J. DIFFERENTIAL GEOMETRY
37 (1993) 651-668

BRUHAT CELLS IN THE NILPOTENT VARIETY AND THE INTERSECTION RINGS OF SCHUBERT VARIETIES

JAMES B. CARRELL

1. Introduction

Let G be a complex semisimple Lie group with fixed opposite Borel subgroups B and B^-, and let H be the maximal torus $B \cap B^-$. $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ denote the Lie algebras of G, B, H respectively and $W = N(H)/H$ is the Weyl group of (G, H). A famous result in Lie theory says that the cohomology algebra $H^{\cdot}(G/B; \mathbb{C})$ of the flag variety G/B of G is isomorphic to the coordinate ring $A(\mathscr{N} \cap \mathfrak{h})$ of the scheme-theoretic intersection of the nilpotent variety $\mathscr{N} \subset \mathfrak{g}$ and the Cartan subalgebra $\mathfrak{h}$. The purpose of this paper is to extend this result to Schubert varieties $X_w := \overline{BwB/B}$ in G/B, where $w \in W$.

We introduce a locally closed stratification $\mathscr{B}_w$ of $\mathscr{N}$ by "Bruhat cells" defined by putting $\mathscr{B}_w = \mathrm{Ad}(Bw^{-1}B)\mathfrak{u}$, where $\mathfrak{u}$ is the nilradical of $\mathfrak{b}$. $\mathscr{N}_w := \overline{\mathscr{B}}_w$ is a Zariski closed irreducible cone in $\mathfrak{g}$ such that $\mathscr{N}_w \subseteq \mathscr{N}_y$ if and only if $X_w \subseteq X_y$. Recall that the scheme-theoretic intersection of varieties Z_1 and Z_2 in $\mathfrak{g}$ is the scheme $Z_1 \cap Z_2$ defined by the ideal $I(Z_1) + I(Z_2)$ where $I(Z_i)$ is the ideal of Z_i in the coordinate $A(\mathfrak{g})$ of $\mathfrak{g}$. By definition, the coordinate ring $A(Z_1 \cap Z_2)$ of $Z_1 \cap Z_2$ is $A(\mathfrak{g})/(I(Z_1) + I(Z_2))$. We will prove

Theorem 1. *For each $w \in W$, there exists a surjective degree doubling homomorphism of graded $\mathbb{C}$-algebras $\psi_w \colon A(\mathscr{N}_w \cap \mathfrak{h}) \to H^{\cdot}(X_w; \mathbb{C})$ such that if $X_w \subseteq X_y$, the diagram*

$$
\begin{array}{ccc}
A(\mathscr{N}_y \cap \mathfrak{h}) & \xrightarrow{\psi_y} & H^{\cdot}(X_y; \mathbb{C}) \\
\downarrow & & \downarrow \\
A(\mathscr{N}_w \cap \mathfrak{h}) & \xrightarrow{\psi_w} & H^{\cdot}(X_w; \mathbb{C})
\end{array}
\tag{1.1}
$$

Received April 30, 1990 and, in revised form, December 4, 1991. The author's work was partially supported by a grant from the National Sciences and Engineering Research Council of Canada.

commutes, where the vertical maps are induced by the natural inclusions. If X_w is smooth, then ψ_w is an isomorphism.

We remark that if w_0 is the longest element of W, then $\mathcal{N}_{w_0} = \mathcal{N}$ and ψ_{w_0} is the classical isomorphism. The homomorphisms ψ_w are constructed by relating $\mathcal{N}_w \cap H$ and the zero scheme of the algebraic vector field V_e on G/B studied in [1], where e is a homogeneous principal nilpotent in $\mathfrak{b}$ (see §2). V_e has exactly one zero, the coordinate ring $A(Z_e)$ of the zero scheme Z_e of V_e is [known to be] a graded $\mathbb{C}$-algebra, and there exists an isomorphism of graded $\mathbb{C}$-algebras $\alpha\colon A(Z_e) \to H^{\cdot}(G/B; \mathbb{C})$. Moreover, V_e is tangent to X_w at all smooth points, so one can consider the scheme-theoretic intersection $Z_e \cap X_w$. The coordinate ring $A(Z_e \cap X_w)$ is also graded, and an application of a result in [3] gives the existence of a surjective graded $\mathbb{C}$-algebra morphism $\alpha_w\colon A(Z_e \cap X_w) \to H^{\cdot}(X_w; \mathbb{C})$ such that the analog of diagram (1.1) commutes when $X_w \subset X_y$ and such that α_w is an isomorphism if X_w is smooth.

The key to proving Theorem 1 is thus to produce isomorphisms β_w : $A(\mathcal{N}_w \cap \mathfrak{h}) \to A(Z_e \cap X_w)$ having the usual naturality properties. To do so, we consider the morphism $\phi_e\colon U^- \to \mathfrak{g}$ given by $\phi_e(u) = \mathrm{Ad}(u^{-1})e$, U^- being the unipotent radical of B^-. Letting $\rho\colon U^- \to G/B$ be the isomorphism $\rho(u) = u \cdot B$ of U^- onto the open cell U centered at B and noting that $\rho^{-1}(X_w \cap U) = \phi_e^{-1}(\mathcal{N}_w)$, we can state

Theorem 2. *The comorphism $(\phi_e\rho^{-1})^*\colon A(\mathfrak{g}) \to A(U)$ induces, for each $w \in W$, a degree-doubling isomorphism $\beta_w\colon A(\mathcal{N}_w \cap h) \to A(Z_e \cap X_w)$ so that if $X_w \subset X_y$, then β_w and β_y commute with the natural restrictions.*

It has been an open question whether the homomorphisms α_w : $A(Z_e \cap X_w) \to H^{\cdot}(X_w; \mathbb{C})$ are isomorphisms. Recently this question has been partially answered by the following two results.

Theorem [5]. *If $G = \mathrm{SL}_n(\mathbb{C})$, every α_w is an isomorphism.*

On the other hand, Dale Peterson has shown

Theorem 3. *Suppose ω is a nonminuscule fundamental dominant weight for $\mathfrak{h}$ and let $r \in W$ be the reflection corresponding to the simple root associated to ω. Let $w = w_0 r$, where w_0 is W's longest element. Then $\dim_{\mathbb{C}} A(Z_e \cap X_w) = \dim_{\mathbb{C}} A(Z_e)$, and consequently α_w is not injective.*

Theorem 3 is proved in the Appendix. Note that $\mathrm{SL}_n(\mathbb{C})$ is the only simple group in which all fundamental dominant weights are minuscule. Hence for G simple not of type A_n there exist codimension-one X_w for which α_w is not an isomorphism.

As a consequence of the fact that all α_w are isomorphisms if $G = \mathrm{SL}_n(\mathbb{C})$, one obtains the

Corollary. *For* $G = \mathrm{SL}_n(\mathbb{C})$, *all* $\psi_w : A(\mathcal{N}_w \cap \mathfrak{h}) \to H^*(X_w; \mathbb{C})$ *are isomorphisms.*

There is a conjectured definition of ψ_w not involving $A(Z_e \cap X_w)$ which we discuss since it yields information on when α_w is an isomorphism. Let $t \in \mathfrak{h}$ and set $\mathcal{B}_{w,t} := \overline{\mathrm{Ad}(Bw^{-1}B)t}$. Using a result of [5], we show in Theorem 8 that $H^*(X_w; \mathbb{C})$ is isomorphic with the graded ring $A(\mathfrak{g})/\mathrm{gr}(I(\mathcal{B}_{w,t}) + I(\mathfrak{h}))$. Here $\mathrm{gr}\, I$ denotes the ideal generated by the leading terms of the ideal I. Since $\mathrm{gr}(I_1 + I_2) \supseteq \mathrm{gr}\, I_1 + \mathrm{gr}\, I_2$, if $I(\mathcal{N}_w) = \mathrm{gr}\, I(\mathcal{B}_{w,t})$, we obtain a natural map from $A(\mathcal{N}_w \cap \mathfrak{h})$ onto $H^*(X_w, \mathbb{C})$, which turns out to be ψ_w. Furthermore, we then obtain that ψ_w is an isomorphism exactly when $\mathrm{gr}(I(\mathcal{B}_{w,t}) + I(\mathfrak{h})) = I(\mathcal{N}_w) + I(\mathfrak{h})$.

