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INDEX THEORY FOR CERTAIN COMPLETE KÄHLER MANIFOLDS

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1. Introduction and notation

Let $\overline{M}$ be a compact Kähler manifold of real dimension n with Kähler form ω', and let $\mathscr{D} = \mathscr{D}_1 \cup \cdots \cup \mathscr{D}_N \subset \overline{M}$ be a divisor with simple normal crossings. The noncompact manifold $M = \overline{M} - \mathscr{D}$ may be endowed with a complete finite volume metric h with Poincaré growth at the $\mathscr{D}_i$ (see for example [2]) determined by the Kähler form

(1)
$$\omega = T\omega' - \sum_{j=1}^{N} \partial\overline{\partial} \log\log^2 |\sigma_j|^2.$$

Here $|\cdot|$ denotes a Hermitian norm on the line bundle $[\mathscr{D}_j]$, σ_j is a section of $[\mathscr{D}_j]$ defining $\mathscr{D}_j$, and T is a large real constant. We normalize the Kähler form so that the Kähler form on $\mathbf{C}$ corresponding to the usual metric $dx^2 \oplus dy^2$ is $\frac{1}{2} dz \wedge d\overline{z} = -i\, dx \wedge dy$. Thus the metric determined by a Kähler form ω is given by $(v_1, v_2) = i\omega(v_1, Jv_2)$, where J is the complex structure operator. For a multi-index I, set

(2)
$$\mathscr{D}_I = \bigcap_{i \in I} \mathscr{D}_i.$$

The manifold $\mathscr{D}_I' \equiv \mathscr{D}_I - \bigcup_{J \supset I,\ J \neq I} \mathscr{D}_J$ inherits a complete metric h_I determined by $\omega|_{D_I}$.

Let E be a unitary flat bundle over M, and F a hermitian holomorphic bundle over M. Denote by $H_2^{\bullet}(M, h, E)$ the L^2 cohomology of (M, h) with coefficients in E. The cup product pairing defines a quadratic form Q on $H_2^{n/2}(M, h, E)$. When this group is finite-dimensional, we call the signature of Q the L^2-signature of (M, h, E). We define the L^2-Euler characteristic of (M, h) to be $\sum_p (-1)^p \dim H_2^p(M, h, \mathbf{C})$, when each of these groups is finite dimensional. Similarly, given a hermitian

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holomorphic vector bundle F over M with finite dimensional $\bar{\partial}$ cohomology, we call the alternating sum of the dimensions of the $\bar{\partial}$-cohomology groups the L^2 holomorphic Euler characteristic of (M, h, F), and denote it by $\chi_2(M, h, F)$. In this paper, we will establish index theorems which will allow us to calculate these L^2 characteristic numbers for a restricted class of bundles.

Given a hermitian vector bundle E, let $T(E)$, $e(E)$, $A(E)$, and $L(E)$ denote the Todd, Euler, and stable Hirzebruch A and L classes of E interpreted as polynomials in the curvature form of E determined by the given metric. Let ν_j denote the first Chern class of $[\mathscr{D}_j]$, and $\operatorname{ch}(E)$ the Chern character of E. Our main result is the following theorem.

Theorem 1.1. *Let E be a unitary flat vector bundle with logarithmic connection along $\mathscr{D}$. Then the L^2-signature of (M, h, E) equals*

$$2^{n/2}\int_M L(TM) + 2^{n/2}\sum_I \int_{\mathscr{D}_I} L(T\mathscr{D}_I)\wedge \operatorname{tr}_E \prod_{i\in I}(L(\nu_i, A_i) - 1)/\nu_i,$$

where $L(\nu_i, A_i)$ is an $\operatorname{End} E$ valued class defined in (54), *and tr_E denotes the trace over E.*

Let F be a holomorphic vector bundle with a good connection in the sense of (4.2). *Then*

$$\begin{aligned}\chi_2(M, h, F) = &\int_M \operatorname{ch}(F)\wedge T(TM)\\ &-\sum_I (-1)^{|I|}\chi_2(\mathscr{D}_I', h_I, F_I)\\ &+\sum_I \int_{\mathscr{D}_I} \operatorname{ch}(F)\wedge T(T\mathscr{D}_I)\\ &\qquad\qquad \wedge (\dim F)^{-1}\operatorname{tr}_F \prod_{i\in I}(L(\nu_i, A_i) - 1)/\nu_i,\end{aligned}$$

where F_I denotes the restriction of F to $\mathscr{D}_I'$. The L^2-Euler characteristic of (M, h) equals

$$\int_M e(TM) + \sum_I \int_{D_I} e(T\mathscr{D}_I).$$

The index of the Dirac operator on M with coefficients in a bundle F with a Dirac-good connection (*see* 4.3) *is given by*

$$\int_M \mathrm{ch}(E) \wedge A(TM)$$
$$+ \sum_I \int_{\mathcal{D}_I} \mathrm{ch}(E) \wedge A(T\mathcal{D}_I) \wedge (\dim E)^{-1} \mathrm{tr}_E \prod_{i \in I} (L(\nu_i, A_i) - 1)/\nu_i.$$

We remark that Theorem 1.4 of [11] implies that the curvature integrals arising in the above theorem compute topological invariants of $\overline{M}$ and the D_I.

Let E be a unitary flat vector bundle with logarithmic connection along $\mathcal{D}$. Suppose that E is the canonical extension of a unitary local system $\mathcal{V}$ on M (see [3] and [4]). Let $j: M \to \overline{M}$ denote the inclusion map. Then we have the following proposition of Timmerscheidt.

Proposition 1.2 [4, D.4].

$$H_2^{\boldsymbol{\cdot}}(M, h, E) \cong H^{\boldsymbol{\cdot}}(\overline{M}, j_*\mathcal{V}).$$

Hence for such E, Theorem 1.1 yields a signature theorem for $H^{\boldsymbol{\cdot}}(\overline{M}, j_*\mathcal{V})$. Similarly, one can obtain Riemann-Roch theorems for the sheaves $\tilde{\Omega}^p_{\overline{M}}(E)$ defined in [4, Appendix D] (but with $\overline{M}$ denoted X and E denoted $\mathcal{M}$). These sheaves arise in the Hodge decomposition of $H^{\boldsymbol{\cdot}}(\overline{M}, j_*\mathcal{V})$ obtained in [4, D.2]. When $E = \mathbf{C}$ is trivial, $H_2^{\boldsymbol{\cdot}}(M, h, \mathbf{C}) \cong H^{\boldsymbol{\cdot}}(\overline{M}, \mathbf{C})$, and a subset of the above results should also follow from more elementary arguments involving the study of variation of Chern-Weil representatives of characteristic classes under certain changes of metric.

The proof of Theorem 1.1, which occupies the remainder of this paper, is a variation and improvement of the techniques of [12]. The improvement lies in the fact that the techniques in this paper can be used to extend the results of [12] to Q-reducible spaces. In [14], the boundary contribution to the L^2-signature of an arithmetic variety is expressed in terms of curvature integrals and special values of certain Sato-Shintani zeta functions. Such a result may be viewed as a generalization of results of Hirzebruch [8], Atiyah, Donnelly, and Singer [1], and Muller [10] relating signature defects of Hilbert modular varieties to special values of Shimizu L-functions. In [8], Hirzeburch initially obtained a formula for the signature defect of Hilbert modular surfaces directly in terms of geometric data associated to the divisor at ∞ of a smooth compactification. The main results in this paper are, in a sense, a return to this original geometric point of view—although for a different collection of spaces. It would be interesting to carry out the computations leading to Theorem 1.1 in the locally symmetric case in order to express the special values of Sato-Shintani zeta functions in terms of geometric data associated to the divisor at ∞ of a smooth compactification.

2. Algebraic preliminaries and index formalism

We recall here notation and elementary algebraic results from [14, §1]. Let V be a real oriented $2m$-dimensional vector space with inner product $(\cdot, \cdot)$. Let $\Lambda^{\cdot} V^*$ denote the full exterior algebra of V^*. Given $X \in V$, let $X^* \in V^*$ denote the covector dual to X, determined by the inner product. Let $\varepsilon(X)$ denote exterior multiplication by X^* on the left and $\varepsilon^*(X)$ its adjoint. We extend ε to $V \otimes \mathbf{C}$ by complex linearity and extend ε^* antilinearly. We let

$$C(X) = \varepsilon(X) - \varepsilon^*(X)$$

denote left Clifford multiplication by X. We also define

$$\widehat{C}(X) = \varepsilon(X) + \varepsilon^*(X).$$

Given distinct orthonormal vectors $X_1, \cdots, X_r \in V$, we call $C(X_1) \cdots C(X_r)$ (respectively $\widehat{C}(X_1) \cdots \widehat{C}(X_r)$) real (respectively imaginary) Clifford multiplication by $X_1^* \wedge \cdots \wedge X_r^*$. We say that an endomorphism of this form has real (respectively imaginary) Clifford degree r. An endomorphism which is the product of an endomorphism of real Clifford degree r_1 and one of imaginary Clifford degree r_2 will be said to have Clifford bidegree (r_1, r_2) and complex degree $r_1 + r_2$. For an endomorphism which is not homogeneous with respect to the Clifford grading, we define the real and complex Clifford degrees to be the maximum of the corresponding degrees of its graded components.

Suppose now that V has a complex structure. Let $V_{\mathbf{R}}$ be a subspace of V such that $V = V_{\mathbf{R}} \oplus jV_{\mathbf{R}}$, where j denotes the complex structure operator. Let $\{X_i\}_{i=1}^m$ be an orthonormal basis of V, and set $Z_i = 2^{-1/2}(X_i - ijX_i)$.

Let $\{T_i\}_{i=1}^{2m}$ be any oriented orthonormal basis of V. We define involutions τ_V, $\hat{\tau}_V$, and τ_R by

$$\tau_V = i^m C(T_1) \cdots C(T_{2m}),$$
$$\hat{\tau}_V = i^{-m} \widehat{C}(T_1) \cdots \widehat{C}(T_{2m}),$$
$$\tau_R = C(\overline{Z}_1) \cdots C(\overline{Z}_m) \widehat{C}(\overline{Z}_1) \cdots \widehat{C}(\overline{Z}_m).$$

We will need the following standard trace identities (see, for example, [12, (4.4.2)]).

Proposition 2.1. *Let Y_1 be an endomorphism of $\Lambda^{\cdot} V^*$ of real Clifford degree r_1, and Y_2 an endomorphism of imaginary Clifford degree r_2. Then*

$$\operatorname{trace} Y_1 = 0 \quad \textit{unless } r_1 = 0,$$

and

$$\operatorname{trace} Y_1 Y_2 = 0, \quad \textit{unless } r_1 = 0 \textit{ and } r_2 = 0.$$

Let X be a smooth complex manifold with hermitian metric. For every $x \in X$, there exist involutions τ_V, $\hat{\tau}_V$, and τ_R of $(\Lambda^{\cdot} V^*) \otimes \mathbf{C}$, with $V = T_x X$. These involutions piece together to give involutions of the L^2 sections of the corresponding bundles. We denote these involutions by τ, $\hat{\tau}$, and τ_R, and set $\tau^e = \tau\hat{\tau}$. Let $\Omega_{\pm}$ denote the ± 1 eigenspaces of τ, and let Ω^e and Ω^o denote respectively the $+1$ and -1 eigenspaces of τ^e. Then Ω^e and Ω^o are the even and odd forms respectively. One easily checks that the elliptic operator

$$D \equiv d + d^*$$

anticommutes with τ, and τ^e, and we denote the restrictions of D to Ω_+ and Ω^e respectively by D_+ and D^e. When D_+ and D^e are Fredholm, the index of D_+ computes the signature of the L^2-cohomology of X, and the index of D^e computes the Euler characteristic of the L^2-cohomology.

Let $L_2^{0,*}(X, F)$ denote the L^2 forms of type $(0, *)$ with coefficients in a holomorphic vector bundle F, and consider the associated Dolbeault complex. Let $\bar{\partial}^*$ denote the formal adjoint of $\bar{\partial}$. The elliptic operator $D_{CF} \equiv 2^{1/2}(\bar{\partial} + \bar{\partial}^*)$ anticommutes with τ_R. Let D^e_{CF} denote the restriction of D_{CF} to the $+1$ eigenspace of τ_R. When D^e_{CF} is Fredholm, its index equals $\sum(-1)^i \dim H_2^{0,i}(X, F)$, where $H_2^{0,*}(X, F)$ denotes the $L^2 - \bar{\partial}$-cohomology of X with coefficients in F.

For simplicity of notation, we primarily restrict our attention to the index of D_+ in this paper and merely indicate where modifications are required for computing the index of other interesting operators.

