VON NEUMANN INDEX THEOREMS
FOR MANIFOLDS WITH BOUNDARY

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1. Introduction

The index theorem of Atiyah and Singer relates the index of elliptic operators on closed manifolds to characteristic numbers on the manifold. In the case of compact manifolds with boundary, Atiyah, Patodi, and Singer [3] showed that the index of a first-order elliptic operator of Dirac type is not the usual characteristic number; instead the difference is a term depending only on the boundary called the eta invariant.

In the case of noncompact manifolds without boundary the index of elliptic operators is not well defined since these operators in general are not Fredholm. In the case of elliptic differential operators, on infinite Galois coverings of closed manifolds, equivariant with respect to the Galois group, Atiyah (cf. [1]) introduced a real-valued index given by replacing the notion of dimension by a generalized dimension introduced by von Neumann. He proceeded to prove an index theorem for elliptic operators in the above context. In the case of noncompact manifolds arising as leaves of a foliation of a closed manifold, with holonomy invariant transverse measure, Connes (cf. [10]) proved a von Neumann index theorem for differential operators elliptic along the leaves of the foliation.

This then brings us to the question that is answered in this paper. What are the corresponding results in the case of noncompact manifolds with boundary, i.e., is there an analog of the Atiyah-Patodi-Singer index theorem for infinite Galois coverings of compact manifolds with boundary? Is there a similar index theorem in the case of foliations of compact manifolds with boundary, with the leaves intersecting the boundary transversally, and equipped with a holonomy invariant transverse measure? As in the usual Atiyah-Patodi-Singer index theorem one would like to know the analog of the usual eta invariant on the boundary. In the case of infinite Galois coverings these generalized eta invariants were introduced

by Cheeger and Gromov [8]. In the case of foliations the foliation eta invariant was introduced by the author [26] and Peric [25] independently.

We prove new von Neumann index theorems for first-order elliptic operators of Dirac type on manifolds with boundary, thus answering the question raised in the previous paragraph, under a mild assumption that the Dirac operators on the boundary satisfy a local cancellation condition. This condition is satisfied by all Dirac operators that arise in geometric contexts, as was proved by Bismut and Freed (cf. [5]).

We now state our main theorem for geometric Dirac operators. See §6 for a more detailed statement.

Theorem 1.1. *Let D be a Dirac operator on a compact manifold M with boundary acting on sections of a graded Clifford bundle S, with grading operator ε. We assume that the data (D, S, ε) has a product structure near the boundary (in the sense of Definition 2.1.1). Let $\widetilde{M}$ be a Galois covering with Galois group Γ. Let $(\widetilde{D}, \widetilde{S}, \widetilde{\varepsilon})$ be the lift of the data (D, S, ε) to $\widetilde{M}$. Let $\widetilde{B}$ be the Atiyah-Patodi-Singer boundary condition (see Definition 2.1.3) associated to $\widetilde{D}$. Then the Γ index (in the sense of §5) of the Breuer Fredholm operator $\widetilde{D}$ with boundary condition $\widetilde{B}$ is given by*

$$\operatorname{ind}_\Gamma(\widetilde{D}) = \int_M \operatorname{ch}(\sigma_D) Td(M) - \frac{\eta_\Gamma(0) + h}{2},$$

where $\eta_\Gamma(0)$ is the Γ-eta invariant on the boundary of $\widetilde{M}$.

We now state our index theorem for foliations.

Theorem 1.2. *Let $(M, \mathscr{F})$ be a compact foliated manifold with boundary, with the foliation $\mathscr{F}$ transverse to the boundary. Let $D_{\mathscr{F}}$ be a leafwise Dirac operator acting leafwise on S a Clifford bundle over $T\mathscr{F}$, the tangent bundle to the foliation. We assume S is graded with grading operator ε. Let ν be a holonomy invariant transverse measure for the foliation $\mathscr{F}$. Further assume that the data $(D_{\mathscr{F}}, S, \varepsilon)$ has a product structure near the boundary (see Definition 2.3.2). Let $B_{\mathscr{F}}$ be the family of Atiyah-Patodi-Singer boundary conditions corresponding to the family $D_{\mathscr{F}}$. Then the ν-index of this family of Breuer Fredholm operators (see §5) is finite and*

$$\operatorname{ind}_\nu(D_{\mathscr{F}}) = \langle \operatorname{ch}(\sigma_{D_{\mathscr{F}}}) Td(M), \nu \rangle - \frac{\eta_\nu(0) + h}{2},$$

where $\eta_\nu(0)$ is the foliation eta invariant of §2.

The proofs of Theorems 1 and 2 follow in outline that of [3] making use of a reformulation of [3] by Roe [31]. The extension of this approach to our case requires several technical modifications. The proof in [3] cannot be directly used to compute the index because the eta invariants in our context do not admit meromorphic continuation to the right half plane.

The paper is organized as follows. Section 1 describes the basic properties of nonlocal Sobolev spaces on manifolds with boundary. The restriction properties of these Sobolev spaces is proved by using the spectral decomposition theorem for selfadjoint elliptic operators due to Browder, Mautner, and Garding. This theorem also plays a key role in constructing boundary parametrices for the Atiyah-Patodi-Singer boundary value problem.

Section 2 deals with type II eta invariants. The proof of existence and the Cheeger-Gromov estimates for these eta invariants is proved by methods in Ramachandran [26]. We remark that these Cheeger-Gromov estimates play an important role in the computation of the index. Sections 4 and 5 deal with the construction of parametrices which are used to prove the Breuer Fredholmness of the Atiyah-Patodi-Singer boundary value problem. The spectral theorem of Browder, Mautner, and Garding plays a key role in the construction of these parametrices. In §6 we formulate the Breuer index of these boundary value problems. The novelty in the foliation case is that we work with the equivalence relation given by the foliation rather than the foliation groupoid. In fact the boundary value problems are most naturally formulated on the leaves of the foliation rather than on their holonomy covers. We complete the calculation of the index in §7. It should be remarked here that Bismut and Cheeger [4] have proved a much stronger theorem when the foliation is a fibration with compact fibres.

2. Preliminaries

We introduce the terminology and prove the basic facts about nonlocal Sobolev spaces on manifolds with boundary. We state the spectral decomposition theorem of Browder and Garding for selfadjoint elliptic operators.

2.1. The Atiyah-Patodi-Singer boundary condition. Our data will be the following. M will denote a C^∞ complete Riemannian manifold with C^∞ boundary N. By the data (D, S, ε) we mean a Dirac operator D acting on smooth sections of a graded Clifford bundle S, with grading operator ε. For definitions see Roe [32], [30], Lawson and Michelson [20].

Definition 2.1.1. By a product structure on the given data (D, S, ε) in a neighborhood of the collar $[0, 1] \times N$, we mean the following:

1. The Riemannian metric on M is a product in a neighborhood of $[0, 1] \times N$.

2. The Dirac operator has the special form $D = \sigma(\partial/\partial y + Q)$ where σ is the Clifford multiplication by the unit normal to the boundary N, y is the coordinates normal to the boundary, and Q is a Dirac operator on N and independent of y.

We assume henceforth that the data (D, S, ε) has a product structure as in Definition 2.1.1, and also that on the geometric double of M which we denote by dbM, the doubled data $(\hat{D}, \hat{S}, \hat{\varepsilon})$ satisfy the bounded geometry hypotheses in §2 of Roe [32]. Such examples naturally arise in the study of Dirac operators on Galois coverings of compact manifolds with boundary and leafwise Dirac operators on foliations of compact manifolds with boundary with the foliation transverse to the boundary. All manifolds with boundary in this paper will be smooth.

$C_c^\infty(M; S)$ will denote compactly supported smooth sections of S smooth up to the boundary. Then we have the following Green's formula:

$$(2.1.1) \qquad (s_1, Ds_2) - (Ds_1, s_2) = \int_N (\sigma bs_1, s_2),$$

where $(,)$ denotes the L^2 inner product on sections of S, s_1 and s_2 belong to $C_c^\infty(M; S)$, and bs denotes the restriction of the section $s \in C_c^\infty(M; S)$ to the boundary N.

Following Roe [31] we make the following definition.

Definition 2.1.2. We say $B: C_c^\infty(M; S) \to C^\infty(M; S)$ defines a self-adjoint boundary condition if

1. B extends to a bounded operator on $L^2(N; S)$,
2. $B = B^*$ and $\sigma B + B\sigma = \sigma$.

If $s_1, S_2 \in C_c^\infty(M; S)$ satisfying $Bbs_1 = 0$ and $Bbs_2 = 0$, then

$$(2.1.2) \qquad \langle s_1, Ds_2 \rangle = \langle Ds_1, s_2 \rangle.$$

Remark. If we analyze the interaction of the product structure of the data (D, S, ε) with the grading operator ε, we find that Q is essentially selfadjoint on $C_c^\infty(N; S)$ and commutes with ε. Since ε is an involution, diagonalizing it splits $S = S^+ \oplus S^-$ and Q preserves this decomposition. We label Q restricted to sections of $S^\pm$ by $Q_\pm$ respectively.

Definition 2.1.3. The Atiyah-Patodi-Singer (A.P.S.) boundary condition is the operator B which restricts to the projection onto the nonnegative part of the spectrum of Q_+ on the $+$ part of the grading, and onto the positive part of the spectrum of Q_- on the $-$ part of the grading. One can easily check that B satisfies the conditions of Definition 2.1.1.

2.2. Sobolev spaces. We now define nonlocal Sobolev spaces for manifolds with boundary. From now on the assumptions of §2.1 carry over

to the rest of this paper. The main references for this section will be Roe [32], [31] and Lions and Magenes [21].

Definition 2.2.1. Let k be a nonnegative integer. The Sobolev space $W^k(M;S)$ is the completion of $C_c^\infty(M;S)$ in the norm

$$\|s\|_k = \{\|s\|^2 + \|Ds\|^2 + \cdots + \|D^k s\|^2\}^{1/2},$$

where $\|s\| = \langle s, s \rangle$. For k a negative integer $W^k(M;S)$ is the dual of $W^{-k}(M;S)$ considered as a space of distributional sections.

Also $W^{-\infty}(M;S) = \bigcup W^k(M;S)$ and $W^\infty(M;S) = \bigcap W^k(M;S)$; $W^\infty(M;S)$ has the obvious Frechet topology, and $W^{-\infty}(M;S)$ is equipped with weak topology that it inherits as the dual of $W^\infty(M;S)$.

For k nonnegative we observe that any element of $W^k(M;S)$ can be extended to $db(M)$ with control over the norm.

Proposition 2.2.1. *There is a bounded linear operator* $E_k: W^k(M;S) \to W^k(dB(M); \widehat{S})$ *for every integer* $k \geq 0$ *with the property that* $E_k f$ *is restricted to* M *in* `f`.

Proof. If $s \in W^k(M;S)$ vanishes in a neighborhood of the collar $[0, \frac{1}{2}) \times N$, we define $E_k s = 0$. By using a bump function it is enough to define E_k for sections supported in the collar $[0, 1] \times N$. Let $s \in W^k(M;S)$ be supported in $[0, 1] \times N$. Then define

$$E_k s(y, n) = \begin{cases} s(y, n) & \text{if } y \geq 0, \\ \sum_{j=1}^{k+1} \alpha_j s(-jy, n) & \text{if } y < 0, \end{cases}$$

where α_j are chosen so that the first k derivatives in the y direction match at $y = 0$. This implies that the α_j's satisfy the system of equations

$$\sum_{l=1}^{k+1} (-1)^l j^l \alpha_l = 1 \quad \text{for } 0 \leq j \leq k - 1.$$

The determinant of this linear equation is not zero, so the appropriate α's can be found.

