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ON TORI EMBEDDED IN FOUR-MANIFOLDS

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1. Introduction

The genus of a smooth curve C inside a complex surface S is related to the self-intersection of C via the adjunction formula:

$$C.C + k_S.C = 2\,\mathrm{genus}(C) - 2,$$

where k_S is the canonical class of S. If S is a minimal irrational surface then $k_S.C \geq 0$. Therefore smooth complex curves inside minimal irrational surfaces satisfy

$$(*) \qquad C.C \leq 2\,\mathrm{genus}(C) - 2.$$

Moreover, there is a long-standing conjecture (originally stated by René Thom for the projective plane) which says that if $F \hookrightarrow S$ is a smoothly embedded Riemann surface homologous to C, then $\mathrm{genus}(F) \geq \mathrm{genus}(C)$. So it is natural to conjecture that $(*)$ is satisfied by smoothly embedded Riemann surfaces $F \hookrightarrow S$. When S is a Dolgachev surface and F is a 2-sphere, this has been verified by Friedman and Morgan [6], [7] using the Γ-invariant introduced by Donaldson in [4]. Also, Morgan, Mrówka and Ruberman [9] proved that if M is a closed, oriented, simply connected, smooth 4-manifold whose intersection form has positive part $b_2^+ > 1$ odd and M has some nonzero Donaldson invariant, then the following hold:

(1) if $S^2 \hookrightarrow M$ is a smoothly embedded 2-sphere representing a nontrivial homology class in M, then $S^2.S^2 < 0$ ([8]),

(2) if $T^2 \hookrightarrow M$ is a smoothly embedded 2-torus representing a nontrivial homology class, then $T^2.T^2 < 2$ ([10]).

By a result of Donaldson every smooth simply connected complex projective surface has nonvanishing Donaldson polynomial invariants; hence (1) and (2) give slightly weaker inequalities than $(*)$ for smooth projective surfaces.

To prove (2) the idea is to pull apart the 4-manifold along the boundary Y of a tubular neighborhood of an embedded sphere (or torus) violating

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the above inequality. This yields a description of the moduli space, and hence of the invariants, in terms of the moduli spaces of the two pieces and of the space of representations of $\pi_1(Y)$ into SU(2). The absence of irreducible representations then implies that the Donaldson invariants vanish. The argument breaks down if the torus has square $+1$ because there are irreducible representations of $\pi_1(Y)$ into SU(2). Although it does not seem possible in this case to prove, at least by these methods, that the invariants γ_k vanish, we show that they must vanish on the orthogonal complement of the class represented by T. This is the main technical result of the paper:

Theorem 7.1. *Let M be a smooth simply connected closed 4-manifold with $b_2^+(M) > 1$, odd and k an integer in the stable range. Suppose $T \hookrightarrow M$ is a smoothly embedded 2-torus with self-intersection $+1$. Then for all d-tuples $\alpha_1, \cdots, \alpha_d \in H_2(M, \mathbf{Z})$ satisfying $\alpha_i.[T] = 0$ for all i, $\gamma_k(\alpha_1, \cdots, \alpha_d) = 0$.*

The same sort of restriction applies to the generalized Donaldson polynomials of Friedman and Morgan (Theorem 7.3). Using these results we are able to improve (2) for certain complex surfaces:

Theorem 8.7. *Let $\widehat{S}$ be a smooth simply connected complex surface with geometric genus greater than zero whose minimal model is either elliptic or a complete intersection. Then there is no smoothly embedded 2-torus $T \hookrightarrow \widehat{S}$ with self-intersection $+1$.*

It follows that given a smoothly embedded 2-torus $T \hookrightarrow \widehat{S}$ then $T.T \leq 0$; i.e., T satisfies $(*)$. Thus, in view of the conjecture, Theorem 8.7 represents the best possible improvement of (2) for these surfaces.

In §2 we fix the notation and recall the results needed later. In §3 we consider an S^1-bundle Y over a 2-torus with Euler number $+1$, describe the variety of representations of $\pi_1(Y)$ into SU(2) modulo conjugation, and compute certain cohomology groups. §§4 and 5 contain a computation of the ρ-invariants and Chern-Simons invariants of the representations of $\pi_1(Y)$ into SU(2). In §6 the results of §§3, 4 and 5 are used to describe a stratification for the L^2-moduli space $\mathscr{M}(X)$, where X is the complement, inside a 4-manifold, of a closed tubular neighborhood of a 2-torus T with self-intersection $+1$. In §7 we use this stratification to obtain the main results on Donaldson polynomials. §8 gives the applications to algebraic surfaces.

It seems possible that the same techniques used in this paper could give useful information about smooth tori of square 0 inside surfaces with geometric genus greater than zero, and maybe about smooth tori with square $+1$ inside surfaces with geometric genus equal to zero.

2. Notation and background material

Definition 2.1. We say that a smooth oriented Riemannian manifold X has one cylindrical end if

(1) there exists a smoothly embedded compact connected submanifold (with boundary) $K \hookrightarrow X$,

(2) there exist a smooth closed oriented Riemannian manifold Y and an orientation-preserving diffeomorphism $\psi: [0, +\infty) \times Y \to X \setminus \mathrm{int}(K)$,

(3) the Riemannian metric g on X converges exponentially fast to the product metric on $X \setminus \mathrm{int}(K)$.

A metric as in (3) is said to be *asymptotically cylindrical*. Let P be a (necessarily trivial) principal SU(2)-bundle over a smooth oriented Riemannian 4-manifold X with one cylindrical end. The *L^2-moduli space* $\mathscr{M}(X)$ is the quotient under the gauge group of P of the space of anti-self-dual SU(2)-connections whose curvatures have finite L^2-norm. Let $\mathscr{R}(Y) = \mathrm{Hom}(\pi_1(Y), \mathrm{SU}(2))$ be the variety of representations of $\pi_1(Y)$ into SU(2), and $\chi(Y) = \mathscr{R}(Y)/\mathrm{SU}(2)$, where SU(2) acts by conjugation. Upon the choice of a suitable topology on $\mathscr{M}(X)$, there is a continuous map

$$\partial_X : \mathscr{M}(X) \to \chi(Y),$$

which sends every connection on P to the limit value, down the end of X, of its restriction to the Y-direction. There is also a based version of the above construction, i.e., a continuous SU(2)-equivariant map from the based moduli space

$$\partial_X^0 : \mathscr{M}^0(X) \to \mathscr{R}(Y)$$

which descends to ∂_X (see [12] and [10] for details). Recall that $R(Y)$ is the space of real points of an affine algebraic variety, having a natural SO(3)-equivariant stratification. The following theorem shows how, at least in some cases, the stratification on $R(Y)$ induces a stratification on the moduli spaces.

Theorem 2.2 [12], [10]. *Let X be a smooth oriented Riemannian manifold with one cylindrical end modelled on $[0, +\infty) \times Y$, where Y is a principal $U(1)$-bundle over a smooth 2-manifold with negative Euler number. Then, for generic asymptotically cylindrical metrics on X, there exists a smooth* SO(3)*-equivariant stratification on $\mathscr{M}^0(X)$ with respect to the filtration induced by ∂_X^0 from the natural one on $\mathscr{R}(Y)$.*

Let M be a smooth, closed, oriented 4-manifold, and $Y \hookrightarrow M$ a smoothly embedded separating 3-manifold. Let $M \setminus Y = X_1 \amalg X_2$. Choose on X_1 and X_2 structures of smooth Riemannian 4-manifolds with cylindrical ends diffeomorphic respectively to $[0, +\infty)\times\partial X_1$ and $[0, +\infty) \times \partial X_2$. Identify $X_i \setminus \mathrm{int}(K_i)$ with $[0, +\infty) \times \partial X_i$, and let $t_i\colon X_i \to \mathbf{R}$, $i = 1, 2$, be smooth functions extending the projections onto the first factors on $[0, +\infty) \times \partial X_i$. For all $s > 0$ let $X_i^s = t_i^{-1}((\infty, s])$. It is possible to choose on $M_s = X_1^{s+1} \amalg X_2^{s+1} / t_1^{-1}[s, s+1] = t_2^{-1}[s, s+1]$ a smooth structure such that M_s is diffeomorphic to M. Then, if g_s is a C^∞ metric defined on M_s by interpolating g_1 and g_2, there is the following result:

Lemma 2.3 [10]. *Let $\{P_n \to M_{s_n}\}$ be a sequence of principal* SU(2)-*bundles with Chern class k, with $\{s_n\} \to \infty$, and for each n let A_n be an anti-self-dual connection on E_n. Then, up to passing to a subsequence, there are*

(1) *an* SU(2)-*bundle $P \to X_1 \amalg X_2$;*

(2) *finite-energy anti-self-dual connections A_i on P restricted to X_i;*

(3) *points $x_1, \cdots, x_r \in X_1 \amalg X_2$ and positive integers $n_1, \cdots, n_r$ such that*

(i) *for any fixed $s > 0$, A_n restricted to $X_i^s \setminus \{x_j\}$ converges to A_i in the C^∞ topology up to a gauge transformation;*

(ii) *at each point x_i the curvature concentrates with energy $8\pi^2 n_i$. Moreover, the following hold:*

(4) $c_2(A_1) + c_2(A_2) + \sum n_i \leq k$;

(5) *if equality holds in* (4), *then $\partial A_1 = \partial A_2$.*

Let X be a smooth oriented Riemannian 4-manifold with one cylindrical end diffeomorphic to $[0, +\infty) \times Y$ for some smooth 3-manifold Y, and let $t\colon X \to \mathbf{R}$ be a smooth function extending the projection onto the first factor over the end of X. The *L^2_δ-moduli space* $\mathscr{M}_\delta(X) \subset \mathscr{M}(X)$ is the space of $[A] \in \mathscr{M}(X)$ such that the L^2_δ-energy of $[A]$, i.e., the integral over X of $e^{-t\delta}|F_A|^2$, is finite. There is, of course, a based version $\mathscr{M}^0_\delta(X) \subset \mathscr{M}^0(X)$. Given $[A] \in \mathscr{M}(X)$, let $c_2([A]) = (1/8\pi^2)\int_X |F_A|^2$. For $r \in [0, +\infty)$ and given a subset $V \subset \chi(Y)$ we use the notation $\mathscr{M}_r(X) = c_2^{-1}(r)$, $\mathscr{M}(X; V) = \partial_X^{-1}(V)$ and the corresponding based ones; the same notation will be used for the analogous subsets of the L^2_δ-moduli spaces.