For combinatorial reasons, we will depart from the modified heat equation techniques of [12], [13], and [14] for computing L^2-indices, and return to an earlier well-known formalism. Let Q be a bounded operator satisfying

$$D_+ Q = I - S_1, \qquad Q D_+ = I - S_0, \tag{3}$$

with S_0 and S_1 trace class. Then

$$\text{Index}\, D = \text{Tr}\, S_0 - \text{Tr}\, S_1, \tag{4}$$

where we use Tr to denote the trace over spaces of L^2 sections, and will use tr to denote the pointwise trace. Such a Q exists if and only if D_+ is Fredholm. Unlike the heat equation approach, this method of computing the index applies equally to compact and noncompact manifolds. We observe that on a compact manifold, the heat equation approach may be embedded into the parametrix method. To see this, we observe that one candidate for Q is given by

$$Q_t = \int_0^t \tfrac{1}{2} D e^{-s\Delta}(1-\tau)\, ds. \tag{5}$$

Then

$$D_+ Q_t = (I - e^{-t\Delta})(1-\tau)/2,$$
$$Q_t D_+ = (I - e^{-t\Delta})(1+\tau)/2,$$
$$\text{Tr}\, S_0 - \text{Tr}\, S_1 = \text{Tr}\, \tau e^{-t\Delta}.$$

This returns us to the heat equation formalism when Q_t is bounded and the two heat operators are trace class. The complete manifold (M, h) introduced in the introduction is noncompact, and it is easy to check that the heat operators are not trace class. In order to take advantage of the well-known heat equation asymptotics, we would like to incorporate Q_t into the construction of a parametrix with appropriate modifications to deal with the problems associated to the noncompactness of M. Such a parametrix construction is carried out in §6.

3. Preliminary computations in the Poincaré punctured disk

Let E be a unitary flat vector bundle on the Poncaré punctured disk. In this section we compute the Laplace operator $\Delta_{P,E}$ acting on E-valued forms in a suitable frame. We parametrize the punctured disk Δ^* as $[0, \infty) \times S^1$ with coordinates r and θ. The Poincaré metric is $dr^2 + e^{-2r} d\theta^2$. Let $w = e^r \partial/\partial\theta$ and $w^* = e^{-r} d\theta$. Observe that $dw^* = -dr \wedge w^*$. The vectors $\partial/\partial r$, w generate orthonormal frames for the Riemannian bundles associated to the tangent bundle of Δ^*. Choose a unitary frame for E which is covariant constant in the r direction. In these frames, the exterior derivative d has the form

$$d = \varepsilon(\partial/\partial r)(\partial/\partial r - \varepsilon(w)\varepsilon^*(w)) + \varepsilon(w)(w + \gamma_E(w)), \tag{6}$$

where γ_E is an End(E) valued 1-form which is skew with respect to the flat hermitian metric on E. We assume that

$$\gamma_E(w) = e^r \gamma_0 + O(e^{-2r}), \tag{7}$$

where γ_0 is a constant matrix in the given frame, and the $O(e^{-2r})$ term is a matrix whose coefficients have the desired decrease. It is easy to see that we can choose a frame such that (7) is satisfied (with room to spare) if the connection on E extends to a connection with logarithmic singularities on the disk. It is this special case which motivates this assumption.

Given a form f, let $\tilde{f} = e^{-r/2} f$. We metrize the image of $\sim$ by making it an isometry. Define $\tilde{d} := e^{-r/2} d e^{r/2}$. Then

$$\tilde{d} = \varepsilon(\partial/\partial r)(\partial/\partial r + 1/2 - \varepsilon(w)\varepsilon^*(w)) + \varepsilon(w)(w + \gamma_E(w)). \tag{8}$$

The transformation $f \to \tilde{f}$ is useful because it makes $\partial/\partial r$ skew (when acting on compactly supported forms). In particular, we have

$$\tilde{d}^* = \varepsilon^*(\partial/\partial r)(-\partial/\partial r + 1/2 - \varepsilon(w)\varepsilon^*(w)) - \varepsilon^*(w)((w + \gamma_E(w)),$$

and

(9)

$$D_P := \tilde{d} + \tilde{d}^* = C(\partial/\partial r)\partial/\partial r + \widehat{C}(\partial/\partial r)\widehat{C}(w)C(w)/2 + C(w)(w + \gamma_E(w)).$$

Henceforth when working with the punctured Poincaré disk, we will always use this frame unless otherwise stated and will omit the $\sim$ from our notation.

We may decompose dr into its $(1, 0)$ and $(0, 1)$ components $dr = \partial r + \overline{\partial} r$, where

$$\partial r = \tfrac{1}{2}(dr + iw^*) \quad \text{and} \quad \overline{\partial} r = \tfrac{1}{2}(dr - iw^*).$$

Let Z and $\overline{Z}$ be the vectors dual to $2^{1/2}\partial r$ and $2^{1/2}\overline{\partial} r$. Then acting on $(0, *)$ forms,

$$\begin{aligned} 2^{1/2}\overline{\partial} &= \varepsilon(\overline{Z})(\partial/\partial r + i(w + \gamma_E(w)) + 1/2), \\ 2^{1/2}\overline{\partial}^* &= \varepsilon^*(\overline{Z})(-\partial/\partial r + i(w + \gamma_E(w)) + 1/2), \\ D_{\mathbf{C}} := 2^{1/2}(\overline{\partial} + \overline{\partial}^*) &= C(\overline{Z})\partial/\partial r + i\widehat{C}(\overline{Z})(-i/2 + w + \gamma_E(w)). \end{aligned} \tag{10}$$

The Laplacian takes the form

$$\begin{aligned}\Delta_P = & -\partial^2/\partial r^2 + 1/4 - (w + \gamma_E(w))^2 \\ & + 2(\varepsilon^*(w)\varepsilon(\partial/\partial r) + \varepsilon(w)\varepsilon^*(\partial/\partial r))(w + \gamma_E(w)) \\ = & -\partial^2/\partial r^2 + 1/4 - (w + \gamma_E(w))^2 \\ & + (C(\partial/\partial r)C(w) - \widehat{C}(\partial/\partial r)\widehat{C}(w))(w + \gamma_E(w)).\end{aligned}$$

The $\overline{\partial}$-Laplacian may be written

$$2\Box_P = -\frac{\partial^2}{\partial r^2} + \frac{1}{4} - (w + \gamma_E(w))^2 + i(C(\overline{Z})\widehat{C}(\overline{Z}) + 1)(w + \gamma_E(w)).$$

It is also useful to consider Dirac operators on Δ^*. Let S be the bundle of spinors over Δ^* with its canonical Riemannian connection ∇^S. In the frame for $S \otimes E$ determined by $\{\partial/\partial r, w\}$ and the given frame on E, $\nabla_w^{S\otimes E} = w + \frac{1}{2}C(w)C(\partial/\partial r) + \gamma_E(w)$, and $\nabla_{\partial/\partial r}^{S\otimes E} = \partial/\partial r$. (We are not yet using the map $f \to \tilde{f}$.) The corresponding Dirac operator D_E is given by

$$D_E = C(\partial/\partial r)(\partial/\partial r - 1/2) + C(w)(w + \gamma_E(w)).$$

The spinor Laplacian D_E^2 is given by

$$D_E^2 = -(\partial/\partial r - 1/2)^2 - (w + \gamma_E(w))^2 - (w + \gamma_E(w))C(w)C(\partial/\partial r).$$

Now replacing f by $\tilde{f}$, this becomes

$$-\partial^2/\partial r^2 - (w + \gamma_E(w))^2 - (w + \gamma_E(w))C(w)C(\partial/\partial r). \tag{11}$$

Assume now that the eigenvalues of $i\gamma_E$ are not integers. Then the restriction of D_E^2 to forms supported in a region with r sufficiently large is strictly positive; hence, D_E (with appropriate boundary values) is Fredholm.

4. Frames and connections

As in the first section, let $\mathscr{D}_1, \cdots, \mathscr{D}_N$ be the components of the divisor $\mathscr{D}$, and let σ_j be a defining section of the line bundle $[\mathscr{D}_j]$. For $I = \{i_1, \cdots, i_k\}$, let $\mathscr{D}_I = \mathscr{D}_{i_1} \cap \cdots \cap \mathscr{D}_{i_k}$. We consider only those I for which this intersection is nonempty. A neighborhood of $\mathscr{D}_I$ is covered by sets of the form

$$V_\alpha = (\Delta^*)^k \times U_\alpha, \tag{12}$$

with $\{U_\alpha\}$ a finite open cover of $\mathscr{D}_I$. Let $z = (z_1, \cdots, z_k)$ denote the coordinate functions for $(\Delta^*)^k$. Then there exist smooth functions u_j on

V_α so that $|\sigma_{i_j}|^2 = e^{u_j}|z_j|^2$. Relabel the divisors so that $I = \{1, \cdots, k\}$. Recall that the metric on M is determined by the Kähler form

$$\omega = T\omega' - \sum_{j=1}^{N} \partial\overline{\partial} \log\log^2 |\sigma_j|^2.$$

Set $\omega_{I,T} = T\omega' - \sum_{j=k+1}^{N} \partial\overline{\partial} \log\log^2 |\sigma_j|^2$. Then

$$\begin{aligned}
\omega &= \omega_{I,T} - \sum_{j=1}^{k} \partial[\overline{\partial} \log^2 |\sigma_j|^2] / \log^2 |\sigma_j|^2 \\
&= \omega_{I,T} + 2\sum_{j=1}^{k} \left\{ \frac{dz_j \wedge d\overline{z}_j}{|z_j|^2(u_j + \log|z_j|^2)^2} - \partial\overline{\partial} u_j/(u_j + \log|z_j|^2) \right. \\
&\qquad \left. + (\partial u_j \wedge \overline{\partial} u_j + \frac{dz_j}{z_j} \wedge \overline{\partial} u_j + \partial u_j \wedge \frac{d\overline{z}_j}{\overline{z}_j} / (u_j + \log|z_j|^2)^2 \right\}.
\end{aligned}$$

Let $z_j = \rho_j e^{i\theta_j}$, with $\rho_j = |z_j|$. Set $r_j = \log|\log \rho_j|$. Observe that in these coordinates $J\, dr_j = e^{-r_j}\, d\theta_j$. Hence we may write ω as

$$\begin{aligned}
\omega = \omega_{I,T} + \sum_j &\left\{ -i \frac{dr_j \wedge e^{-r_j} d\theta_j}{(1 + 1/2e^{-r_j} u_j/2)^2} + 2e^{-r_j} \partial\overline{\partial} u_j (1 + \frac{1}{2} e^{-r_j} u_j) \right. \\
&\left. + \left(e^{-2r_j} \partial u_j \wedge \overline{\partial} u_j - i\, dr_j \wedge e^{-r_j} d_c u_j + \frac{ie^{-r_j}\, d\theta_j \wedge e^{-r_j}\, du_j}{2(1 + e^{-r_j} u_j/2)^2} \right) \right\},
\end{aligned}$$

where we recall that $d_c = i(\overline{\partial} - \partial)$. Thus

$$\omega - \omega_{I,T} = -\sum_j i\, dr_j \wedge e^{-r_j}\, d\theta_j + O(e^{-r}),$$

where $O(e^{-r})$ denotes the terms which for some j are $O(e^{-r_j})$ (with respect to an h orthonormal basis). Restated in metric terms we see that the difference between the metrics corresponding to ω and $\omega_{I,T}$ is

$$\sum_j (dr_j^2 \oplus e^{-2r_j}\, d\theta_j^2) + O(e^{-r}).$$

Surprisingly, the terms of order e^{-r_j} cannot be neglected, and the above approximation is too crude for our purposes. We will find, however, that we may neglect terms of order e^{-2r_j}. Let $\omega'_\alpha(z)$ denote the restriction of $\omega_{I,T}$ to the tangent space of $z \times U_\alpha$; then $\omega_{I,T} - \omega'_\alpha(z) = O(e^r e^{-e^r})$, and with respect to some trivialization, $\omega'_\alpha(z) - \omega'_\alpha(0) = O(e^r e^{-e^r})$. (Observe that $|z_i|\,|\log(|z_i|^2)|\partial u_i/\partial z_j$ is $O(|z_i|\,|\log(|z_i|^2)|)$. Hence we record here for later use:

$$\begin{aligned}\omega = \omega'_\alpha(z) &+ e^{-r_j}\partial\bar\partial u_j/2 \\ &+ \sum_j \{-i\,dr_j \wedge e^{-r_j}\,d\theta_j/(1+e^{-r_j}u_j/2)^2 \\ &\qquad - \tfrac{1}{2}e^{-r_j}[dr_j \wedge i\,d_c u_j - ie^{-r_j}\,d\theta_j \wedge du_j]\} + O(e^{-2r}).\end{aligned}$$

Moreover the order of growth of the error is clearly preserved by all derivatives by smooth unit vector fields.

Remark 4.1. It is important to note that in the error term $O(e^{-2r})$ every term involving a dr_i or $e^{-r_i}\,d\theta_i$ is in fact $O(e^{-2r_i})$ and not merely $O(e^{-2r_j})$ for some j. This follows from elementary computations.

