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PINCHING AND CONCORDANCE THEORY

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Abstract

It is known that a complete simply connected Riemannian manifold M whose sectional curvature $\sec(M)$ satisfies $1/4 < \sec(M) \leq 1$ is homeomorphic to a sphere. Beyond that, the diffeomorphism type of M is subject to a symmetry condition formulated in this paper. Methods from concordance theory and algebraic K-theory show that many exotic spheres do not satisfy the condition.

0. Introduction

The sphere theorem of Rauch [20], Berger [1], and Klingenberg [18] states that a complete simply connected Riemannian manifold M whose sectional curvature $\sec(M)$ satisfies $1/4 < \sec(M) \leq 1$ everywhere is homeomorphic to a sphere. Grove and Shiohama [12] have obtained the same conclusion from a weaker hypothesis on the Riemannian metric (details below). Should it not be possible to keep the original hypothesis and get a stronger conclusion? In connection with this question, the notion of *Morse perfection* seems to be useful.

Let N^n be a closed smooth manifold and let $W(N)$ be the set of all smooth Morse functions on N having only two critical points (necessarily of index 0 and n). Of course, this may well be empty. In any case, $Z/2$ acts freely on $W(N)$ by $f \mapsto -f$ (for $f \in W(N)$).

0.1 Definition. The *Morse perfection* of N is $\geq k$ if there exists a smooth $Z/2$-map $q\colon S^k \to W(N)$ where $Z/2$ acts on S^k by the antipodal action. (By definition, q is *smooth* if its adjoint $q^{\#}\colon S^k \times N \to \mathbb{R}$ is smooth.)

First examples:

(i) Any N has Morse perfection ≥ -1.

(ii) The Morse perfection of N^n is ≥ 0 if and only if $W(N) \neq \varnothing$, and in this case N is homeomorphic to S^n.

(iii) The standard sphere S^n has Morse perfection $\geq n$. (Define q by $q^{\#}(z, y) = \langle z, y\rangle$ for $z, y \in S^n$, using the Euclidean scalar product in $\mathbb{R}^{n+1} \supset S^n$.)

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(iv) The Borsuk-Ulam theorem (see [22]) implies that the Morse perfection of any N^n is $\leq n$. Otherwise, take a smooth $q\colon S^k \to W(N)$ with $k > n$. Let $u\colon S^k \to N$ be given by $u(z) =$ critical point of index 0 of $q(z) \in W(N)$. Then u is continuous and satisfies $u(z) \neq u(-z)$ for all $z \in S^k$. This is impossible in view of (ii).

From now on, any *complete* Riemannian manifold is understood to be *connected.*

0.2. Theorem A. *Let M^n be a complete simply Riemannian manifold with $1/4 < \sec(M) \leq 1$ everywhere. Then M has Morse perfection n.*

This is interesting because it is possible to give nontrivial upper bounds for the Morse perfection of some homotopy spheres. Specifically, assume that $N^{4m-1} = \partial V$, where V^{4m} is a compact smooth parallellized manifold, $m \geq 2$, and N is a homotopy sphere. Then the signature of V is divisible by 8, and the following implications hold:

signature (V) $=$ odd $\cdot$ 8	$\Rightarrow$	Morse perfection of $N = 1$
signature (V) $=$ odd $\cdot$ 16	$\Rightarrow$	Morse perfection of $N \leq 5$
signature (V) $=$ odd $\cdot$ 32	$\Rightarrow$	Morse perfection of $N \leq 7$
signature (V) $=$ odd $\cdot$ 64	$\Rightarrow$	Morse perfection of $N \leq 8$
signature (V) $=$ odd $\cdot$ 128	$\Rightarrow$	Morse perfection of $N \leq 1 + 8$
. .	.	. $\leq 5 + 8$
. .	.	. $\leq 7 + 8$
. .	.	. $\leq 8 + 8$
		$\leq 1 + 16$
		.
		.
		.

In particular (from Kervaire and Milnor [17]), if $N = \partial V$ above has even order in the group of oriented homotopy spheres modulo diffeomorphism, then

$$\text{Morse perfection of } N \leq \dim(N) - 2\,.$$

Consequently, N does not admit a metric satisfying the conditions in Theorem A.

These upper bounds for Morse perfection are much harder to establish than Theorem A, but they are not really the subject of this paper. Still, §5 and §6 give some explanations, and §7 is a guide through the published parts of the proof (which is part of a larger theory, developed jointly with Bruce Williams).

The diameter sphere theorem of Grove and Shiohama [12] states the following. If M is a complete Riemannian manifold which satisfies $1 \leq \sec(M)$ everywhere, and the diameter $\operatorname{diam}(M)$ is $> \pi/2$, then M is homeomorphic to a sphere. (M is a metric space, with the geodesic distance; $\operatorname{diam}(M)$ is the maximum of the distance function on $M \times M$.) This admits a generalization which involves Morse perfection.

0.3. Theorem B. *Let M be a complete Riemannian manifold satisfying $1 \leq \sec(M)$ everywhere. Assume that there exists a map $v\colon S^k \to M$ such that*

$$\operatorname{dist}(v(x), v(-x)) > \pi/2$$

for all $x \in S^k$. Then M has Morse perfection $\geq k$.

(The case $k = 0$ is the diameter sphere theorem.)

Example. Suppose that N is a Riemannian manifold such that $1 \leq \sec(M)$ everywhere, and (if possible) that N is diffeomorphic to the Milnor homotopy sphere (i.e., $N = \partial V$ for some smooth compact parallelized V^{4m} of signature 8, where $m \geq 2$). Then, for any (continuous) map $v\colon S^2 \to N$, there exists $z \in S^2$ such that $\operatorname{dist}(v(z), v(-z)) \leq \pi/2$.

Finally, Karsten Grove suggested a "metric explanation" of Theorem A. Recall that the radius of a compact Riemannian manifold N is

$$\operatorname{rad}(N) = \min_{x \in N} \max_{y \in N} \operatorname{dist}(x, y).$$

(See also [11].)

0.4. Theorem C. *Let M^n be a complete Riemannian manifold satisfying $1 \leq \sec(M)$ everywhere, and $\operatorname{rad}(M) > \pi/2$. Then M has Morse perfection n.*

The point of view taken in this paper is not so very different from that taken by Gromoll in [9]. Gromoll's starting point is the notion of *Gromoll filtration* of a homotopy sphere. (The terminology is due to Hitchin [15], not of course to Gromoll.) This is defined as follows. Let N^n be an oriented homotopy sphere, and let $\varphi\colon S^{n-1} \to S^{n-1}$ be a clutching diffeomorphism giving N (so that N is diffeomorphic to the union of two copies of the n-disk with their boundaries identified under φ). We can assume that φ is the identity near the north pole. Deleting the north pole, and identifying its complement with $\mathbb{R}^{n-1}$, we get $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, a diffeomorphism which agrees with the identity outside a compact set. The Gromoll filtration of N, an integer between 1 and n, is given by

$$(\text{Gromoll filtration of } N) > k$$

$\Leftrightarrow \varphi$ can be arranged to respect projection to the last k coordinates:

$$\begin{array}{ccc} \mathbb{R}^{n-1} & \xrightarrow{\varphi} & \mathbb{R}^{n-1} \\ \downarrow \text{projection} & & \downarrow \text{projection} \\ \mathbb{R}^k & \xrightarrow{\text{id}} & \mathbb{R}^k \end{array}$$

commutes.
("Can be arranged" means that it is permitted to change φ by an isotopy with compact support.)

First examples:

(i) N^n has Gromoll filtration n if and only if it is diffeomorphic to S^n.

(ii) Cerf's theorem on pseudo-isotopies alias concordances [5] implies that any homotopy sphere N^n $(n \geq 7)$ has Gromoll filtration > 1.

0.5. Proposition. *The inequality*

$$\textit{Morse perfection of } N \geq \textit{(Gromoll filtration of } N) - 1$$

holds for any smooth homotopy sphere N^n, *with* $n \geq 7$.

(This is proved in §4.) In particular, any upper bound on the Morse perfection of some N implies an upper bound on the Gromoll filtration of N. For example, if N^{4m-1} is the Milnor homotopy sphere mentioned earlier, then

$$(\text{Morse perfection of } N) = 1,$$
$$(\text{Gromoll filtration of } N) \leq 2$$

by 0.5, and

$$(\text{Gromoll filtration of } N) = 2$$

by the result of Cerf quoted just above. Estimates of this type were not available to Gromoll in 1964, and this is hardly surprising because they involve much algebraic K-theory.

This paper is organized as follows. In §1, the hierarchy

$$\text{Theorem B} \Rightarrow \text{Theorem C} \Rightarrow \text{Theorem A}$$

is established. (The first implication uses critical point theory as in Grove and Shiohama [12] and Grove [10], [11], and the second uses a well-known theorem due to Klingenberg [18].) The proof of theorem B occupies §§2 and 3. Again, this uses critical point theory and comparison theory (Toponogov's theorem [23]). §4 relates Morse perfection to Gromoll filtration. §5 relates both Morse perfection and Gromoll filtration to concordance theory. Algebraic K-theory enters in the not-so-rigorous §6, which is an attempt to explain where the upper bounds for the Morse perfection of specific homotopy spheres come from. Finally, §7 points out that certain claims made in §6 can be reduced to results stated in the three long papers by Weiss and Williams [29], [30], [31].

1. The hierarchy

Let M^n be a complete Riemannian manifold satisfying $\sec(M) \geq 1$ everywhere, and $\operatorname{rad}(M) > \pi/2$. For $x \in M$, let

$$L_x = \{y \in M \mid \operatorname{dist}(x, y) > \pi/2\}.$$

By assumption, $L_x \neq \emptyset$ for all x.

1.1. Lemma. *Let $z \in L_x$ have the maximum distance from x. Then $L_x - \{z\}$ consists of points which are regular for the distance function* dist_x.

(Here dist_x is given by $\operatorname{dist}_x(y) = \operatorname{dist}(x, y)$. *Regular* and *critical* points for dist_x are defined as in Grove and Shiohama [12], Grove [10], [11].)