Let su(2) be the Lie algebra of SU(2). Given a representation $\alpha \in \mathscr{R}(Y)$, let $\mathrm{su}(2)_{\mathrm{ad}(\alpha)}$ be the flat su(2)-bundle associated with $\mathrm{ad}(\alpha)\colon \pi_1(Y) \to \mathrm{Aut}(\mathrm{su}(2)) \cong \mathrm{SO}(3)$, and denote by ρ_α the Atiyah-Patodi-Singer ρ-invariant for the signature complex of Y twisted by $\mathrm{su}(2)_{\mathrm{ad}(\alpha)} \otimes \mathbf{C}$. Let $(\Omega^*_\alpha, d_\alpha)$ be the de Rham complex of Y twisted by $\mathrm{su}(2)_{\mathrm{ad}(\alpha)}$, $H^*_\alpha =$

$H^*(\Omega^*_\alpha)$ and let $\chi(X)$ be the Euler characteristic of X.

Theorem 2.4 [10]. *Let $A \in \mathscr{M}^0_\delta(X, \alpha)$. If $0 < \delta < 2\mu$, where μ is the first positive eigenvalue of the operator $-*d_\alpha : \Omega^1_\alpha \to \Omega^1_\alpha$, then for generic asymptotically cylindrical metrics on X the moduli space $\mathscr{M}^0_\delta(X, \alpha)$ is a smooth manifold of dimension*

$$d(A) = 8c_2(A) - \frac{3}{2}(\chi(X) + \operatorname{sign}(X)) - \frac{1}{2}(\dim H^1_\alpha - \dim H^0_\alpha) + \frac{\rho_\alpha}{2}.$$

Theorem 2.5 [10]. *Let $V \subset \chi(Y)$ be an open subset such that the preimage $V^0 \subset \mathscr{R}(Y)$ consists of the smooth points of $\mathscr{R}(Y)$. Suppose that there is a $\delta > 0$ with $\delta/2$ not in the spectrum of any $-*d_\alpha$, for $\alpha \in V$, and fix $k > 0$. Then, if the intersection form of X is not negative definite, for generic asymptotically cylindrical metrics on X the moduli space $\mathscr{M}^0_{\delta,k}(X; V^0)$ is a smooth manifold. Moreover, if $A \in \mathscr{M}^0_{\delta,k}(X; V^0)$, then the dimension of the component of $\mathscr{M}^0_{\delta,k}(X; V^0)$ containing A is $d(A) + \dim(V^0)$.*

Let M be a smooth closed oriented 4-manifold decomposed as a union $X_1 \bigcup_\Phi X_2$, where X_1 and X_2 are oriented compact 4-manifolds with boundary and $\Phi: \partial X_1 \to \partial X_2$ is an orientation-reversing diffeomorphism. Let A_1 and A_2 be two anti-self-dual connections of finite L^2_δ-energy on the interiors respectively of X_1 and X_2; in [11] is analyzed the problem of "gluing" together A_1 and A_2 to construct an anti-self-dual connection on M. Let $V \subseteq \chi(Y)$ be an open set such that the closure of the preimage $V^0 \subset \mathscr{R}(Y)$ is contained in the set of smooth points of $\mathscr{R}(Y)$, and put structures of manifolds with cylindrical end on X_i for $i = 1, 2$. Let $\varepsilon > 0$ and fix half-infinite tubes $T_{X_i} \cong [T, +\infty) \times Y$ inside X_i. Let $\mathscr{M}_\delta(X_i, T_{X_i})$ be the subspace of $\mathscr{M}_\delta(X_i)$ of all connections which are "ε-close" (i.e., within ε with respect to a suitable norm) to a flat connection in V on T_{X_i}. Let $\delta > 0$ be smaller than twice the smallest positive eigenvalue of $-*d_\alpha$ for all $\alpha \in V^0$.

Theorem 2.6 [11]. *Let k_1 and k_2 be positive integers. For generic choices of asymptotically cylindrical metrics on X_1 and X_2 the following is true: given $T > 0$ sufficiently large and $\varepsilon > 0$ sufficiently small, for all s large enough there is a map*

$$\gamma^0_s = \gamma^0_s(k_1, k_2): \mathscr{M}^0_{\delta,k_1}(X_1, T_{X_1}) \times_{V^0} \mathscr{M}^0_{\delta,k_2}(X_2, T_{X_2}) \to \mathscr{M}^0_{k_1+k_2}(M_s)$$

which is an SO(3)*-equivariant diffeomorphism onto an open set. Furthermore, given $\mu > 0$, for s sufficiently large and $\varepsilon > 0$ sufficiently small, $g_s(A_1, A_2)$ differs by less than μ in the L^2_2-topology from A_i on X^s_i.*

When the energy on one of the moduli spaces is zero, there is another version of Theorem 2.6. Suppose for instance $k_1 = 0$. Then $\mathscr{M}^0_{\delta,0}(X_1)$ can be identified with the representation variety $\mathscr{R}(X_1)$. Let $V \subset \chi(Y)$ be a subset as above, and $\mathscr{R}(X_1, T_{X_1}) \subset \mathscr{R}(X_1)$ be the subspace of representations ε-close to a flat connection in V^0 on T_{X_1}. An SO(3)-equivariant bundle (the Taubes obstruction bundle) $\Xi^0 \to \mathscr{R}(X_1)$ with fiber $H^2_{\alpha,+}(X_1)$ over $\alpha \in \mathscr{R}(X_1)$ enters in the picture as described by the following theorem. Let $\delta > 0$ as in Theorem 2.6.

Theorem 2.7 [11]. *Let k be a positive integer. For generic choices of asymptotically cylindrical metrics on X_1 and X_2 the following is true: given $T > 0$ sufficiently large and $\varepsilon > 0$ sufficiently small, for all s large enough there is an* SO(3)*-equivariant smooth map*

$$\gamma^0_s : \mathscr{R}(X_1, T_{X_1}) \times_{V^0} \mathscr{M}^0_{\delta,k}(X_2, T_{X_2}) \to \mathscr{B}^0_k(M_s)$$

and an SO(3)*-equivariant section ξ^0 of the pullback to $\mathscr{R}(X_1, T_{X_1}) \times_{V^0} \mathscr{M}^0_{\delta,k}(X_2, T_{X_2})$ of Ξ^0 such that $\gamma^0_s((\xi^0)^{-1}(0)) \subset \mathscr{M}^0_k(M_s)$, and*

$$\gamma^0_s|_{(\xi^0)^{-1}(0)} : (\xi^0)^{-1}(0) \to \mathscr{M}^0_k(M_s)$$

is an SO(3)*-equivariant diffeomorphism onto an open set. Moreover, given $\mu > 0$, for all s sufficiently large and ε sufficiently small $\gamma^0_s|_{(\xi^0)^{-1}(0)}(A^0_1, A^0_2)$ differs less than μ in the L^2_2-topology from A^0_i on X^s_i.*

3. The character variety

Given a smooth 3-manifold Y, let $\mathscr{R}(Y) = \mathrm{Hom}(\pi_1(Y), \mathrm{SU}(2))$ be the space of representations of $\pi_1(Y)$ into SU(2). The *character variety* $\chi(Y)$ is the quotient of $\mathscr{R}(Y)$ under the action of SU(2) by conjugation. We assume for the rest of the section that Y is a principal S^1-bundle over a 2-torus with Euler number $+1$.

Proposition 3.1. *Let $p: Y \to T$ be a principal S^1-bundle over a 2-torus with Euler number $+1$. Then the fundamental group of Y has the presentation*

$$\langle a, b, c \mid c \text{ is central, } [a, b] = c \rangle .$$

Proof. The exact sequence of the fibration $S^1 \to Y \to T$ yields

$$\{1\} \to \pi_1(S^1) \xrightarrow{i_*} \pi_1(Y) \xrightarrow{p_*} \pi_1(T) \to \{1\}.$$

Let $c = i_*(g)$, with g a generator of $\pi_1(S^1)$, and let $a, b \in \pi_1(Y)$ be lifts of generators of $\pi_1(T)$. Monodromy around $p_*(a)$ and $p_*(b)$ gives the relations

(1) $a^{-1}ca = c$,

(2) $b^{-1}cb = c$.

Consider a section $s: T\backslash\{x\} \to Y$ such that, if Δ_x is a little disk around x and $\pi: p^{-1}(\Delta_x) \to S^1$ denotes projection onto the fiber with respect to some trivialization, then $\pi \circ s|_{\partial\Delta_x} : \partial\Delta_x \to S^1$ has degree $+1$. s defines a homotopy between $aba^{-1}b^{-1}$ and c; hence we have

(3) $[a, b] = c^n$.

It is easy to check that (1), (2) and (3) are the only relations that can occur. q.e.d.

Proposition 3.2. *Let Y be a principal S^1-bundle over a 2-torus with Euler number $+1$. Let $\mathscr{R}(Y) = \mathrm{Hom}(\pi_1(Y); \mathrm{SU}(2))$ be the variety of representations of $\pi_1(Y)$ into $\mathrm{SU}(2)$. Then $\mathscr{R}(Y)$ has two connected components: The irreducible representations $\mathscr{R}(Y)^{\mathrm{irr}}$ and the reducible representations $\mathscr{R}(Y)^{\mathrm{red}}$. The first component of $\chi(Y) = \mathscr{R}(Y)^{\mathrm{irr}}/\mathrm{SU}(2) \amalg \mathscr{R}(Y)^{\mathrm{red}}/\mathrm{SU}(2)$ is homeomorphic to a single point, and the second one to $S^1 \times S^1/(z_1, z_2) \sim (\bar{z}_1, \bar{z}_2)$, i.e., to a 2-sphere with four singular points.*

Proof. Fix a presentation for $\pi_1(Y)$ as in Proposition 3.1. Then, since $c = [a, b]$, any reducible representations $\alpha \in \mathscr{R}(Y)$ must send the central element c to the identity, because $\mathrm{Im}(\alpha)$ is abelian. On the other hand suppose that α is irreducible. Since c is in the center of $\pi_1(Y)$, $\alpha(c)$ must be in the centralizer of $\mathrm{Im}(\alpha)$, which is $\{\pm 1\}$. But $\mathrm{Im}(\alpha)$ is not abelian, so we must have $\alpha(c) = [\alpha(a), \alpha(b)] \neq 1$. Hence $\alpha(c) = -1$. This proves that $\mathscr{R}(Y)^{\mathrm{irr}}$ and $\mathscr{R}(Y)^{\mathrm{red}}$ are both the union of connected components. It is an exercise to see that, if α is irreducible, the only possible values, up to conjugation, for $\alpha(a)$ and $\alpha(b)$ are, respectively, i and j. Therefore there is a unique orbit of irreducible representations, so $\mathscr{R}(Y)^{\mathrm{irr}}$ is connected, and when we pass to the quotient it collapses to a point. Up to conjugation any reducible representation has image contained in the standard circle subgroup $S^1 \subset \mathrm{SU}(2)$. Moreover, for each $z \in S^1$, $\mathrm{orbit}(x) \cap S^1 = \{x, \bar{x}\}$. Therefore a presentation of $\pi_1(Y)$ as in Proposition 3.1 gives a homeomorphism of $\chi(Y)^{\mathrm{red}}$ with $S^1 \times S^1/(z_1, z_2) \sim (\bar{z}_1, \bar{z}_2)$, which is a 2-sphere with four singular points corresponding to the points $\{(\pm 1, \pm 1)\}$ on $S^1 \times S^1$ fixed under conjugation. Since $S^1 \times S^1/(z_1, z_2) \sim (\bar{z}_1, \bar{z}_2)$ is connected, $\mathscr{R}(Y)^{\mathrm{red}}$ must be connected as well. q.e.d.

