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ON THE GENERALIZED CYCLE MAP

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Abstract

We relate Friedlander-Mazur cycle map [14] for projective varieties with Almgren's isomorphism [1] for integral currents. As a consequence we obtain the naturality of the F-M map, extend it to quasiprojective varieties, show its compatibility with localization sequences and pull-backs, and use it to compute several examples. As a corollary of our main result, we give a characterization of those varieties for which the cycle map is an isomorphism, as the ones whose space of p-dimensional algebraic cycles is weakly homotopy equivalent to the space of $2p$-dimensional topological cycles, for all p.

1. Introduction

The aim of this paper is to study and extend, under the light of geometric measure theory, some properties of the "Lawson homology" of quasiprojective varieties.

The Lawson homology $L_pH_n(X)$ of a closed complex projective variety $X \subseteq \mathbb{P}^N$ was first defined by E. Friedlander in [11], who was building on the fundamental work of H. B. Lawson [20]. The definition was subsequently extended to include quasiprojective varieties in [24]. For a closed projective variety X, the Lawson homology group $L_pH_n(X)$, $n \geq 2p$, was originally defined as the homotopy group $\pi_{n-2p}(\mathscr{C}_p(X)^+)$, where $\mathscr{C}_p(X)$ is the Chow monoid of effective p-cycles supported in X and $\mathscr{C}_p(X)^+$ is a "homotopy group completion" of $\mathscr{C}_p(X)$. Lawson homology is a *covariant* functor from the category of quasiprojective varieties and proper morphisms to the category of abelian groups, and a *contravariant* functor from the category of quasiprojective varieties and flat maps to the category of abelian groups. In this paper we only consider the Lawson homology of varieties over the complex numbers and its behavior under proper maps. In particular, when we assert that Lawson homology is functorial, we mean

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that it is a covariant functor from the category of quasiprojective varieties over $\mathbb{C}$ to the category of abelian groups.

It is shown in [23] that one can use the Grothendieck group $\widetilde{\mathscr{C}}_p(X)$ (naïve group completion), endowed with the quotient topology from $\mathscr{C}_p(X) \times \mathscr{C}_p(X)$, as a model for the "homotopy group completion". This provides an equivalent (and more geometric) definition of Lawson homology which, as shown in [24], allows the theory to be extended for quasiprojective varieties.

In [14] E. Friedlander and B. Mazur used Lawson's main theorem in [20], together with "ruled join" constructions to obtain, among other beautiful results, a *generalized cycle map*

$$(*) \qquad s_X^{(p)}: L_pH_n(X) \to H_n(X\,;\mathbb{Z}),$$

where $L_pH_n(X)$ is Lawson homology of a closed projective variety X and $H_n(X, \mathbb{Z})$ is singular homology. For $n = 2p$ this maps is the classical cycle map [15] from the Chow group of cycles modulo algebraic equivalence to homology. Their construction of $s_X^{(p)}$ is algebraic, and depends fundamentally on the projective embedding of X. They conjecture that the map itself is independent of the embedding.

In this paper we give a more geometric and natural interpretation of Friedlander-Mazur's (F-M) map, and we prove this conjecture and show that the induced map between Lawson homology groups is independent of the projective embedding of the varieties.

In order to analyze the F-M map, we use two fundamental ingredients. The first one is the embedding $e: \widetilde{\mathscr{C}}_p(X) \hookrightarrow \mathscr{Z}_{2p}(X)$ of the group of algebraic p-cycles into the space of the integral cycles (integral currents with zero boundary) endowed with the flat norm topology. The second one is the natural isomorphism [1]

$$\mathscr{A}: \pi_i(\mathscr{Z}_k(X)) \to H_{i+k}(X, \mathbb{Z}),$$

which we shall henceforth refer to as *Almgren's isomorphism*. Our main result, found in §4, is the following:

Theorem 4.3. *The F-M map coincides with the composition*

$$L_pH_n(X) \stackrel{\text{def}}{=} \pi_{n-2p}(\widetilde{\mathscr{C}}_p(X)) \xrightarrow{e_*} \pi_{n-2p}(\mathscr{Z}_{2p}(X)) \xrightarrow{\mathscr{A}} H_n(X, \mathbb{Z}),$$

where e_* is the homomorphism induced by the inclusion e, and $\mathscr{A}$ is Almgren's isomorphism.

The geometric meaning of the above theorem is, essentially, the following. Given a representative $f: S^{n-2p} \to \widetilde{\mathscr{C}}_p \to \widetilde{\mathscr{C}}_p(X)$ of a class $\alpha \in L_pH_n(X)$, the cycle map assigns to α the fundamental class in

$H_n(X, \mathbb{Z})$ of the $(n-2p)$-dimensional family of p-cycles parametrized by f. In other words, $s_X^{(p)}$ is a "cycle map" in a proper sense.

Restricting the homology functors from the category of spaces to the category of projective varieties and regular maps, one obtains the following important consequence.

Corollary 4.4 [22]. *The cycle map $s_*^{(*)}$ is a natural transformation of covariant functors. In particular, it is independent of projective embeddings.*

In the same section we extend the main result so as to define the cycle map for a quasiprojective variety U, taking values in its Borel-Moore homology $H_*^{\mathrm{BM}}(U)$. We prove that $s_U^{(p)}$ is naturally covariant with respect to proper morphisms, commutes with vector bundle projections and preserves localization exact sequences. More precisely, we prove the following corollary and proposition.

Corollary 4.7. *Given an algebraic vector bundle $E \xrightarrow{\pi} X$ of rank r over a projective variety X, the cycle map commutes with pullback under the bundle projection; i.e., the following diagram commutes:*

$$\begin{array}{ccc} L_pH_n(X) & \xrightarrow{\pi^*} & L_{p+r}H_{n+2r}(E) \\ {\scriptstyle s_X^{(p)}}\downarrow & & \downarrow{\scriptstyle s_E^{(p+r)}} \\ H_N(X) & \xrightarrow[\pi^*]{} & H_{n+2r}^{\mathrm{BM}}(E)\,. \end{array}$$

Proposition 4.9. *For a pair of quasiprojective varieties (U, V), with V closed in U, the cycle map gives a morphism of localization long exact sequences*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & L_pH_n(U) & \longrightarrow & L_pH_n(U-V) & \longrightarrow & L_pH_{n-1}(V) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H-n^{\mathrm{BM}}(U) & \longrightarrow & H_n^{\mathrm{BM}}(U-V) & \longrightarrow & H_{n-1}^{\mathrm{BM}}(V) & \longrightarrow & \cdots \end{array}$$

The above proposition is the key ingredient in characterizing those varieties for which the spaces of topological and algebraic cycles are weakly homotopy equivalent.

This paper is organized as follows. In §2 we establish notation, provide the basic ingredients from geometric measure theory needed throughout the paper, and review Almgren's and Federer's fundamental results.

We must point out here that Almgren's isomorphism extended classical results of A. Dold and R. Thom [8], and served as partial inspiration for Lawson's work [20]. Lawson's main result in [20] will be referred to as the "Complex Suspension Theorem":

Theorem 1.1 (CST) [20]. *The complex suspension map*

$$\Sigma\!\!\!\!/\,\colon \widetilde{\mathscr{C}}_p(X) \to \widetilde{\mathscr{C}}_{p+1}(\Sigma\!\!\!\!/\,X)$$

is a homotopy equivalence.

Here the complex suspension $\Sigma\!\!\!\!/\,V$ of an irreducible subvariety $V \subseteq \mathbb{P}^n$ is the ruled join of V to a point $p_\infty \in \mathbb{P}^{n+1} - \mathbb{P}^n$, where $\mathbb{P}^n$ is embedded linearly into $\mathbb{P}^{n+1}$. One extends the suspension linearly to arbitrary cycles. In our discussion of quasiprojective varieties in §4 we reinterpret the CST in terms of pullback of cycles to hyperplane bundles.

In §3 we discuss ruled (complex) joins of algebraic cycles and show that their definition can be extended to integral currents. We then proceed to relate joints with Thom isomorphisms, both via Federer's isomorphism between the homology of the complex of integral currents and singular homology, and via Almgren's isomorphism. Identifying the complex suspension $\Sigma\!\!\!\!/\,X$ with the Thom space of the hyperplane bundle $\mathscr{O}_X(1)$, one has the following propositions.

Proposition 3.6. *Let X be a projective algebraic variety. Then, the map induced by the complex suspension in the homology of the integral currents coincides with the Thom isomorphism τ_X for the hyperplane bundle $\mathscr{O}_X(1)$;*

Proposition 3.7. *The homomorphism $\Sigma\!\!\!\!/\,_*$ induced on homotopy groups by $\Sigma\!\!\!\!/\,\colon \mathscr{Z}_k(X) \to \mathscr{Z}_{k=2}(\Sigma\!\!\!\!/\,X)$ corresponds to the Thom isomorphism via the isomorphism $\mathscr{A}$. In other words, the following diagram commutes:*

$$\begin{array}{ccc}
\pi_m(\mathscr{Z}_k(X)) & \xrightarrow{\Sigma\!\!\!\!/\,_*} & \pi_m(\mathscr{Z}_{k+2}(\Sigma\!\!\!\!/\,X)) \\
\mathscr{A}\big\downarrow & & \big\downarrow\mathscr{A} \\
H_{m+k}(X;\mathbb{Z}) & \xrightarrow[\tau_X]{} & H_{m+k+2}(\Sigma\!\!\!\!/\,X;\mathbb{Z})
\end{array}$$

Those are key ingredients for our analysis of the cycle map and the main results of §4.