Definition 2.2.2. Let r be a nonnegative integer. The uniform C^r space $UC^r(M;S)$ is the Banach space of C^r sections S, C^r up to the boundary of M such that the norm

$$\|\|s\|\|_r \sup\{|\nabla_{v_1} \cdots \nabla_{v_q} s(m)|\}$$

is finite, where the supremum is taken over all $m \in M$ and collections $v_1, \cdots, v_q$ $(0 \leq q \leq r)$ of unit tangent vectors at m. Also $UC^\infty(M;S) = \bigcap_r UC^r(M;S)$.

Proposition 2.2.2. *The Frechet space* $W^\infty(M; S)$ *is continuously included in* $UC^\infty(M; S)$.

Proof. By Proposition 2.2.1, $W^\infty(M; S)$ is continuously included in $W^\infty(db(M); \widehat{S})$. Proposition 2.8 of Roe [32] now implies the result.

Proposition 2.2.3. *A continuous linear opeator from* $W^{-\infty}(M; S)$ *to* $W^\infty(M; S)$ *is represented by a smoothing kernel smooth up to the corners on* $M \times M$. *Also the kernel and all its covariant derivatives are uniformly bounded.*

Proof. The strategy for the proof is the same as in Proposition 2.9 of Roe [32]. We use Proposition 2.2.2 instead of Proposition 2.8 used in Roe [32]. q.e.d.

In the remark following Definition 2.1.2 we mentioned that Q is essentially selfadjoint on $C_c^\infty(N; S)$. This follows immediately by a minor modification of the proof in Chernoff [9]. The main difference between Chernoff [9] and our context is that N need not be connected, but the fact that N is a disjoint union of countable many complete Riemannian manifolds implies that the proof in [9] can be used here. We leave the details to the reader.

Since Q is essentially selfadjoint, it has a unique closure which we denote by Q. By the spectral theorem we can define Sobolev spaces on the boundary N as follows.

Definition 2.2.3. Let k be a nonnegative half integer. Then define

$$W^k(N; S) = \text{domain}(Q^k).$$

If k is a nonnegative integer $W^k(N; S)$, it coincides with the closure of $C_c^\infty(N; S)$ under the norm

$$\{\|s\|^2 + \|Qs\|^2 + \cdots + \|Q^k s\|\}^{1/2}.$$

We now state the generalized eigenfunction expansion theorem for Q due to Browder and Garding.

Theorem 2.2.1 (Browder and Garding). *There exists a sequence of smooth sectional maps* $e_j \colon \mathbb{R} \times N \to S$, *namely* e_j *is measurable and for each* $\lambda \in \mathbb{R}$, $e_j(\lambda, \ )$ *is a smooth section of* S *over* N, *and measures* μ_j *on* $\mathbb{R}$ *such that*

$$(2.2.1) \qquad\qquad Qe_j(\lambda, n) = \lambda e_j(\lambda, n).$$

Further, the map

$$(2.2.2) \qquad\qquad (Vs)_j(\lambda) = \int_N (s(n) | e_j(\lambda, n)) \, d\operatorname{vol}_N$$

defined on $C_c^\infty(N; S)$ *extends to an isometry of Hilbert spaces*

$$(2.2.3) \qquad V: L^2(N; S) \to \bigoplus_j L^2(\mu_j),$$

where the sum on the right-hand side of (2.2.3) *is the Hilbert direct sum, and also the bar inside the integral sign of* (2.2.2) *is the fiber wise inner product on* S. *Further* V *intertwines the operator* $f(Q)$ *with multiplication by* $f(\lambda)$,

$$(2.2.4) \qquad \text{domain } f(Q) = \left\{ s \mid \sum_j \int_{\mathbb{R}} |f(\lambda)|^2 |(Vs)_j(\lambda)|^2 \, d\mu_j(\lambda) < \infty \right\}$$

and

$$(2.2.5) \qquad \int_N |s(n)|^2 \, d\mathrm{vol}_N = \sum_j \int_{\mathbb{R}} |(Vs)_j(\lambda)|^2 \, d\mu_j(\lambda).$$

Proof. See Dieudonne [11, 300–307] and Dunford and Schwartz [13].

We now sketch the proof of a restriction theorem for the Sobolev spaces defined before in this section.

Theorem 2.2.2. $b: C_c^\infty(M; S) \to C_c^\infty(N; S)$ *extends to a bounded operator* $b: W^k(M; S) \to W^{k-1/2}(N; S)$ *for any natural number* k.

Proof. Again it is enough to consider elements of $W^k(M; S)$ supported in the collar $[0, 1) \times N$. We extend them to elements of $W^k(db(M); \widehat{S})$ supported in $(-1, 1) \times N$. Note that our data $(\widehat{D}, \widehat{S}, \hat{\varepsilon})$ has product structure on $(-1, 1) \times N$. Hence we can consider them as elements of $W^k((-\infty, \infty) \times; \widehat{S})$. Therefore it is enough to prove that

$$(2.2.6) \qquad W^k((-\infty, \infty) \times N; \widehat{S}) \xrightarrow{r} W^{k-1/2}(0 \times N; S),$$

where r is the restriction map is bounded. Because our data has product structure on $(-\infty, \infty) \times N$, we have $D^2 = -\partial^2/\partial y^2 + Q^2$. Hence $W^k((-\infty, \infty) \times N; \widehat{S})$ is the closure of $C_c^\infty((-\infty, \infty) \times N; \widehat{S})$ with respect to the norm

$$\left\{ \langle s, s \rangle + \left\langle \left(-\frac{\partial^2}{\partial y^2} + q \right) s, s \right\rangle + \cdots + \left\langle \left(-\frac{\partial^2}{\partial y^2} + Q^2 \right)^k s, s \right\rangle \right\}^{1/2}.$$

Using the map V given in (2.2.3) in the N direction and the Fourier transform in the y direction we see that

$$W^k((-\infty, \infty) \times N; \widehat{S})$$

$$(2.2.7) \qquad = \left\{ s \in L^2 \mid \sum_{\mathbb{R}} \int_{\mathbb{R}} (1 + \lambda^2 + \xi^2)^k |(\widehat{V}s)_j(\lambda, \xi)|^2 \, d\mu \, d\xi \right\},$$

where the "hat" denotes the Fourier transform in the y direction. We call the operator V, the spectral transform. The rest of the proof is the same as the classical proof. See the appendix in Hormander [18].

2.3. Foliations of manifolds with boundary.

Definition 2.3.1. A C^∞ *manifold with boundary* and the foliation transverse to the boundary is a C^∞ manifold with boundary and a collection of open sets $\{U_\alpha\}$ covering M and homeomorphisms $\varphi_\alpha\colon U_\alpha \to V_\alpha \times W_\alpha$ with V_α open in $\mathbf{H}^p = \{(x_1, \cdots, x_p) \in \mathbb{R}^p | x_1 \geq 0\}$ and W_α open in $\mathbb{R}^q$ which satisfies the following condition:

1. If we write $\varphi_\alpha = (v, w)$, the coordinate changes are given by C^∞ map φ and local diffeomorphism ψ, namely,

$$v' = \varphi(v, w) \quad \text{and} \quad w' = \psi(w).$$

Further, the collection $\{U_\alpha\}$ is assumed maximal among all such collections. Since the coordinate changes smoothly transform level surfaces $w = $ constant to $w' = $ constant, the level sets coalesce to form maximal connected sets called leaves and the manifold M is foliated by these leaves which intersect the boundary transversely to give a smooth foliation of the boundary with same codimension as the foliation of the interior of M. We denote the foliation by $\mathscr{F}$, and let $(M, \mathscr{F})$ be the manifold with the foliation. If we consider the tangent bundle to the leaves of $\mathscr{F}$, then we get a smooth vector bundle over M, which is a subbundle of the tangent bundle of M. We denote this subbundle by $T\mathscr{F}$.

We say the foliation $\mathscr{F}$ is transversely orientable if the quotient bundle of $TM/T\mathscr{F}$ is orientable.

From now on we assume that our foliation is transversely oriented.

Fact 2.3.1. *Let $(M, \mathscr{F})$ be as above. Then there is a collar W on N such that $\mathscr{F}|_W$ is diffeomorphic to $[0, 1) \times (\mathscr{F}|_N)$.*

Proof. See Hector and Hirsch [16, p. 43].

Let $D_{\mathscr{F}}$ be a leafwise Dirac operator on S where S is a Clifford bundle over $T\mathscr{F}$ (cf. Roe [29]).

Definition 2.3.2. We say the data $(D_{\mathscr{F}}, S, \varepsilon)$ where S is a graded Clifford bundle over $T\mathscr{F}$ with grading ε has a product structure in a neighborhood of the foliation collar $[0, 1) \times \mathscr{F}|_N$ if

1. the Riemannian metric on M is a product in a neighborhood of $[0, 1] \times N$, and

2. the leafwise Dirac operators $D_{\mathscr{F}}$ has the form $D_{\mathscr{F}} = \sigma(\partial/\partial y + Q_{\mathscr{F}})$ where $Q_{\mathscr{F}}$ is a leafwise Dirac operator on $\mathscr{F}|_N$, and σ is the Clifford multiplication of the unit normal along the leaves to the boundary foliation.

We remark that, implicitly included in our data $(D_{\mathscr{G}}, S, \varepsilon)$ is the fact that M has a Riemannian metric.

3. Type II eta invariants

This section proves the existence of eta invariants for Dirac operators on coverings of compact manifolds, and for leafwise Dirac operators on foliations of compact manifolds. Cheeger and Gromov [7] proved the existence of eta invariants for the signature opeator on coverings of compact manifolds. Peric in his thesis [25] proved the existence of the foliation eta invariant. In our study of Cheeger-Gromov estimates and generalizations of A.P.S. theorem to foliation we discovered a proof of existence of these eta invariants which also gave the Cheeger-Gromov estimates [26]. These Cheeger-Gromov estimates were applied in Douglas, Hurder, and Kaminker [15] and by Hurder [19]. This section presents the proof in [26]. §3.1 deals with the covering eta invariant, and §3.2 with the foliation eta invariant.

3.1. Eta invariant for coverings. Let N be a compact Riemannian manifold without boundary. Let D be a Dirac operator on S, a Clifford bundle. Now D is essentially selfadjoint, and De^{-tD^2} defined by the spectral theorem is a smoothing opeator. See Roe [30] for a proof of these statements. We assume that the pointwise trace of De^{-tD^2} is $O(t^{1/2})$. This local cancellation property was first observed by Bismut and Freed [5] for Dirac operators arising in geometric situations. Following them we call this local cancellation property the Bismut-Freed cancellation property.

Let $\widetilde{N}$ be a Γ principal bundle over N, where Γ is a discrete countable group. Let $\widetilde{D}$ and $\widetilde{S}$ be the lifts of D and S respectively to $\widetilde{N}$. By Atiyah [1], $\widetilde{D}$ acting on $C_c^\infty(\widetilde{N}; \widetilde{S})$ is essentially selfadjoint. Let

$$\text{End}_\Gamma(L^2(\widetilde{N}; \widetilde{S})) = \{\text{bounded operator on } L^2(\widetilde{N}; \widetilde{S}) \text{ commuting with } \Gamma\}.$$

If $T \in \text{End}_\Gamma(L^2(\widetilde{N}; \widetilde{S}))$ is an integral operator with smooth kernel k_T, then we define the Γ trace of T as follows.

Definition 3.1.1. $\text{tr}_\Gamma(T) = \int_F \text{Tr}_x k_T(x, x)\, dx$ where F is a fundamental domain for the Γ action on $\widetilde{N}$, and Tr_x is the matrix trace on $\text{End}(\widetilde{S}_x)$.