Some of the results in this paper have been generalized by Peterson [16] to the Kac-Moody setting. Moreover, he has shown that the maps α_w etc. are all defined over $\mathbb{Z}$ (instead of $\mathbb{C}$). An account of these results is included in the expository article [9].

The paper is organized as follows. In §2, the basic theorem (Theorem 5) on the zero scheme of the homogeneous principal nilpotent is proven and an example to illustrate the result is given. In §3, we prove the basic result that ϕ_e induces a degree-doubling isomorphism from $A(\mathcal{N} \cap \mathfrak{h})$ onto $A(Z_e)$ and in §4 we extend this to the relative case $\phi_{e,w} : A(\mathcal{N}_w \cap \mathfrak{h}) \to A(Z_e \cap X_w)$. In §5 we consider a semisimple deformation as a path to an alternate definition of the morphisms ψ_w. In the Appendix, examples that show the ψ_w are not always injective are given.

The author would like to thank Dale Peterson for contributing the result in the Appendix. He would also like to thank Hanspeter Kraft and Shrawan Kumar for their comments.

2. A description of $I(Z_v)$

2.1. The starting point of this paper is the problem of giving a geometric description of the zero scheme of an algebraic vector field on $X = G/B$ obtained by exponentiating a principal nilpotent $v \in \mathfrak{g}$. Such a vector field has its only zero at the unique Borel subgroup of G whose Lie algebra contains v. We will suppose $v \in \mathfrak{b}$, so the vector field V_v assigned to v vanishes only at $B \in X$. Recall U denotes the big open cell in X centered at B. Then the zero scheme Z_v of V_v is the affine punctual scheme in U supported at B associated to the ideal $I(Z_v)$ generated by the functions $V_v(f)$, where $f \in A(U)$, the affine coordinate ring of U, and V_v is viewed as a derivation of $A(U)$. By definition, the coordinate ring of Z_v is $A(Z_v) := A(U)/I(Z_v)$. We will solve our problem in $A(U^-)$, where U^- is the unipotent radical of B^-, using the isomorphism $\rho: U^- \to U$ defined by $\rho(u) = uB$. Let $\Omega = U^-B$ be the corresponding big cell in G and $\Pi: \Omega \to U^-$ the canonical map defined by the composition

$$\Omega \xrightarrow{m^{-1}} U^- \times B \xrightarrow{\pi_1} U^-$$

where $m(u, b) = ub$ and $\pi_1(u, b) = u$.

Theorem 4. *Let $v \in \mathfrak{b}$ be a principal nilpotent. Then $\rho^*(I(Z_v)) \subset A(U^-)$ is generated by the components (with respect to any basis) of the $\mathfrak{u}^-$-valued map $u \mapsto \Pi_* \mathrm{Ad}(u^{-1})v$ where Π_* denotes the differential of Π at the identity 1_G of G.*

Proof. For $y \in \mathfrak{g}$, let W_y be the corresponding right invariant vector field on G. Thus

$$W_y(g) = R_{g^*}y = \frac{d}{dt}(\exp(ty)g)|_{t=0}.$$

The holomorphic vector field V_y on X induced by W_y is

$$V_y(gB) = \frac{d}{dt}(\exp(ty)gB)|_{t=0}.$$

Let $\widetilde{W}_y$ be the holomorphic vector field on U^- defined at $u \in U^-$ by $\widetilde{W}_y(u) = \Pi_*(W_y(u))$ where $\Pi_*: T'_u\Omega \to T'_uU^-$ stands for the holomorphic differential of Π on the holomorphic tangent space $T'_u\Omega$ to Ω at u. Let $\mu: \Omega \to U$ be the quotient map $\mu(g) = gB$. Then $\rho\Pi = \mu$, and consequently

$$V_y = \mu_* W_y = (\rho\Pi)_* W_y = \rho_*(\Pi_* W_y) = \rho_* \widetilde{W}_y,$$

where μ_* and ρ_* are analogous to Π_*. Now let $y = v$ and define $I_v \subset A(U^-)$ to be the ideal generated by all $\widetilde{W}_v(f)$ where $f \in A(U^-)$.

Lemma 1. *$\rho^*(I(Z_v)) = I_v$, and ρ^* induces an isomorphism $\tilde{\rho}: A(Z_v) \to A_v := A(U^-)/I_v$.*

Proof. This is obvious from (2.1).

We now compute the ideal I_v. Note that for $u \in U^-$,

$$W_v(u) = \frac{d}{dt}(uu^{-1}(\exp tv)u)|_{t=0} = L_{u*}\mathrm{Ad}(u^{-1})v,$$

where $L_u(g) = ug$. Since $\Pi L_u = L_u\Pi$ for $u \in U^-$,

$$\widetilde{W}_v(u) = \Pi_* W_v(u) = \Pi_* L_{u_*} \operatorname{Ad}(u^{-1})v = L_{u_*} \Pi_* \operatorname{Ad}(u^{-1})v.$$

Thus, $L_{u^{-1}*}(\widetilde{W}_v(u)) = \Pi_* \operatorname{Ad}(u^{-1})v$. Now suppose that $v_1, v_2, \cdots, v_k$ form a basis of left invariant vector fields on U^- and write $\widetilde{W}_v = \sum_{i=1}^k a_i v_i$, where $a_1, \cdots, a_k$ are in $A(U^-)$. Then

$$L_{u^{-1}*}(\widetilde{W}_v(u)) = \sum_{i=1}^{k} a_i(u) L_{u^{-1}*}(v_i(u)) = \sum_{i=1}^{k} a_i(u) v_i(1_G).$$

Hence $a_1, \cdots, a_k$ are the components with respect to $v_1(1_G), \ldots, v_k(1_G)$ of $\Pi_* \operatorname{Ad}(u^{-1})v$. Since $a_1, \cdots, a_k$ generate I_v, the theorem is proved.

2.2. We now bring in the homogeneous principal nilpotent e. Let $\Phi \subset \mathfrak{h}^*$ be the root system of $(\mathfrak{g}, \mathfrak{h})$ and let Φ^+ be the set of positive roots, i.e., the roots of $(\mathfrak{b}, \mathfrak{h})$. Denote the set of simple roots in Φ^+ by $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ and choose $e_i \in \mathfrak{g}_{\alpha_i} \backslash 0$, where $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the root subspace corresponding to each $\alpha \in \Phi$. We set $e = e_1 + \cdots + e_l$. Recall $\phi_e(u) = \operatorname{Ad}(u^{-1})e$ and note $\phi_e(u) \in e + \mathfrak{h} + \mathfrak{u}^-$. Thus

$$\phi_e(u) = e + k_e(u) + \sum_{\alpha>0} v_{-\alpha}(u) e_{-\alpha}, \tag{2.2}$$

where every $e_{-\alpha} \in \mathfrak{g}_{-\alpha} \backslash 0$ and $k_e(u) \in \mathfrak{h}$. Hence I_e is defined by the condition $\phi_e(u) \in e + \mathfrak{h}$.

