J. DIFFERENTIAL GEOMETRY
37 (1993) 535-568

THE LIMITING ETA INVARIANTS OF COLLAPSED THREE-MANIFOLDS

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In this paper, we study the limiting eta invariants of collapsed Riemannian manifolds. These invariants were defined and previously studied in [9]. In particular, we prove a conjecture of Cheeger and Gromov which asserts their rationality in the three-dimensional case, provided that the collapse has bounded covering geometry.

0. Introduction

Let M be an n-dimensional complete Riemannian manifold with sectional curvature bounded in absolute value, say $|K| \leq 1$. Let $\alpha(g)$ denote one of the following geometric quantities associated to g: the diameter of M, the supremum of injectivity radii at all points of M, or the volume of M. Roughly speaking, M is said to be *sufficiently* $\alpha(g)$*-collapsed*, if $\alpha(g)$ is smaller than a sufficiently small constant depending only on dimension of M. M is said to admit an $\alpha(g)$*-collapse*, if there exists a family of metrics $\{g_\delta\}$ on M, $0 < \delta \leq 1$, such that the sequence $\{\alpha(g_\delta)\}$ converges to zero as $\delta \to 0$ (here we assume the sectional curvatures of all g_δ are bounded in absolute value by one).

The basic questions about the interplay between the collapsing geometry and the topology of M are the following:

(1) What kind of structures and invariants can be attached to a sufficiently α-collapsed metric or to an α-collapse ?

(2) Does a sufficiently α-collapsed metric imply the existence of an α-collapse ?

Starting with [21], there has been considerable progress on the above questions; for instance, Gromov's theorem of almost flat manifolds [21], i.e., manifolds whose diameter is sufficiently collapsed, the F-structure theory for sufficiently collapsed injectivity radii [10], [11], the bundle structure theorems and their applications for manifolds which collapse to a

Received September 25, 1991 and, in revised form, March 25, 1992.

manifold of lower dimension [17]–[20] and more recently, the Nil-structure theorem [7].

Important progress has been made, concerning the implications of the existence of a volume collapse, in the work of [8], [9], [12] and [31]. In [8], [9] and [12], Cheeger and Gromov generalized the type of Chern-Gauss-Bonnet theorem to open manifolds and studied topological properties of the η-invariant in the sense of Atiyah-Patodi-Singer [1], [2].

A volume collapse $\{g_\delta\}$ is said to have *bounded covering geometry* (briefly, BCG), if the family of pullback metrics $\{\tilde{g}_\delta\}$ on the universal covering of M has a uniform lower bound on the injectivity radius.

Theorem 0.1 [9]. *Let N be a closed oriented $(4k-1)$-dimensional manifold. Suppose M admits a volume collapse $\{g_\delta\}$ with BCG. Then the limit of the η-invariants associated to the volume collapse,*

$$\eta_{(2)}(N) = \lim_{\delta \to 0} \eta(N, g_\delta) \tag{0-1}$$

exists (thus $\eta_{(2)}(N)$ is a topological invariant of N).

Note that without the condition of BCG Theorem 0.1 fails completely (see [36, Example 4]). Concerning the value of $\eta_{(2)}(N)$, the following conjecture is due to Cheeger and Gromov.

Conjecture 0.2. *$\eta_{(2)}(N)$ is a rational (for a volume collapse with BCG).*

Now we start to state our results as follows:

Theorem 0.3. *For $k = 1$, under the conditions of Theorem 0.1, $\eta_{(2)}(N)$ is a rational.*

Our approach to Conjecture 0.2 is to show that the existence of some volume collapsed with BCG implies the existence of a "nice volume collapse with BCG", i.e., on N for which $\eta_{(2)}(N)$ is computable.

To begin with, the work of [11] implies that, on a three-manifold, the existence of a volume collapse is equivalent to the existence of a positive rank F-structure (see below). We find that the injectivity (see below) of such a structure is equivalent to the additional assumption, BCG, on the volume collapse (Theorem 0.4). Furthermore, from an injective F-structure, one is able to construct an *invariant* volume collapse with BCG, i.e., a volume collapse that is compatible with the structure. It turns out that this invariance of the volume collapse with BCG enables one to use the residue formula of [36] to compute $\eta_{(2)}(N)$ explicitly, and therefore concludes $\eta_{(2)}(N)$ is rational.

The F-structure was defined in [10] and [11]. Roughly, an F-structure on a manifold M can be thought of as a family of local torus actions on M satisfying certain consistency conditions on overlaps so that M is partitioned into orbits of these local torus actions (see §1). The existence

of an F-structure on a manifold is equivalent to existence of an injectivity radius collapse [10], [11]. An F-structure is said to be *injective*, if the fundamental group of each orbit injects into the fundamental group of the total space. As shown below, in dimension three, an injective F-structure fully encodes the topological information associated to a volume collapse with BCG.

Theorem 0.4. *There exists a constant $\epsilon > 0$ such that if a closed three-manifold N admits a Riemannian metric g satisfying*

(0.1) $|K(N, g)| \leq 1$,

(0.2) $\mathrm{Injrad}(x, g) < \epsilon$, *for* $x \in N$,

(0.3) $\widetilde{\mathrm{geo}}(N, g) \geq 1$ (BCG),

then either N admits an injective F-structure for infinite $\pi_1(N)$ or N (or double cover of N) is homeomorphic to a lens space for finite $\pi_1(N)$. In particular, if N admits a volume collapse with BCG, then N admits an injective F-structure.

Remark 0.5. The proof of Theorem 0.4 uses some theorems from three-dimensional topology. We point out that an injective F-structure cannot be obtained from a metric satisfying (0.1)–(0.3) by means of the *general* geometrical constructions using either the local short geodesic loops as in [11] or the frame bundle technique as in [18] and [19] (see Example 4.9). Further work is required .

An F-structure is said to be *polarized*, if the local orbits have the same dimension as the tori which act locally. Using a polarized F-structure, one is able to construct an invariant volume collapse ([10], also Theorem 1.8). An injective F-structure is automatically a polarized F-structure. The basic feature of an injective F-structure is the local splitting property (Proposition 2.4). This property guarantees that the invariant volume collapse constructed by a slight modification of procedure in [10] has BCG (Theorem 2.5). A consequence of this fact and Theorem 0.4 is that, in computing the invariant $\eta_{(2)}(N)$, one can assume that the volume collapse g_δ is compatible with an injective F-structure, $\mathscr{F}$, of N. The advantage of this invariance is that one is able to use the result of [36] to explicitly compute $\eta_{(2)}(N)$.

In [36], the residue formulas for characteristic numbers of closed manifolds in [4] and [5] have been generalized to compact manifolds whose boundary supports a polarized F-structure $\mathscr{F}$. A minor extension of the result of [36], when restricted to the case of signature form, asserts that the limiting η-invariant $\eta(N, \mathscr{F})$, associated to the invariant volume collapse constructed as in [10] by using a polarized F-structure $\mathscr{F}$ on N, exists and is independent of the particular invariant volume collapse. In

addition, if N is the boundary of some compact $4k$-manifold M and $\mathscr{F}$ can be extended to M with all orbits closed submanifolds, then $\eta(N, \mathscr{F})$ is rational (Theorem 1.11).

In our special circumstances, by using the invariant plumbing technique [28] it is not hard to construct a compact 4-manifold M, $\partial M \simeq N$, and extend $\mathscr{F}$ to M simultaneously (Theorem 6.1). As a consequence of above results, we obtain

Theorem 0.3′. *Let the assumptions be as in Theorem* 0.3, *and let* $\mathscr{F}$ *be an injective F-structure on* N. *Then*

$$\eta_{(2)}(N) = \eta(N, \mathscr{F}). \tag{0-2}$$

In particular, $\eta_{(2)}(N)$ *is rational.*

In higher dimensions, we also verify Conjecture 0.2 under the additional strong assumption that an injective F-structure exists (Theorem 2.8).

From [36], $\eta(N, \mathscr{F})$ is a cobordism invariant depending not only on N but also on $\mathscr{F}$. Thus (0-2) may suggest that in a certain sense at most one injective F-structure can exist on a given three-manifold. To be precise, two F-structures of N, $\mathscr{F}_1$ and $\mathscr{F}_2$, are said to be *weakly equivalent*, if there is a polarized F-structure which has both $\mathscr{F}_1$ and $\mathscr{F}_2$ as substructures. By [36], $\eta(N, \mathscr{F})$ is determined only by the weak isomorphism class of $\mathscr{F}$. By making use of the classification of graph manifolds given in [34] and [35], we prove the following rigidity result for injective F-structures.

Theorem 0.6. *Let* N *be a closed three-manifold whose finite cover is not homeomorphic to a solvable manifold or to* $S^2 \times S^1$. *Then up to a weak isomorphism,* N *admits at most one injective F-structure.*

Remark 0.7. Combining (0-2) with Theorem 0.6, we conclude that, in dimension three, the limiting η-invariant $\eta_{(2)}(N)$ is a cobordism invariant of the weak isomorphism class of the injective F-structure of N (which is unique except in the above-mentioned cases).

Starting from (0-2), one is able to write down an explicit formula for $\eta_{(2)}(N)$ in terms of $(N, \mathscr{F})$. In particular, the noninteger part of $\eta_{(2)}(N)$ is contributed by the exceptional orbits of $\mathscr{F}$ (see (3.4)). The result of Theorem 0.6 implies that the residue formula for $\eta(N, \mathscr{F})$ is also intrinsic in most cases. In this paper, we only give the explicit residue formula for an oriented injective Seifert manifold with Seifert invariants (see [28])

$$N = \{b; (o, g); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}. \tag{0-3}$$

Theorem 0.8. *Let* N *be an oriented injective Seifert manifold whose Seifert invariant is given in* (0-3). *Then*

$$\text{(0-4)}\quad \eta_{(2)}(N) = -\sum_{i=1}^{r} s_i + \delta(b, r) + \frac{1}{3}(b+r) + \frac{1}{3}\sum_{i=1}^{r}\sum_{j=1}^{s_i} b_{ij} + \frac{1}{3}\sum_{i=1}^{r}\frac{\alpha_{i,s_i-1}}{\alpha_i},$$

where $\frac{\alpha_i}{\alpha_i - \beta_i} = [b_{i1}, \cdots, b_{is_i}]$ *is the continued fraction* $(b_{ij} \geq 2)$, $\alpha_{ij} = b_{ij}\alpha_{i,j-1} - \alpha_{i,j-2}$ *with* $\alpha_{i,0} = 1$, $\alpha_{i1} = b_{i1}$ *and*

$$\delta(b, r) = \begin{cases} 1, & \text{if } b + r \leq 0, \\ -1, & \text{if } b + r > 0. \end{cases}$$

For a real number $D > 0$, let $\mathscr{M}^3(D)$ be the collection of closed orientable three-manifolds which admit a volume collapse with BCG and for which the diameters are bounded uniformly by D. Our last result concerns the finiteness of the noninteger part of $\eta_{(2)}(N)$ for $N \in \mathscr{M}^3(D)$.

Theorem 0.9. *For each* $D > 0$,

$$\{\eta_{(2)}(N) \bmod Z \mid N \in \mathscr{M}^3(D)\}$$

is a finite set.

Remark 0.10. Note that for each $D > 0$, $\mathscr{M}^3(D)$ contains infinitely many topological types, for instance, Gromov's almost flat manifolds of dimension three [21]. Also, the size of $\{\eta_{(2)}(N) \bmod Z \mid N \in \mathscr{M}^3(D)\}$ depends on D. For instance, if $D \leq \exp(-\exp(9))$, then $\mathscr{M}^3(D)$ consists of Gromov's almost flat manifolds [21]. From Theorem 0.8, we have

$$\{\eta_{(2)}(N) \bmod Z \mid N \in \mathscr{M}^3(D)\} = \{0, \tfrac{1}{3}, \tfrac{2}{3}\}.$$

This paper is organized as follows.