Definition 3.1.2. Let $RB(\mathbb{R}) = $ Borel functions on $\mathbb{R}$ which are rapidly decreasing. By rapidly decreasing we mean

$$\sup_{x\in\mathbb{R}}(1+|x|)^k|f(x)| < C_k \quad \text{for every positive integer } k.$$

Remark. By the spectral theorem and Sobolev lemma, $f(\widetilde{D}) \in$ $\mathrm{End}_\Gamma(L^2(\widetilde{N}; \widetilde{S}))$ and $f(\widetilde{D})$ is an integral operator with smooth kernel for $f \in RB(\mathbb{R})$. See Chapter 13 of Roe [30] for more details. Hence $\mathrm{tr}_\Gamma f(\widetilde{D})$ is finite for $f \in RB(\mathbb{R})$.

Let the kernel of the integral operator $e^{-t\widetilde{D}^2}$ be $K_t(x, y)$, and the kernel $e^{-t\Delta_g}$ be $k_t(x, y)$ where Δ_g is the Laplace-Beltrami operator on $\widetilde{N}$. Then we have

Lemma 3.1.1.

$$(3.1.2) \qquad\qquad |\mathrm{Tr}_x \widetilde{D} K_t(x, x)| < At^{1/2},$$

where A is a constant depending on the local geometry of $\widetilde{N}$, $\widetilde{S}$, the dimension of $\widetilde{N}$, and the rank of $\widetilde{S}$.

Proof. Let $P(x, y, t)$ be a smooth parametrix for the kernel $K_t(x, y)$, supported in an ε neighborhood of the diagonal of $\widetilde{N} \times \widetilde{N}$ where ε is one-half the injectivity radius of $\widetilde{N}$. We also assume that $P(x, y, t)$ satisfies the following additional properties:

$$(3.1.3) \qquad\qquad P(x, y, t) \in \mathrm{Hom}(\widetilde{S}_y, \widetilde{S}_x),$$

$$(3.1.4) \qquad\qquad (\partial/\partial t + \widetilde{D}^2)P(x, y, t) \text{ is } O(t^m),$$

$$(3.1.5) \qquad\qquad \widetilde{D}(\partial/\partial t + \widetilde{D}^2)P(x, y, t) \text{ is } O(t^{m-1}).$$

m is chosen so that

$$(3.1.6) \qquad \int_0^t (t-s)^{-n/2}s^{m-1}\,ds \text{ is } O(t^{1/2}) \quad \text{where } n = \dim N,$$

$$(3.1.7) \qquad\qquad \|P(x, y, t)\|_{x,y} \le At^{-n/2}, \qquad 0 \le t \le 1.$$

In this section A will denote a constant depending on the data described in the statement of the lemma. If the constant A appears in two places repeated by some prose, then they are different. For a construction of such a parametrix P see Patodi [24]. By Theorem 3.5 of Rosenberg [33, p. 294] we have

$$(3.1.8) \qquad\qquad \|K_t(x, y)\|_{x,y} \le e^{ct}k_t(x, y),$$

where c depends only on the local geometry. The proof of (3.1.8) in Rosenberg uses probabilistic methods. For a proof of (3.1.8) based on the work of Dodziuk [14] see Ramachandran [26].

By Duhamel's principle (cf. Roe [30]) we obtain
(3.1.9)
$$\text{Tr}_x(\widetilde{D}K_t(x, x)) = \text{Tr}_x(\widetilde{D}P(x, x, t))$$
$$+ \int_0^t ds\, \text{Tr}_{x'} \left(\int_{\widetilde{N}} K_{t-s}(x, y) \left(\frac{\partial}{\partial s} + \widetilde{S}^2 \right) \widetilde{D}P(y, x, s)\, d\, \text{vol}_{\widetilde{N}}(y) \right).$$

Bismut and Freed [5] showed that

$$(3.1.10) \qquad\qquad |\text{Tr}_x(\widetilde{D}P(x, x, t))| \leq A t^{1/2},$$

so that

$$\left| \text{Tr}_x \int_{\widetilde{N}} K_{t-s}(x, y)\widetilde{D} \left(\frac{\partial}{\partial s} + \widetilde{D}^2 \right) P(y, x, s)\, d\, \text{vol}_{\widetilde{N}}(y) \right|$$
$$\leq A \int_{B(x, \varepsilon)} \|K_t(x, y)\|_{x, y} s^{m-1}\, d\, \text{vol}_{\widetilde{N}}(y)$$
$$\leq A\, \text{vol}(B(x, \varepsilon))(t - s)^{-n/2} s^{m-1},$$

where $B(x, \varepsilon)$ is the metric ball of radius ε centered at x, and we have used standard estimates heat kernel for the Laplace Beltrami operator $0 \leq k_t(x, y) \leq A t^{-n/2}$; see Chavel [6] and 3.1.5. The bounded geometry of $\widetilde{N}$ implies that
(3.1.11)
$$\left| \int_0^t \text{Tr}_x \int_{\widetilde{N}} K_{t-s}(x, y)\widetilde{D} \left(\frac{\partial}{\partial s} + \widetilde{D}^2 \right) P(y, x, s)\, d\, \text{vol}_{\widetilde{N}}(y) \right| \leq A t^{1/2}.$$

The estimate of (3.1.2) is completed by using estimates (3.1.10) and (3.1.11). q.e.d.

By the remark following Definition 3.1.2 for any $f \in S(\mathbb{R})$, $S(\mathbb{R})$ denotes the Schwartz space where we have $\text{tr}_\Gamma f(\widetilde{D}) \geq 0$. Consider the linear functional
$$I(f) = \text{tr}_\Gamma f(\widetilde{D}) \quad \text{for } f \in S(\mathbb{R}).$$

By standard methods in harmonic analysis we have

$$(3.1.13) \qquad\qquad I(f) = \int_{\mathbb{R}} f\, dm_\Gamma,$$

where m_Γ is a tempered measure on $\mathbb{R}$, namely, there exists a positive integer l such that
$$\int_{\mathbb{R}} \frac{1}{(1 + |x|)^l}\, dm_\Gamma \text{ is finite}.$$

We now proceed to the main theorem of this section. Let

$$(3.1.14) \qquad \eta_\Gamma(0) = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2}\, \text{tr}_\Gamma(\widetilde{D}e^{-t\widetilde{D}^2})\, dt.$$

Theorem 3.1.1.

(3.1.15) $$|\eta_\Gamma(0)| \le A\operatorname{vol}(N),$$

where A is a constant satisfying the properties described in Lemma 3.1.1.

   Proof. As in Cheeger and Gromov [7] we split the integral in (3.1.14) into the following integrals and estimate them separately.

$$\frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} \operatorname{tr}_\Gamma(\widetilde{D} e^{-t\widetilde{D}^2})\, dt = \frac{1}{\Gamma(1/2)} \int_0^1 t^{-1/2} \operatorname{tr}_\Gamma(\widetilde{D} e^{-t\widetilde{D}^2})\, dt$$
$$+ \frac{1}{\Gamma(1/2)} \int_1^\infty t^{-1/2} \operatorname{tr}_\Gamma(\widetilde{D} e^{-t\widetilde{D}^2})\, dt.$$

Now

(3.1.16)
$$\left| \frac{1}{\Gamma(1/2)} \int_0^1 t^{-1/2} \operatorname{tr}_\Gamma(\widetilde{D} e^{-t\widetilde{D}^2})\, dt \right|$$
$$\le \int_0^1 \int_F t^{-1/2} |\operatorname{Tr}_x(\widetilde{D} K_t(x,x))|\, d\operatorname{vol}_{\widetilde{N}}\, dt$$
$$\le A \int_0^1 t^{-1/2} t^{1/2}\, dt \int_F d\operatorname{vol}_{\widetilde{N}} = A\operatorname{vol}(N),$$

where the final inequality follows from (3.1.2).

   We estimate

(3.1.17)
$$\left| \frac{1}{\Gamma(1/2)} \int_1^\infty t^{-1/2} \operatorname{tr}_\Gamma(\widetilde{D} e^{-t\widetilde{D}^2})\, dt \right|$$
$$\le \left| \frac{1}{\Gamma(1/2)} \int_1^\infty t^{-1/2} \int_{\mathbb{R}} \lambda e^{-t\lambda^2}\, dm_\Gamma(\lambda)\, dt \right|$$
$$\le \frac{1}{\Gamma(1/2)} \int_1^\infty t^{-1/2} \int_{\mathbb{R}} |\lambda| e^{-t\lambda^2}\, dm_\Gamma(\lambda)\, dt$$
$$= \frac{1}{\Gamma(1/2)} \int_{\mathbb{R}} |\lambda| e^{-\lambda^2} \int_1^\infty t^{-1/2} e^{-(t-1)\lambda^2}\, dt\, dm_\Gamma(\lambda)$$
$$\le \int_{\mathbb{R}} e^{-\lambda^2}\, dm_\Gamma(\lambda) = \operatorname{tr}_\Gamma(e^{-\widetilde{D}^2}).$$

By (3.1.8), $\operatorname{tr}(e^{-\widetilde{D}^2}) \le A\operatorname{vol}(N)$. Combining (3.1.16) and (3.1.17) we have (3.1.15).

   3.2. Eta invariants for foliations. Let N be a C^∞ compact Riemannian manifold, and $\mathscr{F}$ a smooth foliation on N. We assume that N does not have a boundary. For each leaf L of $\mathscr{F}$ we denote the volume element of the induced Riemannian metric by $d\operatorname{vol}_L$. Let ν be a holonomy invariant transverse measure, and $D_\mathscr{F}$ a leafwise Dirac operator acting

on a Clifford bundle δ (cf. Roe [29], Moore and Schochet [23]). Further we assume that $D_{\mathscr{F}}$ restricted to each leaf satisfies the Bismut Freed cancellation property. We now state the main theorem of this section.

Theorem 3.2.1. *The integral*

$$(3.2.1) \qquad \eta_{\mathscr{F}}(0) = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{1/2} \, \mathrm{tr}_\nu(D_{\mathscr{F}} e^{-tD^2\mathscr{F}}) \, dt$$

exists and satisfies the inequality

$$(3.2.2) \qquad |\eta_{\mathscr{F}}(0)| \leq A\eta(N),$$

where tr_ν *is the foliation trace given by the holonomy invariant transverse measure* ν, *A is a constant depending on the rank of the vector bundle S and the local geometry of the leaves, and* μ *is the total measure on N given by combining the leafwise volume elements with transverse measure* ν.

Proof. The main references for the theorem are Roe [32], [29], Moore and Schochet [23], and Connes [10]. The Dirac operator D_L on each leaf L is essentially selfadjoint on $C_c^\infty(L; S_L)$. By the spectral theorem $f(D_L)$ is a bounded operator for f a bounded Borel function on $\mathbb{R}$. If $f \in S(\mathbb{R})$, then the $f(D_L)$ is an integral operator with smooth kernel (cf. Roe [32]).

On the leaf L we can define a smooth measure $\mathrm{Tr}_{pt} f(D_L) \, d\,\mathrm{vol}_L$ if $f \in S(\mathbb{R})$, where $\mathrm{Tr}_{pt} f(D_L)$ is the pointwise trace of the integral kernel of $f(D_L)$. The family of measures $\{\mathrm{Tr}_{pt} f(D_L) \, d\,\mathrm{vol}_L\}$, by the parameterized version of the spectral theorem, is a Borel family of tangential measures on the equivalence relation corresponding to the foliation $\mathscr{F}$. By the uniform geometry of the leaves, $\mathrm{Tr}_{pt} f(D_L)$ is uniformly bounded over all leaves. By Proposition 4.22 in Moore and Schochet [23], the integral $\mathrm{tr}_\nu f(D_{\mathscr{F}}) = \int_N \lambda \, dv$ is well defined and finite, where $\lambda = \{\lambda_L\}_{L \in \mathscr{F}}$ denotes the tangential measures $\{\mathrm{Tr}_{pt} f(D_L) \, d\,\mathrm{vol}_L\}$. If $f \geq 0$, then $\mathrm{tr}_{pt} f(D_L) \, d\,\mathrm{vol}_L$ is a positive measure. This implies that $\mathrm{tr}_\nu f(D_{\mathscr{F}}) \geq 0$ if $f \geq 0$.

