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THE BRUNN-MINKOWSKI-FIREY THEORY I: MIXED VOLUMES AND THE MINKOWSKI PROBLEM

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The Brunn-Minkowski theory is the heart of quantitative convexity. It had its origins in Minkowski's joining his notion of mixed volumes with the Brunn-Minkowski inequality. One of Minkowski's major contributions to the theory was to show how this theory could be developed from a few basic concepts: support functions, Minkowski combinations, and mixed volumes. Thirty years ago, Firey [8] (see Burago and Zalgaller [4, §24.6]) extended the notion of a Minkowski combination, and introduced, for each real $p \geq 1$, what he called p-sums.

It is the aim of this series of articles to show that these Firey combinations lead to a Brunn-Minkowski theory for each $p \geq 1$.

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean n-space, $\mathbb{R}^n$. Let $\mathscr{K}_o^n$ denote the set of convex bodies containing the origin in their interiors. For $K \in \mathscr{K}^n$, let $h_K = h(K, \cdot)\colon S^{n-1} \to \mathbb{R}$ denote the support function of K; i.e., for $u \in S^{n-1}$, $h_K(u) = h(K, u) = \max\{u \cdot x : x \in K\}$, where $u \cdot x$ denotes the standard inner product in $\mathbb{R}^n$. The set $\mathscr{K}^n$ will be viewed as equipped with the usual Hausdorff metric, d, defined by $d(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ is the sup (or max) norm on the space of continuous functions on the unit sphere, $C(S^{n-1})$.

For $K, L \in \mathscr{K}^n$, and $\alpha, \beta \geq 0$ (not both zero), the Minkowski linear combination $\alpha K + \beta L \in \mathscr{K}^n$ is defined by

$$h(\alpha K + \beta L, \cdot) = \alpha h(K, \cdot) + \beta h(L, \cdot).$$

Firey [8] introduced, for each real $p \geq 1$, new linear combinations of convex bodies: For $K, L \in \mathscr{K}_o^n$, and $\alpha, \beta \geq 0$ (not both zero), the Firey combination $\alpha \cdot K +_p \beta \cdot L \in \mathscr{K}_o^n$ can be defined by

$$h(\alpha \cdot K +_p \beta \cdot L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p.$$

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Note that " $\cdot$ " rather than " $\underset{p}{\cdot}$ " is written for Firey scalar multiplication. This should create no confusion. Obviously, Firey and Minkowski scalar multiplications are related by $\alpha{\cdot}K = \alpha^{1/p}K$.

For $Q \in \mathscr{K}^n$, let $W_0(Q), W_1(Q), \cdots, W_n(Q)$ denote the Quermassintegrals of Q. Thus, $W_0(Q) = V(Q)$, the volume of Q, $nW_1(Q) = S(Q)$, the surface area of Q, $W_n(Q) = V(B) = \omega_n$, where B denotes the unit ball in $\mathbb{R}^n$, while $(2/\omega_n)W_{n-1}(Q) = W(Q)$, the mean width of Q.

The mixed Quermassintegrals $W_0(K, L), W_1(K, L), \cdots, W_{n-1}(K, L)$ of $K, L \in \mathscr{K}^n$ are defined by

$$\text{(I)} \qquad (n-i)W_i(K, L) = \lim_{\varepsilon\to 0^+} \frac{W_i(K+\varepsilon L) - W_i(K)}{\varepsilon}.$$

Since $W_i(\lambda K) = \lambda^{n-i}W_i(K)$, it follows that $W_i(K, K) = W_i(K)$, for all i. Since the Quermassintegral W_{n-1} is Minkowski linear, it follows that $W_{n-1}(K, L) = W_{n-1}(L)$, for all K. The mixed Quermassintegral $W_0(K, L)$ will usually be written as $V_1(K, L)$.

The fundamental inequality for mixed Quermassintegrals states that: For $K, L \in \mathscr{K}^n$ and $0 \le i < n-1$,

$$\text{(II)} \qquad W_i(K, L)^{n-i} \ge W_i(K)^{n-i-1} W_i(L),$$

with equality if and only if K and L are homothetic.

Good general references for this material are Busemann [5] and Leichtweiß [15].

Mixed Quermassintegrals are, of course, the first variation of the ordinary Quermassintegrals, with respect to Minkowski addition. Define the mixed p-Quermassintegrals $W_{p,0}(K, L), W_{p,1}(K, L), \cdots, W_{p,n-1}(K, L)$, as the first variation of the ordinary Quermassintegrals, with respect to Firey addition: For $K, L \in \mathscr{K}_o^n$, and real $p \ge 1$, define

$$\text{(Ip)} \qquad \frac{n-i}{p} W_{p,i}(K, L) = \lim_{\varepsilon\to 0^+} \frac{W_i(K +_p \varepsilon{\cdot}L) - W_i(K)}{\varepsilon}.$$

Of course for $p = 1$, the mixed p-Quermassintegral $W_{p,i}(K, L)$ is just $W_i(K, L)$. Obviously, $W_{p,i}(K, K) = W_i(K)$, for all $p \ge 1$. It will be shown that for these new mixed Quermassintegrals, there is an extension of inequality (II): If $K, L \in \mathscr{K}_o^n$, $0 \le i < n-1$, and $p > 1$, then

$$\text{(IIp)} \qquad W_{p,i}(K, L)^{n-i} \ge W_i(K)^{n-i-p} W_i(L)^p,$$

with equality if and only if K and L are dilates.

Aleksandrov [1] and Fenchel and Jessen [7] (see Busemann [5] or Schneider [17]) have shown that for $K \in \mathscr{K}^n$, and $i = 0, 1, \cdots, n-1$, there exists a regular Borel measure $S_i(K, \cdot)$ on S^{n-1}, such that the mixed Quermassintegral $W_i(K, L)$ has the following integral representation:

$$\text{(III)} \qquad W_i(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)\, dS_i(K, u),$$

for all $L \in \mathscr{K}^n$. The measure $S_{n-1}(K, \cdot)$ is independent of the body K, and is just ordinary Lebesgue measure, S, on S^{n-1}. The ith surface area measure of the unit ball, $S_i(B, \cdot)$, is independent of the index i. In fact, $S_i(B, \cdot) = S$, for all i. The surface area measure $S_0(K, \cdot)$ will frequently be written simply as $S(K, \cdot)$. If ∂K is a regular C^2-hypersurface with everywhere positive principal curvatures, then $S(K, \cdot)$ is absolutely continuous with respect to S, and the Radon-Nikodym derivative,

$$dS(K, \cdot)/dS \colon S^{n-1} \longrightarrow \mathbb{R},$$

is the reciprocal Gauss curvature of ∂K (viewed as a function of the outer normals).