In §1, we define the notion of F-structure and state the main results of [10] and [11] on collapses and F-structures. We also derive a residue formula for a certain volume collapse. This is an easy consequence of the work of [36]. The remainder of the paper is based on these results.

In §2 we study both topological and geometrical aspects of an injective F-structure in general. As an application, we verify Conjecture 0.3 in higher dimensions by assuming the existence of an injective F-structure.

In §§3–5, we systematically study injective F-structures on a three-manifold. Theorem 0.4 is proved in §4, and Theorem 0.6 in §5.

In §6, we will give the proof for Theorem 0.3′.

§7 is devoted to showing Theorem 0.7.

In §8, we prove Theorem 0.9.

1. Preliminaries

In this section we will briefly recall the main results of [10] and [11] and some results of [36].

a. Collapses and F-structures.

Definition 1.1. Let M be a manifold. An F-structure $\mathscr{F}$ of M is determined by a collection $\{(\tilde{U}_i, U_i, T^{k_i}, \phi_i, \psi_i)\}$ (called an atlas of $\mathscr{F}$), which satisfies the following conditions

(1.1) $\{U_i\}$ is a locally finite open cover of M,

(1.2) $\pi_i: \tilde{U}_i \to U_i$ is a finite Galois covering with Galois (deck transformation) group G_i,

(1.3) T^{k_i} is a k_i-dimensional torus, and $\phi_i: T^{k_i} \to \mathrm{Diff}(\tilde{U}_i)$ is an effective and smooth action,

(1.4) $\psi_i: G_i \to \mathrm{Aut}(T^{k_i})$ is a homeomorphism satisfying

$$g_i(\phi_i(\gamma_i)(x)) = \phi_i((\psi_i(g_i)(\gamma_i))(g_i x))$$

for each $g_i \in G_i$, $\gamma_i \in T^{k_i}$ and $x \in \tilde{U}_i$,

(1.5) if $U_i \cap U_j \neq \varnothing$, there is a common covering V_{ij} of $\pi_i^{-1}(U_i \cap U_j)$ and $\pi_j^{-1}(U_i \cap U_j)$ so that the lifting actions of T^{k_i} and T^{k_j} on V_{ij} commute.

Remark 1.2. From (1.4), the orbits of local action ϕ_i on U_i are well defined. By (1.5), we define the *orbit* $\mathscr{O}_x$ of $\mathscr{F}$ at $x \in M$ as the union of all orbits of ϕ_i through x. The *rank* of $\mathscr{F}$ at x is defined as $\dim(\mathscr{O}_x)$.

Definition 1.3. Let $\mathscr{F} = \{(\tilde{U}_i, U_i, T^{k_i}, \phi_i, \psi_i)\}$ be an F-structure.

(1.6) $\mathscr{F}$ is said to have *positive rank*, if it has positive rank at every point.

(1.7) $\mathscr{F}$ is said to be a *T-structure*, if $\tilde{U}_i = U_i$ for all i.

(1.8) $\mathscr{F}$ is said to be *pure*, if $k_i = k_j$ for all i, j.

(1.9) $\mathscr{F}$ is said to be *polarized*, if the for each i, the local action ϕ_i has a finite isotropy group at each point.

Definition 1.4. Let $\mathscr{F}$ be an F-structure (not necessarily of positive rank). An orbit $\mathscr{O}$ of $\mathscr{F}$ is said to be *singular*, if $\mathscr{O}$ is a singular orbit of some local torus action at x. The singular set $Z(\mathscr{F})$ of $\mathscr{F}$ is defined as the union of all singular orbits of $\mathscr{F}$.

Definition 1.5. A *polarization* of a positive rank F-structure is a collection of connected subgroups $H_i \subset T^{k_i}$ such that the dimension of each H_i-orbit is equal to $\dim(H_i)$. If H_i is a compact subgroup of T^i for all i, we call this polarization a *polarized substructure.*

The existence of a sufficiently injectivity radius collapsed metric is equivalent to the existence of a positive rank F-structure.

Theorem 1.6 [10], [11]. *Let M be a complete n-dimensional manifold with $|K| \leq 1$. Then there exists a constant $\epsilon_n > 0$ depending only on n, such that if the injectivity radii of M are smaller than ϵ_n at every point, then M admits a positive rank F-structure $\mathscr{F}$ which is almost compatible with the metric. Conversely, suppose a manifold admits a positive rank F-structure. Then it admits an invariant injectivity radius collapse.*

Remark 1.7. The first part of Theorem 1.6 has been generalized considerably in a recent paper [7].

The geometric consequence of the existence of a polarized F-structure is

Theorem 1.8 [10]. *Let M be a manifold.*

(1.10) *Suppose M admits a polarized F-structure $\mathscr{F}$ outside some compact subset C. Then M admits a complete metric of $|K| \leq 1$ and finite volume which is compatible with $\mathscr{F}$.*

(1.11) *As in* (1.10), *if C is the empty set, then M admits an invariant volume collapse.*

Remark 1.9. By inspecting the proof of Theorem 1.8 in [10] one sees that Theorem 1.8 remains valid if "polarized F-structure" is replaced by "polarization".

Remark 1.10. It is an open question whether the converse of Theorem 1.6 is true. However, in dimension three, the affirmative answer follows directly from Theorem 1.6 (see Proposition 3.1).

For examples of collapsing and F-structures, we refer to [10], [11] and [20].

b. Limiting eta invariants associated to a polarized F-structure. Based on the work of [35] on secondary geometric invariants, we will easily obtain a residue formula for the limiting eta-invariant with respect to a certain invariant volume collapse for $\mathscr{F}$.

Let (N, g) be a closed orientable $(2n-1)$-dimensional Riemannian manifold which is the boundary of some compact orientable $2n$-manifold M. For any extension $\tilde{g}$ of g to M such that $\tilde{g}$ is the product metric near N and any $O(n)$-invariant polynomial P such that the associated cohomology class is integral, the associated secondary geometric P-invariant of N, defined by

$$SP(N, g) = \frac{1}{(2\pi)^n} \int_{(M, \tilde{g})} P(\tilde{\Omega}) \mod Z,$$

depends only on the metric of N and P. The secondary geometric

invariants were studied in [13], [14], [1], [2], etc. The results of [4] and [5] imply that if N admits a nonvanishing Killing vector field, then the above secondary geometric invariants can be made into topological invariants by modifying (0-3) by the integration of a closed form α_X (which is determined by the Killing field) over N ($P(\Omega) - \alpha_X$ is called a Bott-form). This fact was generalized in [36] to the situation where N admits a polarized F-structure. Yang constructed the generalized Bott-form $\widetilde{P}(\Omega)$ ($= P(\Omega)$ modified by a canonical boundary form) in terms of $\mathscr{F}$ and proved its value on M,

$$P[M, \mathscr{F}] = \int_M \widetilde{P}(\Omega), \tag{1-1}$$

is the topological invariant depending only M, P and $\mathscr{F}$. We refer to $P[M, \mathscr{F}] \bmod Z$ as a *secondary topological invariant.* In particular, if $\mathscr{F}$ has an extension $\widetilde{\mathscr{F}}$ to M with singularity $Z(\widetilde{\mathscr{F}})$, then $\widetilde{P}(\Omega)$ is actually exact away from $Z(\widetilde{\mathscr{F}})$, i.e., $\widetilde{P}(\Omega) = d\alpha$ on $M - Z(\widetilde{\mathscr{F}})$. Thus, (1-1) can be written as

$$P[M, \mathscr{F}] = \int_M \widetilde{P}(\Omega) = \sum_i \operatorname{Res}(\alpha, Z_i), \tag{1-2}$$

where Z_i is a component of $Z(\widetilde{\mathscr{F}})$, and $\operatorname{Res}(\alpha, Z_i)$ is the residue of α on Z_i (see [36]).

There is an another approach to (1-2) by using the collapsing theorems of [10]. (This was actually the original idea of Cheeger and Gromov.) Put $M_\infty = M \cup (N \times [0, +\infty))$. First, using $\mathscr{F}$ one is able to construct a complete invariant metric g on M_∞ which satisfies (0.1) and (0.2) [10] and

(1.12) $|\mathrm{II}(N \times \{r\})| \leq C_2$, where $\mathrm{II}(\ )$ is the second fundamental form of $(\ )$.

(1.13) $\operatorname{Vol}(N \times \{r\}) \to 0$ as $r \to +\infty$.

It turns out that the "collapsing" ((1.12) and (1.13)) on the level sets, $N \times \{r\}$, kills the modified boundary term. Hence the integral

$$P[M, \mathscr{F}] = \int_{(M_\infty, g)} P(\Omega) \tag{1-3}$$

is a topological invariant determined only by M, P and $\mathscr{F}$ (compare (0-1)). Define the map $\phi_r : N \to M_\infty$ by $\phi_r(x) = (x, r) \in N \times [0, +\infty) \subset M_\infty$, and denote the pullback metric $g_\delta = \phi_r^*(g_\infty)$, $r = \delta^{-1}$. If we restrict attention to $P = P_L$, Hirzebruch's L-polynomial, and apply the Atiyah-Patodi-Singer index formula to $M_r = M \cup (N \times [0, r])$, from (1.12), (1.13)

we deduce

$$\begin{aligned}\int_{(M_\infty, g_\infty)} P_L(\Omega) &= \lim_{r\to\infty}\int_{(M_r, g_\infty|_{M_r})} P_L(\Omega)\\ &= \lim_{\delta\to 0}(\sigma(M_{\delta^{-1}}) + \eta(N, g_\delta) + \mathrm{II}_\sigma(N, g_\delta))\\ &= \sigma(M) + \lim_{\delta\to 0}\eta(N, g_\delta).\end{aligned} \tag{1-4}$$

Combining (1-2) with (1-3) and (1-4) gives

Theorem 1.11. *Let N be a closed oriented $(4k-1)$-manifold which is the boundary of some compact $4k$-manifold, and let $\mathscr{F}$ be a polarized F-structure on N. Then for the invariant volume collapse g_δ, the associated limiting η-invariant, $\eta(N, \mathscr{F}) = \lim_{\delta\to 0}\eta(N, g_\delta)$, exists and is a topological invariant depending only on N and $\mathscr{F}$. Moreover, if there is an F-structure $\widetilde{\mathscr{F}}$ of M such that $\widetilde{\mathscr{F}}|_N = \mathscr{F}$, then*

$$\lim_{\delta\to 0}\eta(N, g_\delta) = \sigma(M) + \sum_i \mathrm{Res}(\alpha, Z_i). \tag{1-5}$$

In particular, if all orbits of $\mathscr{F}$ are closed submanifolds of M, then $\eta(N, \mathscr{F})$ is rational.

Remark 1.12. We point out that Theorem 1.11 is true for *arbitrary* invariant volume collapse (see [31] for details).

2. Injective F-structures and collapsing with BCG

In order to study the topological implications of the existence of a volume collapse with BCG, we introduce the notion of the *injectivity* of an F-structure. If we assume the existence of an injective F-structure, Conjecture 0.2 can be easily proved (Theorem 2.8).

Definition 2.1. Let $\mathscr{F} = \{(\widetilde{U}_i, U_i, T^{k_i}, \phi_i, \psi_i)\}$ be a positive rank F-structure on N. For any $x \in M$ and $(\widetilde{U}_i, U_i, T^{k_i}, \phi_i, \psi_i) \in \mathscr{F}$, $x \in U_i(\dim(\mathscr{O}_x) = k_i)$, consider the following diagram:

$$\begin{array}{ccc}(T^{k_i}, e) & \xrightarrow{\phi_i} & (\widetilde{U}_i, \tilde{x})\\ & & \big\downarrow \pi_i\\ & & (M, x)\end{array}$$

$\mathscr{F}$ is said to be *injective* if the induced map $(\tilde{\phi}_{i_0})_*: \pi_1(T^{k_{i_0}}, e) \to \pi_1(M, x)$ is injective at every $x \in M$.