We next construct a frame which allows us to treat the Laplacian corresponding to the above metric as a perturbation of an operator with which we can compute. Fix a base point $(x, r, \theta) \in U_\alpha \times \mathbf{R}^k_+ \times (S^1)^k$. Let $\{Y_i\}$ be an orthonormal moving frame defined in a neighborhood of $(x, r) \times (S^1)^k \subset V_\alpha$ of the form $\{X_j\}_{j=1}^{2n-2k} \cup \{R_j, W_j\}_{j=1}^k$ obtained in the following manner. Let $\widetilde{X}_1, \cdots, \widetilde{X}_{2n-k}$ be a geodesic normal frame at $(x, r) \times \{\theta\} \subset U_\alpha \times \mathbf{R}^k_+ \times \{\theta\}$ endowed with the metric obtained by restriction. Assume moreover that at $(x, r) \times \{\theta\}$, $\{\widetilde{X}_{2n-2k+i}\}_{i=1}^k$ is obtained from $\{\partial/\partial r_i\}$ by applying Gram-Schmidt. Near $(x, r) \times (S^1)^k$, use the product structure to extend this frame to a frame $\{\widetilde{Y}_i\}_{i=1}^{2n-k}$ for the image of the tangent space of $U_\alpha \times \mathbf{R}^k_+$. We may use the Gram-Schmidt process to make this an orthonormal frame $\{\widetilde{Y}_i\}_{i=1}^{2n-k} = \{X_a\}_{a=1}^{2n-2k} \cup \{\widetilde{R}_i\}_{k=1}^k$. We modify the $\widetilde{R}_i$ so that $[X_a$, modified $\widetilde{R}_i] = O(|x|e^{-r})$. Applying the Gram-Schmidt process to $\{X_a\}_{a=1}^{2n-2k} \cup$ the modification of $\{\widetilde{R}_i\}_{i=1}^k$ yields $\{Y_i\}_{i=1}^{2n-k} = \{X_i\}_{i=1}^{2n-2k} \cup \{R_i\}_{i=1}^k$. (Because $\partial u_j/\partial\theta_a = O(e^{-e^r})$ these applications of the Gram-Schmidt process will only modify vectors and their covariant derivatives by $O(e^{-e^r})$ in the $e^{r_j}\partial/\partial\theta - j$ directions.) Extend this frame to the complete tangent space by applying the Gram-Schmidt process to the frame $\{Y_1, \cdots, Y_{2n-k}, \partial/\partial\theta_1, \cdots, \partial/\partial\theta_k\}$ to obtain $\{Y_i\}_{i=1}^{2n} = \{Y_1, \cdots, Y_{2nb-=k}, W_1, \cdots, W_k\}$. In particular, we have

$$(13)\quad W_j = \left(1 - \frac{1}{2}e^{-r_j}u_j\right)e^{r_j}\frac{\partial}{\partial\theta_j} + \sum_{i=1}^{2n-2k}\frac{1}{2}e^{-r_j}du_j(JX_i)X_i + O(e^{-2r}).$$

We record the following commutation relations:

$$(14)\quad [R_i, W_j] = \delta_{ij}W_j + \sum_a O(e^{-r_j})X_a + \delta_{ij}O(e^{-r_j})W_j.$$

$$(15)\quad [X_i, X_j] = O(|x| + |r - r(0)|e^{-r} + e^{-2r}).$$

$$(16)\quad \begin{aligned}[X_i, W_j] = &\sum_{a=1}^{2n-2k} e^{-r_j}(X_iJX_au_j)X_a/2 + O(e^{-r_j})W_j \\ &+ O((|x| + |r - r(0)|)e^{-r} + e^{-2r}).\end{aligned}$$

We rewrite this as

$$(17)\quad \begin{aligned}&([X_i, W_j], X_a) - ([X_a, W_j], X_i) \\ &= \tfrac{1}{2}e^{-r_j}d_c du_j(X_i, X_a)|_{TU_a} + O((|x| + |r - r(0)|)e^{-r} + e^{-2r}).\end{aligned}$$

We now recall the formula for the Levi-Civita connection. Let $\{Z_i\}$ be orthonormal vector fields. Then

$$2\langle\nabla_1 Z_2, Z_3\rangle = \langle[Z_1, Z_2], Z_3\rangle + \langle[Z_3, Z_1], Z_2\rangle - \langle[Z_2, Z_3], Z_1\rangle.$$

Thus we see that

$$\langle\nabla_{Y_i}Y_j, Y_k\rangle = O(|x| + |r - r(0)| + e^{-2r}), \quad \text{for } i, j, k \le 2n - k.$$

$$(18)\quad \langle\nabla_{W_j}W_j, R_j\rangle = 1 + O(e^{-2r}).$$

$$(19)\quad \begin{aligned}\langle\nabla_{X_i}W_j, X_k\rangle = &\langle[X_i, W_j], X_k\rangle/2 + \langle[X_k, W_j], X_i\rangle/2 \\ &+ O(|x|e^{-r} + |r - r(0)|e^{-r} + e^{-2r}) = O(e^{-r_j}).\end{aligned}$$

$$(20)\quad \begin{aligned}\langle\nabla_{W_j}X_i, X_k\rangle = &-\langle[X_i, W_j], X_k\rangle/2 + \langle[X_k, W_j], X_i\rangle/2 \\ &+ O((|x| + |r - r(0)|)e^{-r} + e^{-r}) \\ = &\, e^{-r_j}d_c du_j(X_k, X_i)/4 + O(|x| + |r - r(0)|e^{-r} + e^{-2r}).\end{aligned}$$

Hence,

$$\sum_{i<a} C(X_i)C(X_a)\langle\nabla_{W_j}X_i, X_a\rangle$$

$$(21)\qquad \begin{aligned}&= \text{Clifford multiplication by } -e^{-r_j}d_c du_j/4|_{TU_a} \\ &\quad + O((|x| + |r - r(0)|)e^{-r} + e^{-2r}) \\ &= -e^{-r_j}C(d_c du_j/4|_{T\mathscr{D}_I}) + O((|x| + |r - r(0)|)e^{-r} + e^{-2r})\end{aligned}$$

in an obvious notation for Clifford multiplication which we now adopt (and extend to $\widehat{C}$ also). For $j \neq k$,

$$(22)\qquad 2\langle\nabla_{X_i}W_j, W_k \text{ or } R_k\rangle = O((|x| + |r - r(0)|)e^{-r} + e^{-2r}).$$

Suppose now that E is a flat unitary bundle over M which is the restriction to M of a bundle on $\overline{M}$ with logarithmic connection along $\mathscr{D}$. (See [4] and [3] for a discussion of logarithmic connections.) Then in every neighborhood $(\Delta^*)^k \times U_\alpha$, we may choose a unitary $\partial/\partial r$ invariant frame for E such that writing ∇^E in this frame as $\nabla^E = d + \Gamma^E$, we have

$$(23)\qquad \Gamma^E - W_j^* \wedge e^{r_j}A_j = e_\alpha,$$

where e_α is a smooth differential form on $\overline{M}$, and with respect to the given frame,

$$(24)\qquad \begin{aligned}&A_j = A_j(E) \text{ is a matrix which is constant} \\ &\qquad \text{in the } r_j \text{ and } \theta_j \text{ directions},\end{aligned}$$

and

$$(25)\qquad [A_j, A_k] = 0.$$

Definition 4.2. Let F be a hermitian vector bundle. We say that F has *good connection* if, in a neighborhood of each component of $\mathscr{D}$, the metric and connection on F are induced by a holomorphic decomposition

$$F = \bigoplus_i (F_i \otimes F^i),$$

where each F_i is a unitary flat bundle with logarithmic connection along $\mathscr{D}$, and each F^i is the restriction to M of a hermitian bundle on $\overline{M}$ with smooth hermitian connection.

In particular, unitary flat bundles with logarithmic connections along $\mathscr{D}$ and restrictions of smooth bundles on $\overline{M}$ have good connections.

Definition 4.3. We say that (F, h) has a Dirac-good connection if it has a good connection, and the endomorphisms $iA_j(F_i)$ have no integral eigenvalues.

In the following sections, we will study generalized Dirac operators coupled to bundles with good connections.

5. Mass

We must modify to fit our present context the notion of mass introduced in [12] and [14]. Fix a coordinate system on U_α. Let $Y(x, r, \theta)$ be a partial differential endomorphism defined on $U_\alpha \times \mathbf{R}_+^k \times (S^1)^k$ with a finite expansion of the form

$$Y(x, r, \theta) = \sum x^I (r - r_0)^J e^{-L\cdot r} Y_{IJL}(r, x, \theta),$$

where Y_{IJL} is bounded, has real (respectively complex) Clifford degree $d(I, J, L)$, and degree $p(I, J, L)$ as a partial differential operator. We define the provisional real (respectively complex) mass of Y at $(0, r_0, \theta)$ to be

$$\begin{aligned}&\text{provisional real (respectively complex) mass}(Y)\\ &\quad = \max_{I,J,L}\{d(I, J, L) + p(I, J, L) - |I| - |J| - |L|\}.\end{aligned}$$

We use the multi-index notation $|L| = l_1 + \cdots + l_k$, for $L = \{l_1, \cdots, l_k\}$. The provisional mass depends on the choice of expansion. Hence we give our final definition of mass by setting

$$\begin{aligned}\text{real mass}(Y) &= \inf\ \text{provisional real mass}(Y),\\ \text{complex mass}(Y) &= \inf\ \text{provisional complex mass}(Y).\end{aligned}$$

Here the inf is taken over all possible expansions satisfying the stated conditions. This definition of mass places e^{-r_i} on the same footing as x. From Proposition 2.1 we may immediately deduce the following proposition.

Proposition 5.1. (i) *If real mass* $Y(x) < 2m$, *then*

$$\lim_{r\to\infty} \int_r^\infty \operatorname{tr} \tau_V Y(0, s, \theta)\, ds = 0.$$

(ii) *If complex mass* $Y(x) < 4m$, *then*

$$\lim_{r\to\infty}\int_r^\infty \operatorname{tr}\tau_V\hat{\tau}_V Y(0, s, \theta)\, ds = 0.$$

(iii) *If complex mass* $Y(x) < 2m$, *then*

$$\lim_{r\to\infty}\int_r^\infty \operatorname{tr}\tau_R;\ Y(0, s, \theta)\, ds = 0.$$

Suppose now that D is a signature operator with coefficients in a unitary flat vector bundle with logarithmic connection. Let D_+ denote the restriction of D to the $+1$ eigenspace of τ. In order to study the mass of the parametrix for D_+ to be constructed below, we must understand the mass of $\Delta = D^2$ in the given frame. Recall the expression for Δ in an orthonormal basis $\{Z_i\}_i$,

$$\Delta = -\nabla_i\nabla_i + \nabla_{\nabla_i e_i} + R, \tag{26}$$

where R is the curvature operator (see [7, p. 111]). We first consider R. R has real Clifford degree 2, complex Clifford degree 2 for the Riemann-Roch complex, and 4 for the Gauss-Bonnet complex. See [9, p. 7], for the Riemann-Roch result. As the curvature is bounded (see [2]) mass(R) $\leq$ degree(R). Next we consider ∇_i. In the given frame we may write

$$\begin{aligned}\nabla_i &= Z_i + \frac{1}{2}\sum_{\alpha<\beta}\langle\nabla_i Z_\alpha, Z_\beta\rangle(C(Z_\alpha)C(Z_\beta) - \hat{C}(Z_\alpha)\hat{C}(Z_\beta)) + \Gamma_i^E \\ &= Z_i + \Gamma_i^S + \Gamma_i^E.\end{aligned}$$

Z_i and Γ_i^E have mass one. Γ_i^S has Clifford degree ≤ 2, but has mass > 1 only for those terms involving only R_j's and W_j's. In particular we record:

$$\begin{aligned}\Gamma_{W_j}^S &= \tfrac{1}{2}(C(W_j)C(R_j) - \hat{C}(W_j)\hat{C}(R_j)) \\ &\quad - e^{-r_j}C(\pi\nu_j/2|_{T\mathscr{D}_l}) + e^{-r_j}\hat{C}(\pi\nu_j/2|_{T\mathscr{D}_l}) \\ &\quad + O(|x| + |r - r(0)|e^{-r} + e^{-2r}),\end{aligned} \tag{28}$$

where ν_j is a representative of the first Chern class of $[\mathscr{D}_j]$ (see [6, pp. 141–142]) given by

$$\nu_j = \frac{i}{2\pi}\bar{\partial}\partial u_j = \frac{1}{4\pi}d_c du_j.$$

We remark that one may also obtain the expression for the covariant derivative in terms of the first Chern class of the normal bundle using directly the relation between the Euler class of a circle bundle and the global angular form. Write

$$(29)\quad \Delta \equiv \Delta_2 - \sum (W_j + A_j)(C(W_j)C(R_j) - \hat{C}(W_j)\hat{C}(R_j)) = \Delta_2 + \Delta_3.$$

Then Δ_2 has mass ≤ 2, and Δ_3 has mass 3.

Remark 5.2. It is important for our later computations to note that $C(W_j)$ and $C(R_j)$ only occur in Δ_2 in terms that are $O(e^{-r_j})$. This follows from (3.9b) and the analogous formula for R_j.

6. Parametrix

Recall that away from $\mathscr{D}_I \cap \mathscr{D}_J$, $J \supset I$, each $\mathscr{D}_I$ is covered by sets quasi-isometric to $(\Delta^*)^{|I|} \times U_\alpha$ with U_α a finite open cover of D_I and Δ^* given the Poincaré punctured disk metric. Moreover, we have shown that this identification is even asymptotically an isometry for an appropriate choice of identification. Fix such a system of identifications with U_α coordinate neighborhoods, and let Δ^*_R denote $[R, \infty) \times S^1 \subset \Delta^*$ with coordinates as in §3. Let $\{V_{I,\alpha}(R)\}_{I,\alpha}$ be an open cover of $\mathscr{D}$, with $V_{I,\alpha}(R) = (\Delta^*_R)^{|I|} \times U_{I,\alpha}(R)$ and with $\{U_{I,\alpha}(R)\}_\alpha \cup \mathscr{D}_I \cap \{\bigcup_{J\supset I} \bigcup_\beta V_{J,\beta}(R)\}$ an open cover of $\mathscr{D}_I$. Complete this to a cover of M with some $V_\varnothing(R)$ chosen so that its injectivity radius is greater than Ce^{-R} for some $C > 0$ independent of R. The existence of $V_\varnothing(R)$ satisfying the desired lower bound on its injectivity radius follows from the corresponding statement for the Poincaré punctured disk. Let $\{\rho_{I,\alpha,R}\}_{I,\alpha}$ be a partition of unity adapted to this cover in the following way. For each I, let $\{\rho_{I,\alpha,R}\}_{\alpha,R} \cup \bigcup_{J\supset I} \{\rho_{J,\alpha,R}\}_{\alpha,R}$ restricted to $\bigcup_\alpha V_{I,\alpha}(R) \cup \bigcup_{J\supset I} \bigcup_\beta V_{J,\beta,R}$ be a partition of unity, and suppose that support $\rho_{J,\alpha,R} \subset V_{J,\alpha}(R-2)$. Let $\rho_R = \sum_{I,\alpha} \rho_{I,\alpha,R}$, and $\rho_{\varnothing,R} = 1 - \rho_R$.