It will be shown that the mixed Quermassintegral $W_{p,i}$ has a similar integral representation: For $p \geq 1$, and $i = 0, 1, \cdots, n-1$, and each $K \in \mathscr{K}_o^n$, there exists a regular Borel measure $S_{p,i}(K, \cdot)$ on S^{n-1}, such that the mixed Quermassintegral $W_{p,i}(K, L)$ has the following integral representation:

$$\text{(IIIp)} \qquad W_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p\, dS_{p,i}(K, u),$$

for all $L \in \mathscr{K}_o^n$. It turns out that the measure $S_{p,i}(K, \cdot)$ is absolutely continuous with respect to $S_i(K, \cdot)$, and has Radon-Nikodym derivative

$$\frac{dS_{p,i}(K, \cdot)}{dS_i(K, \cdot)} = h(K, \cdot)^{1-p}.$$

Of course, the case $p = 1$ of the representation (IIIp) is just the representation (III).

In §1 of this article, the integral representation (IIIp) and the inequalities (IIp) are established. The reader familiar with the classical development of the Brunn-Minkowski theory of mixed volumes might find the direction taken in establishing inequality (IIp) unusual. Indeed, a proof of the inequality (IIp) could be given by using Firey's [8] extension of the Brunn-Minkowski inequality. The independent approach taken in this article seems preferable for two reasons. As a byproduct of this approach, the integral representation (IIIp) is obtained along the way. Another advantage is that the article is reasonably self-contained in that Firey's extension of the Brunn-Minkowski inequality is obtained as a corollary of inequality (IIp).

In [18] and [19], Simon obtained a number of characterizations of relative spheres. These characterizations are strong extensions and generalizations of characterizations previously obtained by Grotemeyer [12] and Süss [20]. As an application of some of the results obtained in §1, it will be shown, in §2, that there is a strong connection between the work of Simon and Firey's p-combinations. Some extensions of Simon's characterizations will also be presented in this section.

The solution of a generalization of the classical Minkowski problem is given in §3. (See [2], [5], [6], [16] for a discussion of the Minkowski problem.) If the surface area measure, $S(K, \cdot)$, of a body K is absolutely continuous with respect to S, then the Radon-Nikodym derivative, $dS(K, \cdot)/dS$, is called the curvature function of K, and is denoted by f_K. A special case of the solution of the Minkowski problem states that given a continuous even function $g\colon S^{n-1} \to (0, \infty)$, there exists a unique (up to translation) convex body $K \in \mathscr{K}^n$ such that $f_K = g$. An extension presented in §3 states that given $\alpha \in \mathbb{R}$, such that $1 - n \neq \alpha \leq 0$, and a continuous even function $g\colon S^{n-1} \to (0, \infty)$, there exists a unique convex body $K \in \mathscr{K}_o^n$ such that $h_K^\alpha f_K = g$.

1. The integral representation and inequalities

The proof of inequality (IIp) becomes easy once the integral representation (IIIp) is established.

Theorem (1.1). *If $p \geq 1$, and $K, L \in \mathscr{K}_o^n$, then for each $i = 0, 1, \cdots, n-1$,*

$$\lim_{\varepsilon \to 0^+} \frac{W_i(K +_p \varepsilon \cdot L) - W_i(K)}{\varepsilon} = \frac{n-i}{pn} \int_{S^{n-1}} h(L, u)^p h(K, u)^{1-p} \, dS_i(K, u).$$

To prove the theorem, for brevity let

$$K_\varepsilon = K +_p \varepsilon \cdot L,$$

and define $g\colon [0, \infty) \to (0, \infty)$ by

$$g(\varepsilon) = W_i(K_\varepsilon)^{1/(n-i)}.$$

Let

$$l_{\text{inf}} = \liminf_{\varepsilon \to 0^+} \frac{W_i(K_\varepsilon) - W_i(K_\varepsilon, K)}{\varepsilon},$$

and

$$l_{\sup} = \limsup_{\varepsilon \to 0^+} \frac{W_i(K, K_\varepsilon) - W_i(K)}{\varepsilon}.$$

Since obviously $K_\varepsilon \supset K$, it follows from the monotonicity of mixed Quermassintegrals that $l_{\inf}$ is the lim inf of a function which is nonnegative (for all ε), and $l_{\sup}$ is the lim sup of a function which is nonnegative (for all ε). Inequality (II) shows that

$$\liminf_{\varepsilon \to 0^+} W_i(K_\varepsilon)^{(n-i-1)/(n-i)} \frac{W_i(K_\varepsilon)^{1/(n-i)} - W_i(K)^{1/(n-i)}}{\varepsilon} \geq l_{\inf},$$

and

$$\limsup_{\varepsilon \to 0^+} W_i(K)^{(n-i-1)/(n-i)} \frac{W_i(K_\varepsilon)^{1/(n-i)} - W_i(K)^{1/(n-i)}}{\varepsilon} \leq l_{\sup}.$$

That g is continuous at 0 follows from the continuity of $W_i\colon \mathscr{K}^n \to \mathbb{R}$ and the easily established fact that in $\mathscr{K}_o^n$, $\lim_{\varepsilon \to 0^+} K_\varepsilon = K$ (see, for example, Firey [8, p. 19]). Since g is continuous at 0, the previous inequalities may be rewritten as

$$W_i(K)^{(n-i-1)/(n-i)} \liminf_{\varepsilon \to 0^+} \frac{W_i(K_\varepsilon)^{1/(n-i)} - W_i(K)^{1/(n-i)}}{\varepsilon} \geq l_{\inf},$$

and

$$W_i(K)^{(n-i-1)/(n-i)} \limsup_{\varepsilon \to 0^+} \frac{W_i(K_\varepsilon)^{1/(n-i)} - W_i(K)^{1/(n-i)}}{\varepsilon} \leq l_{\sup}.$$

If it can be shown that $l_{\inf} \geq l_{\sup}$, then the last two inequalities will imply that g is differentiable at 0, and $g'(0)\, g(0)^{n-i-1} = l_{\inf} = l_{\sup}$. But the differentiability of g at 0 would imply that g^{n-i} is differentiable at 0, and that

$$\lim_{\varepsilon \to 0^+} \frac{g(\varepsilon)^{n-i} - g(0)^{n-i}}{\varepsilon} = (n-i) g(0)^{n-i-1} \lim_{\varepsilon \to 0^+} \frac{g(\varepsilon) - g(0)}{\varepsilon},$$

or

$$\frac{1}{n-i} \lim_{\varepsilon \to 0^+} \frac{W_i(K_\varepsilon) - W_i(K)}{\varepsilon} = l_{\inf} = l_{\sup}.$$

In fact, a bit more than $l_{\inf} \geq l_{\sup}$ will be proven. What will be shown is that

$$(*) \qquad l_{\sup} = l_{\inf} = \frac{1}{pn} \int_{S^{n-1}} h(L, u)^p h(K, u)^{1-p}\, dS_i(K, u),$$

which will complete the proof of the theorem.