In the last section we compute several examples which are contained in a vast class (called class $\mathscr{L}$) consisting of those varieties X for which the cycle map $(*)$ is an isomorphism for all $0 \le p \le \dim X$ and $n \ge 2p$. It follows from Theorem 4.3 that the varieties in class $\mathscr{L}$ are characterized by the fact that

> their space of algebraic p-cycles is naturally weakly homotopy equivalent to the space of topological $2p$-cycles (closed integral currents) for all $p \ge 0$.

Examples of these are varieties admitting "algebraic cellular decompositions", generalized flag varieties, compact hermitian symmetric spaces and smooth varieties admitting a suitable reductive group action.

Later developments that unfolded from Lawson's initial work turned from the differential geometric point of view of [1] and [20] to a prominently algebraic geometric one, as in [11]. Several other works had the same origin, such as [21], [14], [13] and [6], where geometry and algebraic topology are beautifully combined. The present work takes up the original differential geometric approach.

2. Basics

Throughout this work, *projective varieties* are reduced (not necessary irreducible) algebraic varieties which admit a closed embedding into some complex projective space, and *quasiprojective varieties* are Zariski open subsets of some projective variety.

Definition 2.1. An algebraic cycle of dimension p in the projective variety X is a finite formal sum $\sigma = \sum_\lambda n_\lambda V_\lambda$, where $n_\lambda \in \mathbb{Z}$ and V_λ is a p-dimensional irreducible subvariety of X. A cycle σ is *effective* if each n_λ is nonnegative. We denote by $C_{p,d}(X)$ the space of all effective p-cycles in X of a fixed degree d, and by $\mathscr{C}_p(X)$ the disjoint union $\coprod_{d\geq 0} C_{p,d}(X)$.

It is well known that $C_{p,d}(X)$ has the structure of a projective variety and that $\mathscr{C}_p(X)$ is an abelian topological monoid under cycle addition, which we call the *Chow monoid* of p-cycles in X. See [20], [26] and [23] for details.

We devote the rest of this section to describing the basic facts and results from geometric measure theory which will be needed. We use [1], [10] and [9] as the main references and recommend [25] as an additional and accessible source of information for this material. Throughout the remainder of this section, X will always be a compact Lipschitz neighborhood retract (CLNR) in some Euclidean space, unless otherwise stated.

Let $\mathscr{E}_k(X)$ be the space of *k-currents* with compact support contained in X. Denote by $(\mathscr{I}_*(X), \partial)$ the chain complex of *integral currents* supported in X. Given a k-current $T \in \mathscr{I}_k(X)$, the *mass* and *flat norms* of T are denoted $M(T)$ and $F(T)$, respectively. Define $N(T) = M(T) + M(\partial T)$. Let $\mathscr{E}_k(X)_d$ be the subset of $\mathscr{E}_k(X)$ consisting of those currents T satisfying $N(T) \leq d$, and define $\mathscr{I}_k(X)_d$ similarly. Recall that $\mathscr{I}_k(X)_d$ is compact in the topology induced by the flat norm; cf. Theorem 4.2.17(2) of [9] or Theorem 5.5 of [25].

Definition 2.2. Let (X, X') be a pair of CLNR's. Define

$$\mathcal{Z}_k(X, X') \stackrel{\text{def}}{=} \mathcal{I}_k(X) \cap \{T\colon \text{support } (\partial T) \subset X'\}$$

and

$$\mathcal{B}_k(X, X') \stackrel{\text{def}}{=} \{T + \partial S\colon T \in \mathcal{Z}_k(X') \text{ and } S \in \mathcal{Z}_{k+1}(X, X')\} \subset \mathcal{Z}_k(X, X').$$

When endowed with the flat norm topology, $Z_k(X, X')$ becomes a compactly generated topological group. When X' is empty, $\mathcal{Z}_k(X, X')$ is simply denoted $\mathcal{Z}_k(X)$ and is called the group of *integral k-cycles* in X.

Remark 2.3. For a projective variety X there is a natural embedding

$$e\colon \widetilde{\mathscr{C}}_p(X) \to \mathcal{Z}_{2p}(X)$$

of the space of algebraic p-cycles into the space of integral $2p$-cycles; cf. [20], [18].

The spaces of interest to use are the projective algebraic varieties, which are always triangulable spaces when endowed with the analytic topology (cf. [17]) and hence are immediately seen to be CLNR's. It is worth mentioning that if (X, X') is a pair of algebraic varieties, i.e., $X' \subseteq X$, then $\widetilde{\mathscr{C}}_*(X')$ is a closed subgroup of $\widetilde{\mathscr{C}}_*(X)$, and $\mathcal{Z}_*(X')$ is a closed subgroup of $\mathcal{Z}_*(X)$.

Denote the singular homology groups by $H_k(X, X'; \mathbb{Z})$. Recall the following two important results which are crucial to our work. The first one is due to H. Federer and W. Fleming:

Theorem 2.4 [10]. *The group $\mathcal{Z}_k(X, X')/\mathcal{B}_k(X, X')$ is naturally isomorphic to $H_k(X, X'; \mathbb{Z})$. In particular, the homology of the complex $\mathcal{I}_*(X)$ is naturally isomorphic to the singular homology of X.*

The second result is concerned with the topological groups $\mathcal{Z}_*(X)$ and is due to F. Almgren. It turns out to be a generalization of the classical Dold-Thom theorem [8] for the abelian group $AG(X)$ on X. Here, $AG(X)$ is the free abelian group generated by X and suitably topologized so as to coincide with the group of integral 0-currents. See [23] for further details.

Theorem 2.5 [1]. *There is a natural isomorphism (in the locally Lipschitz category)*

$$\mathcal{A}\colon \pi_i(\mathcal{Z}_k(X, X')/\mathcal{I}_k(X')) \to H_{i+k}(X, X'; \mathbb{Z}),$$

where $\mathcal{Z}_k(X, X')/\mathcal{I}_k(X')$ has the quotient group topology.

For the sake of completeness we state here the basic technical results which we need throughout the paper.

1. Isoperimetric choices [1], [9]. Given a *compact Lipschitz neighborhood retract* (CLNR) $A \subset \mathbb{R}^n$, an integral cycle $T \in \mathscr{Z}_k(A)$ with small mass possesses a fundamental property which, roughly speaking, asserts that T bounds an integral current satisfying an isoperimetric inequality. More precisely, one has

Fact 2.6. *For a CLNR $A \subset \mathbb{R}^n$ there are constants $\nu_A > 0$, $\lambda_A < \infty$, depending on A, such that for every $T \in \mathscr{Z}_k(A)$, $k > 0$, satisfying $M(T) < \nu_A$, there exists $S \in \mathscr{I}_{k+1}(A)$ with the following properties*:

$$\partial S = T \quad \text{and} \quad M(S) \leq \lambda_A M(T)^{1+1/k}.$$

A proof of this fact is found in [10, Remark 6.2], and a more general result is proved in [9, 4.4.2(1)]. If T and S are as above, then S is called an *M-isoperimetric choice* for T.

It follows from the compactness properties of integral currents and lower semicontinuity of the mass norm ([9, Theorem 4.2.17(2)], see second paragraph of the present section) that one can make an isoperimetric choice S, for T as above, with the following additional property:

(2.6.1) $$M(S) = \inf\{M(Q) \colon Q \in \mathscr{I}_{k+1}(A) \text{ and } \partial Q = T\}.$$

Such an isoperimetric choice is called a *mass minimizing* choice.

We also need the following corollary of Fact 2.6:

Fact 2.7. *For each positive integer l there exists a constant $\nu_A(l)$ depending on A and l, which satisfies the following*: *Given $T_i \in \mathscr{I}_k(A)$, $k > 0$ and $i = 1, \cdots, l$ such that*

$$\sum_{i=0}^{l} \partial T_i = 0, \qquad \sup\{M(T_i) \colon i = 1, 2, \cdots, l\} \leq \nu_A(l),$$

there is $S \in \mathscr{I}_{k+1}(A)$ which is an M-isoperimetric choice for $\sum_{i=0}^{l} T_i$ (there is even a mass minimizing isoperimetric choice) with

$$M(S) \leq \sup\{m(T_i) \colon i = 1, 2, \cdots, l\}.$$

Recall that the flat norm $F_A(T)$ of an integral current $T \in \mathscr{I}_k(A)$, for a CLNR A, is defined as

(2.7.1) $$F_A(T) = \inf\{M(T + \partial S) + M(S) \colon S \in \mathscr{I}_{k+1}(A)\}.$$

The last basic result we need is a straightforward consequence of the definition of F_A and the above facts:

Fact 2.8. *Given a CLNR A there exists a constant μ_A, depending on A, such that if $T \in \mathscr{Z}_k(A)$ satisfies $F_A(T) < \mu_A$, then*

(2.8.1) $$F_A(T) = \inf\{M(Q) \colon Q0 \in I_{k+1}(A),\ \partial Q = T\}.$$

Furthermore, there is some $S \in \mathscr{I}_{k+1}(A)$ such that $\partial S = T$ and $M(S) = F_A(T)$, in which case S is called a *F_A-isoperimetric choice* for T.