Proof. Fix $y \in L_x - \{z\}$. We have to show that there exists a nonzero tangent vector $w \in T_yM$ making an angle $> \pi/2$ with any minimal geodesic segment joining y to x (in M).

Let $(\gamma_1, \gamma_2, \alpha)$ be a "hinge" in M consisting of minimal (unit speed) geodesic segments $\gamma_i\colon [0, \lambda_i] \to M$ for $i = 1, 2$, where $\gamma_1(0) = z$, $\gamma_1(\lambda_1) = \gamma_2(0) = y$, $\gamma_2(\lambda_2) = x$, and α is the angle enclosed by $-\dot{\gamma}_1(\lambda_1)$ and $\dot{\gamma}_2(0)$ in T_yM. (See Cheeger and Ebin [6, Chapter 2] or Grove [11].) We have to prove that $\alpha > \pi/2$ (because then $w = -\dot{\gamma}_1(\lambda_1)$ will do). By Toponogov's theorem (the hinge version), there exists a hinge $(\overline{\gamma}_1, \overline{\gamma}_2, \alpha)$ in the standard sphere S_1^2 of constant curvature 1, such that length $\overline{\gamma}_i = \lambda_i$ for $i = 1, 2$, and such that $\operatorname{dist}(\overline{x}, \overline{z}) \geq \operatorname{dist}(x, z)$, where $\overline{x}$ and $\overline{z}$ are the hinge endpoints in S^2. We have $\lambda_2 > \pi/2$ by assumption, and $\pi \geq \lambda_1, \lambda_2$ by Myers theorem. If also $\alpha \leq \pi/2$, then we conclude $\operatorname{dist}(\overline{x}, \overline{z}) < \lambda_2$, in contradiction to

$$\operatorname{dist}(\overline{x}, \overline{z}) \geq \operatorname{dist}(x, z) \geq \lambda_2 = \operatorname{dist}(x, y).$$

1.2. Corollary. *There is only one point $z \in L_x$ at the maximum distance from x. (We shall write $z = \Delta(x)$.)*

Proof. By Berger's lemma (6.2 in Cheeger and Ebin [6], 1.6 in Grove [11]), such a z is critical for dist_x.

1.3. Lemma (see also Gromoll [9, Lemma 3]). *The map $\Delta\colon M \to M$ defined in Corollary* 1.2 *is continuous.*

Proof. Its graph is closed by definition.

Now let $E := \{(x, y) \in M \times M \mid \operatorname{dist}(x, y) > \pi/2\}$ and define $f\colon M \to E$ by $f(x) = (x, \Delta(x))$.

1.4. Lemma. *There exists a smooth vector field μ on $E - f(M)$ with the following properties:*

(i) $\mu_1(x, y) = 0$, $\mu_2(x, y) \neq 0$ *for all* $(x, y) \in E - f(M)$, *where* $\mu_1(x, y)$ *and* $\mu_2(x, y)$ *are the components of* $\mu(x, y)$ *in*

$$T_x M \times T_y M \cong T_{(x,y)} E.$$

(ii) *For* $(x, y) \in E - f(M)$, *the tangent vector* $\mu_2(x, y) \in T_y M$ *makes an angle* $> \pi/2$ *with any minimal geodesic segment from* y *to* x.

Proof. It follows from Lemma 1.1 that such a vector field can be constructed locally, in a small neighborhood of any point $(x, y) \in E - f(M)$. Use partitions of unity to construct a global example.

1.5. Corollary. *The set* $f(M)$ *is a deformation retract of* E.

Proof. Fix $(x, y) \in E - f(M)$, and let $\sigma_{(x,y)}\colon [0, c[\to E - f(M)$ be the partial solution curve of μ with $\sigma_{(x,y)}(0) = (x, y)$ and c maximal. By Lemma 1.4(i), the values $\sigma_{(x,y)}(t)$ belong to $\mathrm{pr}_1^{-1}(x)$, where $\mathrm{pr}_1\colon E - f(M) \to M$ is the projection to the first coordinate. So we can regard $\sigma_{(x,y)}$ as a solution curve of a certain vector field on

$$\mathrm{pr}_1^{-1}(x) \cong L_x - \{\Delta(x)\}.$$

By Lemma 1.4(ii), this vector field $\mu_2(x, -)$ is gradient-like (see Grove [11]) for the distance function dist_x, and it follows easily that $\sigma_{(x,y)}(t)$ converges to $\Delta(x) \in L_x$ as t approaches c. Thus the map

$$h\colon [0, +\infty] \times E \to E,$$
$$(t, (x, y)) \mapsto \begin{cases} \sigma_{(x,y)}(t) & \text{if defined,} \\ (x, \Delta(x)) & \text{otherwise} \end{cases}$$

is continuous. Here $[0, +\infty]$ is the one point compactification of $[0, +\infty[$. Note that h is the required deformation retraction.

1.6. Corollary. *The inclusion* $j\colon E \to M \times M -$diagonal *is a homotopy equivalence (and a* $Z/2$*-map).*

Proof. The composition

$$f(M) \hookrightarrow E \xrightarrow{j} M \times M - \text{diagonal}$$

is a homotopy equivalence since M is homeomorphic to S^n, by Grove-Shiohama [12].

Now choose a homeomorphism $h\colon S^n \to M^n$. Then

$$\bar{h}\colon S^n \to M \times M\text{-diagonal},$$
$$z \mapsto (h(z), h(-z))$$

is another homotopy equivalence (and a $Z/2$-map, where S^n has the antipodal involution). Using this fact, Corollary 1.6, and induction over the skeletons $S^k \subset S^n$, construct a $Z/2$-map $\overline{v}: S^n \to E$ such that $j \circ \overline{v}$ and $\overline{h}$ are $Z/2$-homotopic. Finally, let $v: S^n \to M$ be the composition

$$S^n \xrightarrow{\overline{v}} E \xrightarrow{\mathrm{pr}_1} M.$$

Then v satisfies $\mathrm{dist}(v(z), v(-z)) > \pi/2$ for all z, because $(v(z), v(-z)) = \overline{v}(z) \in E$. What is more, the degree of v is ± 1 by construction; see Remark 1.8.

We have now shown that the hypotheses in Theorem C imply those in Theorem B, with $k = n$. Therefore Theorem B implies Theorem C.

1.7. Remark. Theorems C and A would be easier to prove if the map Δ in Corollary 1.2 were involutory. But it is not.

1.8. Remark. Let $c_0: W(M^n) \to M$ be given by

$$c_0(f) = \text{critical point of index } 0 \text{ of } f.$$

When we construct a $Z/2$-map $q: S^n \to W(M^n)$ in §§2 and 3, say from the assumptions in Theorem C, then we construct it in such a way that $c_0 \circ q$ is a smooth approximation to $\Delta \cdot v$. We have just seen that v can be chosen to have degree ± 1. Also, Δ has degree $(-1)^{n+1}$ since it is fixed point free. Thus we conclude that q can be chosen in such a way that $c_0 \circ q$ has degree ± 1.

Presumably this imposes further restrictions on the diffeomorphism type of M.

To prove that Theorem C implies Theorem A, we first have to change scales. Then we only need to prove that a manifold M satisfying the hypothesis in Theorem A has radius $\geq \pi$. This follows at once from a theorem due to Klingenberg [18], stating that the exponential map $T_xM \to M$ has injectivity radius $\geq \pi$ for any $x \in M$.

2. A useful flow

Assume throughout this section that M^n is a complete Riemannian manifold such that $\sec(M) < 1$ everywhere (strict inequality). Let

$$E = \{(x, y) \in M \times M \mid \mathrm{dist}(x, y) > \pi/2\},$$
$$U = \{(x, y) \in E \mid B_{\pi/2}(x) \cup B_{\pi/2}(y) = M\},$$

where $B_{\pi/2}(x)$ is the open metric ball in M with radius $\pi/2$ and center x. Clearly U is open in E. What we do in this section is to construct a smooth action of the semigroup $\mathbb{R}_+$ (with addition) on E, such that

(a) $t \cdot U \subset U$ for all $t \geq 0$,

(b) for any compact $K \subset E$, there exists $t \geq 0$ such that $t \cdot K \subset U$,

(c) the action commutes with the standard involution on E. Of course, such an action is the positive semiflow of some smooth vector field on E.

2.1. Proposition. *There exists a smooth vector field ξ on E satisfying the following conditions*:

(i) *For any $(x, y) \in E$, the components $\xi_1(x, y) \in T_xM$ and $\xi_2(x, y) \in T_yM$ of $\xi(x, y) \in T_{(x,y)}E$ have scalar product ≤ 0 with the appropriate boundary tangent vectors (explanation just below) of any minimal geodesic segment joining x and y.*

(ii) *For $(x, y) \in E$ and $z \in M$ such that*

$$\operatorname{dist}(x, z) \geq \pi/2 \quad \textit{and} \quad \operatorname{dist}(y, z) \geq \pi/2,$$

the vectors $\xi_1(x, y) \in T_xM$ and $\xi_2(x, y) \in T_yM$ have scalar product > 0 with the appropriate boundary tangent vectors of arbitrary minimal geodesic segments joining x and z, or y and z.

(iii) *ξ is invariant under the involution $(x, y) \mapsto (y, x)$ on E.*

Terminology. The boundary tangent vectors of a smooth curve $\gamma\colon [0, \lambda] \to M$ are $\dot\gamma(0)$ and $-\dot\gamma(\lambda)$.

Proof of Proposition 2.1. Conditions (i) and (ii) make sense locally in E. They are also *convex* in the following sense: if ξ, ζ are smooth vector fields defined in an open set $V \subset E$, satisfying (i) and (ii), and $f, g\colon V \to \mathbb{R}$ are nonnegative smooth functions such that $f + g \equiv 1$, then the vector field $f\xi + g\zeta$ also satisfies (i) and (ii). Further, if ξ is defined on all of E and satisfies (i) and (ii), then $\xi + I^*\xi$ satisfies (i), (ii) and (iii). Here $I\colon E \to E$ is the involution.