Recall that e induces gradings on $A(U)$ and $A(U^-)$ [13]. We now show that $I(Z_e)$ and I_e are homogeneous and that ρ^* determines the graded isomorphism. Let $s \in \mathfrak{h}$ be the unique element such that $\langle \alpha_i, s \rangle = 2$ if $1 \le i \le l$, and $\gamma: \mathbb{C}^* \to G$ the one-parameter group such that $\gamma'(1) = s$. Since U^- is H-invariant, γ determines a $\mathbb{C}^*$-action $\mathbb{C}^* \times U^- \to U^-$ via $(t, u) \mapsto t \cdot u = \gamma(t) u \gamma(t)^{-1}$ and this determines the grading of $A(U^-)$ by setting $A(U^-)_k := \{f \in A(U^-) \mid t \cdot f = t^k f \text{ for all } t \in \mathbb{C}^*\}$. Likewise, $A(U)$ has a grading—namely the one associated with the $\mathbb{C}^*$-action $(t, gB) \to \gamma(t) gB$ on U. Clearly, ρ is γ-equivariant so ρ^* is a graded isomorphism, and the homogeneity of $I_e = \rho^*(I(Z_e))$ and $I(Z_e)$ are equivalent. Recall the height of $\alpha \in \Phi^+$ is $\operatorname{ht}(\alpha) = \frac{1}{2}\langle \alpha, s \rangle$.

Lemma 2. *Each* $v_{-\alpha}$ $(\alpha > 0)$ *is homogeneous of degree* $2(1 + \operatorname{ht}(\alpha))$, *and the map* k_e *is homogeneous of degree* 2.

Proof. We must show that $t \cdot v_{-\alpha} = t^{2(1+\operatorname{ht}(\alpha))} v_{-\alpha}$ where as usual $t \cdot v_{-\alpha}(u) = v_{-\alpha}(t^{-1} \cdot u)$. Now

$$\begin{aligned} t\cdot\phi_e(u) &= \phi_e(\gamma(t)^{-1}u\gamma(t)) = \mathrm{Ad}(\gamma(t)^{-1}u^{-1}\gamma(t))e \\ &= \mathrm{Ad}(\gamma(t)^{-1})\,\mathrm{Ad}(u^{-1})\,\mathrm{Ad}(\gamma(t))e \\ &= t^2\,\mathrm{Ad}(\gamma(t))^{-1}\,\mathrm{Ad}(u^{-1})e \\ &= t^2\,\mathrm{Ad}(\gamma(t))^{-1}\left(e + k_e(u) + \sum v_{-\alpha}(u)e_{-\alpha}\right) \\ &= e + t^2\left(k_e(u) + \sum v_{-\alpha}(u)t^{\langle\alpha,s\rangle}e_{-\alpha}\right) \\ &= e + t^2k_e(u) + \sum t^{2(1+\mathrm{ht}(\alpha))}v_{-\alpha}(u)e_{-\alpha}. \end{aligned}$$

This establishes the lemma. q.e.d.

To summarize, we state

Theorem 5. *Let e be the principal homogeneous nilpotent $e = e_1 + \cdots + e_l$. Then the ideals I_e and $I(Z_e)$ are homogeneous in $A(U^-)$ and $A(U)$ respectively, and $\tilde{p}\colon A(Z_e) \to A_e$ is an isomorphism of graded $\mathbb{C}$-algebras. Consequently, $A_e \cong H^{\cdot}(X;\mathbb{C})$. If $\phi_e\colon U^- \to \mathfrak{g}$ is the embedding $\phi_e(u) = \mathrm{Ad}(u^{-1})e$, then I_e is generated by the functions $v_{-\alpha} = \pi_{-\alpha}\phi_e$ $(\alpha > 0)$, where $\pi_{-\alpha}\colon \mathfrak{g} \to \mathfrak{g}_{-\alpha} \cong \mathbb{C}$ is the canonical projection.*

Example. Let

$$u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u_1 & 1 & 0 & 0 \\ u_2 & u_4 & 1 & 0 \\ u_3 & u_5 & u_6 & 1 \end{pmatrix}$$

denote an arbitrary element of U^-. We have

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and so

$$\mathrm{Ad}(u^{-1})\cdot e = \begin{pmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \tag{2.3}$$

where

$$\begin{aligned}
&a_{11} = u_1, \quad a_{22} = u_4 - u_1, \quad a_{33} = u_6 - u_4, \quad a_{44} = -u_6, \\
&a_{21} = u_2 - u_1^2, \quad a_{31} = u_3 - u_2u_4 + u_1(u_1u_4 - u_2), \\
&a_{41} = -u_3u_6 + u_2(u_4u_6 - u_5) + u_1(-u_3 + u_1u_5 + u_2u_6 - u_1u_4u_6), \\
&a_{32} = u_5 - u_2 + u_4(u_1 - u_4), \\
&a_{42} = -u_3 + u_6(u_2 - u_5) + (u_1 - u_4)(u_5 - u_4u_6), \\
&a_{43} = -u_5 + u_6(u_4 - u_6).
\end{aligned}$$

The a_{ij} with $i > j$ generate I_e. On the other hand, I_e is also generated by the coefficients of the vector field

$$\widetilde{W}_e = \sum_{i=1}^{6} b_i \frac{\partial}{\partial u_i},$$

which are given as follows:

$$\begin{aligned}
&b_1 = u_2 - u_1^2, \quad b_2 = u_3 - u_1u_2, \quad b_3 = -u_1u_3, \\
&b_4 = u_5 - u_2 + u_4(u_1 - u_4), \quad b_5 = -u_3 + u_5(u_1 - u_4), \\
&b_6 = -u_5 + u_6(u_4 - u_6).
\end{aligned}$$

Note that the coefficients b_1, b_4, b_6 of $\widetilde{W}_e$ are matrix entries in (2.3). $b_1, \cdots, b_6$ give a simpler but theoretically less interesting set of generators of I_e than the entries in (2.3).

3. The fundamental isomorphism $\tilde{\phi}_e$

3.1. In this section we show

Theorem 6. *ϕ_e induces a degree-doubling isomorphism of graded algebras*

$$\tilde{\phi}_e : A(\mathcal{N} \cap \mathfrak{h}) \to A_e.$$

Proof. We first show ϕ_e induces a surjective homomorphism $\phi_e^{\#}$: $A(\mathcal{N} \cap (e + \mathfrak{h})) \to A_e$. By Theorem 5, it suffices to show ϕ_e^* is surjective. This follows easily from the result of Kostant [13] that if $\{e, s, f\}$ is an sl_2-triplet, then the map $U^- \times \mathfrak{g}^f \to \mathfrak{g}$ sending (u, x) to $\mathrm{Ad}(u)(x + f)$ is an isomorphism onto $e + \mathfrak{h} + \mathfrak{u}^-$. Here $\mathfrak{g}^f$ denotes the centralizer of f. Notice that $A(\mathcal{N} \cap (e + \mathfrak{h}))$ is not graded; however the usual grading of $A(\mathfrak{g}) = \mathbb{C}[z_\alpha, z_{-\beta}, x_i \mid \alpha, \beta > 0, 1 \le i \le l]$ ($\{x_i, z_\alpha, z_{-\beta}\}$ denoting the usual dual basis of $\mathfrak{g}^*$) defines a filtration $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_i \subseteq F_{i+1} \subseteq \cdots$ of $A(\mathcal{N} \cap (e + \mathfrak{h}))$ such that $F_iF_j \subseteq F_{i+j}$, where $F_i := \mathrm{Im} \bigoplus_{j \le i} A(\mathfrak{g})_j$.