Remark 2.2. The notion of injective F-structure is a kind of generalization of the notion of injective torus action studied in [15], [16] and [24].

Definition 2.3. A polarization, $\mathscr{H}$, of a positive rank F-structure, $\mathscr{F}$, is said to be injective, if $\dim(H_x) = \operatorname{rank}(\operatorname{Im}(\tilde{\phi}_{i*}))$ at every point x.

Injective F-structures are polarized. The basic topological feature of injectivity is the following local splitting phenomenon which occurs in the universal covering space.

Proposition 2.4. *Let $\mathscr{F}$ be an injective F-structure of M, and let $\pi: \widetilde{M} \to M$ be the universal covering. For any $x \in M$ there exists an invariant tubular neighborhood U of $\mathscr{O}_x$ such that $\pi^{-1}(V)$ is homeomorphic to $S_x \times R^k$, where S_x is a slice of $\mathscr{O}_x$ at x which is homeomorphic to a $(n-k)$-dimensional ball $(k = \dim(\mathscr{O}_x))$.*

Proof. In this proof we shall use some elementary facts from compact transformation group theory (see [6] for reference). For $x \in M$, let $(\tilde{U}_i, U_i, \phi, \psi)$ be a chart of $\mathscr{F}$ containing x, and let $\tilde{x} \in \tilde{U}$ with $\pi(\tilde{x}) = x$. Let Γ be the finite isotropy subgroup of T^k at $\mathscr{O}_{\tilde{x}}$, and choose $S_{\tilde{x}}$ to be the slice of $\mathscr{O}_{\tilde{x}}$ at $\tilde{x}$ which is homeomorphic to a $(n-k)$-ball. Then $S_{\tilde{x}}$ determines the invariant tubular neighborhood $\tilde{V}$, $\tilde{V} \approx S_{\tilde{x}} \times_\Gamma T^k$. The universal covering of $\tilde{V}$ is $S_x \times R^k$, and the lifting group R^k of T^k acts on $S_x \times R^k$ by the addition in R^k. We have the following commutative diagram:

$$\begin{array}{ccc} R^k \times (S_x \times R^k) & \xrightarrow{\tilde{\phi}} & S_x \times R^k \\ \downarrow{\scriptstyle \pi_1 \times \pi'} & & \downarrow{\scriptstyle \pi'} \\ T^k \times (S_x \times_\Gamma T^k) & \xrightarrow{\phi} & S_x \times_\Gamma T^k \end{array}$$

Let $\pi: \widetilde{M} \to M$ be the universal cover. Put $V = \pi_i(\tilde{V})$. To see that V is the desired invariant neighborhood, it suffices to verify that $\pi^{-1}(V)$ is simply connected. If not, pick $\tilde{x}' \in \pi^{-1}(V)$ with $\pi(\tilde{x}') = x$, and let γ be any nontrivial loop in $\pi^{-1}(V)$ at $\tilde{x}'$. Then $\pi \circ \gamma$ determines a nontrivial element in $\pi_1(V, x)$. Since V is homotopically equivalent to $\mathscr{O}_x$, $\pi \circ \gamma$ is also nontrivial in $\pi_1(\mathscr{O}_x, x)$. Note that the injectivity of the local T^k-action implies that $\pi_1(\mathscr{O}_x, x)$ is a torsion free subgroup of $\pi_1(M, x)$. Thus $\pi \circ \gamma$ has infinite order in $\pi_1(M, x)$. Consequently, the lift γ of $\pi \circ \gamma$ is not a closed path. This contradicts our choice for γ, and the proof is complete. q.e.d.

The geometric consequence of the existence of injective F-structures is (compare with Theorem 1.8)

Theorem 2.5. *Let M be a manifold.*

(2.1) *Suppose M admits an injective F-structure $\mathscr{F}$ outside some compact subset C of M. Then M admits a complete metric which is compatible with $\mathscr{F}$, with $|K| \leq 1$, finite volume and BCG near infinity.*

(2.2) *As in (2.1), if C is the empty set, then M admits an invariant volume collapse with BCG.*

Proof. First, note that $\mathscr{F}$ is polarized. The construction of the metric as in (2.1) and the volume collapse $\{g_\delta\}$ $(0 < \delta \leq 1)$ as in (2.2) are given in the proof of Theorem 1.8 [10]. What we shall do is to check BCG, which is actually a consequence of the local splitting property of $\mathscr{F}$ (Proposition 2.4). Here we also need to use the result of [10] to construct a suitable initial metric (see below). Since the proofs of (2.1) and (2.2) are essentially the same, we only give the proof of (2.2).

Let $\pi: \widetilde{M} \to M$ be the universal covering, and let $\tilde{g}_\delta = (\pi)^*(g_\delta)$ be the pullback metrics. Pick $\tilde{x} \in \widetilde{M}$ with $\pi(\tilde{x}) = x$. Let $T_x^\perp(\mathscr{O}_x)$ be the orthogonal complement of the tangent space $T_x\mathscr{O}_x$ of $\mathscr{O}_x$ in T_xM, the tangent space of M at x, with respect to the initial metric g_1. Clearly, for each x, we can find $\rho_x > 0$ such that the slice of $\mathscr{O}_x$ as in Proposition 2.4 is given by $S_x = \exp_x(B_{\rho_x}^\perp)$, where $B_{\rho_x}^\perp$ is the ball of radius ρ_x in $T_x^\perp\mathscr{O}_x$. By Proposition 2.4, $\pi^{-1}(V) \simeq S_x \times R^k$. For each δ, let $g_\delta^\perp$ be the restriction of g_δ on $T_x^\perp\mathscr{O}_x$. Consider the convergence of pointed metric space $(\widetilde{M}, \tilde{x}, \tilde{g}_\delta)$ with respect to the Gromov-Hausdorff distance [23]. Following the proof of Theorem 1.7 as in [10] one sees that when restricting to the $(\pi^{-1}(V), \tilde{x})$, the sequence $\{\tilde{g}_\delta\}$ converges to a C^∞ product metric $\tilde{g}_0$ in $\pi^{-1}(V)$, $\tilde{g}_0 = \tilde{g}_0' + \tilde{g}_0''$, where $\tilde{g}_0'' = fg_e$ (f is a C^∞ function on S_x, g_e is the *Euclidean metric* and $\tilde{g}_0'$ is the limit of $\{g_0^\perp\}$). If $\mathscr{F}$ is pure F-structure, then $\tilde{g}_0' = g_\delta^\perp = g_1^\perp$ $(0 < \delta \leq 1)$. It follows that $\operatorname{Injrad}(\tilde{x}, \tilde{g}_\delta) \geq \rho_x$ as $\delta \to 0$. In case $\mathscr{F}$ is not pure, one still has $\operatorname{Injrad}(\tilde{x}, \tilde{g}_\delta) \geq \rho_x$ since there $\tilde{g}_\delta^\perp$ is constructed basically by spanning $g_1^\perp$.

By the above discussion it is clear that if the initial metric is chosen such that $\rho_x \geq 1$ for all $x \in M$, then $\{g_\delta\}$ has BCG. The existence of such metric is obvious if M is compact. In the case M is noncompact the construction of the metric was given in [10]. Thus the proof is complete.

Remark 2.6. It is easy to check that Theorem 2.5 remains valid if one modifies the statement by replacing the injective F-structure by an injective polarization.

Example 2.7. The following are some examples of the manifolds which admit injective F-structures:

(2.3) The injective Seifert fiber space with fiber flat manifolds which is defined (see [24]). In fact, this Seifert fibration actually coincides with a pure injective F-structure. In particular, a three-dimensional Seifert fiber space with infinite fundamental group is injective [32].

(2.4) A flat n-dimensional manifold is covered by an n-torus. The multiplication on T^n is injective. An almost flat manifold has a finite cover which is diffeomorphic to a nilmanifold [21]. The center of a nilmanifold acts injectively on itself. Thus an almost flat manifold admits pure injective F-structure.

(2.5) A complete manifold with $-b^2 \leq K \leq -a^2 (a \neq 0)$ and finite volume admits a pure injective F-structure outside some compact subset. This is because this manifold is homeomorphic to the interior of a compact manifold whose boundary components are infranilmanifolds (see [5] for details).

(2.6) [10] Let M^n be a closed oriented manifold, and let $f: K(\pi, 1)$ be the classifying map, where $\pi \simeq \pi_1(M^n)$. We call M^n *essential*, if the fundamental class $[M^n] \in H_n(M, R)$ satisfies $f_*([M^n]) \neq 0$. Suppose $\mathscr{F}$ is a pure positive rank F-structure on an essential manifold M^n. Then $\mathscr{F}$ is injective.

We conclude this section by giving an application of Theorem 2.5.

Theorem 2.8. *Let N be a closed oriented $(4k-1)$-manifold which is the boundary of some compact orientable manifold M. Suppose M admits an F-structure $\mathscr{F}$ satisfying the following conditions:*

(2.7) *$\mathscr{F}|N$ is injective.*

(2.8) *Each component of $Z(\mathscr{F})$ is a closed submanifold of M.*

Then $\eta_{(2)}(N)$ is rational.

Proof. First, from (2.2) of Theorem 2.5, we construct an invariant volume collapse with BCG, $\{g_\delta\}$, on N. $\eta_{(2)}(N) = \lim_{\delta \to 0} \eta(N, g_\delta)$ by Theorem 0.1. Using the invariance of $\{g_\delta\}$ and (2.8), we get

$$\eta_{(2)}(N) = \lim_{\delta \to 0} \eta(N, g_\delta) = \eta(N, \mathscr{F}|_N),$$

and conclude that $\eta(N, \mathscr{F}|_N)$ is rational (Theorem 1.11).

3. Polarized T-structures on three-manifolds

As a general preparation for the sequel, we restrict attention to F-structures on collapsed three-manifolds. A basic and well-known fact for positive rank F-structures on a three-manifold is

Proposition 3.1. *Let $\mathscr{F}$ be a positive rank F-structure on a three-manifold. Then $\mathscr{F}$ has a substructure which is polarized.*

Proof. Let $\mathscr{F} = \{(\widetilde{U}_\alpha, U_\alpha, T^{k_\alpha}, \phi_\alpha)\}$. If there is a chart, $(\widetilde{U}_\alpha, U_\alpha, T^{k_\alpha}, \phi_\alpha)$, with $k_\alpha = 3$, then $U_\alpha \simeq N$ and thus $N \simeq T^3$ up to a finite cover.

Now we assume that $k_\alpha \leq 2$ for all α. Let Z be a component of the singular set of $\mathscr{F}$, and let $(\widetilde{U}_1, U_1, T^2, \phi_1, \psi_1), \cdots, (\widetilde{U}_r, U_r, T^2, \phi_r, \psi_r)$ be all the charts which contain Z. By taking a common cover U for $\widetilde{U}_1, \cdots, \widetilde{U}_r$ and lifting the T^2-action to U, we may assume $\mathscr{F}$ is locally a pure T^2-structure. By $\dim(N) = 3$ and $\dim(Z) \geq 1$ it is easy to see that Z consists of a single S^1-orbit; that is, $Z \simeq S^1$ is isolated. Clearly, $\mathscr{F}$ contains a polarized substructure near Z. q.e.d.

For a positive rank F-structure $\mathscr{F}$ of M, the basic questions about the topology of $\mathscr{F}$ concern the existence axiom (in particular those which are polarized, pure and injective). Let $\pi: \widetilde{M} \to M$ be any finite covering. Then $\widetilde{M}$ has a natural F-structure $\widetilde{\mathscr{F}}$ induced by π which has the same properties as $\mathscr{F}$ does (one may think $\widetilde{\mathscr{F}}$ is the pullback sheaf, $\pi^*(\mathscr{F})$, by using the sheaf-theoretic definition of F-structure as in [10], [11]). Thus, as far as the basic properties are concerned, one is free to work on a finite covering space. In particular, one can assume the base manifold is orientable. In the three-dimensional case, this principle can be strengthened to reduce the study of F-structures to that of T-structures. To be precise, we introduce the following.