Let $\mathrm{su}(2)_{\mathrm{ad}(\alpha)}$ be the flat su(2)-valued bundle associated to $\mathrm{ad}(\alpha)\colon \pi_1(Y) \to \mathrm{su}(2)$, ($\mathrm{su}(2)_{\mathrm{ad}(\alpha)}$ depends only on the conjugacy class of α), and let $(\Omega^*_\alpha, d_\alpha)$ be the de Rham complex of Y twisted by $\mathrm{su}(2)_{\mathrm{ad}(\alpha)}$.

Proposition 3.3. *Let* $\alpha \in \mathscr{R}(Y)$ *and* $H^0_\alpha = H^0(\Omega^*_\alpha)$. *Then*

$$\dim H^0_\alpha = \begin{cases} 3 & \textit{if } \mathrm{ad}(\alpha) \textit{ is trivial}, \\ 1 & \textit{if } \alpha \textit{ is reducible and } \mathrm{ad}(\alpha) \textit{ is nontrivial}, \\ 0 & \textit{if } \alpha \textit{ is irreducible.} \end{cases}$$

Proof. Y is a $K(\pi_1(Y), 1)$; thus $H^*(Y; \mathrm{su}(2)_{\mathrm{ad}(\alpha)})$ can be identified with the cohomology of $\pi_1(Y)$ with coefficients in the Lie algebra su(2) considered as a $\pi_1(Y)$-module via $\mathrm{ad}(\alpha)$. By definition

$$H^0(\pi_1(Y); \mathrm{su}(2)_{\mathrm{ad}(\alpha)}\} = \{v \in \mathrm{su}(2) \mid \mathrm{ad}(\alpha)(g)v = v \text{ for all } g \in \pi_1(Y)\};$$

hence there is also an identification of $H^0(\pi_1(Y); \mathrm{su}(2)_{\mathrm{ad}(\alpha)})$ with the Lie algebra of the centralizer of $\mathrm{Im}(\alpha)$ in $SU(2)$. Therefore $\dim H^0(Y; \mathrm{su}(2)_{\mathrm{ad}(\alpha)})$ equals the rank of the centralizer of $\mathrm{Im}(\alpha)$. So if $\mathrm{ad}(\alpha)$ is trivial, i.e., $\mathrm{Im}(\alpha) \subseteq \{\pm 1\}$, $\dim H^0_\alpha = 3$; if α is reducible and $\mathrm{ad}(\alpha)$ is nontrivial so that $\mathrm{Im}(\alpha)$ is not contained in $\{\pm 1\}$ but is contained in a circle subgroup of $SU(2)$, $\dim H^0_\alpha = 1$; if α is irreducible, $\dim H^0_\alpha = 0$.

Proposition 3.4. *Let* $\alpha \in \mathscr{R}(Y)$ *and let* $H^1_\alpha = H^1(\Omega^*_\alpha)$. *Then*

$$\dim H^1_\alpha = \begin{cases} 6 & \textit{if } \mathrm{ad}(\alpha) \textit{ is trivial}, \\ 2 & \textit{if } \alpha \textit{ is reducible and } \mathrm{ad}(\alpha) \textit{ is nontrivial}, \\ 0 & \textit{if } \alpha \textit{ is irreducible.} \end{cases}$$

Proof. If $\mathrm{ad}(\alpha)$ is trivial, it is clear that $\dim H^1_\alpha = 3 \dim H^1(Y; \mathbf{R}) = 6$. In the other cases, as in the proof of Proposition 3.3, we compute the dimension of $H^1(\pi_1(Y); \mathrm{su}(2)_{\mathrm{ad}(\alpha)})$. The space of of su(2)-valued 1-cocycles is

$$Z^1_\alpha = \{z : \pi_1(Y) \to \mathrm{su}(2) \mid z(xy) = z(x) + \mathrm{ad}(\alpha(x))z(y)\}.$$

Fix as presentation for $\pi_1(Y)$ as in Proposition 3.1.

First case: α *is reducible,* $\mathrm{ad}(\alpha)$ *nontrivial.* Since $\mathrm{Im}(\alpha)$ is abelian, $\alpha(c) = \alpha([a, b]) = [\alpha(a), \alpha(b)] = 1$. Therefore

(1) $ac = ca$ implies $z(a) + \mathrm{ad}(\alpha(a))z(c) = z(c) + z(a)$, i.e.,

$$\mathrm{ad}(\alpha(a))z(c) = z(c).$$

(2) $bc = cb$ implies

$$\mathrm{ad}(\alpha(b))z(c) = z(c).$$

(3) $ab = cba$ implies

$$z(a) + \mathrm{ad}(\alpha(a))z(b) = z(c) + (z(b) + \mathrm{ad}(\alpha(b))z(a)).$$

Let $S^1 \subset \mathrm{SU}(2)$ be a circle subgroup containing $\mathrm{Im}(\alpha)$. The first two relations imply that $z(c)$ belongs to the line $l = \mathrm{Lie}(S^1) \subseteq \mathrm{su}(2)$. Therefore by the third relation Z^1_α can be identified with the kernel of the map

$$\mathrm{su}(2) \times \mathrm{su}(2) \times l \to \mathrm{su}(2),$$
$$(v, w, z) \mapsto v + \mathrm{ad}(\alpha(a))w - w - \mathrm{ad}(\alpha(b))v - z,$$

which is surjective for $(\alpha(a), \alpha(b)) \notin \{(\pm 1, \pm 1)\}$. Hence $\dim Z^1_\alpha = 4$. The 1-coboundaries are given by

$$B^1_\alpha = \{z : \pi_1(Y) \to \mathrm{su}(2) | z(x) = \mathrm{ad}(\alpha(x))v - v, \text{ for some } v \in \mathrm{su}(2)\}.$$

Since $\ker(\mathrm{ad}(\alpha(x)) - \mathrm{id}) = l$ for all $z \in \pi_1(Y)$, $\dim B^1_\alpha = \dim(l^\perp) = 2$; hence $\dim H^1_\alpha = \dim Z^1_\alpha / B^1_\alpha = 2$.

Second case: α *irreducible*. Since H^1_α does not depend on the conjugacy class of α, we may assume, without loss of generality, $\alpha(a) = i$, $\alpha(b) = j$. Since $\alpha(c) = -1$, the relations $ac = ca$ and $bc = cb$ imply $\mathrm{ad}(i)(z(c)) = z(c)$ and $\mathrm{ad}(j)(z(c)) = z(c)$ for each 1-cocycle z. Hence we must have $z(c) = 0$. The relation $ab = cba$ gives $z(a) + \mathrm{ad}(i)z(b) = z(b) + \mathrm{ad}(j)z(a)$. Thus Z^1_α can be identified with the kernel of the linear map

$$\mathrm{su}(2) \times \mathrm{su}(2) \to \mathrm{su}(2),$$
$$(v, w) \mapsto \mathrm{ad}(i)v + w - \mathrm{ad}(j)w - v.$$

Since this map is surjective, $\dim Z^1_\alpha = 3$. B^1_α is 3-dimensional as well, because $\mathrm{ad}(\alpha)$ is irreducible; hence $\dim H^1_\alpha = \dim Z^1_\alpha / B^1_\alpha = 0$.

4. Computation of the ρ-invariants

Throughout the section Y will be a principal S^1-bundle over a 2-torus with Euler number $+1$. Let $\phi\colon \pi_1(Y) \to \mathrm{U}(n)$ be a representation which factors through a finite subgroup $G \subset \mathrm{U}(n)$, and $\widetilde{Y} \to Y$ a finite G-covering. Let $\rho(\alpha)$ be the Atiyah-Patodi-Singer ρ-invariant of the signature complex of Y twisted by the flat $\mathrm{U}(n)$-vector bundle associated with

ϕ. In [1] it is proved that

$$\rho(\phi) = \frac{1}{|G|} \sum_{g \in G \setminus \{1\}} \sigma_g(\widetilde{Y})(\mathrm{tr}(\phi(g)) - n), \tag{4.1}$$

where $\sigma_g(Y)$ is defined in [3], and is given by

$$\sigma_g(Y) = L(g, X) - \mathrm{sign}(g, X),$$

under the assumption that g acts on the oriented smooth 4-manifold X with no fixed points on $\partial X = Y$; $L(g, X)$ denotes the number obtained via the cohomological formula appearing in the G-signature theorem and involving the fixed points of g, while $\mathrm{sign}(g, X)$ is the g-signature.

Given $\alpha \in \mathscr{R}(Y)$ we may consider the representation $\phi_\alpha\colon \pi_1(Y) \to U(3)$ given by the composition $\pi_1(Y) \overset{\alpha}{\to} \mathrm{SU}(2) \overset{\mathrm{ad}}{\to} \mathrm{SO}(3) \subset U(3)$. It is clear that $\rho_\alpha = \rho(\phi_\alpha)$, where ρ_α is defined in §2.