A_e, being graded, also has a filtration $F_i' \subseteq F_{i+1}'$ such that $F_i'F_j' \subseteq F_{i+j}'$. Since ϕ_e^* is not graded, we must show that for all $m \geq 0$, $\phi_e^{\#}(F_m) \subseteq F_{2m}'$. A typical monomial M in $A(\mathfrak{g})$ of degree at most m has the form

$$M = \prod_{\alpha \in \Delta} (z_\alpha)^{i_\alpha} \prod_{j=1}^{l} (x_j)^{r_j},$$

where $\sum i_\alpha + \sum r_j \leq m$, and all i_α and r_j are ≥ 0. Then $\phi_e^*(M) = 0$ if $i_\alpha > 0$ for some α such that $\mathrm{ht}(\alpha) > 1$, and $\phi_i^*(M) \in I_e$ if $i_\alpha > 0$ for some $\alpha < 0$. Hence we may assume

$$M = \prod_{i=1}^{l} (z_{\alpha_i})^{i_{\alpha_i}} \prod_{j=1}^{l} (x_j)^{r_j},$$

and then

$$\phi_e^*(M) = k_e^* \left(\prod_{i=1}^{l} (x_j)^{r_j} \right),$$

which has degree $2\sum r_j \leq 2m$. Therefore $\phi_e^{\#}(F_m) \subseteq F_{2m}'$ as claimed. Let $\mathrm{Gr}\, A(\mathscr{N} \cap (e + \mathfrak{h})) := F_0 + \sum F_i/F_{i-1}$ be the graded ring associated with the filtration F. Thus $\phi_e^{\#}$ induces

$$\mathrm{Gr}\, \phi_e^{\#} \colon \mathrm{Gr}\, A(\mathscr{N} \cap (e + \mathfrak{h})) \to \mathrm{Gr}\, A_e = A_e.$$

The final step in the proof is to define $\tilde{\phi}_e$. By [14, p. 134], there exists a canonical isomorphism

$$j \colon A(\mathfrak{g}) / \mathrm{gr}(I(\mathscr{N}) + I(e + \mathfrak{h})) \to \mathrm{Gr}\, A(\mathscr{N} \cap (e + \mathfrak{h})) \tag{3.1}$$

such that if $f \in A(\mathfrak{g})$ has degree $\leq p$ and residue class $\bar{f}$, then $j(\bar{f})$ is the element of F_p/F_{p-1} determined by f. Now $\mathrm{gr}\, I(\mathscr{N}) + \mathrm{gr}\, I(e + \mathfrak{h})$ is clearly $I(\mathscr{N}) + I(\mathfrak{h})$, so we can define $\tilde{\phi}_e$ by the composition

$$\begin{aligned} A(\mathscr{N} \cap \mathfrak{h}) &\to A(\mathfrak{g}) / \mathrm{gr}(I(\mathscr{N}) + I(e + \mathfrak{h})) \\ &\to \mathrm{Gr}\, A(\mathscr{N} \cap (e + \mathfrak{h})) \to A_e. \end{aligned} \tag{3.2}$$

Since $\dim_{\mathbb{C}} A(\mathscr{N} \cap \mathfrak{h}) = \dim_{\mathbb{C}} A_e$, $\tilde{\phi}_e$ is as claimed and the theorem is proved. q.e.d.

Consequently, we have an isomorphism

$$\tilde{\nu}\tilde{\phi}_e \colon A(\mathscr{N} \cap \mathfrak{h}) \to A(Z_e),$$

where $\tilde{\nu}$ is the inverse of $\tilde{\rho}$.

3.2. Let $I_+^W \subset A(\mathfrak{h})$ be the ideal generated by the homogeneous W-invariants. The inclusion map $i \colon \mathfrak{h} \to \mathfrak{g}$ induces an isomorphism $\tilde{\imath}$:

$A(\mathscr{N} \cap \mathfrak{h}) \to S_W := A(\mathfrak{h})/I_+^W$. In [1], it was shown that if π denotes the projection of $\mathfrak{g}$ onto $\mathfrak{h}$, then the map $\tau_e: \mathfrak{u}^- \to \mathfrak{h}$ given by $\tau_e(n) = \pi[e, n]$ induces an isomorphism of graded algebras

$$\tilde{\tau}_e: S_W \to A(\mathfrak{u}^-)/\exp^*(I_e). \tag{3.3}$$

It seems worthwhile to use the above results to reprove this.

Corollary. *$\tilde{\tau}_e$ is a degree-doubling isomorphism of graded algebras. If $\omega_1, \cdots, \omega_l$ are the fundamental dominant weights of $\mathfrak{h}$ with respect to $\alpha_1, \cdots, \alpha_l$, then $\tilde{\tau}_e(\bar{\omega}_i) = \bar{z}_{-i}$ if $1 \leq i \leq l$, where "bars" denote residue classes.*

Proof. In fact, $\tau_e = \tau\phi_e \exp = k_e \exp$. We show π^* induces the inverse of $\tilde{i}$. To see that $\pi^*(I_+^W) \subseteq I(\mathscr{N}) + I(\mathfrak{h})$, use the fact that for any $f \in A(\mathfrak{h})^W$, there exists a $g \in A(\mathfrak{g})^G$ such that $i^*g = f$. Then $\pi^*f - g$ vanishes on $\mathfrak{h}$; hence π induces a morphism $\tilde{\pi}: S_W \to A(\mathscr{N} \cap \mathfrak{h})$ which is the inverse of $\tilde{i}$. Thus $\tilde{\tau}_e$ is an isomorphism. Next, let $a = \sum a_\alpha e_{-\alpha} \in \mathfrak{u}^-$. We may suppose $\{e_i, a_i^\vee, e_{-i}\}$ forms an sl_2-triplet. Hence

$$[e, a] = \sum a_{\alpha_j}[e_j, e_{-j}] = \sum a_{\alpha_j}\alpha_j^v.$$

Now for any fundamental dominant weight ω_i,

$$\tau_e^*(\omega_i)(a) = \omega_i(\tau_e(a)) = \sum a_{\alpha_j}\omega_i(h_j) = a_{\alpha_j}.$$

This shows that $\tau_e^*(\omega_i) = z_{-i}$, and completes the proof.

3.3. To summarize the maps that have been introduced, we note the following commutative diagram of isomorphisms:

$$\begin{array}{ccccc} A(\mathscr{N} \cap \mathfrak{h}) & & \xrightarrow{\tilde{i}} & & S_W \\ & & & \nearrow{\scriptstyle \tilde{\tau}_e} & \\ {\scriptstyle \tilde{\phi}_e}\downarrow & & A(\mathfrak{u}^-)/\exp^* I_e & & \downarrow{\scriptstyle \beta} \\ & \swarrow{\scriptstyle \widetilde{\exp}} & & & \\ A_e & & \xrightarrow{\alpha\tilde{p}} & & H^\cdot(X; \mathbb{C}) \end{array} \tag{3.4}$$

It is well known that as a W-module, S_W is the regular representation. There is no obvious W-module structure on A_e however. Recently, Dale Peterson showed that there exists an action of W on U^- such that the functions $v_{-\alpha}$ of (2.2) are a fundamental system of generators for $A(U^-)^W$, which is a polynomial ring. Moreover, $\tilde{\phi}_e\tilde{\pi}: S_W \to A_e$ is W-equivariant.