Definition 3.2. Let M be a manifold, and let $\mathscr{F}_i$ $(i = 1, 2)$ be two F-structures of M. $\mathscr{F}_1$ is said to have the same orbit structure as $\mathscr{F}_2$, if there is a homeomorphism of N which preserves the orbits.

If two F-structures have the same orbit structure, then roughly speaking they have the same basic properties.

Let $\mathscr{F}$ be a polarized F-structure on an orientable three-manifold N. Let N', N'' be the union of one-dimensional, respectively, two-dimensional, orbits of $\mathscr{F}$. Write $\bar{N}' = \bigcup N_i$ and $\bar{N}'' = \bigcup X_j$ (N_i, X_j are connected components).

Assume every two-dimensional orbit of $\mathscr{F}$ is a topological torus. Since N is orientable, $X_j \approx I_j \times T^2$, I_j a closed interval. We observe the following:

(3.1) each N_i is a Seifert fiber space with torus boundary components such that the Seifert fiber structure is trivial near the boundary ∂N_i,

(3.2) by identifying the corresponding boundary components of N_i''s in pairs via suitable gluing maps, ϕ_k, one obtains N again.

From [32], it is easy to see that any Seifert fibration on a compact three-manifold can be viewed as the orbits of some pure S^1-structure. This is because any S^1-fibered solid torus (and also solid Klein bottle) admits an S^1-action leaving the fibration invariant. Thus, if we let $\mathscr{T}_i$ be the pure S^1 T-structure of N_i, then obviously these $\mathscr{T}_i$ generate a mixed T-structure $\mathscr{T}$ of N. Clearly, $\mathscr{T}$ has the same orbit structure as $\mathscr{F}$ does. For convenience, in the remainder of the discussion, we shall use the following terminology.

(3.3) We shall call the set $\mathscr{D}(N, \mathscr{T}) = \{(N_i, \mathscr{T}_i), \phi_k\}$ the *natural decomposition of* $\mathscr{T}$. Sometimes we just denote it by $\mathscr{D}(N, \mathscr{T}) = \{N_i, \phi_k\}$. Each N_i is called a *piece* of $\mathscr{D}(N, \mathscr{T})$.

(3.4) The exceptional orbits of the N_i are called the *exceptional orbits of* $\mathscr{F}$.

Now, assume $\mathscr{F}$ contains a two-dimensional orbit $\mathscr{O}_x \simeq K^2$, a Klein bottle. By the assumption on orientation, a neighborhood U of $\mathscr{O}_x$ is a twisted I-bundle over $\mathscr{O}_x$. Clearly, N is homeomorphic to a double of U. Further, T^3 double covers N.

Summarizing the above discussion, we have the following.

Proposition 3.2. *Let N be an orientable three-manifold, and let $\mathscr{F}$ be a polarized F-structure on N. Then there is a polarized T-structure $\mathscr{T}$ on N which has the same orbits as $\mathscr{F}$, provided N is not a double of twisted I-bundle over K^2.*

We now return to (3.3) and (3.4). If each N_i is a S^1-fiber bundle, i.e., N_i has no exceptional orbits, then the family of the embedded two-torus, $\{\partial N_i\}$, without counting multiplicities, determines a so-called *graph structure* of N in the sense of [34] and [35] (see also [28]). Given an exceptional orbit, say, $\mathscr{O}_k$, $k = 1, 2, \cdots, r$, we cut out a small tubular neighborhood V_i, around each $\mathscr{O}_k$. Then the family of embedded tori, $\{\partial N_i, \partial V_k\}$, without counting mutiplicities, determines a graph structure for N. A three-manifold which possesses a graph structure is called a *graph manifold.* Thus we identify N as the graph manifold. On the other hand, an orientable graph manifold admits a decomposition as (3.3). These S^1-fiber bundle structures on the N_i determine an obvious T-structure of N. We have actually found a one-to-one correspondence between the following two sets:

$$\{\text{graph manifolds}\} \Leftrightarrow \{\text{three-manifolds admitting an F-structure}\}.$$

The topological classification for graph manifolds was obtained in [34] and [35]. We shall use these results to explore the rigidity of injective T-structures on three-manifolds in §5.

4. Injective T-structures on three-manifolds

In this section, we will prove Theorem 0.4. Our basic tools are some results from three-dimensional topology and combinational group theory. The main work is to establish a criterion which enables us to modify a polarized T-structure so as to obtain one which is injective (Theorem 4.3).

Let Σ be a closed surface embedded in a three-manifold N. Σ is said to be incompresssible if Σ is not S^2 or P^2, and the induced map $i_*: \pi_1(\Sigma) \to \pi_1(N)$ is injective ($i: \Sigma \to N$).

Lemma 4.1. *The boundary of a Seifert fiber space is incompressible unless it is homeomorphic to a solid torus or a solid Klein bottle.*

Proof. See [32].

Theorem 4.2. *Let N be a closed three-manifold, and $\mathscr{T}$ be a polarized T-structure of N. Then $\mathscr{T}$ is injective if and only if the natural decomposition $\mathscr{D}(N, \mathscr{F})$ contains no piece which is a solid torus.*

Proof. In one direction the proof is obvious. Assume that $\mathscr{F}$ is injective. This amounts to saying that the boundaries of every piece in $\mathscr{D}(N, \mathscr{F})$ is incompressible. From Lemma 4.1 we conclude that no piece is a solid torus.

On the other hand, assuming that no N_i is a solid torus is equivalent to assuming that the boundaries of every N_i are incompressible. Let $i: \partial N_i \to N$ be the natural inclusion ($i = 1, 2, \cdots, r$). It suffices to show that the induced map $i_*: \pi(\partial N_i) \to \pi(N)$ is injective.

We start with N_1 and glue the component $(\partial N_1)_1$ to its partner. We denote by $\widetilde{N}_1$ the result. Note that $\widetilde{N}_1$ is formed by gluing $(\partial N_1)_1$ with either some $(\partial N_1)_j$, another component of ∂N_1, or $(\partial N_i)_j$ ($i \neq 1$).

In the former case, again using the induced map, $(\phi_{1,1}^{1,j})_*: \pi_1((\partial N_1)_1) \to \pi_1((\partial N_1)_j)$ is an isomorphism since the components of ∂N are incompressible. Thus, the fundamental group, $\pi_1(\widetilde{N}_1)$, is actually an HNN extension of $\pi_1(N_1)$ relative to the subgroups $\pi_1((\partial N_1)_1)$, $\pi_1((\partial N_1)_j)$ and $(\phi_{1,1}^{1,j})_*$ (see Chapter IV of [25] for details). Denote by $\tilde{i}$ the natural inclusion from N_1 to $\widetilde{N}_1$; from Theorem 4.2 of [25] we see that $\tilde{1}_*: \pi_1(N_1) \to \pi_1(\widetilde{N}_1)$ is injective. Therefore the map $(\tilde{1} \circ 1)_*: \pi_1(\partial N_1) \to \pi_1(\widetilde{N}_1)$ is injective.

In the second case, still from the incompressibility of every ∂N_k in N_k it is easy to see that $\pi_1(\widetilde{N}_1)$ is the free product of $\pi_1(N_1)$ and $\pi_1(N_i)$ with amalgamation $(\phi_{1,i}^{1,j})_*$ (see [25]). Consequently, $\tilde{k}_*: \pi_1(N_k) \to \pi_1(\widetilde{N}_1)$ are injective for $k = 1, i$ [25, Theorem 2.6]. As before, we see that $(\tilde{k} \circ k)_*: \pi_1(\partial N_k) \to \pi_1(\widetilde{N}_1)$ are injective ($k = 1, i$).

Note that in the above two cases, $\partial \widetilde{N}_1$ are incompressible in $\widetilde{N}_1$. Thus we are able to replace N_1 or N_1 and N_i by $\widetilde{N}_1$ and continue the above gluing process. Apparently, by an obvious induction argument we then finish the proof.

Theorem 4.3. *Under the same assumption as in Theorem* 4.2, *if every* S^1*-orbit of* $\mathscr{F}$ *is not homotopically contractible, then either* N *admits an injective T-structure when* $\pi_1(N)$ *is infinite or* N *is homeomorphic to a lens space up to a double covering when* $\pi_1(N)$ *is finite.*

Let $\mathscr{D}(N, \mathscr{T}) = \{N_1, \cdots, N_r\}$ be a natural decomposition of $(N, \mathscr{T})$. According to Theorem 4.2, if no N_i is homeomorphic to a solid torus, then $\mathscr{F}$ is already injective. Now we assume that some of N_i's are solid tori. Basically, what we want to do is to modify $\mathscr{T}$ to be injective. We first deal with the following simple case.

Lemma 4.4. *Theorem* 4.3 *is true if* $N_1, \cdots, N_{r-1}$ *are all solid tori.*

Proof. Clearly, N has a Seifert fibration by extending (uniquely) the Seifert fibration of N_r to each N_i, $1 \leq i \leq n-1$. Here we emphasize the fact that S^1-orbits of the Seifert fibration are not homotopically trivial is crucial for the extension. From the classification result for Seifert manifolds as in [28] we immediately obtain the desired result. q.e.d.

Note that in the above proof, if $\pi_1(N)$ is infinite, then we have actually modified $\mathscr{T}$ on N_i ($i = 1, 2, \cdots, r-2$) to be injective. In general, the modification process turns out to be more complicated. We illustrate this by presenting the following example.

Example 4.5. This example shows that a solid torus may support mixed T-structures whose natural decompositions are rather complicated.

Let Σ be the surface formed by deleting three disjoint disks from a sphere. Take two solid tori $D_i \times S^1$ ($i = 1, 2$) and attach them to $\Sigma \times S^1$ in such a way that the S^1-fiber of $\Sigma \times S^1$ is not identified with ∂D_i. Clearly, the result is a Seifert fiber space with compressible boundary. Thus it is a solid torus (Lemma 4.1).

For any natural number k, take k ($k \geq 2$) copies of $\Sigma \times S^1$, $\Sigma_1 \times S^1, \cdots, \Sigma_k \times S^1$, and perform $k+1$ solid tori, $D_0 \times S^1, D_1 \times S^1, \cdots, D_k \times S^1$; we proceed with the following gluing. The first step is to attach $D_0 \times S^1$ and $D_1 \times S^1$ to $\Sigma_1 \times S^1$ as above. We denote the result by $D_{0,1} \times S^1$. The second step is to attach $D_{0,1} \times S^1$ and $D_1 \times S^1$ to $D_2 \times S^1$ in the same manner. Clearly, by successively attaching, we finally obtain a solid torus, $D \times S^1$. Note that the S^1-rotation on every $\Sigma_i \times S^1$ generates a

mixed T-structure $\mathscr{T}$. It is apparent that $\mathscr{T}$ has a natural decomposition $\mathscr{D}(D \times S^1, \mathscr{F}) = \{D_i \times S^1, \Sigma_j \times S^1, 0 \le i \le k, 1 \le j \le k\}$.

Motivated by the above example, we introduce the following:

Definition 4.6. Let $\mathscr{D}(N, \mathscr{T})$ be a natural decomposition of $(N, \mathscr{F})$. A *solid torus chain* $\mathscr{T}_{\max}$ is a maximal subset of $\mathscr{D}(N, \mathscr{F})$ whose total space is homeomorphic to a solid torus.

Remark 4.7. We point out that a solid torus chain may appear naturally in the construction for $\mathscr{T}_g$ as in [10] from a sufficiently collapsed metric.