Consider the positive linear functional $I: S(\mathbb{R}) \to \mathbb{C}$ defined as

$$(3.2.3) \qquad I(f) = \mathrm{tr}_\nu f(D_{\mathscr{F}}).$$

There exists a tempered measure $m_{\mathscr{F}}$ on $f\mathbb{R}$ such that

$$(3.2.4) \qquad I(f) = \int_{\mathbb{R}} f \, dm_{\mathscr{F}}.$$

Let

$$(3.2.5) \qquad \eta_{\mathscr{F}}^\infty(0) = \frac{1}{\Gamma(1/2)} \int_1^\infty t^{-1/2} \, \mathrm{tr}_\nu(D_{\mathscr{F}} e^{-tD_{\mathscr{F}}^2}) \, dt.$$

From 3.2.4 replacing m_Γ by $m_{\mathscr{F}}$ in (3.1.17) we have $|\eta_{\mathscr{F}}^\infty(0)| \le A\mu(V)$. To deal with the integral

$$(3.2.6) \qquad \int_0^1 t^{-1/2} \, \mathrm{tr}_\nu (D_{\mathscr{F}} e^{-tD_{\mathscr{F}}^2}) \, dt,$$

we observe that (3.1.2) implies that

$$(3.2.7) \qquad |\mathrm{Tr}_{pt}(D_L e^{-tD_L^2}(x, x))| < A_L t^{1/2}, \qquad 0 \le t \le 1.$$

Since we have global bounds on the local geometry of the leaves in terms of the geometry of V and S, we have a uniform bound for the A_L's in (3.2.7). Thus

$$(3.2.8) \qquad |\mathrm{Tr}_{pt}(D_L e^{-tD_L^2}(x, x))| < At^{1/2}, \qquad 0 \le t \le 1,$$

for all leaves L.

From (3.2.8) we get

$$(3.2.9) \qquad |\mathrm{tr}_\nu(D_{\mathscr{F}} e^{-tD_{\mathscr{F}}^2})| = \left| \int_N \mathrm{Tr}_{pt}(D_L e^{-tD_L^2}(x, x)) \, d\,\mathrm{vol}_L \, d\nu \right|$$

$$\le At^{1/2}\mu(N).$$

Therefore

$$\left| \int_0^1 t^{-1/2} \, \mathrm{tr}_\nu(D_{\mathscr{F}} e^{-tD_{\mathscr{F}}^2}) \, dt \right| \le A\mu(N),$$

which completes the proof of the theorem.   q.e.d.

We remark that in the case of the signature operator with co-efficients in a flat bundle the methods in Cheeger and Gromov [8] can be adapted to our situation to insure that the constants A in Theorems 3.1.1 and 3.2.1 depend only on the metric data and not on its derivatives.

4. Selfadjointness of the boundary value problem

Our assumptions are the same as in §§2.1 and 2.2. Henceforth B will denote the boundary condition satisfying Definitions 2.1.2 and 2.1.3. This boundary condition will be known as the A.P.S. boundary condition. We prove in this section that D acting on $W^\infty(M; S)_B$ is essentially selfadjoint where

$$W^\infty(M; S)_B = \{ S \in W^\infty(M; S) | Bbf = 0 \}.$$

Our approach to the problem of essential selfadjointness was inspired by Roe [31]. This will involve the construction of bounded linear operators

R_1 and R_2 satisfying the properties:

(4.1) $$R_i \colon W^k(M\,;\,S) \to W^{k+1}(M\,;\,S)_B\,,$$

for $i = 1, 2$ and k a nonnegative integer, is continuous. Moreover,

(4.2) $$DR_1 - \mathrm{Id} = S_1\,, \qquad R_2D - \mathrm{Id} = S_2\,,$$

where S_1 and S_2 are smoothing operators, and

(4.3) $$W^k(M\,;\,S)_B = \{s \in W^k(M\,;\,S) | Bbs = 0\} \quad \text{for } k \geq 1\,.$$

Assuming the existence of operators R_i and their adjoints R_i satisfying (4.1)–(4.3) we prove the essential selfadjointness of the boundary value problem (B.V.P.).

Theorem 4.1. *The unbounded operator* $D \colon L^2(M\,;\,S) \to L^2(M\,;\,S)$ *with domain* $W^\infty(M\,;\,S)_B$ *is essentially selfadjoint.*

Proof. We first prove that the minimal domain of D is $W^1(M\,;\,S)_B$. This proof follows Roe [31]. The closure of $W^\infty(M\,;\,S)_B$ in the graph norm is clearly contained in $W^1(M\,;\,S)_B$. To prove the two spaces to be the same, we show that any element in $W^1(M\,;\,S)_B$ can be approximated by elements in $W^\infty(M\,;\,S)_B$. It is enough to show that elements in $W^1(M\,;\,S)_B$ supported in a collar $[0,1) \times N$ can be approximated by elements in $W^\infty(M\,;\,S)_B$. Let $s \in W^1(M\,;\,S)_B$ be supported in $[0,1) \times N$. Then we can think of s as an element of $W^1([0,\infty) \times N\,;\,S)_B$. We regularize s in the boundary direction by the operator

(4.4) $$H_t s = be^{-tQ^2}Bs + (I - B)e^{-tQ^2}(I - B)s\,.$$

By reflection we can extend s to $W^1((-\infty, \infty) \times N\,;\,\widehat{S})$, so in the cylinder direction we regularize s by

(4.5) $$s_\varepsilon = \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \varphi\left(\frac{x-y}{\varepsilon}\right)s\,dy\,,$$

where ϕ is a smooth compactly supported positive function on $[-1,1]$ with $\int \phi = 1$ and even ϕ. Combining (4.4) and (4.5) gives the required approximation.

We now prove that

$$\text{domain of } D^* = W^1(M\,;\,S)_B\,,$$

where D^* is the Hilbert adjoint of D. Now

(4.7) $$\mathrm{Dom}(D^*) = \{s \in L^2(M\,;\,S)|f \mapsto \langle S, Df\rangle \text{ for } f \in W^1(M\,;\,S)_B$$

$$\text{extends to a bounded linear functional on } L^2\}\,,$$

(4.8) $$\langle f, Ds\rangle = \langle g, s, \rangle \quad \text{for all } s \in W^1(M\,;\,S)_B\,.$$

Since $DR_1s + S_1s = s$, by (4.8) we have

$$\langle f, s \rangle = \langle f, DR_1S + S_1s \rangle = \langle g, R_1s \rangle + \langle f, S_1s \rangle$$
$$= \langle R_1^* g, s \rangle + \langle S_1^* f, s \rangle.$$

Hence $f = R_1^* g + S_1^* f \in W^1(M; S)$ and bf is defined. Also if $s \in W^1(M; S)_B$, then

$$(4.9) \qquad \langle f, Ds \rangle = \langle Df, s \rangle + \langle bf, \sigma bs \rangle = \langle g, s \rangle.$$

By choosing s compactly supported in the interior of M we see that $Df = g$, so that by (4.9) we have

$$\langle bf, \sigma bs \rangle = 0 \quad \text{for all } s \in W^1(M; S)_B.$$

Therefore $bf \in (\sigma \ker B)^\perp = \ker B$. Note $B^2 = B$. This implies that $f \in W^1(M; S)_B$.   q.e.d.

The construction of the operators R_i is based on the following procedure. We first construct an interior parametrix for D. Following the notation in §2.1 we consider the data $(\hat{D}, \hat{S}, \hat{\varepsilon})$ on the complete manifold without boundary $db(M)$. By Chernoff [9], $\hat{D}$ is essentially selfadjoint on $C_c^\infty(db(M); \hat{S})$. We construct parametrices I_1 and I_2 with appropriate Sobolev regularity, namely

$$(4.10) \qquad I_l : W^k(db(M) : \hat{S}) \to W^{k+1}(dbM; \hat{S}), \qquad l = 1, 2,$$

and k a nonnegative integer and

$$(4.11) \qquad\qquad\qquad DI_1 - \text{Id} = S_1',$$

$$(4.12) \qquad\qquad\qquad I_2 D - \text{Id} = S_2',$$

where S_1' and S_2' are smoothing operators.

The next step is to construct a boundary parametrix and then to patch the two together to get operators R_i. The construction of boundary parametrix is very similar to that in §2 of [3]. The main tools in the construction of boundary parametrix are separation of variables and Theorem 2.2.1.

Since we will be first constructing the boundary parametrix in a neighborhood of the collar $[0, 1) \times N$ and then multiplying this parametrix by bump functions, we will assume without loss of generality that the collar is $[0, \infty) \times N$, with the product metric. For the rest of this section let $M = [0, \infty) \times N$.

Diagonalizing ε the grading operator we have $S = S^+ \oplus S^-$ where S^+ and S^- are the $+1$ and -1 eigenspaces of ε respectively. Then

$$(4.13) \; D : W^\infty(M; S^+) \oplus W^\infty(M; S^-) \to W^\infty(M; S^+) \oplus W^\infty(M; S^-).$$

Since $D = \sigma(\partial/\partial y + Q)$ where $\sigma^2 = -1$, σ anticommutes with ε and Q commutes with ε. Therefore we have

$$(4.14) \qquad Q = \begin{pmatrix} Q_+ & 0 \\ 0 & Q_- \end{pmatrix} \quad \text{and} \quad \sigma = \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix},$$

where $\beta : S^+ \to S^-$ is an isomorphism given by the Clifford action. Identifying S^- with S^+ and using β we can assume

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$(4.15) \qquad Q = \begin{pmatrix} Q_+ & 0 \\ 0 & -Q_+ \end{pmatrix} : W^\infty(M; S^+) \oplus W^\infty(M; S^+).$$

Further,

$$(4.16) \qquad B = \begin{pmatrix} P & 0 \\ 0 & I - P \end{pmatrix} \quad \text{where } P = \chi_{[0,\infty)}(Q_+).$$

Further, Q_+ and all its powers are essentially selfadjoint. Hence we can apply Theorem 2.2.1. Under these identifications our B.V.P. is

$$(4.17) \quad D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\frac{\partial}{\partial y} + \begin{pmatrix} Q_+ & 0 \\ 0 & -Q_+ \end{pmatrix} \right), \quad B = \begin{pmatrix} P & 0 \\ 0 & I - P \end{pmatrix}.$$

This application of the spectral transform to prove Sobolev regularity of the boundary parametrix is based on the observation

$$f \in W^k(M; S) \Leftrightarrow f \in L^2([0, \infty); W^k(N; S))$$

and

$$\frac{\partial^j f}{\partial y^j} \in W^j([0, \infty); W^{k-j}(N; S)), \quad 1 \leq j \leq k, \ k \geq 0, \ ke \in \mathbb{Z}.$$

Now

$$(4.18) \quad W^k([0, \infty); H) = \left\{ f \mid \frac{\partial^j f}{\partial y^j} \in L^2([0, \infty); H), \ 0 \leq j \leq k \right\},$$

where H is a Hilbert space.