2. Notation (from [1]**).** For each $n = 0, 1, 2, \cdots$, let $\mathrm{I}(1, n)$ be the cell decomposition of the interval $\mathrm{I} = [0, 1]$ whose 1-cells are the subintervals

$$[0, 1 \cdot 2^{-n}], [1 \cdot 2^{-n}, 2 \cdot 2^{-n}], \cdots, [(2^n - 1) \cdot 2^{-n}, 1],$$

and whose 0-cells are the endpoints $[0], [1 \cdot 2^{-n}], [2 \cdot 2^{-n}], \cdots, [1]$. One has the usual boundary homorphism

$$\begin{aligned} d\colon \mathrm{I}(1, n) &\to \mathrm{I}(1, n), \\ [a, b] &\mapsto [b] - [a] \quad \text{for each 1-cell } [a, b], \\ [a] &\mapsto 0 \quad \text{for each 0-cell } [a]. \end{aligned}$$

For each $m = 1, 2, 3, \cdots$ and each $n = 0, 1, 2, \cdots$

$$\mathrm{I}(m, n) = \mathrm{I}(1, n) \otimes \cdots \otimes \mathrm{I}(1, n) \quad (m \text{ times})$$

is a cell complex on I^m, where $\alpha = \alpha_1 \cdots \alpha_m \in \mathrm{I}(m, n)$ is a p-cell and $\dim(\alpha) = p$ if and only if for each $i = 1, \cdots, m$, α_i is a cell in $\mathrm{I}(1, n)$ and $\sum_{i=1}^m \dim(\alpha_i) = p$. We denote by $\mathrm{I}(m, n)_p$ the direct summand of $\mathrm{I}(m, n)$ generated by cells of dimension p. The boundary homomorphism d is given on each cell by

$$d(\alpha) = d(\alpha_1 \otimes \cdots \otimes \alpha_i \otimes \cdots \otimes \alpha_m) = \sum_{i=1}^{m} (-1)^{\sigma(i)} \alpha_1 \otimes \cdots \otimes d\alpha_i \otimes \cdots \otimes \alpha_m,$$
$$\sigma(i) = \sum_{j<i} \dim(\alpha_j).$$

A cell β is a *face* of a cell α if and only if for each $i = 1, \cdots, m$, either $\beta_i = \alpha_i$ or β_i is an endpoint of α_i. The *vertex set* of α consists of all 0-dimensional faces of α.

3. Joins of currents and Thom isomorphisms

3.1. Joins. Let $\mathbb{P}^n$ and $\mathbb{P}^m$ be complex projective spaces embedded as disjoint linear subspaces of $\mathbb{P}^{n+m+1}$. The *complex linear join*, or simply the *join*, of two algebraic subvarieties $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$, is the subvariety $X \sharp Y \subseteq \mathbb{P}^{n+m+1}$ obtained as the union of all projective lines joining points of X to points of X to points of Y. In the case $m = 0$, the join $X \sharp \mathbb{P}^0$

of X to a point $\mathbb{P}^0 \in (\mathbb{P}^{n+1} - \mathbb{P}^n)$ is denoted by $\Sigma\!\!\!/ X$ and is called the *complex suspension* of X.

In what follows we show how to extend the joins of subvarieties to suitable "joins" of currents in projective spaces.

Let π_i, $i = 1, 2$, be the projections onto the first and second factors of $\mathbb{P}^n \times \mathbb{P}^m$, respectively and let $H_1 \oplus H_2$ be the Whitney sum of the bundles $H_1 = \pi_1^* \mathscr{O}_{\mathbb{P}^n}(-1)$ and $H_2 = \pi_2^* \mathscr{O}_{\mathbb{P}^m}(-1)$, where $\mathscr{O}_{\mathbb{P}^n}(-1)$ and $\mathscr{O}_{\mathbb{P}^m}(-1)$ are the tautological bundles over $\mathbb{P}^n$ and $\mathbb{P}^m$, respectively.

Proposition 3.1. *The total space of the smooth* $\mathbb{P}^1$*-bundle* $\mathbb{P}(H_1 \oplus H_2) \xrightarrow{\pi} \mathbb{P}^n \times \mathbb{P}^m$ *is the blow-up of* $\mathbb{P}^n \# \mathbb{P}^m$ *along* $\mathbb{P}^n$ *and* $\mathbb{P}^m$.

We refer the reader to [19, §2], for a proof of the above proposition and for further details on the geometry of this construction. Let b: $\mathbb{P}(H_1 \oplus H_2) \to \mathbb{P}^n \# \mathbb{P}^m$ denote the blow-up map.

It is shown in [7] that for any smooth bundle $F \to E \to B$ with compact, n-dimensional fiber F, there is a unique linear map

$$L_\pi : \mathscr{E}_*(B) \cap \{T : M(T) < \infty\} \to \mathscr{E}_{*+n}(E)$$

which is characterized by

P1. It is natural with respect to bundle maps;

P2. If $E = B \times F$ is a product bundle, then

$$L_\pi(T) = T \times F.$$

This map has the following additional properties:

P3. L_π is continuous (in the weak topology) on $\mathscr{E}_k(B) \cap \{T : M(T) < r\}$, for all $r \in \mathbb{R}$;

P4. There are constants c_0, c_1 such that

$$c_0 M(T) \le M(L_\pi(T)) \le c_1 M(T),$$

whenever $M(T) < \infty$;

P5. L_π takes integral currents into integral currents;

P6. L_π commutes with the boundary operator ∂.

Suppose that $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover of B, and that for each $\lambda \in \Lambda$, $\phi_\lambda : U_\lambda \times F \to \pi^{-1}(U_\lambda)$ is a trivialization of E over U_λ. If $\{U_\lambda, f_\lambda\}_{\lambda \in \Lambda}$ is a partition of unity subordinate to the cover $\{U_\lambda\}_{\lambda \in \Lambda}$, then

$$L_\pi(T) = \sum_\lambda \phi_{\lambda\#}((T \wedge f_\lambda) \times F).$$

Remark 3.2. In our particular case $H_1 \oplus H_2 \xrightarrow{\pi} \mathbb{P}^n \times \mathbb{P}^m$, properties P4 and P6 imply that L_π takes $\mathscr{I}_k(\mathbb{P}^n \times \mathbb{P}^m)_d$ into $\mathscr{I}_{k+2}((\mathbb{P}(H_1 \oplus H_2))_D$, for some D; and P3 implies that L_π is continuous on $\mathscr{I}_k(\mathbb{P}^n \times \mathbb{P}^m)_d$ for all d, since the flat and weak topologies agree in those spaces (cf. [10, Corollary 7.3]). Furthermore, P6 implies that L_π is a chain map when restricted to the complex of integral currents.

Definition 3.3. The (complex) *join* $T\sharp S$ of the integral currents $T \in \mathscr{I}_k(\mathbb{P}^n)$ and $S \in \mathscr{I}_r(\mathbb{P}^m)$ is the current

$$b_\sharp(L_\pi(T \times S)) \in \mathscr{I}_{k+r+2}(\mathbb{P}^{n+m+1}),$$

where $\times$ denotes the Cartesian product of currents, and $b_\sharp$ is the push-forward under the blow-up map described above.

The following properties hold for the complex join of currents:

Proposition 3.4. *Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be algebraic varieties.*

(a) *The complex join map $\sharp$ induces a continuous homomorphism*

$$\sharp : \mathscr{I}_k(X) \wedge \mathscr{I}_s(Y) \to \mathscr{I}_{r+s+1}(X\sharp Y),$$

defined for all $r, s \geq 0$, and satisfying

$$\partial(\sigma\sharp\tau) = (\partial\sigma)\sharp\tau + (-1)^r\sigma\sharp(\partial\tau).$$

When the currents T and S are algebraic cycles, then $T\sharp S$ coincides with the algebraic linear join of cycles.

(b) *In particular the complex suspension $\Sigma\!\!\!/$ provides a continuous homomorphism*

$$\Sigma\!\!\!/ : \mathscr{I}_k(X) \to \mathscr{I}_{k+2}(\Sigma\!\!\!/ X)$$

which is also a chain map of degree two for the complex of integral currents.

Proof. (a) Observe that whenever T or S is the zero current, then $S \times T = 0$, and hence $\sharp = b_\sharp \circ l_\pi \circ \times$ descends to the smash product $\mathscr{I}_k(X) \wedge \mathscr{I}_s(Y)$. The continuity of $\sharp$ for the case $X = \mathbb{P}^n$ and $Y = \mathbb{P}^m$ follows from the fact that $\mathscr{I}_*(\mathbb{P}^n)$ and $I_*(\mathbb{P}^m)$ are topologized as the inductive limit of their subsets of bounded mass and from Remark 3.2. The general case follows by restriction of domains.

The boundary formula is obtained by using the fact that both L_π and $b_\sharp$ commute with ∂ and by the corresponding formula for the product of currents [9].

Since $\pi : \mathbb{P}(H_1 \otimes H_2) \to \mathbb{P}^n \times \mathbb{P}^m$ is a flat map in the sense of algebraic geometry, it induces an "algebraic" pullback map

$$\pi^\sharp : \widetilde{\mathscr{C}}_p(\mathbb{P}^n \times \mathbb{P}^m) \to \widetilde{\mathscr{C}}_{p+1}(\mathbb{P}(h_1 \oplus H_2))$$

between the corresponding spaces of algebraic cycles; cf. [15] and [11]. The restriction of L_π to algebraic cycles is easily seen to coincide with $\pi^\sharp$, and hence the join of currents restricts to the ruled join, since $\sharp = b_\sharp \circ \pi^\sharp \circ \times$ on algebraic cycles. For details on this geometric setting see [19], for example.

(b) Follows from (a) when Y is taken to be a point.