Therefore Proposition 2.1 will follow (by means of partitions of unity) from the local statement: For any $(x, y) \in E$, there exist an open neighborhood V of (x, y) in E and a smooth vector field ξ on V satisfying conditions (i) and (ii), in so far as they make sense.

Fix $(x, y) \in E$. If $(x, y) \in U$, then U is an open neighborhood of (x, y) and the zero vector field ξ on U satisfies conditions (i) and (ii) insofar as they make sense. From now on, we can assume that (x, y) belongs to $E - U$. We consider three classes of minimal geodesic segments:

Class A consisting of those joining x and y;

Class B consisting of those joining x with a point z which satisfies $\operatorname{dist}(x, z) \geq \pi/2$ and $\operatorname{dist}(y, z) \geq \pi/2$;

Class C consisting of those joining y with a point z which satisfies $\operatorname{dist}(x, z) \geq \pi/2$ and $\operatorname{dist}(y, z) \geq \pi/2$.

Claim. Any member of class A makes an angle $> \pi/2$ with any member of class B (at the point x). Any member of class A makes an angle $> \pi/2$ with any member of class C (at the point y). (The angles here are angles between the appropriate boundary tangent vectors.)

This follows easily from Toponogov's comparison theorem (again the hinge version). Note that the comparison manifold should not be the standard sphere S_1^2 of constant curvature 1, but the sphere $S_{1+\varepsilon}^2$ of constant curvature $1+\varepsilon$. Here $\varepsilon > 0$ is such that $\sec(M) \geq 1+\varepsilon$ everywhere, and it exists because we assumed $\sec(M) > 1$ everywhere.

The claim implies that there exists a linear form $\varphi_x \colon T_xM \to \mathbb{R}$ such that $\varphi_x(w) > 0$ if w is the boundary tangent vector at x of any member of class B, and $\varphi_x(w) < 0$ if w is the boundary tangent vector at x of any member of class A. (This is a not-so-trivial exercise.) Similarly, there exists a linear form $\varphi_y \colon T_yM \to \mathbb{R}$ such that $\varphi_y(w) > 0$ if w is the boundary tangent vector at y of any member of class C, and $\varphi_y(w) < 0$ if w is the boundary tangent vector at y of any member of class A.

Now choose any neighborhood V^* of (x, y) in E, and a smooth vector field ξ on V^* such that

$$\xi_1(x, y) = \text{gradient of } \varphi_x,$$
$$\xi_2(x, y) = \text{gradient of } \varphi_y.$$

Then ξ satisfies conditions (i) and (ii) at the point (x, y), by construction. In fact, the scalar products mentioned in condition (i) (now evaluated only at x and y) are < 0, not just ≤ 0. It follows (from an easy continuity argument) that ξ still satisfies conditions (i) and (ii) in a smaller open neighborhood $V \subset V^*$ of (x, y). This completes the local part of the proof of Proposition 2.1, and thereby the entire proof.

2.2. Observation. *The function* $(x, y) \mapsto \operatorname{dist}(x, y)$ *on* E *is nondecreasing along solution curves of the vector field* ξ *in Proposition* 2.1.

Proof. Let $\beta \colon [0, \varepsilon] \to E$ be a (partial) solution curve of ξ, and assume $\operatorname{dist}(\beta(0)) > \operatorname{dist}(\beta(\varepsilon))$ if possible (where dist is still the distance function on $E \subset M \times M$). Then

$$\operatorname{dist}(\beta(0)) - \operatorname{dist}(\beta(\varepsilon)) > a \cdot \varepsilon$$

for some $a > 0$. By Dirichlet's "drawer" principle (Schubfachprinzip) we can find arbitrarily short subintervals $[t, t+\delta] \subset [0, \varepsilon]$ such that

$$\operatorname{dist}(\beta(t)) - \operatorname{dist}(\beta(t+\delta)) > a \cdot \delta$$

for the same $a > 0$. On the other hand, a simple variational argument together with condition (i) in Proposition 2.1 shows that we cannot find arbitrarily short subintervals of this type.

For $c > \pi/2$ let

$$E_c = \{(x, y) \in E \mid \operatorname{dist}(x, y) \geq c\}.$$

The sets E_c are compact, and E is the union of the E_c for $c > \pi/2$. By Observation 2.2, any solution curve $\beta\colon [0, \varepsilon] \to E$ of ξ will stay in E_c where $c = \operatorname{dist}(\beta(0))$. Therefore the domain of the solution curve extends from $[0, \varepsilon]$ to $[0, +\infty[$.

2.3. Corollary. *There exists a unique smooth semigroup action of* $\mathbb{R}_+$ *on* E *such that*

$$\left.\frac{d}{dt}\right|_{t=0} \cdot (x, y) = \xi(x, y) \in T_{(x,y)}E \quad (\textit{from Proposition } 2.1)$$

for all $t \geq 0$ *and all* $(x, y) \in E$.

It remains to check that the action has the properties stated at the beginning of this section. Clearly it respects the standard involution on E.

2.4. Proposition. *We have* $t \cdot U \subset U$ *for all* $t \geq 0$. *Moreover, if* $K \subset E$ *is compact, then* $t \cdot K \subset U$ *for some* $t \geq 0$.

Proof. Assume first that $t \cdot U \not\subset U$ for some $t > 0$. Choose $(x_0, y_0) \in U$ such that $t \cdot (x_0, y_0) \notin U$. Also choose $s > 0$ minimal such that $(x_s, y_s) := s \cdot (x_0, y_0)$ does not belong to U. Then there exists $z \in M$ such that $\operatorname{dist}(x_s, z) \geq \pi/2$ and $\operatorname{dist}(y_s, z) \geq \pi/2$. Apply condition Proposition 2.1(ii) to conclude that $(x_{s-\tau}, y_{s-\tau}) = (s-\tau)\cdot(x_0, y_0)$ does not belong to U for sufficiently small τ, where $0 < \tau < s$. This contradicts the minimality of s.

Next, let $K \subset E$ be compact. Without loss of generality, assume $K = E_c$ for some $c > \pi/2$; then we have $t \cdot K \subset K$ for all $t \geq 0$. Let $K^! \subset M \times M$ consist of all $((x, y), z)$ such that $\operatorname{dist}(x, z)$ and $\operatorname{dist}(y, z)$ are $\geq \pi/2$. Note that $K^!$ is compact. Using condition (ii) of Proposition 2.1 we can find, for any $((x_0, y_0), z)$ in $K^!$, a number $\varepsilon > 0$ such that the difference quotients $(\operatorname{dist}(x_0, z) - \operatorname{dist}(x_t, z))/t$, $(\operatorname{dist}(y_0, z) - \operatorname{dist}(y_t, z))/t$ are $> \varepsilon$ for all sufficiently small $t > 0$; here

$$(x_t, y_t) = t \cdot (x_0, y_0).$$

Note that if ε works for $((x_0, y_0), z) \in K^!$, then $\varepsilon/2$ works for all $((x, y), z')$ in a small neighborhood of $((x_0, y_0), z)$ in $K^!$. Since $K^!$ is compact, we can also find a $\delta > 0$ which works for all $((x, y), z)$ in $K^!$ simultaneously. In fact, this means the following. If $(x_0, y_0) \in K$ and $((x_s, y_s), z) \in K^!$, where $(x_s, y_s) := s \cdot (x_0, y_0)$, *then*

$$((x_t, y_t), z) \in K^! \quad \text{for all } t \in [0, s],$$

and the functions $t \mapsto \operatorname{dist}(x_t, z)$, $t \mapsto \operatorname{dist}(y_t, z)$ on $[0, s]$ decrease with speed faster than δ.

If we now let $s := \operatorname{diam}(M)/\delta$, then we certainly have $s \cdot K \subset U$.

3. Constructing Morse functions

In order to prove Theorem B, we should start with the following.

3.1. Assumption. The smooth connected manifold M^n admits a complete Riemannian metric g with $\sec(M) \geq 1$ everywhere, and there exists a continuous map $v\colon S^k \to M$ such that $\operatorname{dist}(v(z), v(-z)) > \pi/2$ for all $z \in S^k$. Here dist is the geodesic distance, measured with respect to g.

The conclusion should be that M has Morse perfection $\geq k$. Now the main result of the previous section shows that it is permitted to add another assumption:

3.2. Addendum. Furthermore,

$$B_{\pi/2}(v(z)) \cup B_{\pi/2}(v(-z)) = M$$

for all $z \in S^k$.

Justification. If Assumption 3.1 can be fulfilled for M, then it can also be fulfilled with $\sec(M) > 1$ everywhere (strict inequality), by scaling. We thus have the smooth action of $\mathbb{R}_+$ on E described at the beginning of §2. If $v\colon S^k \to M$ in Assumption 3.1 does not fulfill Addendum 3.2, let $\overline{v}\colon S^k \to E$ be given by $\overline{v}(z) = (v(z), v(-z))$. This respects standard involutions. Let $\overline{v}_t\colon S^k \to E$ be given by $\overline{v}_t(z) = t \cdot \overline{v}(z)$ for $t \in \mathbb{R}_+$ and $z \in S^k$. This also respects standard involutions. Let $v_t\colon S^k \to M$ be the first component of $\overline{v}_t$. For sufficiently large $t > 0$, the map v_t satisfies Addendum 3.2. (End of justification.)

As in §1, let

$$L_x = \{y \in M \mid \operatorname{dist}(x, y) > \pi/2\}$$

for $x \in M$. This time our assumptions do not imply that $L_x \neq \varnothing$ for all x. But if $L_x \neq \varnothing$ for a particular $x \in M$, then there exists a unique element $\Delta(x) \in L_x$ having maximum distance from x. This is proved as in §1. The map $x \mapsto \Delta(x)$ is continuous where defined; it is defined on an open subset of M. Continuity is also proved as in §1.

3.3. Terminology. A smooth vector field η defined on an open subset $G \subset S^k \times M$ is *vertical* if $\eta(z, x)$ belongs to

$$\{0\} \times T_xM \subset T_zS^k \times T_xM \cong T_{(z,x)}(S^k \times M)$$

for all $(z, x) \in G$. In this case the vector field η restricts to a smooth vector field η_z on $G_z = G \cap (\{z\} \times M)$, for all $z \in S^k$. This η_z may be called the *slice* of η over z. Further, G_z will usually be identified with the open subset $\{x \in M | (z, x) \in G\} \subset M$.