Theorem 4.2. *There exists an inverse T to D acting on $C_c^\infty([0, \infty); W^\infty(N; S))$ with the following properties:*

$$(4.19) \ T : C_c^\infty([0, \infty); W^\infty(N; S)) \to C^\infty([0, \infty); W^\infty(N; S))_B,$$
$$(4.20) \qquad\qquad DTf = f \quad \text{and} \quad TDf = f.$$

Further T extends to a continuous operator

(4.21) $T: W^k(M; S) \to W_{y\,\text{loc}}^{k+1+}(M; S), \qquad k \geq 0, \; k \in Z,$

where

$W_{y\,\text{loc}}^k(M; S) = \{f \text{ a distribution} | \phi f \in W^k(M; S), \; \phi \in C_c^\infty([0, \infty))\}.$

Proof. Our construction is the same as that in [3] except that we use Theorem 2.2.1 instead of eigenfunction expansions.

We need to solve the equations

$$(\partial/\partial y + Q_+)f_1 = g_1 \quad \text{with } Pf_1(0) = 0,$$
$$(\partial/\partial y - Q_+)f_2 = g_2 \quad \text{with } (I - P)f_2(0) = 0.$$

Applying the spectral transform we find

$$\left(\frac{\partial}{\partial y} + \lambda\right)(Vf_1)(\lambda, y) = (Vg_1)(\lambda, y) \quad \text{with } (Vf_1)_j(\lambda, 0) = 0$$

$$\text{for } \lambda \geq 0,$$

(4.22)

$$\left(\frac{\partial}{\partial y} - \lambda\right)(Vf_2)_j(\lambda, y) = (Vg_2)_j(\lambda, y) \quad \text{with } (Vf_2)_j(\lambda, 0) = 0$$

$$\text{for } \lambda < 0.$$

Denote the Fourier Laplace transform by

$$\tilde{g}(\xi) = \int_0^\infty e^{iy\xi} g(y)\, dy.$$

Then the solutions of (4.22) are

$$(Vf_1)_j(\lambda, y) = \int_0^y e^{\lambda(x-y)}(Vg_1)_j(\lambda, x)\, dx \quad \text{if } \lambda \geq 0$$

$$= -\int_y^\infty e^{\lambda(x-y)}(Vg_1)_j(\lambda, x)\, dx \quad \text{if } \lambda < 0,$$

(4.23)

$$(Vf_2)_j(\lambda, y) = -\int_y^\infty e^{\lambda(y-x)}(Vg_2)_j(\lambda, x)\, dx \quad \text{for } \lambda \geq 0$$

$$= \int_0^y e^{\lambda(y-x)}(Vg_2)_j(\lambda, x)\, dx \quad \text{for } \lambda < 0.$$

Apply the Fourier Laplace transform with respect to the y variable to (4.22), we get

$$(\lambda + i\xi)(\widetilde{V}f_1)_j(\lambda, \xi) = (\widetilde{V}g_1)_j(\lambda, \xi) + (Vf_1)_j(\lambda, 0),$$

$$\text{where } (Vf_1)_j(\lambda, 0) = 0 \text{ for } \lambda \geq 0$$

(4.24)

$$(-\lambda + i\xi)(\widetilde{V}f_2)_j(\lambda, \xi) = (\widetilde{V}g_2)_j(\lambda, \xi) + (Vf_2)_j(\lambda, 0),$$

$$\text{where } Vf_2)_j(\lambda, 0) = 0 \text{ for } \lambda < 0.$$

Also

$$(Vf_1)_j(\lambda, 0) = -\int_0^\infty e^{\lambda x}(Vg_1)_j(\lambda, x)\,dx \quad \text{if } \lambda < 0,$$

(4.25)

$$(Vf_2)_j(\lambda, 0) = -\int_0^\infty e^{-\lambda x}(Vg_2)_j(\lambda, x)\,dx \quad \text{if } \lambda \geq 0.$$

When estimating the L^2 norms, we find that f_1 and f_2 are in $L^2_{y\,\text{loc}}$. We now prove that

(4.26) $$T^{\pm}: L^2 \to W^1_{y\,\text{loc}},$$

where

$$T^+ g_1 = V^{-1}(Vf_1) \quad \text{and} \quad T^- g_2 = V^{-1}(Vf_2).$$

(Vf_1) and (Vf_2) being defined by (4.23) are continuous. The higher derivative estimates are similar. From (4.24) it follows that

$$\lambda^2 \int |(\widetilde{V}f_i)_j(\lambda, \xi)|^2\,d\xi \leq \int |(\widetilde{V}g_i)_j(\lambda, \xi)|^2\,d\xi \quad \text{for } \lambda \geq 0$$

$$\leq 2\left\{\int |(\widetilde{V}g_i)_j(\lambda, \xi)|^2\,d\xi + \lambda^2|(Vf_i)_j(\lambda, 0)|\int_{-\infty}^\infty \frac{d\xi}{\lambda^2 + \xi^2}\right\}$$

$$\leq 4\int |(Vg_i)_j(\lambda, y)|^2\,dy,$$

where $i = 1, 2$.

Using (4.22), we obtain

$$\frac{\partial}{\partial y}(Vf_1)_j(\lambda, y) = -\lambda(Vf_1)_j(\lambda, y) + (Vg_1)_j(\lambda, y),$$

(4.27)

$$\frac{\partial}{\partial y}(Vf_2)_j(\lambda, y) = \lambda(Vf_2)_j(\lambda, y) + (Vg_2)_j(\lambda, y).$$

Thus

$$\int \left|\frac{\partial}{\partial y}(Vf_i)_j(\lambda, y)\right|^2\,dy \leq 9\int |(Vg_i)_j(\lambda, y)^2\,dt.$$

We consider

$$\int |(Vf_1)_j(\lambda, y)|^2\,dy = \int \left|\int_0^y e^{\lambda(x-y)}(Vg_1)_j(\lambda, x)\,dx\right|^2\,dy \quad \text{for } \lambda \geq 0.$$

Applying the Cauchy-Schwartz inequality yields

(4.28) $$\int e^{-2\lambda y}\left[\left(\int_0^y e^{2\lambda x}\,dx\right)\int_0^y |(Vg_1)_j(\lambda, x)|^2\,dx\right]\,dy$$

$$\leq \int e^{-2\lambda y}\left[\frac{e^{2\lambda y} - 1}{2\lambda}\right]\,dy\int_0^\infty |(Vg_1)_j(\lambda, x)|^2\,dx.$$

One can estimate uniformly in λ, $(1 - e^{-2\lambda y})/f2\lambda$. Hence this implies that $(Vf_i)_j(\lambda, y) \in L^2_{y \, loc}$.

Putting these estimates together we have

$$T_1 = \begin{pmatrix} T^+ & 0 \\ 0 & T^- \end{pmatrix} : L^2 \to W^1_y \, loc,$$

where T_1 is the solution to (4.22). Higher derivative estimates are made in the same fashion as in §2 of [3]. We get the inverse of D:

$$T = \begin{pmatrix} T^+ & 0 \\ 0 & T^- \end{pmatrix} \begin{pmatrix} 0 & -\beta^{-l} \\ 0 & \beta \end{pmatrix} . \quad \text{q.e.d.}$$

Now we patch T and I_l together as in §3 of [3]. Let $p(a, b)$ denote an increasing C^∞ function of the real variable y with $p = 0$ for $t \leq a$ and $p = 1$ for $y \geq b$. Define C^∞ functions ϕ_1, ϕ_2, ψ_1, ψ_2 by

(4.29)
$$\phi_2 = p\left(\frac{1}{4}, \frac{1}{3}\right), \qquad \psi_2 = p\left(\frac{1}{2}, \frac{3}{4}\right),$$
$$\phi_1 = 1 - p\left(\frac{5}{6}, 1\right), \qquad \psi_1 = 1 - \psi_2.$$

Then $R_l = \phi_1 T \psi_1 + \phi_2 I_l \psi_2$. Putting the estimates for T and I_l together we see that R_l satisfies the properties (4.1)–(4.3) and so does its adjoint.

5. Parametrix for the parabolic initial boundary value problem

In §4 we showed that $D: L^2(M; S) \to L^2(M; S)$ densely defined with domain $W^1(M; S)_B$ is selfadjoint. Further from the regularity properties of the parametrix constructed in §4 it follows that $\text{Dom}(D^k) \subseteq W^k(M; S)$ for k a positive integer. Therefore by duality and the spectral theorem we have

Proposition 5.1. *If* $f \in RB(\mathbb{R})$, *then* $f(D): W^{-k_1}(M; S) \to W^{k_2}(M; S)$ *for all* k_1 *and* k_2 *positive integers. Hence* $f(D)$ *is represented by a smooth kernel.*

Proof. Use functional calculus and Proposition 2.2.3.   q.e.d.

By Proposition 5.1, e^{-tD^2} is a smoothing operator. The rest of this section focuses on constructing a parametrix for this initial boundary value problem. As in [3] we construct an interior parametrix by considering the heat kernel of $\hat{D}$ on the $db(M)$ and restricting to the interior of M. As in §4, in the construction of the boundary parametrix we can assume that our collar is $M = [0, \infty) \times N$. Under the identifications (4.13)–(4.15), by

the spectral transform the initial boundary value problem

$$(5.1) \quad \left(\frac{\partial}{\partial t} + D^2\right) f(\ , t) = 0, \quad f(\ , 0) = g(\ ), \quad Bbf = 0, \quad BbDf = 0$$

reduces to

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial y^2} + \lambda^2\right)(Vf_1)_j(\lambda, y, t) = 0,$$

$$(Vf_1)_j(\lambda, 0, t) + \lambda(Vf_1)_j(\lambda, 0, t) = 0 \quad \text{for } \lambda \ge 0,$$

(5.2)

$$\left(\frac{\partial}{\partial y}(Vf_1)_j(\lambda, 0, t) + \lambda(Vf_1)_j(\lambda, 0, t)\right) = 0 \quad \text{for } \lambda < 0,$$

$$(Vf_1)_j(\lambda, y, 0) = (Vg_1)_j(\lambda, y),$$

and

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial y^2} + \lambda^2\right)(Vf_2)_j(\lambda, y, t) = 0,$$

$$(Vf_2)_j(\lambda, 0, t) = 0 \quad \text{for } \lambda < 0,$$

(5.3)

$$-\frac{\partial}{\partial y}(Vf_2)_j(\lambda, 0, t) + \lambda(Vf_2)_j(\lambda, 0, t) = 0 \quad \text{for } \lambda \ge 0,$$

$$(Vf_2)_j(\lambda, y, 0) = (Vg_2)_j(\lambda, y),$$

where

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

The solution to (5.2) is given by

$$(Vf_1)_j(\lambda, x, t) = \int_0^\infty a_\lambda(x, y, t)(Vg_1)_j(\lambda, y)\, dt \quad \text{for } \lambda \ge 0$$

(5.4)

$$= \int_0^\infty b_\lambda(x, y, t)(Vg_1)_j(\lambda, y)\, dy \quad \text{for } \lambda < 0,$$

where

$$(5.5) \quad a_\lambda(x, y, t) = \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}}\left[\exp\left(-\frac{(x-y)^2}{4t}\right) - \exp\left(-\frac{(x+y)^2}{4t}\right)\right]$$

and

$$(5.6) \quad b_\lambda(x, y, t) = a_\lambda(x, y, t) + \lambda e^{-\lambda(x+y)} \operatorname{erfc}\left\{\frac{x+y}{2\sqrt{t}} - \lambda\sqrt{t}\right\},$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2}\, d\xi.$$

Similarly the solution of (5.3) is

$$
(5.7) \quad (Vf_2)_j(\lambda, x, t) = \int_0^\infty a_\lambda(x, y, t)(Vg_2)_j(\lambda, y)\, dy \quad \text{for } \lambda < 0
$$
$$
= \int_0^\infty b_{-\lambda}(x, y, t)(Vg_1)_j(\lambda, y)\, dy \quad \text{for } \lambda \geq 0.
$$

These solutions can be found in [3].