To show this, we shall make use of the following trivial observation: If $f_0, f_1, \cdots \in C(S^{n-1})$, with $\lim_{j \to \infty} f_j = f_0$, uniformly on S^{n-1}, and $\mu_0, \mu_1, \cdots$ are finite measures on S^{n-1} such that $\lim_{j\to\infty}\mu_j = \mu_0$, weakly on S^{n-1}, then

$$\lim_{j\to\infty}\int_{S^{n-1}} f_j(u)\,d\mu_j(u) = \int_{S^{n-1}} f_0(u)\,d\mu_0(u).$$

Using the definition of a Firey linear combination, we have

$$\lim_{\varepsilon\to 0^+}\frac{h_{K_\varepsilon}-h_K}{\varepsilon} = \lim_{\varepsilon\to 0^+}\frac{(h_K^p+\varepsilon h_L^p)^{1/p}-h_K}{\varepsilon}.$$

Hence,

$$\lim_{\varepsilon\to 0^+}\frac{h_{K_\varepsilon}-h_K}{\varepsilon} = \frac{1}{p}h_L^p h_K^{1-p}, \quad \text{uniformly on } S^{n-1}.$$

That

$$\lim_{\varepsilon\to 0^+} S_i(K_\varepsilon, \cdot) = S_i(K, \cdot), \quad \text{weakly on } S^{n-1},$$

follows from the weak continuity of surface area measures (see, for example, Schneider [17]) and the fact that $\lim_{\varepsilon\to 0^+} K_\varepsilon = K$, in $\mathscr{K}^n$. Since $W_i(K_\varepsilon) = W_i(K_\varepsilon, K_\varepsilon)$, it follows from (III) that

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{W_i(K_\varepsilon)-W_i(K_\varepsilon, K)}{\varepsilon} &= \lim_{\varepsilon\to 0^+}\frac{1}{n}\int_{S^{n-1}}\frac{h_{K_\varepsilon}(u)-h_K(u)}{\varepsilon}\,dS_i(K_\varepsilon, u)\\ &= \frac{1}{np}\int_{S^{n-1}} h_L(u)^p h_K(u)^{1-p}\,dS_i(K, u).\end{aligned}$$

Similarly, $W_i(K) = W_i(K, K)$, and (III) yield

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{W_i(K, K_\varepsilon)-W_i(K)}{\varepsilon} &= \lim_{\varepsilon\to 0^+}\frac{1}{n}\int_{S^{n-1}}\frac{h_{K_\varepsilon}(u)-h_K(u)}{\varepsilon}\,dS_i(K, u)\\ &= \frac{1}{np}\int_{S^{n-1}} h_L(u)^p h_K(u)^{1-p}\,dS_i(K, u).\end{aligned}$$

Thus $(*)$ is established. q.e.d.

Define the Borel measure $S_{p,i}(K, \cdot)$, on S^{n-1}, by

$$S_{p,i}(K, \omega) = \int_\omega h(K, u)^{1-p}\,dS_i(K, u),$$

for each Borel $\omega \subset S^{n-1}$. Then, with definition (Ip), identity (IIIp) is proved.

With the aid of the integral representation (IIIp), inequality (IIp) is easily established.

Theorem (1.2). *If $p > 1$, and $K, L \in \mathscr{K}_o^n$, while $0 \le i < n$, then*

$$W_{p,i}(K, L)^{n-i} \ge W_i(K)^{n-i-p} W_i(L)^p,$$

with equality if and only if K and L are dilates.

Proof. First suppose $i < n-1$. The Hölder inequality [13, p. 140], together with the integral representations (III) and (IIIp), yield

$$\begin{aligned} W_{p,i}(K, L) &= \frac{1}{n}\int_{S^{n-1}} h(L, u)^p h(K, u)^{1-p}\, dS_i(K, u) \\ &\ge W_i(K, L)^p W_i(K)^{1-p}. \end{aligned}$$

When this is combined with inequality (II):

$$W_i(K, L)^p \ge W_i(K)^{p(n-i-1)/(n-i)} W_i(L)^{p/(n-i)},$$

the result is the inequality of the theorem.

To obtain the equality conditions, note that there is equality in Hölder's inequality precisely when $W_i(K, L)h_K = W_i(K)h_L$, almost everywhere, with respect to the measure $S_i(K, \cdot)$, on S^{n-1}. Equality in inequality (II) holds precisely when there exists an $x \in \mathbb{R}^n$ such that

$$W_i(K, L)h_K(u) = x \cdot u + W_i(K)h_L(u),$$

for all $u \in S^{n-1}$. Since the body K has interior points, the support of the measure $S_i(K, \cdot)$ cannot be contained in the great sphere of S^{n-1} orthogonal to x. Hence $x = 0$, and

$$W_i(K, L)h_K = W_i(K)h_L$$

everywhere.

The case $i = n-1$ is even simpler. Since $W_{n-1}(K, L) = W_{n-1}(L)$, only half of the preceding argument will be needed—the Hölder inequality by itself will yield the inequality. To obtain the equality conditions, recall that $S_{n-1}(K, \cdot) = S$, Lebesgue measure on S^{n-1}. q.e.d.

From the integral representation (IIIp) it is easily seen that the mixed p-Quermassintegral is linear in its second argument, with respect to Firey p-sums; i.e., for $Q, K, L \in \mathscr{K}_o^n$,

$$W_{p,i}(Q, K +_p L) = W_{p,i}(Q, K) + W_{p,i}(Q, L).$$

This together with inequality (IIp) shows that

$$W_{p,i}(Q, K +_p L) \ge W_i(Q)^{(n-i-p)/(n-i)}[W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)}],$$

with equality if and only if K and L are dilates of Q. Now take $K +_p L$ for Q, recall that $W_{p,i}(Q, Q) = W_i(Q)$, and the result is

Corollary (1.3). *If* $p > 1$, *and* $K, L \in \mathscr{K}_o^n$, *while* $0 \le i < n$, *then*

$$W_i(K +_p L)^{p/(n-i)} \ge W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)},$$

with equality if and only if K *and* L *are dilates.*

This is Firey's [8] (see also Burago-Zalgaller [4, p. 162]) extension of the Brunn-Minkowski inequality. The case $p = 1$ and $i = 0$ of this inequality is the well-known Brunn-Minkowski inequality.