3.4. Terminology. Let $x \in M$, $y \in M - \{x\}$, and $w \in T_yM$. Call w *gradient-like* for dist_x if the scalar product $\langle w, \dot{\gamma}(0)\rangle$ is < 0 for any minimal geodesic segment $\gamma: [0, \lambda] \to M$ with $\gamma(0) = y$ and $\gamma(\lambda) = x$. If such a w exists in T_yM, then y is regular for dist_x.

Here is some motivation for the construction just below, which essentially completes the proof of Theorem B. We have M and $v: S^k \to M$ as in Assumption 3.1 and Addendum 3.2. In particular, for any $z \in S^k$ we have points $v(z)$, $v(-z) \in M$ such that

$$\mathrm{dist}(v(z), v(-z)) > \pi/2, \qquad B_{\pi/2}(v(z)) \cup B_{\pi/2}(v(-z)) = M.$$

Using this configuration we should somehow produce a Morse function $f_z: M \to \mathbb{R}$ with only two critical points. We should ensure that this depends continuously on z, and that $-f_z = f_{-z}$. So far, so good; but where in M should the Morse function f_z have its two critical points? At $v(z)$ and at $v(-z)$? *No*; rather, at $\Delta(v(z))$ and at $\Delta(v(-z))$. That is what experience suggests, e.g., the experience from Lemma 1.1. Further, but this is less surprising, we construct the Morse function f_z by first constructing (approximately) its gradient field. We do not lose much by regarding this as a nonzero vector field η_z on $M - \{\Delta(v(z)), \Delta(v(-z))\}$.

3.5. Proposition. *Assumptions being as in Assumption* 3.1 *and Addendum* 3.2, *let*

$$G := \{(z, x) \in S^k \times M | x \neq \Delta(v(z)), x \neq \Delta(v(-z))\}.$$

Then there exists a smooth vertical vector field η on $G \subset S^k \times M$ whose slices η_z (on $G_z \subset M$) satisfy the following:

(i) $-\eta_z = \eta_{-z}$ *on* $G_z = G_{-z}$, *for all* $z \in S^k$.

(ii) *The vector* $\eta_z(x)$ *is gradient-like for* $\mathrm{dist}_{v(-z)}$ *if* $x \in B_{\pi/2}(v(z))$.

(iii) *The vector* $\eta_z(x)$ *is gradient-like for* $\mathrm{dist}_{\Delta(v(z))}$ *if* $x \notin B_{\pi/2}(v(z))$.

Proof. Note that (i) and (ii), (iii) together imply:

(ii)* the vector $-\eta_z(x)$ is gradient-like for $\mathrm{dist}_{v(z)}$ if $x \in B_{\pi/2}(v(-z))$,

(iii)* the vector $-\eta_z(x)$ is gradient-like for $\mathrm{dist}_{\Delta(v(-z))}$ if $x \notin B_{\pi/2}(v(-z))$.

Conversely, if η is a smooth vertical vector field on G which satisfies (ii), (iii), (ii)*, and (iii)*, then $\hat{\eta}$ given by

$$\hat{\eta}_z = \eta_z - \eta_{-z} \quad \text{for } z \in S^k$$

satisfies (ii), (iii), and (i). So it is sufficient to construct an η satisfying (ii), (iii), (ii)*, and (iii)*. For the usual reasons it suffices to do this locally, in a small neighborhood of any point $(z, x) \in G \subset S^k \times M$. Fix therefore $(z, x) \in G$. There are three cases to consider.

Case 1. $x \in B_{\pi/2}(v(z))$ and $x \in B_{\pi/2}(v(-z))$. We then have to watch conditions (ii) and (ii)*. For the usual reasons, it suffices to show:

Fact 1. If $\gamma_1: [0, \lambda_1] \to M$ and $\gamma_2: [0, \lambda_2] \to M$ are minimal geodesic segments with $\gamma_1(0) = v(z)$, $\gamma_1(\lambda_1) = \gamma_2(0) = x$, $\gamma_2(\lambda_2) = v(-z)$, then the angle between the boundary tangent vectors of γ_1, γ_2 at x is $> \pi/2$.

(Thus one can find a linear form $\varphi: T_xM \to \mathbb{R}$ whose value on the appropriate boundary tangent vector of any minimal geodesic segment from $v(z)$ to x is > 0, and from $v(-z)$ to x is < 0. One can then define η in a neighborhood of (z, x) in such a way that $\eta_z(x) =$ gradient of φ.)

Fact 1 follows at once from Toponogov's comparison theorem (the hinge version) and our assumption $\operatorname{dist}(v(x), v(-x)) > \pi/2$, by assuming the angle α under investigation to be $\leq \pi/2$ and comparing the hinge $(\gamma_1, \gamma_2, \alpha)$ with a suitable hinge in S_1^2.

Case 2. $x \notin B_{\pi/2}(v(z))$ and $x \in B_{\pi/2}(v(-z))$. We now have to watch conditions (iii) and (ii)*. Moreover, if $\operatorname{dist}(x, v(z)) = \pi/2$, then any neighborhood of x will intersect $B_{\pi/2}(v(z))$, so we also need to watch condition (ii) in this very special case. As in the proof of Proposition 2.1, we consider two classes of minimal geodesic segments.

Class A: those joining x and $\Delta(v(z))$. If the distance between x and $v(z)$ is $\pi/2$, allow also those joining x and $v(-z)$.

Class B: those joining x and $v(z)$. For the usual reasons, it suffices to show:

Fact 2. Any member of class A makes an angle $> \pi/2$ with any member of class B, at x.

Proof of Fact 2. If γ_1 joins $\Delta(v(z))$ and x, and γ_2 joins x and $v(z)$, then form the hinge $(\gamma_1, \gamma_2, \alpha)$ in M and a comparison hinge $(\overline{\gamma}_1, \overline{\gamma}_2, \alpha)$ in S_1^2. Note that $\lambda_2 := \operatorname{length} \gamma_2 \geq \pi/2$ (because we are in Case 2), and the distance between the hinge endpoints in M is $> \lambda_2$ (because $\Delta(v(z))$ has maximum distance from $v(z)$). The same must be true for the comparison hinge in S_1^2, and therefore $\alpha > \pi/2$.

If $\operatorname{dist}(x, v(z)) = \pi/2$, γ_1 joins $v(-z)$ and x, and γ_2 joins x and

$v(z)$, then form again the hinge $(\gamma_1, \gamma_2, \alpha)$ and a comparison hinge $(\overline{\gamma}_1, \overline{\gamma}_2, \alpha)$ in S_1^2. Thus

$$\lambda_2 := \operatorname{length} \gamma_2 = \pi/2, \qquad \lambda_1 := \operatorname{length} \gamma_1 < \pi/2,$$

because

$$B_{\pi/2}(v(z)) \cup B_{\pi/2}(v(-z)) = M.$$

The distance between the hinge endpoints in M is still $> \pi/2$. The same must be true for the comparison hinge in S_1^2, and therefore $\alpha > \pi/2$.

Case 3. $x \in B_{\pi/2}(v(z))$ and $x \notin B_{\pi/2}(v(-z))$. This is analogous to Case 2 by interchanging z and $-z$.

3.6. Lemma. *Let $\gamma:]a, b[\to G_z \subset M$ be a maximal solution curve of the vector field η_z in Proposition* 3.5, *for some $z \in S^k$. Then*

$$\lim_{t\to a} \gamma(t) = \Delta(v(z)), \qquad \lim_{t\to b} \gamma(t) = \Delta(v(-z)),$$

where $t \in]a, b[$, a can be $-\infty$, and b can be $+\infty$. In other words: γ is injective, alias nonperiodic, and has closed image.

Proof. If $\gamma(t_0) \in B_{\pi/2}(v(z))$ for some t_0 with $a < t_0 < b$, then $\gamma(t)$ belongs to a fixed compact subset of $B_{\pi/2}(v(z))$ for all t with $t_0 \le t < b$, by Proposition 3.5(ii)* and Addendum 3.2, since Proposition 3.5(ii)* leaves only one way to escape, and that is barred by Addendum 3.2. Thus by Proposition 3.5(ii), the values $\gamma(t)$ must converge to $\Delta(v(-z))$ for $t \to b$; otherwise, $b = +\infty$ and the function $\operatorname{dist}_{v(-z)}$ has no upper bound.

On the other hand, if the solution curve never meets $B_{\pi/2}(v(z))$, then Proposition 3.5(iii) applies. This shows that the curve stays away from $\Delta(v(z))$ for $t \to b$, that $b = +\infty$, and that the function $\operatorname{dist}_{\Delta(v(z))}$ has no upper bound. This is absurd, so

$$\lim_{t\to b} \gamma(t) = \Delta(v(-z)).$$

Now apply Proposition 3.5(i) to complete the proof.

The next lemma is related to uniqueness of collars (see 13.7 in Bröcker and Jänich [3]). It will facilitate the construction of Morse functions from the vector fields η_z in Proposition 3.5.

3.7. Lemma. *Let N be a closed smooth manifold, let $Y \subset N \times [0, 1]$ be a neighbourhood of $N \times \{0\}$, and let $f: Y \to [0, 1]$ be a smooth regular function (no critical points) such that $f^{-1}(0) = N \times \{0\}$. Then there exists a smooth regular function $F: N \times [0, 1] \to [0, 1]$ which agrees with f near $N \times \{0\}$, and with the projection to $[0, 1]$ near $N \times \{1\}$.*

Proof. Choose $\varepsilon > 0$ small, and $k > 0$ large. Also choose a smooth nondecreasing function $g\colon [0, 1] \to [0, 1]$ such that $g(t) = 0$ for t near 0, and $g(t) = 1$ for $t \in [\varepsilon, 1]$. Then $\overline{f}\colon N \times [0, 1] \to [0, k]$ given by

$$\overline{f}(x, t) = (1 - g(t)) \cdot f(x, t) + g(t) \cdot kt$$

is smooth regular if ε is small and k is large. $\overline{f}$ agrees with f near $N \times \{0\}$, and with $k \cdot$ projection near $N \times \{1\}$. Compose $\overline{f}$ with a diffeomorphism from $[0, k]$ to $[0, 1]$, which is the identity near 0, and multiplication by $1/k$ near k. The composite map is F.