Proposition 4.1. *Let Y be a principal S^1-bundle over a 2-torus with Euler number $+1$, and let $\alpha \in \mathscr{R}(Y)$. Then*

$$\rho_\alpha = \begin{cases} 0 & \text{if } \mathrm{ad}(\alpha) \text{ is trivial}, \\ 2 & \text{if } \alpha \text{ is reducible and } \mathrm{ad}(\alpha) \text{ is nontrivial}, \\ 3 & \text{if } \alpha \text{ is irreducible}. \end{cases}$$

Proof. $\mathscr{R}(Y)$ has a natural stratification induced by the function $\alpha \mapsto \dim(H^0_\alpha)$, and it is well known that the ρ-invariant is constant on components of its strata. If $\mathrm{ad}(\alpha)$ is trivial, it is immediate from the definition that $\rho_\alpha = 0$. Fix a presentation for $\pi_1(Y)$ as in Proposition 3.1, and let α be a reducible non-ad-trivial representation. Since the ρ-invariant is locally constant, to compute $\rho(\phi_\alpha)$ we can assume $\alpha(a) = e^{2\pi i/3}$, $\alpha(b) = \alpha(c) = 1$, so $\mathrm{Im}(\mathrm{ad}(\alpha)) = \mathbf{Z}_3 \subseteq \mathrm{SU}(3)$. Let $T = S^1 \times S^1$ be the 2-torus base of Y. The map $T \to T$ given by $(x, y) \mapsto (x^3, y)$ lifts to a $\mathbf{Z}_3$-covering $Y \to Y$, so we can take $\widetilde{Y} = Y$ in formula 4.1. In order to compute $\sigma_g(Y)$ for $g \in \mathbf{Z}_3 \setminus \{0\}$, observe that the $\mathbf{Z}_3$-action on Y is induced by the inclusion $\mathbf{Z}_3 \subset S^1$ and the principal bundle action of S^1 on Y, which extends to the corresponding disk-bundle X. This shows that each $g \in \mathbf{Z}_3$ is homotopic to the identity map. Hence, since no $g \neq 0$ has fixed points, $\sigma_g(Y) = -1$ for all $g \in \mathbf{Z}_3 \setminus \{0\}$. It is straightforward to compute

$$\mathrm{tr}(\mathrm{ad}(e^{2\pi i/3})) = \mathrm{tr}(\mathrm{ad}(e^{4\pi i/3})) = 0.$$

Since $n = 3$ in (4.1), we have

$$\rho_\alpha = \rho(\varphi_\alpha) = \frac{1}{3}\sum_{i=1}^{2}(-1)(0-3) = 2.$$

Now let $\alpha \in \mathscr{R}(Y)$ be irreducible. Again we may assume $\alpha(a) = i$, $\alpha(b) = j$. Such an α is irreducible because $[i, j] = -1$ implies that $\operatorname{Im}\alpha$ is not abelian. Let $G = \operatorname{ad}(\pi_1(Y)) \subseteq SU(3)$. It is easy to compute that

$$\begin{aligned} G &= \{1, \operatorname{ad}(i), \operatorname{ad}(j), \operatorname{ad}(k)\} \\ &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \\ &\cong \mathbf{Z}_2 \times \mathbf{Z}_2. \end{aligned}$$

The G-covering of Y is the pullback $\widetilde{Y}$ of Y under the natural $\mathbf{Z}_2 \times \mathbf{Z}_2$-covering of T. $\widetilde{Y}$ is an S^1-bundle over T with Euler number $+4$. Also, $\widetilde{Y} = \partial\widetilde{X}$, where $\widetilde{X}$ is the corresponding disk-bundle, and the G-action extends to $\widetilde{X}$. Hence $L(\operatorname{ad}(i), \widetilde{X}) = L(\operatorname{ad}(j), \widetilde{X}) = L(\operatorname{ad}(k), \widetilde{X}) = 0$ because $\operatorname{ad}(i)$, $\operatorname{ad}(j)$, $\operatorname{ad}(k)$ have no fixed points. Let $\pi\colon \widetilde{X} \to T$ be the bundle projection. $H^2(\widetilde{X}, \widetilde{Y})$ has rank one, and is generated by the orientation class of $X \to T$, whose restriction to $\widetilde{X}$ is the pullback of the Euler class $e(\widetilde{X})$ of $\widetilde{X}$ itself, thus $\operatorname{Im}(H^2(\widetilde{X}, \widetilde{Y}) \to H^2(\widetilde{X})) = \langle \pi^*(e(\widetilde{X}))\rangle$. Moreover, since $\operatorname{ad}(i)$, $\operatorname{ad}(j)$ and $\operatorname{ad}(k)$ all commute with π and act as orientation-preserving homeomorphisms on T, the actions induced on $H^2(T)$ and $\pi^*(e(\widetilde{X}))$ are all equal to the identity. Therefore $\operatorname{sign}(\operatorname{ad}(i), \widetilde{X}) = \operatorname{sign}(\operatorname{ad}(j), \widetilde{X}) = \operatorname{sign}(\operatorname{ad}(k), \widetilde{X}) = 1$. Putting all this together, we have

$$\sigma_{\operatorname{ad}(i)}(\widetilde{Y}) = \sigma_{\operatorname{ad}(j)}(\widetilde{Y}) = \sigma_{\operatorname{ad}(k)}(\widetilde{Y}) = -1.$$

Clearly $\operatorname{tr}(\operatorname{ad}(i)) = \operatorname{tr}(\operatorname{ad}(j)) = \operatorname{tr}(\operatorname{ad}(k)) = -1$. Hence by formula (4.1),

$$\rho_\alpha = \frac{1}{4}[-3(-1-3)] = 3.$$

5. Chern-Simons invariants

We use the notation $\operatorname{cs}\colon \chi(Y) = \mathscr{R}(Y)/SU(2) \to \mathbf{R}/\mathbf{Z}$ for the Chern-Simons function.

Lemma 5.1. *Let Y be a principal S^1-bundle over a 2-torus with Euler number $+1$. Let τ be the product connection on $Y \times SU(2)$ considered as an $SU(2)$-bundle over Y, and let $\alpha \in \mathscr{R}(Y)$ be irreducible. Let a be a flat $SU(2)$-connection on the trivial $SU(2)$-bundle over Y with holonomy conjugated to α, and let $p\colon \widetilde{Y} \to Y$ be the covering corresponding to $\ker(\alpha) \triangleleft \pi_1(Y)$. Suppose $\varphi\colon \widetilde{Y} \to SU(2)$ is a smooth map inducing a* *gauge transformation* φ^* *on* $\widetilde{Y} \times \mathrm{SU}(2)$ *such that* $\varphi^*(\tau) = p^*(a)$. *Given the* SU(2)-*bundle* $E = \widetilde{Y} \times [0, 1] \times \mathrm{SU}(2)/(y, 0, g) \sim (y, 1, \varphi(y)g)$ *over* $\widetilde{Y} \times S^1$, *we have*

$$\mathrm{cs}([\alpha]) = \frac{1}{8} c_2(E)([\widetilde{Y} \times S^1]) \qquad \bmod \mathbf{Z},$$

where $[\alpha]$ *denotes the conjugacy class of* α.

Proof. Observe that a map φ as in the statement exists because, since $p_*(\pi_1(\widetilde{Y})) = \ker \alpha$, $p^*(a)$ has trivial holonomy, and hence it is gauge equivalent to the product connection.

Choose a smooth path of connections joining a to the trivial connection. This determines a connection $A \in \mathscr{A}(Y \times [0, 1])$ and

$$(*) \qquad \frac{1}{8\pi^2} \int_{Y \times [0,1]} \mathrm{tr}(F_A \wedge F_A) = \mathrm{cs}(a) - \mathrm{cs}\,(\tau) = \mathrm{cs}([\alpha]) \qquad \bmod \mathbf{Z}$$

by Stokes' formula. Lift A to $p^*(A)$ on the trivial SU(2)-bundle over $\widetilde{Y} \times [0, 1]$. Then $p^*(A)|_{\widetilde{Y} \times \{0\}} = p^*(a)$. Since $\varphi: \widetilde{Y} \to \mathrm{SU}(2)$ gives a gauge transformation carrying the trivial connection to $p^*(a)$, $p^*(A)$ lives on the SU(2)-bundle

$$E = \widetilde{Y} \times [0, 1] \times \mathrm{SU}(2)/(y, 0, g) \sim (y, 1, \varphi(y)g).$$

Notice that since α is irreducible, we have $|\mathrm{Im}\,\alpha| = 8$ (see the proof of Proposition 3.2 for instance). Hence p has degree 8, and therefore

$$\begin{aligned} c_2(E)([\widetilde{Y} \times S^1]) &= \frac{1}{8\pi^2} \int_{\widetilde{Y} \times [0,1]} \mathrm{tr}(F_{p^*(A)} \wedge F_{p^*(A)}) \\ &= 8 \left(\frac{1}{8\pi^2} \int_{Y \times [0,1]} \mathrm{tr}(F_A \wedge F_A) \right), \end{aligned}$$

which, together with $(*)$, gives the result.

Lemma 5.2. *Let* Z *be a smooth* 3-*manifold, and* $f: Z \to \mathrm{SU}(2)$ *a smooth map. Consider the* SU(2)-*bundle* $P = Z \times [0, 1] \times \mathrm{SU}(2)/(z, 1, g) \sim (z, 0, f(z)g) \to Z \times S^1$. *Then* $c_2(P)([Z \times S^1]) = \deg(f)$.

Proof. There is a bundle map

$$\begin{aligned} &P \to Q = \mathrm{SU}(2) \times [0, 1] \times \mathrm{SU}(2)/(h, 1, g) \sim (h, 0, hg), \\ &[y, t, g] \mapsto [f(y), t, g]. \end{aligned}$$

Therefore $c_2(P)([Z \times S^1]) = \deg(f) c_2(Q)([\mathrm{SU}(2) \times S^1])$. We are going to show that $c_2(Q)([\mathrm{SU}(2) \times S^1]) = +1$. Consider the associated quaternionic bundle

$$\begin{aligned} E &= \mathrm{SU}(2) \times [0, 1] \times \mathbf{H}/(g, 1, v) \sim (g, 0, gv) \to S^3 \times S^1 \\ &= \mathrm{SU}(2) \times [0, 1]/(g, 1) \sim (g, 0). \end{aligned}$$

The map $[g, t] \mapsto [g, t, t+(1-t)g]$ defines a section of E vanishing only at $[-1, \frac{1}{2}]$, and it is easy to show that it vanishes transversely with degree $+1$.

Lemma 5.3. *Let $f\colon S^1 \to S^1$ be a smooth map. Consider the S^1-bundle $P = S^1 \times [0, 1] \times S^1/(s_1, 1, s_2) \sim (s_1, 0, f(s_1)s_2) \to S^1 \times S^1 = S^1 \times [0, 1]/(s, 1) \sim (s, 0)$. Then $c_1(P)([S^1 \times S^1]) = \deg(f)$.*

Proof. It is an almost word-for-word repetition of the proof of Lemma 5.2.

Corollary 5.4. *Let $T = S^1 \times S^1$, and $P = T \times [0, 1]/(t, 1) \sim (b(t), 0)$, where $b\colon T \to T$ is the homeomorphism specified by the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Then P is a principal S^1-bundle over the 2-torus with Euler number $+1$ with respect to the projection $p\colon P \to S^1 \times S^1 = S^1 \times [0, 1]/(s, 1) \sim (s, 0)$ given by $p([s_1, s_2, t]) = [s_2, t]$.*

Proof. $S^1 \times S^1 \times [0, 1]/(s_1, s_2, 1) \sim (s_1 s_2, s_2, 0) = S^1 \times [0, 1] \times S^1/(s_2, 1, s_1) \sim (s_1, 0, s_2)$. Notice that since we switched two factors the equality above holds for orientations as well. Now apply Lemma 5.3.