2. Simon's characterization of relative spheres

Recall that the functional

$$W_{p,i}\colon \mathscr{K}_o^n \times \mathscr{K}_o^n \longrightarrow (0, \infty)$$

is (Minkowski) homogeneous of degree $n - i - p$ in its first argument and (Minkowski) homogeneous of degree p in its second argument; i.e., for $K, L \in \mathscr{K}_o^n$ and $\alpha, \beta > 0$,

$$W_{p,i}(\alpha K, \beta L) = \alpha^{n-i-p} \beta^p W_{p,i}(K, L).$$

Thus, when $p = n - i$,

$$W_{p,i}(\alpha K, L) = W_{p,i}(K, L),$$

for all $\alpha > 0$.

A helpful consequence of Theorem (1.2) is contained in

Lemma (2.1). *Suppose* $0 \le i < n$, *and* $K, L \in \mathscr{K}_o^n$ *are bodies such that* $W_i(K) \le W_i(L)$.

(2.1.1) *If* $W_i(K) \ge W_{p,i}(K, L)$, *for some* $p > 1$, *then* $K = L$.

(2.1.2) *If* $W_i(K) \ge W_{p,i}(L, K)$, *for some* p *such that* $n - i > p > 1$, *then* $K = L$.

(2.1.3) *If* $W_i(L) \ge W_{p,i}(K, L)$, *for some* $p > n - i$, *then* $K = L$.

All three parts have similar proofs, so only the proof of (2.1.1) is given. Since $W_i(K) \ge W_{p,i}(K, L)$, it follows from Theorem (1.2) that

$$W_i(K)^{n-i} \ge W_{p,i}(K, L)^{n-i} \ge W_i(K)^{n-i-p} W_i(L)^p,$$

with equality in the right inequality if and only if K and L are dilates. But the hypothesis, $W_i(K) \le W_i(L)$, shows there is in fact equality in both inequalities and that $W_i(K) = W_i(L)$. Hence $K = L$.

An immediate consequence of Lemma (2.1) is

Theorem (2.2). *Suppose* $K, L \in \mathcal{K}_o^n$, *and* $\mathscr{C} \subset \mathcal{K}_o^n$ *is a class of bodies such that* $K, L \in \mathscr{C}$. *If* $0 \le i < n$, *and* $n - i \ne p > 1$, *and if*

$$W_{p,i}(K, Q) = W_{p,i}(L, Q), \quad \text{for all } Q \in \mathscr{C},$$

then $K = L$.

To see this take $Q = K$, and get: $W_i(K) = W_{p,i}(K, K) = W_{p,i}(L, K)$. Take $Q = L$, and get: $W_i(L) = W_{p,i}(L, L) = W_{p,i}(K, L)$. Lemmas (2.1.2) and (2.1.3) now yield the desired result.

The following corollary is an immediate consequence of Theorem (2.2) and the integral representation (IIIp).

Corollary (2.3). *Suppose* $K, L \in \mathcal{K}_o^n$ *and* $0 \le i < n$. *If* $n - i \ne p > 1$, *and*

$$S_{p,i}(K, \cdot) = S_{p,i}(L, \cdot),$$

then $K = L$.

The following companion to Lemma (2.1) will prove helpful.

Lemma (2.4). *Suppose* $K, L \in \mathcal{K}_o^n$ *and* $0 \le i < n - 1$. *If* $p = n - i$ *and* $W_i(K) \ge W_{p,i}(L, K)$, *then* K *and* L *are dilates.*

Proof. The hypothesis together with Theorem (1.2) gives

$$W_i(K)^{n-i} \ge W_{p,i}(L, K)^{n-i} \ge W(L)^{n-i-p} W(K)^p,$$

with equality in the right inequality implying that K and L are dilates. But since $p = n - i$, the terms on the left and right are identical and thus K and L must be dilates. q.e.d.

An obvious consequence of Lemma (2.4) and the fact that $W_i(K) = W_{p,i}(K, K)$ is

Theorem (2.5). *Suppose* $K, L \in \mathcal{K}_o^n$, *and* $\mathscr{C} \subset \mathcal{K}_o^n$ *is a class of bodies such that* $K \in \mathscr{C}$. *If* $0 \le i < n - 1$, $p = n - i$, *and*

$$W_{p,i}(K, Q) \ge W_{p,i}(L, Q), \quad \text{for all } Q \in \mathscr{C},$$

then K *and* L *are dilates, and hence*

$$W_{p,i}(K, Q) = W_{p,i}(L, Q), \quad \text{for all } Q \in \mathcal{K}_o^n.$$

Theorem (2.5) and the integral representation (IIIp) yield directly

Corollary (2.6). *Suppose* $K, L \in \mathcal{K}_o^n$ *and* $0 \le i < n - 1$. *If* $p = n - i$, *and*

$$S_{p,i}(K, \cdot) \ge S_{p,i}(L, \cdot),$$

then K *and* L *are dilates, and hence*

$$S_{p,i}(K, \cdot) = S_{p,i}(L, \cdot).$$

Using Lemma (2.1.1) we immediately obtain

Theorem (2.7). *Suppose* $K, L \in \mathcal{K}_o^n$, *and* $\mathcal{C} \subset \mathcal{K}_o^n$ *is a class of bodies such that* $K, L \in \mathcal{C}$. *If* $p > 1$, $0 \le i < n-1$, *and*

$$W_{p,i}(Q, K) = W_{p,i}(Q, L), \quad \text{for all } Q \in \mathcal{C},$$

then $K = L$.