3.8. Addendum. *If N in Lemma 3.7 is the total space of a smooth fiber bundle $p\colon N \to B$ where B is another smooth closed manifold, and $f\colon Y \to [0, 1]$ is fiberwise regular, i.e., still regular as a smooth function on $Y \cap (p^{-1}(x) \times [0, 1])$, for all $x \in B$, then F can also be chosen fiberwise regular.*

Proof. As above.

Now, to conclude the proof of Theorem B, let us return to Assumption 3.1 and Addendum 3.2.

Step 1. *Choose a small* $\delta > 0$ such that

$$B_\delta(\Delta(v(z))) \cap B_{\pi/2}(v(z)) = \varnothing$$

for all $z \in S^k$, so that

$$B_\delta(\Delta(v(z))) \subset B_{\pi/2}(v(-z)),$$

by Addendum 3.2, and therefore

$$B_\delta(\Delta(v(z))) \cap B_\delta(\Delta(v(-z))) = \varnothing.$$

Further, δ should be smaller than the injectivity radius of $\exp\colon T_xM \to M$ at all points $x \in M$.

To see what δ does for us, fix $z \in S^k$. The vector field η_z in Proposition 3.5 is transverse to the spheres $S_\delta(\Delta(v(z)))$ and $S_\delta(\Delta(v(-z)))$, consisting of all points at distance δ from $\Delta(v(z))$ and $\Delta(v(-z))$, respectively. This follows from Proposition 3.5(iii) and 3.5(iii)* in the proof of Proposition 3.5. Further, by Lemma 3.6 any solution curve of η_z which starts at time $t = 0$ somewhere in $S_\delta(\Delta(v(z)))$ will stay (for $t \geq 0$) in $M - B_\delta(\Delta(v(z))) - B_\delta(\Delta(v(-z)))$ until, after a finite time interval, it reaches $S_\delta(\Delta(v(-z)))$.

Step 2. *Choose a smooth map $u\colon S^k \to M$ close to $\Delta \circ v\colon S^k \to M$.* (Note that Δ need not be smooth, and v was never assumed to be smooth.) This map u should be at least $\delta/2$-close to $\Delta \circ v$, and such that, for all $z \in S^k$, the vector field η_z is still transverse to $S_\delta(u(z))$

and $S_\delta(u(-z))$. Then any solution curve of η_z, starting at time $t = 0$ somewhere in $S_\delta(u(z))$, will still stay in $M - B_\delta(u(z)) - B_\delta(u(-z))$ for positive t until it reaches $S_\delta(u(-z))$ after a finite time interval.

Fix $z \in S^k$ again. We can construct a smooth regular function $f_z\colon G_z \to \mathbb{R}$ with $G_z = M - \{\Delta(v(z)), \Delta(v(-z))\}$ such that

$$f_z^{-1}(-1) = S_\delta(u(z)) \quad \text{and} \quad f_z^{-1}(+1) = S_\delta(u(-z)),$$

as follows. For $x \in G_z$ let $\gamma\colon]a, b[\to G_z$ be the unique maximal solution curve of η_z passing through x and crossing $S_\delta(u(z))$ at time $t = 0$, so $a < 0 < b$. This curve will pass through x at time t_0, say, and will cross $S_\delta(u(-z))$ at time $t_1 > 0$, say. Let

$$f_z(x) = (2t_0/t_1) - 1.$$

All we need to know about G_z in the following is that it is a neighborhood of $M - B_\delta(u(z)) - B_\delta(u(-z))$. Note that $f_{-z} = -f_z$.

Step 3. Apply Lemma 3.7 and Addendum 3.8. To this end, fix $z \in S^k$ again. Let $A_{\delta/2,\delta}(u(z)) \subset M$ be the closed annulus bounded by $S_\delta(u(z))$ and $S_{\delta/2}(u(z))$. Let $r_z\colon B_\delta(u(z)) \to \mathbb{R}$ be given by $r_z(x) = (\mathrm{dist}(x, u(z))^2 - 2$. (This is a smooth Morse function with one critical point of index 0: prove it by composing with $\exp\colon T_{u(z)}M \to M$.) Since δ is small, the map $\exp\colon T_{u(z)}M \to M$ gives rise to a canonical diffeomorphism

$$A_{\delta/2,\delta}(u(z)) \xrightarrow{\cong} S_\delta(u(z)) \times [\delta/2, \delta].$$

Using Lemma 3.7, it is now easy to find a smooth regular function

$$h_z\colon A_{\delta/2,\delta}(u(z)) \to \mathbb{R},$$

which agrees with f_z near $S_\delta(u(z))$, and with r_z near $S_{\delta/2}(u(z))$. Using the parametrized, alias fibered, version Addendum 3.8, we can even construct h_z simultaneously for all $z \in S^k$, in such a way that the map $(z, x) \mapsto h_z(x)$ is smooth where defined. Now let $f_z^!\colon M \to \mathbb{R}$ be such that

$$f_z^! \equiv \begin{cases} r_z & \text{on } B_{\delta/2}(u(z)), \\ h_z & \text{on } A_{\delta/2,\delta}(u(z)), \\ f_z & \text{on } M - B_\delta(u(z)) - B_\delta(u(-z)), \\ -h_{-z} & \text{on } A_{\delta/2,\delta}(u(-z)), \\ -r_{-z} & \text{on } B_{\delta/2}(u(z)). \end{cases}$$

Then $f_z^!$ is a smooth Morse function on M with exactly two critical points, at $u(z)$ and at $u(-z)$. The map $S^k \to W(M)$; $z \mapsto f_z^!$ is smooth and respects $Z/2$-actions (because $f_{-z}^! = -f_z^!$). This proves Theorem B.

4. Gromoll filtration and Morse perfection

The proof of Proposition 0.5 involves some manipulations with subsets of $\mathbb{R}^{n+1}$; the first and last coordinates of the points in $\mathbb{R}^{n+1}$ will play a special role. Here is some notation:

$$\begin{aligned} &S^n \subset \mathbb{R}^{n+1} \text{ is the standard sphere},\\ &V_+ = S^n - \{(0, 0, \dots, 0, -1)\},\\ &V_- = S^n - \{(0, 0, \dots, 0, +1)\},\\ &C = S^{n-1} \times]-1, +1[\subset \mathbb{R}^n \times \mathbb{R} \cong \mathbb{R}^{n+1},\\ &P = (1, 0, 0, \dots, 0) \in S^{n-1} \subset \mathbb{R}^n. \end{aligned}$$

Further, $j_+ : C \to V_+$ and $j_- : C \to V_-$ are the smooth embeddings given by one and the same formula

$$(z, t) \to ((1 - t^2)^{1/2} \cdot z, t) \in \mathbb{R}^n \times \mathbb{R},$$

for $z \in S^{n-1}$ and $t \in]-1, +1[$.

Observe that S^n is the pushout (in the category $\mathscr{D}$ of smooth manifolds and smooth maps) of the diagram $V_- \overset{j_-}{\leftarrow} C \overset{j_+}{\rightarrow} V_+$. Any diffeomorphism $\varphi: S^{n-1} \to S^{n-1}$ gives rise to a homotopy sphere Σ^n in the usual way (glueing two disks). For our purposes Σ^n is best described as the pushout of

$$V_- \overset{j_-}{\leftarrow} C \xrightarrow{j_+ \circ (\varphi \times \mathrm{id})} V_+$$

where

$$\begin{array}{ccc} \varphi \times \mathrm{id}: S^{n-1} \times]-1, +1[& \to & S^{n-1} \times]-1, +1[\\ \| & & \| \\ C & & C \end{array}$$

sends (z, t) to $(\varphi(z), t)$.

We can always assume that φ is the identity in the complement of a small disk around $P \in S^{n-1}$. If Σ^n has Gromoll filtration $> k$, then we may also assume that the diagram

$$(*) \qquad \begin{tikzcd} S^{n-1} \arrow[r, "\varphi"] \arrow[d, hook] & S^{n-1} \arrow[d, hook] \\ \mathbb{R}^n \arrow[d, "\mathrm{proj}"] & \mathbb{R}^n \arrow[d, "\mathrm{proj}"] \\ \mathbb{R}^k \arrow[r, "\mathrm{id}"] & \mathbb{R}^k \end{tikzcd}$$

commutes. Here "proj" is the projection to the *last* k coordinates:

$$(x_1, \dots, x_n) \mapsto (x_{n-k+1}, x_{n-k+2}, \dots, x_n).$$

Suppose now that $\lambda\colon \mathbb{R}^{n+1} \to \mathbb{R}$ is a linear map of norm 1 which vanishes on $\mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$ (the "first $n-k$ coordinates"). Then with our assumptions including commutativity of $(*)$, the diagram

$$\begin{tikzcd} C \arrow[rr, "j_+\circ(\varphi\times\mathrm{id})"] \arrow[ddd, "j_-"] & & V_+ \arrow[d, hook] \\ & & \mathbb{R}^{n+1} \arrow[dd, "\lambda"] \\ & & \\ V_- \arrow[r, hook] & \mathbb{R}^{n+1} \arrow[r, "\lambda"] & \mathbb{R} \end{tikzcd}$$

commutes. By the pushout description of Σ^n, this fact gives rise to a smooth map $\lambda_!\colon \Sigma^n \to \mathbb{R}$. Since the critical points of $\lambda_!$ on the patches $V_+, V_- \subset \Sigma^n$ are all nondegenerate of index 0 and n respectively, we see that $\lambda_!$ is a smooth Morse function having critical points of index 0 and n only. Using 7.4 in Milnor [19], it is easily seen that $\lambda_!$ has just two critical points. Therefore the rule $\lambda \mapsto \lambda_!$ is a map from the unit sphere in $(\mathbb{R}^{n+1}/\mathbb{R}^{n-k})^*$ to the space $W(\Sigma^n)$ of Morse functions with just two critical points. Since the map is smooth, and respects standard $Z/2$-actions, we conclude:

$$\text{Morse perfection of } \Sigma^n \geq k$$

by the assumption

$$\text{Gromoll filtration of } \Sigma^n > k.$$

Hence Proposition 0.5 is proved.