From the proof of Theorem 4.2 one observes that every solid torus chain contains at least one piece that is a solid torus, $D_{i_0} \times S^1$. To form a solid torus chain, one can start with $D_{i_0} \times S^1$ and proceed as follows. If the partner of $D_{i_0} \times S^1$ is also a solid torus, then the solid torus chain is $D_{i_0} \times S^1$ itself. Otherwise, let $\widetilde{N}_1$ be the gluing result of $D_{i_0} \times S^1$ with its partner. Next, one attaches all the solid tori which are the partners of $\widetilde{N}_1$ and denotes the result by $\widetilde{N}_2$. If $\widetilde{N}_2$ is not a solid torus, then the solid torus chain is again $D_{i_0} \times S^1$ (Theorem 4.2), otherwise one repeats the same process starting at $\widetilde{N}_2$. Clearly, after finitely many steps the process ends with the desired solid torus chain containing $D_{i_0} \times S^1$. Also, from the above construction, it is clear that if two distinct torus solid torus chains of $\mathscr{D}(N, \mathscr{F})$ have nonempty intersection, then N, formed by gluing two solid tori along their boundaries, is homeomorphic to a lens space. Thus we have actually proved the following lemma.

Lemma 4.8. *Let N be orientable, and let $\mathscr{T}_{\max}$ and $\mathscr{T}'_{\max}$ be two distinct solid torus chains in $\mathscr{D}(N, \mathscr{T})$. If $\mathscr{T}_{\max} \cap \mathscr{T}'_{\max} \neq \varnothing$, then N is homeomorphic to a lens space.*

Proof of Theorem 4.3. Let $\{\mathscr{T}_{\max,k}\}$ be the collection of all solid torus chains of $\mathscr{D}(N, \mathscr{T})$. We construct a new decomposition, $\mathscr{D}_1(N) = \{\widetilde{N}_1, \cdots, \widetilde{N}_s\}$, which is obtained by simply replacing the $\{\mathscr{T}_{\max,k}\}$ by their total spaces $D_k \times S^1$ and attaching every $D_k \times S^1$ to its partner N_{i_k}. If $s = 1$, i.e., $\widetilde{N}_1 \approx N$, then using Lemma 4.4 we complete the proof. In the case of $s \ge 2$, from our construction of $\mathscr{D}_1(N)$ we observe that no $\widetilde{N}_i$ is a solid torus and every $\widetilde{N}_i$ has the Seifert fiber structure from extending those of N_{i_k}. Finally by applying Theorem 4.2 to $\mathscr{D}_1(N)$ we complete the proof. q.e.d.

With the above topological preliminaries, now we are able to give a simple proof of Theorem 0.4.

Proof of Theorem 0.4. We choose the constant ϵ to be the critical injective radius ϵ_3 as in Theorem 1.5. Let g be the Riemannian metric satisfying conditions (0.1), (0.2) and (0.3). Then we can find a polarized T-structure $\mathscr{T}_g$ (Theorem 1.5). According to Theorem 4.3, we only need to know that any S^1-orbit of $\mathscr{T}_g$ is not homotopically trivial.

Suppose that there is a point $x \in N$ with $\mathscr{O}_x$ homotopically trivial in N. Note that according to [11], locally $\mathscr{O}_x$ is not homotopically trivial. More precisely, there exists a constant r, $\epsilon < r < 1$, such that $\mathscr{O}_x$ is not homotopically trivial in the metric ball $B_r(x)$ at x.

On the other hand, we look at the universal covering $\pi: \tilde{N} \to N$ with the pullback metric $\tilde{g} = \pi^* g$. Picking a point $\tilde{x} \in \pi^{-1}(x)$ we lift the $\mathscr{O}_x$ at $\tilde{x}$. From BCG and length$(\mathscr{O}_x) < r < 1$ we conclude that $\widetilde{O}_{\tilde{x}}$ is homotopically trivial in $B_r(\tilde{x})$. It follows that $\mathscr{O}_x$ must be homotopically trivial in $B_r(x)$. Since this contradicts our assumption on $B_r(x)$, the proof is complete. q.e.d.

We conclude this section by giving an example showing that, in general, injective T-structures may not be found from a volume collapse with BCG by means of the general geometrical constructions given in [11] or [18] and [19].

Example 4.9. Take the standard S^2 and S^1 and form $S^2 \times S^1$. Let H be the S^1 subgroup of SO(3), defined by

$$H = \left\{ \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; 0 \le t \le 2\pi \right\}.$$

Then $T^2 = H \times S^1$ acts as isometries on $S^2 \times S^1$ by H acting on the first factor and multiplication on the second factor. Take any R^1-subgroup R_θ of T^2 and split the metric as $g = g_0 + g_1$, where g_0 is the restriction of g to the orbits of R^1_θ and g_1 its orthogonal complement. We then construct a volume collapse $\{g_\delta\}$ by multiplying δ^2 to g_0. It is not hard to see that the pullback volume collapse on the universal covering $S^2 \times R^1$ is equivalent to the one obtained by shrinking the metric $\tilde{g}$ along the direction of the R^1-flow lines (the lift action of H on $R^2 \times S^1$) while keeping the metric in the orthogonal direction fixed. Therefore $\{g_\delta\}$ has BCG. Note that the limit space of $(S^2 \times S^1, g_\delta)$ is an interval since R^1_θ is dense in T^2. This implies that for any sufficiently collapsed metric g_δ, the T-structure $\mathscr{T}_{g_\delta}$, constructed by using either the local short geodesics

technique [11] or the frame-bundle technique [18], [19], has to contain T^2-orbits. Clearly, $\mathscr{T}_{g_\delta}$ are not injective for every small δ.

5. Rigidity of injective T-structures on three-manifolds

As pointed out in §0, Corollary 0.8 suggests that the injective T-structures on a three-manifold, if any, are essentially the same in some sense. To be precise, we introduce the following:

Definition 5.1. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two positive rank T-structures and $\mathscr{D}(N, \mathscr{T}_i) = \{N_{i,1}, \cdots, N_{i,r_i}\}$ be the natural decompositions. $\mathscr{T}_1$ is said to be isomorphic to $\mathscr{T}_2$ if $r_1 = r_2$ and there is a homeomorphism ϕ of N, such that $\phi|_{N_{1,j}}: N_{1j} \to N_{2,j}$ preserves the local actions. The exceptional orbits of $\mathscr{T}$ consist of the exceptional orbits of N_i as a Seifert fiber space.

Definition 5.2. Let $\mathscr{T}$ be an injective T-structure of N and $\mathscr{D}(N, \mathscr{T}) = \{N_1, \cdots, N_r\}$ be the natural decomposition. $\mathscr{T}$ is said to be *simple*, if $\mathscr{T}$ contains no piece which is homeomorphic to $I \times T^2$.

We will see that simple injective T-structures are rigid on most three-manifolds. Namely, they are unique up to isomorphism (Lemma 5.4). This generalizes the well-known fact that most three-manifolds admit an unique S^1-fibration (if any) (see [32]). Before proceeding with the proof, let us first look at all the exceptional cases.

Example 5.3. The injective T-structures on $S^1 \times S^2$ are in one-to-one correspondence with fixed-point-free S^1-action on $S^1 \times S^2$. It is obvious that every such structure is a simple injective S^1-structure. In particular, there are infinitely many nonisomorphic classes of simple F-structures.

Example 5.4. Let N be a three-dimensional solve-manifold and let $\mathscr{T}$ be a simple T-structure of N. Then $\mathscr{T}$ is pure. If $\mathscr{T}$ is of rank three, then a finite cover of N is T^3. If $\mathscr{F}$ is of rank two, then N is the total space of a T^2-bundle over S^1. If $\mathscr{T}$ is of rank one, then N is actually a nilmanifold with the center acting on N.

Lemma 5.5. *Suppose N is a closed three-manifold which has no finite cover homeomorphic to $S^2 \times S^1$ or a solve-manifold. Then N admits a simple injective T-structure which is unique up to isomorphism (if any).*

Proof. Let $\mathscr{T}_i$ be simple T-structures on N and

$$\mathscr{D}(N, \mathscr{T}_i) = \{N_{i,1}, \cdots, N_{i,r_i}\} \qquad (i = 1, 2)$$

be the natural decompositions ($i = 1, 2$). It suffices to show that $r_1 = r_2$ and that there is a homeomorphism ϕ of N such that $\phi|_{N_{1,j}}: N_{1,j} \to N_{2,j}$

preserves the Seifert fiber structures $(j = 1, 2, \cdots, r_1)$. Actually this follows from the classification result of [34], [35] and [32] (see also [28, §8]).

To see this, we further decompose $\mathscr{D}(N, \mathscr{T}_i)$ as follows. Let $\{\mathscr{O}_{i,1}, \cdots, \mathscr{O}_{i,k_i}\}$ denote the set of the exceptional orbits of $\mathscr{T}_i$ (see (3.4)). For each $\mathscr{O}_{i,j}$, let $U_{i,j}$ be an open invariant tubular neighborhood of it. We can assume that $\{U_{i,j}\}$ are pairwise disjoint and each $\mathscr{O}_{i,j} \subset N_{i,j'}$ for some $N_{i,j'}$ $(i = 1, 2)$. Put $U_i = \bigcup_1^{k_i} U_{i,j}$. Then the Seifert fiber structure on each $N_{i,j} - U_i$ is actually a S^1-bundle structure. This implies that the boundary components $\{\partial(N_{i,j} - U_i)\}$ (without counting the mutiplicities) give the graph structure of N (see the discussion at the end of §3). Since the $\mathscr{T}_i$ are simple injective T-structures, it is not hard to check that this graph structure is actually simple in the sense of [34] and [35] (see also [28, §8]). Thus, by Theorem 6, §8 of [28], we obtain the following:

(i) The number of boundary components of $\{\partial(N_{1,j} - U_i)\}$ is the same as $\{\partial(N_{1,j} - U_i)\}$ (without counting the multiplicities).

(ii) There is a homeomorphism ϕ_1 of N such that

$$\phi_1(\{\partial(N_{1,j} - U_1)\}) \approx \{\partial(N_{2,j} - U_2)\}.$$

It follows from (i) that $k_1 = k_2$ and hence $r_1 = r_2$. By (ii), we can assume, by properly rearranging the indices, that $\phi_j: N_{1,j} \to N_{2,j}$ is a homeomorphism for $j = 1, \cdots, r_1$. Note that each $N_{i,j}$ is neither a solid torus nor an I-bundle over T^2 or K^2 since the $\mathscr{T}_i$ are simple injective T-structures (Theorem 4.2). Thus, from Theorem 3.9 of [32], we can find a fiber preserving isomorphism $\tilde{\phi}_i$ which is isotopic to ϕ_j $(j = 1, \cdots, r_1)$. Gluing these $\{\tilde{\phi}_i\}$ further in an obvious way, we then obtain the desired homeomorphism ϕ. Consequently, $\mathscr{T}_1 \simeq \mathscr{T}_2$. q.e.d.

From (1-5), one easily concludes the following.

Lemma 5.6. *Let N be a closed oriented three-manifold. Suppose N admits a polarized T-structure $\mathscr{T}$. If $\mathscr{T}'$ is a substructure of $\mathscr{T}$, then $\eta(N, \mathscr{T}') = \eta(N, \mathscr{T})$.*

Motivated by Lemma 5.6, we introduce

Definition 5.7. Let $\mathscr{T}_i$ $(i = 1, 2)$ be two polarized T-structures on N. $\mathscr{T}_1$ is said to be *weakly isomorphic* to $\mathscr{T}_2$, if there is a polarized T-structure $\mathscr{T}$ of N which has both $\mathscr{T}_1$ and $\mathscr{T}_2$ as substructures.

