp. $C^{\infty}(\tilde{f}^{*}H^{0,1}N)$), Proposition 9.2 implies $\Pi_{*}T_{\tilde{\Gamma}}^{0,1}\mathfrak{N} = T_{\Gamma}^{0,1}\mathfrak{M}$, i.e., $\Pi_{*}(\tilde{\Gamma})$ intertwines the almost complex structure tensors $\mathfrak{J}_{\mathfrak{N}}$, $\mathfrak{J}_{\mathfrak{M}}$. This being true for any $\tilde{\Gamma} \in \mathfrak{L}$, it follows that $\mathfrak{L}$ is a complex submanifold (in the sense that $T\mathfrak{L}$ is $\mathfrak{J}_{\mathfrak{N}}$ invariant), and $\Pi|_{\mathfrak{L}}$ and its inverse, Θ, are holomorphic.

Corollary 9.6. *$\mathfrak{M}$ and $\mathfrak{L}$ are biholomorphic via a smooth biholomorphism. If (M, g) is real analytic, then both are locally integrable in a weak sense. For example, any $\Gamma \in \mathfrak{M}$ has a neighborhood $\mathfrak{U} \subset \mathfrak{M}$ such that for any nonzero $\mathfrak{X} \in T_{\Gamma}\mathfrak{M}$ there is a holomorphic $F : \mathfrak{U} \to \mathbb{C}$ with $\mathfrak{X}F \neq 0$.*

This clearly follows from Theorem 4.1 and the fact that the twistor CR manifold of an analytic Riemannian three-manifold is itself real analytic, and hence embeddable into a complex manifold as a hypersurface.

10. Failure of strong integrability

Theorem 10.1. *Assume that (M, g) is real analytic. Then no open subset $\mathfrak{U} \subset \mathfrak{M}$, $\mathfrak{U} \neq \emptyset$ is biholomorphic to an open subset of a Fréchet space.*

Instead of $\mathfrak{M}$ we can work with the manifold $\mathfrak{L}$ of transverse negative Legendrean loops in N. We shall need some information about the tangent bundle of $\mathfrak{L}$. In the following Propositions we do not assume (M, g) to be analytic.

Proposition 10.2. *If $[f]=\Gamma\in\mathfrak{L}$, and $T_\Gamma\mathfrak{N}$ is identified with $C^\infty(f^*HN)$, then $T_\Gamma\mathfrak{L}$ corresponds to the subset $\Lambda_f\subset C^\infty(f^*HN)$ consisting of smooth Legendrean vector fields along f (in the sense of Definition 8.2).*

Proof. If $f_\epsilon : S^1 \to N$ is a one-parameter family of transverse negative Legendrean immersions such that $f_0 = f$, and $\sigma = df_\epsilon/d\epsilon|_{\epsilon=0} \in C^\infty(f^*HN)$ represents a tangent vector in $T_\Gamma\mathfrak{L}$, then by definition σ is a Legendrean vector field. Conversely, if $\sigma\in C^\infty(f^*HN)$ is Legendrean, then its image under the projection $\pi : N \to M$ is

$$\pi_*\sigma = \tau \in C^\infty(\nu_{\pi\circ f}).$$

Let φ_ϵ be a family of immersions $S^1 \to M$ such that $\varphi_0 = \pi\circ f$, and $d\varphi_\epsilon/d\epsilon|_{\epsilon=0} = \tau$. Then the velocity vector of the Legendrean lifts $f_\epsilon = \tilde{\varphi}_\epsilon : df_\epsilon/d\epsilon|_{\epsilon=0}$ is σ, i.e., σ corresponds to a vector in $T_\Gamma\mathfrak{L}$.

Corollary 10.3. *Λ_f is a vector subspace of $C^\infty(f^*HN)$. If $\sigma\in\Lambda_f$, then $J\sigma\in\Lambda_f$. Further, complexified tangent vectors in $\mathbb{C}\otimes T_\Gamma\mathfrak{L}$ correspond to elements of $\mathbb{C}\otimes\Lambda_f$, and tangent vectors in $T_\Gamma^{1,0}\mathfrak{L}$ (resp. $T_\Gamma^{0,1}\mathfrak{L}$) correspond to vector fields of form $\sigma - iJ\sigma$ (resp. $\sigma + iJ\sigma$), $\sigma\in\Lambda_f$.*

We shall denote the space of vector fields of form $\sigma - iJ\sigma$ $(\sigma\in\Lambda_f)$ by $\Lambda_f^{1,0}$.