Remark 3.5. Since the Lipschitz constant $\mathrm{Lip}(b)$ of b equals 1, Property P4 implies that for every k there is a constant γ_k such that whenever $M(T) < \infty$ one has

$$M(\not\Sigma T) = M(b_\sharp \circ L_\pi(T)) \leq c_k M(L_\pi(T)) \leq \gamma_k M(T).$$

3.2. Thom isomorphisms. Consider the hyperplane bundle $H = \mathscr{O}(1) \to \mathbb{P}^n$, and let the *projective closure* of H be the $\mathbb{P}^1$-bundle $\mathbb{P}(H \oplus \mathbf{1}_{\mathbb{P}^n})$, where $\mathbf{1}_{\mathbb{P}^n}$ denotes the trivial line bundle over $\mathbb{P}^n$. Since $H \oplus \mathbf{1}_{\mathbb{P}^n}$ carries two canonical subbundles $0 \oplus \mathbf{1}_{\mathbb{P}^n}$ and $H \oplus 0$, we have a zero section $\mathbb{P}^n \equiv \mathbb{P}(H \oplus 0) \overset{s_0}{\hookrightarrow} \mathbb{P}(H \oplus \mathbf{1}_{\mathbb{P}^n})$ and a section at infinity $\mathbb{P}^n \equiv \mathbb{P}(0 \oplus \mathbf{1}_{\mathbb{P}^n}) \overset{s_\infty}{\hookrightarrow} \mathbb{P}(H \oplus \mathbf{1}_{\mathbb{P}^n})$. Furthermore, $\mathbb{P}(H \oplus \mathbf{1}_{\mathbb{P}^n}) - s_0(\mathbb{P}^n)$ and $\mathbb{P}(H \oplus \mathbf{1}_{\mathbb{P}^n}) - s_\infty(\mathbb{P}^n)$, are, respectively, the total spaces of the bundles $H^* \to \mathbb{P}^n$ and $H \to \mathbb{P}^n$, where $H^* = \mathscr{O}(-1)$ is the dual of H. See last paragraph of §2 in [19] for details.

The Thom space for $H \to \mathbb{P}^n$ is the quotient DH/SH, where DH and SH are the disc and sphere bundle for a metric on H, respectively. It can be seen easily that $DH/SH \cong \mathbb{P}(H \oplus \mathbf{1}_{\mathbb{P}^n})/\mathbb{P}^n \cong \not\Sigma\mathbb{P}^n$, where $\mathbb{P}^n$ is identified here to $s_\infty(\mathbb{P}^n) \subset \mathbb{P}(H \oplus \mathbf{1}_{\mathbb{P}^n})$. In this setting the "Thom isomorphism" is defined as

$$\begin{aligned} \Phi_{\mathbb{P}^n} \colon H_r(\mathbb{P}(H \oplus \mathbf{1}_{\mathbb{P}^n}), \mathbb{P}^n; \mathbb{Z}) &\to H_{r-2}(\mathbb{P}^n; \mathbb{Z}), \\ z &\mapsto \pi_*(z \frown U), \end{aligned} \tag{3.5.1}$$

where $U \in H^2(\mathbb{P}(H \oplus \mathbf{1}_{\mathbb{P}^n}), \mathbb{P}^n; \mathbb{Z})$ is an orientation class for the fiber-bundle pair $(\mathbb{P}(H \oplus \mathbf{1}_{\mathbb{P}^n})$, which can be given by the first Chern class of the "hyperplane" bundle $\xi_H \to \mathbb{P}(H \oplus \mathbf{1}_{\mathbb{P}^n})$.

More generally, let $X \subseteq \mathbb{P}^n$ be any projective algebraic variety, and denote also by H the restriction $\mathscr{O}_X(1)$ of $\mathscr{O}_{\mathbb{P}^n}(1)$ to X. The projectivization $\mathbb{P}(H \oplus \mathbf{1}_X)$ of H is again a $\mathbb{P}^1$-bundle over X with two canonical sections as above. The restriction of U to $\mathbb{P}(H \oplus \mathbf{1}_X)$ gives an orientation class for the fiber bundle pair $(\mathbb{P}(H \oplus \mathbf{1}_X), X)$ where S is identified with the section at infinity. Once again, one has a Thom isomorphism

$$\begin{aligned} \Phi_X \colon H_r(\mathbb{P}(H \oplus \mathbf{1}_X), X; \mathbb{Z}) &\to H_{r-2}(X; \mathbb{Z}), \\ z &\mapsto \pi_*(z \frown U); \end{aligned} \tag{3.5.2}$$

cf. [27, Theorem 7.10].

Observe that the blow-down map

$$b \colon (\mathbb{P}(H \oplus \mathbf{1}_X), X) \to (\not\Sigma X, x_\infty)$$

induces an isomorphism

$$b_*\colon H_*(\mathbb{P}(H\oplus \mathbf{1}_X), X; \mathbb{Z}) \to H_*(\Sigma\!\!\!\!/\, X, x_\infty; \mathbb{Z}) \cong \widetilde{H}_*(\Sigma\!\!\!\!/\, X; \mathbb{Z}),$$

and we also call "Thom isomorphisms", by abuse of language, both

$$\Psi_X = \Phi_X \circ b_*^{-1}\colon H_{r+2}(\Sigma\!\!\!\!/\, X; \mathbb{Z}) \to H_r(X; \mathbb{Z}), \tag{3.5.3}$$

and its inverse

$$\tau_X = b_* \circ \Phi_X^{-1}\colon H_r(X; \mathbb{Z}) \to H_{r-2}(\Sigma\!\!\!\!/\, X; \mathbb{Z}). \tag{3.5.4}$$

Proposition 3.6. *Let X be a projective algebraic variety. Then the map induced by the complex suspension in the homology of the integral currents,*

$$\Sigma\!\!\!\!/\colon \mathcal{I}_*(X) \to \mathcal{I}_{*+2}(\Sigma\!\!\!\!/\, X),$$

coincides with the Thom isomorphism τ_X, i.e.,

$$\Sigma\!\!\!\!/_* = \tau_X\colon H_*(X, \mathbb{Z}) \xrightarrow{\cong} H_{*+2}(\Sigma\!\!\!\!/\, X, \mathbb{Z}).$$

Proof. The isomorphism established in [10] between the homology of the complex of integral currents and singular homology is obtained in the chain level by considering Lipschitz singular chains

$$\sigma = \sum n_i\sigma_i, \qquad \sigma_i\colon \Delta_k \to X$$

as currents. We can go further and consider only singular chains which are subordinate to a coordinate chart $\{U_\lambda, \phi_\lambda\}$ for the bundle $\mathbb{P}(H\oplus \mathbf{1}_X)$ over X.

Now choose a representative σ for a class in $H_k(X; \mathbb{Z})$, and let $\pi\colon \mathbb{P}(H\oplus\mathbf{1}_X) \to X$ be the bundle projection and $\Phi_X\colon H_k(\mathbb{P}(H\oplus\mathbf{1}_X), X; \mathbb{Z}) \xrightarrow{\cong} H_k(X; \mathbb{Z})$ be the isomorphism (3.5.2).

In the level of chains

$$\begin{aligned}
\Phi_X(L_\pi(\sigma)) &= p_\sharp(U \frown L_\pi(\sigma)) \\
&= \sum_i n_i p_\sharp(U \frown L_\pi(\sigma_i)) = \sum_i n_i p_\sharp(U \frown \Phi_{\lambda_i\sharp}(\sigma_i \times \mathbb{P}^1)) \\
&= \sum_i n_i p_\sharp(\Phi_{\lambda_i\sharp}(\Phi^\sharp_{\lambda_i} U \frown (\sigma_i \times \mathbb{P}^1))) = \sum_i n_i p_\sharp(\Phi_{\lambda_i\sharp}(\sigma_i \times x_\infty)) \\
&= \sum_i n_i\sigma_i = \sigma,
\end{aligned}$$

where $\sigma_i(\Delta_k) \subset U_{\lambda_i}$. Therefore $[L_\pi(\sigma)] = \Phi^{-1}([\sigma])$, where brackets mean homology classes. Now, for the blow-up map $b\colon \mathbb{P}(H\oplus\mathbf{1}_X) \to \Sigma\!\!\!\!/\, X$ we have

$$\begin{aligned}
b_*\Phi_X^{-1}[\sigma] &= b_*(L_\pi[\sigma]) = [b_\sharp L_\pi(\sigma)] \\
&= [\Sigma\!\!\!\!/\,\sigma] = \Sigma\!\!\!\!/_*[\sigma],
\end{aligned}$$

and hence, $\Sigma\!\!\!\!/_* = b_* \circ \Phi_X^{-1} \overset{\text{def}}{=} \tau_X$. q.e.d.

Having identified the effect of the complex suspension on the homology of the complex of integral currents, we proceed to identify its effect on the homotopy of the integral cycle groups $\mathscr{Z}_k(X)$, under the light of Almgren's isomorphism.