Proposition 5.5. *Let Y be a principal S^1-bundle over a 2-torus with Euler number $+1$, $\alpha\colon \pi_1(Y) \to \mathrm{SU}(2)$ a representation and $[\alpha]$ its conjugacy class. Then*

$$\mathrm{cs}([\alpha]) = \begin{cases} 0 \bmod \mathbf{Z} & \text{if } \alpha \text{ is reducible}, \\ \frac{1}{4} \bmod \mathbf{Z} & \text{if } \alpha \text{ is irreducible}. \end{cases}$$

Proof. If α is reducible, there is a smooth path of connections $\{a_t\}_{t\in[0,1]} \subseteq \mathscr{A}(Y)$ such that $\{[a_t]\}$ joins $[\alpha]$ to the class of the product connection and it lies inside $\chi(Y)$. This determines a connection $A \in \mathscr{A}(Y \times [0, 1])$, and it is easy to check that $F_A \wedge F_A = 0$. Therefore

$$\mathrm{cs}[\alpha]) = \frac{1}{8\pi^2}\int_{Y\times[0,1]} \mathrm{tr}(F_A \wedge F_A) = 0 \qquad \bmod \mathbf{Z}.$$

Suppose now that α is irreducible. Lemmas 5.1 and 5.2 give $c_2(\alpha) \equiv \frac{1}{8}\deg(\varphi) \bmod \mathbf{Z}$, where $\varphi\colon \widetilde{Y} \to \mathrm{SU}(2)$ is a map as in the statement of Lemma 5.1. Now we describe an explicit map as above and we compute its degree. This is essentially the map constructed in [9] (see in particular equation $(*)$, page 365). Let $a \in \mathscr{A}(Y)$ be flat, with $[a] = [\alpha]$. Since a is flat, it determines a smooth foliation of $Y \times \mathrm{SU}(2)$. Given $\tilde{y}_0 \in \widetilde{Y}$, there is a unique lifting of $p\colon \widetilde{Y} \to Y$ to a covering-space isomorphism $\tilde{p}\colon \widetilde{Y} \to \mathscr{F} \subset Y \times \mathrm{SU}(2)$, where $\mathscr{F}$ is the leaf through $(y_0 = p(\tilde{y}_0), 1)$. Let $\varphi = \pi_2|_{\mathscr{F}} \circ \tilde{p}\colon \widetilde{Y} \to \mathrm{SU}(2)$, where π_2 denotes projection onto the second factor. φ satisfies and is uniquely determined by the two properties:

(i) $\varphi(\tilde{y}_0) = 1$;
(ii) $\varphi(x.\tilde{y}) = \varphi(\tilde{y}).\alpha(x)^{-1}$.

In fact, given a map $\varphi\colon \tilde{Y} \to \mathrm{SU}(2)$ satisfying (i) and (ii), φ determines a foliation $\{(y, \varphi(\tilde{y}))|p(\tilde{y}) = y\}_{y\in Y} \subseteq Y \times \mathrm{SU}(2)$ which coincides with the foliation associated to α, and clearly $\varphi = \pi_2|_{\mathscr{F}} \circ \tilde{p}$ if $\mathscr{F}$ is the leaf through $(y_0, 1)$. Observe that $\mathrm{graph}(\varphi) \subseteq \tilde{Y} \times \mathrm{SU}(2)$ is the pullback under $p\times\mathrm{id}\colon \tilde{Y}\times\mathrm{SU}(2) \to Y\times\mathrm{SU}(2)$ of the leaf $\mathscr{F}$, which shows that, if τ is the product connection, the foliation associated to $\varphi^*(\tau)$ is the pullback under $p\times\mathrm{id}$ of the foliation associated to a, and therefore it is the foliation associated to $p^*(a)$. Hence $\varphi^*(\tau) = p^*(a)$. Now we construct a map satisfying (i) and (ii). Fix a presentation of $\pi_1(Y)$ as in Proposition 3.1. By Corollary 5.4 we can write $Y = T \times [0, 1]/(t, 1) \sim (b(t), 0)$. This shows that the generators a, b and c of $\pi_1(Y)$ act on the universal covering $\mathbf{R}^3$ by

$$\begin{aligned} &a\colon (x, y, t) \mapsto (x, y+1, t),\\ &b\colon (x, y, t) \mapsto (b(x, y), t) = (x+y, y, t+1),\\ &c\colon (x, y, t) \mapsto (x+1, y, t). \end{aligned}$$

We consider the map $\Phi\colon \mathbf{R}^2 \times [0, 1] \to \mathrm{SU}(2)$ given by

$$\Phi(x, y, t) = \cos\left(\frac{\pi t}{2}\right) e^{-\pi i(x+y/2)} + \sin\left(\frac{\pi t}{2}\right) e^{-\pi i(x-y/2)}(-j).$$

It is easy to check that

$$\begin{aligned} &\Phi(a(x, y, t)) = \Phi(x, y, t)\alpha(a)^{-1},\\ &\Phi(c(x, y, t)) = \Phi(x, y, t)\alpha(c)^{-1},\\ &\Phi(b(x, y), 0) = \Phi(x, y, 0)\alpha(b)^{-1} = \Phi(b(x, y), 1). \end{aligned}$$

Therefore Φ can be extended by requiring equivariance to $\Phi\colon \mathbf{R}^3 \to \mathrm{SU}(2)$. It follows that Φ induces a map $\varphi\colon \tilde{Y} = \mathbf{R}^3/\ker\alpha \to \mathrm{SU}(2)$ satisfying (i) and (ii) above, with $\tilde{y}_0 = [0, 0, 0]$. Thus φ gives our gauge transformation. Let $G = \mathrm{Im}(\alpha) \subseteq \mathrm{SU}(2)$. Notice that by G-equivariance the degree of φ is equal to the degree of the induced map $\psi\colon Y = \tilde{Y}/G \to \mathrm{SU}(2)/G$. Take a regular value for ψ, e.g., $\overline{g} = \overline{\sqrt{2}/2 + \sqrt{2}j/2} \in \mathrm{SU}(2)/G$. Since $[0, 1)^3 \subseteq \mathbf{R}^3$ is a fundamental domain for the $\pi_1(Y)$-action, we have

$$\deg(\psi) = \sum_{p\in\psi^{-1}(g)} \mathrm{sign}(\det d\psi_p) = \sum_{h\in gG}\ \sum_{p\in\Phi^{-1}(h)\cap[0,1)^3} \mathrm{sign}(\det d\Phi_p).$$

The points in $[0, 1)^3$ mapped by Φ to

$$\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}j\right) \cdot G = \left\{\pm\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i\right\} \cup \left\{\frac{\sqrt{2}}{2}i \pm \frac{\sqrt{2}}{2}k\right\} \subseteq \mathbf{H}$$

are $p_1 = (0, 0, \frac{1}{2})$, $p_2 = (\frac{1}{2}, 0, \frac{1}{2})$, $p_3 = (\frac{1}{4}, \frac{1}{2}, \frac{1}{2})$ and $p_4 = (\frac{3}{4}, \frac{1}{2}, \frac{1}{2})$. By straightforward calculation $\operatorname{sign}(\det(d\Phi_{p_1})) = \operatorname{sign}\det(d\Phi_{p_3})) = \operatorname{sign}(\det(d\Phi_{p_4})) = +1$, and $\operatorname{sign}(\det(d\Phi_{p_2})) = -1$. Hence $\deg(\varphi) = \deg(\psi) = +2$.

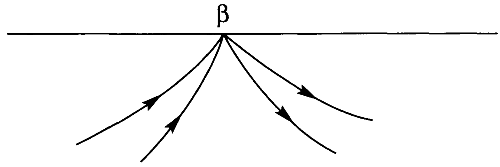
Remark 5.6. To prove that for $\alpha \in \mathscr{R}(Y)$ irreducible $\operatorname{cs}([\alpha]) \equiv \frac{1}{4} \bmod \mathbf{Z}$ we computed the degree of an explicit G-equivariant map $\varphi\colon \widetilde{Y} \to SU(2)$, where $G = \operatorname{Im}\alpha \subset SU(2)$ and $\widetilde{Y}$ is the covering of Y corresponding to $\ker\alpha$. We remark that to prove $\operatorname{cs}([\alpha]) \equiv \pm\frac{1}{4} \bmod \mathbf{Z}$ it is not necessary to compute the degree of a particular map. In fact one can show, as in the proof of Proposition 5.5 for instance, that there exists a G-equivariant map $\varphi\colon \widetilde{Y} \to SU(2)$ satisfying the hypotheses of Lemma 5.1. And it is not difficult to prove, by an algebraic topological argument, that $\deg(\varphi) \equiv 2 \bmod 4$. Hence, by Lemma 5.1, $\operatorname{cs}([\alpha]) \equiv \pm\frac{1}{4} \bmod \mathbf{Z}$. Indeed, for the applications we have in mind all we need is $\operatorname{cs}([\alpha]) \not\equiv 0$ (see Remark 7.2).

6. The stratification of the moduli spaces

Recall our notation $(\Omega^*_\alpha, d_\alpha)$ to denote the de Rham complex twisted by the flat bundle $su(2)_{\operatorname{ad}(\alpha)}$ (see §3).

Lemma 6.1. *Let Z be a principal S^1-bundle with Euler number -1 over a 2-torus. Then there is a $\delta > 0$ such that for all $\alpha \in \mathscr{R}(Z)$ δ is smaller than twice the smallest positive eigenvalue of the operator $- * d_\alpha|_{\operatorname{Im} d^*_\alpha}$.*

Proof. Observe that the eigenvalues of $- * d_\alpha|_{\operatorname{Im} d^*_\alpha}$ do not depend on the conjugacy class of α. Since $\mathscr{R}(Z)$ is the disjoint union of $\mathscr{R}(Z)^{\mathrm{red}}$ and $\mathscr{R}(Z)^{\mathrm{irr}}$, and $\mathscr{R}(Z)^{\mathrm{irr}}$ consists of a single $SU(2)$-orbit, the problem really amounts to finding a $\delta > 0$ which works for all $\alpha \in \mathscr{R}(Z)^{\mathrm{red}}$. This is perfectly analogous to the case where the Euler number of Z is -2, which has been analyzed in [10]. In fact in that case there are no irreducible representations in $\mathscr{R}(Z)$. We reproduce the argument. The only points where there is a spectral flow crossing zero are obviously the ad-trivial connections (for the definition of the spectral flow, see [2]). Fix an orthogonal decomposition $\Omega^1_\alpha = \operatorname{Im} d_\alpha \oplus H^1_\alpha \oplus \operatorname{Im} d^*_\alpha$ for all α and let $\{d_\alpha(\psi_{\mu(\alpha)})\}$, $\{\varphi_i(\alpha)\}_{i=1}^{h^1(\alpha)}$, $\{\phi_{\lambda(\alpha)}\}$ be orthogonal basis respectively for $\operatorname{Im} d_\alpha$, H^1_α and $\operatorname{Im} d^*_\alpha$, with $\{\psi_{\mu(\alpha)}\} \subseteq \Omega^0(Y)$, $\mu(\alpha) > 0$, a complete orthonormal set of $\mu(\alpha)$-eigenvectors for $\Delta_\alpha|_{\operatorname{Im} d^*_\alpha}$, and $\{\phi_{\lambda(\alpha)}\} \subseteq \Omega^1(Y)$, $\lambda(\alpha) \neq 0$, a complete orthonormal set of $\lambda(\alpha)$-eigenvectors for $-*d_\alpha|_{\operatorname{Im} d^*_\alpha}$. Let β be an ad-trivial connection. Since $\dim H^0_\beta = 3$ while $\dim H^0_\alpha = 1$ for any non-ad-trivial α, if α_t, $t \in [0, 1]$, is a path through the ad-trivial connection β with $\alpha_{1/2} = \beta$, $\lim_{t\to 1/2} \mu(\alpha_t) = 0$ for two of the $\mu(\alpha_t)$'s, counting multiplicities. On the other hand $\dim H^1_\beta = 6$, while $\dim H^1_\alpha = 2$ for any non-ad-trivial α. So the $\mu(\alpha_t)$'s going to zero account for only half the jump of $\dim H^1$. By Proposition 4.1 since the ρ-invariant changes sign if the orientation of Y is reversed, ρ_{α_t} passes from -2 to 0 as t reaches $\frac{1}{2}$. Therefore exactly two of the $\lambda(\alpha_t)$'s, counting multiplicities, are going to zero, and they have to be both negative. Thus the spectral flow along the path looks like