Lemma 5.8. *Let $\mathscr{T}$ be an injective T-structure on N. Then $\mathscr{T}$ is weakly isomorphic to a simple injective T-structure $\mathscr{T}_s$ of N.*

Proof. Let $\mathscr{D}(N, \mathscr{T}) = \{N_1, \cdots, N_r\}$ be the natural decomposition of $(N, \mathscr{T})$. If we assume that $\mathscr{T}$ is not simple, then we have the subset

$\mathscr{D}_1 = \{N_{i_1}, \cdots, N_{i_k}\}$ of $\mathscr{D}(N, \mathscr{T})$ consisting of the pieces which are homeomorphic to $I \times T^2$. We reglue the pieces in $\mathscr{D}_1$ together and obtain the set $\mathscr{D}_2 = \{\tilde{N}_{i_1}, \cdots, \tilde{N}_{i_k}\}$, whose elements are also homeomorphic to $I \times T^2$. By replacing $\mathscr{T}|_{\tilde{N}_{i_j}}$ by the obvious T^2-action on $\tilde{N}_{i_j}$ and retaining the rest of $\mathscr{T}$, we obtain an injective T-structure $\tilde{\mathscr{T}}$ which clearly contains $\mathscr{T}$ as a substructure.

In order to find a simple substructure $\mathscr{T}_s$ of $\tilde{\mathscr{T}}$, we modify $\mathscr{T}|_{\tilde{N}_{i_j}}$ as follows: let N_i and N_j be the partners of $\tilde{N}_{i_j}$. (Note that if $\mathscr{D}_2 = \{\tilde{N}_{i_1}\}$ and $\tilde{N}_{i_1}$ has no partner but itself, then $\tilde{\mathscr{T}}$ is a pure T^2 structure on a solve-manifold. In particular, $\tilde{\mathscr{T}}$ is simple.)

(i) If the S^1-fibration on the disjoint union, $N_i \cup N_j$, can be extended to $N_i \cup \tilde{N}_{i_j} \cup N_j$, then we replace $\mathscr{T}|_{N_i \cup \tilde{N}_{i_j} \cup N_j}$ by this extended S^1-fibration,

(ii) otherwise, we return $\mathscr{T}|_{\tilde{N}_{i_j}}$ and the pure T^2-structure as before.

From our construction for $\tilde{\mathscr{T}}$ and $\mathscr{T}_s$ it is routine to check that $\mathscr{T}_s$ is a simple substructure of $\tilde{\mathscr{T}}$.

Proof of Theorem 0.10. Combine Lemma 5.6 and Lemma 5.8.

6. Proof of Theorem 0.3 and filling three-manifolds by equivariant plumbing

In this section we will prove the following result:

Theorem 6.1. *Let N be a closed orientable three-manifold and let $\mathscr{T}$ be a polarized T-structure on N. Then there is a compact manifold M_N and a T-structure $\tilde{\mathscr{T}}$ (possibly with singularities) such that*

(6.1) $\partial M_N \simeq N$,

(6.2) $\tilde{\mathscr{T}}|_N \simeq T$,

(6.3) *every component of the singularity $Z(\tilde{\mathscr{T}})$ is an embedded submanifold.*

If we assume Theorem 6.1 for the moment, then we can finish the proof of Theorem 0.3′ (Theorem 0.3) as follows (compare with Theorem 2.8).

Proof of Theorem 0.3. Let N be as in Theorem 0.3. By Theorem 0.4 and (2.2) of Theorem 2.5, there exist an injective T-structure $\mathscr{T}$ on N and an invariant volume collapse with BCG. According to Theorem 0.1 and Theorem 1.11,

$$\eta_{(2)}(N) = \lim_{\delta \to 0} \eta(N, g_\delta) = \eta(N, \mathscr{T}).$$

From Theorem 6.1, we fill N with the compact 4-manifold M and extend $\mathscr{T}$ to $\widetilde{\mathscr{T}}$ that satisfies (6.1)–(6.3). Consequently, $\eta(N, \mathscr{T})$ is rational (Theorem 1.11). q.e.d.

Note that (6.1) is well known from the fact that the oriented cobordism ring of three-manifolds is trivial [26]. To insure that we can extend $\mathscr{T}$ to M_N in a canonical way (this is needed in order to obtain an intrinsic formula for $\eta_{(2)}(N)$), we will construct M_N by the so-called *equivariant plumbing technique.* We refer to [28] for facts concerning equivariant plumbing. First, we state a lemma of [28] which we shall use in the proof of Theorem 6.1.

Lemma 6.2. *Let $L(p, q)$ be a lens space, and let $M(p, q)$ be the result of the equivariant plumbings according to the graph given by the continued fraction $q/p = \Gamma[b_1, \cdots, b_r]$. Then $\partial M(p, q) = L(p, q)$. Moreover, any S^1-action on $L(p, q)$ extends uniquely to $M(p, q)$.*

Proof. See [28].

Proof of Theorem 6.1. It is natural that the filling must take into account $\mathscr{T}$. Let $\mathscr{D}(N, \mathscr{T}) = \{N_1, \cdots, N_r, \phi_k\}$ be the natural decomposition. Note that each orientable Seifert fiber space has a decomposition into a union of Seifert manifolds. Here Seifert manifold means the three-manifold which admits a fixed point free S^1-action. Further, each Seifert manifold decomposes into the product spaces: $\{\Sigma \times S^1, (D_i \times S^1, \phi_i)\}$, where Σ is a surface (perhaps not orientable), and $\phi_i \in SL(2, Z) : \partial D_i \times \to \partial\Sigma \times S^1$ are the gluing maps (see [29]). Thus, we can start with a decomposition of $(N, \mathscr{T})$, say $\{\Sigma_j \times S^1, D_{ij} \times S^1, \phi_k\}$. We shall construct M_N in three steps below:

(6.4) Fill in each $\Sigma_j \times S^1$ by $\Sigma_j \times D$ and fill in $D_{ij} \times S^1$ by $D_{ij} \times D^2$. We observe that any S^1-action on $\Sigma_j \times S^1$ or $\Sigma_j \times D$ extends uniquely to its filling.

(6.5) Glue these $\Sigma_j \times D$'s and $D_{ij} \times D$'s together via the ϕ_k and denote the result by M_1. Note that M_1 is not a manifold since each gluing map ϕ_k produces a "hole" whose boundary is a lens space, say, $L(p_k, q_k)$. We denote by $\mathscr{T}_k'$ the restriction of $\mathscr{T}$ to $L(p_k, q_k)$.

(6.6) For each k, let $M(p_k, q_k)$ be the filling of $L(p_k, q_k)$ as in Lemma 6.2. We then close up the holes of M_1 by filling in the corresponding $M(p_k, q_k)$'s, and denote the result by M_N. Finally, by extending $\mathscr{T}_k'$ to $M(p_k, q_k) \hookrightarrow M_N$ we obtain $\widetilde{\mathscr{T}}$. From our construction for M_N, it is obvious that $\partial M_N \simeq N$ and (6.3) is satisfied. q.e.d.

By making the additional assumption that each piece in the natural decomposition is a Seifert manifold with orientable base, we are able to give an intrinsic construction of M_N. This can be used to derive an intrinsic residue formula for $\eta_{(2)}(N)$. Here we need a more general lemma than Lemma 6.2.

Lemma 6.3. *Let* $N = \{b; (o, g); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}$, *and let* M_N *be the result of the equivariant plumbings according to the graph determined by the above Seifert invariants of* N. *Then* $\partial M_N \simeq N$. *Moreover, any* S^1*-action on* N *extends uniquely to* M_N.

Remark 6.4. *Let*

$$N = \{b; (o, g); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}.$$

Then

$$-N = \{-b - r; (o, g); (\alpha_1, \alpha_1 - \beta_1), \cdots, (\alpha_r, \alpha_r - \beta_r)\}.$$

The computation for the signatures shows that in general $\sigma(M_N) \neq \sigma(M_{-N})$ *(see* §7*). Thus the filling* M_N *depends on the orientation.*

We begin with the natural decomposition $\mathscr{D}(N, \mathscr{T}) = \{N_1, \cdots, N_r, \phi_k\}$ (see (3.3)). Now we construct M_N in three steps which are similar to (6.4)–(6.6):

(6.7) Consider Seifert manifold N_i. Since every component of ∂N_i can be trivialized as $D_{ik} \times S^1$, we form a closed Seifert manifold $\widetilde{N}_i$ by gluing $D_{ik} \times D$ to each component of ∂N_i. By the classification result of [29], we have

$$\widetilde{N}_i = \{b_i; (o, g_i); (\alpha_{i1}, \beta_{i1}), \cdots, (\alpha_{ir}, \beta_{ir})\}.$$

By applying Lemma 6.3, we fill in $\widetilde{N}_i$ with M_{N_i}.

(6.8) Corresponding to each ϕ_k, there are two attached pieces, $D_{ik} \times D$, $D_{jk} \times D$ (note that it may happen that $i = j$). We may normalize ϕ_k as

$$\phi_k = \begin{pmatrix} u_k & v_k \\ p_k & q_k \end{pmatrix} : (\partial N_i)_k \times S^1 \to (\partial N_j)_k \times S^1; \qquad \det(\phi_k) = -1.$$

Then we construct equivariant plumbings from $D_{ik} \times D$ to $D_{jk} \times D$ according to the graph given by $q_k/p_k = \Gamma[c_{k1}, \cdots, c_{k,t_k}]$. (Note that the result of plumbings is $M(p_k, q_k)$ as in Lemma 6.2.)

(6.9) From our construction, it is routine to check that $M_N \simeq N$. It follows from Lemma 6.2 and Lemma 6.3 that $\mathscr{T}$ extends to M_N, by extending the S^1-action on each N_i to $M_{\widetilde{N}_i}$ and the S^1-actions on N_i

and N_j to $M(p_k, q_k)$. (Here we assume that ϕ_k glues one component of ∂N_i with one of ∂N_j.)

7. Residue formula for limiting eta invariants

This section is devoted to the explicit topological formula for $\eta_{(2)}(N)$, for N a closed oriented Seifert manifold. First, combining Theorem 0.3′, Theorem 6.1 and Theorem 1.11, we have

$$\eta_{(2)}(N) = \sigma(M_N) + \sum \mathrm{Res}(\alpha, \mathscr{T}), \tag{7-1}$$

where M_N is as in Theorem 6.1, $\mathscr{T}$ is a T-structure of M_N which is injective when restricted to N. The rigidity result for injective T-structures (Theorem 0.9) implies that in most cases formula (7–1) will be intrinsic. More precisely, one is able to write $\eta_{(2)}(N)$ in terms of the data of the simple decomposition of $(N, \mathscr{T})$ (of course, $\mathscr{T}$ is simple). However, in this paper, we shall only carry out the computation for $\eta_{(2)}(N)$ for injective Seifert manifolds.

Before computing the residue formula for general Seifert manifolds, let us first do the simpler case where the Seifert manifold has only one exceptional orbit.

a. Residue formula for Seifert manifolds with one exceptional orbit

Lemma 7.1. *For the oriented injective Seifert manifold* $N = \{b; (o, g), (\alpha, \beta)\}$, *we have*

$$\eta_{(2)}(N) = -s + \epsilon(b) + \frac{b+1}{3} + \frac{1}{3}\sum_{i=1}^{s} b_i + \frac{\alpha_{s-1}}{3\alpha}, \tag{7-2}$$

where $\alpha/(\alpha-\beta) = [b_1, \cdots, b_s]$, $\alpha_i = b_i\alpha_{i-1} - \alpha_{i-2}$ *with* $\alpha_0 = 0$, $\alpha_1 = 1$ *and* $\epsilon(b) = 1$ *if* $b \le -1$ *or* $\epsilon(b) = -1$ *otherwise.*

Remark 7.2. In the proof of Lemma 7.1 we find $\sigma(M_N) = -s + \epsilon(b)$. Thus in general, M_N is not homeomorphic to M_{-N} (compare Remark 6.4). This is the reason why the relation $\eta_{(2)}(-N) = -\eta_{(2)}(N)$ cannot be seen directly from (7-2).