5. Some concordances

A nonstandard definition of concordance spaces, very convenient here, is given as follows. Let N^n be a smooth compact manifold. The smooth concordance space $\mathscr{C}(N)$ is the space of all smooth *regular* functions (no critical points)

$$g: N \times [-1, +1] \to [-1, +1]$$

which agree with the projection to $[-1, +1]$ on an infinitesimal neighborhood of $\partial(N \times [-1, +1])$. (The "infinitesimal neighborhood" means that g should have the same values and the same higher derivatives as the projection, at any point of $\partial(N \times [-1, +1])$.) Equip $\mathscr{C}(N)$ with the C^{∞}-topology (see Chapter 2 of Hirsch [14]).

The standard involution on $\mathscr{C}(N)$ sends g to $\overline{g}$, with

$$\overline{g}(x, t) = -g(x, -t) \quad \text{for } x \in N \text{ and } t \in [-1, +1].$$

5.1. Description. The construction below provides, for fixed $n \geq 2$,

(i) a fiber bundle $p: E \to S^{n-2}$ with fibers weakly homotopy equivalent to the concordance space $\mathscr{C}(D^{n-2})$;

(ii) an involution on E, covering the antipodal involution on S^{n-2};

(iii) for any diffeomorphism $\varphi: D^{n-1} \to D^{n-1}$ which is the identity near $\partial D^{n-1} = S^{n-2}$, a section $\omega(\varphi): S^{n-2} \to E$ of p which respects standard involutions.

Item (i). Let E consist of all pairs (x, f) where $x \in S^{n-2}$ and f is a smooth regular function on D^{n-1}, which agrees with the linear form $\lambda_x: z \mapsto \langle z, x\rangle$ on an infinitesimal neighborhood of S^{n-2}. Topologize E as a subspace of $S^{n-2} \times C^{\infty}(D^{n-1})$, where C^{∞} denotes spaces of smooth functions with the C^{∞}-topology. Define the bundle projection p by $p(x, f) = x$.

Item (ii). The standard involution on E sends (x, f) to $(-x, -f)$.

Item (iii). Define the section $\omega(\varphi)$ by

$$\omega(\varphi)(x) = (x, \lambda_x \circ \varphi) \in p^{-1}(x),$$

with λ_x as above.

This construction already appears in Weiss [28], in French and in a paper which contains too many misprints. Note that the section $\omega(\varphi)$ in Item (iii) can also be regarded as a section of a certain fiber bundle over $\mathbb{R}P^{n-2}$, whose total space is the quotient of E by $Z/2$. From this point of view, $\omega(\varphi)$ tells us how much φ fails to respect the various orthogonal projections $D^{n-1} \subset \mathbb{R}^{n-1} \to L$ to one-dimensional linear subspaces L of $\mathbb{R}^{n-1}$. In this connection, note also that $p: E \to S^{n-2}$ has a *trivial* section respecting standard involutions. This is $\omega(\text{id})$, given by $x \mapsto (x, \lambda_x)$. Finally, note that for $n \geq 7$ an isotopy class of diffeomorphisms $\varphi: D^{n-1} \to D^{n-1}$ which agree with the identity near S^{n-2} is worth as much as an oriented diffeomorphism class of homotopy spheres $\sum^n$. We can therefore write $\omega(\Sigma^n)$, meaning $\omega(\varphi)$, a well-defined homotopy class of $Z/2$-sections of $p\colon E \to S^{n-2}$.

Construction 5.1 is a systematic way of extracting concordances (and therefore, by the work of Waldhausen, algebraic K-theory) from homotopy spheres. All we need to know now is that it has something to do with Morse perfection and Gromoll filtration.

5.2. Proposition. *If Σ^n has Gromoll filtration $> k$, then $\omega(\Sigma^n)$ is $Z/2$-homotopic to a section which is trivial over $S^{k-1} \subset S^{n-2}$.*

Proof. If Σ^n has Gromoll filtration $> k$, then we can assume that the glueing diffeomorphism $\varphi\colon D^{n-1} \to D^{n-1}$ respects the projection to the last k coordinates, $D^{n-1} \subset \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}/\mathbb{R}^{n-1-k}$. This means that $\lambda_x \cdot \varphi = \lambda_x$ for any $x \in S^{n-2} \subset \mathbb{R}^{n-1}$ which is orthogonal to the subspace $\mathbb{R}^{n-1-k} \subset \mathbb{R}^{n-1}$. In other words, $\omega(\varphi) = \omega(\Sigma^n)$ is actually trivial over the $(k-1)$-sphere consisting of the unit vectors orthogonal to $\mathbb{R}^{n-1-k}$.

5.3. Proposition. *If $\omega(\Sigma^n)$ is $Z/2$-homotopic to a section which is trivial over $S^{k-1} \subset S^{n-2}$, then the Morse perfection of Σ^n is $\geq k$.*

(Of course, Proposition 0.5 follows from Propositions 5.2 and 5.3, but it would have been unwise to prove Proposition 5.3 before proving Proposition 0.5.)

5.4. Notation (for the proof of Proposition 5.3). As in §4, think of Σ^n as the pushout of

$$V_- \xleftarrow{j_-} C \xrightarrow{j_+\circ(\varphi\times \mathrm{id})} V_+$$

where $\varphi\colon S^{n-1} \to S^{n-1}$ is a diffeomorphism which is the identity outside a small disk D_ε around $P = (1, 0, \dots, 0) \in S^{n-1}$. Points in Σ^n will be denoted by $(z, t)_+$ or $(z, t)_-$, where $(z, t)_+$ and $(z, t)_-$ are the images of

$$(z, t) = (z_1, z_2, \dots, z_n, t) \in V_+ \subset \mathbb{R}^{n+1}$$

and

$$(z, t) = (z_1, z_2, \dots, z_n, t) \in V_- \subset \mathbb{R}^{n+1}$$

under the canonical embeddings $V_+ \to \sum^n$ and $V_- \to \Sigma^n$, respectively.

Proof of Proposition 5.3. Let $S^{:k-1} \subset S^{n-1} \subset \mathbb{R}^n$ consist of all unit vectors whose first $n-k$ coordinates are zero. The assumption in Proposition 5.3 can be recast as follows: There exists a (continuous) family of smooth functions

$$\{\mu_{x,t}\colon S^{n-1} \to \mathbb{R} | x \in S^{:k-1}, t \in [-1, +1]\}$$

such that

$\mu_{x,t} = \lambda_x$ for $t \leq -1/2$ and all x;
$\mu_{x,t} = \lambda_x \circ \varphi$ for $t \geq 1/2$ and all x;
$\mu_{x,t}$ agrees with the linear form λ_x outside D_ε, and is regular inside D_ε, for all x and t;
$\mu_{-x,t} = -\mu_{x,t}$ for all x and t. We may even assume that the family is smooth, i.e., the function

$$S^{:k-1} \times [-1, +1] \times S^{n-1} \to \mathbb{R}; \qquad (x, t, z) \mapsto \mu_{x,t}(z)$$

is smooth.

Now, for $x \in S^{:k-1}$, let $f_x \colon \sum^n \to \mathbb{R}$ be given by

$$f_x((z, t)_-) = \|z\| \cdot \mu_{x,t}(z/\|z\|),$$
$$f_x((z, t)_+) = \|z\| \cdot \mu_{x,t} \circ \varphi^{-1}(z/\|z\|)$$

(to be read as 0 if $z = 0$). See §5.4 for notation. *Claim*: f_x is a smooth Morse function having critical points of index 0 and n only. *Proof*: if $z \neq 0$ and $z/\|z\| \notin D_\varepsilon$, then $f_x((z, t)_-)$ agrees with $f_x((z, t)_+)$ and with $\lambda_x(z, t)$. At points $(z, t)_+$ or $(z, t)_-$ where $z/\|z\|$ belongs to D_ε, the function f_x is regular by our assumptions. Near the north pole $(z, t)_+$ with $z = 0$ and $t = 1$, the function f_x agrees with λ_x on V_+; near the south pole $(z, t)_-$ with $z = 0$ and $t = -1$, the function f_x agrees with λ_x on V_-.

Since also $f_{-x} = -f_x$, we have a smooth $Z/2$-map

$$S^{:k-1} \to W(\Sigma^n); \qquad x \mapsto f_x.$$

This shows that the Morse perfection of Σ^n is $\geq k - 1$, which is a little below the target.

Fortunately, there is another $g \in W(\Sigma^n)$ which we have not used yet: it is given by

$$g((z, t)_+) = t, \qquad g((z, t)_-) = t.$$

This g is *transverse* to the functions f_x above, in the sense that $af_x + bg$ belongs to $W(\Sigma^n)$ for all $x \in S^{:k-1}$ and all $(a, b) \in S^1 \subset \mathbb{R}^2$. (We have proved this for $b = 0$; the general case is similar.) Let $S^{:k} \subset \mathbb{R}^{n+1}$ consist of all unit vectors whose first $n - k$ coordinates vanish. Write points in $S^{:k}$ in the form

$$ax + by \qquad (y = (0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}),$$

where $x \in S^{:k-1}$ and $(a, b) \in S^1 \subset \mathbb{R}^2$. Map such a point $ax + by$ to $af_x + bg \in W(\Sigma^n)$. This gives a continuous $Z/2$-map from $S^{:k}$ to

$W(\Sigma^n)$. (*Continuous* refers to the C^∞-topology on $W(\Sigma^n)$.) It can easily be approximated by a smooth $Z/2$-map. This shows that the Morse perfection of Σ^n is indeed $\geq k$.