This implies that there is no spectral flow on $\mathscr{R}(Y)^{\mathrm{red}}$ coming from above zero, so that $\eta = \inf\{\lambda(\alpha) | \lambda(\alpha) > 0\} > 0$. Let ζ be the smallest positive eigenvalue of $- * d_\alpha|_{\operatorname{Im} d^*_\alpha}$ for $\alpha \in \mathscr{R}(Y)^{\mathrm{irr}}$, and choose $\delta < \min\{2\eta, 2\zeta\}$. q.e.d.

Remark 6.2. Let cs: $\chi(Y) \to \mathbf{R}/\mathbf{Z}$ be the Chern-Simons function. It follows by Stokes' formula that $c_2([A])$ is congruent $\operatorname{mod} \mathbf{Z}$ to $\operatorname{cs}(\partial_X([A]))$.

We refer to §2 for notation on moduli spaces from now on.

Theorem 6.3. *Let M be a smooth, simply connected, closed 4-manifold with $b_2^+(M) > 1$, odd. Let $T \hookrightarrow M$ be a smoothly embedded 2-torus with self-intersection $+1$, and denote by $X_2 \subset M$ the complement of a closed tubular neighborhood of T inside M. Then it is possible to choose on X_2 a structure of Riemannian 4-manifold with cylindrical and modelled on $[0, +\infty) \times Z$, where Z is a principal S^1-bundle with Euler number -1. For generic choices of asymptotically cylindrical metrics on X_2 the following are true:*

(1) *There exists a $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ there is an identification $\mathscr{M}(X_2) = \mathscr{M}_\delta(X_2)$.*

(2) *Let $\chi(Z)^{\mathrm{tr}} \subset \chi(Z)$ be the subset of classes of ad-trivial represen-*

tations, and let $c > 0$ *be an integer.* $\mathscr{M}_c(X_2)$ *is smoothly stratified with respect to the filtration*

$$\mathscr{M}_c(X_2) \supset \mathscr{M}_c(X, \chi(Z)^{\mathrm{tr}}).$$

Moreover,

$$\dim\{\mathscr{M}_c(X_2)\backslash\mathscr{M}_c(X, \chi(Z)^{\mathrm{tr}})\} = 8c - 3(1 + b_2^+(M)) + 1,$$
$$\dim\mathscr{M}_c(X, \chi(Z)^{\mathrm{tr}}) = 8c - 3b_2^+(M) - 6.$$

(3) *Let* $c' > 0$ *be congruent to* $\frac{3}{4}$ $\bmod \mathbf{Z}$. *Then* $\mathscr{M}_{c'}(X_2)$ *is a smooth manifold of dimension* $8c' - 3(1 + b_2^+(M))$.

Proof. The existence of the cylindrical structure is clear from Definition 2.1. By an argument given in [10] if 2δ is smaller than the first positive eigenvalue of $-*d_\alpha$ and the metric is generic, the space $\mathscr{M}(X_2; [\alpha])$ can be identified with $\mathscr{M}_\delta(X_2; [\alpha])$. Hence Lemma 6.1 implies (1). The smooth stratification in (2) follows from Proposition 3.2 and Theorem 2.2. To compute the dimensions, let $\alpha \in \mathscr{R}(Z)$ be reducible non-ad-trivial. Then by Propositions 3.3, 3.4 and 4.1, since ρ changes into $-\rho$ if the orientation of Y is reversed, $\dim H^0_\alpha = 1$, $\dim H^1_\alpha = 2$ and $\rho_\alpha = -2$. Observe that if $\omega \in \mathscr{M}(X_2, \chi(Z)^{\mathrm{red}})$ then $c_2(\omega) = 0 \bmod \mathbf{Z}$ by Proposition 5.5 and Remark 6.2. Hence if $\alpha \in \mathscr{R}(Z)$ is reducible non-ad-trivial, $0 < \delta < \delta_0$, with δ_0 as in (i) and $c > 0$ is an integer, we can apply Theorem 2.4 to get

$$\begin{aligned}\dim\mathscr{M}^0_{\delta,c}(X_2, \alpha) &= 8c - \frac{3}{2}(\chi(M) + \mathrm{sign}(M) - 1) - \frac{1}{2} - 1\\ &= 8c - 3(1 + b_2^+(M)).\end{aligned}$$

By Theorem 2.5 if $V^0 \subset \mathscr{R}(Z)^{\mathrm{red}}$ is an open SO(3)-equivariant subset whose closure consists of smooth points $\dim\mathscr{M}^0_{\delta,c}(X_2, V^0) = \dim\mathscr{M}^0_{\delta,c}(X_2, \alpha) + \dim\mathscr{R}(Z)$. Therefore by (1) we have

$$\begin{aligned}\dim\{\mathscr{M}_c(X_2)\backslash\mathscr{M}_c(X, \chi(Z)^{\mathrm{tr}}\} &= \dim\mathscr{M}_{\delta,c}(X_2, V^0)\\ &= 8c - 3(1 + b_2^+(M)) + 1.\end{aligned}$$

If α is ad-trivial then by Propositions 3.3, 3.4 and 4.1 $\dim H^0_\alpha = 3$, $\dim H^1_\alpha = 6$ and $\rho(\alpha) = 0$. Hence, if $c > 0$ is an integer and $0 < \delta < \delta_0$ with δ_0 as in (1), by Theorem 2.4 we have

$$\begin{aligned}\dim\mathscr{M}^0_{\delta,c}(X_2, \alpha) &= 8c - \frac{3}{2}(\chi(M) + \mathrm{sign}(M) - 1) - \frac{3}{2}\\ &= 8c - 3(1 + b_2^+(M)).\end{aligned}$$

Since $\mathscr{M}_{\delta,c}(X_2, [\alpha]) = \mathscr{M}^0_{\delta,c}(X_2, \alpha)/\operatorname{Stab}(\alpha)$, by (1) this gives the dimension of the lowest stratum in $\mathscr{M}_c(X_2)$ and concludes the proof of (2). To prove (3), observe first that smoothness follows from (1). Then, let $\omega \in \mathscr{M}_c(X_2, \chi(Z)^{\mathrm{irr}})$; $c_2(\omega) = -\frac{1}{4} \bmod \mathbf{Z}$ by Remark 6.2 and Proposition 5.5, because Z has the opposite orientation with respect to the manifold Y of Proposition 5.5. For the same reason, if $\alpha \in \mathscr{R}(Z)^{\mathrm{irr}}$, then $\rho_\alpha = -3$ by Proposition 4.1. Moreover, by Propositions 3.3 and 3.4, $\dim H^0_\alpha = \dim H^1_\alpha = 0$. Hence if $\alpha \in \mathscr{R}(Z)^{\mathrm{irr}}$, $0 < \delta < \delta_0$ with δ_0 as in (1), and $c' > 0$ is congruent to $\frac{3}{4}$ mod $\mathbf{Z}$, then by Theorem 2.4 we have

$$\begin{aligned}\dim \mathscr{M}^0_{\delta,c'}(X_2, \alpha) &= 8c' - \frac{3}{2}(\chi(M) + \operatorname{sign}(M) - 1) - \frac{3}{2} \\ &= 8c' - 3(1 + b_2^+(M)).\end{aligned}$$

Since $\mathscr{M}_{\delta,c}(X_2, [\alpha]) = \mathscr{M}^0_{\delta,c}(X_2, \alpha)/\operatorname{Stab}(\alpha)$, (3) follows by (1).

7. Smoothly embedded tori in 4-manifolds

Let M be a smooth, closed, simply connected 4-manifold with $b_2^+(M) > 1$ odd, and k a nonnegative integer. The *Donaldson polynomial invariants* $\gamma_k(M) \in \operatorname{Sym}^*(H^2(M))$ are defined if $k > \frac{3}{4}(1 + b_2^+(M))$, [5]. When k satisfies this inequality we say, following [8], that it is in the *stable range.* We shall always use the notation $d = d(k) = 4k - \frac{3}{2}(b_2^+(M) + 1)$. Also, $[T] \in H_2(M; \mathbf{Z})$ will denote the 2-homology class represented by T, and if q_M is the intersection form of M, for α, $\beta \in H_2(M; \mathbf{Z})$, we shall denote $q_M(\alpha, \beta)$ by $\alpha.\beta$.