Given a parameter R and a constant α with $0 < \alpha \ll 1$ to be determined in §7, we set $t = e^{-2R/(1-\alpha)}$ and fix a parametrix Q of the following form:

$$(30)\quad Q(x, y) = \int_0^t \frac{1}{2} De^{-s\Delta}(1-\tau)\, ds + \rho_{2R}(x) \int_t^\infty \frac{1}{2} De^{-s\Delta}(1-\tau)\, ds.$$

Then

$$D_+Q = \left\{(1-\rho_{2R}) \int_0^t \Delta e^{-s\Delta}\, ds + \rho_{2R} \int_0^\infty \Delta e^{-s\Delta}\, ds \right.$$
$$\left. + [D, \rho_{2R}] \int_t^\infty De^{-s\Delta}\, ds \right\} \frac{1-\tau}{2}$$

$$= \left\{(1-\rho_{2R})(I-e^{-t\Delta})+\rho_{2R}(I-\Pi) + [D, \rho_{2R}]\int_t^\infty De^{-s\Delta}\,ds\right\}\frac{1-\tau}{2}$$
$$= \left\{I - \rho_{2R}\Pi - (1-\rho_{2R})e^{-t\Delta} + [D, \rho_{2R}]\int_t^\infty De^{-s\Delta}\,ds\right\}\frac{1-\tau}{2},$$

where Π denotes the projection onto the kernel of Δ. Similarly,

$$QD_+ = \{I - \rho_{2R}\Pi - (1-\rho_{2R})e^{-t\Delta}\}(1+\tau)/2.$$

Thus if D_+ is Fredholm, then

$$\begin{aligned}
&\operatorname{index}(D_+) \\
&\quad = \operatorname{Tr}(\rho_{2R}\Pi + (1-\rho_{2R})e^{-t\Delta})(1+\tau)/2 \\
&\quad\quad - \operatorname{Tr}\left(\rho_{2R}\Pi + (1-\rho_{2R})e^{-t\Delta} - [D, \rho_{2R}]\int_t^\infty De^{-s\Delta}\,ds\right)\frac{1-\tau}{2} \\
&\quad = \operatorname{Tr}\tau(\rho_{2R}\Pi + (1-\rho_{2R})e^{-t\Delta}) - \operatorname{tr}[D, \rho_{2R}]\int_t^\infty De^{-s\Delta}\frac{1-\tau}{2}\,ds,
\end{aligned} \tag{31}$$

where we recall that we use Tr to denote the global trace, and tr to denote the local pointwise trace.

Lemma 6.1.

$$-\operatorname{Tr}[D, \rho_{2R}]\int_t^\infty De^{-s\Delta}(1-\tau)/2\,ds = \operatorname{Tr}[D, \rho_{2R}]\int_t^\infty De^{-s\Delta}ds\,\tau/2.$$

Proof. We want to show that

$$\operatorname{Tr}[D, \rho_{2R}]\int_t^\infty De^{-s\Delta}\,ds/2 = 0.$$

This follows from a strictly formal argument. If D were a bounded operator we could write

$$\begin{aligned}
&\operatorname{Tr} D\rho_{2R}\int_t^\infty De^{-s\Delta}ds - \operatorname{Tr}\rho_{2R}D\int_t^\infty De^{-s\Delta}\,ds \\
&\quad = \operatorname{Tr} D\rho_{2R}\int_t^\infty De^{-s\Delta}ds - \operatorname{Tr}\rho_{2R}\int_t^\infty De^{-s\Delta}\,ds = 0,
\end{aligned}$$

using the cyclic nature of the trace and the relation $[D, e^{-s\Delta}] = 0$. In order to obtain the desired conclusion from this argument, we replace D by $De^{-x\Delta}$, $x > 0$, and take the limit as x tends to zero. q.e.d.

We may now rewrite our expression for the index as

$$\begin{aligned}
\operatorname{index}(D_+) &= \operatorname{Tr}\tau(\rho_{2R}\Pi + (1-\rho_{2R})e^{-t\Delta}) \\
&\quad + \tfrac{1}{2}\operatorname{Tr}[D, \rho_{2R}]\int_t^\infty \tau De^{-s\Delta}\,ds\,\tau.
\end{aligned} \tag{32}$$

It will follow from Lemma 6.4 that D_+ is Fredholm. Hence it is clear that

$$\lim_{R\to\infty} \operatorname{Tr} \tau(\rho_{2R}\Pi) = 0.$$

In Proposition 8.1 we will prove that, for the signature complex,

$$\lim_{R\to\infty} \operatorname{Tr}[D, \rho_{2R}] \int_t^\infty De^{-s\Delta}\, ds\, \tau/2 = 0.$$

Hence for the signature operator (and similarly for the Dirac complex with coefficients in a bundle with Dirac-good connection) we may conclude that

$$\text{Index}(D_+) = \lim_{R\to\infty} \operatorname{Tr} \tau(1-\rho_{2R})e^{-t\Delta}. \tag{33}$$

For the Gauss-Bonnet and Riemann-Roch complexes

$$\lim_{R\to\infty} \operatorname{Tr}[D, \rho_{2R}] \int_t^\infty De^{-s\Delta}\, ds\, \tau/2 \neq 0,$$

and this term contributes to the index.

In order to prove that D_+ is Fredholm and to compute

$$\lim_{R\to\infty} \operatorname{Tr} \tau(1-\rho_{2R})e^{-t\Delta} \quad \text{and} \quad \lim_{R\to\infty} \operatorname{Tr}[D, \rho_{2R}] \int_t^\infty De^{-s\Delta}\, ds\, \tau/2,$$

we next construct good approximations to $(\Delta - \lambda)^{-1}$ and $e^{-t\Delta}$. The assumption that $t = e^{-2R/(1-\alpha)}$ implies that on the support of $(1-\rho_R)$, $t/(\text{injective radius})^2 \le Ce^{-2R/(1-\alpha)}e^{2R} = Ce^{-2\alpha R/(1-\alpha)}$. This inequality easily implies that for any $N > 0$, the standard local approximation of the restriction of $e^{-t\Delta}$ to this set can be constructed with error $O(t^N)$. We use such local approximations on this set, and obtain

$$\lim_{R\to\infty} \operatorname{Tr}(1-\rho_R)(1-\rho_{2R})\tau e^{-t\Delta} = 2^{n/2} \int L(TM) \tag{34}$$

for the signature complex and analogous local expressions for other complexes. We are thus left to compute $\lim_{R\to\infty} \operatorname{Tr} \rho_R(1-\rho_{2R})\tau e^{-t\Delta}$ and $\frac{1}{2}\lim_{R\to\infty} \operatorname{Tr}[D, \rho_{2R}] \int_t^\infty De^{-s\Delta}\, ds\, \tau$.

We now construct an approximation $H^{I,\alpha}(\lambda)$ to $(\Delta-\lambda)^{-1}$ in a neighborhood of each open set $V_{I,\alpha} \equiv V_{I,\alpha}(R)$ of the open cover of $\mathscr{D}$. Fix $V_{I,\alpha}$. Given the frame fixed in §4, we may treat forms supported on a neighborhood of $V_{I,\alpha}$ as vector-valued functions. Hence we may write a form f as $f = \sum_m f_m(x, r)e^{im\cdot\theta}$, $\theta \in [0, 2\pi]$. Let T_m denote projection onto the span of $e^{im\cdot\theta}$.

Definition 6.2. We call m *singular* if $im_j + A_j$ is not invertible for some j; otherwise m is *nonsingular.*

Let $\mu_j = -ie^{-im\cdot\theta}W_j e^{im\cdot\theta} - iA_j$. When adding a constant and a matrix, we mean the sum of the constant multiple of the identity and the matrix. We treat μ_j and A_j as the components of matrix-valued vector μ and A and use such vector notation as

$$\|\mu\|^2 := \sum_j \mu_j^2.$$

Let η_j be the $O(e^{-r_j})$ vector field corresponding to W_j minus the projection of W_j onto the span of the $\partial/\partial\theta_a$, where the projection is taken relative to the fixed product structure (and not the metric). In this frame and with these notation we may write Δ as

$$\Delta f = \sum_m e^{im\cdot\theta}\left\{ \sum_{i\le n-k}(-\nabla_i\nabla_i + \nabla_{\nabla_i X_i}) + \sum_j(\mu_j - i\Gamma^S_{W_j} - i\Gamma^E_{W_j} - i\eta_j)^2 \right.$$

$$\left. + \sum_j e^{-im\cdot\theta}\nabla_{\nabla_{W_j}W_j}e^{im\cdot\theta} + R\right\} f_m. \tag{35}$$

We construct the approximation to $(\Delta - \lambda)^{-1}$ inductively. Set

$$h_0(x, x', \lambda, v, \mu) = \text{Identity}.$$

Write

$$(\Delta - \lambda)e^{im\cdot(u-u')}e^{i2\pi(x-x')\cdot v}\left\{\frac{h_l(x, x', \lambda, v, \mu)}{(2\pi)^k(\|2\pi v + \mu\|^2 - \lambda)^{l+1}}\right\}$$

$$= e^{im\cdot(u-u')}e^{i2\pi(x-x')\cdot v}\left\{\frac{h_l(x, x', \lambda, v, \mu)}{(2\pi)^k(\|2\pi v + \mu\|^2 - \lambda)^{l}}\right\}$$

$$- 2\nabla\left\{\frac{e^{im\cdot(u-u')}e^{i2\pi(x-x')\cdot v}}{(2\pi)^k}\right\}\cdot\nabla\left\{\frac{h_l(x, x', \lambda, v, \mu)}{(\|2\pi v + \mu\|^2 - \lambda)^{l+1}}\right\}$$

$$+ \left\{\frac{e^{im\cdot(u-u')}e^{i2\pi(x-x')\cdot v}}{(2\pi)^k}\right\}\Delta\left\{\frac{h_l(x, x', \lambda, v, \mu}{(\|2\pi v + \mu\|^2 - \lambda)^{l+1}}\right\}$$

$$= I_{1,l} + I_{2,l} + I_{3,l}.$$

Set

$$h_{l+1} = -(I_{2,l} + I_{3,l})[e^{im\cdot(u-u')}e^{i2\pi(x-x')\cdot v}(\|2\pi v + \mu\|^2 - \lambda)^{l+1}].$$

Then

$$\aleph_1(\Delta-\lambda)\sum_{i=0}^{N}\int_{\mathbf{R}^{n-k}}(2\pi)^{-k}e^{im\cdot(u-u')}e^{i2\pi(x-x')\cdot v}\left\{\frac{h_l(x,x',\lambda,v,\mu)}{(\|2\pi v+\mu\|^2-\lambda)^{l+1}}\right\}dv\aleph_2$$
$$=\aleph_1\left(T_m+\int_{\mathbf{R}^{n-k}}(I_{2,N}+I_{3,N})dv\right)\aleph_2,$$

for any cutoff functions $\aleph_1$ and $\aleph_2$ with support in a neighborhood of $V_{I,\alpha}$. Clearly, for N sufficiently large, the error term

$$\aleph_1\int_{\mathbf{R}^{n-k}}(I_{2,N}+I_{3,N})\,dv\,\aleph_2$$

and its derivatives are $O(e^{-Nr/2})$, when m is nonsingular. For such estimates, it is useful to recall that

$$2\|2\pi v+\mu\|^2\geq|2\pi v|^2+\sum e^{2r_j}(m_j-iA_j)^2.$$

Similarly, error terms (and their derivatives) associated with derivatives of cutoffs will be exponentially decreasing in r when m is nonsingular. Set

$$H_m^{I,\alpha}(\lambda)\equiv\sum_{i=0}^{N}\int_{\mathbf{R}^{n-k}}(2\pi)^{-k}e^{im\cdot(u-u')}e^{i2\pi(x-x')\cdot v}\times(\|2\pi v+\mu\|^2-\lambda)^{-l-1}h_l(x,x',\lambda,v,\mu)\,dv. \tag{36}$$

Let φ be a cutoff function with support in $V_{I,\alpha}$ and ψ be a cutoff function with $\psi|_{support\,\varphi}\equiv 1$. Then $\psi H_m^{I,\alpha}\varphi$ is a good approximation to $(\Delta-\lambda)^{-1}T_m\varphi$ for m nonsingular; i.e., the error terms (and their derivatives) are exponentially decreasing; hence they must have $O(e^{-R})$ trace norm.