Proposition 3.7. *The homomorphism* $\Sigma\!\!\!\!/_*$ *induced on homotopy groups by*

$$\Sigma\!\!\!\!/: \mathscr{Z}_k(X) \to \mathscr{Z}_{k+2}(\Sigma\!\!\!\!/ X)$$

corresponds to the Thom isomorphism via the isomorphism $\mathscr{A}$. *In other words, the following diagram commutes*:

$$\begin{array}{ccc}
\pi_m(\mathscr{Z}_k(X)) & \xrightarrow{\Sigma\!\!\!\!/_*} & \pi_m(\mathscr{Z}_{k+2}(\Sigma\!\!\!\!/ X)) \\
\mathscr{A}\big\downarrow & & \big\downarrow\mathscr{A} \\
H_{m+k}(X;\mathbb{Z}) & \xrightarrow[\tau_X]{} & H_{m+k+2}(\Sigma\!\!\!\!/ X;\mathbb{Z})
\end{array}$$

Proof. Let $f: (\mathrm{I}^m, \partial\,\mathrm{I}^m) \to (\mathscr{Z}_k(X), \underline{o})$ be a representative for a class $[f] \in \pi_m(\mathscr{Z}_k(X))$, and let $\Sigma\!\!\!\!/ f$ be the corresponding representative for $\Sigma\!\!\!\!/_*[f]$ in $\pi_m(\mathscr{Z}_{k+2}(\Sigma\!\!\!\!/ X))$. For any constant $\lambda > 0$, let $N_f(\lambda) > 0$ be such that

$$F_X(\alpha - \beta) < \lambda \quad \text{whenever } \operatorname{dist}(\alpha, \beta) < 2^{-N_f(\lambda)}.$$

By definition, for any positive integer n sufficiently large (for example $n \geq \max\{N_f(\nu_X(2^m)), N_{\Sigma\!\!\!\!/ f}(\nu_{\Sigma\!\!\!\!/ X}(2^m))\}$), Almgren's construction gives chain maps

$$\phi_X^f: \mathrm{I}(m, n) \to \mathscr{I}_{*+k}(X)$$

and

$$\phi_{\Sigma\!\!\!\!/ X}^{\Sigma\!\!\!\!/ f}: \mathrm{I}(m, n) \to \mathscr{I}_{*+k+2}(\Sigma\!\!\!\!/ X)$$

of degrees k and $k+2$, respectively, unique up to homotopy, characterized by the following

(a) $\phi^f|_{\mathrm{I}(m,n)_0} = f$ and $\phi^{\Sigma\!\!\!\!/ f}|_{\mathrm{I}(m,n)_0} = \Sigma\!\!\!\!/ f$. Here f and $\Sigma\!\!\!\!/ f$ are extended linearly to $\mathrm{I}(m, n)_0$.

(b) For each 1-cell $\alpha \in \mathrm{I}(m, n)_1$, $\phi_X^f(\alpha)$ and $\phi_{\Sigma\!\!\!\!/ X}^{\Sigma\!\!\!\!/ f}(\alpha)$ are F_X and $F_{\Sigma\!\!\!\!/ X}$-isoperimetric choices for $\phi_X^f(d\alpha)$ and $\phi_{\Sigma\!\!\!\!/ X}^{\Sigma\!\!\!\!/ f}(d\alpha)$, respectively.

(c) For each p-cell $\alpha \in \mathrm{I}(m, n)_p$ with $p > 1$, ϕ_X^f and $\phi_{\Sigma\!\!\!\!/ X}^{\Sigma\!\!\!\!/ f}$ are M-isoperimetric choices for $\phi_X^f(d\alpha)$ and $\phi_{\Sigma\!\!\!\!/ X}^{\Sigma\!\!\!\!/ f}(d\alpha)$, respectively, as in (2.7) (with $l \geq 2^m$).

(d) If $\Theta_X = \max\{F_A(f(\alpha), f(\beta)) : \ \alpha$ and β are 0-cells lying in the vertex set of same m-cell in $\mathrm{I}(m, n)\}$ (similarly, $\Theta_{\Sigma\!\!\!/ X}$), then for each p-cell $\alpha \in \mathrm{I}(m, n)$ and $p > 1$, there is a constant $1 < \rho_X < \infty$ (respect. $\rho_{\Sigma\!\!\!/ X}$) such that

$$M(\phi_X^f(\alpha)) \le \rho_X \Theta_X \quad (\text{respect. } M(\phi_{\Sigma\!\!\!/ X}^{\Sigma\!\!\!/ f}(\alpha)) \le \rho_{\Sigma\!\!\!/ X}\Theta_{\Sigma\!\!\!/ X}).$$

Almgren's map is then given by

$$\mathscr{A}_X[f] = \left[\sum_i \phi_X^f(\sigma_i)\right],$$

(similarly for $\mathscr{A}_{\Sigma\!\!\!/ X}$), where the σ_i's are the m-cells of the complex $\mathrm{I}(m, n)$.

Now choose $\delta > 0$ satisfying

$$\delta < \min\{1, \nu_X(2^{m+2}), \nu_{\Sigma\!\!\!/ X}(2^{m+2})\},$$
$$\delta \cdot (1 + \gamma_{k+1}) < \mu_{\Sigma\!\!\!/ X},$$
$$\max\{\rho_X, \rho_{\Sigma\!\!\!/ X}\}(1 + \gamma_{p+k+1} + 2^m)\delta < \mu_{\Sigma\!\!\!/ X}, \qquad p = 0, \cdots, m,$$

and fix $n > \max\{N_f(\delta), N_{\Sigma\!\!\!/ f}(\delta)\}$ large enough to define Almgren's maps ϕ_X^f and $\phi_{\Sigma\!\!\!/ X}^{\Sigma\!\!\!/ f}$. Here the ν's and μ's were defined in the isoperimetric choices of §2, and the γ's in Remark 3.5. Define a new chain map

$$\Psi : \mathrm{I}(m, n) \to \mathscr{I}_{*+k+2}(\Sigma\!\!\!/ X)$$

of degree $k + 2$, as the composition $\Sigma\!\!\!/ \circ \phi^f$. This map has the following properties:

(a) If α is a 0-cell in $\subset (m, n)$, then

$$\Psi(\alpha) = \Sigma\!\!\!/(\phi^f(\alpha)) = \Sigma\!\!\!/(f(\alpha)) = (\Sigma\!\!\!/ f)(\alpha) = \phi^{\Sigma\!\!\!/ f}(\alpha),$$

since $\phi_{|_{\mathrm{I}(m,n)}} \equiv f$ by definition.

(b) If α is a 1-cell in $\mathrm{I}(m, n)$, then

$$\begin{aligned} \partial\Psi(\alpha) &= \partial \circ \Sigma\!\!\!/ \circ \phi^f(\alpha) = \Sigma\!\!\!/ \circ \phi^f(d\alpha) \\ &= \Sigma\!\!\!/\phi^f(\alpha_1 - \alpha_2) = \Sigma\!\!\!/(f(\alpha_1 - \alpha_2)) \\ &= \Sigma\!\!\!/ f(\alpha_1) - \Sigma\!\!\!/ f(\alpha_2) = \phi^{\Sigma\!\!\!/ f}(d\alpha) \\ &= \partial\phi^{\Sigma\!\!\!/ f}(\alpha), \end{aligned}$$

where we write $d\alpha = \alpha_1 - \alpha_0$ with α_1 and α_0 are 0-cells in $\mathrm{I}(m, n)$, and

$$\begin{aligned} M(\Psi(\alpha)) &= M(\Sigma\!\!\!/\phi^f(\alpha)) \le \gamma_{k+1} M(\phi^f(\alpha)) \\ &= \gamma_{k+1} F_x(f(\alpha_1) - f(\alpha_0)) \le \gamma_{k+1}\Theta_X \\ &\le \gamma_{k+1}\delta, \end{aligned}$$

where the second equality comes from the F_X-minimizing choice for ϕ^f when restricted to the 1-cells of $\mathrm{I}(m, n)$, and the last inequality comes from the choice of $n > N_f(\delta)$.

(c) Let α now be a p-cell in $\mathrm{I}(m, n)$ with $p > 1$, and recall that $\phi^f(\alpha)$ was chosen so as to satisfy the conditions of isoperimetric choices, in §2.

In particular, we have the inequality $M(\phi^f(\alpha)) \le \rho_X \Theta_X$, and hence

$$\begin{aligned} M(\Psi(\alpha)) &= M(\Sigma\phi^f(\alpha)) \\ &\le \gamma_{p+k} M(\phi^f(\alpha)) \le \gamma_{p+k}\rho_X\Theta_X \\ &\le \gamma_{p+k}\rho_X\delta . \end{aligned}$$

Define, inductively, homomorphisms $K_i\colon \mathrm{I}(m, n)_i \to \mathscr{I}_{i+k+3}(\Sigma X)$ as follows:

For $i = 0$ let K_0 be the zero homomorphism. Let α be a 1-cell in $\mathrm{I}(m, n)_1$. From item (b) above, we have that $\partial\Psi(\alpha) = \partial\phi^{\Sigma f}$. Also

$$\begin{aligned} M(\phi^{\Sigma f}(\alpha) - \Psi(\alpha)) &\le M(\phi^{\Sigma f}(\alpha)) + M(\Psi(\alpha)) \\ &\le M(\phi^{\Sigma f}(\alpha)) + \gamma_{k+1} M(\phi^f(\alpha)) \\ &= F_{\Sigma X}(\Sigma f(\alpha_1) - \Sigma f(\alpha_0)) + \gamma_{k+1} F_X(\alpha_1 - \alpha_0) \\ &\le \delta + \gamma_{k+1}\delta = (1 + \gamma_{k+1})\delta \\ &< \mu_{\Sigma X} . \end{aligned}$$

Our hypothesis on δ assures the existence of an M-minimizing isoperimetric choice $S \in I_{k+4}(\Sigma X)$ for $\phi^{\Sigma F}(\alpha) - \Psi(\alpha)$ satisfying the conditions of (2.7). Define $K_1(\alpha) = S$ and notice that

$$\phi^{\Sigma f} - \Psi = \partial \circ K_1 + k_0 \circ d$$

and

$$M(K_1(\alpha)) \le \max\{M(\phi^{\Sigma f}(\alpha)), M(\Psi(\alpha))\},$$

the latter inequality coming from (2.7).