Theorem 7.1. *Let M be a smooth simply connected closed 4-manifold with $b_2^+(M) > 1$, odd and k an integer in the stable range. Suppose $T \hookrightarrow M$ is a smoothly embedded 2-torus with self-intersection $+1$. Then $\gamma_k(\alpha_1, \cdots, \alpha_d) = 0$ for all d-tuples $\alpha_1, \cdots, \alpha_d \in H_2(M, \mathbf{Z})$ satisfying $\alpha_i.[T] = 0$ for all i.*

Proof. Let $X_1 \subset M$ be a tubular neighborhood of T, and $X_2 \subset M$ the complement of the closure of X_1. We shall assume throughout the proof to be working with generic asymptotically cylindrical metrics on the X_i's. Suppose $\Sigma_1, \cdots, \Sigma_d \hookrightarrow X_2$ are smoothly embedded surfaces representing the α_i's and meeting transversally in general position in the sense of [5]. It is easy to see that one can define divisors D_{Σ_i} as in [5] inside the space $\mathscr{M}(X_2)$. This is because the sections s_{Σ_i}, defining these divisors depend only the gauge equivalence classes of the restrictions of the connections to tubular neighborhoods of the Σ_i's. Moreover the transversality results continue to hold. Let M_s, for $s > 0$, be the Riemannian 4-manifold M_s diffeomorphic to M defined in §2. We are going to show that for all s large enough if $\Delta_{\Sigma_1}(s), \cdots \Delta_{\Sigma_d}(s) \subset \mathcal{M}_k(M_s)$ are divisors corresponding to the Σ_i's, then $\bigcap_i \Delta_{\Sigma_i}(s)$ consists of a finite set of points with orientations whose algebraic sum is zero. Let $\{s_n\} \subseteq \mathbf{R}$ be a sequence with $s_n \to +\infty$ and let $\{A_n\}$ be anti-self-dual connections on principal SU(2)-bundles $E_n \to M_{s_n}$ with $c_2(E_n) = k$ and $[A_n] \in \mathcal{M}_k(M_{s_n}) \cap \Delta_{\Sigma_1}(s_n) \cap \cdots \cap \Delta_{\Sigma_d}(s_n)$. By Lemma 2.3, up to passing to a subsequence, $[A_n]$ converges weakly to $([A], x_1, \cdots, x_r)$, with $A_i = A|_{X_i}$ anti-self-dual of finite energy. Suppose that the curvature is concentrating at x_i with energy $8\pi^2 n_i$. Then $c_2(A_1)+ c_2(A_2) + \sum n_i \leq k$. Observe that given $i \in \{1, \cdots, d\}$ if $x_j \notin \Sigma_i$ for all $j = 1, \cdots, r$, then $[A_2] \in D_{\Sigma_i}$, because D_{Σ_i} is closed and the sequence $\{A_n|_{X_2}\}$ converges in the C^∞ topology on $X_2 \backslash \bigcup\{x_i\}$. Let $Z = \partial X_2$. We claim that, if the D_{Σ_i} are chosen transverse to the moduli spaces, $\partial_{X_2}([A_2]) \in \chi(Z)^{\text{red}}$. In fact, arguing by contradiction, let us suppose this is not the case. Let $f = c_2([A_2])$. Then $[A_2] \in \mathcal{M}_f(X_2, \chi(Z)^{\text{irr}})$, for $i = 1, \cdots, d$, $A_2 \in D_{\Sigma_i}$ or $x_j \in \sum_i$ for some j, and since the $\sum_i$'s are in general position, there are no points x_j belonging to more than two surfaces. By (3) of Theorem 6.3, $\mathcal{M}_f(X_2, \chi(Z)^{\text{irr}})$ is smooth of dimension $8f - 3(1 + b_2^+(M))$. Therefore, if t is the number of points $x_j \in X_2$, where the curvature is concentrating, by transversality we have $d \leq 2t + \frac{1}{2}\dim \mathcal{M}_f(X_{2,\chi}(Z)^{\text{irr}})$, i.e.,

$$(*) \qquad 2d \leq 4t + 8f - 3(1 + b_2^+(M)).$$

Since $f \leq k - t$ and $f \equiv \frac{3}{4} \mod \mathbf{Z}$ by Proposition 5.5 and Remark 6.2, we have $f < k - t$. Hence by $(*)$ $2d < 2d - 4t$, which gives a contradiction because $t \geq 0$. We conclude that $[A_2] \in \mathcal{M}_f(X_2, \chi(Z)^{\text{red}})$. Next we claim that $[A_2] \in \mathcal{M}_f(X_{2,\chi}(Z)^{\text{red}} \backslash \chi(Z)^{\text{tr}})$, i.e., that $\partial_{X_2}([A_2])$ is not ad-trivial. In fact by Theorem 6.3, $\mathcal{M}_f(X_2, \chi(Z)^{\text{tr}})$ is smooth of dimension $8f - 3b_2^+(M) - 6$, and we can argue as before; so if $\partial_{X_2}([A_2])$ is ad-trivial by transversality, we have $d \leq 2t + \frac{1}{2}\dim \mathcal{M}_f(X_2, \chi(Z)^{\text{tr}})$. Thus, since $f \leq k - t$, $2d \leq 4t + 2d - 3 - 8t$, i.e., $4t + 3 \leq 0$, which gives a contradiction. Furthermore, by Theorem 6.3 and the same dimension counting argument applied to $\mathcal{M}_f(X_2, \chi(Z)^{\text{red}} \backslash \chi(Z)^{\text{tr}})$, $c_2([A_2]) = k$. In particular, there are no points of concentration for the curvature, and $c_2([A_1]) = \lim_{n \to \infty} c_2([A_n|_{X_1}]) = 0$. By (5) in Lemma 2.3, $\partial_{X_1}(A_1) = \partial_{X_2}(A_2)$. This shows that for any given $t > 0$ there is an s large enough so that, for every point $[A] \in \bigcap_{i=1}^{d} \Delta_{\Sigma_i}(s) \cap \mathcal{M}_k(M_s)$, $|A|_{X_1^s}$ is close on X_1^t, up to gauge transformation, to a flat non-ad-trivial connection. Therefore if s is large enough, in order to compute the value of the Donaldson polynomial on $\alpha_1, \cdots, \alpha_d$ we only have to consider the contribution coming from points in $\bigcap_{i=1}^{d} \Delta_{\Sigma_i}(s) \cap \mathcal{M}_k(M_s)$ which are close, on a neighborhood of the torus T, to flat reducible non-ad-trivial connections. But the contribution of these points to the Donaldson invariant is algebraically zero. This follows from the same argument given in [10] for the case of a torus with self-intersection $+2$. Let $V^0 \subset \mathcal{R}(Y)^{\text{red}}$ be a subset of non-ad-trivial connections with closure contained inside the smooth points. With the notation of Theorem 2.7, we have $\mathcal{R}(X_1, T_{X_1}) \cong V^0$ and $V^0 \times_{V^0} \mathcal{M}^0_{\delta,k}(X_2, T_{X_1}) \cong \mathcal{M}^0_{\delta,k}(X_2, T_{X_2}, V^0)$. Therefore by Theorem 2.7 the open set of connections in $\mathcal{M}_k(M_s)$ close on X_1 to a connection in $V = V^0/\mathrm{SO}(3)$ is diffeomorphic, via the Mrówka map, to the zero set of a section ξ of the real line bundle $\Xi = \Xi^0/\mathrm{SO}(3)$ over $\mathcal{M}_{\delta,k}(X_2, T_{X_2}, V)$. Moreover Ξ can be identified with $\bigwedge^{\text{top}} T\mathcal{M}_{\delta,k}(X_2, T_{X_2}, V)$, and by transversality and dimension counting, the intersection of the divisors can be assumed to lie inside $\mathcal{M}_{\delta,k}(X_2, T_{X_2}, V)$ and to be a smooth closed oriented one-dimensional manifold Y with oriented normal bundle N. Thus $w_1(\Xi) = w_1(N) = 0$, which implies that the number of zeros of ξ on Y is algebraically zero.

Remark 7.2. In the proof of Theorem 7.1 we applied the fact that $\dim \mathcal{M}_f(X_2, \chi(Z)^{\text{irr}}) = 8f - 3(1 + b_2^+(M))$, with $f \not\equiv 0 \mod \mathbf{Z}$, without using the exact value of f. So to prove the theorem it is enough to show that if $\alpha \in \mathcal{R}(Z)^{\text{irr}}$, then $\mathrm{cs}([\alpha]) \not\equiv 0 \mod \mathbf{Z}$, which is quite easier than computing the actual value (see Remark 5.6).

Let $\widehat{M} = M \# \overline{\mathbf{CP}}^2 \# \cdots \# \overline{\mathbf{CP}}^2$ be the connected sum of M with n copies of $\overline{\mathbf{CP}}^2$, the complex projective plane with orientation opposite to the standard one. Let $e_1, \cdots, e_n$ be the exceptional classes, i.e., generators for the H^2's of the $\overline{\mathbf{CP}}^2$, inside $H^2(\widehat{M})$, and identify $H^2(M)$ with the orthogonal complement of $E = \langle e_1, \cdots, e_n \rangle$. Since $\mathrm{Sym}^*(H^2(\widehat{M})) \cong \mathrm{Sym}^*(H^2(M)) \otimes \mathrm{Sym}^*(E))$, we can write formally

$$\gamma_k(\widehat{M}) = \sum_I \gamma_{k,I}(M) e^I,$$

where $I = (i_1, \cdots, i_n)$, $e^I = e_1^{i_1} \cdots e_n^{i_n}$ and $\gamma_{k,I}(M) \in \mathrm{Sym}^*(H^2(M))$. The $\gamma_{k,I}(M)$ are the *generalized Donaldson invariants* [8].

Theorem 7.3. *Let M be a smooth simply connected closed 4-manifold with $b_2^+(M)$ odd greater than one, and suppose $T \hookrightarrow M$ is a smoothly embedded 2-torus with self-intersection $+1$. Let k be an integer in the stable range, and $I = (i_1, \cdots, i_n)$ a multi-index with $|I| = \sum_j i_j < d = d(k)$. Then the generalized Donaldson polynomial $\gamma_{k,I}(M)$ of M vanishes on any $(d - |I|)$-tuple of classes $\alpha_1, \cdots, \alpha_{d-|I|} \in H_2(M; \mathbf{Z})$ satisfying $\alpha_i \cdot [T] = 0$ for all i.*

Proof. We use cohomology with complex coefficients. Let $\widehat{M} = M\#\overline{\mathbf{CP}}^2\#\cdots\#\overline{\mathbf{CP}}^2$, and assume all the gluing maps of the connected sums are supported in the complement of a neighborhood of T inside M. We can identify $H^2(M)$ with a subspace of $H^2(\widehat{M})$. Then we have the orthogonal decomposition $H^2(\widehat{M}) = H^2(M) \oplus \langle e_1 \rangle \oplus \cdots \oplus \langle e_n \rangle$, where $e_1, \cdots, e_n$ are dual to the exceptional classes $e^1, \cdots, e^n \in H_2(\widehat{M}; \mathbf{Z})$. Since the gluing maps avoid T, we have $e^i.[T] = 0$ for all $i = 1, \cdots, n$. Therefore given an integer k in the stable range, a multi-index $I = (i_1, \cdots, i_n)$ and classes $\alpha_1, \cdots, \alpha_{d-|I|} \in H_2(M; \mathbf{Z})$ perpendicular to $[T]$, using the identification of $\mathrm{Sym}^*(H^2(M))$ with the algebra of symmetric multilinear functions on $H_2(M)$ we have

$$\gamma_k(\widehat{M})(\alpha_1, \cdots, \alpha_{d-|I|}) \overbrace{e^1, \cdots, e^1}^{i_1}, \cdots, \overbrace{e^n, \cdots, e^n}^{i_n}) = 0.$$

But since

$$\gamma_k(\widehat{M}) = \sum_I \gamma_{k,I}(M) e^I,$$

this clearly implies $\gamma_{k,I}(M)(\alpha_1, \cdots, \alpha_{d-|I|}) = 0$.