Obstruction theory gives a good way to illuminate Proposition 5.3.

5.5. Question. Given an oriented homotopy sphere Σ^n, and a $Z/2$-map (smooth or continuous) $q: S^k \to W(\Sigma^n)$, what is the obstruction to extending q to a $Z/2$-map $\hat{q}: S^{k+1} \to W(\Sigma^n)$?

(We assume $k < n-1$; the case $k = n-1$ is special, and does not have so much to do with Proposition 5.3.)

Answer. The obstruction is an element in $\pi_k(\mathscr{C}(S^{n-1}))$.

Explanation. We use the commutative diagram of $Z/2$-maps

$$\begin{tikzcd} & W(\Sigma^n) \arrow[d, "c"] \\ S^k \arrow[ur, "q"] \arrow[r, "u"] & \Sigma^n \times \Sigma^n - \text{diagonal} \end{tikzcd}$$

where c assigns to $g \in W(\Sigma^n)$ the ordered pair consisting of the two critical points of g (ordered by their index). Let

$$* = (1, 0, \dots, 0) \in S^k,$$
$$f = q(*) \in W(\Sigma^n),$$
$$(x, y) = c(f) = u(*) \in \Sigma^n \times \Sigma^n - \text{diagonal}.$$

We can assume that $f(x) < -1$ and $f(y) > +1$.

The target of u is $(n-1)$-connected, so $u = cq$ extends to a $Z/2$-map

$$\hat{u}: S^{k+1} \to \Sigma^n \times \Sigma^n\text{-diagonal}.$$

The extension is unique up to homotopy $(\operatorname{rel} S^k)$. Further, c is a fiber bundle, and therefore the obstruction to finding the extension $\hat{q}$ is an element in

$$\pi_k(\text{fiber of } c) = \pi_k(c^{-1}(x, y)).$$

We use $f \in c^{-1}(x, y)$ as the base point for the fiber. Write

$$N := f^{-1}([-1, +1])$$

and let

$$Y := \{g \in W(\Sigma^n) | g \text{ agrees with } f \text{ outside } N\}.$$

The inclusion $Y \subset c^{-1}(x, y)$ is a weak homotopy equivalence. Further, Y is identified with the space of smooth regular functions on N, which agree with f on an infinitesimal neighborhood of ∂N. Use the orientation of Σ^n to choose a diffeomorphism of degree $+1$ from $f^{-1}(-1)$ to S^{n-1}

(note that $f^{-1}(-1)$ bounds an embedded disk in Σ^n, containing x). Using the fact that f is regular on N, extend this to a diffeomorphism

$$\psi\colon N \to S^{n-1} \times [-1, +1]$$

such that

$$f = \text{projection} \circ \psi.$$

This identifies Y with the concordance space $\mathscr{C}(S^{n-1})$.

Let $\alpha\colon S^{n-2} \to E$ be a $Z/2$-section of $p\colon E \to S^{n-2}$ in §5.1. Restriction produces a partial section α_k of p over $S^k \subset S^{n-2}$, for $k \le n-2$. By a $Z/2$-nullhomotopy of α over S^k is meant a $Z/2$-homotopy (through partial sections over S^k) deforming α_k into the trivial section.

5.6. Question. Let h be a $Z/2$-nullhomotopy over S^{k-1} of the section $\omega(\varphi) = \omega(\Sigma^n)$ in §5.1. What is the obstruction to extending h to a $Z/2$-nullhomotopy over S^k?

Answer. The obstruction is an element in $\pi_k(\mathscr{C}(D^{n-2}))$. (This is obvious.)

One would expect that the obstruction groups in Questions 5.5 and 5.6 are related by a homomorphism

$$j_k\colon \pi_k(\mathscr{C}(D^{n-2})) \to \pi_k(\mathscr{C}(S^{n-1}))$$

induced by a suitable map

$$j\colon \mathscr{C}(D^{n-2}) \to \mathscr{C}(S^{n-1}).$$

Namely, a $Z/2$-homotopy h as in Question 5.6 determines an obstruction $o(h) \in \pi_k(\mathscr{C}(D^{n-2}))$; it also determines, by the proof of Proposition 5.3, a $Z/2$-map $q_h\colon S^k \to W(\Sigma^n)$ which in turn determines, by Question 5.5, an obstruction $o(q_h) \in \pi_k(\mathscr{C}(S^{n-1}))$. It would be nice to be able to predict

$$(!) \qquad o(q_h) = j_k(o(h)).$$

Now it is not hard to find a candidate for j: the composite of the stabilization map $\mathscr{C}(D^{n-2}) \to \mathscr{C}(D^{n-1})$ (see, e.g., Hatcher [13]) with the map $\mathscr{C}(D^{n-1}) \to \mathscr{C}(S^{n-1})$ induced by an embedding $D^{n-1} \to S^{n-1}$. It is harder to show that (!) works with this choice of j. The proof (left to the reader) follows the lines of Proposition 5.3.

The map above from $\mathscr{C}(D^{n-1})$ to $\mathscr{C}(S^{n-1})$ is about $2n$-connected (see Burghelea, Lashof and Rothenberg [4, Chapter 3, Theorem A′]). The stabilization map from $\mathscr{C}(D^{n-2})$ to $\mathscr{C}(D^{n-1})$ is about $n/3$-connected by Igusa's stability result [16]. So it can happen that

$$j_k\colon \pi_k(\mathscr{C}(D^{n-2})) \to \pi_k(\mathscr{C}(D^{n-1})) \cong \pi_k(\mathscr{C}(S^{n-1}))$$

(for $k < n-1$) loses information: namely, if $k > n/3$. But most of what is known today about homotopy groups of concordance spaces comes from an analysis of the *stabilized* concordance spaces (by means of algebraic K-theory, as in Waldhausen's work [27], [24], [25], [26], or by means of Goodwillie's calculus of functors [7], [8]). In particular, the assumptions

(1) Morse perfection of $\Sigma^n \geq k$,

(2) $\omega(\Sigma^n)$ is $Z/2$-nullhomotopic over $S^{k-1} \subset S^{n-2}$ on a homotopy sphere Σ^n, with $n \geq 7$ and $k < n$, have the same K-theoretic implications.

6. Some algebraic K-theory

In [24]–[27], Waldhausen constructs maps from concordance spaces to algebraic K-theory spaces:

$$\mathscr{C}(N) \to \Omega^2 K(\mathbb{Z}[\pi_1(N)])$$

where N is compact, connected, smooth, and K means the algebraic K-theory functor (from rings to spaces). Of course, Waldhausen does more than that: for example in the case where N is a disk, he does not use the ring $\mathbb{Z}$ but the "ring space" of self-maps from S^m to S^m for large m. This has $\mathbb{Z}$ as ring of components. However, discrete rings are easier to handle and give interesting results, too. So we have a map $\mathscr{C}(D^{n-2}) \to \Omega^2 K(\mathbb{Z})$, for example. This factors through the stabilization maps

$$\mathscr{C}(D^{n-2}) \to \mathscr{C}(D^{n-1}) \to \mathscr{C}(D^n) \to \cdots$$

mentioned at the end of §5.

In the setting of §5.1, one would like to have such a map for each fiber of $p\colon E \to S^{n-2}$. More precisely, one might hope to be able to construct a fibration $\overline{p}\colon \overline{E} \to S^{n-2}$ with fibers homotopy equivalent to $\Omega^2 K(\mathbb{Z})$, and an involution on $\overline{E}$ covering the antipodal map on S^{n-2}, and a fiberwise Waldhausen map $\chi\colon E \to \overline{E}$ respecting the $Z/2$-actions and the projection to S^{n-2}.

All this can be arranged. What follows is, firstly, a description of $\overline{p}\colon \overline{E} \to S^{n-2}$; secondly, for some homotopy spheres Σ^n, a description of the $Z/2$-section $\chi \circ \omega(\Sigma^n)$ of $\overline{p}$. Here $\omega(\Sigma^n)$ comes from §5.1 and sequel.

Description 1. *The fibration* $\overline{p}$.

Dividing by $Z/2$, think of $\overline{p}$ as a fibration over $\mathbb{R}P^{n-2}$, with fibers homotopy equivalent to $\Omega^2 K(\mathbb{Z})$.

One way to construct $K(\mathbb{Z})$ is as follows: Take the groups $\mathrm{GL}(m, \mathbb{Z})$ for all $m \geq 0$, their classifying spaces $\mathrm{BGL}(m, \mathbb{Z})$, and their disjoint union

$$\mathfrak{M} = \coprod_{m \geq 0} \mathrm{BGL}(m, \mathbb{Z}).$$

$\mathfrak{M}$ is an associative topological monoid with a unit. The multiplication is by direct sum, so it maps $\mathrm{BGL}(m_1, \mathbb{Z}) \times \mathrm{BGL}(m_2, \mathbb{Z})$ to $\mathrm{BGL}(m_1 + m_2, \mathbb{Z})$. There is a canonical map $\mathfrak{M} \to \Omega B\mathfrak{M}$, where $B\mathfrak{M}$ is the classifying space of $\mathfrak{M}$, and we let

$$K(\mathbb{Z}) = \Omega B\mathfrak{M}.$$

This space $\Omega B\mathfrak{M}$ is also known as the group completion of $\mathfrak{M}$.

Now $Z/2$ acts on $\mathrm{GL}(m, \mathbb{Z})$ by $A \mapsto (A^{-1})^t$ for an invertible $(m \times m)$-matrix A. Therefore $Z/2$ acts on $\mathrm{BGL}(m, \mathbb{Z})$, on the monoid $\mathfrak{M}$, and on $\Omega B\mathfrak{M} = K(\mathbb{Z})$. Take the projection map $K(\mathbb{Z}) \times S^n \to S^n$ and divide by $Z/2$ on both sides to get

$$\overline{p}_0\colon K(\mathbb{Z}) \times_{Z/2} S^n \to \mathbb{R}P^n,$$

a fibration over $\mathbb{R}P^n$ with fibers homotopy equivalent to $K(\mathbb{Z})$. This is not quite what we want: we want a fibration over $\mathbb{R}P^{n-2}$ with fibers homotopy equivalent to $\Omega^2 K(\mathbb{Z})$. However, think of $\mathbb{R}P^{n-2}$ as a subspace of $\mathbb{R}P^n$, with orthogonal subspace $L \subset \mathbb{R}P^n$ (so $L \cong \mathbb{R}P^1$). Sections of $\overline{p}_0$ which are trivial over L determine sections of a certain fibration over $\mathbb{R}P^{n-2}$ with fibers homotopy equivalent to $\Omega^2 K(\mathbb{Z})$. This fibration is $\overline{p}$.