Proposition 10.4. *Given $[f]=\Gamma\in\mathfrak{L}$, $t_0\in S^1$, and $v\in\mathbb{C}\otimes H_{f(t_0)}N$, there is a (complexified) Legendrean vector field $\sigma\in\mathbb{C}\otimes\Lambda_f$ such that $\sigma(t_0) = v$.*

Proof. It suffices to treat the case $v\in H_{f(t_0)}N$. Issue a smooth curve p_ϵ in N with $p_0 = f(t_0)$, $dp_\epsilon/d\epsilon|_{\epsilon=0} = v$. There is a smooth family of immersions $\varphi_\epsilon : S^1 \to M$ such that $\varphi_0 = \pi\circ f$, $\varphi_\epsilon(t_0) = \pi(p_\epsilon)$, and $\varphi_{\epsilon *}(t_0)\partial/\partial t \in T_{\pi(p_\epsilon)}M$ is in the kernel of $p_\epsilon \in \mathbb{C}\otimes T^*_{\pi(p_\epsilon)}M$, for every ϵ. Denote the (Legendrean) lift of φ_ϵ by f_ϵ. Then $f_0 = f$, and $\sigma = df_\epsilon/d\epsilon|_{\epsilon=0}\in\Lambda_f$ does it.

Proof of Theorem 10.1. If (M, g) is analytic, then so is $(N, H^{0,1}N)$, so that this latter can be embedded as a hypersurface in a complex manifold Q. Given an open set $\mathfrak{U}\subset\mathfrak{M}$, $\mathfrak{U}\neq\varnothing$, fix a nonanalytic embedded loop $\Gamma_0\in\mathfrak{U}$. Then $\Theta(\Gamma_0) = \Gamma = [f]$ is nonanalytic either. Let $t_0\in S^1$ be such that for no $\epsilon > 0$ is $f(t_0-\epsilon, t_0+\epsilon)$ an analytic arc. Choose two independent vectors $v_1, v_2 \in H^{1,0}_{f(t_0)}N$, and $\sigma_1, \sigma_2\in\Lambda_f^{1,0}$ such that $\sigma_j(t_0) = v_j$ $(j = 1, 2)$; cf. Proposition 10.4. σ_1, σ_2 correspond to tangent vectors $X_1, X_2\in T_\Gamma^{1,0}\mathfrak{L}$.

If $\mathfrak{U}$ were biholomorphic to an open subset of a Fréchet space, then so would be $\mathfrak{V} = \Theta(\mathfrak{U})$, which is an open subset of $\mathfrak{L}$. In this case there would exist an open neighborhood D of $0 \in \mathbb{C}^2$ and a holomorphic mapping $\Phi : D \to \mathfrak{V} \subset \mathfrak{N}$ with $\Phi(0) = \Gamma$, $\Phi_*(0)\partial/\partial\zeta_j = X_j$ $(j = 1, 2)$. As in the proof of Proposition 5.3, we could find a neighborhood $U \subset N$ of $f(t_0) \in N$ and a CR submersion $u : U \to \mathbb{C}^2$ (of class C^1) such that $\{u = 0\}$ agrees with the portion of the (range of the) loop Γ in U. As said before, the Levi form of N has signature $(+, -)$, whence any CR function on U extends to a holomorphic function on some neighborhood $G \subset Q$ of $f(t_0)$. In particular u extends to a holomorphic submersion $\tilde{u} : G \to \mathbb{C}^2$. Then $\{\tilde{u} = 0\} = \Sigma$ is a Riemann surface whose transverse intersection with N contains an arc $f(t_0 - \epsilon, t_0 + \epsilon)$. But this is a contradiction, since $N \subset Q$ is an analytic hypersurface, so $\Sigma \cap N$ is an analytic arc. This contradiction proves the theorem.

Similarly as in §6 we can prove a slightly weaker statement for smooth Riemannian manifolds; details will be left to the reader.

Theorem 10.5. *If (M, g) is a smooth Riemannian manifold, then no open subset $\mathfrak{U} \subset \mathfrak{M}$, $\mathfrak{U} \neq \emptyset$, is smoothly biholomorphic to an open subset of a Fréchet space.*

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