Proof. By Lemma 6.3, we fill in N with M_N via the equivariant plumbing. More precisely, M_N is formed by plumbing successively a number of disc bundles $M_{(m)}$ (m, the *Euler number* of the bundle) over surfaces, $\xi = (M_{-b-1}, \pi, Y)$, $\xi_i = (M_{-b_i}, \pi_i, S_i^2)\,(1 \le i \le s)$, where Y is an orientable closed surface of genus g [28]. If we write $S_i^2 = B_{i,1} \cup B_{i,2}$, the union of two discs, then $M_{-b_i} = B_{i,1} \times D \cup_{\phi_i} B_{i,2} \times D$ with the gluing

map

$$\phi_i = \begin{pmatrix} -1 & 0 \\ b_i & 1 \end{pmatrix}.$$

Thus M_N can be expressed as

(7-3)

$$\begin{array}{c} M_N = M_{-b-1} \cup^{\perp} B_{11} \times D \underset{\phi_1}{\cup} B_{12} \times D \\ \Big\downarrow s \\ B_{21} \times D \underset{\phi_2}{\cup} B_{22} \times D \\ \Big\downarrow s \\ \cdots \\ \Big\downarrow s \\ B_{s1} \times D \underset{\phi_s}{\cup} B_{s2} \times D \end{array}$$

where

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha_{s-1} & \beta_{s-1} \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ b_s & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ b_1 & 1 \end{pmatrix}. \tag{7-4}$$

Let X be the vector field generated by the extended S^1-action from N to M_N, and let X_i be the restriction of X to $B_{i,1} \times D (1 \leq i \leq s)$. In the polar coordinates, $((\rho_1, \theta_1), (\rho_2, \theta_2))$, of $B_{i,1} \times D$, we have

$$X_i = \alpha_i \frac{\partial}{\partial \theta_{i,1}} - \alpha_{i+1} \frac{\partial}{\partial \theta_{i,2}}.$$

Note that $\alpha_0 = 0$ and $\alpha_1 = 1$ since the S^1-action becomes a rotation of the fiber when restricted to M_{-b-1}. From (7-4) we solve for α_i, $\alpha_{i+1} = b_i \alpha_i - \alpha_{i-1}$, $1 \leq i \leq r$. Using $b_i \leq 2$ and the initial condition, we conclude by induction that

$$\alpha_{i+1} = b_i \alpha_i - \alpha_{i-1} = \alpha_{i-1} \left(b_i - \frac{\alpha_{i-2}}{\alpha_{i-1}} \right) > \alpha_{i-1} > 0.$$

Consequently, the fixed point set of X is $Z_0 \cup \{0_{1,1} \times 0, 0_{2,1} \times 0, \cdots, 0_{s,1} \times 0, 0_{s,2} \times 0\}$, where $Z_0 \simeq Y$ and $0_{i,j} \times 0$ is the center of $B_{i,j} \times D$.

Therefore, (7-1) can be written as

$$\eta_{(2)}(N) = \sigma(M_N) + \mathrm{Res}(\alpha_X, Z_0) + \sum_{i=1}^{s} \mathrm{Res}(\alpha_X, 0_{i,1} \times 0) + \mathrm{Res}(\alpha_X, 0_{s,2} \times 0), \tag{7-5}$$

where α_X is the closed form determined by X (see [36] for details). Also,

$$\begin{aligned} \mathrm{Res}(\alpha_X, 0_{i,1} \times 0) &= \frac{\alpha_{i+1}}{\alpha_i} + \frac{\alpha_i}{\alpha_{i+1}} = b_i + \frac{\alpha_i}{\alpha_{i+1}} - \frac{\alpha_{i-1}}{\alpha_i}, \\ \mathrm{Res}(\alpha_X, Z_0) &= \frac{b+r}{3}. \end{aligned} \tag{7-6}$$

By substituting (7-6) into (7-5), we get

$$\eta_{(2)}(N) = \sigma(M_N) + \frac{b+r}{3} + \frac{1}{3}\sum_{i=1}^{r} b_i + \frac{\alpha_{s-1}}{3\alpha}.$$

Now the only thing remaining to check is that $\sigma(M_N) = -s + \epsilon(b)$. From [28], $\sigma(M_N)$ is equal to the signature of the matrix

$$A = \begin{pmatrix} -b-1 & 1 & & & \\ 1 & -b_1 & 1 & & \\ & 1 & -b_2 & & \\ & & & \ddots & 1 \\ & & & 1 & -b_s \end{pmatrix},$$

and A is negative definite if and only if $-b-1 < 0$. Thus, here we only need to consider $-b-1 \geq 0$. First, we consider $-b-1 > 0$. It is easy to see that A is congruent to

$$A \sim \begin{pmatrix} -b_1 - 1 & 0 \\ 0 & A_1 \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} -b_1 + \frac{1}{b+1} & 1 & & & \\ 1 & -b_2 & 1 & & \\ & 1 & -b_3 & & \\ & & & \ddots & 1 \\ & & & 1 & -b_s \end{pmatrix}.$$

Note that $-b_1 + \frac{1}{b+1} < 0$ because $\frac{1}{b+1} < 0$. Consequently, A_1 is negative definite, and therefore $\sigma(A) = -s + 1$.

If $b + 1 = 0$, then

$$A \sim \begin{pmatrix} -b_1 & 0 & 0 \\ 0 & \frac{1}{b_1} & 0 \\ 0 & 0 & A_2 \end{pmatrix},$$

where

$$A_2 = \begin{pmatrix} -b_2 + \frac{1}{b_1} & 1 & & & \\ 1 & -b_3 & 1 & & \\ & 1 & -b_4 & & \\ & & & \ddots & 1 \\ & & & 1 & -b_s \end{pmatrix}.$$

Since $-b_2 + 1/b_1 < 0$ $(b_i \geq 2)$, by the same reason we conclude that A_2 is negative definite, and hence that $\sigma(A) = -s + 1$. Now the proof is complete.

b. Residue formula for Seifert manifolds. We shall prove Theorem 0.9. As mentioned in the above part a the anti-invariance of $\eta_{(2)}(N)$ under a change of orientation is not obvious from (0-3). However, it turns out that the anti-invariance of (0-3) can be derived easily from the anti-invariance of (7-2).

Let $N = \{b; (o, g); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}$. Then $-N = \{-b - r; (0, g); (\alpha_1, \alpha_1 - \beta_1), \cdots, (\alpha_r, \alpha_r - \beta_r)\}$. Without loss of generality, we may assume $b + r > 0$. Put

$$N_1 = \{b; (o, g); (\alpha_1, \beta_1)\}, \qquad N_i = \{0; (o, g); (\alpha_i, \beta_i)\}, \qquad 2 \leq i \leq r.$$

Comparing (0-3) with (7-2), we find

$$\eta_{(2)}(N) = \sum_{i=1}^{r} \eta_{(2)}(N_i) + (r-1), \qquad \eta_{(2)}(-N) = \sum_{i=1}^{r} \eta_{(2)}(-N_i) - (r-1).$$

It follows from our assumption that

$$\eta_{(2)}(-N_i) = -\eta_{(2)}(N_i), \qquad \eta_{(2)}(-N) = -\eta_{(2)}(N).$$

Proof of Theorem 0.9. First, by our construction for M_N (Lemma 6.3), essentially the same calculation as in the proof of Lemma 7.1 yields

$$\eta_{(2)}(N) = \sigma(M_N) + \frac{b+r}{3} + \frac{1}{3}\sum_{i=1}^{r}\sum_{j=1}^{s_i} b_{ij} + \sum_{i=1}^{r} \frac{\alpha_{i,s_i-1}}{3\alpha_i}.$$

According to §2 of [28], $\sigma(M_N)$ is equal to the signature of the matrix

$$A = \begin{pmatrix} -b-r & E^1_{12} & E^1_{13} & \cdots & E^1_{1r} \\ E^1_{21} & A_2 & 0 & \cdots & 0 \\ E^1_{31} & 0 & A_3 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & 0 \\ E_{r1} & 0 & 0 & \cdots & A_r \end{pmatrix}$$

with

$$A_{i+1} = \begin{pmatrix} -b_{i1} & 1 & & \\ 1 & -b_{i2} & & \\ & & \ddots & 1 \\ & & 1 & -b_{i,s_i} \end{pmatrix} \qquad (i = 1, 2, \cdots, s-1),$$

$E_{1j} = (1, 0, \cdots, 0)_{1\times s_j}$, $E_{j1} = E_{1j}^{\top}$, and A is negative if and only if $-b - r < 0$. Thus what we shall do is to check $\sigma(A) = -\sum_{i=1}^{r} s_i$ if $-b - r \geq 0$. As preparation, we first prove the following lemma.

Lemma 7.4. *Let*

$$A = \begin{pmatrix} A_1 - \alpha E_{11}^1 & -\alpha E_{12}^1 & \cdots & -\alpha E_{1r}^1 \\ -\alpha E_{21}^1 & A_2 - \alpha E_{22}^1 & \cdots & -\alpha E_{2r}^1 \\ \vdots & \vdots & \cdots & \vdots \\ -\alpha E_{r1}^1 & -\alpha E_{r2}^1 & \cdots & A_r - \alpha E_{rr}^1 \end{pmatrix}$$

with A_i and $E_{ij}^k = (a_{ij})$ is the $s_i \times s_j$ matrix in which entries are zero but $a_{k1} = 1$. Suppose $\alpha > 0$ and $b_i j \geq 2$. Then A is negative definite.

Proof. Lemma 7.4 is known in [28] for $\alpha = 0$. Assume $\alpha > 0$. We proceed by induction on r. The case $r = 1$ is verified in the proof of Lemma 7.1. Assume Lemma 7.4 holds for $r - 1$. In order to use the induction assumption we make a congruence transformation on A so that A_1^1 is diagonalized and E_{1i}^1 and E_{i1}^1 vanish for $i = 1, 2, \cdots, r$. In the first step we see the following

$$A \sim A_1 = \begin{pmatrix} A_1^1 & -\frac{1}{a_1}E_{12}^2 & \cdots & -\frac{1}{a_1}E_{1r}^2 \\ -\frac{1}{a_1}E_{21}^2 & A_2 - (\frac{1}{a_1} - \tilde{a}_1)E_{22}^1 & \cdots & -(\frac{1}{a_1} - \tilde{a}_1)E_{2r}^1 \\ \vdots & \vdots & \cdots & \vdots \\ -\frac{1}{a_1}E_{r1}^2 & -(\frac{1}{a_1} - \tilde{a}_1)E_{r2}^1 & \cdots & A_r - (\frac{1}{a_1} - \tilde{a}_1)E_{rr}^1 \end{pmatrix},$$

where

$$A_1^1 = \begin{pmatrix} -\left(b_{11} + \frac{1}{a_1}\right) & & & & \\ & -\left(b_{12} - \frac{1}{a_2}\right) & 1 & & \\ & 1 & -b_{13} & & \\ & & & \ddots & 1 \\ & & & 1 & -b_{i,s_i} \end{pmatrix}$$

with

$$a_1 = \frac{1}{\alpha}, \quad a_2 = b_{11} + \frac{1}{a_1}, \quad \tilde{a}_1 = \frac{1}{a_1^2 a_2}.$$

In the second step we get

$$A^1 \sim A^2 = \begin{pmatrix} A_1^3 & -\frac{1}{a_1 a_2} E_{12}^3 & \cdots & -\frac{1}{a_1 a_2} E_{1r}^3 \\ -\frac{1}{a_1 a_2} E_{21}^3 & A_2 - (\frac{1}{a_1} - \tilde{a}_2) E_{22}^1 & \cdots & -(\frac{1}{a_1} - \tilde{a}_2) E_{2r}^1 \\ \vdots & \vdots & \cdots & \vdots \\ -\frac{1}{a_1 a_2} E_{r1}^3 & -(\frac{1}{a_1} - \tilde{a}_2) E_{r2}^1 & \cdots & A_r - (\frac{1}{a_1} - \tilde{a}_2) E_{rr}^1 \end{pmatrix},$$

where

$$A_1^2 = \begin{pmatrix} -(b_{11} + \frac{1}{a_1}) & & & & & \\ & -(b_{12} - \frac{1}{a_2}) & & & & \\ & & -(b_{13} - \frac{1}{a_3}) & 1 & & \\ & & 1 & -b_{14} & & \\ & & & & \ddots & 1 \\ & & & & 1 & -b_{i,s_i} \end{pmatrix}$$

with

$$a_3 = b_{12} - \frac{1}{a_2}, \qquad \tilde{a}_2 = \frac{1}{a_1^2 a_2} + \frac{1}{a_1^2 a_2^2 a_3}.$$