If $\phi \in O(n)$, then obviously $h(\phi K, x) = h(K, \phi^{-1}x)$, for all $x \in \mathbb{R}^n$. From this and the definition of a Firey linear combination, it follows immediately that for $p \ge 1$, $\phi \in O(n)$, $K, L \in \mathcal{K}_o^n$, and $\alpha, \beta \ge 0$ (not both zero),

$$\phi(\alpha \cdot K +_p \beta \cdot L) = \alpha \cdot \phi K +_p \beta \cdot \phi L.$$

This, together with the definition of $W_{p,i}$, and the facts that $\phi B = B$, and $W_i(\phi Q) = W_i(Q)$, for all $Q \in \mathcal{K}_o^n$ and $\phi \in O(n)$, immediately yields

Proposition (2.8). *Suppose* $K, L \in \mathcal{K}_o^n$ *and* $p \ge 1$. *If* $0 \le i < n$, *and* $\phi \in O(n)$, *then*

$$W_{p,i}(\phi K, \phi L) = W_{p,i}(K, L).$$

For $K \in \mathcal{K}^n$, let $-K = \{-x : x \in K\}$. A body K is said to be centered if $-K = K$. Write $\mathcal{K}_e^n$ for the class of centered convex bodies. For $\omega \subset S^{n-1}$, let $-\omega = \{-u : u \in \omega\}$. A Borel measure μ on S^{n-1} is said to be even provided $\mu(\omega) = \mu(-\omega)$, for all Borel $\omega \subset S^{n-1}$.

Theorem (2.9). *Suppose* $p > 1$, *and* $0 \le i < n$. *For* $K \in \mathcal{K}_o^n$, *the following statements are equivalent*:

(2.9.1) *The body* K *is centered.*

(2.9.2) *The measure* $S_{p,i}(K, \cdot)$ *is even.*

(2.9.3) $W_{p,i}(K, Q) = W_{p,i}(K, -Q)$, *for all* $Q \in \mathcal{K}_o^n$.

(2.9.4) $W_{p,i}(K, Q) = W_{p,i}(K, -Q)$, *for* $Q = K$.

Proof. To see that (2.9.1) implies (2.9.2), recall that if K is centered, then $h(K, \cdot)$ is an even function, and $S_i(K, \cdot)$ is an even measure. The implication is now a consequence of the fact that $dS_{p,i}(K, \cdot) = h(K, \cdot)^{1-p} dS_i(K, \cdot)$.

That (2.9.2) yields (2.9.3) is a consequence of the integral representation (IIIp) and the fact that, in general, $h(-Q, u) = h(Q, -u)$, for all $u \in S^{n-1}$. Obviously (2.9.4) follows directly from (2.9.3).

To see that (2.9.4) implies (2.9.1), note that (2.9.4), for $Q = K$, gives

$$W_i(K) = W_{p,i}(K, -K).$$

The desired result follows from the fact that $W_i(-K) = W_i(K)$ and the equality conditions of inequality (IIp). q.e.d.

That (2.9.3) implies that K is centrally symmetric, for the case $p = 1$ and $i = 0$, was shown (using other methods) by Goodey [11].

Another consequence of Lemma (2.1) is

Theorem 2.10. *Suppose* $p \in \mathbb{R}$, *such that* $n-i \neq p > 1$, *and* $0 \le i < n$. *Suppose also that* $K, L \in \mathscr{K}_o^n$ *are bodies such that* $S_{p,i}(K, \cdot) \le S_{p,i}(L, \cdot)$.
(2.10.1) *If* $W_i(K) \ge W_i(L)$, *and* $p < n-i$, *then* $K = L$.
(2.10.2) *If* $W_i(K) \le W_i(L)$, *and* $p > n-i$, *then* $K = L$.

To see this, note that since $S_{p,i}(K, \cdot) \le S_{p,i}(L, \cdot)$, it follows from the integral representation (IIIp) that

$$W_{p,i}(K, Q) \le W_{p,i}(L, Q), \quad \text{for all } Q \in \mathscr{K}_o^n.$$

As before, take $Q = L$, and since $W_i(L) = W_{p,i}(L, L) \ge W_{p,i}(K, L)$, Lemmas (2.1.2) and (2.1.3) yield the desired result. q.e.d.

Theorem (2.10), for the case where $p = 1$, is due to Aleksandrov [3] (see Schneider [17, p. 44]). The case $p = n - i$ of Theorem (2.10) is contained in Corollary (2.6).

A body $K \in \mathscr{K}^n$ is said to have a continuous ith curvature function $f_i(K, \cdot): S^{n-1} \to [0, \infty)$ provided that the integral representation

$$W_i(K, Q) = \frac{1}{n}\int_{S^{n-1}} h(Q, u) f_i(K, u)\, dS(u)$$

holds for all $Q \in \mathscr{K}^n$. Let $\mathscr{F}_i^n$ denote the subset of $\mathscr{K}^n$ consisting of all bodies which have continuous ith curvature functions. From the integral representation (III) it follows immediately that $K \in \mathscr{F}_i^n$, if and only if $S_i(K, \cdot)$ is absolutely continuous with respect to S and the Radon-Nikodym derivative

$$\frac{dS_i(K, \cdot)}{dS} = f_i(K, \cdot).$$

If ∂K is a regular C^2-hypersurface with (everywhere) positive principal curvatures, then $K \in \mathscr{F}_i^n$, for all i, and the curvature functions of K are proportional to the elementary symmetric functions of the principal radii of curvature (viewed as functions of the outer normals) of K. Thus, $f_0(K, u)$ is the reciprocal Gauss curvature of ∂K at the point of ∂K whose outer normal is u, while $f_{n-2}(K, u)$ is proportional to the arithmetic mean of the radii of curvature of ∂K at the point whose outer normal is u.

Suppose $K, L \in \mathscr{F}_i^n$, and there exist a $c > 0$ and an $\alpha < 0$, such that

$$h_K^\alpha f_i(K, \cdot) = c\, h_L^\alpha f_i(L, \cdot).$$

By Corollaries (2.3) and (2.6), K and L must be dilates. Corollary (2.6) shows that if there exists a $c \geq 1$ and an $i < n-1$, such that

$$\frac{f_i(K,\cdot)}{h_K^{n-i-1}} \geq c\,\frac{f_i(L,\cdot)}{h_L^{n-i-1}},$$

then $c = 1$ and K and L must be dilates. These results are mild generalizations of Simon's characterization of relative spheres.

If $K \in \mathscr{F}_i^n$, and the function $h(K,\cdot)^\alpha f_i(K,\cdot)$ is even for some $\alpha < 0$, then according to Theorem (2.9) K must be centered. The question of whether this is so when $\alpha > 0$ appears to be an open problem, even for the special case where $\alpha = 1$ and $i = 0$ (see Firey [10]). The case $\alpha = 0$ is well known: K must be centrally symmetric.