Suppose that we have defined homomorphisms K_p for $p \le p_0$, satisfying

$$\phi^{\Sigma f} - \Psi = \partial \circ K_p + K_{p-1} \circ d$$

and

$$M(K_p(\alpha)) \le \max\{M(\phi^{\Sigma f}(\alpha)), M(\Psi(\alpha))\},$$

on p-cells. Let α be a (p_0+1)-cell in $\mathrm{I}(m, n)$ and set $T = \phi^{\Sigma f}(\alpha) - \Psi(\alpha) - K_{p_0}(d\alpha)$. Now, observe that $d\alpha = \sum_i \alpha_i$, where the α_i's are p_0-cells not exceeding 2^m in number. Hence we have

$$M(K_{p_0}(d\alpha)) \le \sum_i M(K_{p_0}(\alpha_i)) \le 2^m \max_i\{M(K_{p_0}(\alpha - i))\}.$$

It is immediate from its construction that $\partial T = 0$, and,

$$\begin{aligned}
M(T) &\le M(\phi^{\not\Sigma f}(\alpha)) + M(\Psi(\alpha)) + M(K_{p_0}(d\alpha)) \\
&\le \rho_{\not\Sigma X}\Theta_{\not\Sigma X} + \gamma_{p_0+k+1}\rho_X\Theta_X + 2^m \max\{M(K_{p_0}(\alpha_i))\} \\
&\le \rho_{\not\Sigma X}\Theta_{\not\Sigma X} + \gamma_{p_0+k+1}\rho_X\Theta_X + 2^m \max\{\rho_X\Theta_X\,,\, \rho_{\not\Sigma X}\Theta_{\not\Sigma X}\} \\
&\le \max\{\rho_X\,,\, \rho_{\not\Sigma X}\}(1 + \gamma_{p_0+k+1} + 2^m)\delta \\
&< \mu_{\not\Sigma X}.
\end{aligned}$$

Our choice of δ again implies that we can make an isoperimetric choice $S \in \mathscr{I}_{k+p_0+4}(\not\Sigma X)$ for T, as in (2.7), and define $K_{p_0+1}(\alpha) = S$. We now see that the maps K_* provide a chain homotopy between $\phi^{\not\Sigma f}$ and Ψ. Finally, let $\{\alpha_i\}$ be the m-cells of $\mathrm{I}(m, n)$. Since $\phi^{\not\Sigma f}$ and Ψ are chain homotopic, we have equality of homology classes:

$$\left[\Psi\left(\sum_i \alpha_i\right)\right] = \left[\phi^{\not\Sigma f}\left(\sum_i \alpha_i\right)\right].$$

However, by definition, the first class is

$$\left[\not\Sigma\left(\sum_i \phi^f(\alpha_i)\right)\right] = \tau_X\left[\sum_i \phi^f(\alpha_i)\right] = \tau_X \circ \mathscr{A}_X[f],$$

while the second one is $[\phi^{\not\Sigma f}(\sum_i \alpha_i)] = \mathscr{A}_{\not\Sigma X}([\not\Sigma f])$. This concludes the proof of the proposition.

4. The cycle map

Let $X \subseteq \mathbb{P}^n$ be a projective variety and consider the complex join

$$\sharp\colon \widetilde{\mathscr{C}}_0(\mathbb{P}^1) \wedge \widetilde{\mathscr{C}}_r(X) \to \widetilde{\mathscr{C}}_{r+1}(\not\Sigma^2 X),$$

where $\mathbb{P}^1$ and $\mathbb{P}^n$ are embedded in $\mathbb{P}^{n+2}$ as disjoint linear subspaces.

Using the canonical identification $(S^2, e_0) \equiv (\mathbb{P}^1, p_\infty)$ and composing with the inclusion $\mathbb{P}^1 \hookrightarrow \widetilde{\mathscr{C}}_0(\mathbb{P}^1)$ sending $p \in \mathbb{P}^1$ to $p - p_\infty$, one obtains a map $S^2 \wedge \widetilde{\mathscr{C}}_r(X) \to \widetilde{\mathscr{C}}_{r+1}(\not\Sigma^2 X)$. Applying the homotopy group functor π_* naturally yields a pairing $\pi_j(S^2) \otimes \pi_k(\widetilde{\mathscr{C}}_r(X)) \to \pi_{k+j}(\widetilde{\mathscr{C}}_{r+1}(\not\Sigma^2 X))$.

Now, let $\mathbf{1}$ be the generator of $\pi_2(S^2)$ (or better $\pi_2(\mathbb{P}^1)$) which corresponds to its canonical orientation. Define the homomorphism

$$\sharp_* : \pi_k(\widetilde{\mathscr{C}}_r(X)) \to \pi_{k+2}(\widetilde{\mathscr{C}}_{r+1}(\Sigma\!\!\!/^2 X)) \tag{4.0.1}$$

as the composite of the above pairing with the map $\alpha \mapsto \mathbf{1} \otimes \alpha$ from $\pi_k(\widetilde{\mathscr{C}}_r(X))$ to $\pi_j(S^2) \otimes \pi_k(\widetilde{\mathscr{C}}_r(X))$.

A further composition with the homotopy inverse of the complex suspension isomorphism (CST) $\Sigma\!\!\!/^2 : \widetilde{\mathscr{C}}_{r-1}(X) \overset{\approx}{\to} \widetilde{\mathscr{C}}_{r+1}(X)$ allows the following definition:

Definition 4.1 (Friedlander-Mazur). Denote the composition $(\Sigma\!\!\!/^2)^{-1} \circ \sharp_*$ by $s : L_pH_n(X) \to L_{p-1}H_n(X)$. We call the iteration

$$s^{(p)} = \underbrace{s \circ \cdots \circ s}_{p\text{-times}} : L_pH_n(X) \to L_pH_0(X) \cong H_n(X, \mathbb{Z})$$

the F-M (Friedlander-Mazur) generalized cycle map.

This map has several interesting properties. For example, it is shown [14] that $s^{(p)} : L_pH_{2p}(X) \to H_n(X, \mathbf{Z})$ coincides with the classical cycle map from the Chow group of cycles modulo algebraic equivalence to the singular homology of X. See [15, Chapter 19]. Those maps are used in [14] to construct an interesting filtration in the ordinary homology of X which relates to Grothendieck's arithmetic filtration and the Hodge filtration.

Our purpose in this section is to associate the F-M cycle map in a natural way with Almgren's map, obtaining, as a consequence, the functoriality of the F-M map. This fact was conjectured in earlier versions of [14] and first proven in [22].

In a similar fashion to Proposition 3.7 we prove:

Lemma 4.2. *The following diagram commutes*:

$$\begin{array}{ccc}
\pi_m(\mathscr{Z}_k(Y)) & \xrightarrow{\mathscr{A}} & H_{m+k}(Y) \\
 & & \downarrow \tau_Y \\
\sharp_* \downarrow & & H_{m+k+2}(\Sigma\!\!\!/ Y) \\
 & & \downarrow \tau_{\Sigma\!\!\!/ Y} \\
\pi_{m+2}(\mathscr{Z}_{k+2}(\Sigma\!\!\!/^2 Y)) & \xrightarrow[\mathscr{A}]{} & H_{m+k+4}(\Sigma\!\!\!/^2 Y)
\end{array}$$

for all m and k.

Proof. After choosing a representative $f : (I^m, \partial I^m) \to (\mathscr{Z}_k(X), \underline{o})$ for a class $\alpha \in \pi_m(\mathscr{Z}_k(X))$, define $\sharp f : (I^{m+2}, \partial I^{m+2}) \to (\mathscr{Z}_{k+2}(\mathbb{P}^1 \sharp X), \underline{o})$ by $\sharp f(x) = \mathbb{P}^1 \sharp f(x)$.

For $n > \max\{N_f(\delta), N_{\sharp f}(\delta)\}$, and δ suitably chosen, one can define the chain maps:

$$\phi^f \colon \mathrm{I}(m, n) \to \mathcal{I}_{*+k}(X)$$

and

$$\phi^{\sharp f} \colon \mathrm{I}(m+2, n) \to \mathcal{I}_{*+k+2}(\mathbb{P}^1 \sharp X).$$

Define another chain map

$$\Psi \colon \mathrm{I}(m+2, n) \to \mathcal{I}_{*+k+2}(\mathbb{P}^1 \sharp X)$$

by sending a p-cell $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_{m+1} \otimes \alpha_{m+2}$ to

$$\Psi(\alpha) = \phi^f(\alpha')\sharp\alpha'',$$

where $\alpha' = \alpha_1 \otimes \alpha_m$, $\alpha'' = \alpha_{m=1}$, and we are identifying $\alpha_{m+1} \otimes \alpha_{m+2}$ with an integral cycle of dimension d, supported in $\mathbb{P}^1 \equiv I^2/\partial I^2$.

Observing that $\Psi(\alpha) = \phi^{\sharp f}$ for 0-cells and also $\partial\Psi(\alpha) = \phi^{\sharp f}(\alpha)$ for 1-chains, one can construct, just as in Proposition 3.7, a chain homotopy between $\phi^{\sharp f}$ and Ψ after a suitable choice of δ.