4. The fundamental isomorphisms $\phi_{e,w}$

4.1. In this section we prove Theorem 2. The main step is to show that ϕ_e induces an isomorphism between $A(\mathcal{N}_w \cap \mathfrak{h})$ and

$$A_{e,w} := A(U^-)/I_e + I(U^- \cap \overline{BwB}).$$

First we will show that $\tilde{\rho}: A(Z_e) \to A_e$ induces an isomorphism $\tilde{\rho}_w$ from $A(Z_e \cap X_w)$ onto $A_{e,w}$. Note that ρ induces an isomorphism of varieties between $U^- \cap \overline{BwB}$ and $U \cap X_w$, hence an isomorphism $\rho_w^*: A(U \cap X_w) \to A(U^- \cap \overline{BwB})$. Thus, by the definition of I_e, we obtain the isomorphism $\tilde{\rho}_w: A(U \cap X_w)/I(Z_e) = A(Z_e \cap X_w) \to A(U^- \cap \overline{BwB})/I_e = A_{e,w}$.

Recall that $\mathcal{N}_w$ is by definition $\overline{\mathrm{Ad}(Bw^{-1}B)e}$. Put $u \cdot e = \phi_e(u)$.

Lemma 3.
(1) $\phi_e^{-1}(\mathcal{N}_w) = U^- \cap \overline{BwB}$ *as varieties.*
(2) $I(U^- \cap \overline{BwB}) \subset A(U^-)$ *is homogeneous.*

Proof. For (1), suppose $u \cdot e = b_1 w b_2 \cdot e$ for some u in U^- and b_1, b_2 in B. Since the centralizer G_e of e is contained in B, $u = b_1' w b_2'$ for some b_1', b_2' in B. Thus $u \in U^- \cap BwB$ so $\phi_e^{-1}(\mathcal{N}_w) = U^- \cap \overline{BwB}$. That ϕ_e is an isomorphism follows from the fact that ϕ_e is a closed immersion. For (2) note that $\overline{BwB}$ is stable under the 1-p.s.g. γ. Using this one proves the homogeneity in the same way it is established for the standard action in [15, p. 21]. This proves the lemma.

It follows that $\phi_e^*(I(\mathcal{N}_w)) = I(U^- \cap \overline{BwB})$ so we obtain a surjective $\mathbb{C}$-algebra homomorphism

$$\phi_{e,w}^{\#}: A(\mathcal{N}_w \cap (e + \mathfrak{h})) \to A_{e,w}.$$

Arguing as in (3.1) and (3.2), using the fact that $I(\mathcal{N}_w)$ is homogeneous, we obtain a morphism of graded rings

$$A(\mathcal{N}_w \cap \mathfrak{h}) \to \mathrm{Gr}\, A(\mathcal{N}_w \cap (e + \mathfrak{h}))$$

which composed with $\mathrm{Gr}\,\phi_{e,w}^{\#}$ yields a morphism

$$\gamma_w: A(\mathcal{N}_w \cap \mathfrak{h}) \to \mathrm{Gr}\, A_{e,w}.$$

However, both ideals I_e and $I(U^- \cap \overline{BwB})$ being homogeneous, it is trivial that $\mathrm{gr}(I_e + I(U^- \cap \overline{BwB})) = I_e + I(U^- \cap \overline{BwB})$, so $\mathrm{Gr}\, A_{e,w} \cong A_{e,w}$ and we finally obtain $\tilde{\phi}_w$ as the composition of surjective morphisms

$$A(\mathcal{N}_w \cap \mathfrak{h}) \xrightarrow{\gamma_w} A_{e,w} \xrightarrow{\tilde{\rho}_w^{-1}} A(Z_e \cap X_w).$$

In order to show that $\tilde{\phi}_w$ is an isomorphism, we will show that γ_w is an isomorphism. Note first that $\dim_{\mathbb{C}} A(\mathcal{N}_w \cap \mathfrak{h}) \geq \dim_{\mathbb{C}} A_{e,w}$. Let $p: G \cdot e \to X$ be the projection defined by $p(g \cdot e) = gB$. Then we have the commutative diagram

$$\begin{array}{ccc} & G \cdot e \cap p^{-1}(U) & \\ \nearrow & & \nwarrow p \\ U^- & \xleftarrow{\breve{p}} & U \end{array}$$

where $\breve{p}(uB) = u^{-1}$. Since $\mathcal{N}$ is normal and $G \cdot e$ has even codimension in $\mathcal{N} = \overline{G \cdot e}$, one knows that $A(G \cdot e \cap p^{-1}(U)) = A(\mathcal{N} \cap p^{-1}(U))$. Therefore, since $p^{-1}(U)$ is a Zariski open neighborhood of e in $G \cdot e$, $\dim_{\mathbb{C}} A(\mathcal{N}_w \cap (e + \mathfrak{h})) = \dim_{\mathbb{C}} A(p^{-1}(U) \cap \mathcal{N}_w \cap (e + \mathfrak{h}))$ for all w in W. Now, p induces a morphism $p_w^{\#}$ sending $\tilde{\rho}^{-1}(A_{e,w})$ onto $A(p^{-1}(U) \cap \mathcal{N}_w \cap (e + \mathfrak{h}))$, the surjectivity holding for each w since it holds for the longest element w_0. Thus

$$\dim_{\mathbb{C}} A_{e,w} = \dim_{\mathbb{C}} \tilde{\rho}^{-1}(A_{e,w}) \geq \dim_{\mathbb{C}} A(\mathcal{N}_w \cap (e + \mathfrak{h})).$$

To finish the proof that γ_w is an isomorphism, it will be sufficient to note that $\operatorname{Gr} A(\mathcal{N}_w \cap (e + \mathfrak{h})) \cong A(\mathcal{N}_w \cap \mathfrak{h})$. Hence it suffices to check that

$$\operatorname{gr}(I(\mathcal{N}_w) + I(e + \mathfrak{h})) = I(\mathcal{N}_w) + I(\mathfrak{h}). \tag{4.1}$$

This is clear, however, since $I(e + \mathfrak{h})$ is generated by linear functions and $I(\mathcal{N}_w)$ is homogeneous. This completes the proof of Theorem 2 except checking commutativity, which is left to the reader.

4.2. We now prove Theorem 1. Since V_e is tangent to the set of regular points of X_w, it follows from [3, Theorem 5] that there exists a filtration of $A(Z_e)$ such that there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Gr} A(Z_e) & \xrightarrow{\alpha} & H^{\bullet}(X; \mathbb{C}) \\ \downarrow & & \downarrow \\ \operatorname{Gr} A(Z_e \cap X_w) & \xrightarrow{\alpha_w} & H^{\bullet}(X_w; \mathbb{C}) \end{array} \tag{4.2}$$

where α is an isomorphism of graded $\mathbb{C}$-algebras, the vertical maps are the restrictions, and the filtration on $A(Z_e \cap X_w)$ defining $\operatorname{Gr} A(Z_e \cap X_w)$ is the image of the filtration on $A(Z_e)$. By [1], $A(Z_e)$ is graded and $\operatorname{Gr} A(Z_e) \cong A(Z_e)$. Similarly, by naturality, $A(Z_e \cap X_w)$ is graded (by §2) and isomorphic with $\operatorname{Gr} A(Z_e \cap X_w)$. Combining this with Theorem 2 finishes the proof of Theorem 1.

5. The semisimple deformation

5.1. In this section we will seek a method for defining the morphisms ψ_w without factoring through $A(Z_e \cap X_w)$. The method is motivated by the deformation argument employed in [8] and the computation of $H^{\cdot}(X_w; \mathbb{C})$ in terms of the orbit $W \cdot t$ of a regular $t \in \mathfrak{h}$. We first recall that computation. Put $W(w) = \{v \in W \mid v \leq w\}$, where $<$ is the partial order on W associated to B.