Let

$$
(5.8) \quad
\begin{aligned}
E^+(t)g(x, n) \\
= \sum_j \int_{\lambda \geq 0} \int_0^\infty a_\lambda(x, y, t)(Vg)_j(\lambda, y)e_j(n, \lambda)\, dy\, d\mu_j(\lambda) \\
+ \sum_j \int_{\lambda < 0} \int_0^\infty b_\lambda(x, y, t)(Vg)_j(\lambda, y)e_j(n, \lambda)\, dy\, d\mu_j(\lambda),
\end{aligned}
$$

$$
(5.9) \quad
\begin{aligned}
E^-(t)g(x, n) \\
= \sum_j \int_{\lambda < 0} \int_0^\infty a_\lambda(x, y, t)(Vg)_j(\lambda, y)e_j(n, \lambda)\, dy\, d\mu_j(\lambda) \\
+ \sum_j \int_{\lambda \geq 0} \int_0^\infty b_{-\lambda}(x, y, t)(Vg)_j(\lambda, y)e_j(n, \lambda)\, dy\, d\mu_j(\lambda).
\end{aligned}
$$

Now the off-diagonal exponential decay, along with the term $e^{-\lambda^2 t}$ of $a_\lambda(x, y, t)$, $b_\lambda(x, y, t)$, and its derivatives imply that $E^+(t)$ and $E^-(t)$ are smoothing operators and are represented by smooth kernels for $t > 0$. Define

$$
(5.10) \quad E_1(t) = \begin{pmatrix} E^+(t) & 0 \\ 0 & E^-(t) \end{pmatrix} \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix}.
$$

Then $E_1(t)$ is the fundamental solution of the initial boundary value problem (5.1) on the cylinder $[0, \infty) \times N$. Let $F(t)$ be the fundamental solution of the heat equation on the double of M for $\hat{D}$. Then we will patch $F(t)$ and $E_1(t)$ to get a parametrix for the initial boundary value problem on M. We defined the functions ϕ_1, ϕ_2 and ψ_1, ψ_2 in §4, and now use them to construct

$$
(5.11) \quad E(t) = \phi_1 E_1(t)\psi_1 + \phi_2 F(t)\psi_2.
$$

Note that $(\partial/\partial t + D^2)E(t)$ is $O(t^k)$ for all $k > 0$. This follows from the off-diagonal exponential decay of $a_\lambda(x, y, t)$, $b_\lambda(x, y, t)$, and $F(t)$.

Theorem 5.1.

$$
(5.12) \quad e^{-tD^2} - E(t): W^{-k_1}(M; S) \to W^{k_2}(M; S)
$$

is a bounded linear operator for all k_1 and k_2 positive integers. For k_1 and k_2 very large positive integers there exists an $\alpha > 0$ such that

$$\|e^{-tD^2} - E(t)\|_{L(W^{-k_1}, W^{k_2})} \le Ct^\alpha \quad \text{for all } 0 \le t \le 1.$$

Proof. This follows from Duhamel's formula

$$e^{-tD^2} - E(t) = \int_0^t e^{-(t-s)D^2} \left(\frac{\partial}{\partial s} + D^2\right) E(s)\, dx$$

and Sobolev estimates. The basic idea is the following:

$$\left(\frac{\partial}{\partial s} + D^2\right) E(s) = \frac{\partial^2 \phi_2}{\partial y^2} F(t) \psi_2 + \frac{\partial \psi_2}{\partial y} \frac{\partial F(t)}{\partial y} \psi_2$$

$$+ \frac{\partial \phi_1}{\partial y} \frac{\partial E_1(t)}{\partial y} \psi_1 + \frac{\partial^2 \psi_1}{\partial y^2} E_{11}(t) \psi_1.$$

Now the off-diagonal exponential decay of $a_\lambda(x, y, t)$, $b_\lambda(x, y, t)$, and $F(t)$ along with the fact that

$$\frac{\partial^2 \phi_2}{\partial y^2}, \psi_2; \quad \frac{\partial \phi_2}{\partial y}, \psi_2; \quad \frac{\partial \phi_1}{\partial y}, \psi_1; \quad \frac{\partial^2 \phi_1}{\partial y^{22}}, \psi_1$$

have disjoint supports proves the theorem.

6. Breuer index for coverings and foliations

This section formulates the Breuer index for elliptic operators on coverings of compact manifolds with boundary and for elliptic operators on leaves of foliations compact manifolds with boundary with leaves transverse to the boundary. Heuristically, in the case of Galois coverings of a compact manifold with boundary M and Galois group Γ, we deal with the von Neumann algebra with a semifinite faithful trace:

$$\text{End}_\Gamma L^2(\widetilde{M}) = \{\text{all bounded linear operators commuting}$$
$$\text{with the } \Gamma \text{ action}\}.$$

This trace is defined on a smaller class of operators. We will work with a dense subalgebra of $\text{End}_\Gamma L^2(\widetilde{M})$ to define finite-Γ dimensionality.

In the case of a foliation $\mathscr{F}$ of M we have the Borel equivalence relation

$$\mathscr{R} = \{(x, y) | x \text{ and } y \text{ are on the same leaf } L \in \mathscr{F}\},$$

which has the structure of a measurable groupoid. See Moore [22] and Moore and Schochet [23] for definitions and more details. Now the

groupoid $\mathscr{R}$ acts on the field of Hilbert spaces $H = \{L^2(L_x)\}_{x \in M}$ naturally. Following Connes [10] we study von Neumann algebra of intertwining endomorphisms of the field H up to suitable equivalence given by a holonomy invariant transverse measure ν. Moreover the transverse measure gives rise to a natural faithful semifinite trace, which is used to define the notion of finite ν dimension. Section 6.1 deals with the case of coverings, and §6.2 the case of foliations.

6.1. Finite dimensionality of the Γ index. Let M be a compact Riemannian manifold with boundary N. Let D be a Dirac operator on a graded Clifford bundle S with grading operator ε. We assume that the data (D, S, ε) has a product structure in a neighborhood of the collar $[0, 1] \times N$; see Definition 2.1.1. Let $\widetilde{M}$ be a Galois covering of M with Galois group Γ. We denote the lifts of D and S to $\widetilde{M}$ by $\widetilde{D}$ and $\widetilde{S}$ respectively. Let $\widetilde{B}$ be the A.P.S. boundary condition associated to $\widetilde{D}$ as in Definitions 2.1.2 and 2.1.3.

In §4 we showed that the densely defined operator $\widetilde{D} : L^2(\widetilde{M}; \widetilde{S}) \to L^2(\widetilde{M}; \widetilde{S})$ with $\mathrm{Dom}(\widetilde{D}) = W^1(\widetilde{M}; \widetilde{S})_{\widetilde{B}}$ is selfadjoint. Further, $\widetilde{D}$ commutes with the action of Γ. As in Roe [30] we introduce a dense subalgebra $\mathfrak{U}$ of $\mathrm{End}_\Gamma(L^2(\widetilde{M}; \widetilde{S}))$.

Definition 6.1.1. $A \in \mathfrak{U}$ if the following hold:

1. A is given by an integral kernel $k(x, y)$ with the properties: There is a constant C such that $\int |k(x, y)|^2 \, d\,\mathrm{vol}_{\widetilde{N}}(y) < C$ and $\int |k(x, y)|^2 \, d\,\mathrm{vol}_{\widetilde{N}}(x) < C$ for every x and $y \in \widetilde{M}$ respectively.

2. A is smoothing, namely,

$$As(x) = \int_{\widetilde{N}} k(x, y)s(y) \, d\,\mathrm{vol}_{\widetilde{N}}(y) \quad \text{for } S \in L^2(\widetilde{M}; \widetilde{S}),$$

and the maps $x \to k(x, \cdot)$ and $y \to k(\cdot, y)$ are smooth maps of $\widetilde{M}$ to the Hilbert space $L^2(\widetilde{M}; \widetilde{S})$.

Proposition 6.1.1. *The set of operators $\mathfrak{U}$ forms an algebra.*

Proof. The proof for manifolds with boundary follows exactly as that of Proposition 13.5 of Roe [30].

Lemma 6.1.1. *There exists a fundamental domain for the Γ action on $\widetilde{M}$, which we label F.*

Proof. The proof is exactly the same as in Atiyah [1] where it is proved for manifolds without boundary.   q.e.d.

We now define a functional $\tau : \mathfrak{U} \to \mathbb{C}$, which we call a trace, as follows.

Definition 6.1.2. Let $A \in \mathfrak{U}$. Then

$$(6.1.1) \qquad \tau(A) = \int_F \mathrm{tr}\, k(x, x) \, d\,\mathrm{vol}_{\widetilde{N}}(x),$$

where F is a fundamental domain for the Γ action. The fact that A commutes with the Γ action implies that the definition of τ is independent of the choice of fundamental domain. We also use the notation tr_Γ for τ.

Proposition 6.1.2. *For* A_1, $A_2 \in \mathfrak{U}$ *we have* $\tau(A_1 A_2) = \tau(A_2 A_1)$.

Proof. The proof for manifolds with boundary is the same as that for manifolds without boundary; see Proposition 13.10 of Roe [30].

Definition 6.1.3. A closed subspace H of $L^2(\widetilde{M}; \widetilde{S})$ is said to be of finite Γ-dimension if the orthogonal projection $P \colon L^2(\widetilde{M}; \widetilde{S}) \to H$ belongs to $\mathfrak{U}$. In this case we define

$$(6.1.2) \qquad\qquad \dim_\Gamma(H) = \tau(P).$$

Proposition 6.1.3. *For any* $f \in RB(\mathbb{R})$, $f(\widetilde{D}) \in \mathfrak{U}$.

Proof. Follows immediately from Proposition 6.1.1.

Theorem 6.1.1. $\widetilde{D} \colon L^2(\widetilde{M}; \widetilde{S}) \to L^2(\widetilde{M}; \widetilde{S})$ *is a closed densely defined operator with* $\mathrm{Dom}(\widetilde{D}) = W^1(\widetilde{M}; \widetilde{S})_{\widetilde{B}}$ *and has finite Γ-dimensional kernel, and the Γ index* $\mathrm{ind}_\Gamma(\widetilde{D}) = \dim_\Gamma(\ker \widetilde{D}^+) - \dim_\Gamma(\ker \widetilde{D}^-)$ *is finite.*

Proof. The selfadjointness of $\widetilde{D}$ implies that $\widetilde{D}^+$ and $\widetilde{D}^-$ are Hilbert space adjoints of each other. By Proposition 6.1.3, the projection onto $\ker(\widetilde{D})$ belongs to $\mathfrak{U}$. Therefore $\dim_\Gamma \ker(\widetilde{D})$ is finite. This implies that $\mathrm{ind}_\Gamma(\widetilde{D})$ is finite. q.e.d.

The next proposition is called the McKean-Singer formula.

Proposition 6.1.4.

$$(6.1.3) \qquad\qquad \mathrm{ind}_\Gamma(\widetilde{D}) = \tau(\varepsilon e^{-t\widetilde{D}^2}).$$

Proof. The proof is the same as that of Proposition 13.14 of Roe [30], if we observe that $\widetilde{D}^+$ and $\widetilde{D}^-$ are Hilbert space adjoints of each other. q.e.d.

In §7.1 we will use Proposition 6.1.3 to identify the Γ-index in terms of topological data and the correction term arising from the eta invariant of §3.1.