3. The p-Minkowski problem

Just as the mixed Quermassintegral $W_0(K, L)$ of $K, L \in \mathscr{K}^n$ is usually written as $V_1(K, L)$, write $V_p(K, L)$ for $W_{p,0}(K, L)$. Recall that

$$\begin{aligned} V_p(K, L) &= \frac{p}{n}\lim_{\varepsilon\to 0^+}\frac{V(K +_p \varepsilon\cdot L) - V(K)}{\varepsilon} \\ &= \frac{1}{n}\int_{S^{n-1}} h(L,u)^p h(K,u)^{1-p}\,dS(K,u), \end{aligned}$$

where $S(K,\cdot) = S_0(K,\cdot)$ is the surface area measure of K.

Let $C^+(S^{n-1})$ denote the set of positive continuous functions on S^{n-1}, endowed with the topology derived from the max norm. Given a function $f \in C^+(S^{n-1})$, the set

$$\{Q \in \mathscr{K}_o^n : h_Q \leq f\}$$

has a unique maximal element, the Aleksandrov body associated with the function f. Obviously if f is the support function of a convex body K, then the Aleksandrov body associated with f is K. Following Aleksandrov [2], define the volume, $V(f)$, of a function $f \in C^+(S^{n-1})$ as the (ordinary) volume of the Aleksandrov body associated with the positive function f.

For $Q \in \mathscr{K}_o^n$, and $f \in C^+(S^{n-1})$, define $V_p(Q, f)$ by

$$V_p(Q, f) = \frac{1}{n}\int_{S^{n-1}} f(u)^p h(Q,u)^{1-p}\,dS(Q,u).$$

Obviously, $V_p(K, h_K) = V(K)$, for all $K \in \mathscr{K}_o^n$.

The following is a trivial extension of a result obtained by Aleksandrov [2] for the case $p = 1$.

Lemma (3.1). *If $p \geq 1$, and K is the Aleksandrov body associated with $f \in C^+(S^{n-1})$, then $V(f) = V(K) = V_p(K, f)$.*

To see this, let $G^*(K, \cdot)$ denote the generalized inverse Gauss map of K; i.e., $G^*(K, \cdot)$ is the set-valued function defined, for $u \in S^{n-1}$, by

$$G^*(K, u) = \{x \in \partial K: u \text{ is an outer unit normal at } x\},$$

or equivalently,

$$G^*(K, u) = \{x \in \partial K: x \cdot u = h(K, u)\}.$$

Let $\omega = \{u \in S^{n-1}: h(K, u) < f(u)\}$. As Aleksandrov [2] shows, it is easy to verify that the points of ω are mapped by $G^*(K, \cdot)$ into singular points of ∂K, and since $S(K, \omega)$ is just the $(n-1)$-dimensional Hausdorff measure of $G^*(K, \omega)$, it follows from Reidemeister's theorem (see [17]) that $S(K, \omega) = 0$. Hence, $h(K, \cdot) = f$ almost everywhere with respect to the measure $S(K, \cdot)$ on S^{n-1}. That $V(f) = V_p(K, f)$ is now seen to be an immediate consequence of the fact that, from the definition of $V(f)$,

$$V(f) = V(K) = \frac{1}{n} \int_{S^{n-1}} h(K, u)^p h(K, u)^{1-p}\, dS(K, u).$$

q.e.d.

For $K \in \mathscr{K}_o^n$, $f \in C^+(S^{n-1})$, $p \geq 1$, and

$$\varepsilon > -\min\{h(K, u)^p / f(u)^p : u \in S^{n-1}\},$$

define

$$h_K +_p \varepsilon \cdot f = (h_K^p + \varepsilon f^p)^{1/p}.$$

In the proof of the next lemma the following convergence lemma of Aleksandrov [2] will be needed: If the functions $f_0, f_1, f_2, \cdots \in C^+(S^{n-1})$, have associated Aleksandrov bodies $K_0, K_1, K_2, \cdots$, and $\lim_{i\to\infty} f_i = f_0$, uniformly on S^{n-1}, then $\lim_{i\to\infty} K_i = K_0$. Among other things this shows that $V: C^+(S^{n-1}) \to (0, \infty)$ is continuous.

Lemma (3.2). *If $K \in \mathscr{K}_o^n$ and $f \in C^+(S^{n-1})$, then, for $p \geq 1$,*

$$\lim_{\varepsilon \to 0} \frac{V(h_K +_p \varepsilon \cdot f) - V(h_K)}{\varepsilon} = \frac{n}{p} V_p(K, f).$$

Proof. Let K_ε denote the Aleksandrov body of $h_K +_p \varepsilon \cdot f$, and let

$$l = V_p(K, f)/p = \frac{1}{np} \int_{S^{n-1}} f(u)^p h(K, u)^{1-p}\, dS(K, u).$$

Since $\lim_{\varepsilon\to 0} h_K +_p \varepsilon\cdot f = h_K$, uniformly on S^{n-1}, it follows from Aleksandrov's convergence lemma that $\lim_{\varepsilon\to 0} K_\varepsilon = K$. Hence,

$$\lim_{\varepsilon\to 0} S(K_\varepsilon, \cdot) = S(K, \cdot), \quad \text{weakly on } S^{n-1}.$$

Also,

$$\lim_{\varepsilon\to 0} \frac{(h_K^p + \varepsilon f^p)^{1/p} - h_K}{\varepsilon} = \frac{1}{p} f^p h_K^{1-p}, \quad \text{uniformly on } S^{n-1}.$$

As in the proof of Theorem (1.1), this yields

$$\begin{aligned}
\lim_{\varepsilon\to 0} & \frac{V_1(K_\varepsilon, h_K +_p \varepsilon\cdot f) - V_1(K_\varepsilon, h_K)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{1}{n} \int_{S^{n-1}} \frac{(h_K(u)^p + \varepsilon f(u)^p)^{1/p} - h_K(u)}{\varepsilon}\, dS(K_\varepsilon, u) \\
&= l,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\varepsilon\to 0} & \frac{V_1(K, h_K +_p \varepsilon\cdot f) - V_1(K, h_K)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{1}{n} \int_{S^{n-1}} \frac{(h_K(u)^p + \varepsilon f(u)^p)^{1/p} - h_K(u)}{\varepsilon}\, dS(K, u) \\
&= l.
\end{aligned}$$