Now consider m singular. We use an iterative construction of an approximation $H_m^{I,\alpha}$ to $(\Delta-\lambda)^{-1}T_m$ following the same recipe as for m nonsingular, except that in the case $\mu=0$, we construct $(\Delta-\lambda)^{-1}$ as a perturbation of $(\Delta_I+\sum_i(-\partial^2/\partial r_i^2+1/4)-\lambda)^{-1}$, where Δ_I is the Laplace operator on $\mathscr{D}_I$ (associated to the complex in question). The point here is that the $k/4$ in the denominator ensures that terms like $\int e^{i(x-y)\cdot u}(|u|^2+k/4-\lambda)^{-j}du$ are bounded independent of λ, for λ in a small neighborhood of $0\in\mathbf{C}$. The error terms from the above construction are smooth, and for T (defined in (1)) sufficiently large can be made to have arbitrarily small sup norm and pointwise trace. Increasing T decreases correspondingly the curvature and connection terms arising in the above iteration.

Because m is singular, we no longer have the rapid decay in r of our integral operators and their error terms which was guaranteed by the terms of the form $[\sum e^{2r_i}(m_i - iA_i)^2]^{-n}$ with all $m_i - iA_i$ invertible. We may have decay in some r_i directions but not all. In particular, the error terms will not be trace class. Nonetheless, we set

$$H^{I,\alpha}(\lambda) = \sum_m H_m^{I,\alpha} = H_{\text{non}}^{I,\alpha} + H_{\text{sing}}^{I,\alpha}, \tag{37}$$

where $H_{\text{sing}}^{I,\alpha} = \sum_{m \text{ singular}} H_m^{I,\alpha}$.

Remark 6.3. It follows form Remark 5.2 that any factors of $C(W_j)$ or $C(R_j)$ which occur in the term in $H_{\text{sing}}^{I,\alpha}$ corresponding to $\mathrm{im}_j + A_j = 0$ occur in terms which are $O(e^{-r_j})$.

When $I = \varnothing$, we use any standard local construction for an approximation $H^{\phi,\alpha}(\lambda)$ to $(\Delta - \lambda)^{-1}$ with

$$\begin{aligned} &\|(1-\rho_{R/2})((\Delta-\lambda)^{-1} - H^{\phi,\alpha}(\lambda))\|_{\text{trace norm}} \\ &\quad = O((\text{injectivity radius of } V_{\phi,\alpha}(R))^{-2N}|\lambda|^{-N}) = O(e^{-2RN}|\lambda|^{-N}), \end{aligned}$$

for any $N > 0$ and any λ with $\arg(\lambda/|\lambda|)$ bounded uniformly from zero. (We will primarily be interested in $|\lambda|^{-1} \le t \le e^{-2R/(1-\alpha)}$.) When $\lambda = 0$, we merely require that the corresponding trace norm be finite. We patch the $H^{I,\alpha}(\lambda)$ together to form

$$\widetilde{H}_n(\lambda) = \sum_{I,\alpha} \psi_{I,\alpha,R} H_{\text{non}}^{I,\alpha}(\lambda)\rho_{I,\alpha,R}, \tag{38}$$

$$\widetilde{H}_s(\lambda) = \sum_{I,\alpha} \psi_{I,\alpha,R} H_{\text{sing}}^{I,\alpha}(\lambda)\rho_{I,\alpha,R}, \tag{39}$$

$$\widetilde{H}(\lambda) = \widetilde{H}_n(\lambda) + \widetilde{H}_s(\lambda), \tag{40}$$

$$P = \sum_{I,\alpha} \psi_{I,\alpha,R} D H^{I,\alpha}(0)\rho_{I,\alpha,R}, \tag{41}$$

where $\psi_{I,\alpha,R}$ is a smooth cutoff function equal to one in a large neighborhood of support $\rho_{I,\alpha,R}$, and satisfying, for $a = 1, 2$,

$$|\nabla_{R_i}^a \psi_{I,a,R}| \le 4R^{-a}, \qquad i \le k,$$

$$|\nabla_{X_i}^a \psi_{I,\alpha,R}| \le (T/2)^{-a}, \qquad i \le 2n - 2k,$$

$$\partial/\partial\theta_i \psi_{I,\alpha,R} = 0, \qquad i \le k.$$

This construction is sufficient to prove that D_+ is Fredholm.

Lemma 6.4. *D_+ is Fredholm.*

Proof. P is a bounded operator because of the $k/4$ in the denominator of the Fourier transform of the $H^{I,\alpha}(0)$. Moreover, we have

$$\begin{aligned} DP &= \sum_{I,\alpha} \psi_{I,\alpha,R} \Delta H^{I,\alpha}(0)\rho_{I,\alpha,R} + \sum [D, \psi_{I,\alpha,R} D] H^{I,\alpha}(0)\rho_{I,\alpha,R} \\ &= I + K_0 + \varepsilon_1, \end{aligned}$$

where ε_1 is the sum of all the error terms coming from noncompact neighborhoods, and K_0 is the error term associated with the compact neighborhood and is therefore a trace class operator. From the above discussion, we may assume that ε_1 has arbitrarily small sup norm. Hence $(I+\varepsilon_1)^{-1}$ is a bounded operator, and

$$DP(I+\varepsilon_1)^{-1} = I + \text{trace class operator}.$$

This implies that D is Fredholm; consequently, D_+ is Fredholm. q.e.d.

We now complete our construction of an approximate to $(\Delta-\lambda)^{-1}$. Set

$$\begin{aligned} &(\Delta-\lambda)\widetilde{H}(\lambda) \\ &\quad = \sum_{I,\alpha} \psi_{I,\alpha,R}(\Delta-\lambda) H^{I,\alpha}(\lambda)\rho_{I,\alpha,R} + \sum_{i,\alpha} [\Delta, \psi_{I,\alpha,R}] H^{I,\alpha}(\lambda)\rho_{I,\alpha,R} \\ &\quad = I + K(\lambda) - \varepsilon_n(\lambda) - \varepsilon_s(\lambda), \end{aligned}$$

where $K(\lambda)$ is again the error associated to the compact neighborhood, $-\varepsilon_n(\lambda)$ is the error associated to the noncompact neighborhoods and the $H^{I,\alpha}_{\text{non}}$, and $-\varepsilon_s(\lambda)$ is the error associated to the $H^{I,\alpha}_{\text{sing}}$. Set $\varepsilon(\lambda) = \varepsilon_n + \varepsilon_s$. The sup norm of $\varepsilon(\lambda)$ is $O(T^{-1})$. We take

$$H(\lambda) = \widetilde{H}(\lambda)(I-\varepsilon(\lambda))^{-1} = \widetilde{H}(\lambda)\sum_{a=0}^{\infty} \varepsilon(\lambda)^a \tag{42}$$

as our approximation to $(\Delta-\lambda)^{-1}$ on the support of ρ_R; the convergence of the sum is guaranteed by the small upper bound on the sup norm of $\varepsilon(\lambda)$. Then

$$(\Delta-\lambda)H(\lambda) = I + K(\lambda)(I-\varepsilon(\lambda))^{-1}.$$

We divide $\varepsilon(\lambda)$ into three types of error terms: perturbation, exterior cutoff, and interior cutoff. Substituting $H^{I,\alpha}_{\text{non}}$ or $H^{I,\alpha}_{\text{sing}}$ for $H^{I,\alpha}$ gives corresponding decompositions of ε_n and ε_s. The perturbation error, ε_p, is the sum of the error terms

$$\varepsilon_{p,I,\alpha,R} \equiv \psi_{I,\alpha,R}(I - (\Delta - \lambda)H^{I,\alpha}(\lambda))\rho_{I,\alpha,R}.$$

This error is $O(e^{-Nr})$ by construction. This is clear for the terms with $\mu \neq 0$. For $\mu = 0$, this follows from the fact that ΔT_0 is an $O(e^{-r})$ perturbation of $\Delta_I + \sum_i \Delta_P$. As $(\Delta - \lambda)^{-1}$ was constructed as a perturbation of $(\Delta_I + \sum_i \Delta_P - \lambda)^{-1}$, the error terms can be taken to be $O(e^{-Nr})$. We remark that, in the singular cases, the e^{-Nr} decay is only guaranteed with respect to some r_i.

In order to define the exterior and interior cutoff terms we refine our choice of $\psi_{I,\alpha,R}$ by assuming that $\psi_{I,\alpha,R} = \psi_{e,I,R}\psi_{i,\alpha,I,R}$, where $\psi_{e,I,R}$ is a cutoff function equal to one in a neighborhood of $\mathscr{D}_I$, with $d(x, y) \geq R/2$, for every $(x, y) \in \operatorname{support} \nabla\psi_{e,I,R} \times \operatorname{support} \rho_{I,\alpha,R}$, and $\psi_{i,\alpha,I,R}$ is the pullback to $(\Delta^*)^{|I|} \times U_\alpha$ of a cutoff function on $\mathscr{D}_I$ supported on a coordinate neighborhood and satisfying $d(x, y) \geq T/2$ for every $(x, y) \in \operatorname{support} \nabla\psi_{i,\alpha,I,R} \times \operatorname{support} \rho_{I,\alpha,R}$. Here $d(x, y)$ denote the distance between x and y.

With these choices we define the exterior cutoff error, ε_e, to consist of the sum of the error terms

$$\varepsilon_{e,I,\alpha,R} = ([\Delta, \psi_{I,\alpha,R}] - \psi_{e,I,R}[\Delta, \psi_{i,\alpha,I,R}])H^{I,\alpha}\rho_{I,\alpha,R}, \tag{43}$$

and the interior cutoff error, ε_i, to be the sum of the error terms

$$\varepsilon_{i,I,\alpha,R} = \psi_{e,I,R}[\Delta, \psi_{i,\alpha,I,R}]H^{I,\alpha}\rho_{I,\alpha,R}.$$

We write $\varepsilon_{n,p}$, $\varepsilon_{n,e}$, $\varepsilon_{n,i}$, $\varepsilon_{s,p}$, $\varepsilon_{s,e}$, and $\varepsilon_{s,i}$ for the corresponding summands of ε_n and ε_s.

Let $\mathscr{C}_t$ be a curve in $\mathbf{C}$ which surrounds the spectrum of Δ, with $\arg(\lambda) \in (b, 2\pi - b)$, for some $b > 0$, for all $\lambda \in \mathscr{C}_t$. Moreover, we assume that $t|\max(-\operatorname{Re}\lambda, 0)| \leq 1$, and $|\lambda| \geq t^{-1}$. For example, we may take $\mathscr{C}_t$ to be the image of $\mathbf{R}$ under the map $u \to |u| + iu - t$. By construction, $\|K(\lambda)\|_{\mathrm{tr}} = O(t^N)$ for $\lambda \in \mathscr{C}_t$. Hence we can ignore this term in our computations.

Remark 6.5. When one uses the functional calculus to compute the heat kernel

$$e^{-t\Delta} = (2\pi i)^{-1} \int_\gamma e^{-t\lambda}(\Delta - \lambda)^{-1}\, d\lambda,$$

the well-known phenomenon that $e^{-t\Delta}$ becomes easier to compute as t tends to zero is realized by choosing $\gamma = \mathscr{C}_t$. As indicated above, this allows one to make the error terms involved in standard local approximations to $(\Delta - \lambda)^{-1}$ to be $O(t^N)$. One cannot, of course, use C_t for t small to compute $e^{-T\Delta}$ for T large. The hypothesis that $T = t$ is used to bound $|e^{-t\lambda}|$, $\lambda \in \mathscr{C}_t$.

Set $\varepsilon_n' = \varepsilon_{s,e} + \varepsilon_n$ and $\varepsilon_s' = \varepsilon_s - \varepsilon_{s,e}$. We will also write $\varepsilon_{s,I,\alpha,R}$ to denote the summand of ε_s' coming from the I, α neighborhood. Expanding $(I - \varepsilon(\lambda))^{-1}$ into a power series in ε_n' an d ε_s', we write

$$(I - \varepsilon(\lambda))^{-1} = I + E_n + E_s,$$

where E_n is the sum of terms in $(I-\varepsilon(\lambda))^{-1}$ where ε_n' occurs to a positive power. We expand $H(\lambda)$ as

$$H(\lambda) = \widetilde{H}(\lambda) + H_n + H_s,$$

where $H_n = \widetilde{H}_n(\lambda)E_s + \widetilde{H}(\lambda)E_n$ and $H_s = \widetilde{H}_s(\lambda)E_s$.

Lemma 6.6. $(2\pi i)^{-1}\mathrm{Tr}\,\tau \int_{\mathscr{C}_t} e^{-t\lambda} H_n(\lambda)\, d\lambda\, \rho_R(1 - \rho_{2R}) = O(R^{-1})$.

Proof. We remark that the $T^{-q-q'}$ bound on the sup norm of $\varepsilon_n^q \varepsilon_s^{q'}$ immediately reduces the estimate of the trace norm of $\widetilde{H}_n(\lambda)E_s$ and $\widetilde{H}(\lambda)E_n$ to that of $\widetilde{H}_n(\lambda)\varepsilon_s'$ and ε_n' respectively. The desired estimate is clear for $\widetilde{H}_n(\lambda)E_s$ and for ε_n. For ε_e (and similarly for its derivatives) we have the estimate

$$|\varepsilon_{e,\alpha,I,R}(\lambda)(x, y)| = O(e^{-|\lambda|^{1/2}|x-y|/B}) \le O(e^{-|\lambda|^{1/2}R/B'}), \tag{44}$$

for $(x, y) \in \mathrm{support}(\nabla \psi_{e,I,R}) \times \mathrm{support}(\rho_{I,\alpha,R})$, and B and $B' > 0$ depending on b. This estimate follows from integrating the explicit form of $H^{I,\alpha}$ and implies the desired estimate of the trace norm.