Therefore, if $\alpha_{ij} = \alpha_i' \otimes \alpha_j''$ are the m-cells of $\mathrm{I}(m+2, n)$, then, by definition,

$$\begin{aligned}\left[\phi^{\sharp f}\left(\sum_{i,j} \alpha_{ij}\right)\right] &= \left[\sum_i \phi^{\sharp f}(\alpha_i')\sharp \sum_j \alpha_j''\right] = \left[\phi^{\sharp f}\left(\sum_i \alpha_i'\right)\sharp\mathbb{P}^1\right] \\ &= \left[\cancel{\Sigma}\left(\phi^{\sharp f}\left(\sum_i \alpha_i'\right)\right)\right] = \tau \circ \tau \left[\phi^{\sharp f}\left(\sum_i \alpha_i'\right)\right] \\ &= \tau \circ \tau \circ \mathcal{A}[f],\end{aligned}$$

and this concludes the proof.

4.1. Projective case. Let X be a projective variety and let $e\colon \widetilde{\mathcal{C}}_p(X) \to \mathcal{Z}_{2p}(X)$ be the natural embedding, cf. Remark 2.3. Our main result is the following

Theorem 4.3. *The composition $\mathcal{A} \circ e_*$ coincides with the F-M cycle map. In other words,*

$$\mathcal{A} \circ e_* = s^{(p)} \colon L_pH_n(X) \to H_n(X, \mathbb{Z}).$$

Proof. We use induction on the dimension of the cycles. For cycles of dimension zero, $\widetilde{\mathcal{C}}_0(Y) \equiv \mathcal{Z}_0(Y)$ for any algebraic set Y. Furthermore, the cycle map from $\pi_i(\widetilde{\mathcal{C}}_0(Y)) \equiv \pi_i(\mathcal{Z}_0(Y))$ into $H_i(Y; \mathbb{Z})$ is actually the Dold-Thom isomorphism [8], as pointed out in [1], and of which Almgren's isomorphism is an extension.

Assume that for any algebraic variety Y the cycle map

$$s^{(r)}\colon L_rH_i(Y) \to H_i(Y\,;\mathbb{Z})$$

equals the composition $\mathscr{A} \circ e_*$, for all $r \le r_0$ and any $i \ge 2r$. Recall that the map

$$s\colon \pi_{i-2r_0-2}(\widetilde{\mathscr{C}}_{r_0+1}(X)) \to \pi_{i-2r_0}(\widetilde{\mathscr{C}}_{r_0}(X))$$

is the composition of

$$\sharp_*\colon \pi_{i-2r_0-2}(\widetilde{\mathscr{C}}_{r_0+1}(X)) \to \pi_{i-2r_0}(\widetilde{\mathscr{C}}_{r_0+2}(\mathbb{P}^1\sharp X)),$$

defined in 4.0.1, with the inverses of the (complex suspension) isomorphisms

$$\Sigma\!\!\!\!/\colon \pi_{i-2r_0}(\widetilde{\mathscr{C}}_{r_0}(X)) \to \pi_{i-2r_0}(\widetilde{\mathscr{C}}_{r_0+1}(\Sigma\!\!\!\!/ X))$$

and

$$\Sigma\!\!\!\!/\colon \pi_{i-2r_0}(\widetilde{\mathscr{C}}_{r_0+1}(\Sigma\!\!\!\!/ X)) \to \pi_{i-2r_0}(\widetilde{\mathscr{C}}_{r_0+1}(\Sigma\!\!\!\!/^2 X)).$$

Gathering the diagrams in Proposition 3.7 and Lemma 4.2 together one obtains the following commutative diagram:

$$\begin{tikzcd}
\pi_{i-2r_0-2}(\widetilde{\mathscr{C}}_{r_0+1}(X)) \arrow[r, "e_4"] \arrow[d, "\sharp_*"] \arrow[ddd, bend right=60, "s"'] & \pi_{i-2r_0-2}(\mathscr{Z}_{2r_0+2}(X)) \arrow[d, "\sharp_*'"] \arrow[dddr, bend left, "\mathscr{A}_4"] & \\
\pi_{i-2r_0}(\widetilde{\mathscr{C}}_{r_0+2}(\Sigma\!\!\!\!/^2(X))) \arrow[r, "e_3"] & \pi_{i-2r_0}(\mathscr{Z}_{2r_0+4}(\Sigma\!\!\!\!/^2 X)) \arrow[r, "\mathscr{A}_3"] & H_{i+4}(\Sigma\!\!\!\!/^2 X) \\
\pi_{i-2r_0}(\widetilde{\mathscr{C}}_{r_0+1}(\Sigma\!\!\!\!/ X)) \arrow[r, "e_2"] \arrow[u, "\cong"] & \pi_{i-2r-0}(\mathscr{Z}_{2r_0+2}(\Sigma\!\!\!\!/ X)) \arrow[r, "\mathscr{A}_2"] \arrow[u, "\Sigma\!\!\!\!/_2"', "\Sigma\!\!\!\!/_2"] & H_{i+2}(\Sigma\!\!\!\!/ X) \arrow[u, "\tau_2"'] \\
\pi_{i-2r_0}(\widetilde{\mathscr{C}}_{r_0}(X)) \arrow[r, "e_1"] \arrow[u, "\cong"] & \pi_{i-2r_0}(\mathscr{Z}_{2r_0}(X)) \arrow[r, "\mathscr{A}_1"] \arrow[u, "\Sigma\!\!\!\!/_1"', "\Sigma\!\!\!\!/_1"] & H_i(X) \arrow[u, "\tau_1"']
\end{tikzcd}$$

By induction, the composition $\mathscr{A}_1 \circ e_1$ (in the bottom line of the diagram) is the cycle map $s_X^{(r_0)}\colon L_{r_0}H_{i+2}(X) \to H_{i+2+2r_0}(X\,;\mathbb{Z})$. A simple diagram chase gives

$$\begin{aligned}
s_X^{(r_0+1)} &= s_X^{(r_0)} \circ s = \mathscr{A}_1 \circ e_1 \circ (\Sigma\!\!\!\!/_1)^{-1} \circ (\Sigma\!\!\!\!/_2)^{-1} \circ \sharp_* \\
&= \tau_1^{-1} \circ \mathscr{A}_2 \circ e_2 \circ \Sigma\!\!\!\!/_1 \circ (\Sigma\!\!\!\!/_1)^{-1} \circ (\Sigma\!\!\!\!/_2)^{-1} \circ \sharp_* \\
&= \tau_1^{-1} \circ \tau_2^{-1} \circ \mathscr{A}_3 \circ e_3 \circ -\Sigma\!\!\!\!/_2 \circ (\Sigma\!\!\!\!/_2)^{-1} \circ \sharp_* \\
&= \tau_1^{-1} \circ \tau_2^{-1} \circ \mathscr{A}_3 \circ \sharp_*' \circ e_4 \\
&= \mathscr{A}_4 \circ e_4,
\end{aligned}$$

as desired. q.e.d.

The following corollary follows from the naturality of Almgren's map in the Lipschitz continuous category and the theorem above.

Corollary 4.4. *The F-M cycle map is a natural transformation of covariant functors. In particular, it is independent of projective embeddings.*

Corollary 4.5. *Given a projective variety $X \hookrightarrow \mathbb{P}^n$ the following diagram commutes:*

$$\begin{tikzcd} L_pH_n(X) \arrow[r, "s_X^{(p)}"] \arrow[d, "\Sigma\!\!\!/"'] & H_n(X;\mathbb{Z}) \arrow[d, "\tau_X"] \\ L_{p+1}H_{n+2}(\Sigma\!\!\!/X) \arrow[r, "s_{\Sigma\!\!\!/X}^{(p+1)}"'] & H_{n+2}(\Sigma\!\!\!/X;\mathbb{Z}), \end{tikzcd}$$

where τ is the Thom isomorphism for the hyperplane bundle over X.

Proof. It follows at once from the Theorem and Proposition 3.7.

4.2. Quasiprojective case. Given a pair (X, X') of projective varieties one can use the natural morphism

$$\widetilde{\mathscr{C}}_p(X)/\widetilde{\mathscr{C}}_p(X') \to \mathscr{Z}_{2p}(X, X')/\mathscr{I}_{2p}(X'),$$

see Definition 2.2, together with the "relative" Almgren's isomorphism

$$\mathscr{A} : \pi_{n-2p}(\mathscr{Z}_{2p}(X, X')/\mathscr{I}_{2p}(X')) \to H_n(X, X'; \mathbb{Z}),$$

to obtain a natural transformation of functors in the category of pairs of projective varieties, which we still denote

$$s_U^{(p)} : L_pH_n(X, X') \to H_n(X, X'; \mathbb{Z}),$$

where $L_pH_n(X, X') \overset{\text{def}}{=} \pi_{n-2p}(\widetilde{\mathscr{C}}_p(X)/\widetilde{\mathscr{C}}_p(X'))$ (see [24]).

It is shown in [24, Theorem 4.3] that the isomorphism type of the topological group $\widetilde{\mathscr{C}}_p(X)/\widetilde{\mathscr{C}}_p(X')$ depends only on the isomorphism type of the quasiprojective variety $X - X'$.

Now, if one denotes by $H_*^{\mathrm{BM}}(U)$ the Borel-Moore homology ([5], [16]) of a quasiprojective variety U, one sees that $H_*^{\mathrm{BM}}(U)$ is naturally isomorphic to the relative singular homology $H_*(\overline{U}, \overline{U}-U; \mathbb{Z})$, for any projective compactification $\overline{U}$ of U. This fact, together with the naturality of $s^{(p)}$ for pairs, makes the following definition independent of choices:

Definition 4.6. Given a quasiprojective variety U, define the cycle map

$$s_U^{(p)} : L_pH_n(U) \to H_n^{\mathrm{BM}}(U) \tag{4.6.1}$$

as $\mathscr{A} \circ e_*$, where the Lawson homology of U is defined as

$$L_pH_n(U) \stackrel{\text{def}}{=} \pi_{n-2p}(\widetilde{\mathscr{C}}_p(\overline{U})/\widetilde{\mathscr{C}}_p(\overline{U}-U)),$$

for a projective compactification $\overline{U}$ of U.