Theorem 7 [3, Theorem 4]. *Let $t \in \mathfrak{h}$ be regular. For each $w \in W$, there exists a degree-doubling isomorphism $\lambda_w: A(\mathfrak{h})/\operatorname{gr} I_{\mathfrak{h}}(W(w^{-1}) \cdot t) \to H^{\cdot}(X_w; \mathbb{C})$ such that if $X_v \subseteq X_w$, the diagram*

$$\begin{array}{ccc} A(h)/\operatorname{gr} I_{\mathfrak{h}}(W(w^{-1}) \cdot t) & \xrightarrow{\lambda_w} & H^{\cdot}(X_w; \mathbb{C}) \\ \downarrow & & \downarrow \\ A(\mathfrak{h})/\operatorname{gr} I_{\mathfrak{h}}(W(v^{-1}) \cdot t) & \xrightarrow{\lambda_v} & H^{\cdot}(X_v; \mathbb{C}) \end{array}$$

commutes, where the vertical maps are the natural restrictions, and $I_{\mathfrak{h}}(W(w) \cdot t)$ denotes the ideal of $W(w) \cdot t$ in $A(\mathfrak{h})$.

Suppose V is an affine variety in $\mathfrak{g}$ with defining ideal $I \subset A(\mathfrak{g})$. The *associated cone* $K(V)$ is by definition the affine variety in $\mathfrak{g}$ determined by the homogeneous ideal $\operatorname{gr} I$. A basic result of Borho and Kraft [8] says that $\operatorname{gr} I(G \cdot t) = I(\mathcal{N})$ for any regular semisimple element t of $\mathfrak{g}$. We will use this to determine the ideals $I(\mathcal{N}_w)$ in a useful way. Recall that s is the unique regular element of $\mathfrak{h}$ characterized by the condition $\langle \alpha_i, s \rangle = 2$ for $i = 1, \cdots, l$. For each $w \in W$, let $B_{w,s}$ denote the Zariski closure of $Bw^{-1}B \cdot s := \operatorname{Ad}(Bw^{-1}B)s$ in $\mathfrak{g}$. We will prove

Theorem 8. *Let w be an arbitrary element of W. Then the following hold*:

(1) *$B_{w,s}$ is a normal, irreducible subvariety of $\mathfrak{g}$ of dimension $l(w) + \dim_{\mathbb{C}} X$. In addition, $B_{w^{-1},s}$ is smooth if X_w is.*

(2) *$\mathcal{N}_w$ is an irreducible component of the associated cone $K(B_{w,s})$. In particular, $\operatorname{gr} I(B_{w,s}) \subseteq I(\mathcal{N}_w)$.*

(3) *$I(W(w^{-1}) \cdot s) = I(B_{w,s}) + I(\mathfrak{h})$, where $I(W(w) \cdot s)$ is the ideal in $A(\mathfrak{g})$ of functions vanishing on $W(w) \cdot s$.*

We obtain therefore a surjective degree-doubling algebra homomorphism

$$\nu_w: A(\mathfrak{g})/\operatorname{gr} I(B_{w,s}) + I(\mathfrak{h}) \to H^{\cdot}(X_w; \mathbb{C})$$

as the composition:

$$A(\mathfrak{g})/\operatorname{gr} I(B_{w,s}) + I(\mathfrak{h}) \xrightarrow[\text{map}]{\text{nat.}} A(\mathfrak{g})/\operatorname{gr}(I(B_{w,s}) + I(\mathfrak{h}))$$
$$= A(\mathfrak{g})/\operatorname{gr} I(W(w^{-1}) \cdot s) \xrightarrow{i^*} A(\mathfrak{h})/\operatorname{gr} I_{\mathfrak{h}}(W(w^{-1}) \cdot s) \xrightarrow{\lambda_w} H^{\cdot}(X_w; \mathbb{C}).$$

Here we have used the fact that $i^*(\operatorname{gr} I(W(w) \cdot s)) \subseteq \operatorname{gr} I_{\mathfrak{h}}(W(w) \cdot s)$. Clearly, i^* is an isomorphism, so ν_w is an isomorphism precisely when

(5.1) $$\operatorname{gr}(I(B_{w,s}) + I(\mathfrak{h})) = \operatorname{gr} I(B_{w,s}) + I(\mathfrak{h}).$$

Remark. Notice that unlike (4.1), (5.1) is not automatically true, the difference being that $I(B_{w,s})$ is neither homogeneous nor generated by linear functions.

Since $\operatorname{gr} I(B_{w,s}) \subseteq I(\mathcal{N}_w)$, there is a similar homomorphism

$$\sigma_w\colon A(\mathfrak{g})/\operatorname{gr} I(B_{w,s}) + I(\mathfrak{h}) \to A(\mathcal{N}_w \cap \mathfrak{h}).$$

Corollary. *For each* $w \in W$ *the following hold*: (1) *The following diagram is commutative*:

(5.2)
$$\begin{array}{ccc} & A(\mathfrak{g})/\operatorname{gr} I(B_{w,s}) + I(\mathfrak{h}) & \\ \sigma_w \swarrow & & \searrow \nu_w \\ A(\mathcal{N}_w \cap \mathfrak{h}) & \overset{\psi_w}{\rightarrow} & H^{\cdot}(X_w; \mathbb{C}). \end{array}$$

(2) ν_w, ψ_w *and* σ_w *are all isomorphisms if and only if* (5.1) *holds.*

(3) *If* $\operatorname{gr} I(B_{w,s}) + I(\mathfrak{h}) = I(\mathcal{N}_w) + I(\mathfrak{h})$, *then* $\psi_w = \nu_w$.

The point of (3) is that when $\operatorname{gr} I(B_{w,s}) = I(\mathcal{N}_w)$, ψ_w can be defined without factoring through $A(Z_e \cap X_w)$. It seems to be an interesting problem to determine when this is so.

Parts (2) and (3) of the corollary follow from the above discussion. We will first prove Theorem 8 and then establish commutativity of (5.2). We begin with a useful fact.

Lemma 4. *For each* $w \in W$, $B_{w,s} = \overline{Bw^{-1}B \cdot s}$.

Proof. Since s is regular and semisimple, $G \cdot s$ is G/H, where H is the algebraic torus corresponding to $\mathfrak{h}$. The orbit map $G \to G \cdot s$ sends closed H-invariant subsets of G onto closed subsets of $G \cdot s$. In particular, $\overline{Bw^{-1}B} \cdot s$ is closed in $G \cdot s$. Since $G \cdot s$ is closed, the lemma is established. q.e.d.

It follows that $B_{w,s}$ is isomorphic with $\overline{Bw^{-1}B/H}$ and is bundle over $\overline{Bw^{-1}B/B} = X_{w^{-1}}$ with fibre B/H. All the statements in Theorem 8 (1) follow immediately from this.