Lemma 6.7. $(2\pi i)^{-1}\mathrm{Tr}\,\tau \int_{\mathscr{C}_t} e^{-t\lambda} H_s(\lambda) d\lambda\, \rho_R(1 - \rho_{2R})$ *is* $O(R^{-1})$.

Proof. We have $\mathrm{Tr}\,\tau H_{\mathrm{sing}}^{\alpha,I} \circ \varepsilon_{s,I,\alpha,R}^N \rho_R(1 - \rho_{2R}) = O(e^{-R})$ for all N, because the factors of $C(R_i)C(W_i)$ which we need for nonzero trace only enter with $O(e^{-2r_i})$ coefficients. (See the remark following (37).) More generally we have

Claim 6.8. *If there exists an* $I_a \in \{I_0, \cdots, I_l\}$ *such that* $I_a \subset I_j$, *for all* $j \in \{0, \cdots, l\}$, *then*

$$\mathrm{Tr}\,\tau H_{\mathrm{sing}}^{\alpha,I_0} \circ \varepsilon_{s,I_1,\alpha_1,R} \circ \cdots \circ \varepsilon_{s,I_l,\alpha_l,R} \rho_R(1 - \rho_{2R}) = O(e^{-R}).$$

Proof of claim. This follows as above because the necessary $C(R_i)C(W_i)$ factors only enter with $O(e^{-2r_i})$ decrease.

In order to pick up the extra factors of $C(R_i)C(W_i)$ required for non-vanishing trace, we must compose error terms coming from distant neighborhoods; thus, we must consider terms of the form

$$\operatorname{Tr} \tau H^{\alpha, I_0}_{\text{sing}} \varepsilon_{s, I_1, \alpha_1, R} \circ \cdots \circ \varepsilon_{s, I_l, \alpha_l, R} \rho_R (1 - \rho_{2R})$$

with some I_j not contained in I_{j+1} and show that this trace is $O(R^{-1})$. Let $(x, y) \in \operatorname{support} \varepsilon_{s, I, \alpha, R} \times \operatorname{support} \varepsilon_{s, J, \beta, R}$, with I not contained in J and J not contained in I. As before, we have the estimate

$$|\varepsilon_{s, I, \alpha, R}(\lambda)(x, y)| \le cT^{-1} e^{-|\lambda|^{1/2}|x-y|/B}.$$

Hence

$$|\varepsilon_{s, I, \alpha, R} \circ \cdots \circ \varepsilon_{s, J, \beta, R}(x, y)|$$
$$\le c \int \cdots \int e^{-|\lambda|^{1/2}|x-t_1|/B} e^{-|\lambda|^{1/2}|t_1-t_2|/B} \cdots e^{-|\lambda|^{1/2}|t_n-y|/B}\, dt_1 \cdots dt_n T^{-n}.$$

Corresponding estimates hold for the derivatives of $\varepsilon_{s, I, \alpha, R} \circ \cdots \circ \varepsilon_{s, J, \beta, R}(x, y)$. The integral takes place over a region with $|x - t_1| + \cdots + |t_n - y| \ge R/2$. Hence the integral is $O(R^{-1})$ and so too is the trace.

Lemma 6.9.

$$(2\pi i)^{-1} \operatorname{Tr} \rho_R (1 - \rho_{2R}) \int_{\mathscr{C}_t} e^{-t\lambda} \tilde{H}(\lambda)\, d\lambda$$
$$= (2\pi i)^{-1} \operatorname{Tr} \rho_R (1 - \rho_{2R}) e^{-t\Delta} + O(R^{-1}).$$

Proof. From Lemmas 6.6 and 6.7, we conclude that

$$\operatorname{Tr} \rho_R (1 - \rho_{2R}) \int_{\mathscr{C}_t} e^{-t\lambda} \tilde{H}(\lambda)\, d\lambda$$
$$= \operatorname{Tr} \rho_R (1 - \rho_{2R}) \int_{\mathscr{C}_t} e^{-t\lambda} H(\lambda)\, d\lambda + O(R^{-1}).$$

Moreover,

$$(\Delta - \lambda)^{-1} - H(\lambda)\rho_R(1 - \rho_{2R}) = (\Delta - \lambda)^{-1}\,(I - (\Delta - \lambda)H(\lambda))\rho_R(1 - \rho_{2R})$$
$$= (\Delta - \lambda)^{-1} K (I - \varepsilon(\lambda))^{-1} \rho_R (1 - \rho_{2R}).$$

The estimate $\|K(\lambda)\|_{\text{tr}} = O(t^N)$ and the boundedness of the sup norms of $(\Delta - \lambda)^{-1}$ and $(I - \varepsilon(\lambda))^{-1}$, for $\lambda \in \mathscr{C}_t$, imply

$$\operatorname{Tr} \rho_R (1 - \rho_{2R}) \int_{\mathscr{C}_t} e^{-t\lambda} \{(\Delta - \lambda)^{-1} - H(\lambda)\}\, d\lambda\, \rho_R (1 - \rho_{2R}) = O(R^{-1}).$$

The result follows.

7. The trace

We continue the notation of §6. In this section we compute $\operatorname{Tr}\tau\rho_R(1-\rho_{2R})e^{-t\Delta}$, using Lemma 6.9. First fix $V_{I,\alpha}$ with $|I|=k$ and consider the contribution of $V_{I,\alpha}$ to the trace. We compute

$$\int_{(S^1)^k}\frac{1}{2\pi i}\int_{\mathscr{C}_t} e^{-t\lambda}\operatorname{tr}\tau H^{I,\alpha}(\lambda)\,d\lambda\,d\theta$$
$$=\frac{1}{2\pi i}\int_{\mathscr{C}_t}\sum_m\sum_{i=0}^{N}\int_{\mathbf{R}^{n-k}}\frac{e^{-t\lambda}\operatorname{tr}\tau h_1(x,x',\lambda,v,\mu)}{(\|2\pi v+\mu\|^2-\lambda)^{l+1}}dv\,d\lambda,$$

where the $d\theta$ integral is the integral over the second factor in

$$V_{i,\alpha}=[R,\infty)^k\times(S^1)^k\times U_{I,\alpha}.$$

Recall that h_l is constructed from h_{l-1} by applications of conjugates of $\Delta=\Delta_2+\Delta_3$, where Δ_2 and Δ_3 have masses 2 and 3 respectively and are defined in (29) by

$$\Delta_3=\sum_j(W_j+A_j)(C(W_j)C(R_j)-\widehat{C}(W_j)\widehat{C}(R_j)),$$

(for Riemann-Roch calculations this is replaced by $\Delta_3'=\sum_j(W_j+A_j)C(\overline{Z}_j)\widehat{C}(\overline{Z}_j))$. All other factors of $C(W_j)C(R_j)$ that arise are $O(e^{-r_j})$. Inductively this implies that

$$\text{the mass of } h_l\le 2l+\min\{l,k\}. \tag{45}$$

Because Δ_3 has mass 3 one might expect the above bound to be mass $h_l\le 3l$, but one observes that after k applications of Δ_3, there are no additional Clifford factors added by higher powers. Write

$$h_l(x,x,\lambda,v,m)=\sum_\sigma(\|2\pi v+\mu\|^2-\lambda)^{-\sigma}h_{l,\sigma}(x,v,\mu),$$

and

$$h_{l,\sigma}(x,v,\mu)=\sum h_{l,\sigma,A,B,C}(x)v^A\mu^Be^{-rC},$$

with $h_{l,\sigma,A,B,C}(x)$ bounded. The $h_{l,\sigma}$ with $\sigma>0$ arise from terms (and their derivatives) of the form

$$(\|2\pi v+\mu\|^2-\lambda)\partial_i(\|2\pi v+\mu\|^2-\lambda)^{-1},$$

and

$$(\|2\pi v+\mu\|^2-\lambda)\partial_i\partial_j(\|2\pi v+\mu\|^2-\lambda)^{-1}$$

arising in the construction of h_l. These operations raise the mass of h_i by less than two. Hence our earlier mass considerations imply

$$\begin{aligned} &|A| + |B| - |C| + \text{Clifford degree of } h_{l,\sigma,A,B,C}(x) \\ &\quad \leq \sigma + 2l + \min\{l, k\}. \end{aligned} \tag{46}$$

Set

$$I'(m) = \{j : m_j = 0\}, \quad \text{and} \quad I'(B) = \{j : B_j = 0\}.$$

Any term with $\mu_j = 0$ or $B_j = 0$ clearly cannot have Δ_3 contributing $C(W_j)$ or $C(R_j)$ in a term of mass greater than 2. Thus we may refine (46) to obtain

$$\begin{aligned} &|A| + |B| - |C| + \text{Clifford degree of } h_{l,\sigma,A,B,C}(x) \\ &\quad \leq \sigma + 2l + \min\{l, k - |I'(B) \cup I'(m)|\}. \end{aligned} \tag{47}$$

Recall that $\operatorname{tr} \tau h_l = 0$ unless the Clifford degree of $h_l \geq n$. Hence if $\operatorname{tr} \tau h_{l,\sigma,A,B,C}(x) \neq 0$, we have

$$n + |A| + |B| - |C| \leq \sigma + 2l + \min\{l, k - |I'(B) \cup I'(m)|\}.$$

We compute.

$$\begin{aligned} &\frac{1}{2\pi i} \int_{(S^1)^k} \int_{\mathscr{C}_t} e^{-t\lambda} \operatorname{tr} \tau H^{I,\alpha}(\lambda)\, d\lambda \\ &= \frac{1}{2\pi i} \int_{\mathscr{C}_t} \sum_m \sum_{l=n/4}^{N} \int_{\mathbf{R}^{n-k}} \frac{e^{-t\lambda} \operatorname{tr} \tau h_{l,\sigma}(x, v, \mu)}{(\|2\pi v + \mu\|^2 - \lambda)^{l+1+\sigma}}\, dv\, d\lambda \\ &= \sum_m \sum_{l=n/4}^{N} [(l+\sigma)!]^{-1} ((-d/ds)^{l+\sigma})|_{s=0} (2\pi i)^{-1} \\ &\quad \times \int_{\mathscr{C}_t} \int_{\mathbf{R}^{n-k}} e^{-t\lambda} \frac{\operatorname{tr} \tau h_{l,\sigma}(x, v, \mu)}{(\|2\pi v + \mu\|^2 - \lambda + s)}\, dv\, d\lambda \\ &= \sum_m \sum_{l=n/4}^{N} \frac{1}{(l+\sigma)!} \int_{\mathbf{R}^{n-k}} e^{-t\|2\pi v + \mu\|^2} t^{l+\sigma} \operatorname{tr} \tau h_{l,\sigma}(x, v, \mu)\, dv \\ &= \sum_m \sum_{l=n/4}^{N} [(l+\sigma)!]^{-1} \int_{\mathbf{R}^{n-k}} \frac{t^{l+\sigma} \operatorname{tr} \tau h_{l,\sigma,A,B,C}(x) v^A \mu^B}{e^{t\|2\pi v + \mu\|^2} e^{rC}}\, dv \\ &= \sum_m \sum_{l=n/4}^{N} \frac{t^{l+\sigma+(k-|A|-n)/2} \operatorname{tr} \tau h_{l,\sigma,A,B,C}(x) \mu^B}{(l+\sigma)! e^{rC} e^{t\|\mu'\|^2}} \\ &\qquad \times \int_{\mathbf{R}^{n-k}} e^{-\|2\pi v\|^2} (v - \tilde{\mu} t^{1/2})^A\, dv. \end{aligned} \tag{48}$$

Here $\mu = 2\pi\tilde{\mu} + \mu'$, with $\tilde{\mu}$ the projection of μ onto $\mathbf{R}^{n-k}$ determined by the norm $\|\cdot\|^2$.