6.2. Finite ν-dimensionality of the foliation index problem. Let $(M, \mathscr{F})$ be a compact foliated manifold with boundary and the foliation transverse to the boundary; see Definition 2.3.1. Let $D_{\mathscr{F}}$ be a leafwise Dirac operator on a graded Clifford bundle S with grading operator. We also assume that the data $(D_{\mathscr{F}}, S, \varepsilon)$ has a product structure in the sense of Definition 2.3.2, near the boundary. Let $\mathscr{R}_M$ denote the measurable equivalence relation on M given by the leaves of the foliation $\mathscr{F}$, and R_N the equivalence relation on the induced foliation of the boundary N.

Let $\mathbf{H} = \{L^2(L_x; S_{L_x})_{x \in M}\}$, where L_x is the leaf through x, is a Borel field of Hilbert spaces. See Dixmier [12, Chapter I, Part II] for definitions. By Proposition 4 of Dixmier [12, p. 167], to prescribe a measure structure on the field of Hilbert spaces H, it is enough to prescribe a countable sequence $\{s_j\}$ of sections of this field of Hilbert spaces with the additional property that for $x \in M$ the countable set $\{s_j(x)\} \subset L^2(L_x; S_{L_x})$ is a complete orthonormal set. We can do this in our context with the property that each $s_j(x)$ is also a smooth section on the leaf L_x. See Appendix of Heitsch and Lazarov [17].

There is a natural representation of the equivalence relation $\mathscr{R}$ on H, namely, if $(x, y) \in \mathscr{R}$, then the unitary isomorphism from $L^2(L_y; S_{L_y})$ to $L^2(L_y; S_y)$ is just the identity map.

A Borel transversal to the foliation $\mathscr{F}$ is a Borel subset of M which intersects every leaf in at most a countable set. The Borel transversals of $\mathscr{F}$ generate a σ-ring S, namely, it is closed under countable unions and relative complementation. Note that the holonomy pseudogroups act on the σ-ring S. For more on this see Hector and Hirsch [16, Chapters III, X].

Definition 6.2.1. A transverse measure ν is a measure ν on the σ-ring S of Borel transversals, and $\nu|_T$ is σ-finite for every $T \in S$.

Definition 6.2.2. A transverse measure ν is holonomy invariant if it is invariant under the action of the holonomy pseudogroup on the σ-ring.

We note that the natural representation of $\mathscr{R}$ on H is "square integrable" in the sense of Connes [10]. We denote

$$\text{End}_{\mathscr{R}}(H) = \{\text{uniformly bounded measurable field of}$$
(6.2.1)   bounded operators intertwining the natural

representation of $\mathscr{R}$ on $H\}$.

Given a holonomy invariant transverse measure ν Connes defines in [10, Chapter V] a von Neumann algebra

$$\text{End}_{\nu}(H) = \{[T] | T \in \text{End}_{\mathscr{R}}(H) \text{ and } T_1 \sim T_2 \text{ if they are}$$
(6.2.2)
equal for ν almost every leaf$\}$.

He also shows that $\text{End}_{\nu}(H)$ is a direct integral of type I and type II von Neumann algebras. In part it has a semifinite faithful trace tr_{ν} obtained from ν. If the field of operators $T \in \text{End}_{\mathscr{R}}(H)$ in the domain of tr_{ν} is implemented by a family of integral operators, one for every leaf $L \in \mathscr{F}$, with the family of leafwise kernels $\{k_L(x, y)\}_{x, y \in L/L \in \mathscr{F}}$, then the

von Neumann trace of T is given by

$$(6.2.3) \qquad \mathrm{tr}_\nu(T) = \int_M k_L(x, x) \, d\,\mathrm{vol}_L \, d\nu.$$

The integral is well defined, and the holonomy invariance of the measure implies that the modular automorphism group generated by this state is trivial.

Lemma 6.2.1. *If the kernels $k_L(x, y)$ are uniformly bounded over all leaves, then $\mathrm{tr}_\nu(T)$ is finite.*

Proof. This follows immediately from (6.2.3).

Definition 6.2.3. We say a measurable field of closed subspaces of H has finite ν-dimension if the corresponding family of orthogonal projections have finite ν trace.

Let $B_{\mathscr{F}} = \{B_L\}_{L \in \mathscr{F}}$ denote the family of A.P.S. boundary conditions, for each leaf $L \in \mathscr{F}$. We thus have a family of closed densely defined unbounded selfadjoint operators

$$(6.2.4) \qquad D_{L_x} : L^2(L_x ; S_{L_x}) \to L^2(L_x ; S_{L_x}), \qquad x \in M,$$

where L_x is the leaf through x, with $\mathrm{Dom}(D_{L_x}) = W^1(L_x ; S_{L_x})_{B_{L_x}}$, B_{L_x} being the A.P.S. boundary condition defined in §2.1 for the leaf L_x. We wish to show that the family $\{D_{L_x}\}_{x \in M}$ is a measurable family of selfadjoint operators. This enables us to use the measurable spectral theorem (cf. Reed and Simon [28, Theorem XIII.85] to prove

Theorem 6.2.1. *If f is a bounded Borel function, then*

$$(6.2.5) \qquad \{f(D_{L_x})\}_{x \in M} \in \mathrm{End}_{\mathscr{R}}(H).$$

Proof. To prove the measurability of the family of operators in (6.2.4) it is enough to show that the family $\{(D_{L_x} + i)^{-1}\}_{x \in M}$ is a measurable family of bounded operators. The family of Hilbert spaces $W^1(L_x ; S_{L_x})_{B_{L_x}}$ has a natural measure structure given by its inclusion into H.

Proposition 6.2.1. *The field of bounded operators*

$$(6.2.6) \qquad \{D_{L_x} + i\}_{x \in M} : \{W^1(L_x ; S_{L_x})_{B_{L_x}}\} \ to \ \{L^2(L_x ; S_{L_x})\}_{x \in M}$$

is measurable, and the leafwise defined inverse is also a measurable family.

Proof. The selfadjointness of D_{L_x} with domain $W^1(L_x ; S_{L_x})_{B_{L_x}}$ implies that $D_{L_x} + i : W^1(L_x ; S_{L_x})_{B_{L_x}}$ is a Hilbert space isomorphism. Let s, t be measurable sections of the domain and range respectively. Following Heitsch and Lazarov [17] we can choose t so that $t(x)$ is smooth

on L_x. Then

$$\langle (D_{L_x} + i)s(x), t(x) \rangle_{L^2(L_x;S_{L_x})} = \langle s(x), (D_{L_x} - i)t(x) \rangle_{L^2(L_x;S_{L_x})} + \int_{\partial L_x} \langle \ , \ \rangle$$

by formula (2.1.1). The measurability of the right-hand side of the above equation as a function of x follows immediately. From Example 2 of Dixmier [20, p. 180] the leafwise inverse is a measurable field of operators. Combining Proposition 6.2.1 with the measurable spectral theorem in Reed and Simon [28] immediately proves the theorem.

Lemma 6.2.2. *If* $f \in RB(\mathbb{R})$, *then* $\text{tr}_\nu f(D_{\mathscr{F}})$ *is finite.*

Proof. Proposition 5.1 and Theorem 6.2.1 imply that $\{f(D_{L_x})\}_{x \in M}$ is a measurable family of integral operators with the integral kernels smooth and uniformly bounded. Therefore by Lemma 6.2.1 the proof is complete. q.e.d.

In particular if we take

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

then we have

(6.2.7) $\dim_\nu(\ker D_{\mathscr{F}}) = \dim_\nu\{\ker D_{L_x}\}$ is finite.

This immediately implies the following theorem.

Theorem 6.2.2. *The family of A.P.S. boundary value problems has finite ν-dimensional kernel, and therefore*

(6.2.8) $\text{ind}_\nu D_{\mathscr{F}} = \dim_\nu \ker(D_{\mathscr{F}}^+) - \dim_\nu \ker(D_{\mathscr{F}}^-)$ *is finite.*

Proposition 6.2.2. $\text{ind}_\nu(D_{\mathscr{F}}) = \text{tr}_\nu(\varepsilon e^{-tD_{\mathscr{F}}^2})$.

Proof. We first observe that on each leaf L, D_L^+ and D_L^- are Hilbert space adjoints of each other. The familywise partial isometries in the polar decomposition of D_L^+ implement the isomorphism of the spectral projections of $D_L^- D_L^+$ and $D_L^+ D_L^-$ on Borel subsets of $(0, \infty)$. See Moore and Schochet [23, Proposition (7.38)] and Connes [10, Corollary 8, p. 134] for more details. q.e.d.

In §7.2 we use Proposition 6.2.2 to compute $\text{ind}_\nu(D_{\mathscr{F}})$ in terms of topological data and a correction term which is a foliation eta invariant defined in §3.2.

7. The index theorem for coverings and foliations

This section completes the proofs of the main theorems stated in the Introduction. In the earlier sections we formulated the appropriate Breuer

index, for boundary value problems. We compute the index in this section, and obtain a topological component in the interior and a boundary spectral component, the eta invariant of §3. The proof is in strategy very similar to the one used by Atiyah, Patodi, and Singer [3]. In §7.1 we prove the index formula in the case of coverings of compact manifolds with boundary. §7.2 gives the proof for the case of foliations. Since this case is very similar to that of coverings in §7.1, we only sketch the proof, indicating where the necessary changes have to be made.

7.1. Proof of index theorem for Galois coverings. Let M be a compact Riemannian manifold with boundary N. Let D be a Dirac operator on a graded Clifford bundle S with grading ε. We assume that the data (D, S, ε) has a product structure in a neighborhood of the collar $[0, 1] \times N$ as in Definition 2.1.1. Let $\widetilde{M}$ be a Galois covering of M with Galois group Γ. We denote the lifts of D and S to $\widetilde{M}$ by $\widetilde{D}$ and $\widetilde{S}$ respectively. Let $\widetilde{B}$ be the A.P.S. boundary condition associated to $\widetilde{D}$ as in Definitions 2.1.1 and 2.1.3. Now we state the index theorem.

Theorem 7.1.1.

$$(7.1.1) \qquad \mathrm{ind}_\Gamma(\widetilde{D}) = \int_M \mathrm{ch}(\sigma_D) Td(M) - \frac{\eta_\Gamma(0) + h}{2},$$

where the integral is the standard formula in the calculation of the index on a manifold without boundary,

$$(7.1.2) \qquad\qquad h = \dim_\Gamma \ker(Q_+),$$

$$(7.1.3) \qquad \eta_\Gamma(0) = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} \, \mathrm{tr}_\Gamma(Q_+ e^{-tQ_+^2}) \, dt,$$

where $\widetilde{D} = \sigma(\partial/\partial y + Q)$ *in a neighborhood of the collar* $[0, 1] \times \partial\widetilde{M}$ *and* $Q = \begin{pmatrix} Q_+ & 0 \\ 0 & Q_- \end{pmatrix}$, *and*

$$Q_+ : L^2(\partial\widetilde{M}; \widetilde{S}^+) \to L^2(\partial\widetilde{M}; \widetilde{S}^+)$$

is a Dirac operator on the boundary satisfying the Bismut-Freed local cancellation property.