Continuing the process till a step numbered s_1, we obtain the desired result:

$$A^{s_1 - 1} \sim A^{s_1} = \begin{pmatrix} A_1^{s_1} & 0 & \cdots & 0 \\ 0 & A_2 - (\frac{1}{a_1} - \tilde{a}_s) E_{22}^1 & \cdots & -(\frac{1}{a_1} - \tilde{a}_s) E_{2r}^1 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & -(\frac{1}{a_1} - \tilde{a}_s) E_{r2}^1 & \cdots & A_r - (\frac{1}{a_1} - \tilde{a}_s) E_{rr}^1 \end{pmatrix},$$

where

$$A_1^{s_1} = \begin{pmatrix} -(b_{11} + \frac{1}{a_1}) & & & \\ & -(b_{12} - \frac{1}{a_2}) & & \\ & & \ddots & \\ & & & -(b_{i,s_i} - \frac{1}{a_{s_1}}) \end{pmatrix}$$

with

$$a_{i+1} = b_{1i} - \frac{1}{a_i} \quad (1 \le i \le s), \qquad \tilde{a}_s = \sum_{i=1}^{s_1} \frac{1}{a_1^2 \cdots a_i^2 a_{i+1}}.$$

We claim the following.

$$b_{1i} - \frac{1}{a_i} \ge \frac{(i+1)a + i}{ia + i - 1} > 0, \tag{7.1}$$

$$\frac{1}{a_1} - \tilde{a}_s > 0. \tag{7.2}$$

Assume (7.1) and (7.2). Then $A_1^{s_1}$ is negative definite and hence A is negative definite (by induction assumption).

To see (7.1), we start with

$$a_2 = b_{11} - \frac{1}{a_1} \ge 2 + \frac{1}{a_1} = \frac{2a_1 + 1}{a}.$$

Then

$$a_3 = b_{12} - \frac{1}{a_2} \ge 2 - \frac{a}{2a_1 + 1} \ge 2 - \frac{a}{2a_1 + 1} = \frac{3a_1 + 2}{2a_1 + 1} > 0.$$

Assuming (7.1) for $i - 1$, we have

$$\begin{aligned} b_{1i} - \frac{1}{a_i} &\ge 2 - \frac{b_{1,i-1} - 1/a_{i-1}}{\ge} 2 - \frac{(i-1)a_1 + i - 2}{ia_1 + i - 1} \\ &= \frac{(i+1)a_1 + i}{ia_1 + i - 1} > 0, \end{aligned}$$

since $a_1 > 0$. Now we prove (7.2). From (7.1),

(7-7)

$$\begin{aligned} \frac{1}{a_1} - \sum_{i=1}^{s_i} \frac{1}{a_1^2 \cdots a_i^2 a_{i+1}} &\ge \frac{1}{a_1}\left(1 - \sum_{i=2}^{s_1} \frac{1}{a_1 a_2^2 \cdots a_i^2 a_{i+1}}\right) \\ &\ge \frac{1}{a_1}\left(1 - \sum_{i=2}^{s_1} \frac{1}{a_1 \left(\frac{2a_1 + 1}{a_1} \cdots \frac{ia_1 + i - 1}{(i-1)a_1 + i - 2}\right)^2 \frac{(i+1)a_1 + i}{ia_1 + i - 1}}\right). \end{aligned}$$

If $a_1 \ge 1$, then from (7-7)

$$\begin{aligned} \frac{1}{a_1} - \sum_{i=1}^{s_i} \frac{1}{a_1^2 \cdots a_i^2 a_{i+1}} &\ge \frac{1}{a_1}\left(1 - \sum_{i=2}^{s_1} \frac{1}{(2i-1)(2i+1)}\right) \\ &\ge \frac{1}{a_1}\left(1 - \frac{1}{2}\right) = \frac{1}{2a_1} > 0. \end{aligned} \tag{7-8}$$

If $0 < a_1 < 1$, then from (7-7)

$$\begin{aligned}\frac{1}{a_1} - \sum_{i=1}^{s_i} \frac{1}{a_1^2 \cdots a_i^2 a_{i+1}} &\geq \frac{1}{a_1}\left(1 - \sum_{i=1}^{s_1} \frac{a_1}{(i-1)(i+1)}\right) \\ &\geq \frac{1}{a_1}\left(1 - \frac{3a_1}{4}\right) > 0.\end{aligned} \tag{7-9}$$

Since (7-8) and (7-9)together give (7.2), the proof is complete. q.e.d.

We continue our proof of Theorem 0.9.

First, we consider $b + r \leq -1$. It is easy to see

$$A \sim A_1 = \begin{pmatrix} -b-r & 0 & \cdots & 0 \\ 0 & A_1 + \frac{1}{b+r}E_{11}^1 & \cdots & \frac{1}{b+r}E_{2r}^1 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \frac{1}{b+r}E_{r1}^1 & \cdots & A_r + \frac{1}{b+r}E_{rr}^1 \end{pmatrix}.$$

Applying Lemma 7.4 we conclude that $\sigma(M_N) = \sigma(A) = -\sum_{i=1}^r s_i + 1$. Now assume $b + r = 0$. After a simple congruence transformation on A (compare the last part in the proof of Lemma 7.1), we find

$$A \sim \begin{pmatrix} -b_{11} & 0 & 0 \\ 0 & \frac{1}{b_{11}} & 0 \\ 0 & 0 & \widetilde{A} \end{pmatrix},$$

where

$$\widetilde{A} = \begin{pmatrix} \widetilde{A}_1 & -E_{12}^1 & -E_{13}^1 & \cdots & -E_{1r}^1 \\ -E_{21} & A_2 - b_{11}E_{22}^1 & -b_{11}E_{23}^1 & \cdots & -b_{11}E_{2r}^1 \\ -E_{31}^1 & -b_{11}E_{32}^1 & A_3 - b_{11}E_{33}^1 & \cdots & -b_{11}E_{3r}^1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -E_{r1}^1 & -b_{11}E_{r2}^1 & -b_{11}E_{r3}^1 & \cdots & A_r - b_{11}E_{rr}^1 \end{pmatrix},$$

$$\widetilde{A}_1 = \begin{pmatrix} -b_{12} & 1 & & & \\ 1 & -b_{13} & 1 & & \\ & & & \ddots & 1 \\ & & & 1 & -b_{i,s_1} \end{pmatrix}.$$

By proceeding with essentially the same diagonalization procedure on $\widetilde{A}_1$ and killing E_{1j}^1 and E_{j1}^1 simultaneously ($1 \leq j \leq r$) as in the proof of Lemma 7.4, we are able to see that $\widetilde{A}$ is conjugate to a matrix satisfying the conditions in Lemma 7.4. Thus, by applying Lemma 7.4, $\sigma(M_N) = \sigma(A) = -\sum_{i=1}^r s_i + 1$ (here we omit the detailed computation).

Now the proof is complete.

8. The proof of Theorem 0.11

Proof. We proceed by contradiction. Assuming the opposite, we have a sequence $\{N_i\}$ in $\mathscr{M}^3(D)$ such that

(8.1) $\{\eta_{(2)}(N_i) \bmod Z\}$ is an infinite set.

For each i, let g_i be a metric of N_i which satisfies conditions (0.1), (0.3) and

(8.2) $\mathrm{Vol}(N_i, g_i) < \frac{1}{i}$,

(8.3) $\mathrm{diam}(N_i, g_i) \leq D$.

From (0.1), (8.2) and (8.3), by Gromov's precompactness theorem [23] we may assume, by passing through a subsequence if necessary, that $\{N_i\}$ converges to a lower-dimensional metric space Y in the Gromov-Hausdorff distance. We split the rest of the proof according to $\dim(Y)$.

(a) $\dim(Y) = 0$. Then all but finitely many N_i are nilmanifolds [21]. Note that a three-dimensional nilmanifold is a Seifert manifold whose Seifert invariants are $\{b; (o, 2)\}$ ($b \in Z$). By Theorem 0.9, $\eta_{(2)}(N_i) \bmod Z = 0, \frac{1}{3}, \frac{2}{3}$. Consequently, $\{\eta_{(2)}(N_i) \bmod Z\}$ is a finite set. This contradicts (8.1).

(b) $\dim(Y) = 1$. Then Y is homeomorphic to either a closed interval or S^1. In the former case, from Theorem 12.8 of [20] we conclude that all but finitely many N_i are actually T^2-manifolds. By the result of [27], we can even identify these T^2-manifolds with either $S^2 \times S^1$ or T^3 (note that $\pi_1(N_i)$ is an infinite group). Consequently, $\{\eta_{(2)}(N_i) \bmod Z\}$ is again a finite set and this contradicts (8.1) again. If $Y \approx S^1$, then all but finitely many N_i are solve-manifolds (Theorem 12.1 of [20]). In fact, the limiting eta-invariant of a solve-manifold is $\frac{1}{3}$ and $\frac{2}{3}$ modulo integers (note that a pure polarized T^2-structure on a solve-manifold has no exceptional orbit). This fact leads to a contradiction to (8.1).

(c) $\dim(Y) = 2$. Then all but finitely many N_i are injective Seifert fiber spaces, and Y is an orbifold. Let $y_1, \cdots, y_k$ be the singular points of the orbifold. Then each N_i (i sufficiently large) has exactly k exceptional orbits which are the preimage of y_i. According to the classification theorem due to [33] (see also [28]), we can write

$$N_i = \{b_i; (\epsilon_i, g); (\alpha_{i,1}, \beta_{i,1}), \cdots, (\alpha_{i,k}, \beta_{i.k})\},$$

where g is the genus of Y, $\epsilon_i = \epsilon$, a constant depending on orientability of Y. For convenience, we define a norm of N_i as follows:

$$||N_i|| = \max\{\alpha_{i,1}, \cdots, \alpha_{i,k}\}.$$

By Theorem 0.9, if $\{||N_i||\}$ is a bounded set, then $\{\eta_{(2)}(N) \bmod Z\}$ is finite. Thus, we may assume that $||N_i|| \to \infty$ as $i \to \infty$. By passing to a subsequence if necessary, we may assume that $\alpha_{i,j} \to \infty$ $(i \to \infty)$ for some fixed j, $1 \leq j \leq k$. To get a contradiction, we choose a small metric ball $B_\delta(y_j)(\subset Y)$ at y_j such that its preimage in N_i is solid torus for sufficiently large i. Note that $B_\delta(y_j)$ can be viewed as the orbit space of the S^1-action on $\pi_i^{-1}(B_\delta(y_j))$ with the isotropy group $Z_{\alpha_{i,j}}$. Since $\operatorname{diam}(\pi_i^{-1}(B_\delta(y_j))) \leq D$ and $\alpha_{i,j} \to \infty$, the limit of the orbifold $\pi_i^{-1}(B_\delta(y_j))/Z_{\alpha_{i,j}}$ has to be of dimension one. This contradicts $\dim(B_\delta(y_j)) = \dim(Y) = 2$.

Now the proof is complete.

Remark 8.1. It would be interesting to know whether or not Theorem 0.11 can be generalized to higher dimensions.

Acknowledgment

The main results in this paper are included in the author's thesis. The author wishes to thank his advisor Jeff Cheeger for introducing him to this field, for his encouragement and many inspiring discussions. He is grateful to G. Gong and B. Maskit for a number of helpful conversations. He also would like to thank the referee for some suggestions.

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