In order to understand the contribution of (48) to the index, we need to compute its integral over $[R, 2R]^k$. (In the limit as $R \to \infty$, there is no difference if $\rho_R(1-\rho_{2R})$ is replaced by the characteristic function of $[R, 2R]^k$.) First we estimate the quantity

$$(49) \qquad \sum_m \int_R^\infty \cdots \int_R^\infty t^{l+\sigma+(k-|A|-n)/2} \mu^B e^{-rC} e^{-t\|\mu'\|^2}\, dr_1 \cdots dr_k,$$

under the assumptions that

$$\begin{gathered} n + |A| + |B| - |C| \le \sigma + 2l + \min\{l, k - |I'(B)|\}, \\ C_i \ge 1, \quad \text{for all } i \in I'(B),\ B_j \text{ is even}, \\ t^\alpha \le te^{2R} \le \varepsilon, \qquad \alpha > 0 \text{ to be determined.} \end{gathered}$$

The condition that $C_i \ge 1$ for $i \in I'(B)$ arises from the fact mentioned above that, for $i \in I'(B)$, factors of $C(W_i)C(R_i)$ only enter with coefficients which are $O(e^{-r_i})$. For the purpose of the estimate, it suffices to replace $\|\mu\|^2$ by $\sum_j e^{2r_j}(m_j - iA_j)^2$. Then we may estimate (49) by

$$t^{l+\sigma+(k-|A|-n)/2} \prod_j \sum_{p\in\mathbf{Z}} (p - iA_j)^{B_j} \int_R^\infty e^{r(B_j - C_j)} e^{-te^{2r}(p-iA_j)^2}\, dr.$$

Consider first the term

$$\begin{aligned} &\sum_{p\in\mathbf{Z}} (p - iA_j)^{B_j} \int_R^\infty e^{r(B_j-C_j)} e^{-te^{2r}(p-iA_j)^2}\, dr \\ (50) \qquad &= t^{(C_j-B_j)/2} \sum_{p\in\mathbf{Z}} (p - iA_j)^{B_j} \int_{te^{2R}}^\infty \frac{r^{(B_j-C_j-2)/2}}{2e^{t(p-iA_j)^2}}\, dr \\ &= t^{(C_j-B_j)/2} \sum_{p\in\mathbf{Z}} (p - iA_j)^{B_j} \int_{te^{2R}}^1 \frac{r^{(B_j-C_j-2)/2}}{e^{r(p-iA_j)^2}}\, dr + O(t^{(C_j-B_j)/2}). \end{aligned}$$

We may use the Poisson summation formula to estimate (50) as

$$\begin{aligned} &t^{(C_j-B_j)/2} c_j \sum_{p\in\mathbf{Z}} e^{2\pi p A_j} \int_{te^{2R}}^1 r^{(B_j-C_j)/2} (d/dp)^{B_j} e^{-ap^2/r} r^{-1/2}\, dr/r \\ &+ O(t^{(C_j-B_j)/2}), \end{aligned}$$

where c_j and a are positive constants. This expression may be rewritten as

$$t^{(C_j-B_j)/2}\left\{O(1)+c_j\sum_{p\in\mathbf{Z}}e^{2\pi pA_j}\int_1^{(te^{2R})^{-1}}r^{(-B_j+C_j-2)/2}\left(\frac{d}{dp}\right)^{B_j}\frac{r^{1/2}}{e^{ap^2r}}dr\right\}$$

$$=t^{(C_j-B_j)/2}\left\{O(1)+c_j'\int_1^{(te^{2R})^{-1}}r^{(C_j-1)/2}dr\right\}$$

$$=t^{(C_j-B_j)/2}\left\{O(1)+c_j''(te^{2R})^{-(C_i+1)/2}\right\},$$

for some positive constants c_j' and c_j'' which vanish if $B_j\neq 0$. Recall that $te^{2R}=e^{-2R/(1-\alpha)}e^{2R}=e^{-2\alpha R/(1-\alpha)}=t^\alpha$. Thus for some $\tilde{c}>0$, (49) can be estimated by

$$\tilde{c}t^{l+\sigma+(k-|A|-n)/2}\prod_j t^{(C_j-B_j)/2}(1+t^{-\alpha(C_j+1)/2})$$

$$=\tilde{c}t^{(\sigma+k-\min\{l,k-|I'(B)|\}+f)/2}\prod_j(1+t^{-\alpha(C_j+1)/2}),$$

where $f\equiv\sigma+2l+\min\{l,\,k-|I'(B)|\}-|A|-|B|+|C|-n$ is nonnegative by assumption. Choose α sufficiently small so that for all C, f pairs which arise in the iteration process, $f-\sum_j\alpha(C_j+1)\geq 1/10$, unless $f=0$, in which case $-\sum_j\alpha(C_j+1)\geq -1/10$. This is possible because f grows as fast as C_j. Thus (49) is dominated by

$$\tilde{c}t^{(\sigma+k-\min\{l,k-|I'(B)|\}+f-\sum_j\alpha(C_j+1))/2},$$

This is shrinking to zero as $t\to 0$ unless

$\sigma=0$,
$l\geq k$, $I'(B)=\varnothing$, and
$n+|A|+|B|-|C|=2l+k$.

The above three relations imply that only the terms in h_l of mass $2l+k$ contribute to the index computation (in the summand $\lim_{R\to\infty}\operatorname{Tr}\rho_R(1-\rho_{2R})\tau e^{-t\Delta}$). These are the terms of maximal mass. In particular, if we discard all terms of mass <2 in Δ when computing h, we see that it suffices to replace $\Delta|_{\operatorname{Im}T_m}$ by

$$\Delta_m=\Delta_I+\sum_j(e^{r_j}(m_j-iA_j)-i\Gamma^S_{W_j})^2-\partial^2/\partial r_j^2.$$

We remark that we have also discarded some terms of mass 2, for example, those of the form $X_a e^{-r_j}C(W_j)C(R_j)$. We can do this because we have shown above that only those terms with $C(W_j)C(R_j)$ entering as factors in mass 3 terms contribute to our computation. Recall (28) that

$$\Gamma^S_{W_j} = -e^{-r_j}\gamma_j - p_j + \text{lower real mass terms},$$

where $\gamma_j = C(\pi\nu_j/2)$, and $p_j = C(R_j)C(W_j)/2$. The maximal mass terms in our parametrix for $e^{-t\Delta}$ are given by the maximal mass terms in

$$\sum_m e^{-t\Delta_I} \prod_j e^{-t(e^{r_j}(m_j - iA_j) + ie^{-r_j}\gamma_j + ip_j)^2}.$$

In particular, up to terms which vanish as te^{2R} tends to zero, our trace is given by the maximal mass terms in

$$\begin{aligned}
&\operatorname{tr} \tau e^{-t\Delta_I} \sum_m \prod_j e^{-t(e^{r_j}(m_j - iA_j) + ie^{-r_j}\gamma_j + ip_j)^2} e^{t\partial^2/\partial r_j^2} \\
&= 2^{k-n/2} \sum_a (2\pi i t)^{(2k-n+a)/2} \\
&\quad \times \operatorname{tr} \tau L_a \sum_m \prod_j (4\pi t)^{-1/2} e^{-t(e^{r_j}(m_j - iA_j) + ie^{-r_j}\gamma_j + ip_j)^2},
\end{aligned}$$

where L_a is Clifford multiplication by $i^{k-n/2}$ times the component of the stable L polynomial (or Todd polynomial etc. for other complexes) of $\mathscr{D}_I$ of Clifford degree a. The orientation of $\mathscr{D}_I$ is determined by the complex structure. We have used the result of the calculation in the main theorem in [5, p. 113] to compute the small t asymptotics of $e^{-t\Delta_I}$. We compute now

$$\begin{aligned}
&\sum_a 2^{-n/2} (2\pi i t)^{(2k-n+a)/2} (\pi t)^{-k/2} \operatorname{tr} \tau L_a \prod_j \int_R^{2R} \sum_{q \in Z} e^{-t(e^r(q - iA_j) + ie^{-r}\gamma_j + ip_j)^2} dr \\
&\quad = \sum_a 2^{-n/2} (2\pi i t)^{(2k-n+a)/2} (\pi t)^{-k/2} \operatorname{tr} \tau L_a \\
&\qquad\qquad \times \prod_j \int_R^\infty \sum_{q \in Z} e^{-t(e^r(q - iA_j) + ie^{-r}\gamma_j + ip_j)^2} dr + o(1).
\end{aligned}$$

We use the Poisson summation formula to rewrite this as

$$\begin{aligned}
&\sum_a 2^{-n/2} (2\pi i t)^{(2k-n+a)/2} (\pi t)^{-k/2} \operatorname{tr} \tau L_a \\
&\quad \times \prod_j \int_R^\infty \sum_q \int_{\mathbf{R}} e^{-i2\pi q x} e^{-te^{2r}(x - iA_j + ie^{-2r}\gamma_j + ie^{-r}p_j)^2} dx\, dr + o(1)
\end{aligned}$$

$$= \sum_a 2^{-n/2}(2\pi it)^{(2k-n+a)/2}(\pi t)^{-k/2}\operatorname{tr}\tau L_a$$
$$\times \prod_j \int_R^\infty \sum_q \int_{\mathbf{R}} e^{-i2\pi q(x+iA_j-ie^{-2r}\gamma_j-ie^{-r}p_j)}e^{-te^{2r}x^2}\,dx\,dr + o(1).$$

The maximal mass terms in the above are given by

$$\sum_a 2^{-n/2}(2\pi it)^{(2k-n+a)/2}t^{-k}\operatorname{tr}\tau L_a$$
$$\times \prod_j \int_R^\infty \sum_{q\neq 0} e^{2\pi q(A_j-e^{-2r}\gamma_j-e^{-r}p_j)}e^{-r}e^{-\pi^2q^2e^{-2r}/t}\,dr + o(1)$$
$$= \sum 2^{-n/2}(2\pi it)^{(2k-n+a)/2}t^{-k}\operatorname{tr}\tau L_a$$
$$(51)\quad \times \prod_j \int_0^\infty \sum_{q\neq 0} e^{2\pi qA_j}e^{-2\pi q\rho(\gamma_j+\rho^{-1/2}p_j)}\frac{\rho^{-1/2}}{2}e^{-\pi^2q^2\rho/t}\,d\rho + o(1)$$
$$= \sum_a 2^{-n/2}(2\pi it)^{(2k-n+a)/2}t^{-k}\operatorname{tr}\tau L_a$$
$$\times \prod_j \int_0^\infty \sum_{q\neq 0} e^{2\pi qA_j}\sum_{b=1}^\infty(-2\pi q\rho)^b\frac{\gamma_j^{b-1}}{2\rho(b-1)!}p_je^{-\pi^2q^2\rho/t}\,d\rho + o(1)$$
$$= \sum_a 2^{-n/2}(2\pi it)^{(2k-n+a)/2}t^{-k}\operatorname{tr}\tau L_a$$
$$\times \prod_j \sum_{q\neq 0} e^{2\pi qA_j}\sum_{b=1}^\infty(-2t/\pi q)^b\gamma_j^{b-1}p_j/2 + o(1).$$

We have used $te^{-4R} < e^{-R}$ to change the limits of integration and have used again the fact that maximal mass terms are those in which each p_j occurs exactly once. Let

$$\zeta_+(s, A_j) = \sum_{n\neq 0} e^{2\pi nA_j}|n|^{-s}/2,$$

and

$$\zeta_-(s, A_j) = \sum_{n\neq 0} e^{2\pi nA_j}\operatorname{sign}(n)|n|^{-s}/2,$$

Then (51) may be rewritten:

$$\sum_a 2^{-n/2} i^{(2k-n+a)/2} \operatorname{tr} \tau L_a (2t\pi)^{(a-n)/2}$$

(52) $$\times \prod_j p_j \sum_{b=1}^{\infty} \{2\zeta_+(2b, A_j)(2t)^{2b} C(\nu_j/2)^{2b-1}$$
$$- 2\zeta_-(2b-1, A_j)(2t)^{2b-1} C(\nu_j/2)^{2b-2}\}.$$

Taking the limit as $t \to 0$, we obtain the following expression for the above trace:

$$2^{k-n/2}(-1)^k \operatorname{tr}_I \tau_I$$

(53) $$\cdot C\left(L(T\mathscr{D}_I) \wedge \left[\prod_j 2 \sum_{b=1}^{\infty} (\zeta_+(2b, A_j)(i\pi)^{-2b} (\nu_j/2)^{\wedge(2b-1)} \right.\right.$$
$$\left.\left. + \zeta_-(2b-1, A_j)(i\pi)^{1-2b} (\nu_j/2)^{2b-2} \right] \right),$$

where τ_I denotes Clifford multiplication by $i^{n/2-k}$ times the volume form of $\mathscr{D}_I$, and tr_I denotes the trace over $\bigwedge^{\cdot} T\mathscr{D}_I \otimes E$. When $A_j = 0$, $\zeta_+(s, A_j)$ reduces to the Riemann zeta function $\zeta(s)$, and $\zeta_-(s, A_j) = 0$. We recall the following well-known formula for $\zeta(2b)$:

$$\zeta(2b) = \pi^{2b} 2^{2b-1} (-1)^{b-1} B_{2b}(0)/(2b)!,$$

where $B_j(x)$ is the jth Bernoulli polynomial. Recall also that the stable L polynomial is the polynomial generated by the power series

$$L(x) = \frac{x/2}{\tanh(x/2)} = 1 + \sum_{b=1}^{\infty} B_{2b}(0) x^{2b}/(2b)!$$
$$= 1 - 2\sum_{b=1}^{\infty} \left(\frac{x}{2\pi i}\right)^{2b} \zeta(2b).$$

Define the twisted L polynomial $L(x, A_j)$ with values in $\operatorname{End}(E)$ by

(54) $$L(x, A_j) = 1 - 2\sum_{b=1}^{\infty} \left[\left(\frac{x}{2\pi i}\right)^{2b} \zeta(2b, A_j) - \left(\frac{x}{2\pi i}\right)^{2b-1} \zeta_-(2b-1, A_j) \right].$$

In fact, we may rewrite this as (see [15, p. 202])

$$L(x, A_j) - 1 = \sum_{b=1}^{\infty} B_{2b}\left(\frac{A_j}{i}\right) \frac{x^{2b}}{(2b)!} + B_{2b-1}\left(\frac{A_j}{i}\right) \frac{x^{2b-1}}{(2b-1)!},$$

with $B_j(x)$, $j > 1$ extended to matrix arguments in the usual manner. For $j = 1$, we define $B_1(A) = (A - 1/2) \circ (I - \pi_A)$, where π_A denotes projection onto the kernel of A. These Bernoulli polynomials can be intepreted as polynomials in Chern classes using [4, (B.3)].

Substituting these equations into the expression (53) we find that the product of the trace and the volume form of $\mathscr{D}_I$ is given by the term of top degree in

$$2^{n/2}\mathrm{tr}_E L(T\mathscr{D}_I) \wedge \prod_j (L(\nu_j, A_j) - 1)/\nu_j, \tag{55}$$

where tr_E denotes the trace over E of the $\mathrm{end}(E)$ valued class.