From Lemma (3.1) it follows that

$$\begin{aligned}
\liminf_{\varepsilon\to 0^+} \frac{V(K_\varepsilon) - V_1(K_\varepsilon, K)}{\varepsilon} &= \liminf_{\varepsilon\to 0^+} \frac{V_1(K_\varepsilon, h_K +_p \varepsilon\cdot f) - V_1(K_\varepsilon, h_K)}{\varepsilon} \\
&= l.
\end{aligned}$$

Since $h_{K_\varepsilon} \le h_K +_p \varepsilon\cdot f$ and $V(K) = V_1(K, h_K)$,

$$\begin{aligned}
\limsup_{\varepsilon\to 0^+} \frac{V_1(K, K_\varepsilon) - V(K)}{\varepsilon} &\le \limsup_{\varepsilon\to 0^+} \frac{V_1(K, h_K +_p \varepsilon\cdot f) - V_1(K, h_K)}{\varepsilon} \\
&= l.
\end{aligned}$$

From this and inequality (II) it follows that

$$\liminf_{\varepsilon\to 0^+} V(K_\varepsilon)^{(n-1)/n} \frac{V(K_\varepsilon)^{1/n} - V(K)^{1/n}}{\varepsilon} \ge l,$$

and

$$\limsup_{\varepsilon\to 0^+} V(K)^{(n-1)/n} \frac{V(K_\varepsilon)^{1/n} - V(K)^{1/n}}{\varepsilon} \le l.$$

Obviously, $\lim_{\varepsilon\to 0} V(K_\varepsilon) = V(K)$, since $\lim_{\varepsilon\to 0} K_\varepsilon = K$. In light of this fact, the previous pair of inequalities can be rewritten as

$$\liminf_{\varepsilon\to 0^+} \frac{V(K_\varepsilon)^{1/n} - V(K)^{1/n}}{\varepsilon} \geq V(K)^{(1-n)/n} l \geq \limsup_{\varepsilon\to 0^+} \frac{V(K_\varepsilon)^{1/n} - V(K)^{1/n}}{\varepsilon}.$$

As in the proof of Theorem (1.1), this shows that

$$\lim_{\varepsilon\to 0^+} \frac{V(K_\varepsilon) - V(K)}{\varepsilon} = nl,$$

or equivalently that,

$$\lim_{\varepsilon\to 0^+} \frac{V(h_K +_p \varepsilon\cdot f) - V(h_K)}{\varepsilon} = \frac{n}{p} V_p(K, f).$$

To prove that

$$\lim_{\varepsilon\to 0^-} \frac{V(h_K +_p \varepsilon\cdot f) - V(h_K)}{\varepsilon} = \frac{n}{p} V_p(K, f),$$

proceed in the same manner. From Lemma (3.1), and $h_{K_\varepsilon} \leq h_K +_p \varepsilon\cdot f$, it follows that

$$\liminf_{\varepsilon\to 0^-} \frac{V(K_\varepsilon) - V_1(K_\varepsilon, K)}{\varepsilon} = \liminf_{\varepsilon\to 0^-} \frac{V_1(K_\varepsilon, h_K +_p \varepsilon\cdot f) - V_1(K_\varepsilon, h_K)}{\varepsilon} = l,$$

and

$$\limsup_{\varepsilon\to 0^-} \frac{V_1(K, K_\varepsilon) - V(K)}{\varepsilon} \geq \limsup_{\varepsilon\to 0^-} \frac{V_1(K, h_K +_p \varepsilon\cdot f) - V_1(K, h_K)}{\varepsilon} = l.$$

The previous inequalities now reverse, and exactly the same argument (used for the case $\varepsilon \to 0^+$) yields the desired result. q.e.d.

The weak solution to the p-Minkowski problem, with even data, is contained in

Theorem (3.3). *If μ is an even positive Borel measure on S^{n-1}, which is not concentrated on a great sphere of S^{n-1}, and $p \in \mathbb{R}$ such that $p > 1$ and $p \neq n$, then there exists a unique centered $K \in \mathscr{K}_o^n$, such that*

$$S_{p,0}(K, \cdot) = \mu,$$

or equivalently,

$$h_K^{1-p}\, dS(K, \cdot) = d\mu.$$

Proof. The function on $\mathbb{R}^n$

$$x \longmapsto \int_{S^{n-1}} |x \cdot v|^p \, d\mu(v)$$

is (by Minkowski's integral inequality) the pth power of a convex function, and is thus continuous. Since μ is not concentrated on a great sphere, this function, when restricted to S^{n-1}, is positive. Choose $c_1 > 0$ such that

$$\int_{S^{n-1}} |u \cdot v|^p \, d\mu(v) \geq c_1 > 0,$$

for all $u \in S^{n-1}$.

Consider the continuous functional

$$\Phi \colon C^+(S^{n-1}) \to (0, \infty),$$

defined, for $f \in C^+(S^{n-1})$, by

$$\Phi(f) = V(f)^{-p/n} \int_{S^{n-1}} f(u)^p \, d\mu(u).$$

We are searching for a function at which Φ attains a minimum. The search can be restricted to support functions of convex bodies in $\mathscr{K}_o^n$. To see this, recall that the Aleksandrov body K corresponding to a given $f \in C^+(S^{n-1})$ has a support function with the property that $0 < h_K \leq f$, and $V(f) = V(h_K)$. Since $\mu \geq 0$, it follows that $\Phi(h_K) \leq \Phi(f)$.

A further restriction to support functions of centered bodies is possible. To see this, note that with each $K \in \mathscr{K}_o^n$, one may associate its p-difference body $\Delta_p K \in \mathscr{K}_e^n$, defined by

$$2 \cdot \Delta_p K = K +_p (-K);$$

i.e., $2\, h(\Delta_p K, u)^p = h(K, u)^p + h(K, -u)^p$, for all $u \in S^{n-1}$. Since μ is an even measure,

$$\int_{S^{n-1}} h(\Delta_p K, u)^p \, d\mu(u) = \int_{S^{n-1}} h(K, u)^p \, d\mu(u).$$

But Firey's extension of the Brunn-Minkowski inequality (Corollary (1.3)) shows that $V(\Delta_p K) \geq V(K)$, and hence $\Phi(h_{\Delta_p K}) \leq \Phi(h_K)$.

Since Φ is positively homogeneous of degree 0, the search may be restricted to functions of unit volume. Let c_2 denote the value of Φ evaluated at the support function of the centered ball of unit volume.