8. Smoothly embedded tori in algebraic surfaces

Let M be a smooth closed 4-manifold, and $P(M) = P[H_2(M, \mathbf{C})]$ the ring of polynomial functions on the vector space $H_2(M; \mathbf{C})$. $P(M)$ will be always identified with $\mathrm{Sym}^*(H^2(M; \mathbf{C}))$. The map induced by evaluation of cocycles on cycles gives an embedding of $H^2(M; \mathbf{C})$ inside $P(M)$ as the subspace of linear functions. There is a natural isomorphism of algebras between $P(M) = \mathrm{Sym}^*(H^2(M; \mathbf{C}))$ and the space $S^*(H_2(M; \mathbf{C}))$ of symmetric multilinear forms on $H_2(M; \mathbf{C})$ endowed with the usual symmetric product. Given $p \in P(M)$ denote by $\tilde{p} \in S^*(H_2(M; \mathbf{C}))$ the corresponding element under this isomorphism. Observe that the intersection form q_M of M corresponds to a second-degree polynomial in $P(M)$.

We shall use the notation $\alpha \cdot \beta = q_M(\alpha, \beta)$, where $\alpha, \beta \in H_2(M; \mathbf{C})$. In the case of a smooth algebraic surface S, we may consider the subring $\mathbf{C}[q_S, k_S] \subset P(S)$ generated by q_S and the canonical class $k_S \in H^2(S; \mathbf{Z}) \subseteq H^2(S; \mathbf{C})$. Given $x \in H_2(M; \mathbf{C})$, denote by $x^\perp \subset H_2(M; \mathbf{C})$ the orthogonal subspace with respect to q_M, i.e., the subspace of all those elements y such that $x.y = 0$.

Lemma 8.1. *Let $\alpha \in H^2(M; \mathbf{C})$ and $p \in P(M)$. Then α divides p in $P(M)$ if and only if p vanishes on $(\alpha^*)^\perp \subseteq H_2(M; \mathbf{C})$, where α^* is the Poincaré dual of α.*

Proof. Suppose $p = q\alpha$. Then for all $x \in \alpha^\perp$ we have $p(x) = q(x)\alpha(x) = q(x)(\alpha.x) = 0$. Conversely if p vanishes on $(\alpha^*)^\perp$, then, say by Hilbert's Nullstellensatz, α divides p^n for some n. But since α has degree 1, α must divide p.

Lemma 8.2. *Let S be a smooth algebraic surface, $k_S \in H^2(S; \mathbf{Z})$ its canonical class and q_S its intersection form. Suppose* $\operatorname{rank}(q_S) \geq 4$, *and let $p \in \mathbf{C}[k_S, q_S] \subset P(S)$ be a nonzero homogeneous polynomial in k_S and q_S. Suppose that $\alpha \in H^2(S; \mathbf{Z})$ divides p in $P(S)$, and $\alpha.\alpha \neq 0$. Then $k_S \in \mathbf{Q}\alpha$.*

Proof. p cannot be a multiple of some power of q_S, because otherwise α would divide q_S which, being nondegenerate and of rank ≥ 3, does not have linear factors. Also, if some power of k_s, e.g. k_s^m, divides p, then either α divides k_s^m, in which case $k_s \in \mathbf{Q}\alpha$, or α divides p/k_s^m. Hence we may assume, without loss of generality, that k_s does not divide p, and we may write

$$(*) \qquad p = a_n q_S^n + a_{n-1} k_S^2 q_S^{n-1} + a_{n-2} k_S^4 q_S^{n-2} + \cdots$$

with $a_n \neq 0$. By contradiction, suppose $k_S \notin \mathbf{Q}\alpha$. Since α^* is not isotropic, $q_S|_{(\alpha^*)^\perp}$ is still nondegenerate and of rank ≥ 3; hence there exists $y \in (k_S^*)^\perp \cap (\alpha^*)^\perp$ with $y.y \neq 0$, where the stars denote Poincaré duals. Since $y \in (\alpha^*)^\perp$ we have

$$\tilde{p}(y, \cdots, y) = 0.$$

On the other hand by $(*)$,

$$\tilde{p}(y, \cdots, y) = a_n (y, y)^n (y.y)^n,$$

which is nonzero because $a_n \neq 0$, $y.y \neq 0$ by our choices. This contradiction proves the lemma.

Theorem 8.3. *Let S be a smooth simply connected minimal elliptic surface with geometric genus greater than zero. Then there is no smoothly embedded 2-torus $T \hookrightarrow S$ with self-intersection $+1$.*

Proof. By the results of Donaldson [5] and Friedman and Morgan [8] we know that $\gamma_k(S)$ is a nonzero polynomial in k_S and q_S for all k sufficiently large. On the other hand by Theorem 7.1 if $T \hookrightarrow S$ is a smoothly embedded 2-torus with $[T].[T] = 1$, then $\gamma_k(S)$ vanishes on $[T]^{\perp}$. Therefore, if $\alpha \in H^2(S; \mathbf{Z})$ is the Poincaré dual of $[T]$, by Lemma 8.1 α must divide $\gamma_k(S)$. Hence, since $\operatorname{rank}(q_S) \geq 4$, by Lemma 8.2 $k_S \in \mathbf{Q}\alpha$. But since S is minimal elliptic, $k_S.k_S = 0$. This could happen only if $k_S = 0$, which would mean that S is a $K3$. But the intersection form of a $K3$ is even, so in this case there is no 2-homology class with odd self-intersection.

Lemma 8.4. *Let $S = S_{d_1} \cap \cdots \cap S_{d_n} \subseteq \mathbf{P}^{n+2}$ be a smooth complete intersection complex surface. Then the hyperplane class of S is primitive in $H^2(S; \mathbf{Z})$.*

Proof. The linear system of hypersurfaces of degree d_1 defines an embedding $\mathbf{P}^{n+2} \subset \mathbf{P}^{N_1}$, for some large N_1 such that $S_{d_1} \subset \mathbf{P}^{n+2}$ is a hyperplane section. By the Lefschetz theorem on hyperplane sections, the map $H_2(S_{d_1}) \to H_2(\mathbf{P}^{n+2})$ is an isomorphism if $n > 1$, and it is onto if $n = 1$. Since $n \geq 1$, it is always onto. Then we embed $S_{d_1} \hookrightarrow \mathbf{P}^{N_2}$ by using the linear system of hypersurfaces of degree d_2 in $\mathbf{P}^{n+2}$. $S_{d_1} \cap S_{d_2}$ is therefore a hyperplane section of S_{d_1}. Again by the Lefschetz theorem $H_2(S_{d_1} \cap S_{d_2}) \to H_2(S_{d_1})$ is onto. If we keep going until we exhaust all the hypersurfaces we get a chain of surjections whose composition is the map $H_2(S) \to H_2(\mathbf{P}^{n+2})$, which is therefore surjective. At this point it is an exercise to finish the proof.

Theorem 8.5. *Let $S = S_{d_1} \cap \cdots \cap S_{d_n} \subseteq \mathbf{P}^n$ be a smooth complete intersection surface with geometric genus greater than zero. Then there is no smoothly embedded 2-torus $T \hookrightarrow S$ with self-intersection $+1$.*

Proof. As in the proof of Theorem 8.3 the existence of a smoothly embedded 2-torus $T \hookrightarrow S$ with $T.T = +1$ implies, by Theorem 7.1, Lemma 8.1 and Lemma 8.2, that k_S is a rational multiple of α, the Poincaré dual of $[T]$. Since α has square 1, it is primitive and k_S is actually an integer multiple of α. On the other hand we know that k_S is a positive integer multiple of the hyperplane class H_S which is also primitive by Lemma 8.4; therefore $H_S = \pm\alpha$. Since $H_S^2 = \prod d_i > 1$, we get a contradiction.

Lemma 8.6. *Let M be a smooth simply connected closed 4-manifold with $b_2^+(M)$ odd greater than one, and suppose some Donaldson polynomial for M is nonzero. Consider $\widehat{M} = M \# \overline{\mathbf{CP}}^2 \# \cdots \# \overline{\mathbf{CP}}^2$, and let $T \hookrightarrow \widehat{M}$ be* *a smoothly embedded 2-torus with self-intersection* $+1$. *If* $\alpha \in H^2(\widehat{M}; \mathbf{Z})$ *is the Poincaré dual of* $[T] \in H_2(\widehat{M}; \mathbf{Z})$, *then* α *belongs to* $H^2(M)$, *and there is an integer* k *and a multi-index* $I = (i_1, \cdots, i_n)$ *such that* $\gamma_{k,I}(M)$ *is nonzero and* α *divides* $\gamma_{k,I}(M)$.

Proof. Certainly $\widehat{M}$ is simply connected and $b_2^+(\widehat{M}) > 1$, odd. Therefore if there is a smoothly embedded 2-torus $T \hookrightarrow \widehat{M}$ with $[T].[T] = +1$, by Theorem 7.3 all the generalized Donaldson polynomials $\gamma_{c,J}(\widehat{M})$ vanish on $[T]^\perp$; hence by Lemma 8.1 α divides all of them. Since some of the Donaldson polynomials $\gamma_k(M)$ are nonzero, by a result of Donaldson (see also [8]) some of the $\gamma_k(\widehat{M})$'s, and therefore also some of the $\gamma_{k,I}(\widehat{M})$'s, are nonzero as well. Let $\gamma_{k,I}(\widehat{M})$ be of minimal degree among those nonzero. By [8, Chapter 4], $\gamma_{k,I}(\widehat{M}) = \gamma_{k,I}(M)$.

Theorem 8.7. *Let* $\widehat{S}$ *be a smooth simply connected complex surface with geometric genus greater than zero, whose minimal model is either elliptic or a complete intersection. Then there is no smoothly embedded 2-torus* $T \hookrightarrow \widehat{S}$ *with self-intersection* $+1$.

By the general theory of surfaces $\widehat{S} = S\#\overline{\mathbf{CP}}^2\#\cdots\#\overline{\mathbf{CP}}^2$, with S either minimal, elliptic or a complete intersection. By Lemma 8.6 if there is a smoothly embedded 2-torus $T \hookrightarrow \widehat{S}$ with self-intersection $+1$, the Poincaré dual α of $[T] \in H_2(\widehat{S}; \mathbf{Z})$ belongs to $H^2(S)$ and divides some nonzero generalized Donaldson polynomial $\gamma_{k,I}(S)$. By [8], since S has big diffeomorphism group with respect to the canonical class k_S, $\gamma_{k,I}(S)$ is a nonzero polynomial in k_S and q_S. Using Lemma 8.2 we may conclude $k_S \in \mathbf{Q} \cdot \alpha$; i.e., $k_S \in \mathbf{Z}.\alpha$ since $\alpha.\alpha = +1$. But this gives a contradiction, as in the proofs of Theorems 8.3 and 8.5.

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