We summarize these computations and definitions with

Proposition 7.1.

$$\begin{aligned}\lim_{R\to\infty} &\mathrm{Tr}\, \tau e^{-t\Delta}(1 - \rho_{2R}) \\ &= 2^{n/2}\int_M L(TM) + \sum_I \int_{\mathscr{D}_I} 2^{n/2} L(T\mathscr{D}_I) \wedge \mathrm{tr}_E \prod_{i\in I}(L(\nu_i, A_i) - 1)/\nu_i.\end{aligned}$$

One could also define a twisted Todd polynomial $T(x, A_j)$ by replacing the coefficients of $\zeta(2b)$ in its expansion by combinations of $\zeta(2b, A_j)$. Instead, we will merely note that

$$L(x) - 1 = T(x) - 1 - x/2,$$

and express our results for the $\bar{\partial}$ and spinor Laplacians in terms of the twisted L polynomials. One has the following.

Proposition 7.2. *Let $\Delta^{S\otimes E}$ denote the spinor Laplacian with coefficients in a bundle E with a connection which is Dirac-good in the sense of* 4.3. *Let $\mathrm{Tr}_{S\otimes E}$ denote the trace over $L^2(M, S \otimes E)$ (S denotes the spinors). Then*

$$\begin{aligned}\lim_{R\to\infty} &\mathrm{Tr}_{S\otimes E}\tau e^{-t\Delta^{S\otimes E}}(1 - \rho_{2R}) \\ &= \int_M \mathrm{ch}(E) \wedge A(TM) + \sum_I \int_{\mathscr{D}_I} \mathrm{ch}(E) \wedge A(T\mathscr{D}_I) \\ &\qquad \wedge (\dim E)^{-1}\mathrm{tr}_E \prod_{i\in I}(L(\nu_i, A_i) - 1)/\nu_i.\end{aligned}$$

Let F be a holomorphic vector bundle with good connection (in the sense of 4.2). *Let Tr_{RF} denote the trace over the square integrable $(0, *)$ forms with coefficients in F, and $\Box$ the $\bar{\partial}$-Laplacian with coefficients in F. Then*

$$\lim_{R\to\infty} \mathrm{Tr}_{RF}\tau_R e^{-t\Box}(1-\rho_{2R})$$
$$= \int_M \mathrm{ch}(F)\wedge T(TM) + \sum_I \int_{\mathscr{D}_I} \mathrm{ch}(F)\wedge T(T\mathscr{D}_I)$$
$$\wedge (\dim F)^{-1}\mathrm{tr}_F \prod_{i\in I}(L(\nu_i, A_i) - 1/\nu_i.$$

Finally, we have

$$\lim_{R\to\infty} \mathrm{Tr}\,\tau^e e^{-t\Delta}(1-\rho_{2R}) = \int_M e(TM).$$

Proof. The demonstration of each of the above cases except the last is the same as for the signature complex. One merely substitutes the correct formula for γ_j and p_j corresponding to the complex at hand. For the auxillary bundles D and F, γ_j and p_j clearly do not change. One merely adds the extra curvature terms which lead to the Chern character contribution. For the Euler characteristic one applies Proposition 5.1(ii) to obtain the additional vanishing.

8. The commutator term

Finally, we are left to evaluate $\lim_{R\to\infty}\mathrm{Tr}[D, \rho_{2R}]\int_t^\infty De^{-s\Delta}ds\tau/2$. We use the construction for $e^{-s\Delta}$ given above. The support of ρ_{2R} may be covered with sets of the form $V_{I,\alpha}(R) = (\Delta_R^*)^{|I|}\times U_{I,\alpha}(R)$ as before. It is elementary to check that

$$\lim_{R\to\infty}\int_{V_{I,\alpha}(R)} \mathrm{tr}[D, \rho_{2R}]\int_t^\infty De^{-s\Delta}ds\,\tau/2T_m = 0, \quad \text{if } m - iA \neq 0.$$

We are thus left to compute the contribution of the $m - iA = 0$ terms. Let $T_S = \sum_{m-iA=0} T_m$. This projection is zero for a Dirac operator with coefficients in a bundle with Dirac-good connection; hence, its commutator term vanishes in the limit. As observed in Lemma 6.7, the $C(R_j)$ and $C(W_j)$ terms in $e^{-s\Delta}T_S$ vanish in the limit as $R\to\infty$. In order for this term to make a nontrivial contribution to the index, the $C(R_j)$ and $C(W_j)$ factors must come from $[D, \rho_{2R}]D$. Thus only those terms with $|I| = 1$ contribute. In the case of the signature operator, it is evident that for appropriate ρ_{2R} (θ_j-invariant), $[D, \rho_{2R}]DT_S$ does not contribute any factor of $C(W_j)$. Hence we have

Proposition 8.1. *For the signature operator with coefficients in a bundle with good connection and the Dirac operator with coefficients in a bundle with Dirac-good connection,*

$$\lim_{R\to\infty} \operatorname{Tr}[D, \rho_{2R}] \int_t^\infty De^{-s\Delta}\, ds\, \tau/2 = 0.$$

We now consider the contribution of the commutator term arising from a neighborhood of a single component $\mathscr{D}_i$ of the divisor $\mathscr{D}$. For the Gauss-Bonnet complex and the Dolbeault complex, we have respectively

$$2[D, \rho_{2R}]D = C(R_i)\widehat{C}(R_i)\widehat{C}(W_i)C(W_i)\partial\rho_{2R}/\partial r + \text{lower mass terms},$$

$$2[D, \rho_{2R}]D = C(\overline{Z}_i)\widehat{C}(\overline{Z}_i)\partial\rho_{2R}/\partial r + \text{lower mass terms}.$$

Let $\tau_i = C(R_i)\widehat{C}(R_i)\widehat{C}(W_i)C(W_i)\tau$ or $C(\overline{Z}_i)\widehat{C}(\overline{Z}_i)\tau$ depending on the complex in question. It is clear that $\partial/\partial r \int_t^\infty \operatorname{tr} e^{-s\Delta} T_S\, ds\, \tau_i$ is rapidly decreasing as R tends to ∞. Thus the contribution of the above term to the index is given by

$$\begin{aligned}&\lim_{R\to\infty} \int_{V_{i,\alpha}(R)} \partial\rho_{2R}/\partial r \int_t^\infty \operatorname{tr} e^{-s\Delta}\, ds\, \tau_i \\ &\quad = \lim_{R\to\infty} \int_{U_{i,\alpha}(R)} \int_t^\infty \operatorname{tr} \tau_i e^{-s\Delta_i} e^{-s/4}\, ds.\end{aligned}$$

Let $U_i(R) = \bigcup_\alpha U_{i,\alpha}(R)$, and let D_i denote the Dirac operator associated to $\mathscr{D}_i$ and our complex. Then for any function f_1 which is the sum of a constant and a rapidly decreasing function,

$$\begin{aligned}&\lim_{R\to\infty} \operatorname{Tr}[D, \rho_{2R}] \int_t^\infty De^{-s\Delta} f_1(s)\, ds\, \tau \\ &\quad = \lim_{R\to\infty} \sum_i \int_{U_i(2R)} \int_t^\infty \frac{1}{2} \operatorname{tr} \tau_i e^{-s\Delta_i} f_1(s) e^{-s/4}\, ds,\end{aligned} \tag{56}$$

and the contribution of the commutator term to the index is obtained by setting $f_1(s) = 1$. Let $F_1(t) = \int_t^\infty f_1(s) e^{-s/4}\, ds/4$. Recall from (32) that

$$\begin{aligned}&\operatorname{Index}(D_i) \\ &\quad = \lim_{R\to\infty} \int_{U_i(2R)} \operatorname{tr} \tau_i e^{-s\Delta_i} + \operatorname{tr}[D_i, \rho_{2R,i}] \int_s^\infty \frac{1}{2} D_i \tau_i e^{-u\Delta_i}\, du,\end{aligned} \tag{57}$$

where in an abuse of notation $\operatorname{index}(D_i)$ denotes the index of the restriction of the selfadjoint operator D_i to the appropriate subdomain (for example, even forms). Incorporating (57) in (56) gives

$$\begin{aligned}
&\lim_{R\to\infty} \operatorname{Tr}[D, \rho_{2R}] \int_t^\infty De^{-s\Delta} f_1(s)\, ds\, \tau/2 \\
&= \lim_{R\to\infty} \int_t^\infty \sum_i \Big\{ \operatorname{Index}(D_i) - \operatorname{Tr}[D_i, \rho_{2R,i}] \\
&\qquad\qquad \cdot \int_s^\infty D_i \tau_i e^{-u\Delta_i}\, du/2 \Big\} \frac{f_1(s)}{4e^{s/4} ds} \\
&= \lim_{R\to\infty} \sum_i \Big\{ \operatorname{Index}(D_i) F_1(t) - \int_t^\infty \operatorname{Tr}[D_i, \rho_{2R,i}] \\
&\qquad\qquad \cdot \int_s^\infty D_i \tau_i e^{-u\Delta_i} du F_1'(s)\, ds/2 \Big\} \\
&= \lim_{R\to\infty} \sum_i \Big\{ \operatorname{Index}(D_i) F_1(t) \\
&\qquad + \int_t^\infty \operatorname{Tr}[D_i, \rho_{2R,i}] D_i \tau_i e^{-s\Delta_i} (F_1(s) - F_1(t))\, ds/2 \Big\}.
\end{aligned}$$

Using the obvious extension of (56) to the operator D_i and the assumption that $t = t(R) \to 0$ as $R \to \infty$, we conclude that the commutator term (58)

$$\begin{aligned}
&\lim_{R\to\infty} \sum_i \int_{U_i(2R)} \int_t^\infty \operatorname{tr} \tau_i e^{-s\Delta_i} f_1(s) e^{-s/4}\, ds/4 \\
&= \sum_i \Big\{ \operatorname{Index}(D_i) F_1(0) \\
&\qquad + \lim_{R\to\infty} \sum_j \int_{U_{ij}(2R)} \int_t^\infty \operatorname{tr} \tau_{ij} e^{-s\Delta_{ij}} (F_1(s) - F_1(0) e^{-s/4})\, ds/4 \Big\},
\end{aligned}$$

where all ij subscripts denote objects associated to the divisor $\mathscr{D}_i \cap \mathscr{D}_j \subset \mathscr{D}_i$ in the same manner as the i-subscripted objects were associated to the divisor $\mathscr{D}_i$. The derivation of (56) may now be iterated, setting $f_1 = 1$, and $f_{i+1}(s) = (F_i(s) - F_i(0))$. We obtain the following expression for the commutator term.

Proposition 8.2.

$$\lim_{R\to\infty} \operatorname{Tr}[D, \rho_{2R}] \int_t^\infty De^{-s\Delta}\, ds\, \tau/2$$
$$= -\sum_I (-1)^{|I|} |I|! \operatorname{Index} D_I 4^{-n}$$
$$\times \int_0^\infty e^{-t_1/4} \int_0^{t_1} e^{-t_2/4} \int_0^{t_2} \cdots \int_0^{t_{|I|-1}} e^{-t_{|I|}/4} dt_{|I|} \cdots dt_1$$
$$= -\sum_I (-1)^{|I|} \operatorname{Index} D_I.$$

Here the factor of $|I|!$ counts the multiplicity of $\mathscr{D}_I$ in the above iteration. For example, $\mathscr{D}_{ij}$ is counted twice because it arises both as a boundary of $\mathscr{D}_i$ and a boundary of $\mathscr{D}_j$. The integral factor arises from the $F_i(0)$ factors defined above. Combining (32) and Propositions 8.2, 7.1, and 7.2 we obtain the following.

Theorem 8.3. *Let E be a unitary flat vector bundle with logarithmic connection along $\mathscr{D}$. Then the L^2-signature of (M, h, E) equals*

$$2^{n/2} \int_M L(TM) + 2^{n/2} \sum_I \int_{\mathscr{D}_I} L(T\mathscr{D}_I) \wedge \operatorname{tr}_E \prod_{i\in I} (L(\nu_i, A_i) - 1)/\nu_i.$$

Let F be a holomorphic vector bundle with a good connection in the sense of (4.2). *Then*

$$\chi_2(M, h, F) = \int_M \operatorname{Ch}(F) \wedge T(TM) - \sum_I (-1)^{|I|} \chi_2(\mathscr{D}_I', h_I, F_I)$$
$$+ \sum_I \int_{\mathscr{D}_I} \operatorname{Ch}(F) \wedge T(T\mathscr{D}_I)$$
$$\wedge (\dim F)^{-1} \operatorname{tr}_F \prod_{i\in I} (L(\nu_i, A_i) - 1)/\nu_i,$$

where F_I denotes the restriction of F to $\mathscr{D}_I'$. The L^2-Euler characteristic of (M, h) equals

$$\int_M e(TM) + \sum_I \int_{\mathscr{D}_I} e(T\mathscr{D}_I).$$

The index of the Dirac operator on M with coefficients in a bundle F with a Dirac-good connection (*see* 4.3) *is given by*

$$\int_M \operatorname{ch}(E) \wedge A(TM) + \sum_I \int_{\mathscr{D}_I} \operatorname{ch}(E) \wedge A(T\mathscr{D}_I)$$
$$\wedge (\dim E)^{-1} \operatorname{tr}_E \prod_{i\in I} (L(\nu_i, A_i) - 1)/\nu_i.$$

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