It follows that in searching for a minimum of Φ, on $C^+(S^{n-1})$, it is sufficient to search among the support functions of the members of the set

$$\mathcal{T} = \{ Q \in \mathcal{K}_e^n : \Phi(h_Q) \le c_2 ,\ V(Q) = 1 \}.$$

Let $K_i \in \mathcal{T}$ be a minimizing sequence for Φ; i.e.,

$$\lim_{i\to\infty} \Phi(h_{K_i}) = \inf\{ \Phi(h_Q) : Q \in \mathcal{T} \}.$$

That the sequence K_i is bounded may be demonstrated as follows: Let $2r_i$ denote the length of the longest segment in K_i. Since the K_i are centered, there exist $u_i \in S^{n-1}$ such that $r_i u_i$, $-r_i u_i$ are the endpoints of this segment. Since this segment is contained in K_i, its support function is dominated by the support function of K_i; i.e., $r_i|u_i\cdot v| \le h(K_i, v)$, for all $v \in S^{n-1}$. Since $\mu \ge 0$, it follows that

$$r_i^p c_1 \le r_i^p \int_{S^{n-1}} |u_i\cdot v|^p \, d\mu(v) \le \int_{S^{n-1}} h(K_i, v)^p \, d\mu(v) = \Phi(h_{K_i}) \le c_2.$$

The Blaschke selection theorem now yields a subsequence of the K_i converging to some compact convex K_0. The body K_0 is easily seen to belong to $\mathcal{T}$.

Abbreviate $h(K_0, \cdot)$ by h_0, and define $c > 0$ by

$$c^{n-p} = \frac{1}{n} \int_{S^{n-1}} h_0(u)^p \, d\mu(u).$$

It will be shown that

$$\mu = S_{p,0}(cK_0, \cdot),$$

by demonstrating that

$$\int_{S^{n-1}} f(u)\, d\mu(u) = \int_{S^{n-1}} f(u) h(cK_0, u)^{1-p} dS(cK_0, u),$$

for all $f \in C^+(S^{n-1})$.

Given $f \in C^+(S^{n-1})$, let $g \in C^+(S^{n-1})$ be defined by $g^p = f$. Consider the function $\varepsilon \longmapsto \Phi(h_0 +_p \varepsilon\cdot g)$, i.e.,

$$\varepsilon \longmapsto V(h_0 +_p \varepsilon\cdot g)^{-p/n} \left[\int_{S^{n-1}} h_0(u)^p \, d\mu(u) + \varepsilon \int_{S^{n-1}} f(u)\, d\mu(u) \right].$$

By Lemma (3.2), this function has a derivative, at $\varepsilon = 0$, equal to

$$-V_p(K_0, g) \int_{S^{n-1}} h_0(u)^p \, d\mu(u) + \int_{S^{n-1}} f(u)\, d\mu(u).$$

But the function has a minimum at $\varepsilon = 0$ (since Φ has a minimum at the point h_0), and hence the derivative of the function, at $\varepsilon = 0$, must equal 0. This yields

$$\int_{S^{n-1}} f(u)\,d\mu(u) = c^{n-p} \int_{S^{n-1}} f(u)h(K_0, u)^{1-p}\,dS(K_0, u),$$

which is the desired result.

The uniqueness part of the theorem is a direct consequence of the case $i = 0$ of Corollary (2.3). q.e.d.

Theorem (3.3) suggests a composition of bodies, in $\mathscr{K}_e^n$, which generalizes the notion of Blaschke addition in the same manner that Firey's addition generalizes the notion of Minkowski addition: For $K, L \in \mathscr{K}_e^n$, and $n \neq p \geq 1$, define $K \mathbin{+_p} L \in \mathscr{K}_e^n$ by

$$S_{p,0}(K \mathbin{+_p} L, \cdot) = S_{p,0}(K, \cdot) + S_{p,0}(L, \cdot).$$

Of course, the existence and uniqueness of $K \mathbin{+_p} L$ are guaranteed by Theorem (3.3).

From the integral representation (IIIp), it follows that for $K, L \in \mathscr{K}_e^n$, and $Q \in \mathscr{K}_o^n$,

$$V_p(K \mathbin{+_p} L, Q) = V_p(K, Q) + V_p(L, Q),$$

which together with inequality (IIp) yields

$$V_p(K \mathbin{+_p} L, Q) \geq V(Q)^{p/n}[V(K)^{(n-p)/n} + V(L)^{(n-p)/n}],$$

with equality (for $p > 1$) if and only if K, L, and Q are dilates. Now take $K \mathbin{+_p} L$ for Q, and since $V_p(Q, Q) = V(Q)$, we get

Theorem (3.4). *If $K, L \in \mathscr{K}_e^n$, and $n \neq p > 1$, then*

$$V(K \mathbin{+_p} L)^{(n-p)/n} \geq V(K)^{(n-p)/n} + V(L)^{(n-p)/n},$$

with equality if and only if K and L are dilates.

The case $p = 1$ of the inequality of Theorem (3.4) is the Kneser-Süss inequality [14].

Theorem (3.3) also suggests a symmetrization procedure, for each $p \in \mathbb{R}$, such that $n \neq p \geq 1$. For $K \in \mathscr{K}_o^n$, define $\nabla_p K \in \mathscr{K}_e^n$, by

$$2S_{p,0}(\nabla_p K, \cdot) = S_{p,0}(K, \cdot) + S_{p,0}(-K, \cdot).$$

Again, existence and uniqueness of $\nabla_p K$ are guaranteed by Theorem (3.3). The same proof used for Theorem (3.4) gives

Proposition (3.5). *If $K \in \mathscr{K}_o^n$, and $n \neq p > 1$, then*

$$V(\nabla_p K) \geq V(K),$$

with equality if and only if K is centered.

For the case $p = 1$, this symmetrization procedure was considered by Firey [9].

Restricting the measure μ, in Theorem (3.3), to be even was necessitated by the method of proof. Certainly, the restriction that $p \neq n$ is essential. To see this, from Corollary (2.6) recall that for $K \in \mathscr{K}_o^n$, the measure $S_{n,0}(K, \cdot)$ cannot strictly dominate $S_{n,0}(B, \cdot)$; i.e., if

$$S_{n,0}(K, \cdot) \geq S_{n,0}(B, \cdot),$$

then the measures $S_{n,0}(K, \cdot)$ and $S_{n,0}(B, \cdot)$ must be identical. If Theorem (3.3) were valid with $p = n$, one could define $K \dotplus_p B$, as above, for $p = n$. But this would imply that $S_{n,0}(K \dotplus_p B, \cdot)$ would strictly dominate $S_{n,0}(B, \cdot)$, which is impossible.

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