Next we will show that $\mathcal{N}_w$ is an irreducible component of $K(B_{w,s})$. In fact, it follows immediately from [14, Satz, p. 133] that

$$K(B_{w,s}) = \overline{\mathbb{C}^* B_{w,s}} \backslash \mathbb{C}^* B_{w,s} = \mathcal{N} \cap \overline{\mathbb{C}^* B_{w,s}}.$$

This implies that $\mathcal{N}_w \subseteq K(B_{w,s})$. For $e \in \mathbb{C}^* B \cdot s$, since $te + s$ and s are conjugate via B for any $t \in \mathbb{C}$, and this implies $\overline{\mathbb{C}^* B w^{-1} B \cdot s} \subseteq K(B_{w,s})$. As $B_{w,s}$ is irreducible, all irreducible components of $K(B_{w,s})$ have the same dimension, namely $\dim_{\mathbb{C}} B_{w,s}$ [14, p. 131]. In particular, as $\mathcal{N}_w$ also has this dimension and is irreducible, it is an irreducible component. We next prove (3). By Lemma 4 and the fact that $G \cdot s \cap \mathfrak{h} = W \cdot s$, it follows from $\overline{BwB} = \bigcup_{x \leq w} BxB$ that $B_{w,s} \cap \mathfrak{h} = W(w^{-1}) \cdot s$. Hence it suffices to show that the scheme theoretic intersection $B_{w,s} \cap \mathfrak{h}$ is reduced. But $B_{w,s} \cap \mathfrak{h}$ is locally closed in $G \cdot s \cap \mathfrak{h}$ and $G \cdot s \cap \mathfrak{h}$ is reduced, so it automatically follows that $B_{w,s} \cap \mathfrak{h}$ is also reduced. This completes the proof of Theorem 8. We omit the proof of the commutativity of (5.2).

Appendix
(after D. Peterson)

The notation of the Appendix is the same as that introduced in §§1 and 2. In particular $Z_e \cap X_w$ will refer to the scheme-theoretic intersection of the zero scheme Z_e, associated to the principal homogeneous nilpotent e in $\mathfrak{b}$, and the Schubert variety $X_w = \overline{BwB/B}$ in the flag variety $X = G/B$ of G. It was conjectured in [1] that the morphisms α_w of §1, sending the coordinate ring $A(Z_e \cap X_w)$ onto $H^{\bullet}(X_w; \mathbb{C})$, are isomorphisms. As mentioned in the Introduction, this has now been established when G is $SL_n(\mathbb{C})$ [5].

The purpose of this Appendix is to prove the following negative result, which impinges on Theorem 1 of this paper. Recall, $\{\alpha_1, \cdots, \alpha_l\}$ is the set of simple roots and $\omega_1, \cdots, \omega_l$ the corresponding fundamental dominant weights satisfying $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ where $(\ ,\ )$ is the inner product on $\mathfrak{h}^*$ and for $\alpha \in \Phi$, $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

Theorem. *Suppose $w \in W$ has length $l(w_0) - 1$ and write $w = w_0 r_i$, where w_0 is the longest element of W and r_i is the reflection corresponding to a simple root α_i. If the fundamental dominant weight ω_i corresponding to α_i is not miniscule, then the inclusion i_w of X_w into X induces an isomorphism between the coordinate rings $A(Z_e)$ and $A(Z_e \cap X_w)$. Consequently,* $\dim_{\mathbb{C}} \ker \alpha_w = \#\{v \in W \mid v \not\leq w\}$.

Recall that a dominant weight λ is called minuscule if $(\lambda, \beta^\vee)$ is 0 or 1 for all positive roots β.

From now on, G will be assumed to be simple. The theorem is based on the

Proposition. *Suppose w is as in the theorem. Then $I(\overline{BwB} \cap U^-)$ in $A(U^-)$ is a principal ideal generated say by $F_w \in A(U^-)$, and if ω_i is not minuscule, then F_w is in the kernel of the homomorphism from $A(U^-)$ onto $H^\cdot(X; \mathbb{C})$ defined in* §1.

Before proving this, let us show how to obtain the main result. Denote by $\overline{F}_w$ in $A(Z_e)$ the residue class of F_w. Since the kernel of the natural map i_w from $A(Z_e)$ into $A(Z_e \cap X_w)$ is the image of $I(\overline{BwB} \cap U^-)$ in $A(Z_e)$, this kernel is generated by $\overline{F}_w$. But $\alpha(\overline{F}_w) = 0$ by the Proposition, and therefore $\overline{F}_w = 0$, so i_w is an isomorphism. Using the long exact sequence of cohomology for the pair (X, X_w) shows that $\dim_{\mathbb{C}} \ker \alpha_w = \dim_{\mathbb{C}} H^\cdot(X, X_w; \mathbb{C})$ which is $\#\{v \in W \mid v \not\leq w\}$ as asserted. This proves the theorem.

We will break the proof of the proposition into a number of steps, the first being the definition of F_w. Assume ω_j is any fundamental dominant weight and $V_j = L(\omega_j)$ is the irreducible G-module with highest weight ω_j. Let V_j^* be the dual G-module. If μ is a weight of V (resp. V^*), let V_μ (resp. V_μ^*) denote the corresponding μ-weight space. Let v^+ and v^* be highest weight vectors in V_i and V_i^* respectively, and define $F_w \in A(U^-)$ as the lowest weight component of $u \cdot v^+$, i.e.,

$$F_w(u) = \langle v^*, u \cdot v^+ \rangle$$

for $u \in U^-$. Since the highest weight of V_i^* is $-w_0(\omega_i)$, $F_w(u)$ is simply the $w_0(\omega_i)$ component of $u \cdot v^+$.

The second step in the proof is to show that the variety $V(F_w)$ of F_w in U^- is $\overline{BwB} \cap U^-$. The proof given here can easily be modified to show that if F_w is viewed in $A(G)$, then $V(F_w) = \overline{BwB}$. Note first that $\overline{BwB} \cap U^-$ is a Zariski closed subset of U^- isomorphic with $X_w \cap U$, where U is the dual open cell in X containing B. The Schubert decomposition $\bigcup_{w' \leq w} Bw'B/B$ is a locally closed decomposition of X_w, and there is a corresponding one $\bigcup_{w' \leq w} (Bw'B \cap U^-)$ for $\overline{BwB} \cap U^-$.

Suppose now that $u \in V(F_w)$. Then $u \in Bw'B$ for some $w' \in W$, so we want to show $w' \leq w$. Since multiplication by w_0 reverses order and $w_0^2 = 1$, it suffices to show $w_0 w' \geq r_i$. If not, there exists a reduced expression $w_0 w' = r_{j_1} \cdots r_{j_k}$ not involving r_i, so $w_0 w'(\omega_i) = \omega_i$, i.e.,

$w'(\omega_i) = w_0(\omega_i)$. Since $u \in Bw'B$, $u \cdot v^+ = bn' \cdot v^+$ for some $b \in B$ and $n' \in w'H$, where H is the maximal torus corresponding to $\mathfrak{h}$. This means $b^{-1}u \cdot v^+$ is a lowest weight vector, which is impossible since, by the definition of $F_w(u)$, the lowest weight component of $u \cdot v^+$ is zero. Hence $w' \leq w$, and $V(F_w) \subseteq \overline{BwB} \cap U^-$.

To see the opposite inclusion, it suffices to show that $BwB \cap U^- \subset V(F_w)$ since, as shown above, $\overline{BwB \cap U^-} = \bigcup_{w' \leq w} Bw'B \cap U^- = \overline{BwB} \cap U^-$. Thus let $u \in BwB \cap U^-$. Then $u \cdot v^+ \in \sum_{\lambda \geq w(\omega_i)} V_\lambda$, so $\langle v^*, u \cdot v^+ \rangle = 0$ since $w(\omega_i) \neq w_0(\omega_i)$. (Indeed, $w_0(\omega_i) = wr_i(\omega_i) = w(\omega_i - \alpha_i)$.) Consequently $u \in V(F_w)$ and we have established that $V(F_w) = \overline{BwB} \cap U^-$.