Description 2. *The sections associated to some homotopy spheres.*

Assume that $\Sigma^n = \partial V$, where V^{n+1} is smooth, compact, with trivialized tangent bundle, and $n+1$ is divisible by 4. Then Σ^n is determined up to oriented diffeomorphism by the signature of V; in particular, by the intersection form on $H^{(n+1)/2}(V)$/torsion. Choosing a basis for this free abelian group, we can express the intersection form by a symmetric matrix A with integer entries and $\det = \pm 1$.

Claim. Such a matrix A gives rise to a section $\sigma(A)$ of

$$\overline{p}_0\colon K(\mathbb{Z}) \times_{Z/2} S^n \to \mathbb{R}P^n.$$

Proof. Assume that A has size $r \times r$. Form the semidirect product $\mathrm{GL}(r, \mathbb{Z}) \rtimes Z/2$, where $Z/2$ acts as before (by transpose $\circ$ inverse). Sending the generator $\mathbf{1} \in Z/2$ to $(A, \mathbf{1})$ defines a homomorphism $Z/2 \to \mathrm{GL}(r, \mathbb{Z}) \rtimes Z/2$ because A is symmetric. This homomorphism splits the projection $\mathrm{GL}(r, \mathbb{Z}) \rtimes Z/2 \to Z/2$. So it gives rise to a section of the map of classifying spaces

$$B(\mathrm{GL}(r, \mathbb{Z}) \rtimes Z/2) \to B(Z/2),$$

which is really the same as the map

$$\mathrm{BGL}(r, \mathbb{Z}) \times_{Z/2} S^\infty \to \mathbb{R}P^\infty.$$

We can regard the section just constructed as a section of $\mathfrak{M} \times_{Z/2} S^\infty \to \mathbb{R}P^\infty$ or of $K(\mathbb{Z}) \times_{Z/2} S^\infty \to \mathbb{R}P^\infty$ or, by restriction, as a section $\sigma(A)$ of $\overline{p}_0 \colon K(\mathbb{Z}) \times_{Z/2} S^n \to \mathbb{R}P^n$.

6.1. Theorem. *In the notation above,*

$$\chi \circ \omega(\Sigma^n) = \sigma(A) - \sigma(H),$$

where $H = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ *is a hyperbolic matrix of the same size as* A.

Notice that each fiber of $\overline{p}_0$ is an infinite loop space in its own right, so that the subtraction of sections is possible and well defined up to homotopy.

In other words, the "Waldhausenization" of the geometric section $\omega(\Sigma^n)$ constructed in §5.1 and sequel is the algebraic section $\sigma(A)$ constructed just before Theorem 6.1, minus a correction term $\sigma(H)$. The purpose of the correction term is to neutralize the rank (alias "size") of A, without spoiling the signature. Also, one can show that $\sigma(A) - \sigma(H)$ is nullhomotopic over $\mathbb{R}P^1 \subset \mathbb{R}P^n$; therefore it can be regarded as a section of the fibration $\overline{p}$ with base space $\mathbb{R}P^{n-2}$ and fibers $\Omega^2 K(\mathbb{Z})$. (Hence Theorem 6.1 makes sense.)

Using the stable classification of symmetric nondegenerate forms over $\mathbb{Z}$ (see Serre [21]), one can get a more explicit formulation:

6.2. Reformulation of Theorem 6.1. *With the same assumptions as in Theorem* 6.1,

$$\chi \circ \omega(\Sigma^n) = 1/2 \cdot \mathit{signature}(V) \cdot (\varepsilon - \eta),$$

where ε *is the "trivial line bundle", and* η *is the "canonical line bundle".*

Explanation. $\varepsilon = \sigma((+1))$ and $\eta = \sigma((-1))$, where $(+1)$ and (-1) are 1×1-matrices.

To see how Theorem 6.2 follows from Theorem 6.1, suppose for example that A in Theorem 6.1 has signature 8 and rank 8. Then arguments involving stabilization show that

$$\sigma(A) = 8\varepsilon \quad \text{and} \quad \sigma(H) = 4\varepsilon + 4\eta.$$

Therefore

$$\sigma(A) - \sigma(H) = 4\varepsilon - 4\eta.$$

6.3. Corollary (same assumptions as in Theorem 6.1). *If the Gromoll filtration of* Σ^n *is* $> k$, *then the section* $1/2 \cdot signature(V) \cdot (\varepsilon - \eta)$ *of* $\overline{p}_0\colon K(\mathbb{Z}) \times_{Z/2} S^n \to \mathbb{R}P^n$ *is nullhomotopic over* $\mathbb{R}P^{k+1}$.

This follows from Theorem 6.2 and Proposition 5.2 (remember also the little difference between $\overline{p}_0$ and $\overline{p}$). With more work, along the lines suggested at the end of §5, the hypothesis can be weakened:

6.4. Corollary (same assumptions as in description 2). *If* Σ^n *has Morse perfection* $k < n$, *then the section* $1/2 \cdot signature(V) \cdot (\varepsilon - \eta)$ *of* $\overline{p}_0$ *is nullhomotopic over* $\mathbb{R}P^{k+1}$.

Example. Let Σ^n be the Milnor homotopy sphere; so $\Sigma^n = \partial V$ where V has signature 8. Let us first calculate in real topological K-theory $K^{\text{top}}(\mathbb{R})$ instead of $K(\mathbb{Z})$. Then

$$\overline{p}_0\colon K^{\text{top}}(\mathbb{R}) \times_{Z/2} S^n \to \mathbb{R}P^n$$

becomes a product fibration because $Z/2$ acts trivially on $K^{\text{top}}(\mathbb{R}) \cong BO \times \mathbb{Z}$, because orthogonal matrices are equal to the transposes of their inverses. Therefore

$$1/2 \cdot \text{signature}(V) \cdot (\varepsilon - \eta) = 4(\varepsilon - \eta)$$

is a map from $\mathbb{R}P^n$ to BO, and ε, η can be interpreted as the trivial line bundle and the canonical line bundle in the usual sense. In this setting, $4(\varepsilon - \eta)$ is trivial over $\mathbb{R}P^3$, but not over $\mathbb{R}P^4$; there is a nonzero obstruction in $\pi_4(BO) \cong \mathbb{Z}$.

If we now return to the $K(\mathbb{Z})$-setting, then we see that $4(\varepsilon - \eta)$ must be *nontrivial* over $\mathbb{R}P^3$. (Assume it is trivial; then the above obstruction in $\pi_4(BO) \cong \mathbb{Z}$ comes from an obstruction in $K_4(\mathbb{Z})$. But this is impossible, because $K_4(\mathbb{Z})$ is finite by Borel's result [2], so $K_4(\mathbb{Z}) \to \pi_4(BO)$ is the zero homomorphism.) Conclusion:

$$\text{Morse perfection of } \Sigma^n \leq 1,$$

and therefore

$$\text{Morse perfection of } \Sigma^n = 1$$

because this is the minimum possible perfection. Also,

$$\text{Gromoll filtration of } \Sigma^n = 2.$$

7. A guide

The labels WW I, WW II, and WW III used below refer to Weiss and Williams [29]–[31], respectively.

For a manifold M, closed for simplicity, let $\widetilde{\mathrm{TOP}}(M)$ be the topological block automorphism space of M, and let $\widetilde{\mathrm{DIFF}}(M)$ be the smooth block automorphism space of M if M is smooth. See the introduction to WW I for details. Write $\mathrm{TOP}(M)$ and $\mathrm{DIFF}(M)$ for the honest topological automorphism and smooth automorphism spaces respectively. In WW I the (group-theoretic) quotients $\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M)$ and $\widetilde{\mathrm{DIFF}}(M)/\mathrm{DIFF}(M)$ are analysed in terms of the concordance theory of M. As $\widetilde{\mathrm{TOP}}(M)$ and $\widetilde{\mathrm{DIFF}}(M)$ are closely related to the L-theory of $\mathbb{Z}[\pi_1(M)]$, this provides a way to get from the L-theory of $\mathbb{Z}[\pi_1(M)]$ to the concordance theory alias K-theory of M.

In WW II, we construct a map from the L-theory of $\mathbb{Z}[\pi_1(M)]$ to a descendant of the algebraic K-theory of M by algebraic methods. This is based on the simple idea (of Thomason, Giffen, Karoubi, etc.) that the quadratic forms which one uses in defining L-groups can also be regarded as "free finitely generated modules identified with their duals", and therefore give rise to homotopy fixed points in the algebraic K-theory under a suitable duality involution. So we now have two ways of getting from L to K, one geometric (WW I) and the other algebraic (WW II).

The main result in WW III is that these two ways give the same result: there is a very large commutative diagram somewhere. This is the commutative cube at the end of the introduction to WW III.

Now specialize by taking M to be a point; this is a smooth manifold. Then $\mathrm{DIFF}(M)$ is also a point, but $\widetilde{\mathrm{DIFF}}(M)$ is not. In fact, it follows directly from the definitions that $\pi_{k-1}(\widetilde{\mathrm{DIFF}}(*))$ is isomorphic to the group of oriented homotopy k-spheres modulo oriented diffeomorphism, if $k \geq 7$. Therefore WW I gives K-theoretic invariants for such homotopy spheres. Moreover, if the homotopy spheres under investigation come from the L-theory of $\mathbb{Z}$ (as in Kervaire and Milnor [17]), then WW III says that their algebraic K-theory invariants can be obtained directly from the L-theory, without any geometry. Of course, this implies that they can sometimes be computed. Hence we have Theorem 6.1 in this paper.

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