Proof. By Proposition 6.1.3 we have $\mathrm{ind}_\Gamma(\widetilde{D}) = \mathrm{tr}_\Gamma(\varepsilon e^{-t\widetilde{D}^2})$. By Theorem 5.1 we can replace $e^{-t\widetilde{D}^2}$ by the parametrix constructed in §4 and denoted by $\widetilde{E}(t)$ as $t \to 0^+$. In the definition of $\widetilde{E}(t)$ we lift cut-off functions ϕ_i, ψ_i from the base M to $\widetilde{M}$. Therefore

$$\mathrm{ind}_\Gamma(\widetilde{D}) \sim \mathrm{tr}_\Gamma(\varepsilon\widetilde{E}(t)),$$
$$\mathrm{tr}_\Gamma(\varepsilon\widetilde{E}(t)) = \mathrm{tr}_\Gamma(\varepsilon\phi_1\widetilde{E}(t)\psi_1) + \mathrm{tr}_\Gamma(\varepsilon\phi_2\widetilde{F}(t)\psi_2).$$

Now

$$\lim_{t\to 0^+} \mathrm{tr}_\Gamma(\varepsilon\phi_2\widetilde{F}(t)\psi_2) = \lim_{t\to 0^+} \int_M F(t,x)\psi_2(x)\,dx\,,$$

where $F(t,x)$ is the local supertrace of the heat kernel on the double of $\widetilde{M}$ considered as a function of M. Since we have a product structure in a neighborhood of the collar, it follows that

$$\lim_{t\to 0^+} F(t,x) = 0 \quad \text{for } x \in \text{ collar},$$

so that

$$\lim_{t\to 0^+} \mathrm{tr}_\Gamma(\varepsilon\phi_2\widetilde{F}(t)\phi_2) = \lim_{t\to 0^+} \int_M F(t,x)\,d\,\mathrm{vol}_M = \mathrm{ch}(\sigma_D)Td(M)$$

by the calculation in Atiyah, Bott, and Patodi [2]. Next we need to evaluate

(7.1.4) $\mathrm{tr}_\Gamma(\varepsilon\phi_1\widetilde{E}_1(t)\psi_1)$.

We note that the operator

$$\phi_1\widetilde{E}(t)\psi_1 \in \mathrm{End}_\Gamma(L^2([0,1]\times\partial\widetilde{M};\widetilde{S}))$$

and has an integral kernel. Now Γ acts only on $\partial\widetilde{M}$ and not on $[0,1]$. Therefore if the integral kernel of $\psi_1\widetilde{E}\psi_1$ is $k_E(x_1, m_1; x_2, m_2)$, then by Fubini's theorem

(7.1.5) $\mathrm{tr}_\Gamma(\varepsilon\phi_1\widetilde{E}_1(t)\psi_1) = \int_0^1 \mathrm{tr}_\Gamma(\varepsilon k_E(x,m;x,m))\,dx\,,$

where tr_Γ under the integral sign is the Γ trace on $\partial\widetilde{M}$. Combining (7.1.5) with the definition of $\widetilde{E}_1(t)$ we have
(7.1.6)
$\mathrm{tr}_\Gamma(\varepsilon\phi_1\widetilde{E}_1(t)\psi_1)$

$$= \int_0^1 \psi_1(y) \int_\mathbb{R} \mathrm{sign}\,\lambda \left\{ \frac{e^{-\lambda^2 t}e^{-y^2/t}}{\sqrt{4\pi t}} \right.$$
$$\left. + |\lambda|e^{2|\lambda|y}\,\mathrm{erfc}\left(\frac{y}{\sqrt{t}} + |\lambda|\sqrt{t}\right) \right\}\,d\mu(\lambda)\,dt\,,$$

where $d\mu(\lambda)$ is the measure on the real line determined by the Γ trace of the spectral measure of the selfadjoint operator Q_+, and $\mathrm{sign}\,\lambda$ is given by

$$\mathrm{sign}\,\lambda = \begin{cases} 1 & \text{if } \lambda \geq 0, \\ -1 & \text{if } \lambda < 0. \end{cases}$$

If we replace $\int_0^1$ and $\psi_1(y)$ by 1 in (7.1.6), the error is estimated by the integral

$$(7.1.7) \qquad C \int_{\mathbb{R}} \frac{1}{2} e^{2|\lambda|} \operatorname{erfc}\left(\frac{1}{\sqrt{t}} + |\lambda|\sqrt{t}\right) d\mu(\lambda),$$

which is bounded above by

$$C \frac{e^{-1/t}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\lambda^2 t}\, d\mu(\lambda) < C e^{-1/t} t^{-n/2},$$

which decays exponentially as $t \to 0^+$. Therefore $\operatorname{tr}_\Gamma(\varepsilon \phi_1 \widetilde{E}(t) \psi_1)$ is asymptotic to the integral

$$(7.1.8) \quad K(t) \overset{\text{def}}{=} \int_0^\infty \int_{\mathbb{R}} \operatorname{sign} \lambda \frac{\partial}{\partial y} \left\{ \frac{1}{2} e^{2|\lambda|y} \operatorname{erfc}\left(\frac{y}{\sqrt{t}} + |\lambda|\sqrt{t}\right) \right\} d\mu(\lambda)\, dy.$$

Changing the order of integration in (7.1.8) we get

$$(7.1.9) \qquad K(t) = - \int_{\mathbb{R}} \operatorname{sign} \lambda \operatorname{erfc}(|\lambda|\sqrt{t})\, d\mu(\lambda).$$

Differentiating with respect to t thus yields

$$(7.1.10) \qquad K'(t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \lambda e^{-\lambda^2 t}\, d\mu(\lambda).$$

By the normality of the Γ trace on the boundary we have $K(t) \to -\frac{1}{2}h$ as $t \to \infty$ where $h = \dim_\Gamma \ker(Q_+)$. Therefore $K(t) + \frac{1}{2}h \to 0$ as $t \to \infty$.

We now consider the following integral. For $\operatorname{Re}(s)$ large and positive,

$$(7.1.11)$$
$$\int_0^T \left(K(t) + \frac{h}{2}\right) t^{s-1}\, dt = \frac{K(T) + h/2}{s} T^s - \frac{1}{s} \int_0^T t^s K'(t)\, dt,$$

$$(7.1.12)$$
$$s \int_0^T \left(K(t) + \frac{h}{2}\right) t^{s-1}\, dt - \left(K(T) + \frac{h}{2}\right) T^s = -\int_0^T t^s K'(t)\, dt.$$

Now

$$-\int_0^T t^s K'(t)\, dt = -\frac{\Gamma(s + 1/2)}{2\sqrt{\pi}} \eta_T(2s),$$

where

$$\eta_T(2s) = \frac{1}{\Gamma(s + 1/2)} \int_0^T t^{s-1/2} \operatorname{tr}_\Gamma(Q_+ e^{-tQ_+^2})\, dt,$$

if

$$K(T) \sim \sum_{k \geq -n} a_k t^{k/f2} \quad \text{as } t \to 0^+.$$

Then taking limit as $s \to 0$ in (7.1.12) gives

$$(7.1.13) \qquad -2(a_0 + h/2) + 2(K(T) + h/2) = \eta_T(0).$$

Letting $T \to \infty$, we get

$$(7.1.14) \qquad -(2a_0 + h) = \lim_{T \to \infty} \eta_T(0) = \eta_\Gamma(0),$$

which follows from Theorem 3.1.1.

We have as $t \to 0^+$

$$(7.1.15) \qquad K(t) \sim \mathrm{ind}_\Gamma(\widetilde{D}) = -\mathrm{tr}_\Gamma(\varepsilon \phi_2 \widetilde{F}(t) \psi_2).$$

(7.1.14) and (7.1.15) imply

$$\mathrm{ind}_\Gamma(\widetilde{D}) = \int_M \mathrm{ch}(\sigma_D) Td(M) - \frac{\eta_\Gamma(0) + h}{2}.$$

7.2. Proof of the index theorem for foliations. Let $(M, \mathcal{F})$ be a compact foliated manifold with boundary and the foliation $\mathcal{F}$ transverse to the boundary. Let $D_{\mathcal{F}}$ be a leafwise Dirac operator on a graded Clifford bundle S over $T\mathcal{F}$, and assume that M and $\mathcal{F}$ are oriented. Let g be a Riemannian metric on M, and assume that data $(D_{\mathcal{F}}, S, \varepsilon)$ has a product structure as in Definition 2.3.2.

Theorem 7.2.1. *Let ν be a holonomy invariant transverse measure. Then*

$$(7.2.1) \qquad \mathrm{ind}_\nu(D_{\mathcal{F}}) = \langle \mathrm{ch}(\sigma_{D_{\mathcal{F}}}) Td(M), \nu \rangle - (\eta_\nu(0) + h)/2,$$

where the first term on the right-hand side is what one gets in the computation of the index of a leafwise elliptic operator on a manifold without boundary,

$$(7.2.2) \qquad h = \dim_\nu(\ker Q_+),$$

$Q_+ = \{Q^+_{\partial L_x}\}$ *the family of Dirac operators on the boundary ∂L_x satisfying the Bismut-Freed cancellation property leafwise, and $\eta_\nu(0)$ is the foliation eta invariant defined in §3.2.*

Proof. Proposition 6.2.2 implies that

$$(7.2.3) \qquad \mathrm{ind}_\nu(D_{\mathcal{F}}) = \mathrm{tr}_\nu(\varepsilon e^{-tD_{\mathcal{F}}^2}).$$

On each leaf we replace the heat operator $e^{-tD_{L_x}^2}$ by the parametrix $E(t)_{L_x}$ constructed in §5. The cut-off functions are functions on M restricted to L_x. By the uniformity of Sobolev estimates and Theorem 5.1, we have

$$(7.2.4) \qquad \|e^{-tD_{L_x}^2} - E(t)_{L_x}\|_{L(W^{-k}, W^k)} < Ct^\alpha, \qquad 0 \le t \le 1, \ \alpha > 0,$$

where C is uniform in x and k a very large positive integer. Hence as $t \to 0^+$

(7.2.5) $$\mathrm{ind}_\nu(D_{\mathscr{F}}) \sim \mathrm{tr}_\nu(E(t)_{\mathscr{F}}),$$

where $E(t)_{\mathscr{F}}$ is the family of operators $\{E(t)_{L_x}\}_{x \in M}$. Therefore

(7.2.6) $$\mathrm{ind}_\nu(D_{\mathscr{F}}) \sim \mathrm{tr}_\nu(\varepsilon \phi_1(t)\psi_1) + \mathrm{tr}_\nu(\varepsilon \phi_2 F(t)\psi_2).$$

Applying the arguments of the previous section leafwise yields

(7.2.7) $$\lim_{t \to 0^+} \mathrm{tr}_\nu(\varepsilon \phi_2 F(t)\psi_2) = \langle \mathrm{ch}(\sigma_{D_{\mathscr{F}}}) Td(M), \nu \rangle.$$

We now study the term $\lim_{t \to 0^+} \mathrm{tr}_\nu(\varepsilon \phi_1 E_1(t)\psi_1)$. $\phi_1 E_1(t)\psi_1$ is a family of leafwise integral operators $\mathrm{End}_{\mathscr{R}}(L^2([0,1] \times \partial L_x; S_{\partial L_x}))$ where $\mathscr{R}$ is the restriction of the equivalence $\mathscr{R}_M$ to $[0,1] \times N$. By applying Fubini's theorem as in the previous section we obtain

(7.2.8)
$$
\begin{aligned}
&\mathrm{tr}_\nu(\varepsilon \phi_1 E_1(t)\psi_1) \\
&= \int_0^1 \psi_1(y) \int_{\mathbb{R}} \left\{ \frac{e^{-\lambda^2 t} e^{-y^2/t}}{\sqrt{4\pi t}} \right. \\
&\qquad\qquad \left. + |\lambda| e^{2|\lambda|y} \, \mathrm{erfc}\left(\frac{y}{\sqrt{t}} + |\lambda|\sqrt{t}\right) \right\} d\mu_\nu(\lambda)\, dy,
\end{aligned}
$$

where $d\mu_\nu(\lambda)$ is the tempered measure on $\mathbb{R}$ given by the foliation trace of the family of spectral projections of the family of Dirac operators $\{Q^+_{aL_x}\}_{x \in N}$. From here on the proof is the same as that in the previous section, except that we replace $d\mu(\lambda)$ by $d\mu_\nu(\lambda)$. This completes the sketch of the proof of the theorem.

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