In order to show that $I(\overline{BwB} \cap U^-)$ is (F_w), observe that since $\overline{BwB} \cap U^-$ is irreducible, it suffices to show that $V(F_w)$ has no multiple components. To establish this, we show that the differential dF_w is nonzero at all points of $BwB \cap U^-$. We begin with a

Lemma. *Let r_j be the simple reflection such that $w = r_j w_0 = w_0 r_i$. Then $w(\omega_i) + r_j(\omega_j) = 0$.*

Proof. r_j exists since $w_0 r_i w_0$ has length one. Clearly $w_0(\alpha_i) = -\alpha_j$, so $w(\omega_i) = w_0 r_i(\omega_i) = w_0(\omega_i) + \alpha_j$. Thus it suffices to show $w_0(\omega_i) = -\omega_j$. This follows since w_0 interchanges the positive and negative roots, $w_0^2 = 1$, and $(w_0(\omega_i), \alpha_j^\vee) = (\omega_i, w_0(\alpha_j^\vee)) = -(\omega_i, \alpha_i^\vee) = -1$. This completes the proof.

Let f_j be a nonzero element of $\mathfrak{g}_{-\alpha_j}$. For $u \in U^-$, we have

$$\left(\frac{d}{dt} F_w((\exp t f_j)u)\right)\bigg|_{t=0} = \langle v^*, f_j \cdot (u \cdot v^+)\rangle$$
$$= -\langle f_j \cdot v^*, u \cdot v^* \rangle.$$

But, for $u \in BwB \cap U^-$, the $w(\omega_i)$-component of $u \cdot v^+$ is nonzero, and, since v^* is a highest-weight vector of weight ω_j, $f_j \cdot v^*$ is a nonzero element of the one-dimensional space $V^*_{\omega_j - \alpha_j} = V^*_{r_j(\omega_j)}$. Since $w(\omega_i) + r_j(\omega_j) = 0$, we deduce that $\langle f_j \cdot v^*, u \cdot v^+ \rangle \neq 0$. Hence $dF_w \neq 0$ at every point of $BwB \cap U^-$, so $V(F_w)$ has no multiple components and we deduce that $I(\overline{BwB} \cap U^-) = (F_w)$.

It remains to show that $\alpha(\overline{F}_w) = 0$ if ω_i is not minuscule. Let P be the stabilizer of the line $\mathbb{C}v^+$. Note that $\dim_{\mathbb{C}} G/P$ is the number of positive roots involving α_i. There exists a commutative diagram of

$\mathbb{C}$-algebra homomorphisms

$$\begin{array}{ccc} A(U^-/U^- \cap P) & \xrightarrow{\alpha_P^\vee} & H^\cdot(G/P; \mathbb{C}) \\ \pi^* \downarrow & & \downarrow \pi^* \\ A(U^-) & \xrightarrow{\alpha^\vee} & H^\cdot(X; \mathbb{C}) \end{array}$$

where the vertical maps π^* are induced by the natural projections π, $\alpha^\vee$ is the obvious lift of α, and $\alpha_P^\vee$ is the lift of the isomorphism

$$\alpha_P \colon A(U^-/U^- \cap P)/I(Z_e) \to H^\cdot(G/P; \mathbb{C})$$

established for G/P in the same way as α for G/B ([4] and [1]). Since $F_w(up) = F_w(u)$ if $p \in U^- \cap P$, there exists an element G_w of $A(U^-/U^- \cap P)$ of the same degree as F_w such that $F_w = \pi^* G_w$. Since $\deg F_w = \mathrm{ht}(\omega_i - w_0(\omega_i))$, it will follow that $\alpha(F_w) = 0$ if $\mathrm{ht}(\omega_i - w_0(\omega_i)) > \dim_{\mathbb{C}} G/P$. Recall that if ω is a positive weight, say $\omega = \sum a_i \alpha_i$, then $\mathrm{ht}(\omega) = \sum a_i$.

Lemma. *Let* $\rho^\vee = \frac{1}{2}\sum_{\beta\in\Phi^+} \beta^\vee$. *Then for any positive weight* λ, $\mathrm{ht}(\lambda) = (\lambda, \rho^\vee)$.

Proof. Since $r_i(\rho^\vee) = \rho^\vee - \alpha_i^\vee$ for $i = 1, \cdots, l$, it follows that $(\rho^\vee, \alpha) = 1$ for all simple roots α. Hence $(\rho^\vee, \sum a_i \alpha_i) = \sum a_i$.

Now

$$\begin{aligned} \deg G_w = \deg F_w &= (\rho^\vee, \omega_i - w_0(\omega_i)) \\ &= (\rho^\vee, \omega_i) - (w_0\rho^\vee, \omega_i) = 2(\rho^\vee, \omega_i) \\ &\geq \#\{\beta \in \Phi^+ \mid (\beta^\vee, \omega_i) > 0\} = \dim_{\mathbb{C}} G/P \end{aligned}$$

with equality if and only if $(\beta^\vee, \omega_i) = 1$ for all β involving α_i. Consequently, if ω_i is not minuscule, then $\alpha_P^\vee(G_w) = 0$ which shows $\alpha(\overline{F}_w) = 0$ and finally completes the proof of the proposition.

Remark. The only simple group for which every fundamental dominant weight is minuscule is $\mathrm{SL}_n(\mathbb{C})$.

References

[1] E. Akyildiz & J. B. Carrell, *Cohomology of projective varieties with regular* SL_2 *actions*, Manuscripta Math. **58** (1987) 473–486.

[2] ——, *A generalization of the Kostant-Macdonald identity*, Proc. Nat. Acad. Sci. U.S.A. **86** (1989) 3934–3937.

[3] E. Akyildiz, J. B. Carrell & D. I. Lieberman, *Zeros of holomorphic vector fields on singular spaces and intersection rings of Schubert varieties*, Compositio Math. **57** (1986) 237–248.

[4] E. Akyildiz, J. B. Carrell, D. I. Lieberman & A. J. Sommese, *On the graded rings associated to holomorphic vector fields with exactly one zero*, Proc. Sympos. Pure Math., Vol. 40, Part 1, Amer. Math. Soc., Providence, RI, 1983, 55–56.
[5] E. Akyildiz, A. Lascoux & P. Pragacz, *Cohomology of Schubert subvarieties of* GL_n/P, J. Differential Geometry **35** (1992) 511–519.
[6] I. N. Bernstein, I. M. Gelfand, & S. I. Gelfand, *Schubert cells and cohomology of the space* G/P, Russian Math. Surveys **28** (1973) 1–26.
[7] A. Borel, *Sur la cohomologie des expaces fibres principaux et des espaces homogenes de groupes de Lie compacts*, Ann. of Math. (2) **57** (1953) 115–207.
[8] W. Borho & H. P. Kraft, *Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen*, Comment. Math. Helv. **54** (1979) 61–104.
[9] J. B. Carrell, *Vector fields, flag varieties and Schubert calculus*, Proc. Hyderabad Conf. Algebraic Groups (S. Ramanan, ed.), Manoj Prakashan, Madras, 1991, 23–58.
[10] J. B. Carrell & D. I. Lieberman, *Holomorphic vector fields and compact Kaehler manifolds*, Invent. Math. **21** (1973) 303–309.
[11] M. Demazure, *Invariants symmetriques des groupes de Weyl et torsion*, Invent. Math. **29** (1973) 287–301.
[12] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963) 327–404.
[13] ——, *On Whittaker vectors and representation theory*, Invent. Math. **48** (1978) 101–184.
[14] H. P. Kraft, *Geometrische Methoden in der Invariantentheorie*, Aspects of Math., Vieweg, 1984.
[15] D. Mumford, *Algebraic geometry* I: *Complex projective varieties*, Grundlehren Math. Wiss. 221, Springer, Berlin, 1976.
[16] D. H. Peterson, *Loop groups, Schubert calculus and the principal nilpotent*, in preparation.

UNIVERSITY OF BRITISH COLUMBIA