Corollary 4.7. *Given an algebraic vector bundle $E \xrightarrow{\pi} X$ of rank r over a projective variety X, the cycle map commutes with pullback under the bundle projection; i.e., the following diagram commutes*:

$$\begin{tikzcd} L_pH_n(X) \arrow[r, "\pi^*"] \arrow[d, "s_X^{(p)}"'] & L_{p+r}H_{n+2r}(E) \arrow[d, "s_E^{(p+r)}"] \\ H_n(X) \arrow[r, "\pi^*"'] & H^{\mathrm{BM}}_{n+2r}(E). \end{tikzcd}$$

Proof. It follows at once from the commutative diagram:

$$\begin{tikzcd} \widetilde{\mathscr{C}}_p(X) \arrow[r, "\pi^\sharp"] \arrow[d] & \widetilde{\mathscr{C}}_{p+r}(\mathbb{P}(E\oplus \mathbf{1})) \arrow[r, "p_\sharp"] \arrow[d] & \widetilde{\mathscr{C}}_{p+r}(\mathbb{P}(E\oplus \mathbf{1}))/\widetilde{\mathscr{C}}_{p+r}(s_\infty X) \arrow[d] \\ \mathscr{Z}_{2p}(X) \arrow[r, "\pi^\sharp"] & \mathscr{Z}_{2(p+r)}(\mathbb{P}(E\oplus \mathbf{1})) \arrow[r, "p_\sharp"] & \mathscr{Z}_{p+r}(\mathbb{P}(E\oplus \mathbf{1}), s_\infty X)/\mathscr{I}_{p+r}(s_\infty X), \end{tikzcd}$$

where the $p^\sharp$'s are pullbacks of currents and cycles and the $p_\sharp$'s are quotient maps.

Remark 4.8. As it is natural to expect (see [24, Remark 4.9]), the pullback of cycles under algebraic vector bundle projections induces an isomorphism in Lawson homology. A proof of this fact has been recently given in [12]. It follows that both horizontal arrows in the corollary's diagram are isomorphisms.

Proposition 4.9. *For a pair of quasiprojective varieties (U, V), with V closed in U, the cycle map gives a morphism of localization long exact sequences*

$$\begin{tikzcd} \cdots \arrow[r] & L_pH_n(U) \arrow[r] \arrow[d] & L_pH_n(U-V) \arrow[r] \arrow[d] & L_pH_{n-1}(V) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^{\mathrm{BM}}_n(U) \arrow[r] & H^{\mathrm{BM}}_n(U-V) \arrow[r] & H^{\mathrm{BM}}_{n-1}(V) \arrow[r] & \cdots \end{tikzcd}$$

The localization exact sequence for Lawson homology was obtained in [24] as a consequence of the results in [23] which show, in particular, that the exact sequence

$$\frac{\widetilde{\mathscr{C}}_p(\overline{V})}{\widetilde{\mathscr{C}}_p(\overline{V}-V)} \to \frac{\check{\mathscr{C}}_p(\overline{U})}{\widetilde{\mathscr{C}}_p(\overline{U}-U)} \to \frac{\widetilde{\mathscr{C}}_p(\overline{U-V})}{\mathscr{C}_p(\overline{U-V}-(U-V))}$$

is a principal fibration. The proof of the proposition follows from definitions and homotopy exact sequences for fibrations.

5. The class $\mathscr{L}$

Using the results obtained so far, together with Lawson's complex suspension theorem it can be easily seen that the cycle maps $s_U^{(p)}$ turn out to be isomorphisms for certain varieties, such as products of projective spaces $\mathbb{P}^n \times \mathbb{P}^m$, affine spaces $\mathbb{A}^n$ and hyperquadrics $\mathscr{Q}_n^k$ of arbitrary rank k in $\mathbb{P}^{n+1}$. See [24] for computations of those specific examples.

In order to simplify some statements, we introduce the following definition.

Definition 5.1. An algebraic variety U lies in the class $\mathscr{L}$ if the cycle maps

$$s_U^{(p)}: L_pH_n(U) \to H_n(U)$$

are isomorphisms, for all p and $n \geq 2p$.

Remark 5.2. An interesting and important characterization of this class is the following fact:

> A variety U is in class $\mathscr{L}$ if and only if the inclusion $e: \widetilde{\mathscr{C}}_p(U) \hookrightarrow \mathscr{Z}_{2p}(U)$ of the space of the algebraic p-cycles $\widetilde{\mathscr{C}}_p(U)$ into the space of topological $2p$-cycles $\mathscr{Z}_{2p}(U)$ $(= \mathscr{Z}_{2p}(\overline{U})/\mathscr{Z}_{2p}(\overline{U} - U))$ is a *weak homotopy equivalence.*

This assertion follows from the definition of $s_U^{(p)}$ as the composition

$$\pi_{n-2p}(\widetilde{\mathscr{C}}_p(U)) \xrightarrow{e_*} \pi_{n-2p}(\mathscr{Z}_{2p}(U)) \xrightarrow{\mathscr{A}} H_n(U)$$

and the fact that the last map is an isomorphism.

It is immediate from its definition that the varieties lying in class $\mathscr{L}$ are very special in nature. If $X \in \mathscr{L}$ is smooth and projective (resp. quasiprojective) for example, its $2p$th singular homology $H_{2p}(X; \mathbb{Z})$ (resp. Borel-Moore homology) is isomorphic via the cycle map to $L_pH_{2p}(X)$ (resp. $L_pH_n(U))$, which in turn is the Chow group of algebraic cycles modulo algebraic equivalence. In particular, the $2p$th cohomology is of type (p, p) in the Hodge decomposition.

The following definition was essentially taken from [15, Example 1.9.1]:

Definition 5.3. Let (X, Y) be a pair of projective varieties. We say that X is an algebraic cellular extension of Y if X has a filtration $X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = Y$ by projective subvarieties, with each

$X_i - X_{i-1}$ a disjoint union of quasiprojective U_{ij} isomorphic to affine spaces $\mathbb{C}^{n_{ij}}$, for $i \geq 0$. In case $Y = \varnothing$, we recover Fulton's definition of a projective variety with a cellular decomposition.

Theorem 5.4. *If X is an algebraic cellular extension of Y, then X lies in the class $\mathscr{L}$ if and only if Y does. In particular, one sees that $\mathscr{L}$ is closed under algebraic cellular extensions.*

Proof. One argues using induction on the height of the filtration, and the result follows from a simple combination of the localization sequence of Proposition 4.9 with the fact that affine spaces are in $\mathscr{L}$ and that $\widetilde{\mathscr{C}}_p(V \amalg W) \cong \widetilde{\mathscr{C}}_p(V) \times \widetilde{\mathscr{C}}_p(W)$.

Corollary 5.5. *If X is a projective variety with a cellular decomposition in the sense of Fulton, (i.e., X is an algebraic cellular extension of $\varnothing$), then it lies in the class $\mathscr{L}$.*

Remark 5.6. Let X and X' both have cellular decompositions $X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \varnothing$ and $X' = X'_m \supset X'_{m-1} \supset \cdots \supset X'_0 \supset X'_{-1} = \varnothing$. It is easy to see that

$$(X \times Y)_n \stackrel{\text{def}}{=} \bigcup_{i+j=n} X_i \times Y_j$$

provides a cellular decomposition for $X \times Y$, and hence the product $X \times Y$ also lies in the class $\mathscr{L}$.

Therefore one sees that the following examples, as well as affine and projective algebraic bundles over them, lie in the class $\mathscr{L}$:

Example 5.7 (Generalized Flag Varieties G/P). Here G is a semisimple linear algebraic group (defined over $\mathbb{C}$), and $P \subset G$ is a parabolic subgroup; i.e., it contains a Borel subgroup B of G.

It is a well-known fact that G/P possesses an algebraic cellular decomposition; see [4] and [2] for details. Consequently all G/P's lie in class $\mathscr{L}$.

This family of examples contains such spaces as the Grassmannians $G(n, k)$ of k-dimensional linear subspaces of $\mathbb{C}^{n+k}$ and the classical flag varieties.

Example 5.8 (Compact hermitian symmetric spaces). These spaces are products of generalized flag varieties, and hence lie in class $\mathscr{L}$, according to Remark 5.6.

Example 5.9 (Varieties with reductive group actions). A smooth projective variety which admits an action by a reductive group G with isolated fixed points possesses an algebraic cellular decomposition in virtue of the Bialyniki-Birula decomposition [3]. More generally, a smooth projective variety having a reductive group action all of whose fixed components lie in class $\mathscr{L}$ must also be in class $\mathscr{L}$. The argument again follows using induction on the Bialyniki-birula stratification together with Corollary 4.7 and Remark 4.8.

We end this paper with some general questions and comments which we hope to address in future work:

(1) Which other characterizations have those varieties in class $\mathscr{L}$? Or, more generally, those for which the cycle map is an isomorphism through a certain range?

(2) What information about Hodge structures and filtrations on the homology (under the image of the cycle map) of a smooth variety can one retrieve, in the spirit of [14], using the current theoretic approach?

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